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ATIYAH CLASSES AND KONTSEVICH–DUFLO TYPE THEOREM FOR DG MANIFOLDS

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To the memory of Professor Qian Min

Abstract. In this expository paper, we survey some recent works on the Atiyah class and other characteristic classes of dg manifolds. In particular, we describe a Kontsevich–Duflo type theorem for dg manifolds: For every finite-dimensional dg manifold $(\mathcal{M}, Q)$, the composition $\mathrm{hkr} \circ (\mathrm{td}_{(\mathcal{M},Q)})^{1/2}$ is an isomorphism of Gerstenhaber algebras from $H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}), \mathcal{Q}))$ to $H^\bullet(\mathrm{tot}(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}} + \mathcal{Q})$ — the square root of the Todd class of the dg manifold $\mathrm{td}^{1/2}_{(\mathcal{M},Q)} \in \prod_{k\geqslant 0} H^k((\Omega^k(\mathcal{M}))^\bullet, \mathcal{Q})$ acts on $H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q})$, by contraction. The Duflo theorem of Lie theory and the Kontsevich theorem regarding the Hochschild cohomology of complex manifolds can both be derived as special cases of this Kontsevich–Duflo type theorem for dg manifolds. The paper ends with a discussion of extensions of this theorem.

Contents

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1. Introduction. The aim of this expository paper is to give an overview of the authors' recent work jointly with several collaborators — mainly Hsuan-Yi Liao — on the Atiyah class and Kontsevich–Duflo type theorems.

In 1997, Kontsevich revolutionized the field of deformation quantization [7] with his formality theorem [59]. See also [21, 24, 22, 26, 116, 119, 117, 130, 129] and references therein for further developments. Beyond deformation quantization, Kontsevich's formality construction found other important applications in several different areas of mathematics. One of them is the extension of the classical Duflo theorem. Given a finite-dimensional Lie algebra $\mathfrak{g}$, the Poincaré–Birkhoff–Witt symmetrization map $\operatorname{pbw}: S(\mathfrak{g}) \xrightarrow{\cong} \mathcal{U}(\mathfrak{g})$ is an isomorphism of $\mathfrak{g}$-modules. It induces an isomorphism $\operatorname{pbw}: S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\cong} \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ between subspaces of $\mathfrak{g}$-invariants. This isomorphism fails to intertwine the obvious multiplications on $S(\mathfrak{g})^{\mathfrak{g}}$ and $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$. Nevertheless, it can be modified so as to become an isomorphism of associative algebras. The Duflo element $J \in \widehat{S}(\mathfrak{g}^\vee)$ is the formal power series on $\mathfrak{g}$ defined by $J(x) = \det\big(\frac{1-\exp(-\operatorname{ad}_x)}{\operatorname{ad}_x}\big)$, for all $x \in \mathfrak{g}$. Considered as a formal linear differential operator on $\mathfrak{g}^\vee$ with constant coefficients, the square root of the Duflo element defines a transformation $J^{1/2}: S(\mathfrak{g}) \to S(\mathfrak{g})$. A remarkable theorem due to Duflo [37] asserts that the composition $\operatorname{pbw} \circ J^{1/2}: S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ is an isomorphism of associative algebras. Duflo's theorem generalizes a fundamental result of Harish-Chandra regarding the center of the universal enveloping algebra of a semi-simple Lie algebra. As an application of his formality construction, Kontsevich proposed a new proof of Duflo's theorem by means of the associative algebra structure carried by the tangent cohomology at a Maurer–Cartan element. Indeed, Kontsevich's approach [59] led to an extension of Duflo's theorem: for every finite-dimensional Lie algebra $\mathfrak{g}$, the map

$$\operatorname{pbw} \circ J^{1/2}: H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\cong} H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$$

is an isomorphism of graded associative algebras. The classical Duflo theorem is simply the isomorphism between the cohomology groups of degree 0. A detailed proof of the above extended Duflo theorem was given by Pevzner–Torossian [95] — see also [81, 82]. Furthermore, Kontsevich discovered a similar phenomenon in complex geometry [59]. Recall that the Hochschild cohomology groups $HH^\bullet(X)$ of a complex manifold X are defined as the groups $\operatorname{Ext}^{\bullet}_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. Gerstenhaber–Schack [41] derived an isomorphism of cohomology groups $\mathrm{hkr} : H^{\bullet}_{\mathrm{sheaf}}(X, \Lambda T_X) \xrightarrow{\cong} HH^{\bullet}(X)$ from the classical Hochschild–Kostant–Rosenberg map. This isomorphism fails to intertwine the multiplications on the two cohomologies but can be tweaked so as to produce an isomorphism of associative algebras. More precisely, Kontsevich [59] obtained the following theorem: the composition

$$\mathrm{hkr} \circ (\mathrm{td}_{T_X^{\mathbb{C}}/T_X^{0,1}})^{1/2} : H^{\bullet}_{\mathrm{sheaf}}(X, \Lambda T_X) \xrightarrow{\cong} HH^{\bullet}(X),$$

where $\mathrm{td}_{T_X^{\mathbb{C}}/T_X^{0,1}}$ denotes the Todd class of the Lie pair $(T_X^{\mathbb{C}}, T_X^{0,1})$ associated to a complex manifold X, is an isomorphism of associative algebras. The multiplications on $H^{\bullet}_{\mathrm{sheaf}}(X, \Lambda T_X)$ and $HH^{\bullet}(X)$ are respectively the wedge product and the Yoneda product. For a detailed proof of Kontsevich's theorem, see the work [17] by Calaque–Van den Bergh, who showed additionally that the map $\mathrm{hkr} \circ (\mathrm{td}_{T_X^{\mathbb{C}}/T_X^{0,1}})^{1/2}$ actually respects the *Gerstenhaber* algebra structures carried by the two cohomologies — see also [19, 18]. A related result was also proved by Dolgushev–Tamarkin–Tsygan [36, 35]. Note that, when X is a compact Kähler manifold, then the Todd class $\mathrm{td}_{T_X^{\mathbb{C}}/T_X^{0,1}}$ of the Lie pair $(T_X^{\mathbb{C}}, T_X^{0,1})$ coincides with the usual Todd class of X.

Thus, Kontsevich's formality construction revealed a hidden connection between complex geometry and Lie theory. Naturally, one would wonder whether a general framework encompassing both Lie algebras and complex manifolds as special cases could be developed in which a Kontsevich–Duflo type theorem would hold. This can indeed be achieved by considering *differential graded* (*dg*) *manifolds*.

By a dg manifold, we mean a $\mathbb{Z}$-graded manifold endowed with a homological vector field, i.e. a vector field Q of degree $+1$ satisfying $[Q, Q] = 0$. Homological vector fields first appeared in physics under the guise of BRST operators used to describe gauge symmetries. Since then, dg manifolds (a.k.a. Q-manifolds) have appeared frequently in the mathematical physics literature, e.g. in the AKSZ formalism [1, 105]. They also arise naturally in many situations in geometry, Lie theory, and mathematical physics. Standard examples of dg manifolds are:

(i) *Lie algebras* — Given a finite-dimensional Lie algebra $\mathfrak{g}$, we write $\mathfrak{g}[1]$ to denote the dg manifold having $C^\infty(\mathfrak{g}[1]) = \Lambda^{\bullet}\mathfrak{g}^{\vee}$ as its algebra of functions and the Chevalley–Eilenberg differential $Q = d_{\mathrm{CE}}$ as its homological vector field.

(ii) *Complex manifolds* — Given a complex manifold X, we write $T_X^{0,1}[1]$ to denote the dg manifold having $C^\infty(T_X^{0,1}[1]) = \Omega^{0,\bullet}(X)$ as its algebra of functions and the Dolbeault operator $Q = \bar{\partial}$ as its homological vector field.

In 1998, Shoikhet [111] proposed a conjecture, known as *Kontsevich–Shoikhet conjecture*, stating that a Kontsevich–Duflo type formula should hold for all finite-dimensional smooth dg manifolds. This conjecture was proved by the authors jointly with Liao [67] — see Theorem 4.6. Applying Theorem 4.6 to the classes of dg manifolds (i) and (ii) mentioned earlier, we recover the Kontsevich–Duflo theorem for Lie algebras (Theorem 4.12) and Kontsevich's theorem for complex manifolds (Theorem 4.16), respectively. Thus we fulfill our stated goal of conceiving a unified framework in which these two important theorems can be understood as one and the same phenomenon.

A key ingredient of the Kontsevich–Duflo type theorem for dg manifolds is the notion of Atiyah class of a dg manifold. As in the classical case of complex manifolds studied by Kapranov [55], the Atiyah 1-cocycle of a dg manifold is the binary bracket of an $L_\infty[1]$ algebra. Indeed, following Kapranov [55], one constructs a dg manifold $(T_\mathcal{M}, D^\nabla)$, which can be considered as the 'formal neighborhood Δ^∞ of the diagonal Δ' of the product dg manifold $(\mathcal{M}, Q) \times (\mathcal{M}, Q)$ [86]. The construction relies on the 'formal exponential map' (introduced in [65]) identifying $T_\mathcal{M}$ to a 'formal neighborhood of the diagonal' of $\mathcal{M} \times \mathcal{M}$ seen as $\mathbb{Z}$-graded manifolds. The homological vector field D^∇ on $T_\mathcal{M}$ is then obtained by pullback of the vector field (Q, Q) on $\mathcal{M} \times \mathcal{M}$ through the formal exponential map [86]. See [32, 106] for some further developments.

A comparison of the present work with that of Calaque–Rossi [16] is in order. In the introduction of their book [16], Calaque–Rossi claimed *"These lecture notes provide a self-contained proof of the Duflo isomorphism and its complex geometric analogue in a unified framework, and gives in particular a unifying explanation of the reason why the series $j(x)$ and its inverse appear."* Let us briefly summarize the actual content of [16] relevant to this matter. First, Calaque–Rossi gave a detailed proof — outlined earlier by Shoikhet in [113, 111] — of a Kontsevich–Duflo type theorem for 'Q-spaces' [16, Theorem 5.3]. A Q-space, according to [16], is a $\mathbb{Z}_2$-graded *vector space* endowed with a homological vector field. Next, following Shoikhet, they applied this result to the Q-space $(\mathfrak{g}[1], d_{\mathrm{CE}})$ so as to recover the Kontsevich–Duflo theorem for a finite-dimensional Lie algebra $\mathfrak{g}$ (Theorem 4.12) — see [113, Section 1.1.1.1] for a clean outline of the argument. (Shoikhet's paper [113] also investigates further properties of the Duflo map in terms of the cup-product property for Tsygan formality [118, 112].) On the complex geometry side of the story, given a complex manifold X, Calaque–Rossi constructed Fedosov 'resolutions' of the complexes $\Omega^{0,\bullet}(X)$, $\Omega^{0,\bullet}(X, T'_{\mathrm{poly}})$, and $\Omega^{0,\bullet}(X, D'_{\mathrm{poly}})$ as introduced in [15]. Then they applied [16, Theorem 5.3] 'fiberwise' to these 'resolutions.' Beyond that, however, further additional steps involving substantial work are required in order to complete the proof of the Kontsevich–Duflo theorem for complex manifolds (Theorem 4.16). Although Calaque–Rossi made use of the same result ([16, Theorem 5.3], the Kontsevich–Duflo type theorem for 'Q-spaces') to prove the two results for Lie algebras and for complex manifolds, they *did not* obtain these two results as special cases of a single generalized Kontsevich–Duflo theorem. Here, however, we describe *a unified Kontsevich–Duflo type theorem* (Theorem 4.6) valid for *all* finite-dimensional smooth dg *manifolds.* Then we specialize this result to two important classes of dg manifolds: $(\mathfrak{g}[1], d_{\mathrm{CE}})$ and $(T_X^{0,1}[1], \bar{\partial})$.

Kontsevich's theorem regarding the Hochschild cohomology of complex manifolds is closely related to homological mirror symmetry [58, 6]. It is natural to expect that Theorem 4.6 will have applications in mirror symmetry, for instance in matrix factorization [57, 38, 69, 96]. This will be investigated somewhere else.

We conclude the paper with an extension of our Kontsevich–Duflo type theorem for dg manifolds to the more general context of dg Lie algebroids and discuss several applications including a specialization of the theorem to Lie pairs. However, the reader only interested in the dg manifold case is encouraged to skip all sections pertaining to '-oids.'

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The paper is dedicated to the memory of Professor Qian Min. Professor Qian was the master thesis director of the second author at Peking University, introduced him to the beautiful subject of 'Deformation Quantization' [97, 98], taught him the importance of scientific integrity, and encouraged him to pursue doctoral studies in the United States. We trust that scientific integrity will eventually prevail in mathematics. Without Professor Qian's generous help, support, and encouragement, this work would never have been possible.

Notation. Some remarks concerning notation are necessary.

By default, in this paper, 'graded' means $\mathbb{Z}$-graded.

Given a module $\mathfrak{M}$ over a ring, the symbol $\widehat{S}(\mathfrak{M})$ denotes the $\mathfrak{m}$-adic completion of the symmetric algebra $S(\mathfrak{M})$, where $\mathfrak{m}$ is the ideal of $S(\mathfrak{M})$ generated by $\mathfrak{M}$.

Let $\mathcal{M}$ be a finite-dimensional graded manifold, let $(x_i)_{i\in\{1,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\dots,n\}}$ be the induced local frame of $T^\vee_{\mathcal{M}}$ regarded as fiberwise linear functions on $T_{\mathcal{M}}$.

We use the symbol $\mathbb{N}$ to denote the set of positive integers and the symbol $\mathbb{N}_0$ for the set of nonnegative integers. Given a multi-index $I = (i_1, i_2, \dots, i_n) \in \mathbb{N}_0^n$, we adopt the following multi-index notation:

$$I! = i_1! i_2! \cdots i_n!$$

$$|I| = i_1 + i_2 + \ldots + i_n$$

$$y^I = (y_1)^{i_1} (y_2)^{i_2} \cdots (y_n)^{i_n}$$

$$\partial_x^I = \underbrace{\partial_{x_1} \odot \ldots \odot \partial_{x_1}}_{i_1 \text{ factors}} \odot \underbrace{\partial_{x_2} \odot \ldots \odot \partial_{x_2}}_{i_2 \text{ factors}} \odot \cdots \odot \underbrace{\partial_{x_n} \odot \ldots \odot \partial_{x_n}}_{i_n \text{ factors}}$$

$$\partial_{\overleftarrow{x}}^I = \underbrace{\partial_{x_n} \odot \ldots \odot \partial_{x_n}}_{i_n \text{ factors}} \odot \underbrace{\partial_{x_{n-1}} \odot \ldots \odot \partial_{x_{n-1}}}_{i_{n-1} \text{ factors}} \odot \cdots \odot \underbrace{\partial_{x_1} \odot \ldots \odot \partial_{x_1}}_{i_1 \text{ factors}}$$

We use the symbol e_k to denote the multi-index all of whose components are equal to 0 except for the k-th which is equal to 1. Thus $\partial_x^{e_k} = \partial_{x_k}$.

The de Rham exterior differential d is an operator of degree $+1$ while the interior product i_X with a homogeneous vector field X of degree $|X|$ is an operator of degree $-1-|X|$. The element

$$dx_{i_1}\wedge\ldots\wedge dx_{i_p}\otimes y^J\frac{\partial}{\partial y_q}$$

of $\Omega^p(\mathcal{M},S^{|J|}(T^\vee_{\mathcal{M}})\otimes T_{\mathcal{M}})$ is of degree

$$\sum_{k=1}^{p}(1+|x_{i_k}|)+\sum_{k=1}^{n}J_k\,|y_k|-|y_q|\,,$$

where $|x_k|$ (resp. $|y_q|$) denotes the degree of the coordinate function x_k (resp. y_q).

2. Dg manifolds and dg vector bundles

2.1. $\mathbb{Z}$-graded manifolds. We use the symbol $\mathbb{K}$ to denote either of the fields $\mathbb{R}$ or $\mathbb{C}$. For a smooth manifold M, denote by $\mathcal{O}_M$, the sheaf of $\mathbb{K}$-valued C^∞-functions over M. A *$\mathbb{Z}$-graded manifold* $\mathcal{M}$ consists of a smooth manifold M (called the support of the graded manifold) and a sheaf $\mathcal{A}$ of $\mathbb{Z}$-graded commutative $\mathcal{O}_M$-algebras over M such that there exists a $\mathbb{Z}$-graded vector space V over $\mathbb{K}$ and a covering of M by open submanifolds $U\subset M$, and for every U in the covering, we have

$$\mathcal{A}|_U\cong C^\infty(U,\mathbb{K})\otimes_{\mathbb{K}}\widehat{S}(V^\vee),$$

where $\widehat{S}(V^\vee)$ denotes the $\mathbb{K}$-algebra of formal power series on V. We say that the graded manifold $\mathcal{M}$ is finite-dimensional if $\dim M$ and $\dim V$ are both finite. We use the notation $C^\infty(\mathcal{M})$ to denote the $\mathbb{Z}$-graded $\mathbb{K}$-algebra $\Gamma(M,\mathcal{A})$ of global sections of $(M,\mathcal{A})$. Let $\mathcal{I}$ denote the sheaf of ideals of the $\mathcal{O}_M$-algebra $\mathcal{A}$ characterized by the property $\mathcal{I}|_U\cong C^\infty(U,\mathbb{K})\otimes_{\mathbb{K}}\widehat{S}^{\geqslant 1}(V)$ for sufficiently small open subsets U of M. We refer the reader to [84, Chapter 2] and [21, 27] for a short introduction to $\mathbb{Z}$-graded manifolds and relevant references. For supermanifolds, see [121]. The word 'graded' means '$\mathbb{Z}$-graded' and, unless otherwise stated, the notation $|-|$ denotes the total degree of its argument.

A *morphism* $\phi:\mathcal{M}\to\mathcal{N}$ from a $\mathbb{Z}$-graded manifold $\mathcal{M}:=(M,\mathcal{A})$ to a $\mathbb{Z}$-graded manifold $\mathcal{N}:=(N,\mathcal{B})$ consists of a differentiable map $f:M\to N$ together with a morphism of sheaves of $\mathbb{Z}$-graded algebras $\psi:f^*\mathcal{B}\to\mathcal{A}$ continuous with respect to the $\mathcal{I}$-adic topology. We also use the notation $\psi:=\phi^*$. It is clear that a morphism of $\mathbb{Z}$-graded manifolds $\phi:\mathcal{M}\to\mathcal{N}$ induces a morphism of $\mathbb{Z}$-graded algebras $\phi^*:C^\infty(\mathcal{N})\to C^\infty(\mathcal{M})$ on global sections.

This defines the category of $\mathbb{Z}$-graded manifolds.

Any $\mathbb{Z}$-graded vector bundle over M determines a $\mathbb{Z}$-graded manifold in a natural way. Let

$$E=E^{-m}\oplus\ldots\oplus E^n \tag{1}$$

be a $\mathbb{Z}$-graded vector bundle over M. Then $(M,\mathcal{A})$, where $\mathcal{A}|_U=\Gamma(U,\widehat{S}(E^\vee))$ for all open submanifolds $U\subseteq M$, is a $\mathbb{Z}$-graded manifold. It is clear that $\mathbb{Z}$-graded vector bundles and morphisms of $\mathbb{Z}$-graded vector bundles form a category.

We have a functor

$$\begin{aligned} (\mathbb{Z}\text{-graded vector bundles}) &\longrightarrow (\mathbb{Z}\text{-graded manifolds}) \\ (M,E) &\longmapsto (M, \mathcal{A}|_U = \Gamma(U, \widehat{S}(E^\vee))\,. \end{aligned} \tag{2}$$

Proposition 2.1. *The functor* (2) *is an equivalence of categories.*

In the supermanifold (i.e. $\mathbb{Z}_2$-graded) case, Proposition 2.1 is known as Batchelor's theorem — see [21, 33].

Definition 2.2. A *dg manifold* is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a homological vector field, i.e. a degree +1 derivation Q on $C^\infty(\mathcal{M})$ satisfying $[Q,Q]=0$.

A dg manifold $(\mathcal{M},Q)$ is said to be of amplitude $[-m,n]$ if $\mathcal{M}$ is the $\mathbb{Z}$-graded manifold associated with a graded vector bundle (1) concentrated in degrees $-m$ to n under the functor (2).

Homological vector fields first appeared in physics under the guise of BRST operators used to describe gauge symmetries. Since then, dg manifolds (a.k.a. Q-manifolds) have appeared frequently in the mathematical physics literature, e.g. in the AKSZ formalism [1, 27].

Let us describe three classes of standard examples of dg manifolds:

Example 2.3. Given a finite-dimensional Lie algebra $\mathfrak{g}$, we write $\mathfrak{g}[1]$ to denote the dg manifold having $C^\infty(\mathfrak{g}[1]) = \Lambda^\bullet \mathfrak{g}^\vee$ as its algebra of functions and the Chevalley–Eilenberg differential $Q = d_{\mathrm{CE}}$ as its homological vector field. This construction admits an 'up to homotopy' version: Given a $\mathbb{Z}$-graded finite-dimensional vector space $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$, the graded manifold $\mathfrak{g}[1]$ is a dg manifold, i.e. admits a homological vector field, if and only if the graded vector space $\mathfrak{g}$ admits a structure of curved L_∞ algebra.

Example 2.4. Given a smooth manifold M, we write $T_M[1]$ to denote the dg manifold having $C^\infty(T_M[1]) = \Omega^\bullet(M)$ as its algebra of functions and the de Rham differential $Q = d_{\mathrm{dR}}$ as its homological vector field. Likewise, given a complex manifold X, we write $T_X^{0,1}[1]$ to denote the dg manifold having $C^\infty(T_X^{0,1}[1]) = \Omega^{0,\bullet}(X)$ as its algebra of functions and the Dolbeault operator $Q = \bar\partial$ as its homological vector field.

Example 2.5. Given a smooth section s of a vector bundle $E \to M$, we write $E[-1]$ to denote the dg manifold having $C^\infty(E[-1]) = \Gamma(\Lambda^{-\bullet} E^\vee)$ as algebra of functions and $Q = \iota_s$, the interior product with s, as homological vector field. This dg manifold can be thought of as a smooth model for the (possibly singular) intersection of s with the zero section of the vector bundle E, and is often called a 'derived intersection', or a *quasi-smooth derived manifold* [8].

Both situations in Example 2.4 are special instances of Lie algebroids.

According to a theorem of Vaĭntrob [120], given a $\mathbb{K}$-vector bundle A over a smooth manifold M, the homological vector fields on $A[1]$ are in one-one correspondence with the Lie algebroid structures on A. Indeed, the homological vector field on $A[1]$ is the Chevalley–Eilenberg differential $d_A : \Gamma(\Lambda^\bullet A^\vee) \to \Gamma(\Lambda^{\bullet+1} A^\vee)$. In other words, we have the following

Proposition 2.6 ([120]). *Dg manifolds of amplitude* $[-1,-1]$ *are in one-one correspondence with Lie algebroids.*

More generally, the dg manifolds of amplitude $[-n,-1]$ can be thought of as *Lie n-algebroids* [125, 127, 107, 43, 11, 51, 104, 54] and [109, Letters 7 and 8]. They can be considered as the infinitesimal counterparts of higher groupoids — see [42, 47, 3, 63, 108, 87, 110]. On the other hand, dg manifolds of amplitude $[1,n]$ are derived manifolds [8]. Hence a general dg manifold of amplitude $[-m,n]$ can encode both stacky and derived singularities in differential geometry.

A *morphism of dg manifolds* from $(\mathcal{M},Q)$ to $(\mathcal{M}',Q')$ is a morphism of $\mathbb{Z}$-graded manifolds $\phi:\mathcal{M}\to\mathcal{M}'$ such that the induced map on global sections $\phi^*:C^\infty(\mathcal{M}')\to C^\infty(\mathcal{M})$ is a morphism of differential graded algebras, i.e. $\phi^*\circ Q'=Q\circ\phi^*$.

Definition 2.7. A morphism of dg manifolds $\phi:(\mathcal{M},Q)\to(\mathcal{M}',Q')$ is said to be a *quasi-isomorphism* if the induced morphism of differential graded algebras of global sections

$$\phi^*:(C^\infty(\mathcal{M}'),Q')\to(C^\infty(\mathcal{M}),Q)$$

is a quasi-isomorphism.

2.2. Formal exponential map

Definition 2.8. Let $\mathcal{E}\to\mathcal{M}$ be a vector bundle in the category of $\mathbb{Z}$-graded manifolds. A connection on $\mathcal{E}\to\mathcal{M}$ is a $\mathbb{K}$-linear map

$$\nabla:\Gamma(T_{\mathcal{M}})\otimes\Gamma(\mathcal{E})\to\Gamma(\mathcal{E})$$

of degree 0 such that

$$\nabla_{fX}S=f\nabla_XS,$$
$$\nabla_X(fS)=X(f)S+(-1)^{|X||f|}f\nabla_XS,$$

for all $f\in C^\infty(\mathcal{M})$, $X\in\Gamma(T_{\mathcal{M}})$ and $S\in\Gamma(\mathcal{E})$.

