Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 294 (2021) 1–39

Journal of Differential Equations

www.elsevier.com/locate/jde

Stochastic functional Kolmogorov equations II: Extinction ☆

Dang H. Nguyen [a], Nhu N. Nguyen [b], George Yin [b,*]

[a] *Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487, USA*
[b] *Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA*

Received 12 February 2021; revised 15 May 2021; accepted 17 May 2021
Available online 21 May 2021

Abstract

This work, Part II, together with its companion-Part I develops a new framework for stochastic functional Kolmogorov equations, which are nonlinear stochastic differential equations depending on the current as well as the past states. Because of the complexity of the problems, it is natural to divide our contributions into two parts to answer a long-standing question in biology and ecology. What are the minimal conditions for long-term persistence and extinction of a population? Part I of our work provides characterization of persistence, whereas in this part, extinction is the main focus. The techniques used in this paper are combination of the newly developed functional Itô formula and a dynamic system approach. Compared to the study of stochastic Kolmogorov systems without delays, the main difficulty is that infinite dimensional systems have to be treated. The extinction is characterized after investigating random occupation measures and examining behavior of functional systems around boundaries. Our characterizations of long-term behavior of the systems reduce to that of Kolmogorov systems without delay when there is no past dependence. A number of applications are also examined.

MSC: 34K50; 60J60; 60J70; 92B99

☆ The research of D.H. Nguyen was supported in part by the National Science Foundation under grant DMS-1853467. The research of N.N. Nguyen and G. Yin was supported in part by the National Science Foundation under grant DMS-2114649.

* Corresponding author.
E-mail addresses: dangnh.maths@gmail.com (D.H. Nguyen), nguyen.nhu@uconn.edu (N.N. Nguyen), gyin@uconn.edu (G. Yin).

https://doi.org/10.1016/j.jde.2021.05.043
0022-0396/

Keywords: Stochastic functional equation; Kolmogorov system; Ecological and biological application; Invariant measure; Extinction; Persistence

Contents

1. Introduction . . . 2
2. Formulation and main results . . . 4
3. Technical results . . . 10
   3.1. Well-posedness of the problem . . . 10
   3.2. Random occupation measures: tightness and convergence . . . 12
4. Extinction . . . 17
5. Applications . . . 26
   5.1. Stochastic delay Lotka-Volterra competitive model . . . 26
   5.2. Stochastic delay Lotka-Volterra predator-prey model . . . 28
   5.3. Stochastic delay replicator equation . . . 30
   5.4. Stochastic delay epidemic SIR model . . . 33
   5.5. Stochastic delay chemostat model . . . 35

References . . . 37

1. Introduction

Motivated by a wide variety of applications in ecology and biology, we aim to develop a new framework of stochastic functional Kolmogorov equations. To keep our work with a manageable length, we divide the contributions to two parts. We aim to answer the long-standing question of fundamental importance pertaining to biology and ecology: What are the minimal (necessary and sufficient) conditions for long-term persistence and extinction of a population? This work, Part II, focuses on extinction, whereas its companion-Part I [46] concentrates on persistence. The extinction and persistence are phenomena go far beyond biological and ecological systems. Such long-term properties are shared by all processes of Kolmogorov type. One of the main difficulties is that we need to deal with infinite dimensional processes.

Taking random fluctuations of the environment into consideration, a stochastic Kolmogorov differential equation is given by

$$dx_i(t) = x_i(t)f_i(x_1(t), \dots, x_n(t))dt + x_i(t)g_i(x_1(t), \dots x_n(t))dB_i(t), \; i = 1, \dots, n, \qquad (1.1)$$