The covariant differential associated to a connection ∇ is the map

$$d^\nabla:\Omega^\bullet(\mathcal{M},\mathcal{E})\to\Omega^{\bullet+1}(\mathcal{M},\mathcal{E})$$

of degree $+1$ satisfying

$$\nabla_XS=\iota_X(d^\nabla S),$$

for all $X\in\Gamma(T_{\mathcal{M}})$ and $S\in\Gamma(\mathcal{E})$, and

$$d^\nabla(\alpha\cdot S)=d\alpha\cdot S+(-1)^{|\alpha|}\alpha\cdot d^\nabla S,$$

for all homogeneous $\alpha\in\Omega(\mathcal{M})$ and $S\in\Omega(\mathcal{M},\mathcal{E})$.

The curvature of a connection ∇ is the 2-form $R^\nabla\in\Omega^2(\mathcal{M},\operatorname{End}(\mathcal{E}))$ defined by

$$R^\nabla(X,Y)=(-1)^{|Y|-1}\big\{\nabla_X\nabla_Y-(-1)^{|X||Y|}\nabla_Y\nabla_X-\nabla_{[X,Y]}\big\},$$

for all homogeneous $X,Y\in\Gamma(T_{\mathcal{M}})$ so that $(d^\nabla)^2=R^\nabla$.

A connection on $T_{\mathcal{M}}$ is called an *affine connection* on $\mathcal{M}$. The torsion of an affine connection ∇ is the (1,2)-tensor $T^\nabla:T_{\mathcal{M}}\otimes T_{\mathcal{M}}\to T_{\mathcal{M}}$ of degree 0 defined by

$$T^\nabla(X,Y)=\nabla_XY-(-1)^{|X||Y|}\nabla_YX-[X,Y],$$

for all homogeneous $X, Y \in \Gamma(T_{\mathcal{M}})$. Given any affine connection, one can define its opposite affine connection ∇^{op}, given by

$$\nabla^{\mathrm{op}}_X Y = \nabla_X Y - T(X,Y) = [X,Y] + (-1)^{|X||Y|}\nabla_Y X. \tag{3}$$

The average $\frac{1}{2}(\nabla + \nabla^{\mathrm{op}})$ is a torsion-free affine connection. This shows that torsion-free affine connections always exist on $\mathbb{Z}$-graded manifolds.

For an *ordinary* manifold M, an affine connection ∇ determines an exponential map

$$\exp^{\nabla} : T_M \to M \times M, \tag{4}$$

which is a local diffeomorphism of fiber bundles

$$\begin{array}{ccc} T_M & \xrightarrow{\exp^{\nabla}} & M \times M \\ \pi \downarrow & & \downarrow \mathrm{pr}_1 \\ M & \xrightarrow[\mathrm{id}]{} & M \end{array} \tag{5}$$

from a neighborhood of the zero section of T_M to a neighborhood of the diagonal Δ in $M \times M$. The space of fiberwise distributions on the vector bundle $\pi : T_M \to M$ with support the zero section can be identified, as a $C^\infty(M)$-coalgebra, to $\Gamma(S(T_M))$, while the space of fiberwise distributions on the fiber bundle $\mathrm{pr}_1 : M \times M \to M$ with support the diagonal Δ can be identified, as a $C^\infty(M)$-coalgebra, to $\mathcal{D}(M)$. Pushing distributions forward through the exponential map (4), we obtain an isomorphism of $C^\infty(M)$-coalgebras

$$\mathrm{pbw}^{\nabla} : \Gamma(ST_M) \to \mathcal{D}(M),$$

which we call *Poincaré–Birkhoff–Witt map*. In other words, pbw^{∇} is the fiberwise ∞-order jet (along the zero section) of the exponential map $\exp^{\nabla} : T_M \to M \times M$ associated to the connection ∇ — whence the terminology 'formal exponential map.'

More precisely, we have

$$\mathrm{pbw}^{\nabla}(X_0 \odot \ldots \odot X_k)(f) = \left.\frac{d}{dt_0}\right|_0 \left.\frac{d}{dt_1}\right|_0 \cdots \left.\frac{d}{dt_k}\right|_0 f(\exp(t_0X_0 + t_1X_1 + \ldots + t_kX_k))$$

for all $X_0, X_1, \ldots, X_k \in \Gamma(T_M)$ and $f \in C^\infty(M)$.

Remark 2.9. The inverse map $(\mathrm{pbw}^{\nabla})^{-1} : \mathcal{D}(M) \to \Gamma(ST_M)$ is also known as a complete symbol map. It plays an important role in quantizing the cotangent symplectic manifold $T^{\vee}_M$ [46].

It turns out that the map pbw^{∇} admits a nice recursive characterization [61, 62], which can be described in a purely algebraic way:

Theorem 2.10 ([61, 62]). *The map* pbw^{∇} *is entirely determined by the identities* $\mathrm{pbw}^{\nabla}(1) = 1$, $\mathrm{pbw}^{\nabla}(X) = X$, *and* $\mathrm{pbw}^{\nabla}(X^{n+1}) = X \cdot \mathrm{pbw}^{\nabla}(X^n) - \mathrm{pbw}^{\nabla}(\nabla_X(X^n))$, *for all* $X \in \Gamma(T_M)$ *and* $n \in \mathbb{N}$, *where* X^n *stands for the symmetric product* $X \odot X \odot \ldots \odot X$ *of* n *copies of* X.

Such a purely algebraic description extends readily to the context of $\mathbb{Z}$-*graded* manifolds.

Definition 2.11 ([65]). Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold and let $\mathcal{D}(\mathcal{M})$ denote its algebra of differential operators. The formal exponential map associated to an affine connection ∇ on $\mathcal{M}$ is the morphism of left $C^\infty(\mathcal{M})$-modules

$$\mathrm{pbw}^\nabla : \Gamma(ST_{\mathcal{M}}) \to \mathcal{D}(\mathcal{M}), \tag{6}$$

inductively defined by the relations

$$\mathrm{pbw}^\nabla(f) = f, \quad \forall\, f \in C^\infty(\mathcal{M})$$
$$\mathrm{pbw}^\nabla(X) = X, \quad \forall\, X \in \Gamma(T_{\mathcal{M}}),$$

and, for all $n \in \mathbb{N}$ and any homogeneous elements $X_0, \ldots, X_n$ of $\Gamma(T_{\mathcal{M}})$,

$$\mathrm{pbw}^\nabla(X_0 \odot \ldots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} \epsilon_k \left\{ X_k \cdot \mathrm{pbw}^\nabla(X^{\{k\}}) - \mathrm{pbw}^\nabla(\nabla_{X_k}(X^{\{k\}})) \right\}, \tag{7}$$

where $\epsilon_k = (-1)^{|X_k|(|X_0|+\ldots+|X_{k-1}|)}$ and $X^{\{k\}} = X_0 \odot \ldots \odot X_{k-1} \odot X_{k+1} \odot \ldots \odot X_n$.

The algebra $\mathcal{D}(\mathcal{M})$ of differential operators on $\mathcal{M}$ admits a natural filtration by the order of the differential operators — see [102, 88, 93]. It is straightforward to prove by induction on n that

$$\mathrm{pbw}^\nabla(X_1 \odot \ldots \odot X_n) \in \mathcal{D}^{\leqslant n}(\mathcal{M}),$$

for all $n \in \mathbb{N}$ and $X_1, \ldots, X_n \in \Gamma(T_{\mathcal{M}})$. In other words, the map pbw^∇ respects the natural filtrations on $\Gamma(ST_{\mathcal{M}})$ and $\mathcal{D}(\mathcal{M})$. By Gr, we denote the functor which takes a filtered vector space

$$\ldots \subset \mathscr{A}^{\leqslant k-1} \subset \mathscr{A}^{\leqslant k} \subset \mathscr{A}^{\leqslant k+1} \subset \ldots$$

to the associated graded vector space

$$\mathrm{Gr}(\mathscr{A}) = \bigoplus_k \frac{\mathscr{A}^{\leqslant k}}{\mathscr{A}^{\leqslant k-1}}\,.$$

It is well known [102] that the symmetrization map

$$\mathrm{sym} : \Gamma(S^\bullet(T_{\mathcal{M}})) \to \mathrm{Gr}^\bullet(\mathcal{D}(\mathcal{M})),$$

defined by

$$X_1 \odot \ldots \odot X_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma; X)\ X_{\sigma(1)} \cdots X_{\sigma(n)},$$

for all homogeneous $X_1, \ldots, X_n \in \Gamma(T_{\mathcal{M}})$, is an isomorphism of graded vector spaces. Here, $\epsilon(\sigma; X)$ denotes the Koszul sign of the permutation σ of the homogeneous elements $X_1, \ldots, X_k \in \Gamma(T_{\mathcal{M}})$. It is clear that

$$\mathrm{Gr}(\mathrm{pbw}^\nabla) = \mathrm{sym}\,.$$

Note that both $\Gamma(ST_{\mathcal{M}})$ and $\mathcal{D}(\mathcal{M})$ are coalgebras over $\mathcal{R} := C^\infty(\mathcal{M})$.

The comultiplication

$$\Delta : \mathcal{D}(\mathcal{M}) \to \mathcal{D}(\mathcal{M}) \otimes_{\mathcal{R}} \mathcal{D}(\mathcal{M}) \tag{8}$$

is characterized by the identities

$$\begin{aligned}\Delta(1) &= 1 \otimes 1; \\ \Delta(X) &= 1 \otimes X + X \otimes 1, \quad \forall X \in \Gamma(T_{\mathcal{M}}); \\ \Delta(U \cdot V) &= \Delta(U) \cdot \Delta(V), \quad \forall U, V \in \mathcal{D}(\mathcal{M}),\end{aligned} \tag{9}$$

and is compatible with the natural filtration of $\mathcal{D}(\mathcal{M})$. Here the symbol $\otimes_{\mathcal{R}}$ in (8) denotes the tensor product of left $\mathcal{R}$-modules, the symbol 1 denotes the constant function 1, and the symbol $\cdot$ denotes the multiplication in $\mathcal{D}(\mathcal{M})$. See [131, equation (15) and the remark following Definition 3.1] for the precise meaning of equation (9).

More explicitly, for all homogeneous elements $X_1, \dots, X_k \in \Gamma(T_{\mathcal{M}})$, we have

$$\Delta(X_1 \cdots X_k) = 1 \otimes (X_1 \cdots X_k) + (X_1 \cdots X_k) \otimes 1 + \sum_{\substack{p+q=k \\ p,q \in \mathbb{N}}} \sum_{\sigma \in \mathfrak{S}_p^q} \epsilon(\sigma; X)\ (X_{\sigma(1)} \cdots X_{\sigma(p)}) \otimes (X_{\sigma(p+1)} \cdots X_{\sigma(k)})$$

where $\mathfrak{S}_p^q$ denotes the space of (p, q)-shuffles.

Similarly, the comultiplication

$$\Delta : \Gamma(ST_{\mathcal{M}}) \to \Gamma(ST_{\mathcal{M}}) \otimes_{\mathcal{R}} \Gamma(ST_{\mathcal{M}})$$

is given by

$$\Delta(X_1 \odot \dots \odot X_k) = 1 \otimes_{\mathcal{R}} (X_1 \odot \dots \odot X_k) + (X_1 \odot \dots \odot X_k) \otimes_{\mathcal{R}} 1 + \sum_{\substack{p+q=k \\ p,q \in \mathbb{N}}} \sum_{\sigma \in \mathfrak{S}_p^q} \epsilon(\sigma; X)\ (X_{\sigma(1)} \odot \dots \odot X_{\sigma(p)}) \otimes_{\mathcal{R}} (X_{\sigma(p+1)} \odot \dots \odot X_{\sigma(k)}).$$

The symbol $\odot$ denotes the symmetric product in $\Gamma(ST_{\mathcal{M}})$.

THEOREM 2.12 ([65]).

- *The formal exponential map* (6) *is a well-defined isomorphism of filtered left* $C^\infty(\mathcal{M})$*-modules.*
- *It is also an isomorphism of filtered coalgebras over* $C^\infty(\mathcal{M})$*.*

REMARK 2.13. The formal exponential map (6) induces a deformation quantization of the $\mathbb{Z}$-graded symplectic manifold $T^\vee_{\mathcal{M}}$. Since pbw^∇ is a morphism of $C^\infty(\mathcal{M})$-modules, it is also called *normal ordering quantization map.* Many quantization maps $\Gamma(ST_{\mathcal{M}}) \to \mathcal{D}(\mathcal{M})$ have appeared in the literature — for instance, see [45] and references therein. The significant feature of the map (6) appearing in Definition 2.11 is that it can be computed explicitly by iteration. This is crucial in the discussion that follows.

2.3. Fedosov dg manifolds. Theorem 2.12 has a number of important applications in graded geometry. First, we describe its application in the construction of Fedosov dg manifolds. Fedosov dg manifolds are closely related to formal geometry [40]. Intuitively, given a $\mathbb{Z}$-graded manifold $\mathcal{M}$, a Fedosov dg manifold 'for $\mathcal{M}$' is a dg manifold of the form $(T_{\mathcal{M}}[1] \oplus T_{\mathcal{M}}, D)$ which encodes the formal neighborhood of 'each point of $\mathcal{M}$' and whose algebra of functions is a resolution of $C^\infty(\mathcal{M})$.

Let $\mathcal{M}$ be a finite-dimensional $\mathbb{Z}$-graded manifold with support M. Choose an affine connection ∇ on $\mathcal{M}$. Let $\mathrm{pbw}^\nabla : \Gamma(ST_\mathcal{M}) \to \mathcal{D}(\mathcal{M})$ be the corresponding formal exponential map, i.e. the PBW map associated to ∇, as defined in Definition 2.11. Multiplication in $\mathcal{D}(\mathcal{M})$ from the left by elements of $\Gamma(T_\mathcal{M})$ defines an infinitesimal action of $T_\mathcal{M}$ on the $C^\infty(\mathcal{M})$-coalgebra $\mathcal{D}(\mathcal{M})$ by coderivations. Pulling back this infinitesimal action through pbw^∇, we obtain an infinitesimal $T_\mathcal{M}$-action on $\Gamma(S(T_\mathcal{M}))$ by coderivations. The latter defines a *flat* connection $\nabla^\natural$ on $S(T_\mathcal{M})$:

$$\nabla^\natural_X S := (\mathrm{pbw}^\nabla)^{-1}(X \cdot \mathrm{pbw}^\nabla(S)) \tag{10}$$

for all $X \in \Gamma(T_\mathcal{M})$ and $S \in \Gamma(ST_\mathcal{M})$. This in turn induces a flat connection on the dual bundle $\widehat{S}(T^\vee_\mathcal{M})$, denoted by the same symbol $\nabla^\natural$. We denote the corresponding Chevalley–Eilenberg differential by

$$D = d^{\nabla^\natural} : \Omega^\bullet(\mathcal{M}, \widehat{S}(T^\vee_\mathcal{M})) \to \Omega^{\bullet+1}(\mathcal{M}, \widehat{S}(T^\vee_\mathcal{M})). \tag{11}$$

Since the covariant derivative

$$\nabla^\natural_X : \Gamma(ST_\mathcal{M}) \to \Gamma(ST_\mathcal{M})$$

is a *co*derivation of $\Gamma(ST_\mathcal{M})$ for all $X \in \Gamma(T_\mathcal{M})$, the covariant derivative

$$\nabla^\natural_X : \Gamma(\widehat{S}(T^\vee_\mathcal{M})) \to \Gamma(\widehat{S}(T^\vee_\mathcal{M}))$$

is a derivation of the completed symmetric algebra $\Gamma(\widehat{S}(T^\vee_\mathcal{M}))$. Therefore D is a derivation of $\Omega^\bullet(\mathcal{M}, \widehat{S}(T^\vee_\mathcal{M}))$ of degree $(+1)$ satisfying $D^2 = 0$. Therefore it is a homological vector field on $T_\mathcal{M}[1] \oplus T_\mathcal{M}$. In other words, $(T_\mathcal{M}[1] \oplus T_\mathcal{M}, D)$ is a dg manifold (with support M). Note that, by construction, the homological vector field D depends on the choice of an affine connection ∇ on $\mathcal{M}$.

Alternatively, the homological vector field D can be constructed explicitly by way of Fedosov's iterative method. We need to introduce some notation.

Let $(x_i)_{i\in\{1,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\dots,n\}}$ be the induced local frame of $T^\vee_\mathcal{M}$ regarded as fiberwise linear functions on $T_\mathcal{M}$. Define

$$\delta : \Omega^p(\mathcal{M}, S^q T^\vee_\mathcal{M}) \to \Omega^{p+1}(\mathcal{M}, S^{q-1} T^\vee_\mathcal{M})$$

and

$$\delta^{-1} : \Omega^p(\mathcal{M}, S^q T^\vee_\mathcal{M}) \to \Omega^{p-1}(\mathcal{M}, S^{q+1} T^\vee_\mathcal{M})$$

by

$$\delta = \sum_{i=1}^n dx_i \otimes \frac{\partial}{\partial y_i} \qquad \text{and} \qquad \delta^{-1} = \frac{1}{p+q} \sum_{i=1}^n \iota_{\partial/\partial x_i} \otimes y_i$$

or, more precisely,

$$\delta(\omega \otimes f) = \sum_{i=1}^n (-1)^{|\partial/\partial y_i||\omega|}\ dx_i \wedge \omega \otimes \frac{\partial}{\partial y_i}(f)$$

and

$$\delta^{-1}(\omega \otimes f) = \frac{1}{p+q} \sum_{i=1}^n (-1)^{|y^i||\omega|} \iota_{\partial/\partial x_i} \omega \otimes y_i \cdot f$$

for all homogeneous $\omega \in \Omega^p(\mathcal{M})$ and for all $f \in \Gamma(S^q T^\vee_{\mathcal{M}})$. It is not difficult to check that the operators δ and δ^{-1} are well defined, i.e. independent of the choice of local coordinates, and can be extended to $\Omega^\bullet(\mathcal{M}, \operatorname{End}(\widehat{S}(T^\vee_{\mathcal{M}})))$. The operator δ has degree $+1$ while the operator δ^{-1} has degree -1. Note that the operators δ and δ^{-1} are *not* inverse of each other.

Proposition 2.14 ([65]). *Let $\mathcal{M}$ be a finite-dimensional $\mathbb{Z}$-graded manifold, and ∇ a* torsion-free *affine connection on $\mathcal{M}$. The homological vector field D of equation* (11) *decomposes as the sum*

$$D = -\delta + d^\nabla + X^\nabla,$$

where

$$X^\nabla = \sum_{i=1}^{n} \sum_{\substack{J \in \mathbb{N}_0^n \\ |J| \geqslant 2}} \sum_{k=1}^{n} X^i_{J,k}\, dx_i \otimes y^J \frac{\partial}{\partial y_k}$$

is an element of degree $(+1)$ *in $\Omega^1(\mathcal{M}, \widehat{S}^{\geqslant 2}(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}})$ satisfying $\delta^{-1}(X^\nabla) = 0$. This element X^∇ can be thought of as a 1-form on $\mathcal{M}$ valued in fiberwise formal vector fields on $T_{\mathcal{M}}$ and hence acts by derivation on $\Omega^\bullet(\mathcal{M}, \widehat{S}(T^\vee_{\mathcal{M}}))$, the differential forms on $\mathcal{M}$ valued in fiberwise formal functions on $T_{\mathcal{M}}$. Indeed, X^∇ is a vector field on $T_{\mathcal{M}}[1] \oplus T_{\mathcal{M}}$.*

Remark 2.15. The vector field X^∇ on $T_{\mathcal{M}}[1] \oplus T_{\mathcal{M}}$ can also be constructed explicitly [106] by way of an iterative method due to Fedosov [40]. Fedosov's iterative method is a standard and very effective procedure for assembling global objects out of local building blocks. The version of Fedosov's method relevant to our purpose is the one which is applicable to arbitrary ordinary smooth manifolds and was developed by Emmrich–Weinstein [39] and later refined by Dolgushev [34]. Given an (ordinary) smooth manifold M, Dolgushev obtained resolutions of $C^\infty(M)$, $\mathcal{T}^\bullet_{\mathrm{poly}}(M)$ and $\mathcal{D}^\bullet_{\mathrm{poly}}(M)$, which he employed to globalize Kontsevich's formality theorem from $\mathbb{R}^d$ to M. The construction relies on the choice of a torsion-free affine connection on M.

Remark 2.16. An analogue, in the context of $\mathbb{Z}$-graded manifolds, of Dolgushev–Fedosov resolution can also be found in Cattaneo–Felder [23, Appendix]. However, rather than resolving the entire algebra of functions on the graded manifold at hand, Cattaneo–Felder consider the underlying $\mathbb{Z}$-graded manifold as a $\mathbb{Z}$-graded vector bundle over an ordinary smooth manifold M, and resolve only the subalgebra of functions on the base manifold M.

Consider the linear map $\sigma : \Omega^\bullet(\mathcal{M}, \widehat{S}(T^\vee_{\mathcal{M}})) \to C^\infty(\mathcal{M})$ of degree 0 characterized by the relations

$$\begin{gathered} \sigma(f \otimes 1) = f, \quad \forall\, f \in C^\infty(\mathcal{M}); \\ \sigma(\omega \otimes y^J) = 0, \quad \forall\, \omega \in \Omega^{\geqslant 1}(\mathcal{M}),\ \forall\, J \in \mathbb{N}_0^n; \\ \sigma(f \otimes y^J) = 0, \quad \forall\, f \in \Omega^0(\mathcal{M}),\ \forall\, J \in \mathbb{N}_0^n \text{ such that } |J| \geqslant 1. \end{gathered} \tag{12}$$

Lemma 2.17. *There exists a unique map*

$$\breve{\tau} : C^\infty(\mathcal{M}) \to \Omega^0(\mathcal{M}, \widehat{S}(T^\vee_{\mathcal{M}}))$$

of degree 0 *satisfying $\sigma \circ \breve{\tau} = \mathrm{id}_{C^\infty(\mathcal{M})}$ and $D \circ \breve{\tau} = 0$.*

One can easily check that $\breve{\tau}$ is a morphism of algebras. Obviously, $\breve{\tau}$ is a chain map from $C^\infty(\mathcal{M})$ seen as a complex concentrated in degree 0 to $(\Omega^\bullet(\mathcal{M}, \widehat{S}(T^\vee_\mathcal{M})), D)$.

PROPOSITION 2.18 ([65]). *The chain map $\breve{\tau}$ defines a quasi-isomorphism of dg manifolds from $(T_\mathcal{M}[1] \oplus T_\mathcal{M}, D)$ to $(\mathcal{M}, 0)$.*

As in the case for the operator D, the map $\breve{\tau}$ may also be obtained directly from the formal exponential map $\operatorname{pbw}^\nabla$.

PROPOSITION 2.19 ([65]). *Let $(x_j)_{j\in\{1,2,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,2,\dots,n\}}$ be the induced local frame of $T^\vee_\mathcal{M}$ regarded as fiberwise linear functions on $T_\mathcal{M}$. For all $f \in C^\infty(\mathcal{M})$, we have*

$$\breve{\tau}(f) = \sum_{I\in\mathbb{N}_0^n} \frac{1}{I!} y^I \otimes \operatorname{pbw}^\nabla\big(\underset{\leftarrow}{\partial_x^I}\big)(f),$$

where

$$\underset{\leftarrow}{\partial_x^I} = \underbrace{\partial_{x_n} \odot \dots \odot \partial_{x_n}}_{i_n \text{ factors}} \odot \underbrace{\partial_{x_{n-1}} \odot \dots \odot \partial_{x_{n-1}}}_{i_{n-1} \text{ factors}} \odot \dots \odot \underbrace{\partial_{x_1} \odot \dots \odot \partial_{x_1}}_{i_1 \text{ factors}}$$

for $I = (i_1, i_2, \dots, i_n) \in \mathbb{N}_0^n$.

2.4. Dg vector bundles. A dg-vector bundle is a vector bundle in the category of dg manifolds. We refer the reader to [85, 60] for details on dg vector bundles.

Many familiar notions in geometry and representation theory can be interpreted in terms of dg manifolds in a unified way.

EXAMPLE 2.20. Consider Example 2.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let V be a finite-dimensional vector space. A structure of $\mathfrak{g}$-module on V is equivalent to a structure of dg vector bundle on $\mathfrak{g}[1] \times V \to \mathfrak{g}[1]$. Similarly, given an L_∞ algebra $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$, saying that a $\mathbb{Z}$-graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is an L_∞ module over $\mathfrak{g}$ is equivalent to saying that $\mathfrak{g}[1] \times V \to \mathfrak{g}[1]$ is a dg vector bundle.

EXAMPLE 2.21. Consider Example 2.4. Let X be a complex manifold, let $E \to X$ be a complex vector bundle, and let π^*E denote the pullback of the complex vector bundle $E \to X$ through the canonical projection $\pi : T_X^{0,1}[1] \to X$. Then $E \to X$ is a holomorphic vector bundle if and only if $\pi^*E \to T_X^{0,1}[1]$ is a dg vector bundle. Similarly, given a complex of holomorphic vector bundles over X of finite length

$$0 \to E^{-m} \to \dots \to E^{i-1} \to E^i \to E^{i+1} \to \dots \to E^n \to 0, \tag{13}$$

the pullback $\pi^*E \to T_X^{0,1}[1]$ of $E = \bigoplus_{i=-m}^n E^i$ through π is a dg vector bundle. However, note that a dg vector bundle structure on the pullback $\pi^*E \to T_X^{0,1}[1]$ of a finite-length complex (13) of vector bundles over $\mathbb{C}$ is not necessarily equivalent to a structure of complex of holomorphic vector bundles. It is related to Quillen's flat superconnections [99, 10]. For more details on this, see [52].

EXAMPLE 2.22 ([60]). Let $A \to M$ be a gauge Lie algebroid with anchor ρ. Then $A[1] \to T_M[1]$ is a dg vector bundle with the Chevalley–Eilenberg differentials as homological vector fields on $A[1]$ and $T_M[1]$.

Example 2.22 is a special case of a general fact [85]: LA-vector bundles give rise to dg vector bundles. An LA-vector bundle [73, 74, 77] (a.k.a. VB-algebroid [44]) is a vector bundle in the category of Lie algebroids. More precisely, an LA-vector bundle is a double vector bundle

$$\begin{array}{ccc} D & \xrightarrow{t} & B \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle r} \\ A & \xrightarrow[b]{} & M \end{array} \tag{14}$$

in which the vector bundles $D \xrightarrow{t} B$ and $A \xrightarrow{b} M$ carry Lie algebroid structures and the 'vertical projections' l and r realize a morphism of Lie algebroids from $D \xrightarrow{t} B$ to $A \xrightarrow{b} M$. The notion of LA-vector bundle was introduced by Mackenzie in his extensive study of 'double structures' [73, 74, 77]. It was later reformulated by Gracia-Saz and Mehta in terms of VB-algebroids [44].

PROPOSITION 2.23 ([44]). *The double vector bundle* (14) *is an LA-vector bundle if and only if* $D[1] \xrightarrow{l} A[1]$ *is a dg vector bundle.*

Going back to Example 2.22, a gauge Lie algebroid $A \to M$ with anchor ρ yields an LA-vector bundle

$$\begin{array}{ccc} A & \longrightarrow & M \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ T_M & \longrightarrow & M \end{array}$$

and thence, according to Proposition 2.23, a dg vector bundle $A[1] \to T_M[1]$.

Given a vector bundle object $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ in the category of $\mathbb{Z}$-graded manifolds, its space of sections $\Gamma(\mathcal{E})$ is defined to be the direct sum $\bigoplus_{j\in\mathbb{Z}} \Gamma(\mathcal{E})^j$, where $\Gamma(\mathcal{E})^j$ denotes the space of degree preserving maps $s \in \operatorname{Hom}(\mathcal{M}, \mathcal{E}[-j])$ such that $(\pi[-j]) \circ s = \mathrm{id}_{\mathcal{M}}$. Here $\pi[-j] : \mathcal{E}[-j] \to \mathcal{M}$ is the natural map induced by $\pi : \mathcal{E} \to \mathcal{M}$ — see [85] for more details. When $\mathcal{E} \to \mathcal{M}$ is a dg vector bundle, the homological vector fields on $\mathcal{E}$ and $\mathcal{M}$ naturally induce an operator $\mathcal{Q}$ of degree $(+1)$ on $\Gamma(\mathcal{E})$, making $\Gamma(\mathcal{E})$ a dg module over $C^\infty(\mathcal{M})$. Since $C^\infty(\mathcal{M})$ and the space $\Gamma(\mathcal{E}^\vee)$ of linear functions on $\mathcal{E}$ together generate the algebra $C^\infty(\mathcal{E})$, the converse is also true: the homological vector field on $\mathcal{M}$ and the operator $\mathcal{Q}$ on $\Gamma(\mathcal{E})$ determine a dg structure on $\mathcal{E}$.

LEMMA 2.24 ([86]). *Let* $\mathcal{M}$ *be a* $\mathbb{Z}$*-graded manifold endowed with a homological vector field* Q*. Given a vector bundle object* $\mathcal{E} \to \mathcal{M}$ *in the category of* $\mathbb{Z}$*-graded manifolds,* $\mathcal{E}$ *admits a dg manifold structure making* $\mathcal{E} \to \mathcal{M}$ *into a dg vector bundle if and only if the space of sections* $\Gamma(\mathcal{E})$ *admits a structure of dg module over the dg algebra* $(C^\infty(\mathcal{M}), Q)$*. Indeed, the category of dg vector bundles over the dg manifold* $(\mathcal{M}, Q)$ *is equivalent to the category of locally free dg modules over the dg algebra* $(C^\infty(\mathcal{M}), Q)$*.*

Given a dg vector bundle $\mathcal{E} \to \mathcal{M}$, the induced operator $\mathcal{Q}$ on $\Gamma(\mathcal{E})$ is the coboundary operator of a cochain complex

$$\ldots \to \Gamma(\mathcal{E})^{i-1} \xrightarrow{\mathcal{Q}} \Gamma(\mathcal{E})^{i} \xrightarrow{\mathcal{Q}} \Gamma(\mathcal{E})^{i+1} \to \ldots,$$

whose cohomology group will be denoted $H^\bullet(\Gamma(\mathcal{E}), \mathcal{Q})$.

In particular, the space $\mathfrak{X}(\mathcal{M})$ of vector fields on a dg manifold $(\mathcal{M}, Q)$ (i.e. graded derivations of $C^\infty(\mathcal{M})$), which can be regarded as the space of sections $\Gamma(T_\mathcal{M})$, is naturally a dg module over the dg algebra $(C^\infty(\mathcal{M}), Q)$ with the Lie derivative

$$\mathcal{L}_Q : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$

playing the role of the operator $\mathcal{Q}$. As a consequence, $T_\mathcal{M}$ is naturally a dg manifold and $T_\mathcal{M} \to \mathcal{M}$ a dg vector bundle according to Lemma 2.24.

Definition 2.25. Given a dg manifold $(\mathcal{M}, Q)$, the homological vector field on $T_\mathcal{M}$ corresponding to the operator $\mathcal{Q} = \mathcal{L}_Q$ on $\Gamma(T_\mathcal{M})$ is called the *complete lift* of Q (it was called the tangent lift in [86]).

See [132] for an analogue of the complete lift of vector fields in classical differential geometry.

3. Atiyah class and characteristic classes of a dg vector bundle

3.1. Dg Lie algebroids. Dg Lie algebroids are Lie algebroid objects in the category of dg manifolds. Below, we briefly recall their precise definition. For more details, we refer the reader to [85], where dg Lie algebroids are called Q-algebroids.

A Lie algebroid object in the category of $\mathbb{Z}$-graded manifolds consists of a vector bundle object $\mathcal{A} \to \mathcal{M}$ in the category of $\mathbb{Z}$-graded manifolds together with a bundle map $\rho : \mathcal{A} \to T_\mathcal{M}$ of degree 0, called anchor, and a structure of graded Lie algebra on $\Gamma(\mathcal{A})$ with Lie bracket satisfying

$$[X, fY] = \rho(X)(f)Y + (-1)^{|X||f|} f[X, Y]$$

for all homogeneous $X, Y \in \Gamma(\mathcal{A})$ and $f \in C^\infty(\mathcal{M})$.

Definition 3.1 ([85]). A *dg Lie algebroid* is a dg vector bundle $\mathcal{A} \to \mathcal{M}$ endowed with an additional structure of Lie algebroid object in the category of $\mathbb{Z}$-graded manifolds such that the dg and the Lie structures are compatible in the following sense:

$$[d_\mathcal{A}, \mathcal{Q}] = 0, \tag{15}$$

where $d_\mathcal{A}$ is the Chevalley–Eilenberg differential corresponding to the Lie algebroid structure on $\mathcal{A} \to \mathcal{M}$ seen as a homological vector field on $\mathcal{A}[1]$, while the symbol $\mathcal{Q}$ denotes the homological vector field on $\mathcal{A}[1]$ induced by the homological vector field $Q^{(\mathcal{A})}$ on $\mathcal{A}$ which is part of the dg vector bundle structure on $\mathcal{A} \to \mathcal{M}$. Here, the bracket (15) stands for the Lie bracket on $\mathfrak{X}(A[1])$.

The compatibility condition (15) is equivalent to the requirement that the Chevalley–Eilenberg differential

$$d_\mathcal{A} : \Gamma(\Lambda^\bullet \mathcal{A}^\vee) \to \Gamma(\Lambda^{\bullet+1} \mathcal{A}^\vee)$$

of the Lie algebroid $\mathcal{A} \to \mathcal{M}$ commute with the differential (of internal degree $(+1)$)

$$\mathcal{Q} : \Gamma(\Lambda^\bullet \mathcal{A}^\vee) \to \Gamma(\Lambda^\bullet \mathcal{A}^\vee)$$

induced by the dg vector bundle structure on $\mathcal{A} \to \mathcal{M}$. As a consequence, the pair of differentials $d_\mathcal{A}$ and $\mathcal{Q}$ make the algebra $\Gamma(\Lambda^\bullet \mathcal{A}^\vee)$ — which is double-graded by the

'•-degree' and the 'internal' degree — a double complex. According to Mehta [85], the total cohomology of this double complex is the dg Lie algebroid cohomology of $\mathcal{A} \to \mathcal{M}$.

Example 3.2. Let $\mathcal{A} \to \mathcal{M}$ be a Lie algebroid object in the category of $\mathbb{Z}$-graded manifolds with anchor map $\rho : \mathcal{A} \to T_{\mathcal{M}}$ and let $s \in \Gamma(\mathcal{A})$ be a section of degree +1 satisfying $[s, s] = 0$. Then $\mathcal{A} \to \mathcal{M}$ admits a structure of dg Lie algebroid: the homological vector field on $\mathcal{M}$ is $\rho(s)$ while the operator of degree +1 on $\Gamma(\mathcal{A})$ is $[s, -]$.

One important class of dg Lie algebroids arise from Mackenzie's *double Lie algebroids* [73], the infinitesimal counterparts of double Lie groupoids [72, 75, 74]. The following is essentially due to Ted Voronov [126].

Theorem 3.3 ([126]). *A double vector bundle structure*

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \tag{16}$$

can be upgraded to a double Lie algebroid in the sense of Mackenzie [73, 72, 75, 74] *if and only if* $D[1] \to A[1]$ *is a dg Lie algebroid. Here* $D[1]$ *and* $A[1]$ *denote the graded manifolds obtained by shifting the degree of the fiberwisely linear functions on the total spaces* D *and* A *of the vector bundles* $D \to B$ *and* $A \to M$*.*

Definition 3.4 ([89, 78]). Two $\mathbb{K}$-Lie algebroids A and B over the same base manifold M and with respective anchors ρ_A and ρ_B are said to form a *matched pair* if there exists an action ∇ of A on B and an action Δ of B on A such that the identities

$$\begin{aligned} [\rho_A(X), \rho_B(Y)] &= -\rho_A(\Delta_Y X) + \rho_B(\nabla_X Y), \\ \nabla_X [Y_1, Y_2] &= [\nabla_X Y_1, Y_2] + [Y_1, \nabla_X Y_2] + \nabla_{\Delta_{Y_2} X} Y_1 - \nabla_{\Delta_{Y_1} X} Y_2, \\ \Delta_Y [X_1, X_2] &= [\Delta_Y X_1, X_2] + [X_1, \Delta_Y X_2] + \Delta_{\nabla_{X_2} Y} X_1 - \Delta_{\nabla_{X_1} Y} X_2, \end{aligned}$$

hold for all $X_1, X_2, X \in \Gamma(A)$ and $Y_1, Y_2, Y \in \Gamma(B)$.

Lemma 3.5 ([89, 78]). *Given a matched pair* (A, B) *of Lie algebroids, there is a Lie algebroid structure* $A \bowtie B$ *on the direct sum vector bundle* $A \oplus B$*, with anchor*

$$X \oplus Y \mapsto \rho_A(X) + \rho_B(Y)$$

and bracket

$$[X_1 \oplus Y_1, X_2 \oplus Y_2] = ([X_1, X_2] + \Delta_{Y_1} X_2 - \Delta_{Y_2} X_1) \oplus ([Y_1, Y_2] + \nabla_{X_1} Y_2 - \nabla_{X_2} Y_1).$$

Conversely, if $A \oplus B$ *carries a Lie algebroid structure for which* $A \oplus 0$ *and* $0 \oplus B$ *are Lie subalgebroids, then the representations* ∇ *and* Δ *defined by*

$$[X \oplus 0, 0 \oplus Y] = -\Delta_Y X \oplus \nabla_X Y$$

endow the couple (A, B) *with a structure of matched pair.*

In fact, the representation ∇ (resp. Δ) can be identified with the Bott representation of A (resp. B) on $L/A \cong B$ (resp. $L/B \cong A$). See (52).

Example 3.6 (matched pair of Lie algebroids). Let (A, B) be a matched pair of Lie algebroids. Then

$$\begin{array}{ccc} A \bowtie B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

is a double Lie algebroid and, according to Theorem 3.3, $(A[1] \oplus B, d_A^{\mathrm{Bott}})$ is a dg Lie algebroid over $(A[1], d_A)$. Here d_A^{Bott} denotes the Chevalley–Eilenberg differential of the Lie algebroid A corresponding to the Bott representation ∇ of A on B.

Example 3.7 (Lie bialgebroid). Let $(A, A^\vee)$ be a Lie bialgebroid [79]. Then

$$\begin{array}{ccc} T^\vee A (\cong T^\vee A^\vee) & \longrightarrow & A^\vee \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

is a double Lie algebroid and, according to Theorem 3.3, $T^\vee A^\vee[1] \to A[1]$ is a dg Lie algebroid.

Differential graded foliations constitute another important class of examples of dg Lie algebroids.

Proposition 3.8. *Let $(\mathcal{M}, Q)$ be a dg manifold.*

- *Then $T_{\mathcal{M}}$ is a dg manifold with the complete lift of Q as homological vector field.*
- *Furthermore, $T_{\mathcal{M}} \to \mathcal{M}$ is a dg Lie algebroid.*
- *More generally, if $\mathcal{D} \subset T_{\mathcal{M}}$ is an integrable distribution on $\mathcal{M}$ and $\Gamma(\mathcal{D})$ is stable under $\mathcal{L}_Q$, then $\mathcal{D} \to \mathcal{M}$ is a dg Lie algebroid with the inclusion $\mathcal{D} \hookrightarrow T_{\mathcal{M}}$ as its anchor map. We say that $\mathcal{D}$ is a dg foliation of $\mathcal{M}$.*

An example of dg foliation in the sense of Proposition 3.8 is Fedosov dg Lie algebroids.

Example 3.9 (Fedosov dg foliation [67]). Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold and, as in Section 2.3, let $(\mathcal{N} = T_{\mathcal{M}}[1] \oplus T_{\mathcal{M}}, D)$ be a Fedosov dg manifold for $\mathcal{M}$. Consider the pullback $\mathcal{F} \to \mathcal{N}$ of the vector bundle $T_{\mathcal{M}} \to \mathcal{M}$ through the canonical projection $\mathcal{N} \to \mathcal{M}$. It is a vector bundle in the category of $\mathbb{Z}$-graded manifolds whose total space $\mathcal{F}$ is a graded manifold with support M. Its space of sections $\Gamma(\mathcal{N}; \mathcal{F})$ is the $C^\infty(\mathcal{N})$-module $C^\infty(\mathcal{N}) \otimes_{\mathcal{R}} \mathfrak{X}(\mathcal{M}) = \Omega^\bullet(\mathcal{M}, \widehat{S}(T_{\mathcal{M}}^\vee) \otimes T_{\mathcal{M}})$. It can be identified canonically to a $C^\infty(\mathcal{N})$-submodule of $\mathfrak{X}(\mathcal{N})$ as follows. Let $\partial_1, \dots, \partial_m$ and $\chi_1, \dots, \chi_m$ denote the dual local frames for $T_{\mathcal{M}}$ and $T_{\mathcal{M}}^\vee$ arising from a choice of local coordinates $(x_1, \dots, x_m)$ on $\mathcal{M}$. To $1 \otimes \partial_k \in C^\infty(\mathcal{N}) \otimes_{\mathcal{R}} \mathfrak{X}(\mathcal{M}) = \Gamma(\mathcal{N}; \mathcal{F})$ we associate the (graded) derivation of $C^\infty(\mathcal{N})$ mapping $\chi_j \in \Omega^0(\mathcal{M}, S^1(T_{\mathcal{M}}^\vee)) \subset C^\infty(\mathcal{N})$ to $\delta_{k,j}$ and $\omega \in \Omega^p(\mathcal{M}, S^0(T_{\mathcal{M}}^\vee)) \subset C^\infty(\mathcal{N})$ to 0. Thus $\mathcal{F} \to \mathcal{N}$ is a vector subbundle of $T_{\mathcal{N}} \to \mathcal{N}$. One can check that $\mathcal{F}$ is indeed a dg foliation of the dg manifold $(\mathcal{N}, D)$. Hence $\mathcal{F} \to \mathcal{N}$ is a dg Lie algebroid, which we elect to call a *Fedosov dg Lie algebroid.*

Fedosov dg Lie algebroids play an important role in the globalization to dg manifolds of Kontsevich's formality theorem for $\mathbb{R}^d$. The Fedosov dg Lie algebroid $\mathcal{F} \to \mathcal{N}$ of

Example 3.9 arising from a $\mathbb{Z}$-graded manifold $\mathcal{M}$ is somehow 'homotopy equivalent' to the tangent Lie algebroid $T_{\mathcal{M}} \to \mathcal{M}$. Given a homological vector field Q on the $\mathbb{Z}$-graded manifold $\mathcal{M}$, one can modify the dg structure on $\mathcal{F} \to \mathcal{N}$ so as to take the homological vector field Q on $\mathcal{M}$ into consideration in such a way that the resulting (modified) Fedosov dg Lie algebroid $\mathcal{F} \to \mathcal{N}$ is 'homotopy equivalent' to the *dg* Lie algebroid $T_{\mathcal{M}} \to \mathcal{M}$ of the *dg* manifold $(\mathcal{M}, Q)$. See [67] for more details.

3.2. Dg Lie algebroids associated with Lie algebroid morphisms. This subsection outlines a work in progress [115].

Let A and L be $\mathbb{K}$-Lie algebroids over the same base manifold M, and $\phi : A \to L$ a Lie algebroid morphism. There exists a double Lie algebroid due to Jotz Lean and Mackenzie [53] (see also [115]), called a *comma double Lie algebroid.* In the case that $\mathbb{K}$ is $\mathbb{R}$, any Lie algebroid morphism arises from a morphism of local Lie groupoids. A comma double Lie algebroid is the infinitesimal of the *comma double Lie groupoid* associated to a morphism of Lie groupoids with the same base manifolds due to Brown–Mackenzie [13, Example 1.8] (see also [72, Example 2.5] and [115]).

We recall its construction briefly below. Let $D = TA \times_{TM,\rho_L} L$. Then $D \xrightarrow{\breve{\varpi}} A$ is naturally a Lie algebroid, the pullback Lie algebroid — see [76] — of $L \xrightarrow{\varpi} M$ through the surjective submersion $\pi : A \to M$.

On the other hand, D inherits a second Lie algebroid structure over L, the transformation Lie algebroid, corresponding to the action of the tangent Lie algebroid $TA \to TM$ on $\rho_L : L \to TM$. To define the action, note that $\Gamma(TM, TA)$ is generated, over $C^\infty(TM)$, by two types of sections: core sections $\hat{X}$ and tangent sections TX, for all $X \in \Gamma(A)$ [79, 80]. Indeed their brackets completely determine the Lie bracket on $\Gamma(TM, TA)$ [79, equation (27)]:

$$[TX, TY] = T([X, Y]), \quad [TX, \hat{Y}] = 0, \quad [\hat{X}, \hat{Y}] = 0,$$

for $X, Y \in \Gamma(A)$. Recall that the core section $\hat{X} \in \Gamma(TM, TA)$, for any $X \in \Gamma(A)$ is defined as a map [79, 80]

$$\hat{X} : TM \to TA, \quad \hat{X}(v_m) = v_m + X|_m \in T_mM \oplus A|_m \cong T_{0_m}A, \quad \forall\, v_m \in T_mM.$$

For any section $X \in \Gamma(A)$, there also associates two vector fields on A, the vertical lift vector field $X^\uparrow$ and the morphic vector field $\tilde{X}$, defined respectively by [80]

$$X^\uparrow(f \circ \pi) = 0, \qquad X^\uparrow(l_\xi) = \langle \xi, X \rangle \circ \pi, \tag{17}$$

$$\tilde{X}(f \circ \pi) = \rho_A(X)(f) \circ \pi, \qquad \tilde{X}(\ell_\xi) = \ell_{L_X(\xi)}, \tag{18}$$

for $f \in C^\infty(M)$, $\xi \in \Gamma(A^*)$. Here $\ell_\xi \in C^\infty(A)$ is the fiberwise linear function determined by ξ.

Define $\Phi : \Gamma(TM, TA) \to \mathfrak{X}(L)$ by

$$\Phi(TX) = \widetilde{\phi(X)}, \qquad \Phi(\hat{X}) = \phi(X)^\uparrow \tag{19}$$

for any $X \in \Gamma(A)$.

One proves that (19) defines uniquely an action of the tangent Lie algebroid $TA \to TM$ on $\rho_L : L \to TM$ [53]. Thus one can form the transformation Lie algebroid $D = TA \times_{TM,\rho_L} L \xrightarrow{\breve{\pi}} L$.

Proposition 3.10 ([53]). *D is a double Lie algebroid*

$$\begin{array}{ccc} D & \xrightarrow{\breve{\pi}} & L \\ {\scriptstyle\breve{\varpi}}\downarrow & & \downarrow{\scriptstyle\varpi} \\ A & \xrightarrow[\pi]{} & M. \end{array} \tag{20}$$

Therefore, according to the Voronov theorem [126]: Theorem 3.3, $T_{A[1]} \times_{T_M} L \to A[1]$ is a dg Lie algebroid, where the Lie algebroid is the pull-back Lie algebroid of $L \to M$ through the surjective submersion $\pi : A[1] \to M$. To describe the dg structure, note that a general section of $T_{A[1]} \times_{T_M} L \xrightarrow{\breve{\varpi}} A[1]$ consists of a pair (X, ν), where $X \in \mathfrak{X}(A[1])$ is a vector field on $A[1]$ and ν is a map $A[1] \to L$ satisfying

$$\pi_* \circ X = \rho_L \circ \nu : A[1] \to T_M. \tag{21}$$

Let

$$s_\phi = (d_A, \mu), \quad \mu : A[1] \to A \xrightarrow{\phi} L. \tag{22}$$

If one thinks of $A \xrightarrow{\phi} L$ as a bundle of two term complex over M, then $\mu : A[1] \to L$ is the same one by assigning A with degree -1 and L degree 0. One proves that [115] s_ϕ is indeed a degree $+1$ section of $T_{A[1]} \times_{T_M} L \xrightarrow{\breve{\varpi}} A[1]$ satisfying

$$[s_\phi, s_\phi] = 0.$$

Proposition 3.11 ([115]). *Let $\phi : A \to L$ be a Lie algebroid morphism. Then*

$$T_{A[1]} \times_{T_M} L \to A[1]$$

is a dg Lie algebroid. As a ($\mathbb{Z}$-graded) Lie algebroid, it is the pull-back (in the Lie algebroid sense) of the Lie algebroid $L \to M$ through the surjective submersion $A[1] \to M$. On the other hand, the dg structure arises, as in Example 3.2, from the section s_ϕ of degree $+1$ defined by equation (22).

3.3. Characteristic classes of a dg vector bundle relative to a dg Lie algebroid.

Let $\mathcal{E} \to \mathcal{M}$ be a dg vector bundle and let $\mathcal{A} \to \mathcal{M}$ be a dg Lie algebroid with anchor $\rho : \mathcal{A} \to T_{\mathcal{M}}$.

An $\mathcal{A}$-connection on $\mathcal{E}$ is a degree 0 map $\nabla : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ satisfying the pair of relations

$$\nabla_{fX} s = f \nabla_X s,$$
$$\nabla_X (fs) = \rho_X(f) s + (-1)^{|X||f|} f \nabla_X s,$$

for all homogeneous elements $f \in C^\infty(\mathcal{M})$, $X \in \Gamma(\mathcal{A})$, and $s \in \Gamma(\mathcal{E})$. Such connections always exist since the standard partition of unity argument holds in the context of graded manifolds. Given a dg vector bundle $\mathcal{E}$ and an $\mathcal{A}$-connection ∇ on it, we can consider the bundle map $\operatorname{At}^\nabla_{\mathcal{E}} : \mathcal{A} \otimes \mathcal{E} \to \mathcal{E}$ of degree $+1$ defined by

$$\operatorname{At}^\nabla_{\mathcal{E}}(X, s) = \mathcal{Q}(\nabla_X s) - \nabla_{\mathcal{Q}(X)} s - (-1)^{|X|} \nabla_X (\mathcal{Q}(s)), \quad \forall X \in \Gamma(\mathcal{A}),\ s \in \Gamma(\mathcal{E}).$$

The bundle map $\operatorname{At}^\nabla_{\mathcal{E}}$ can be regarded as a section of degree $+1$ of $\mathcal{A}^\vee \otimes \operatorname{End} \mathcal{E}$, and hence as a 1-cochain in the cochain complex $(\Gamma(\mathcal{A}^\vee \otimes \operatorname{End} \mathcal{E})^\bullet, \mathcal{Q})$.

LEMMA 3.12 ([86]). *The 1-cochain* $\mathrm{At}_{\mathcal{E}}^{\nabla}$ *is a cocycle:* $\mathcal{Q}(\mathrm{At}_{\mathcal{E}}^{\nabla}) = 0$. *Its cohomology class is independent of the choice of the connection* ∇.