where $B_1(t), \dots, B_n(t)$ are n real-valued Brownian motions (independent or not). Such stochastic Kolmogorov system are used extensively in the modeling and analysis of ecological and biological systems such as Lotka-Volterra predator-prey models, Lotka-Volterra competitive model, replicator dynamic systems, stochastic epidemic models, stochastic tumor-immune system, and stochastic chemostat models, among others. Apart from ecological and biological systems, numerous problems arising in mathematical physics, statistical mechanics, and many related fields, use Kolmogorov nonlinear stochastic differential equations. For example, the generalized Ginzburg-Landau equation is used for bistable systems, chemical turbulence, phase transitions in non-equilibrium systems, among other. Because of its prevalence in applications, Kolmogorov equations (1.1) have attracted much attention in the past decades; substantial progress has been made. To proceed, let us briefly recall some important works of the developments to date. Some early mathematical formulations were introduced by Verhulst [63] for logistic models, by Lotka and Volterra [39,64] for Lotka-Volterra systems, and by Kermack and McKendrick [29,30] for infectious diseases. By now, Kolmogorov stochastic population systems (using stochastic differential equations or difference equations) together with their longtime behavior have been relatively well understood; see [5,55,57] for Kolmogorov stochastic systems in compact domains and [4,25] for Kolmogorov systems in non-compact domains, [9,13,16,17,40,45,48,62] for variants of Kolmogorov systems such as epidemic models, migration and spatial heterogeneity on single and multiple species, chemostat models, tumor-immune system, and [28,27] for replicator dynamics. For the most recent developments and substantial progress, we refer to Benaïm [4], Henning and Nguyen [25], Schreiber and Benaïm [57], and references therein. For related works on Markov processes, stochastic differential equation, stochastic equations with switching, delay, and jumps, we mention [3,6,21,31,42,43,47,54], among others.

In contrast to existing works, our work in this paper and its companion [46] aim to substantially advance the existing literature for a class of stochastic functional Kolmogorov systems allowing delay and past dependence, so as to provide essential utility to a wide range of applications. Clearly, the delays or past dependence are unavoidable natural phenomena in dynamic systems; the framework of stochastic functional differential equations is more realistic, more effective, and more general for the population dynamics in real life than a stochastic differential equation counterpart. In population dynamics, some delay mechanisms studied in the literature include age structure, feeding times, replenishment or regeneration time for resources [12]. Although there are many excellent treatises of Kolmogorov stochastic differential equations, the work on Kolmogorov stochastic differential equations with delay is relatively scarce. In addition, other than the specific models and applications treated, there has not been a unified framework and a systematic treatment for Kolmogorov stochastic functional differential systems yet, and most of the existing results involving delay are not as sharp as desired. There is a strong motivation and pressing need to develop a unified framework for stochastic functional Kolmogorov systems. New methods and techniques need to be developed to carry out the analysis.

While the models with functional stochastic differential equations are more realistic and more general, the analysis of such systems is far more difficult. Perhaps, part of the difficulties in studying stochastic delay systems is that there had been virtually no bona fide operators and functional Itô formulas except some general setup in a Banach space such as [44] before 2009. In [18], Dupire generalized the Itô formula to a functional setting by using pathwise functional derivatives. The Itô formula developed has substantially eased the difficulties and encouraged subsequent development with numerous applications. His work was developed further by Cont and Fournié [10,11]. Using the newly developed functional Itô formula enables us to analyze effectively the segment processes in the stochastic functional Kolmogorov equations. Moreover, while the solutions of stochastic differential equations (without delays) are Markovian processes, the solutions of stochastic differential equations with delay are non-Markov. One uses the so-called segment processes for the delay equations. Nevertheless, such segment processes live in an infinite dimensional space. Many of the known results in the usual stochastic differential equation setup are no longer applicable. Because Kolmogorov is highly nonlinear, analyzing such systems with delay becomes even more difficult.