The cohomology class $\alpha_{\mathcal{E}} := [\mathrm{At}_{\mathcal{E}}^{\nabla}]$ in $H^1(\Gamma(\mathcal{A}^\vee \otimes \operatorname{End}\mathcal{E})^\bullet, \mathcal{Q})$ is called the *Atiyah class* of the dg vector bundle $\mathcal{E} \to \mathcal{M}$ relative to the dg Lie algebroid $\mathcal{A} \to \mathcal{M}$ [86].

The *Todd cocycle* and $\widehat{A}$ *cocycle* of the dg vector bundle $\mathcal{E}$ associated with the $\mathcal{A}$-connection ∇ are the elements $\mathrm{Td}_{\mathcal{E}}^{\nabla}$ and $\widehat{A}_{\mathcal{E}}^{\nabla}$ of $\prod_{k\geqslant 0}(\Gamma(\Lambda^k\mathcal{A}^\vee))^k$ defined, respectively, by

$$\mathrm{Td}_{\mathcal{E}}^{\nabla} = \mathrm{Ber}\left(\frac{\mathrm{At}_{\mathcal{E}}^{\nabla}}{1 - e^{-\mathrm{At}_{\mathcal{E}}^{\nabla}}}\right) \qquad \text{and} \qquad \widehat{A}_{\mathcal{E}}^{\nabla} = \mathrm{Ber}\left(\frac{\mathrm{At}_{\mathcal{E}}^{\nabla}}{e^{(1/2)\,\mathrm{At}_{\mathcal{E}}^{\nabla}} - e^{-(1/2)\,\mathrm{At}_{\mathcal{E}}^{\nabla}}}\right),$$

where Ber denotes the Berezinian [9, 21] and $\Lambda^k\mathcal{A}^\vee$ denotes the dg vector bundle $S^k(\mathcal{A}^\vee[-1])[k] \to \mathcal{M}$. Both $\mathrm{Td}_{\mathcal{E}}^{\nabla}$ and $\widehat{A}_{\mathcal{E}}^{\nabla}$ are cocycles: $\mathcal{Q}(\mathrm{Td}_{\mathcal{E}}^{\nabla}) = 0 = \mathcal{Q}(\widehat{A}_{\mathcal{E}}^{\nabla})$. Note that every element of $(\Gamma(\Lambda^k\mathcal{A}^\vee))^k$ is a finite sum $\sum \alpha_1 \wedge \ldots \wedge \alpha_k$ with $\alpha_1, \ldots, \alpha_k \in \Gamma(\mathcal{A}^\vee)$ homogeneous and satisfying the degree condition $|\alpha_1| + \ldots + |\alpha_k| = k$. The cohomology classes $\mathrm{td}_{\mathcal{E}}$ and $\widehat{\mathcal{A}}_{\mathcal{E}}$ in $\prod_{k\geqslant 0} H^k((\Gamma(\Lambda^k\mathcal{A}^\vee))^\bullet, \mathcal{Q})$ of the cocycles $\mathrm{Td}_{\mathcal{E}}^{\nabla}$ and $\widehat{A}_{\mathcal{E}}^{\nabla}$ are independent of the choice of the connection ∇ and are respectively called the *Todd class* and $\widehat{A}$ *class* of the dg vector bundle $\mathcal{E}$ relative to the dg Lie algebroid $\mathcal{A}$. Hence, the Todd class and the $\widehat{A}$ class of a dg vector bundle $\mathcal{E}$ relative to a dg Lie algebroid $\mathcal{A}$ are respectively the elements

$$\mathrm{td}_{\mathcal{E}} = \mathrm{Ber}\left(\frac{\alpha_{\mathcal{E}}}{1 - e^{-\alpha_{\mathcal{E}}}}\right) \qquad \text{and} \qquad \widehat{\mathcal{A}}_{\mathcal{E}} = \mathrm{Ber}\left(\frac{\alpha_{\mathcal{E}}}{e^{(1/2)\,\alpha_{\mathcal{E}}} - e^{-(1/2)\,\alpha_{\mathcal{E}}}}\right) \tag{23}$$

in $\prod_{k\geqslant 0} H^k(\Gamma(\Lambda^k\mathcal{A}^\vee)^\bullet, \mathcal{Q})$.

Both $\mathrm{td}_{\mathcal{E}}$ and $\widehat{\mathcal{A}}_{\mathcal{E}}$ can be expressed in terms of the scalar Atiyah classes

$$c_k = \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k \mathrm{str}(\alpha_{\mathcal{E}}^k) \in H^k(\Gamma(\Lambda^k\mathcal{A}^\vee)^\bullet, \mathcal{Q}).$$

Here $\mathrm{str} : \operatorname{End}(\mathcal{E}) \to C^\infty(\mathcal{M})$ denotes the supertrace. Note that $\mathrm{str}(\alpha_{\mathcal{E}}^k) \in \Gamma(\Lambda^k\mathcal{A}^\vee)$ since $\alpha_{\mathcal{E}}^k \in \Gamma(\Lambda^k\mathcal{A}^\vee) \otimes_{C^\infty(\mathcal{M})} \Gamma(\operatorname{End}(\mathcal{E}))$. For details, see [86].

3.4. Characteristic classes of a dg manifold. Let $(\mathcal{M}, Q)$ be a finite-dimensional dg manifold. According to Proposition 3.8, its tangent bundle $T_{\mathcal{M}} \to \mathcal{M}$ is naturally a dg Lie algebroid. By definition, the *Atiyah class of the dg manifold* $(\mathcal{M}, Q)$ is the Atiyah class of the dg vector bundle $T_{\mathcal{M}}$ relative to the dg Lie algebroid $T_{\mathcal{M}}$. More precisely, given a dg manifold $(\mathcal{M}, Q)$ and an affine connection ∇ on $\mathcal{M}$, the degree $+1$ section of $T_{\mathcal{M}}^\vee \otimes \operatorname{End}(T_{\mathcal{M}})$ corresponding to the bundle map $\mathrm{At}_{(\mathcal{M},Q)}^{\nabla} : T_{\mathcal{M}} \otimes T_{\mathcal{M}} \to T_{\mathcal{M}}$ defined by

$$\mathrm{At}_{(\mathcal{M},Q)}^{\nabla}(X, Y) = \mathcal{L}_Q(\nabla_X Y) - \nabla_{\mathcal{L}_Q(X)} Y - (-1)^{|X|}\nabla_X(\mathcal{L}_Q(Y)), \quad \forall\, X, Y \in \Gamma(T_{\mathcal{M}}),$$

is a cocycle whose cohomology class

$$\alpha_{(\mathcal{M},Q)} = [\mathrm{At}_{(\mathcal{M},Q)}^{\nabla}] \in H^1(\Gamma(T_{\mathcal{M}}^\vee \otimes \operatorname{End} T_{\mathcal{M}})^\bullet, \mathcal{Q})$$

is independent of the choice of the connection ∇ and is called the Atiyah class of the dg manifold $(\mathcal{M}, Q)$.

The Atiyah class of a dg manifold is the obstruction to the existence of connections compatible with the homological vector field. It was first investigated by Shoikhet [111] in relation with Kontsevich's formality theorem and Duflo's formula.

The *Todd cocycle* and the $\widehat{A}$ *cocycle* of the dg manifold $(\mathcal{M}, Q)$ associated with the affine connection ∇ are the elements

$$\operatorname{Td}^{\nabla}_{(\mathcal{M},Q)} = \operatorname{Ber}\left(\frac{\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}}{1-e^{-\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}}}\right) \quad \text{and} \quad \widehat{A}^{\nabla}_{(\mathcal{M},Q)} = \operatorname{Ber}\left(\frac{\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}}{e^{(1/2)\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}} - e^{-(1/2)\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}}}\right)$$

of $\prod_{k\geqslant 0}(\Gamma(\Lambda^k T^{\vee}_{\mathcal{M}}))^k \cong \prod_{k\geqslant 0}(\Omega^k(\mathcal{M}))^k$.

Their respective cohomology classes $\operatorname{td}_{(\mathcal{M},Q)}$ and $\widehat{\mathcal{A}}_{(\mathcal{M},Q)}$ in $\prod_{k\geqslant 0} H^k((\Omega^k(\mathcal{M}))^{\bullet}, \mathcal{Q})$ are independent of the choice of the connection ∇ and will be referred to as the *Todd class* and the $\widehat{A}$ *class* of the dg manifold $(\mathcal{M}, Q)$, respectively.

Given a finite-dimensional Lie algebra $\mathfrak{g}$, consider the dg manifold $(\mathcal{M}, Q)$, where $\mathcal{M} = \mathfrak{g}[1]$ and Q is the Chevalley–Eilenberg differential d_{CE}. The following result can be easily verified using the canonical trivialization $T_{\mathcal{M}} \cong \mathfrak{g}[1] \times \mathfrak{g}[1]$.

LEMMA 3.13 ([86]). *Let $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ be the dg manifold arising from a finite-dimensional Lie algebra $\mathfrak{g}$. There are canonical isomorphisms*

$$H^k(\Gamma(T^{\vee}_{\mathcal{M}} \otimes \operatorname{End} T_{\mathcal{M}})^{\bullet}, \mathcal{Q}) \cong H^{k-1}_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{g}^{\vee} \otimes \mathfrak{g}^{\vee} \otimes \mathfrak{g})$$

and

$$H^k(\Omega^k(\mathcal{M})^{\bullet}, \mathcal{Q}) \cong (S^k(\mathfrak{g}^{\vee}))^{\mathfrak{g}}.$$

Recall that the Duflo element $J \in (\widehat{S}(\mathfrak{g}^{\vee}))^{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is the invariant formal power series on $\mathfrak{g}$ defined by

$$J(x) = \det\left(\frac{1-e^{-\operatorname{ad}_x}}{\operatorname{ad}_x}\right)$$

for all $x \in \mathfrak{g}$.

PROPOSITION 3.14 ([86]). *Let $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ be the dg manifold arising from a finite-dimensional Lie algebra $\mathfrak{g}$.*

- *Its Atiyah class $\alpha_{(\mathfrak{g}[1],d_{\mathrm{CE}})}$ is precisely the Lie bracket of $\mathfrak{g}$ regarded as an element of*

$$(\mathfrak{g}^{\vee} \otimes \mathfrak{g}^{\vee} \otimes \mathfrak{g})^{\mathfrak{g}} \cong H^0_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{g}^{\vee} \otimes \mathfrak{g}^{\vee} \otimes \mathfrak{g}) \cong H^1(\Gamma(T^{\vee}_{\mathcal{M}} \otimes \operatorname{End} T_{\mathcal{M}})^{\bullet}, \mathcal{Q}).$$

- *Consequently, the isomorphism*

$$\prod_k H^k(\Omega^k(\mathcal{M})^{\bullet}, \mathcal{Q}) \xrightarrow{\cong} (\widehat{S}(\mathfrak{g}^{\vee}))^{\mathfrak{g}}$$

maps the Todd class $\operatorname{td}_{(\mathfrak{g}[1],d_{\mathrm{CE}})}$ onto the Duflo element $J \in (\widehat{S}(\mathfrak{g}^{\vee}))^{\mathfrak{g}}$ of $\mathfrak{g}$.

EXAMPLE 3.15. Let $(x_1, \dots, x_m; x_{m+1} \cdots x_{m+n})$ be coordinate functions on $\mathbb{R}^{m|n}$ and let $Q = \sum_k Q_k \frac{\partial}{\partial x_k}$ be a homological vector field on $\mathbb{R}^{m|n}$. The Atiyah 1-cocycle of the dg manifold $(\mathbb{R}^{m|n}, Q)$ associated with the trivial connection $\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = 0$ is then

$$\operatorname{At}^{\nabla}_{(\mathbb{R}^{m|n},Q)}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (-1)^{|x_i|+|x_j|} \sum_k \frac{\partial^2 Q_k}{\partial x_i \partial x_j} \frac{\partial}{\partial x_k}\,. \tag{24}$$

The Atiyah 1-cocycle $\operatorname{At}^{\nabla}_{(\mathbb{R}^{m|n},Q)}$ captures the components of the homological vector field Q of second and higher orders. See [106] for more details.

Remark 3.16. The Atiyah class of a dg manifold was introduced independently by Lyakhovich–Mosman–Sharapov [71, footnote 6]. Furthermore, characteristic classes for tangent bundles of dg manifolds were studied in [92, 71, 70]. It would be interesting to explore the relation between these characteristic classes and those introduced earlier in the present paper.

3.5. Example: Atiyah class and dg manifolds associated with integrable distributions. Consider a regular foliation $\mathcal{F}$ on a smooth manifold M. The tangent bundle of $\mathcal{F}$ is a subbundle of T_M, denoted $T_{\mathcal{F}}$, whose sections are closed under the Lie bracket of vector fields. Then $T_{\mathcal{F}} \subseteq T_M$ is an integrable distribution and $(T_{\mathcal{F}}[1], d_{T_{\mathcal{F}}})$ is a dg manifold according to Proposition 2.6. In what follows, we compute the Atiyah class and Todd class of this dg manifold.

First, let us recall the construction of the Molino class of a foliation $\mathcal{F}$, i.e. the Atiyah class of the Lie (algebroid) pair $(T_M, T_{\mathcal{F}})$ — see [29].

Denote the normal bundle $T_M/T_{\mathcal{F}}$ to the foliation $\mathcal{F}$ by $N_{\mathcal{F}}$ and the conormal bundle $(T_M/T_{\mathcal{F}})^\vee$ by $N_{\mathcal{F}}^\vee$ or $T_{\mathcal{F}}^\perp$. The Bott flat $T_{\mathcal{F}}$-connection on $N_{\mathcal{F}}$ is defined by

$$\nabla_a^{\mathrm{Bott}} q(l) = q([a, l]), \quad \forall\, a \in \Gamma(T_{\mathcal{F}}),\ l \in \mathfrak{X}(M),$$

where $q : T_M \to T_M/T_{\mathcal{F}}$ denotes the canonical projection. The Chevalley–Eilenberg Lie algebroid cohomology $H^\bullet_{\mathrm{CE}}(T_{\mathcal{F}}, \mathfrak{M})$ with coefficients in a $T_{\mathcal{F}}$-module $\mathfrak{M}$ coincides exactly with the leafwise de Rham cohomology $H^\bullet_{\mathrm{dR}}(\mathcal{F}, \mathfrak{M})$ of the foliation $\mathcal{F}$ with coefficients in the module $\mathfrak{M}$.

Let us choose a splitting $j : N_{\mathcal{F}} \to T_M$ of the short exact sequence of vector bundles (over M)

$$0 \longrightarrow T_{\mathcal{F}} \xrightarrow{\ i\ } T_M \underset{j}{\overset{q}{\rightleftarrows}} N_{\mathcal{F}} \longrightarrow 0$$

and a T_M-connection ∇ on $N_{\mathcal{F}}$ extending the Bott $T_{\mathcal{F}}$-connection on $N_{\mathcal{F}}$. The associated Atiyah 1-cocycle

$$R^\nabla \in \Gamma(M; T_{\mathcal{F}}^\vee \otimes T_{\mathcal{F}}^\perp \otimes \mathrm{End}(N_{\mathcal{F}}))$$

is defined by the relation

$$R^\nabla(X, V)W = \nabla_X \nabla_{j(V)} W - \nabla_{j(V)} \nabla_X W - \nabla_{[X, j(V)]} W,$$

for all $X \in \Gamma(T_{\mathcal{F}})$ and $V, W \in \Gamma(N_{\mathcal{F}})$. It is simple to check that $R^\nabla \in \Gamma(T_{\mathcal{F}}^\vee \otimes T_{\mathcal{F}}^\perp \otimes \mathrm{End}(N_{\mathcal{F}}))$ is a leafwise de Rham closed 1-form with values in the $T_{\mathcal{F}}$-module $T_{\mathcal{F}}^\perp \otimes \mathrm{End}(N_{\mathcal{F}})$. Its cohomology class

$$\alpha_{T_M/T_{\mathcal{F}}} = [R^\nabla] \in H^1_{\mathrm{dR}}(\mathcal{F}, T_{\mathcal{F}}^\perp \otimes \mathrm{End}(N_{\mathcal{F}}))$$

does not depend on the choice of j and ∇. We call it the *Molino class* of the foliation $\mathcal{F}$ or the *Atiyah class of the Lie pair* $(T_M, T_{\mathcal{F}})$ — see [90, 91, 29]. This generalization of Atiyah's class for holomorphic vector bundles [2] to the context of connections 'transverse to a foliation' was introduced by Molino in [90, 91]. Molino's class measures the obstruction to the 'projectability' of connections 'transverse to a foliation,' i.e. whether the connection is stable under the parallel transport along any path tangent to the foliation.

The Todd class of the Lie pair $(T_M, T_{\mathcal{F}})$ is the cohomology class

$$\mathrm{td}_{T_M/T_{\mathcal{F}}} = \det\left(\frac{\alpha_{T_M/T_{\mathcal{F}}}}{1-\exp(-\alpha_{T_M/T_{\mathcal{F}}})}\right) \in \bigoplus_{k\geqslant 0} H^k_{\mathrm{dR}}(\mathcal{F}, \Lambda^k T^{\perp}_{\mathcal{F}}). \tag{25}$$

PROPOSITION 3.17 ([30]). *Let $(\mathcal{M}, Q) = (T_{\mathcal{F}}[1], d_{T_{\mathcal{F}}})$ be the dg manifold arising from the foliation $\mathcal{F}$ of the manifold M. Then there exist canonical isomorphisms, for all $k \geq 0$ and $l \geq 0$,*

$$\Phi^{k,l} : H^\bullet\big(\Gamma((T_{\mathcal{M}})^{\otimes k} \otimes (T^\vee_{\mathcal{M}})^{\otimes l}), \mathcal{Q}\big) \xrightarrow{\cong} H^\bullet_{\mathrm{dR}}\big(\mathcal{F}, (N_{\mathcal{F}})^{\otimes k} \otimes (N^\vee_{\mathcal{F}})^{\otimes l}\big) \tag{26}$$

such that

1. $\Phi^{1,2}(\alpha_{(\mathcal{M},Q)}) = \alpha_{T_M/T_{\mathcal{F}}}$,
2. $\Phi^{0,\bullet}(\mathrm{td}_{(\mathcal{M},Q)}) = \mathrm{td}_{T_M/T_{\mathcal{F}}}$.

It is well known that a complex manifold can be considered as a kind of '$\mathbb{C}$-foliation' of the underlying real smooth manifold. More precisely, given a complex manifold X, the subbundle $T^{0,1}_X$ of the complexified tangent bundle $T^{\mathbb{C}}_X = T_X \otimes \mathbb{C}$ is an integrable distribution, and $(T^{\mathbb{C}}_X, T^{0,1}_X)$ is a Lie algebroid pair (over $\mathbb{C}$) [29]. The Atiyah class of this Lie pair is precisely the Atiyah class of the holomorphic tangent bundle T_X defined by Atiyah in [2].

Consider the canonical flat $T^{0,1}_X$-connection $\nabla^{\overline{\partial}}$ on $T^{1,0}_X$ induced by the holomorphic vector bundle structure on T_X: a local section of $T^{1,0}_X$ is $\nabla^{\overline{\partial}}$-horizontal if and only if it is holomorphic. Since $T^{\mathbb{C}}_X \cong T^{0,1}_X \oplus T^{1,0}_X$, picking any $T^{1,0}_X$-connection $\nabla^{1,0}$ on $T^{1,0}_X$ and adding it to $\nabla^{\overline{\partial}}$, one obtains a $T^{\mathbb{C}}_X$-connection $\nabla = \nabla^{\overline{\partial}} + \nabla^{1,0}$ on $T^{1,0}_X$. The Atiyah 1-cocycle

$$R^\nabla \in \Omega^{0,1}((T^{1,0}_X)^\vee \otimes \mathrm{End}(T^{1,0}_X)) \tag{27}$$

of the complex Lie pair $(T^{\mathbb{C}}_X, T^{0,1}_X)$ associated with the connection ∇ is defined by

$$R^\nabla(Z, V)W = \nabla_Z\nabla_V W - \nabla_V\nabla_Z W - \nabla_{[Z,V]}W,$$

for all $Z \in \Gamma(T^{0,1}_X)$ and $V, W \in \Gamma(T^{1,0}_X)$. Its cohomology class

$$\alpha_{T_X} = [R^\nabla] \in H^{1,1}(X, \mathrm{End}(T_X)) \cong H^1_{\mathrm{sheaf}}(X, \Omega^1 \otimes \mathrm{End}(T_X))$$

does not depend on the choice of $\nabla^{1,0}$ and is called the *Atiyah class* of the holomorphic tangent bundle T_X [2, 55]. Here $H^{1,1}(X, \mathrm{End}(T_X))$ denotes the degree $(1,1)$ Dolbeault cohomology of the holomorphic vector bundle $\mathrm{End}(T_X)$ [49, 128]. The Atiyah class of a holomorphic vector bundle was introduced by Atiyah as the cohomological obstruction to the existence of a global holomorphic connection [2].

The Todd class of the Lie pair $(T^{\mathbb{C}}_X, T^{0,1}_X)$ is the cohomology class

$$\mathrm{td}_{T^{\mathbb{C}}_X/T^{0,1}_X} = \det\left(\frac{\alpha_{T_X}}{1-\exp(-\alpha_{T_X})}\right) \in \bigoplus_{k\geqslant 0} H^{k,k}(X) \cong \bigoplus_{k\geqslant 0} H^k_{\mathrm{sheaf}}(X, \Omega^k). \tag{28}$$

PROPOSITION 3.18 ([30]). *Let $(\mathcal{M}, Q) = (T^{0,1}_X[1], \bar{\partial})$ be the dg manifold arising from a complex manifold X as in Example* 2.4. *Then there exist canonical isomorphisms, for all $k \geq 0$, $l \geq 0$,*

$$\Phi^{k,l} : H^\bullet\big((\Gamma(T_{\mathcal{M}})^{\otimes k} \otimes (T^\vee_{\mathcal{M}})^{\otimes l}), \mathcal{Q}\big) \xrightarrow{\cong} H^\bullet_{\mathrm{sheaf}}\big(X, (T_X)^{\otimes k} \otimes (T^\vee_X)^{\otimes l}\big) \tag{29}$$

such that

1. $\Phi^{1,2}(\alpha_{(\mathcal{M},Q)}) = \alpha_{T_X}$,
2. $\Phi^{0,\bullet}(\mathrm{td}_{(\mathcal{M},Q)}) = \mathrm{td}_{T_X^{\mathbb{C}}/T_X^{0,1}}$.

REMARK 3.19. Given a compact Kähler manifold X, we have an isomorphism

$$\bigoplus_k H^k_{\mathrm{sheaf}}(X, \Omega^k_X) \xrightarrow{\cong} \bigoplus_k H^{2k}(X, \mathbb{C}).$$

This isomorphism maps the Todd class $\mathrm{td}_{T_X^{\mathbb{C}}/T_X^{0,1}}$ of the Lie pair $(T_X^{\mathbb{C}}, T_X^{0,1})$ to the ordinary Todd class td_X of X. Note that the Kähler condition is crucial here. For an arbitrary complex manifold, the Todd class $\mathrm{td}_{T_X^{\mathbb{C}}/T_X^{0,1}}$ of the Lie pair $(T_X^{\mathbb{C}}, T_X^{0,1})$ may depend on the complex structures on X, while the Todd class td_X of X is a purely topological invariant [10, 48].

3.6. Atiyah class and homotopy Lie algebras. A celebrated theorem of Kapranov states that for a complex manifold X, the complex of sheaves $T_X[-1]$ is a Lie algebra object in the derived category $D(X)$ of coherent sheaves on X with the Atiyah class α_{T_X} playing the role of the Lie bracket [55, 103, 100]. If X is Kähler, Kapranov proved an even stronger result by describing explicitly an $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$. Let us recall it briefly below.

If X is a Kähler manifold, the Levi-Civita connection ∇^{LC} induces a $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ as follows. First, extend the Levi-Civita connection $\mathbb{C}$-linearly to a $T_X^{\mathbb{C}}$-connection ∇ on $T_X^{\mathbb{C}}$. Since X is Kähler, the almost complex structure J on X is parallel and ∇ restricts to a $T_X^{\mathbb{C}}$-connection on $T_X^{1,0}$. It is easy to check that the induced $T_X^{0,1}$-connection on $T_X^{1,0}$ is the canonical flat connection $\nabla^{\overline{\partial}}$ encoding the holomorphic vector bundle structure on T_X while the induced $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ is flat and torsion-free. Thus $\nabla = \nabla^{\overline{\partial}} + \nabla^{1,0}$. Since $\nabla^{1,0}$ is torsion-free, the Dolbeault representative R^∇ of the Atiyah 1-cocycle belongs to $\Omega^{0,1}(S^2(T_X^{1,0})^\vee \otimes T_X^{1,0})$ — see equation (27).

Let

$$R_2 = R^\nabla \in \Omega^{0,1}(S^2(T_X^{1,0})^\vee \otimes T_X^{1,0})$$

and, for $k \geqslant 2$,

$$R_{k+1} = d^{\nabla^{1,0}} R_k \in \Omega^{0,1}(S^{k+1}(T_X^{1,0})^\vee \otimes T_X^{1,0}).$$

THEOREM 3.20 ([55, Theorem 2.6]). *Given a Kähler manifold X, the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ admits a structure of $L_\infty[1]$ algebra whose unary bracket λ_1 is the Dolbeault operator $\overline{\partial} : \Omega^{0,j}(T_X^{1,0}) \to \Omega^{0,j+1}(T_X^{1,0})$ and whose k-th multibracket λ_k for $k \geqslant 2$ is the composition of the wedge product*

$$\Omega^{0,j_1}(T_X^{1,0}) \otimes \ldots \otimes \Omega^{0,j_n}(T_X^{1,0}) \to \Omega^{0,j_1+\ldots+j_k}((T_X^{1,0})^{\otimes k})$$

with the map

$$\Omega^{0,j_1+\ldots+j_k}((T_X^{1,0})^{\otimes k}) \to \Omega^{0,j_1+\ldots+j_n+1}(T_X^{1,0})$$

induced by

$$R_k \in \Omega^{0,1}(S^k(T_X^{1,0})^\vee \otimes T_X^{1,0}) \subset \Omega^{0,1}(\mathrm{Hom}((T_X^{1,0})^{\otimes k}, T_X^{1,0})).$$

In [62], Theorem 3.20 was extended to all complex manifolds.

THEOREM 3.21 ([62, Theorem 5.24]). *Given a complex manifold X, each torsion-free $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ determines an $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ such that*

- *the unary bracket λ_1 is the Dolbeault operator*
$$\overline{\partial} : \Omega^{0,j}(T_X^{1,0}) \to \Omega^{0,j+1}(T_X^{1,0});$$
- *the binary bracket λ_2 is the map*
$$\lambda_2 : \Omega^{0,j_1}(T_X^{1,0}) \otimes \Omega^{0,j_2}(T_X^{1,0}) \to \Omega^{0,j_1+j_2+1}(T_X^{1,0})$$
induced by the Dolbeault $R_2 := R^\nabla$ representative of the Atiyah 1-cocycle;
- *for every $k \geqslant 3$, the k-th multibracket λ_k is the composition of the wedge product*
$$\Omega^{0,j_1}(T_X^{1,0}) \otimes \ldots \otimes \Omega^{0,j_n}(T_X^{1,0}) \to \Omega^{0,j_1+\ldots+j_k}((T_X^{1,0})^{\otimes k})$$
with the map
$$\Omega^{0,j_1+\ldots+j_k}((T_X^{1,0})^{\otimes k}) \to \Omega^{0,j_1+\ldots+j_n+1}(T_X^{1,0})$$
induced by an element R_k of $\Omega^{0,1}(S^k(T_X^{1,0})^\vee \otimes T_X^{1,0}) \subset \Omega^{0,1}(\mathrm{Hom}((T_X^{1,0})^{\otimes k}, T_X^{1,0}))$ arising as an algebraic function of R_2, the curvature of $\nabla^{1,0}$, their higher covariant derivatives, and compositions thereof.

In what follows, following Kapranov [55], we show that the Atiyah 1-cocycle of a dg manifold gives rise to an interesting homotopy Lie algebra in a similar fashion.

Let $(\mathcal{M}, Q)$ be a dg manifold and let ∇ be an affine connection on $\mathcal{M}$.

The Lie derivative along the homological vector field Q is a coderivation of degree $+1$ of the coalgebra $\mathcal{D}(\mathcal{M})$ of differential operators on $\mathcal{M}$:
$$\mathcal{L}_Q(X_1 \cdots X_n) = \sum_{k=1}^{n} (-1)^{|X_1|+\ldots+|X_{k-1}|} X_1 \cdots X_{k-1} [Q, X_k] X_{k+1} \cdots X_n. \tag{30}$$

Transferring $\mathcal{L}_Q$ from $\mathcal{D}(\mathcal{M})$ to $\Gamma(S(T_\mathcal{M}))$ by way of the isomorphism of coalgebras pbw^∇ — see equation (6), we obtain a coderivation δ^∇ of degree $+1$ of $\Gamma(S(T_\mathcal{M}))$:
$$\delta^\nabla := (\mathrm{pbw}^\nabla)^{-1} \circ L_Q \circ \mathrm{pbw}^\nabla. \tag{31}$$
Finally, dualizing δ^∇, we obtain an operator
$$D^\nabla : \Gamma(\widehat{S}(T_\mathcal{M}^\vee)) \to \Gamma(\widehat{S}(T_\mathcal{M}^\vee))$$
on
$$\Gamma(\widehat{S}(T_\mathcal{M}^\vee)) \cong \mathrm{Hom}_{C^\infty(\mathcal{M})}(\Gamma(S(T_\mathcal{M})), C^\infty(\mathcal{M})).$$

THEOREM 3.22 ([86]). *Let $(\mathcal{M}, Q)$ be a dg manifold and let ∇ be a torsion-free affine connection on $\mathcal{M}$.*

1. *The operator D^∇, dual to $(\mathrm{pbw}^\nabla)^{-1} \circ \mathcal{L}_Q \circ \mathrm{pbw}^\nabla$, is a derivation of degree $+1$ of the graded algebra $\Gamma(\widehat{S}(T_\mathcal{M}^\vee))$ satisfying $(D^\nabla)^2 = 0$.*
2. *There exists a sequence $(R_k)_{k\geqslant 2}$ of homomorphisms $R_k \in \mathrm{Hom}(S^k T_\mathcal{M}, T_\mathcal{M}[-1])$, whose first term R_2 is precisely the Atiyah 1-cocycle $\mathrm{At}^\nabla_{(\mathcal{M},Q)}$ and the operator D^∇ is the sum $D^\nabla = \mathcal{L}_Q + \sum_{k=2}^\infty \widetilde{R_k}$, where $\widetilde{R_k}$ denotes the $C^\infty(\mathcal{M})$-linear operator on $\Gamma(\widehat{S}(T_\mathcal{M}^\vee))$ induced by R_k.*

REMARK 3.23. One proves that all R_k, $k \geq 3$, arise as algebraic functions of $\mathrm{At}^{\nabla}_{(\mathcal{M},Q)}$, the curvature of ∇, their higher covariant derivatives, and compositions thereof [106].

As an immediate consequence of Theorem 3.22, we have

COROLLARY 3.24 ([86]). *Let $(\mathcal{M}, Q)$ be a dg-manifold and let ∇ be a torsion-free affine connection on $\mathcal{M}$. There exists a sequence $(\lambda_k)_{k\geqslant 2}$ of maps $\lambda_k \in \mathrm{Hom}(S^k(T_{\mathcal{M}}), T_{\mathcal{M}}[-1])$ starting with $\lambda_2 := \mathrm{At}^{\nabla}_{(\mathcal{M},Q)} \in \mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}[-1])$ which, together with $\lambda_1 := \mathcal{L}_Q : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, satisfy the $L_\infty[1]$ algebra axioms. As a consequence, the space of vector fields $\mathfrak{X}(\mathcal{M})$ on a dg manifold $(\mathcal{M}, Q)$ admits an $L_\infty[1]$ algebra structure with the Lie derivative $\mathcal{L}_Q$ as unary bracket λ_1 and the Atiyah cocycle $\mathrm{At}^{\nabla}_{(\mathcal{M},Q)}$ as binary bracket λ_2.*

REMARK 3.25. It follows from Theorem 3.22 that D^∇ can be considered as a homological vector field on $T_{\mathcal{M}}$ and, therefore, $(T_{\mathcal{M}}, D^\nabla)$ is a dg manifold. Indeed, one should consider $(T_{\mathcal{M}}, D^\nabla)$ as the 'formal neighborhood' $\Delta^{(\infty)}$ of the diagonal Δ of the product dg manifold $(\mathcal{M} \times \mathcal{M}, (Q, Q))$: the PBW map pbw^∇ is by construction a formal exponential map identifying $T_{\mathcal{M}}$ to a 'formal neighborhood of the diagonal' of $\mathcal{M} \times \mathcal{M}$ as $\mathbb{Z}$-graded manifolds and equation (31) asserts that D^∇ is the homological vector field obtained on $T_{\mathcal{M}}$ by pullback of the vector field (Q, Q) on $\mathcal{M} \times \mathcal{M}$ through this formal exponential map. The readers are invited to compare Theorem 3.22 with [55, Theorem 2.8.2]. In fact, the construction in Theorem 3.22 was very much inspired by Kapranov's construction [55, Theorem 2.8.2].

We can prove the following

THEOREM 3.26 ([106]). *The Atiyah class $\alpha_{(\mathcal{M},Q)}$ of a dg manifold $(\mathcal{M}, Q)$ vanishes if and only if there exists a torsion-free affine connection ∇ on $\mathcal{M}$ such that*

$$\mathrm{pbw}^\nabla : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{D}(\mathcal{M})$$

is an isomorphism of dg coalgebras over $C^\infty(\mathcal{M})$.

4. Kontsevich–Duflo type theorems

4.1. Polyvector fields and polydifferential operators. Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold over $\mathbb{K}$. We use the symbol $(\mathcal{T}^0_{\mathrm{poly}}(\mathcal{M}))^q$ to denote the space of smooth functions of degree q on $\mathcal{M}$ and the symbol $(\mathcal{T}^p_{\mathrm{poly}}(\mathcal{M}))^q$ to denote the space $(\Gamma(\Lambda^p T_{\mathcal{M}}))^q = \Gamma\big(S^p(T_{\mathcal{M}}[-1])[p]\big)^q$ of p-vector fields of degree q on $\mathcal{M}$.[1] In other words, an element in $(\mathcal{T}^p_{\mathrm{poly}}(\mathcal{M}))^q$ is a finite sum $\sum X_1 \wedge \ldots \wedge X_p$, where $X_1, \ldots, X_p \in \Gamma(T_{\mathcal{M}})$ are homogeneous vector fields on $\mathcal{M}$ with $|X_1| + \ldots + |X_p| = q$. The bigraded left $\mathcal{R}$-module

$$(\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M}))^\bullet = \bigoplus_{\substack{p,q\in\mathbb{Z} \\ p\geqslant 0}} (\mathcal{T}^p_{\mathrm{poly}}(\mathcal{M}))^q$$

[1] Note that the symbols $(\mathcal{T}^p_{\mathrm{poly}}(\mathcal{M}))^q$ and $(\mathcal{D}^p_{\mathrm{poly}}(\mathcal{M}))^q$ in this paper mean something slightly different than in [67]. Essentially, there is a degree shift between the conventions used in the two papers.

is called the *space of polyvector fields on* $\mathcal{M}$. We are most interested in the graded left $\mathcal{R}$-module $\operatorname{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))$ defined by

$$\operatorname{tot}^n_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})) = \bigoplus_{p+q=n} (\mathcal{T}^p_{\mathrm{poly}}(\mathcal{M}))^q.$$

When endowed with the graded commutator $[-,-]$, the space $(\mathcal{T}^1_{\mathrm{poly}}(\mathcal{M}))^\bullet = (\operatorname{Der}(\mathcal{R}))^\bullet$ of graded derivations of $\mathcal{R}$ is a graded Lie algebra. This Lie bracket can be extended to the space $(\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M}))^\bullet$ of graded polyvector fields on $\mathcal{M}$ in such a way that the triple

$$(\operatorname{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), [-,-], \wedge)$$

becomes a Gerstenhaber algebra:

$$[\xi, \eta_1 \wedge \eta_2] = [\xi, \eta_1] \wedge \eta_2 + (-1)^{(|\xi|-1)|\eta_1|} \eta_1 \wedge [\xi, \eta_2],$$

for $\xi \in (\mathcal{T}^{p_0}_{\mathrm{poly}}(\mathcal{M}))^{q_0}$, $\eta_1 \in (\mathcal{T}^{p_1}_{\mathrm{poly}}(\mathcal{M}))^{q_1}$, $\eta_2 \in (\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M}))^\bullet$ so that $|\xi| = p_0 + q_0$ and $|\eta_1| = p_1 + q_1$. This extended bracket is called Schouten bracket. Note that, under our degree convention, $\operatorname{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[1]$ is a graded Lie algebra under the Schouten bracket.

Finally, throwing in the zero differential

$$0 : \operatorname{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})) \to \operatorname{tot}^{\bullet+1}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})),$$

we obtain the dg Gerstenhaber algebra

$$(\operatorname{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), 0, [-,-], \wedge).$$

A linear differential operator of degree q on $\mathcal{M}$ is a $\mathbb{K}$-linear endomorphism of $\mathcal{R}$ that can be obtained as a finite sum $\sum X_1 \circ \ldots \circ X_k$ of compositions of graded derivations $X_1, \ldots, X_k$ of $\mathcal{R}$ with $|X_1| + \ldots + |X_k| = q$. We use the symbol $(\mathcal{D}(\mathcal{M}))^q$ to denote the space of linear differential operators of degree q on $\mathcal{M}$.

The space $\mathcal{D}^p_{\mathrm{poly}}(\mathcal{M})$ of p-differential operators on $\mathcal{M}$ admits a canonical identification with the tensor product of p copies of the left $\mathcal{R}$-module $\mathcal{D}(\mathcal{M})[-1]$. We use the symbol $(\mathcal{D}^p_{\mathrm{poly}}(\mathcal{M}))^q$ to denote the subspace of $\mathcal{D}^p_{\mathrm{poly}}(\mathcal{M})$ consisting of elements of degree $p+q$.

The bigraded left $\mathcal{R}$-module $(\mathcal{D}^\bullet_{\mathrm{poly}}(\mathcal{M}))^\bullet = \bigoplus_{\substack{p,q\in\mathbb{Z}\\ p\geqslant 0}} (\mathcal{D}^p_{\mathrm{poly}}(\mathcal{M}))^q$ is called the *space of polydifferential operators on* $\mathcal{M}$. We are most interested in the graded left $\mathcal{R}$-module $\operatorname{tot}^\bullet_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))$ defined by

$$\operatorname{tot}^n_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})) = \bigoplus_{p+q=n} (\mathcal{D}^p_{\mathrm{poly}}(\mathcal{M}))^q.$$

As in the classical case, endowing the space of polydifferential operators $\operatorname{tot}^\bullet_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))$ with the Gerstenhaber bracket $[\![-,-]\!]$ and the Hochschild differential

$$d_{\mathscr{H}} := [\![m,-]\!] : (\mathcal{D}^p_{\mathrm{poly}}(\mathcal{M}))^q \to (\mathcal{D}^{p+1}_{\mathrm{poly}}(\mathcal{M}))^q$$

makes $\operatorname{tot}^\bullet_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[1]$ into a differential graded Lie algebra (dgla in short). The tensor product of left $\mathcal{R}$-modules determines a cup product

$$(\mathcal{D}^p_{\mathrm{poly}}(\mathcal{M}))^q \times (\mathcal{D}^{p'}_{\mathrm{poly}}(\mathcal{M}))^{q'} \xrightarrow{\smile} (\mathcal{D}^{p+p'}_{\mathrm{poly}}(\mathcal{M}))^{q+q'},$$

which descends to Hochschild cohomology. When endowed with the cup product and the Gerstenhaber bracket, the cohomology of the cochain complex $(\mathrm{tot}_\oplus^\bullet(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}})$ becomes a Gerstenhaber algebra [23, Appendix].

For more details, the reader might wish to consult [67, 21, 119].

4.2. Formality and Kontsevich–Duflo type theorem for dg manifolds. Let $(\mathcal{M}, Q)$ be a finite-dimensional dg manifold. Since Q is a homological vector field of degree $+1$, it is a Maurer–Cartan element in the dgla of polyvector fields

$$(\mathrm{tot}_\oplus^\bullet(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[1], 0, [-,-]).$$

Therefore, we can consider the *tangent dgla* at the homological vector field Q:

$$(\mathrm{tot}_\oplus^\bullet(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[1])_Q := (\mathrm{tot}_\oplus^\bullet(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[1], [Q,-], [-,-]).$$

The associated (shifted) cohomology $H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q})$ is again a Gerstenhaber algebra with the associative multiplication induced by the wedge product. Here, by abuse of notation, we denote the differential $[Q,-]$ by $\mathcal{Q}$.

PROPOSITION 4.1 ([30]). *Let $(\mathcal{M}, Q) = (T_{\mathcal{F}}[1], d_{T_{\mathcal{F}}})$ be the dg manifold arising from a foliation $\mathcal{F}$ of the manifold M. Then the isomorphisms $\Phi^{\bullet,0}$ defined in* (26) *induce an isomorphism*

$$H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q}) \xrightarrow{\cong} \mathbb{H}_{\mathrm{dR}}^\bullet(\mathcal{F}, \Lambda N_{\mathcal{F}}). \tag{32}$$

REMARK 4.2. The hypercohomology group $\mathbb{H}_{\mathrm{dR}}^\bullet(\mathcal{F}, \Lambda N_{\mathcal{F}})$ should be understood as the space of polyvector fields on the leaf space of the foliation $\mathcal{F}$ or the space of *transversal polyvector fields.* A priori, it is not obvious whether $\mathbb{H}_{\mathrm{dR}}^\bullet(\mathcal{F}, \Lambda N_{\mathcal{F}})$ admits a Gerstenhaber algebra structure. However, it turns out that the obvious Gerstenhaber algebra structure carried by $H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q})$ can be transferred to $\mathbb{H}_{\mathrm{dR}}^\bullet(\mathcal{F}, \Lambda N_{\mathcal{F}})$ by way of the isomorphism (32). For more details, see [30, 122, 124, 123, 4] and also [5], where use is made of Fedosov dg Lie algebroids.

PROPOSITION 4.3 ([30]). *Let $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar\partial)$ be the dg manifold arising from a complex manifold X as in Example* 2.4. *Then the isomorphisms $\Phi^{\bullet,0}$ defined in* (29) *induce an isomorphism of Gerstenhaber algebras*

$$H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q}) \xrightarrow{\cong} \mathbb{H}_{\mathrm{sheaf}}^\bullet(X, \Lambda T_X).$$

Likewise, we can consider the *tangent dgla* at the Maurer–Cartan element Q of the dgla $\mathrm{tot}_\oplus^\bullet(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[1]$ of polydifferential operators:

$$(\mathrm{tot}_\oplus^\bullet(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[1])_Q := \big(\mathrm{tot}_\oplus^\bullet(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[1], d_{\mathscr{H}} + [\![Q,-]\!], [\![-,-]\!]\big).$$

The associated (shifted) cohomology $H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}} + \mathcal{Q})$ is a Gerstenhaber algebra with the cup product as associative multiplication. Again, to simplify the notation, we denote the differential $[\![Q,-]\!]$ by $\mathcal{Q}$.

The Hochschild–Kostant–Rosenberg map hkr is the natural inclusion of $(\mathcal{T}_{\mathrm{poly}}^\bullet(\mathcal{M}))^\bullet$ into $(\mathcal{D}_{\mathrm{poly}}^\bullet(\mathcal{M}))^\bullet$ defined by skew-symmetrization:

$$\mathrm{hkr}(X_1 \wedge \ldots \wedge X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \kappa(\sigma)\ X_{\sigma(1)} \otimes \ldots \otimes X_{\sigma(p)},$$

for all homogeneous vector fields $X_1, \dots, X_p \in (\mathcal{T}^1_{\mathrm{poly}}(\mathcal{M}))^\bullet$ — the skew Koszul sign $\kappa(\sigma)$ is the scalar defined by the relation $X_1 \wedge \dots \wedge X_p = \kappa(\sigma)\ X_{\sigma(1)} \wedge \dots \wedge X_{\sigma(p)}$. The Hochschild–Kostant–Rosenberg map is a morphism of double complexes

$$\mathrm{hkr} : ((\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M}))^\bullet, 0, \mathcal{Q}) \to ((\mathcal{D}^\bullet_{\mathrm{poly}}(\mathcal{M}))^\bullet, d_{\mathscr{H}}, \mathcal{Q}). \tag{33}$$

The following Hochschild–Kostant–Rosenberg theorem for dg manifolds follows from the HKR theorem for graded manifolds [23, Lemma A.2] and a spectral sequence argument.

PROPOSITION 4.4 ([67]). *Let $(\mathcal{M}, Q)$ be a finite-dimensional dg manifold. The Hochschild–Kostant–Rosenberg map* (33) *induces an isomorphism of vector spaces*

$$\mathrm{hkr} : H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q}) \xrightarrow{\cong} H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}} + \mathcal{Q})$$

on the cohomology level.

REMARK 4.5. Proposition 4.4 holds for direct sum total cohomology. The analogous assertion for direct product total cohomology is false; a counterexample can be found in [25].

The next theorem was conjectured by Shoikhet [111] and was known as *the Kontsevich–Shoikhet conjecture.*

THEOREM 4.6 (Kontsevich–Duflo type theorem for dg manifolds [67]). *For any finite-dimensional dg manifold $(\mathcal{M}, Q)$, the composition*

$$\mathrm{hkr} \circ (\mathrm{td}_{(\mathcal{M},Q)})^{1/2} : H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q}) \xrightarrow{\cong} H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}} + \mathcal{Q})$$

of (i) *the action of* $(\mathrm{td}_{(\mathcal{M},Q)})^{1/2} \in \prod_{k\geqslant 0} H^k((\Omega^k(\mathcal{M}))^\bullet, \mathcal{Q})$ *on* $H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q})$, *by contraction, with* (ii) *the Hochschild–Kostant–Rosenberg map* (*on cohomology*) *is an isomorphism of Gerstenhaber algebras.*

We also have

THEOREM 4.7 (formality theorem for dg manifolds [67]). *Let $(\mathcal{M}, Q)$ be a finite-dimensional dg manifold. Given a torsion-free affine connection ∇ on $\mathcal{M}$, there exists an L_∞ quasi-isomorphism of dglas*

$$\mathcal{I} : (\mathrm{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[1])_Q \rightsquigarrow (\mathrm{tot}^\bullet_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[1])_Q$$

with first Taylor coefficient

$$\mathcal{I}_1 = \mathrm{hkr} \circ (\mathrm{Td}^\nabla_{(\mathcal{M},Q)})^{1/2} : \mathrm{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[1] \to \mathrm{tot}^\bullet_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[1],$$

where

$$(\mathrm{Td}^\nabla_{(\mathcal{M},Q)})^{1/2} \in \prod_{k\geqslant 0} (\Gamma(\Lambda^k T^\vee_{\mathcal{M}}))^k \cong \prod_{k\geqslant 0} (\Omega^k(\mathcal{M}))^k$$

acts on $\mathrm{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[1]$ *by contraction.*

A formality theorem for $\mathbb{Z}$-graded manifolds was obtained by Cattaneo–Felder [23], who applied to the quantization of coisotropic submanifolds of Poisson manifolds.

REMARK 4.8. Given a pair of torsion-free affine connections ∇ and ∇' on $(\mathcal{M}, Q)$ with corresponding Todd cocycles $\mathrm{Td}^\nabla_{(\mathcal{M},Q)}$ and $\mathrm{Td}^{\nabla'}_{(\mathcal{M},Q)}$, there exists an L_∞ automorphism of the dgla $(\mathrm{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[1])_Q$ having the operator $(\mathrm{Td}^\nabla_{(\mathcal{M},Q)})^{-1/2} \circ (\mathrm{Td}^{\nabla'}_{(\mathcal{M},Q)})^{1/2}$ as first Taylor coefficient.

Theorem 4.7 can be used to study deformation quantization of (0-shifted) derived Poisson manifolds or P_∞-manifolds [94, 23, 4].

4.3. Application of the Kontsevich–Duflo type theorem. Theorem 4.6 can be specialized to various geometric situations. In particular, we can recover the Kontsevich–Duflo theorem for Lie algebras [37, 59, 95] and the Kontsevich theorem for complex manifolds [59, 17] and unify them in a common framework by considering two special classes of dg manifolds.

4.3.1. *Kontsevich–Duflo theorem for Lie algebras.* Let $(\mathcal{M}, Q)$ be the dg manifold $(\mathfrak{g}[1], d_{\mathrm{CE}})$ arising from a finite-dimensional Lie algebra $\mathfrak{g}$. By definition,

$$H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}} + \mathcal{Q}) \xrightarrow{\cong} HH^\bullet_\oplus(\Lambda\mathfrak{g}^\vee, \Lambda\mathfrak{g}^\vee) \tag{34}$$

is the direct sum Hochschild cohomology of the commutative differential graded algebra (cdga in short) $(\Lambda\mathfrak{g}^\vee, d_{\mathrm{CE}})$. Following a similar method of Shoikhet [114] using Keller dg category, or more precisely Keller admissible triple [56], one constructs a canonical isomorphism of Gerstenhaber algebras [64]

$$HH^\bullet_\oplus(\Lambda\mathfrak{g}^\vee, \Lambda\mathfrak{g}^\vee) \xrightarrow{\cong} HH^\bullet(\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g})). \tag{35}$$

According to the Cartan–Eilenberg theorem [20, Theorem 5.1],

$$HH^\bullet(\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g})) \xrightarrow{\cong} H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g})), \tag{36}$$

as associative algebras where the $\mathfrak{g}$-action on $\mathcal{U}(\mathfrak{g})$ is induced by the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$. Thus, by composing isomorphisms (34)–(36), one obtains an isomorphism of associative algebras

$$\phi : H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}} + \mathcal{Q}) \xrightarrow{\cong} H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g})). \tag{37}$$

Remark 4.9. An explicit chain map from the cochain complex computing $HH^\bullet_\oplus(\Lambda\mathfrak{g}^\vee, \Lambda\mathfrak{g}^\vee)$ to the cochain complex computing $H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$ was constructed in [16, Theorem 4.10 or indeed Lemma 4.12]. It is however not clear whether the isomorphism in cohomology induced by the map in [16] and the isomorphism (37) are the same isomorphism or not.