Our goal in this paper, is to characterize the long-term behavior focusing on extinction. The results of this paper combined with the persistence presented in [46] provide a complete long-term characterization for the stochastic functional Kolmogorov system. We show that our results will cover, improve, and outperform many existing results of Kolmogorov systems (with and without delays) such as the study on stochastic delay Lotka-Volterra competitive models [2,35], the work on stochastic delay Lotka-Volterra predator-prey models [23,32–34,67], the treatment of stochastic delay epidemic SIR models [8,36–38,41], and the study on stochastic delay chemostat models [58,59,68] and the delayed replicator models. It should be mentioned that for replicator dynamics, it seems to be no investigation of stochastic delay systems to date to the best of our knowledge. It is also worth noting that our sufficient conditions for extinction in this part and the conditions of persistence in [46] are almost necessary in the sense that excluding critical cases, if the system is not persistent, the extinction will happen and vice versa.

By combining the newly developed functional Itô formula and the dynamic system point of view, we advance knowledge to treat infinite dimensional systems. Characterization of the extinction is obtained after investigating the random occupation measures and examining carefully the behavior of a functional system around the boundary. It is worth noting that handling random occupation measures in an infinite dimensional space to obtain the tightness and its limit is far more challenging. Examining the behavior of solutions near the boundary for functional Kolmogorov systems requires more delicate analysis than that for systems without delay because even if the state is close to the boundary, its history may not be.

The rest of the paper is organized as follows. Section 2 presents the formulation of our problem and main results of stochastic functional Kolmogorov systems. Section 3 recalls basic properties of Kolmogorov equations with delays, including well-posedness of the system, positivity of solutions, which have been proved in [46]. We then study the tightness of families of random occupation measures and the convergence properties. Next, Section 4 studies the conditions for extinction of Kolmogorov systems. Finally, Section 5 provides several applications involving Kolmogorov dynamic systems and detailed accounts on how to use our results on stochastic functional Kolmogorov equations to treat each of the application examples.

2. Formulation and main results

We first provide a glossary of symbols and notation to be used in this paper to help the reading.

r	a fixed positive number
$\lvert\cdot\rvert$	Euclidean norm
$\mathcal{C}([a;b];\mathbb{R}^n)$	set of $\mathbb{R}^n$-valued continuous functions defined on $[a;b]$
$\mathcal{C}$	$:=\mathcal{C}([-r;0];\mathbb{R}^n)$
$\boldsymbol{\varphi}$	$=(\varphi_1,\dots,\varphi_n)\in\mathcal{C}$
$\mathbf{x}$	$=(x_1,\dots,x_n):=\boldsymbol{\varphi}(0)\in\mathbb{R}^n$
$\lVert\boldsymbol{\varphi}\rVert$	$:=\sup\{\lvert\boldsymbol{\varphi}(t)\rvert : t\in[-r,0]\}$
$\mathbf{X}_t$	$:=\mathbf{X}_t(s):=\{\mathbf{X}(t+s): -r\le s\le 0\}$ (segment function)
$X_{i,t}$	$:=X_{i,t}(s):=\{X_i(t+s): -r\le s\le 0\}$
$\mathcal{C}_+$	$:=\{\boldsymbol{\varphi}=(\varphi_1,\dots,\varphi_n)\in\mathcal{C}:\varphi_i(s)\ge 0\ \forall s\in[-r,0], i=1,\dots,n\}$
$\partial\mathcal{C}_+$	$:=\{\boldsymbol{\varphi}=(\varphi_1,\dots,\varphi_n)\in\mathcal{C}:\lVert\varphi_i\rVert=0$ for some $i=1,\dots,n\}$
$\mathcal{C}_+^\circ$	$:=\{\boldsymbol{\varphi}\in\mathcal{C}_+:\varphi_i(s)>0, \forall s\in[-r,0], i=1,\dots,n\}\ne\mathcal{C}_+\setminus\partial\mathcal{C}_+$
$\lVert\boldsymbol{\varphi}\rVert_\alpha$	$:=\lVert\boldsymbol{\varphi}\rVert+\sup_{-r\le s<t\le 0}\frac{\lvert\boldsymbol{\varphi}(t)-\boldsymbol{\varphi}(s)\rvert}{(t-s)^\alpha}$, for some $0<\alpha<1$
$\mathcal{C}^\alpha$	space of Hölder continuous functions endowed with the norm $\lVert\cdot\rVert_\alpha$
Γ	$n\times n$ matrix
$\Gamma^\top$	transpose of Γ
$\mathbf{B}(t)$	$=(B_1(t),\dots,B_n(t))^\top$, a n-dimensional standard Brownian motion
$\mathbf{E}(t)$	$=(E_1(t),\dots,E_n(t))^\top:=\Gamma^\top\mathbf{B}(t)$