Proposition 4.10 ([64]). *Let $(\mathcal{M}, Q) := (\mathfrak{g}[1], d_{\mathrm{CE}})$ be the dg manifold corresponding to a finite-dimensional Lie algebra $\mathfrak{g}$. Then the diagrams*

$$\begin{array}{ccc} H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q}) & \xrightarrow{\mathrm{hkr}} & H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}} + \mathcal{Q}) \\ \cong \downarrow & & \phi \downarrow \cong \\ H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) & \xrightarrow[\mathrm{pbw}]{} & H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g})) \end{array}$$

and

$$\begin{array}{ccc} H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q}) & \xrightarrow{\mathrm{td}^{1/2}_{(\mathcal{M},Q)}} & H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q}) \\ \cong \downarrow & & \downarrow \cong \\ H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) & \xrightarrow[J^{1/2}]{} & H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) \end{array}$$

commute and the two vertical maps are isomorphisms of associative algebras.

COROLLARY 4.11 ([64]). *Let* $(\mathcal{M}, Q) := (\mathfrak{g}[1], d_{\mathrm{CE}})$ *be the dg manifold corresponding to a finite-dimensional Lie algebra* $\mathfrak{g}$. *Then the diagram*

$$\begin{array}{ccc} H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q}) & \xrightarrow{\mathrm{hkr}\circ \mathrm{td}^{1/2}_{(\mathcal{M},Q)}} & H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}} + \mathcal{Q}) \\ \cong\downarrow & & \downarrow\cong \\ H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) & \xrightarrow[\mathrm{pbw}\circ J^{1/2}]{} & H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g})) \end{array} \tag{38}$$

commutes and its two vertical maps are isomorphisms of associative algebras.

Theorem 4.6, together with Corollary 4.11, thus implies

THEOREM 4.12 (Kontsevich–Duflo theorem for Lie algebras [59, 95]). *For every finite-dimensional Lie algebra* $\mathfrak{g}$, *the map*

$$\mathrm{pbw}\circ J^{1/2} : H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\cong} H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$$

is an isomorphism of associative algebras.

Restriction of this isomorphism to the subalgebras consisting solely of the cohomology groups of degree 0 yields the classical Duflo theorem [37]: the composition $\mathrm{pbw}\circ J^{1/2} : S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ is an isomorphism of associative algebras. Duflo's theorem generalizes a fundamental result of Harish-Chandra regarding the center of the universal enveloping algebra of a semi-simple Lie algebra.

REMARK 4.13. Theorem 4.12 is due to Kontsevich [59]. See Pevzner–Torossian [95] for a detailed proof. (See also Manchon–Torossian [81, 82].) The approach followed in [59, 95] relies on the formality quasi-isomorphism $\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}^\vee)[1] \rightsquigarrow \mathcal{D}_{\mathrm{poly}}(\mathfrak{g}^\vee)[1]$ for the dual of a Lie algebra $\mathfrak{g}$ and its tangent map at $\pi_{\mathfrak{g}^\vee}$, the Lie–Poisson bivector on $\mathfrak{g}^\vee$ seen as a Maurer–Cartan element of $\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}^\vee)[1]$. In the present survey, however, we follow Shoikhet's approach [111, 113].

4.3.2. *Kontsevich theorem for complex manifolds.* Let $(\mathcal{M}, Q)$ be the dg manifold $(T^{0,1}_X[1], \bar\partial)$ arising from a complex manifold X as in Example 2.4. Recall that the Hochschild cohomology groups $HH^\bullet(X)$ of the complex manifold X are defined as the groups $\mathrm{Ext}^\bullet_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ [41, 19, 18, 58, 6, 83, 100, 101]. In fact, the Hochschild cohomology is indeed a Gerstenhaber algebra: its associative multiplication is the Yoneda product while its Lie bracket is the Gerstenhaber bracket. From the classical Hochschild–Kostant–Rosenberg map, Gerstenhaber–Schack [41] derived an isomorphism of cohomology groups

$$\mathrm{hkr} : \mathbb{H}^\bullet_{\mathrm{sheaf}}(X, \Lambda T_X) \xrightarrow{\cong} HH^\bullet(X).$$

PROPOSITION 4.14 ([30, 31]). *Let* $(\mathcal{M}, Q) = (T^{0,1}_X[1], \bar\partial)$ *be the dg manifold arising from a complex manifold* X. *Then the diagrams*

$$\begin{array}{ccc} H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q}) & \xrightarrow{\mathrm{hkr}} & H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), d_{\mathscr{H}} + \mathcal{Q}) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{H}^\bullet_{\mathrm{sheaf}}(X, \Lambda T_X) & \xrightarrow[\mathrm{hkr}]{} & HH^\bullet(X) \end{array}$$

and

$$\begin{array}{ccc} H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})),\mathcal{Q}) & \xrightarrow{\mathrm{td}^{1/2}_{(\mathcal{M},Q)}} & H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})),\mathcal{Q}) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{H}^\bullet_{\mathrm{sheaf}}(X,\Lambda T_X) & \xrightarrow[\mathrm{td}^{1/2}_{T^{\mathbb{C}}_X/T^{0,1}_X}]{} & \mathbb{H}^\bullet_{\mathrm{sheaf}}(X,\Lambda T_X) \end{array}$$

commute and the two vertical maps are isomorphisms of associative algebras (*and indeed isomorphisms of Gerstenhaber algebras*)*.*

Corollary 4.15 ([31]). *Let* $(\mathcal{M},Q) := (T^{0,1}_X[1],\bar\partial)$ *be the dg manifold arising from a complex manifold* X*. Then the diagram*

$$\begin{array}{ccc} H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})),\mathcal{Q}) & \xrightarrow{\mathrm{hkr}\circ\mathrm{td}^{1/2}_{(\mathcal{M},Q)}} & H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})),d_{\mathscr{H}}+\mathcal{Q}) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{H}^\bullet_{\mathrm{sheaf}}(X,\Lambda T_X) & \xrightarrow[\mathrm{hkr}\circ\mathrm{td}^{1/2}_{T^{\mathbb{C}}_X/T^{0,1}_X}]{} & HH^\bullet(X) \end{array} \tag{39}$$

commutes and its two vertical maps are isomorphisms of associative algebras (*and indeed isomorphisms of Gerstenhaber algebras*)*.*

Combining Theorem 4.6 with Corollary 4.15, we recover

Theorem 4.16 (Kontsevich–Duflo theorem for complex manifolds [59, 17, 67, 68]). *For every complex manifold* X*, the composition*

$$\mathrm{hkr}\circ(\mathrm{td}_{T^{\mathbb{C}}_X/T^{0,1}_X})^{1/2} : \mathbb{H}^\bullet_{\mathrm{sheaf}}(X,\Lambda T_X) \xrightarrow{\cong} HH^\bullet(X)$$

is an isomorphism of associative algebras. It is understood that the square root of the Todd class

$$\mathrm{td}_{T^{\mathbb{C}}_X/T^{0,1}_X} \in \bigoplus_{k=0} H^{k,k}(X) \cong \bigoplus_{k=0} H^k_{\mathrm{sheaf}}(X,\Omega^k_X)$$

acts on $\mathbb{H}^\bullet_{\mathrm{sheaf}}(X,\Lambda T_X)$ *by contraction.*

Remark 4.17. The Kontsevich theorem for complex manifolds is due to Kontsevich [59] — the theorem pertains to the associative algebra structures only. A detailed proof appeared later in [17], where the additional Gerstenhaber algebra structures were also addressed. The approach followed in the present survey yields a different proof in terms of dg manifolds [67, 31].

4.4. Kontsevich–Duflo type theorem for dg Lie algebroids. The same way Lie algebroids can be seen as generalizations of tangent bundles, dg Lie algebroids can be considered as generalizations of the tangent bundle $T_{\mathcal{M}} \to \mathcal{M}$ of a dg manifold $(\mathcal{M},Q)$. In particular, one can make sense of 'polyvector fields' and 'polydifferential operators' for dg Lie algebroids just as one does for dg manifolds.

More precisely, a k-vector field on a dg Lie algebroid $\mathcal{L}\to\mathcal{M}$ is a section of the vector bundle $\Lambda^k\mathcal{L}\to\mathcal{M}$, while a k-differential operator is an element of $(s\mathcal{U}(\mathcal{L}))^{\otimes k}$, the tensor product (as left $C^\infty(\mathcal{M})$-modules) of k copies of the shifted universal enveloping algebra $s\mathcal{U}(\mathcal{L}) := \mathcal{U}(\mathcal{L})[-1]$. Denote by $\operatorname{tot}^\bullet_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L}))$ the graded left $\mathcal{R}$-module defined by

$$\operatorname{tot}^n_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})) = \bigoplus_{p+q=n} (\Gamma(\Lambda^p\mathcal{L}))^q,$$

where $(\Gamma(\Lambda^p\mathcal{L}))^q$ is the space of p-vector fields on $\mathcal{L}$ of degree q. Similarly, denote by $\operatorname{tot}^\bullet_\oplus(\mathscr{D}_{\mathrm{poly}}(\mathcal{L}))$ the graded left $\mathcal{R}$-module defined by

$$\operatorname{tot}^n_\oplus(\mathscr{D}_{\mathrm{poly}}(\mathcal{L})) = \bigoplus_{p+q=n} ((s\mathcal{U}(\mathcal{L}))^{\otimes p})^q,$$

where $((s\mathcal{U}(\mathcal{L}))^{\otimes p})^q$ is understood as the space of p-differential operators on $\mathcal{L}$ of degree q.

It is clear that the differential $\mathcal{Q} : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})$ and the homological vector field Q on $\mathcal{M}$ extend naturally to a differential

$$\mathcal{Q} : (\Gamma(\Lambda^k\mathcal{L}))^\bullet \to (\Gamma(\Lambda^k\mathcal{L}))^{\bullet+1}$$

of degree $(+1)$ and the Lie algebroid structure on $\mathcal{L}$ yields a Schouten bracket

$$[-,-] : \Gamma(\Lambda^u\mathcal{L}) \otimes \Gamma(\Lambda^v\mathcal{L}) \to \Gamma(\Lambda^{u+v-1}\mathcal{L}).$$

The universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of a dg Lie algebroid $\mathcal{L} \to \mathcal{M}$ is a (left) dg coalgebra over the differential graded algebra $\mathcal{R} := C^\infty(\mathcal{M})$ [131]. Its comultiplication

$$\Delta : \mathcal{U}(\mathcal{L}) \to \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{R}} \mathcal{U}(\mathcal{L})$$

is characterized by the identities

$$\Delta(1) = 1 \otimes 1;$$
$$\Delta(b) = 1 \otimes b + b \otimes 1, \quad \forall\, b \in \Gamma(\mathcal{L});$$
$$\Delta(u \cdot v) = \Delta(u) \cdot \Delta(v), \quad \forall\, u, v \in \mathcal{U}(\mathcal{L}),$$

where the symbol $\cdot$ denotes the multiplication in $\mathcal{U}(\mathcal{L})$. We refer the reader to [131, equation (15) and the remark following Definition 3.1] for the precise meaning of the last equation above. Explicitly, we have

$$\begin{aligned}\Delta(b_1 \cdot b_2 \cdots b_n) &= 1 \otimes (b_1 \cdot b_2 \cdots b_n)\\ &+ \sum_{\substack{p+q=n\\ p,q\in\mathbb{N}}} \sum_{\sigma\in\mathfrak{S}_p^q} \epsilon(\sigma; b_1, \dots, b_n)(b_{\sigma(1)} \cdots b_{\sigma(p)}) \otimes (b_{\sigma(p+1)} \cdots b_{\sigma(n)}) + (b_1 \cdot b_2 \cdots b_n) \otimes 1,\end{aligned}$$

for all $b_1, \dots, b_n \in \Gamma(\mathcal{L})$. Here $\epsilon(\sigma; b_1, \dots, b_n)$ denotes the Koszul sign of the permutation σ of the homogeneous elements $b_1, \dots, b_n \in \Gamma(\mathcal{L})$ and $\mathfrak{S}_p^q$ denotes the space of (p, q)-shuffles.

The differential $\mathcal{Q} : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})$ and the homological vector field $Q : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ naturally induces a differential $\mathcal{Q} : \mathcal{U}(\mathcal{L}) \to \mathcal{U}(\mathcal{L})$ of degree $(+1)$, which is compatible with both the algebra and coalgebra structures on $\mathcal{U}(\mathcal{L})$. Indeed, $\mathcal{U}(\mathcal{L})$ is a *dg Hopf algebroid* over the cdga $\mathcal{R} := C^\infty(\mathcal{M})$. As a consequence, we obtain a differential

$$\mathcal{Q} : (s\mathcal{U}(\mathcal{L})^{\otimes k})^\bullet \to (s\mathcal{U}(\mathcal{L})^{\otimes k})^{\bullet+1}$$

of degree $(+1)$ for all $k \geqslant 0$. A Hochschild coboundary differential

$$d_{\mathscr{H}} : s\mathcal{U}(\mathcal{L})^{\otimes k} \to s\mathcal{U}(\mathcal{L})^{\otimes k+1}$$

and a Gerstenhaber bracket

$$[\![-,-]\!] : s\mathcal{U}(\mathcal{L})^{\otimes u} \otimes s\mathcal{U}(\mathcal{L})^{\otimes v} \to s\mathcal{U}(\mathcal{L})^{\otimes u+v-1} \tag{40}$$

can be defined by the following explicit algebraic expressions:

$$d_{\mathscr{H}}(u_1 \otimes \ldots \otimes u_k) = (\pm)1 \otimes u_1 \otimes \ldots \otimes u_k + \sum_{i=1}^{k} (\pm) u_1 \otimes \ldots \otimes \widehat{\Delta}(u_i) \otimes \ldots \otimes u_k + (\pm) u_1 \otimes \ldots \otimes u_k \otimes 1, \tag{41}$$

and

$$[\![\phi, \psi]\!] = \phi \star \psi - (\pm)\psi \star \phi \in s\mathcal{U}(\mathcal{L})^{\otimes u+v-1}, \tag{42}$$

where $\phi \star \psi \in \mathcal{U}(\mathcal{L})^{\otimes u+v-1}$ is defined by

$$\phi \star \psi = \sum_{k=0}^{u} (\pm) d_1 \otimes \ldots \otimes d_{k-1} \otimes (\widehat{\Delta}^{v-1} d_k) \cdot \psi \otimes d_{k+1} \otimes \ldots \otimes d_u$$

if $\phi = d_1 \cdots d_u$ for some $d_1, \ldots, d_u \in s\mathcal{U}(\mathcal{L})$. Here $\widehat{\Delta} : s\mathcal{U}(\mathcal{L}) \to s\mathcal{U}(\mathcal{L}) \otimes_{\mathcal{R}} s\mathcal{U}(\mathcal{L})$ is the map induced by the coproduct Δ on $\mathcal{U}(\mathcal{L})$.

Again we refer the reader to [131, equation (15) and the remark following Definition 3.1] for the precise meaning of the product $(\widehat{\Delta}^{v-1} d_k) \cdot \psi$ in $s\mathcal{U}(\mathcal{L})^{\otimes v}$ appearing in the last equation above.

PROPOSITION 4.18. *Let $\mathcal{L}$ be a dg Lie algebroid over $\mathcal{M}$.*

- *When endowed with the wedge product and the Schouten bracket, the cohomology $H^\bullet(\mathrm{tot}_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})), \mathcal{Q})$ is a Gerstenhaber algebra.*
- *When endowed with the cup product (i.e. the tensor product $\otimes_{C^\infty(\mathcal{M})}$) and the Gerstenhaber bracket, the Hochschild cohomology $H^\bullet(\mathrm{tot}_\oplus(\mathscr{D}_{\mathrm{poly}}(\mathcal{L})), d_{\mathscr{H}} + \mathcal{Q})$, is a Gerstenhaber algebra.*

The Kontsevich–Duflo type theorem for dg manifolds (Theorem 4.6) can be extended to this general context. It suffices to adapt the proof outlined in [67].

Define the Hochschild–Kostant–Rosenberg map

$$\mathrm{hkr} : (\Gamma(\Lambda^p \mathcal{L}))^q \hookrightarrow (s\mathcal{U}(\mathcal{L})^{\otimes p})^q$$

by skew-symmetrization:

$$\mathrm{hkr}(X_1 \wedge \ldots \wedge X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \kappa(\sigma)\ X_{\sigma(1)} \otimes \ldots \otimes X_{\sigma(p)}$$

for all homogeneous vector fields $X_1, \ldots, X_p \in (\Gamma(\Lambda^1 \mathcal{L}))^\bullet$ — the skew Koszul sign $\kappa(\sigma)$ is the scalar defined by the relation $X_1 \wedge \ldots \wedge X_p = \kappa(\sigma) X_{\sigma(1)} \wedge \ldots \wedge X_{\sigma(p)}$. The Hochschild–Kostant–Rosenberg map is a morphism of double complexes

$$\mathrm{hkr} : ((\Gamma(\Lambda^\bullet \mathcal{L}))^\bullet, 0, \mathcal{Q}) \to ((s\mathcal{U}(\mathcal{L})^{\otimes \bullet})^\bullet, d_{\mathscr{H}}, \mathcal{Q}). \tag{43}$$

Therefore it induces a chain map between total complexes

$$\mathrm{hkr} : (\mathrm{tot}_\oplus^\bullet(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})), \mathcal{Q}) \to (\mathrm{tot}_\oplus^\bullet(\mathscr{D}_{\mathrm{poly}}(\mathcal{L})), d_{\mathscr{H}} + \mathcal{Q}). \tag{44}$$

We have the following Kontsevich–Duflo type theorem for dg Lie algebroids.

THEOREM 4.19. *For every dg Lie algebroid $\mathcal{L}$, the composition*

$$\mathrm{hkr}\circ(\mathrm{td}_{\mathcal{L}})^{1/2}: H^\bullet(\mathrm{tot}_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})),\mathcal{Q}) \xrightarrow{\cong} H^\bullet(\mathrm{tot}_\oplus(\mathscr{D}_{\mathrm{poly}}(\mathcal{L})), d_{\mathscr{H}}+\mathcal{Q})$$

is an isomorphism of Gerstenhaber algebras. It is understood that

$$(\mathrm{td}_{\mathcal{L}})^{1/2}\in\prod_{k\geqslant 0} H^k((\Gamma(\Lambda^k\mathcal{L}^\vee))^\bullet,\mathcal{Q})$$

acts on $H^\bullet(\mathrm{tot}_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})),\mathcal{Q})$ by contraction.

REMARK 4.20. While Theorem 4.19 looks similar to [17, Corollary 1.4], it is a very different result. Theorem 4.19 pertains to *dg* Lie algebroids, while [17, Corollary 1.4] applies to Lie algebroids understood as sheaves of Lie–Rinehart algebras — standard Lie–Rinehart algebras rather than dg Lie–Rinehart algebras.

4.5. Kontsevich–Duflo type theorem for matched pairs. Let (A,B) be a matched pair of Lie algebroids over $\mathbb{K}$. According to Example 3.6, $A[1]\oplus B\to A[1]$ is a dg Lie algebroid. Denote by $(\mathcal{L},Q)$ the dg manifold $(A[1]\oplus B, d_A^{\mathrm{Bott}})$. The space of sections of $\mathcal{L}\to A[1]$ can be naturally identified with $\Gamma(\Lambda^\bullet A^\vee\otimes B)$. The bracket on $\Gamma(\Lambda^\bullet A^\vee\otimes B)$ is defined in terms of the Bott B-connection on $\Lambda A^\vee$ by

$$[\xi_1\otimes b_1,\xi_2\otimes b_2]=\xi_1\wedge\xi_2\otimes[b_1,b_2]+\xi_1\wedge(\nabla^{\mathrm{Bott}}_{b_1}\xi_2)\otimes b_2-(\nabla^{\mathrm{Bott}}_{b_2}\xi_1)\wedge\xi_2\otimes b_1 \tag{45}$$

for all $\xi_1,\xi_2\in\Gamma(\Lambda^\bullet A^\vee)$ and $b_1,b_2\in\Gamma(B)$, while the anchor map $\Gamma(\Lambda^\bullet A^\vee\otimes B)\xrightarrow{\bar\rho}\mathrm{Der}(\Lambda^\bullet A^\vee)$ is defined by

$$\bar\rho_{\xi\otimes b}(\eta)=\xi\wedge\nabla^{\mathrm{Bott}}_b\eta, \tag{46}$$

for all $\xi,\eta\in\Gamma(\Lambda^\bullet A^\vee)$ and $b\in\Gamma(B)$. Finally, the induced differential $\mathcal{Q}$ on the space of sections of $\mathcal{L}\to A[1]$ is the Chevalley–Eilenberg differential $d_A^{\mathrm{Bott}}:\Gamma(\Lambda^\bullet A^\vee\otimes B)\to\Gamma(\Lambda^{\bullet+1}A^\vee\otimes B)$ corresponding to the Bott A-connection on B. It is clear that

$$\mathrm{tot}^n_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L}))\cong\bigoplus_{p+q=n}\Gamma(\Lambda^q A^\vee\otimes\Lambda^p B).$$

Hence the induced differential on $\mathrm{tot}^\bullet_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L}))$, the space of polyvector fields for the dg Lie algebroid $\mathcal{L}\to A[1]$, is the Chevalley–Eilenberg differential

$$d_A^{\mathrm{Bott}}:\Gamma(\Lambda^q A^\vee\otimes\Lambda^p B)\to\Gamma(\Lambda^{q+1}A^\vee\otimes\Lambda^p B) \tag{47}$$

corresponding to the Bott A-connection on ΛB and the Lie bracket on $\mathrm{tot}^\bullet_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L}))$ is the Schouten bracket of the dg Lie algebroid $\mathcal{L}\to A[1]$ — essentially the extension of equations (45) and (46) by the graded Leibniz rule. Hence we obtain the following

LEMMA 4.21 ([5]). *Let (A,B) be a matched pair of Lie algebroids, and let $\mathcal{L}:=A[1]\oplus B$ be the corresponding dg Lie algebroid over $(A[1],d_A)$. When endowed with the wedge product and the Schouten bracket, the cohomology $\mathbb{H}^\bullet_{\mathrm{CE}}(A,\Lambda B)$ is a Gerstenhaber algebra, and we have an isomorphism of Gerstenhaber algebras*

$$H^\bullet(\mathrm{tot}_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})),\mathcal{Q})\cong\mathbb{H}^\bullet_{\mathrm{CE}}(A,\Lambda B). \tag{48}$$

Next, consider the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of the dg Lie algebroid $\mathcal{L}\to A[1]$, which is a dg Hopf algebroid over $(\Gamma(\Lambda^\bullet A^\vee),d_A)$. It is clear that $\mathcal{U}(\mathcal{L})\cong\Gamma(\Lambda^\bullet A^\vee)\otimes_R\mathcal{U}(B)$

and $s\mathcal{U}(\mathcal{L})^{\otimes p} \cong \Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{U}(B)^{\otimes p}$, where $R = C^\infty(M)$. Under this identification, the differential $\mathcal{Q} : s\mathcal{U}(\mathcal{L})^{\otimes p} \to s\mathcal{U}(\mathcal{L})^{\otimes p}$ becomes the Chevalley–Eilenberg differential

$$d_A^{\mathcal{U}} : \Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{U}(B)^{\otimes p} \to \Gamma(\Lambda^{\bullet+1} A^\vee) \otimes_R \mathcal{U}(B)^{\otimes p}. \tag{49}$$

Here the A-module structure on $\mathcal{U}(B)$ follows from the canonical identification of $\mathcal{U}(B)$ with $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ — the Lie algebroid A acts on the latter by multiplication in $\mathcal{U}(L)$ from the left — and extends to an A-module structure on $\mathcal{U}(B)^{\otimes p}$ in the natural way. Moreover, we have the isomorphism

$$\mathrm{tot}_\oplus^n(\mathscr{D}_{\mathrm{poly}}(\mathcal{L})) \cong \bigoplus_{p+q=n} \Gamma(\Lambda^q A^\vee) \otimes_R \mathcal{U}(B)^{\otimes p}. \tag{50}$$

Under the isomorphism (50), the total differential $\mathcal{Q} + d_{\mathscr{H}}$ on $\mathrm{tot}_\oplus^\bullet(\mathscr{D}_{\mathrm{poly}}(\mathcal{L}))$ corresponds to $d_A^{\mathcal{U}} + \mathrm{id} \otimes d_{\mathscr{H}}$ on $\bigoplus_{p+q=\bullet} \Gamma(\Lambda^q A^\vee) \otimes_R \mathcal{U}(B)^{\otimes p}$, where $d_{\mathscr{H}} : \mathcal{U}(B)^{\otimes k} \to \mathcal{U}(B)^{\otimes k+1}$ is the Hochschild differential for the Lie algebroid B defined by equation (41). Recall that, the degree of the operator $d_{\mathscr{H}}$ being $+1$, the usual sign convention for the tensor product of linear maps in the presence of gradings dictates that

$$(\mathrm{id} \otimes d_{\mathscr{H}})(\omega \otimes u) = (-1)^q \omega \otimes d_{\mathscr{H}}(u), \quad \forall\, \omega \in \Gamma(\Lambda^q A^\vee),\ \forall\, u \in \mathcal{U}(B)^{\otimes p}.$$

The cohomology of the total complex

$$\Big(\bigoplus_{p+q=\bullet} \Gamma(\Lambda^q A^\vee) \otimes_R \mathcal{U}(B)^{\otimes p}, d_A^{\mathcal{U}} + \mathrm{id} \otimes d_{\mathscr{H}}\Big)$$

will be denoted by

$$\mathbb{H}_{\mathrm{CE}}^\bullet\big(A, \mathcal{U}(B)^{\otimes \diamond} \xrightarrow{d_{\mathscr{H}}} \mathcal{U}(B)^{\otimes \diamond + 1}\big).$$

Summarizing the discussion above, we have the following

Lemma 4.22 ([5]). *Let (A, B) be a matched pair of Lie algebroids, and let $\mathcal{L} := A[1] \oplus B$ be the corresponding dg Lie algebroid over $(A[1], d_A)$. When endowed with the cup product and the Gerstenhaber bracket, the Hochschild hypercohomology $\mathbb{H}_{\mathrm{CE}}^\bullet\big(A, \mathcal{U}(B)^{\otimes \diamond} \xrightarrow{d_{\mathscr{H}}} \mathcal{U}(B)^{\otimes \diamond+1}\big)$ is a Gerstenhaber algebra, and we have an isomorphism of Gerstenhaber algebras*

$$H^\bullet(\mathrm{tot}_\oplus(\mathscr{D}_{\mathrm{poly}}(\mathcal{L})), d_{\mathscr{H}} + \mathcal{Q}) \cong \mathbb{H}_{\mathrm{CE}}^\bullet\big(A, \mathcal{U}(B)^{\otimes \diamond} \xrightarrow{d_{\mathscr{H}}} \mathcal{U}(B)^{\otimes \diamond+1}\big). \tag{51}$$

Remark 4.23. Note that the Gerstenhaber bracket on $\bigoplus_{p+q=\bullet} \Gamma(\Lambda^q A^\vee) \otimes_R \mathcal{U}(B)^{\otimes p}$ is *not* the obvious extension of the Gerstenhaber bracket on $\mathcal{U}(B)^{\otimes \bullet}$ obtained by tensoring with the commutative associative algebra $\Gamma(\Lambda^\bullet A^\vee)$. The explicit formula for the Gerstenhaber bracket is quite complicated and involves the Bott representation of B on $\Gamma(\Lambda^\bullet A^\vee)$.

Proposition 4.24. *Let (A, B) be a matched pair of Lie algebroids, and let $\mathcal{L} := A[1] \oplus B$ be the corresponding dg Lie algebroid over $(A[1], d_A)$. The diagrams*

$$\begin{array}{ccc}
H^\bullet(\mathrm{tot}_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})), \mathcal{Q}) & \xrightarrow{\mathrm{hkr}} & H^\bullet(\mathrm{tot}_\oplus(\mathscr{D}_{\mathrm{poly}}(\mathcal{L})), d_{\mathscr{H}} + \mathcal{Q}) \\
\cong \downarrow & & \downarrow \cong \\
\mathbb{H}_{\mathrm{CE}}^\bullet(A, \Lambda B) & \xrightarrow[\mathrm{hkr}]{} & \mathbb{H}_{\mathrm{CE}}^\bullet\big(A, \mathcal{U}(B)^{\otimes \diamond} \xrightarrow{d_{\mathscr{H}}} \mathcal{U}(B)^{\otimes \diamond+1}\big)
\end{array}$$

and

$$\begin{array}{ccc} H^\bullet(\mathrm{tot}_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})),\mathcal{Q}) & \xrightarrow{(\mathrm{td}_{\mathcal{L}})^{1/2}} & H^\bullet(\mathrm{tot}_\oplus(\mathscr{D}_{\mathrm{poly}}(\mathcal{L})),d_{\mathscr{H}}+\mathcal{Q}) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{H}^\bullet_{\mathrm{CE}}(A,\Lambda B) & \xrightarrow[(\mathrm{td}_{A\bowtie B/A})^{1/2}]{} & \mathbb{H}^\bullet_{\mathrm{CE}}\big(A,\mathcal{U}(B)^{\otimes\diamond}\xrightarrow{d_{\mathscr{H}}}\mathcal{U}(B)^{\otimes\diamond+1}\big), \end{array}$$

commute and the two vertical maps are isomorphisms of Gerstenhaber algebras. Here $\mathrm{td}_{A\bowtie B/A}\in\bigoplus_{k\geqslant 0}H^k_{\mathrm{CE}}(A,\Lambda^k B^\vee)$ *denotes the Todd class of the matched pair* (A,B)*, i.e. the Todd class of the Lie pair* $(A\bowtie B,A)$ *— see equation* (54) *below.*

Combining Theorem 4.19 with Proposition 4.24, we obtain

Theorem 4.25 (Kontsevich–Duflo type theorem for matched pairs [68]). *For every matched pair of Lie algebroids* (A,B)*, the composition*

$$\mathrm{hkr}\circ(\mathrm{td}_{A\bowtie B/A})^{1/2}:\mathbb{H}^\bullet_{\mathrm{CE}}(A,\Lambda B)\xrightarrow{\cong}\mathbb{H}^\bullet_{\mathrm{CE}}\big(A,\mathcal{U}(B)^{\otimes\diamond}\xrightarrow{d_{\mathscr{H}}}\mathcal{U}(B)^{\otimes\diamond+1}\big)$$

where $(\mathrm{td}_{A\bowtie B/A})^{1/2}\in\bigoplus_{k\geqslant 0}H^k_{\mathrm{CE}}(A,\Lambda^k B^\vee)$ *acts on* $H^\bullet_{\mathrm{CE}}(A,\Lambda B)$ *by contraction, is an isomorphism of Gerstenhaber algebras.*

As an example of matched pair of Lie algebroids, consider a smooth manifold M (with algebra of smooth functions $R:=C^\infty(M)$) endowed with an infinitesimal action $\mathfrak{g}\ni a\mapsto\hat{a}\in\mathfrak{X}(M)$ of a finite-dimensional Lie algebra $\mathfrak{g}$, i.e. a $\mathfrak{g}$-manifold. Then $(A:=\mathfrak{g}\ltimes M, B:=T_M)$ is a matched pair of Lie algebroids. Its Atiyah class

$$\alpha_{M/\mathfrak{g}}\in H^1_{\mathrm{CE}}(A,B^\vee\otimes\mathrm{End}\,B)\cong H^1_{\mathrm{CE}}(\mathfrak{g},\Gamma(T^\vee_M\otimes\mathrm{End}\,T_M))$$

is the cohomology class of the Atiyah 1-cocycle

$$\mathrm{At}^\nabla_{M/\mathfrak{g}}:\mathfrak{g}\times\mathfrak{X}(M)\to\mathrm{End}_R\,\mathfrak{X}(M)$$

corresponding to any affine connection ∇ on M, which is defined by the relation

$$\mathrm{At}^\nabla_{M/\mathfrak{g}}(a,X)=\mathcal{L}_{\hat{a}}\circ\nabla_X-\nabla_X\circ\mathcal{L}_{\hat{a}}-\nabla_{\mathcal{L}_{\hat{a}}X},$$

for all $a\in\mathfrak{g}$ and $X\in\mathfrak{X}(M)$. Its Todd class is

$$\mathrm{td}_{M/\mathfrak{g}}=\det\left(\frac{\alpha_{M/\mathfrak{g}}}{1-\exp(-\alpha_{M/\mathfrak{g}})}\right)\in\bigoplus_{k=0}H^k_{\mathrm{CE}}(\mathfrak{g},\Omega^k(M)).$$

Note that the Hochschild cochain complex

$$\ldots\to\mathcal{D}^k_{\mathrm{poly}}(M)\xrightarrow{d_{\mathscr{H}}}\mathcal{D}^{k+1}_{\mathrm{poly}}(M)\to\ldots$$

of the $\mathfrak{g}$-manifold M is a complex of $\mathfrak{g}$-modules.

Applying Theorem 4.25 to matched pairs of the type $(\mathfrak{g}\ltimes M,T_M)$, we obtain

Theorem 4.26 (Kontsevich–Duflo type theorem for $\mathfrak{g}$-manifolds [66, 68]). *Given a* $\mathfrak{g}$*-manifold* M*, the map*

$$\mathrm{hkr}\circ\mathrm{td}^{1/2}_{M/\mathfrak{g}}:\mathbb{H}^\bullet_{\mathrm{CE}}(\mathfrak{g},\mathcal{T}_{\mathrm{poly}}(M))\xrightarrow{\cong}\mathbb{H}^\bullet_{\mathrm{CE}}\big(\mathfrak{g},\mathcal{D}^\diamond_{\mathrm{poly}}(M)\xrightarrow{d_{\mathscr{H}}}\mathcal{D}^{\diamond+1}_{\mathrm{poly}}(M)\big)$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\mathrm{td}^{1/2}_{M/\mathfrak{g}}$ *of the Todd class* $\mathrm{td}_{M/\mathfrak{g}}\in\bigoplus_{k=0}H^k_{\mathrm{CE}}(\mathfrak{g},\Omega^k(M))$ *acts on* $\mathbb{H}^\bullet_{\mathrm{CE}}(\mathfrak{g},\mathcal{T}_{\mathrm{poly}}(M))$ *by contraction.*

4.6. Kontsevich–Duflo type theorem for Lie pairs. In this section, we extend the Kontsevich–Duflo type theorem for matched pairs (Theorem 4.25) to arbitrary Lie pairs (L, A), i.e. inclusions $i : A \to L$ of Lie algebroids over the same base manifold M. Though $A[1] \oplus L/A \to A[1]$ is no longer a dg Lie algebroid in this case, it is still an L_∞ algebroid [4]. Hence, we cannot apply Theorem 4.19 directly to $A[1] \oplus L/A \to A[1]$ as in the case of a matched pair, but we must first replace the L_∞ algebroid $A[1] \oplus L/A \to A[1]$ by a 'homotopy equivalent' dg Lie algebroid. The dg Lie algebroid $T_{A[1]} \times_{T_M} L \to A[1]$ associated to the Lie algebroid morphism $i : A \to L$ as described in Section 3.2 is a natural candidate.

Denoting the algebra of smooth functions on M by R, we set $\mathcal{T}^k_{\mathrm{poly}}(L/A) = \Gamma(\Lambda^k(L/A))$ for $k \geqslant 0$, and $\mathcal{T}^\bullet_{\mathrm{poly}}(L/A) = \bigoplus_{k=0}^{\infty} \mathcal{T}^k_{\mathrm{poly}}(L/A)$[2]. The Bott flat A-connection on L/A is defined by [29]

$$\nabla^{\mathrm{Bott}}_a q(l) = q([a, l]), \quad \forall\, a \in \Gamma(A),\ l \in \Gamma(L/A), \tag{52}$$

where $q : L \to L/A$ denotes the canonical projection. The Bott A-connection on L/A makes every $\mathcal{T}^k_{\mathrm{poly}}(L/A)$ an A-module. We can thus consider the complex of A-modules with trivial differential

$$0 \longrightarrow \mathcal{T}^0_{\mathrm{poly}}(L/A) \xrightarrow{0} \mathcal{T}^1_{\mathrm{poly}}(L/A) \xrightarrow{0} \mathcal{T}^2_{\mathrm{poly}}(L/A) \xrightarrow{0} \mathcal{T}^3_{\mathrm{poly}}(L/A) \xrightarrow{0} \cdots$$

The Chevalley–Eilenberg hypercohomology $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A))$ of the Lie algebroid A with coefficients in this complex of A-modules is the cohomology of the cochain complex

$$\big(\mathrm{tot}(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(L/A)), d_A^{\mathrm{Bott}}\big).$$

Similarly, we set $\mathcal{D}^\bullet_{\mathrm{poly}}(L/A) = \bigoplus_{k=0}^{\infty} \mathcal{D}^k_{\mathrm{poly}}(L/A)$, where

$$\mathcal{D}^0_{\mathrm{poly}}(L/A) = R, \qquad \mathcal{D}^1_{\mathrm{poly}}(L/A) = \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)},$$

and $\mathcal{D}^k_{\mathrm{poly}}(L/A)$ with $k \geqslant 1$ is the tensor product $\mathcal{D}^1_{\mathrm{poly}}(L/A) \otimes_R \ldots \otimes_R \mathcal{D}^1_{\mathrm{poly}}(L/A)$ of k copies of the left R-module $\mathcal{D}^1_{\mathrm{poly}}(L/A)$. Multiplication in $\mathcal{U}(L)$ from the left by elements of $\Gamma(A)$ induces an A-module structure on the quotient $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$. This action of A on $\mathcal{D}^1_{\mathrm{poly}}(L/A)$ extends naturally to an action of A on $\mathcal{D}^k_{\mathrm{poly}}(L/A)$ for each k. In fact, $\mathcal{D}^1_{\mathrm{poly}}(L/A)$ is a cocommutative coassociative coalgebra over R whose comultiplication

$$\Delta : \mathcal{D}^1_{\mathrm{poly}}(L/A) \to \mathcal{D}^1_{\mathrm{poly}}(L/A) \otimes_R \mathcal{D}^1_{\mathrm{poly}}(L/A)$$

is a morphism of A-modules. Therefore the Hochschild complex

$$0 \longrightarrow \mathcal{D}^0_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^1_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^2_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^3_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \cdots$$

determined by the comultiplication $\Delta : \mathcal{D}^1_{\mathrm{poly}}(L/A) \to \mathcal{D}^1_{\mathrm{poly}}(L/A) \otimes_R \mathcal{D}^1_{\mathrm{poly}}(L/A)$ is a complex of A-modules. The Chevalley–Eilenberg hypercohomology

$$\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\diamond_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(L/A))$$

[2]Note that the symbols $\mathcal{T}^k_{\mathrm{poly}}(L/A)$ and $\mathcal{D}^k_{\mathrm{poly}}(L/A)$ in this paper mean something slightly different than in [68]. Essentially, there is a degree shift between the conventions used in the two papers.

of the Lie algebroid A with coefficients in this complex of A-modules is the cohomology of the cochain complex

$$\big(\operatorname{tot}(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}(L/A)), d^{\mathcal{U}}_A + \mathrm{id}\otimes d_{\mathscr{H}}\big).$$

We elect to call it the Hochschild cohomology of the Lie pair (L, A).

It is simple to see that $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A))$ and $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\diamond_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(L/A))$ are graded commutative associative algebras under the wedge and the cup product, respectively. However, at priori, neither $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A))$ nor $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\diamond_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(L/A))$ has an obvious graded Lie algebra structure. Nevertheless, we prove

Proposition 4.27 ([5]). *For any Lie pairs (L, A) both*

$$\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A)) \quad \text{and} \quad \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\diamond_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(L/A))$$

admit canonical Gerstenhaber algebra structures.

The natural inclusion $\Gamma(L/A) \hookrightarrow \mathcal{D}^1_{\mathrm{poly}}(L/A)$ extends to a morphism of complexes of A-modules

$$\mathrm{hkr} : \mathcal{T}^\bullet_{\mathrm{poly}}(L/A) \to \mathcal{D}^\bullet_{\mathrm{poly}}(L/A)$$

by skew-symmetrization:

$$\mathrm{hkr}(b_1\wedge\ldots\wedge b_n) = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma) b_{\sigma(1)}\otimes b_{\sigma(2)}\otimes\ldots\otimes b_{\sigma(n)}, \quad \forall\, b_1,\ldots,b_n\in\Gamma(L/A).$$

The map

$$\mathrm{id}\otimes\mathrm{hkr} : (\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(L/A), d^{\mathrm{Bott}}_A, 0) \to (\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}(L/A), d^{\mathcal{U}}_A, \pm\,\mathrm{id}\otimes d_{\mathscr{H}})$$

is a morphism of double complexes and therefore induces a morphism of hypercohomology groups

$$\mathrm{hkr} : \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A)) \to \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\diamond_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(L/A)). \tag{53}$$

The Atiyah class and the Todd class of a Lie pair (L, A) are defined similarly to the Molino class and the Todd class of a foliation $\mathcal{F}$ — see Section 3.5 and [29, 62, 50, 28].

Let us choose a splitting $j : L/A \to L$ of the short exact sequence of vector bundles over M

$$0 \longrightarrow A \xrightarrow{i} L \xrightarrow{q} L/A \longrightarrow 0$$

and an L-connection ∇ on L/A extending the Bott A-connection. The associated Atiyah 1-cocycle

$$R^\nabla \in \Gamma(M; A^\vee\otimes A^\perp\otimes \mathrm{End}(L/A)),$$

where $A^\perp = (L/A)^\vee$, is defined by

$$R^\nabla(V, Z)W = \nabla_V\nabla_{j(Z)}(W) - \nabla_{j(Z)}\nabla_V(W) - \nabla_{[V,j(Z)]}(W),$$

for all $V\in\Gamma(A)$ and $Z, W\in\Gamma(L/A)$. It is easily seen that it does not actually depend on the choice of the splitting j. The cohomology class

$$\alpha_{L/A} = [R^\nabla] \in H^1_{\mathrm{CE}}(A, A^\perp\otimes\mathrm{End}(L/A))$$

does not depend on the choice of the connection ∇ and is called the *Atiyah class of the Lie pair* (L, A) [29]. See also [62, 50].

The Todd class of the Lie pair (L, A) is the cohomology class

$$\operatorname{td}_{L/A} = \det\left(\frac{\alpha_{L/A}}{1-\exp(-\alpha_{L/A})}\right) \in \bigoplus_{k\geqslant 0} H^k_{\mathrm{CE}}(A, \Lambda^k A^\perp). \tag{54}$$

Proposition 4.28. *Let (L, A) be a Lie pair, and let $\mathcal{L} \to \mathcal{M}$, with $\mathcal{L} = T_{A[1]} \times_{T_M} L$ and $\mathcal{M} = A[1]$, be the dg Lie algebroid associated with the Lie algebroid morphism $i : A \to L$ as in Proposition* 3.11. *The diagrams*

$$\begin{array}{ccc} H^\bullet(\operatorname{tot}_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})), \mathcal{Q}) & \xrightarrow{\ \mathrm{hkr}\ } & H^\bullet(\operatorname{tot}_\oplus(\mathscr{D}_{\mathrm{poly}}(\mathcal{L})), d_{\mathscr{H}} + \mathcal{Q}) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A)) & \xrightarrow[\mathrm{hkr}]{} & \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\diamond_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(L/A)) \end{array}$$

and

$$\begin{array}{ccc} H^\bullet(\operatorname{tot}_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})), \mathcal{Q}) & \xrightarrow{(\operatorname{td}_{\mathcal{L}})^{1/2}} & H^\bullet(\operatorname{tot}_\oplus(\mathscr{T}_{\mathrm{poly}}(\mathcal{L})), \mathcal{Q}) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A)) & \xrightarrow[(\operatorname{td}_{L/A})^{1/2}]{} & \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A)) \end{array}$$

commute and both vertical maps are isomorphisms of Gerstenhaber algebras.

Combining Proposition 4.28 with Theorem 4.19, we obtain

Theorem 4.29 (Kontsevich–Duflo type theorem for Lie pairs [68]). *Given a Lie pair (L, A), the map*

$$\mathrm{hkr} \circ \operatorname{td}^{1/2}_{L/A} : \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A)) \xrightarrow{\cong} \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\diamond_{\mathrm{poly}}(L/A) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(L/A))$$

is an isomorphism of Gerstenhaber algebras — the square root of the Todd class $\operatorname{td}^{1/2}_{L/A} \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(A, \Lambda^k A^\perp)$ acts on $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}_{\mathrm{poly}}(L/A))$ by contraction.

Remark 4.30. Theorem 4.29 was proved in [68] without resorting to the dg Lie algebroid $\mathcal{L} \to \mathcal{M}$ (with $\mathcal{L} = T_{A[1]} \times_{T_M} L$) appearing in Proposition 4.28.

Below we consider two important examples of Lie pairs.

Let $\mathcal{F}$ be a regular foliation of a smooth manifold M. Consider the Lie pair $(T_M, T_{\mathcal{F}})$. Let $N_{\mathcal{F}} = T_M/T_{\mathcal{F}}$ be the normal bundle to the foliation $\mathcal{F}$, and $N^\vee_{\mathcal{F}}$ or $T^\perp_{\mathcal{F}}$ the conormal bundle $(T_M/T_{\mathcal{F}})^\vee$. Then $\mathcal{T}^\bullet_{\mathrm{poly}}(N_{\mathcal{F}}) = \bigoplus_{k\geqslant 0} \Gamma(\Lambda^k N_{\mathcal{F}})$ can be considered as the space of polyvector fields transversal to the foliation $\mathcal{F}$ [122, 124]. Similarly, $\mathcal{D}^\bullet_{\mathrm{poly}}(N_{\mathcal{F}}) = \bigoplus_{k\geqslant 0} \mathcal{D}^k_{\mathrm{poly}}(N_{\mathcal{F}})$ can be considered as the space of polydifferential operators transversal to $\mathcal{F}$. Here $\mathcal{D}^0_{\mathrm{poly}}(N_{\mathcal{F}})$ denotes the algebra R of smooth functions on the manifold M, $\mathcal{D}^1_{\mathrm{poly}}(N_{\mathcal{F}})$ denotes the left R-module $\frac{\mathcal{U}(T_M)}{\mathcal{U}(T_M)\cdot\Gamma(T_{\mathcal{F}})} \cong \frac{\mathcal{D}(M)}{\mathcal{D}(M)\cdot\Gamma(T_{\mathcal{F}})}$ of 'transverse differential operators,' and $\mathcal{D}^k_{\mathrm{poly}}(N_{\mathcal{F}})$ denotes the tensor product $\mathcal{D}^1_{\mathrm{poly}}(N_{\mathcal{F}}) \otimes_R \ldots \otimes_R \mathcal{D}^1_{\mathrm{poly}}(N_{\mathcal{F}})$ of k copies of the left R-module $\mathcal{D}^1_{\mathrm{poly}}(N_{\mathcal{F}})$. (Should there exist a foliation $\mathcal{F}'$ transverse to $\mathcal{F}$, the space $\mathcal{D}^1_{\mathrm{poly}}(N_{\mathcal{F}})$ would be isomorphic to the space $\mathcal{U}(T_{\mathcal{F}'})$ of leafwise differential operators in the direction of $\mathcal{F}'$.)

Theorem 4.29 implies

THEOREM 4.31 (Kontsevich–Duflo type theorem for foliations [68]). *Given a regular foliation $\mathcal{F}$ on a smooth manifold M, the map*

$$\operatorname{hkr}\circ \operatorname{td}^{1/2}_{T_M/T_{\mathcal{F}}} : \mathbb{H}^\bullet_{\mathrm{dR}}(\mathcal{F}, \mathcal{T}_{\mathrm{poly}}(N_{\mathcal{F}})) \xrightarrow{\cong} \mathbb{H}^\bullet_{\mathrm{dR}}\big(\mathcal{F}, \mathcal{D}^\diamond_{\mathrm{poly}}(N_{\mathcal{F}}) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(N_{\mathcal{F}})\big)$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\operatorname{td}^{1/2}_{T_M/T_{\mathcal{F}}}$ *of the Todd class* $\operatorname{td}_{T_M/T_{\mathcal{F}}} \in \bigoplus_{k=0} H^k_{\mathrm{dR}}(\mathcal{F}, \Lambda^k T^\perp_{\mathcal{F}})$ *acts on* $\mathbb{H}^\bullet_{\mathrm{dR}}(\mathcal{F}, \mathcal{T}_{\mathrm{poly}}(N_{\mathcal{F}}))$ *by contraction.*

Next we consider a Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$, i.e., a Lie algebra $\mathfrak{g}$ and a Lie subalgebra $i : \mathfrak{h} \to \mathfrak{g}$. A $\mathfrak{g}$-connection on $\mathfrak{g}/\mathfrak{h}$ is simply a bilinear map $\nabla : \mathfrak{g} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. Let ∇ be a $\mathfrak{g}$-connection on $\mathfrak{g}/\mathfrak{h}$ which extends the Bott $\mathfrak{h}$-connection: $\nabla^{\mathrm{Bott}}_a q(l) = q([a, l])$, for all $a \in \mathfrak{h}$ and $l \in \mathfrak{g}$. Here the map $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection. The Atiyah cocycle associated with ∇ is the bilinear map

$$R^\nabla : \mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} \to \operatorname{End}(\mathfrak{g}/\mathfrak{h})$$

defined by

$$R^\nabla(a; q(l)) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{[a,l]}, \qquad \forall\, a \in \mathfrak{h},\ l \in \mathfrak{g}.$$

Then the element $R^\nabla \in \mathfrak{h}^\vee \otimes \mathfrak{h}^\perp \otimes \operatorname{End}(\mathfrak{g}/\mathfrak{h})$ is a Chevalley–Eilenberg 1-cocycle for the Lie algebra $\mathfrak{h}$ with values in the $\mathfrak{h}$-module $\mathfrak{h}^\perp \otimes \operatorname{End}(\mathfrak{g}/\mathfrak{h})$. Its cohomology class $\alpha_{\mathfrak{g}/\mathfrak{h}} \in H^1_{\mathrm{CE}}(\mathfrak{h}, \mathfrak{h}^\perp \otimes \operatorname{End}(\mathfrak{g}/\mathfrak{h}))$ is independent of the choice of $\mathfrak{g}$-connection ∇ and is called the Atiyah class of the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ [12, 14, 29]. The Todd class of the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ is the corresponding Chevalley–Eilenberg cohomology class

$$\operatorname{td}_{\mathfrak{g}/\mathfrak{h}} = \det\left(\frac{\alpha_{\mathfrak{g}/\mathfrak{h}}}{1 - \exp(-\alpha_{\mathfrak{g}/\mathfrak{h}})}\right) \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(\mathfrak{h}, \Lambda^k \mathfrak{h}^\perp).$$

The Bott $\mathfrak{h}$-connection on $\mathfrak{g}/\mathfrak{h}$ extends by the Leibniz rule to an $\mathfrak{h}$-action on $\mathcal{T}^\bullet_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}) = \bigoplus_{k\geqslant 0} \Lambda^k(\mathfrak{g}/\mathfrak{h})$. Let $d^{\mathrm{Bott}}_{\mathfrak{h}} : \Lambda^p \mathfrak{h}^\vee \otimes \Lambda^q(\mathfrak{g}/\mathfrak{h}) \to \Lambda^{p+1}\mathfrak{h}^\vee \otimes \Lambda^q(\mathfrak{g}/\mathfrak{h})$ be the corresponding Chevalley–Eilenberg differential. According to Proposition 4.27, its hypercohomology $\mathbb{H}^\bullet_{\mathrm{CE}}(\mathfrak{h}, \mathcal{T}_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}))$ is a Gerstenhaber algebra. Similarly, the Lie algebra $\mathfrak{h}$ acts on $\mathcal{D}^1_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}) = \frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot \mathfrak{h}}$ by left multiplication and henceforth it acts on $\mathcal{D}^\bullet_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}) = \bigoplus_{k\geqslant 0}\big(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}\big)^{\otimes k}$ as well. The Chevalley–Eilenberg differential associated with this action is denoted by

$$d^{\mathcal{U}}_{\mathfrak{h}} : \Lambda^p \mathfrak{h}^\vee \otimes \mathcal{D}^q_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}) \to \Lambda^{p+1}\mathfrak{h}^\vee \otimes \mathcal{D}^q_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}).$$

Meanwhile, the Hochschild differential $d_{\mathscr{H}} : \mathcal{D}^q_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}) \to \mathcal{D}^{q+1}_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h})$ extends to

$$\mathfrak{d}_{\mathscr{H}} : \Lambda^p \mathfrak{h}^\vee \otimes \mathcal{D}^q_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}) \to \Lambda^p \mathfrak{h}^\vee \otimes \mathcal{D}^{q+1}_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h})$$

by graded linearity. According to Proposition 4.27, the corresponding hypercohomology $\mathbb{H}^\bullet_{\mathrm{CE}}(\mathfrak{h}, \mathcal{D}^\diamond_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}))$ is a Gerstenhaber algebra.

The natural map induced by skew-symmetrization

$$\operatorname{hkr} : \operatorname{tot}(\Lambda^\bullet \mathfrak{h}^\vee \otimes \Lambda^\bullet(\mathfrak{g}/\mathfrak{h})) \to \operatorname{tot}\left(\Lambda^\bullet \mathfrak{h}^\vee \otimes \left(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}\right)^{\otimes \bullet}\right)$$

is a quasi-isomorphism of cochain complexes.

Theorem 4.29 implies

THEOREM 4.32 (Kontsevich–Duflo type theorem for Lie algebra pairs [68]). *Given a Lie algebra pair* $(\mathfrak{g}, \mathfrak{h})$*, the map*

$$\mathrm{hkr} \circ \mathrm{td}^{1/2}_{\mathfrak{g}/\mathfrak{h}} : \mathbb{H}^{\bullet}_{\mathrm{CE}}(\mathfrak{h}, \mathcal{T}_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h})) \xrightarrow{\cong} \mathbb{H}^{\bullet}_{\mathrm{CE}}(\mathfrak{h}, \mathcal{D}^{\diamond}_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{\diamond+1}_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}))$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\mathrm{td}^{1/2}_{\mathfrak{g}/\mathfrak{h}}$ *of the Todd class* $\mathrm{td}_{\mathfrak{g}/\mathfrak{h}} \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(\mathfrak{h}, \Lambda^k(\mathfrak{g}/\mathfrak{h})^{\vee})$ *acts on* $\mathbb{H}^{\bullet}_{\mathrm{CE}}(\mathfrak{h}, \mathcal{T}_{\mathrm{poly}}(\mathfrak{g}/\mathfrak{h}))$ *by contraction.*

References

[1] M. Alexandrov, A. Schwarz, O. Zaboronsky, M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, Internat. J. Modern Phys. A 12 (1997), 1405–1429.

[2] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. 85 (1957), 181–207.

[3] R. Bandiera, *Descent of Deligne–Getzler* ∞*-groupoids*, Arxiv: 1705.02880 [math.AT].

[4] R. Bandiera, Z. Chen, M. Stiénon, P. Xu, *Shifted derived Poisson manifolds associated with Lie pairs*, Comm. Math. Phys. 375 (2020), 1717–1760.

[5] R. Bandiera, M. Stiénon, P. Xu, *Polyvector fields and polydifferential operators associated with Lie pairs*, J. Noncommut. Geom., to appear; Arxiv: 1901.04602 [math.QA].

[6] S. Barannikov, M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Internat. Math. Res. Notices 4 (1998), 201–215.

[7] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures*, Ann. Physics 111 (1978), 61–110.

[8] K. Behrend, H.-Y. Liao, P. Xu, *Derived differential geometry*, Arxiv: 2006.01376.

[9] F. A. Berezin, *Introduction to Superanalysis*, Math. Phys. Appl. Math. 9, D. Reidel, Dordrecht, 1987.

[10] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Grundlehren Text Ed., Springer, Berlin, 2004.

[11] G. Bonavolontà, N. Poncin, *On the category of Lie* n*-algebroids*, J. Geom. Phys. 73 (2013), 70–90.

[12] M. Bordemann, *Atiyah classes and equivariant connections on homogeneous spaces*, in: Travaux mathématiques, Volume XX, Trav. Math. 20, Fac. Sci. Technol. Commun. Univ. Luxemb., Luxembourg, 2012, 29–82.

[13] R. Brown, K. C. H. Mackenzie, *Determination of a double Lie groupoid by its core diagram*, J. Pure Appl. Algebra 80 (1992), 237–272.

[14] D. Calaque, A. Căldăraru, J. Tu, PBW *for an inclusion of Lie algebras*, J. Algebra 378 (2013), 64–79.

[15] D. Calaque, V. Dolgushev, G. Halbout, *Formality theorems for Hochschild chains in the Lie algebroid setting*, J. Reine Angew. Math. 612 (2007), 81–127.

[16] D. Calaque, C. A. Rossi, *Lectures on Duflo Isomorphisms in Lie Algebra and Complex Geometry*, EMS Ser. Lect. Math., Eur. Math. Soc., Zürich, 2011.

[17] D. Calaque, M. Van den Bergh, *Hochschild cohomology and Atiyah classes*, Adv. Math. 224 (2010), 1839–1889.

[18] A. Căldăraru, *The Mukai pairing. II. The Hochschild–Kostant–Rosenberg isomorphism*, Adv. Math. 194 (2005), 34–66.

[19] A. Căldăraru, S. Willerton, *The Mukai pairing. I. A categorical approach*, New York J. Math. 16 (2010), 61–98.

[20] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton Landmarks in Math., Princeton Univ. Press, Princeton, NJ, 1999.

[21] A. S. Cattaneo, *From topological field theory to deformation quantization and reduction*, in: International Congress of Mathematicians, Vol. III, Eur. Math. Soc., Zürich, 2006, 339–365.

[22] A. S. Cattaneo, G. Felder, *A path integral approach to the Kontsevich quantization formula*, Comm. Math. Phys. 212 (2000), 591–611.

[23] A. S. Cattaneo, G. Felder, *Relative formality theorem and quantisation of coisotropic submanifolds*, Adv. Math. 208 (2007), 521–548.

[24] A. S. Cattaneo, G. Felder, T. Willwacher, *The character map in deformation quantization*, Adv. Math. 228 (2011), 1966–1989.

[25] A. S. Cattaneo, D. Fiorenza, R. Longoni, *On the Hochschild–Kostant–Rosenberg map for graded manifolds*, Int. Math. Res. Not. 2005, 3899–3918.

[26] A. Cattaneo, B. Keller, C. Torossian, A. Bruguières, *Déformation, quantification, théorie de Lie*, Panor. Synthèses 20, Soc. Math. France, Paris, 2005.

[27] A. S. Cattaneo, F. Schätz, *Introduction to supergeometry*, Rev. Math. Phys. 23 (2011), 669–690.

[28] Z. Chen, Z. Liu, M. Xiang, *Kapranov's construction of sh Leibniz algebras*, Homology Homotopy Appl. 22 (2020), 141–165.

[29] Z. Chen, M. Stiénon, P. Xu, *From Atiyah classes to homotopy Leibniz algebras*, Comm. Math. Phys. 341 (2016), 309–349.

[30] Z. Chen, M. Xiang, P. Xu, *Atiyah and Todd classes arising from integrable distributions*, J. Geom. Phys. 136 (2019), 52–67.

[31] Z. Chen, M. Xiang, P. Xu, *Hochschild cohomology of dg manifolds associated to integrable distributions*, Arxiv: 2103.08096 [math.DG].

[32] J. Cheng, Z. Chen, D. Ni, *Hopf algebras arising from dg manifolds*, Arxiv: 1911.01388 [math.DG].

[33] P. Deligne et al. (eds.), *Quantum Fields and Strings. A Course for Mathematicians*, Amer. Math. Soc., Providence, RI, 1999.

[34] V. Dolgushev, *Covariant and equivariant formality theorems*, Adv. Math. 191 (2005), 147–177.

[35] V. Dolgushev, D. Tamarkin, B. Tsygan, *Formality of the homotopy calculus algebra of Hochschild (co)chains*, Arxiv: 0807.5117 [math.KT].

[36] V. Dolgushev, D. Tamarkin, B. L. Tsygan, *Formality theorems for Hochschild complexes and their applications*, Lett. Math. Phys. 90 (2009), 103–136.

[37] M. Duflo, *Caractères des groupes et des algèbres de Lie résolubles*, Ann. Sci. École Norm. Sup. (4) 3 (1970), 23–74.

[38] T. Dyckerhoff, D. Murfet, *Pushing forward matrix factorizations*, Duke Math. J. 162 (2013), 1249–1311.

[39] C. Emmrich, A. Weinstein, *The differential geometry of Fedosov's quantization*, in: Lie Theory and Geometry, Progr. Math. 123, Birkhäuser, Boston, MA, 1994, 217–239.

[40] B. V. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. 40 (1994), 213–238.

[41] M. Gerstenhaber, S. D. Schack, *Algebraic cohomology and deformation theory*, in: Deformation Theory of Algebras and Structures and Applications (Il Ciocco, 1986) NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 247, Kluwer Acad. Publ., Dordrecht, 1988, 11–264.

[42] E. Getzler, *Lie theory for nilpotent L_∞-algebras*, Ann. of Math. (2) 170 (2009), 271–301.

[43] K. Grabowska, J. Grabowski, *n-tuple principal bundles*, Int. J. Geom. Methods Mod. Phys. 15 (2018), 1850211.

[44] A. Gracia-Saz, R. A. Mehta, *Lie algebroid structures on double vector bundles and representation theory of Lie algebroids*, Adv. Math. 223 (2010), 1236–1275.

[45] M. Grützmann, J.-P. Michel, P. Xu, *Weyl quantization of degree 2 symplectic graded manifolds*, Arxiv: 1410.3346 [math.DG].

[46] S. Gutt, *An explicit $\star$-product on the cotangent bundle of a Lie group*, Lett. Math. Phys. 7 (1983), 249–258.

[47] A. Henriques, *Integrating L_∞-algebras*, Compos. Math. 144 (2008), 1017–1045.

[48] F. Hirzebruch, T. Berger, R. Jung, *Manifolds and Modular Forms*, Aspects Math. E20, Friedr. Vieweg & Sohn, Braunschweig, 1992.

[49] D. Huybrechts, *Complex Geometry. An Introduction*, Universitext, Springer, Berlin, 2005.

[50] M. Jotz Lean, *Infinitesimal ideal systems and the Atiyah class*, Arxiv: 1910.04492.

[51] M. Jotz Lean, *Lie 2-algebroids and matched pairs of 2-representations: a geometric approach*, Pacific J. Math. 301 (2019), 143–188.

[52] M. Jotz Lean, *Obstructions to representations up to homotopy and ideals*, Arxiv: 1905.10237 [math.DG].

[53] M. Jotz Lean, K. C. H. Mackenzie, *Transitive double Lie algebroids via core diagrams*, Arxiv: 2103.13189 [math.DG].

[54] M. Jotz Lean, R. A. Mehta, T. Papantonis, *Modules and representations up to homotopy of Lie n-algebroids*, Arxiv: 2001.01101 [math.DG].

[55] M. Kapranov, *Rozansky–Witten invariants via Atiyah classes*, Compositio Math. 115 (1999), 71–113.

[56] B. Keller, *Derived invariance of higher structures on the Hochschild complex*, preprint, 2003.

[57] B. Kim, A. Polishchuk, *Atiyah class and Chern character for global matrix factorizations*, Arxiv: 1907.11441 [math.AG].

[58] M. Kontsevich, *Homological algebra of mirror symmetry*, in: Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, 120–139.

[59] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003), 157–216.

[60] A. Kotov, T. Strobl, *Characteristic classes associated to Q-bundles*, Int. J. Geom. Methods Mod. Phys. 12 (2015), 1550006.

[61] C. Laurent-Gengoux, M. Stiénon, P. Xu, *Exponential map and L_∞ algebra associated to a Lie pair*, C. R. Math. Acad. Sci. Paris 350 (2012), 817–821.

[62] C. Laurent-Gengoux, M. Stiénon, P. Xu, *Poincaré–Birkhoff–Witt isomorphisms and Kapranov dg-manifolds*, Adv. Math., to appear; Arxiv: 1408.2903 [math.DG].

[63] D. Li-Bland, P. Ševera, *Integration of exact Courant algebroids*, Electron. Res. Announc. Math. Sci. 19 (2012), 58–76.

[64] H.-Y. Liao, S. Seol, *Keller admissible triples and Duflo theorem*, preprint 2021.

[65] H.-Y. Liao, M. Stiénon, *Formal exponential map for graded manifolds*, Int. Math. Res. Not. IMRN 3 (2019), 700–730.

[66] H.-Y. Liao, M. Stiénon, P. Xu, *Formality theorem for $\mathfrak{g}$-manifolds*, C. R. Math. Acad. Sci. Paris 355 (2017), 582–589.

[67] H.-Y. Liao, M. Stiénon, P. Xu, *Formality theorem for differential graded manifolds*, C. R. Math. Acad. Sci. Paris 356 (2018), 27–43.

[68] H.-Y. Liao, M. Stiénon, P. Xu, *Formality and Kontsevich–Duflo type theorems for Lie pairs*, Adv. Math. 352 (2019), 406–482.

[69] K. H. Lin, D. Pomerleano, *Global matrix factorizations*, Math. Res. Lett. 20 (2013), 91–106.

[70] S. L. Lyakhovich, E. A. Mosman, A. A. Sharapov, *On characteristic classes of Q-manifolds*, Funktsional. Anal. i Prilozhen. 42 (2008), 88–91; Funct. Anal. Appl. 42 (2008), 75–77.

[71] S. L. Lyakhovich, E. A. Mosman, A. A. Sharapov, *Characteristic classes of Q-manifolds: classification and applications*, J. Geom. Phys. 60 (2010), 729–759.

[72] K. C. H. Mackenzie, *Double Lie algebroids and second-order geometry. I*, Adv. Math. 94 (1992), 180–239.

[73] K. C. H. Mackenzie, *Double Lie algebroids and the double of a Lie bialgebroid*, Arxiv: math/9808081 [math.DG].

[74] K. C. H. Mackenzie, *Drinfel'd doubles and Ehresmann doubles for Lie algebroids and Lie bialgebroids*, Electron. Res. Announc. Amer. Math. Soc. 4 (1998), 74–87.

[75] K. C. H. Mackenzie, *Double Lie algebroids and second-order geometry. II*, Adv. Math. 154 (2000), 46–75.

[76] K. C. H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, London Math. Soc. Lecture Note Ser. 213, Cambridge Univ. Press, Cambridge, 2005.

[77] K. C. H. Mackenzie, *Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids*, J. Reine Angew. Math. 658 (2011), 193–245.

[78] K. C. H. Mackenzie, T. Mokri, *Locally vacant double Lie groupoids and the integration of matched pairs of Lie algebroids*, Geom. Dedicata 77 (1999), 317–330.

[79] K. C. H. Mackenzie, P. Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. 73 (1994), 415–452.

[80] K. C. H. Mackenzie, P. Xu, *Classical lifting processes and multiplicative vector fields*, Quart. J. Math. Oxford Ser. (2) 49 (1998), 59–85.

[81] D. Manchon, C. Torossian, *Cohomologie tangente et cup-produit pour la quantification de Kontsevich*, Ann. Math. Blaise Pascal 10 (2003), 75–106.

[82] D. Manchon, C. Torossian, *Erratum: Cohomologie tangente et cup-produit pour la quantification de Kontsevich*, Ann. Math. Blaise Pascal 11 (2004), 129–130.

[83] N. Markarian, *The Atiyah class, Hochschild cohomology and the Riemann–Roch theorem*, J. Lond. Math. Soc. (2) 79 (2009), 129–143.

[84] R. A. Mehta, *Supergroupoids, double structures, and equivariant cohomology*, Thesis (Ph.D.), University of California, Berkeley, 2006. ProQuest LLC, Ann Arbor, MI, 133, Arxiv: math/0605356 [math.DG].

[85] R. A. Mehta, *Q-algebroids and their cohomology*, J. Symplectic Geom. 7 (2009), 263–293.

[86] R. A. Mehta, M. Stiénon, P. Xu, *The Atiyah class of a dg-vector bundle*, C. R. Math. Acad. Sci. Paris 353 (2015), 357–362.

[87] R. A. Mehta, X. Tang, *Symplectic structures on the integration of exact Courant algebroids*, J. Geom. Phys. 127 (2018), 68–83.

[88] I. Moerdijk, J. Mrčun, *On the universal enveloping algebra of a Lie algebroid*, Proc. Amer. Math. Soc. 138 (2010), 3135–3145.

[89] T. Mokri, *Matched pairs of Lie algebroids*, Glasgow Math. J. 39 (1997), 167–181.

[90] P. Molino, *Classe d'Atiyah d'un feuilletage et connexions transverses projetables*, C. R. Acad. Sci. Paris Sér. A-B 272 (1971), A779–A781.

[91] P. Molino, *Propriétés cohomologiques et propriétés topologiques des feuilletages à connexion transverse projetable*, Topology 12 (1973), 317–325.

[92] E. Mosman, A. Sharapov, *All stable characteristic classes of homological vector fields*, Lett. Math. Phys. 94 (2010), 243–261.

[93] V. Nistor, A. Weinstein, P. Xu, *Pseudodifferential operators on differential groupoids*, Pacific J. Math. 189 (1999), 117–152.

[94] Y.-G. Oh, J.-S. Park, *Deformations of coisotropic submanifolds and strong homotopy Lie algebroids*, Invent. Math. 161 (2005), 287–360.

[95] M. Pevzner, C. Torossian, *Isomorphisme de Duflo et la cohomologie tangentielle*, J. Geom. Phys. 51 (2004), 487–506.

[96] A. Polishchuk, A. Vaĭntrob, *Chern characters and Hirzebruch–Riemann–Roch formula for matrix factorizations*, Duke Math. J. 161 (2012), 1863–1926.

[97] M. Qian, P. Xu, *Generalized Weyl transformation*, Acta Math. Appl. Sinica (English Ser.) 4 (1988), 275–288.

[98] M. Qian, P. Xu, *Coherent states, inverse mapping formula for Weyl transformations and application to equations of KdV type*, Acta Math. Appl. Sinica (English Ser.) 6 (1990), 193–204.

[99] D. Quillen, *Superconnections and the Chern character*, Topology 24 (1985), 89–95.

[100] A. C. Ramadoss, *The big Chern classes and the Chern character*, Internat. J. Math. 19 (2008), 699–746.

[101] A. C. Ramadoss, *The Mukai pairing and integral transforms in Hochschild homology*, Mosc. Math. J. 10 (2010), 629–645.

[102] G. S. Rinehart, *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. 108 (1963), 195–222.

[103] J. Roberts, S. Willerton, *On the Rozansky–Witten weight systems*, Algebr. Geom. Topol. 10 (2010), 1455–1519.

[104] D. Roytenberg, *On the structure of graded symplectic supermanifolds and Courant algebroids*, in: Quantization, Poisson Brackets and Beyond (Manchester, 2001), Contemp. Math. 315, Amer. Math. Soc., Providence, RI, 2002, 169–185.

[105] A. Schwarz, *Geometry of Batalin–Vilkovisky quantization*, Comm. Math. Phys. 155 (1993), 249–260.

[106] S. Seol, M. Stiénon, P. Xu, *Dg manifolds, formal exponential maps and homotopy Lie algebras*, preprint 2021.

[107] P. Ševera, *Some title containing the words "homotopy" and "symplectic", e.g. this one*, in: Travaux mathématiques. Fasc. XVI, Trav. Math. 16, Univ. Luxemb., Luxembourg, 2005, 121–137.

[108] P. Ševera, *L_∞ algebras as 1-jets of simplicial manifolds (and a bit beyond)*, Arxiv: math/0612349 [math.DG].

[109] P. Ševera, *Letters to Alan Weinstein about Courant algebroids*, Arxiv: 1707.00265.

[110] Y. Sheng, C. Zhu, *Higher extensions of Lie algebroids*, Commun. Contemp. Math. 19 (2017), 1650034.

[111] B. Shoikhet, *On the Duflo formula for L_∞-algebras and Q-manifolds*, Arxiv: math/9812009 [math.QA].

[112] B. Shoikhet, *A proof of the Tsygan formality conjecture for chains*, Adv. Math. 179 (2003), 7–37.

[113] B. Shoikhet, *Tsygan formality and Duflo formula*, Math. Res. Lett. 10 (2003), 763–775.

[114] B. Shoikhet, *Koszul duality in deformation quantization and Tamarkin's approach to Kontsevich formality*, Adv. Math. 224 (2010), 731–771.

[115] M. Stiénon, L. Vitagliano, P. Xu, *A_∞-algebras from Lie pairs*, work in progress.

[116] D. E. Tamarkin, *Operadic proof of M. Kontsevich's formality theorem*, Thesis (Ph.D.) – The Pennsylvania State Univ., ProQuest LLC, Ann Arbor, MI, 1999, 51.

[117] D. E. Tamarkin, B. L. Tsygan, *Cyclic formality and index theorems*, Lett. Math. Phys. 56 (2001), 85–97.

[118] B. L. Tsygan, *Formality conjectures for chains*, in: Differential Topology, Infinite-Dimensional Lie Algebras, and Applications, Amer. Math. Soc. Transl. Ser. 2 194, Amer. Math. Soc., Providence, RI, 1999, 261–274.

[119] B. L. Tsygan, *Noncommutative calculus and operads*, in: Topics in Noncommutative Geometry, Clay Math. Proc. 16, Amer. Math. Soc., Providence, RI, 2012, 19–66.

[120] A. Y. Vaĭntrob, *Lie algebroids and homological vector fields*, Uspekhi Mat. Nauk 52 (1997), no. 2(314), 161–162; Russian Math. Surveys 52 (1997), 428–429.

[121] V. S. Varadarajan, *Supersymmetry for Mathematicians: an Introduction*, Courant Lecture Notes in Math. 11, New York Univ., Courant Inst. of Math. Sciences, New York; Amer. Math. Soc., Providence, RI, 2004.

[122] L. Vitagliano, *On the strong homotopy Lie–Rinehart algebra of a foliation*, Commun. Contemp. Math. 16 (2014), 1450007.

[123] L. Vitagliano, *On the strong homotopy associative algebra of a foliation*, Commun. Contemp. Math. 17 (2015), 1450026.

[124] L. Vitagliano, *Representations of homotopy Lie–Rinehart algebras*, Math. Proc. Cambridge Philos. Soc. 158 (2015), 155–191.

[125] T. T. Voronov, *Q-manifolds and higher analogs of Lie algebroids*, in: XXIX Workshop on Geometric Methods in Physics, AIP Conf. Proc. 1307, Amer. Inst. Phys., Melville, NY, 2010, 191–202.

[126] T. T. Voronov, *Q-manifolds and Mackenzie theory*, Comm. Math. Phys. 315 (2012), 279–310.

[127] T. T. Voronov, *Graded Geometry, Q-Manifolds, and Microformal Geometry*, Fortschr. Phys. 67 (2019), 1910023.

[128] R. O. Wells, Jr., *Differential Analysis on Complex Manifolds*, Grad. Texts in Math. 65, Springer, New York, 2008.

[129] T. Willwacher, *Formality of cyclic chains*, Int. Math. Res. Not. IMRN 17 (2011), 3939–3956.

[130] T. Willwacher, *The homotopy braces formality morphism*, Duke Math. J. 165 (2016), 1815–1964.

[131] P. Xu, *Quantum groupoids*, Comm. Math. Phys. 216 (2001), 539–581.

[132] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles: Differential Geometry*, Pure Appl. Math. 16, Marcel Dekker, New York, 1973.