Σ	$=(\sigma_{ij})_{n\times n}:=\Gamma^\top\Gamma$
$\mathcal{M}$	set of ergodic invariant probability measures of $\mathbf{X}_t$ supported on $\partial\mathcal{C}_+$
$\mathrm{Conv}(\mathcal{M})$	convex hull of $\mathcal{M}$
$\mathbf{0}$	the zero constant function in $\mathcal{C}$
$\boldsymbol{\delta}^*$	the Dirac measure concentrated at $\mathbf{0}$
$\mathbf{1}_A$	the indicator function of set A
$D_{\varepsilon,R}$	$:=\{\boldsymbol{\varphi}\in\mathcal{C}_+:\|\boldsymbol{\varphi}\|\le R, x_i\ge\varepsilon\ \forall i;\mathbf{x}:=\boldsymbol{\varphi}(0)\}, \varepsilon, R>0$
$\mathbb{D}$	space of Cádlág functions mapping $[-r,0]$ to $\mathbb{R}^n$
A_0, A_1, A_2	constants satisfying Assumption 2.1
γ_0, γ_b, M	constants satisfying Assumption 2.1
$\mathbf{c}, h(\cdot), \mu$	vector, function and probability measure satisfying Assumption 2.1
$\widetilde{K}, b_1, b_2$	constants satisfying Assumption 2.2
$h_1(\cdot), \mu_1$	function and probability measure satisfying Assumption 2.2
B_0, B_1, B_2	constants satisfying Assumption 2.5
B_3, p_2	constants satisfying Assumption 2.5
I	a subset of $\{1,\dots,n\}$
I^c	$:=\{1,\dots,n\}\setminus I$
$\mathcal{C}_+^I$	$:=\{\boldsymbol{\varphi}\in\mathcal{C}_+:\|\varphi_i\|=0 \text{ if } i\in I^c\}$
$\mathcal{C}_+^{I,\circ}$	$:=\{\boldsymbol{\varphi}\in\mathcal{C}_+:\|\varphi_i\|=0 \text{ if } i\in I^c \text{ and } \varphi_i(s)>0\ \forall s\in[-r,0] \text{ if } i\in I\}$
$\partial\mathcal{C}_+^I$	$:=\{\boldsymbol{\varphi}\in\mathcal{C}_+:\|\varphi_i\|=0 \text{ if } i\in I^c \text{ and } \|\varphi_i\|=0 \text{ for some } i\in I\}$
$\mathcal{M}^I$	sets of ergodic invariant probability measure on $\mathcal{C}_+^I$
$\mathcal{M}^{I,\circ}$	sets of ergodic invariant probability measure on $\mathcal{C}_+^{I,\circ}$
$\partial\mathcal{M}^I$	sets of ergodic invariant probability measure on $\partial\mathcal{C}_+^I$
I_π	the subset of $\{1,\dots,n\}$ such that $\pi(\mathcal{C}_+^{I_\pi,\circ})=1, \pi\in\mathcal{M}$
γ, p_0, A	constants satisfying the condition in Lemma 3.1
$\boldsymbol{\rho}$	$=(\rho_1,\dots,\rho_n)$ vector satisfying the condition in Lemma 3.1
$V_{\boldsymbol{\rho}}(\boldsymbol{\varphi})$	$:=\Big(1+\mathbf{c}^\top\mathbf{x}\Big)\prod_{i=1}^n x_i^{\rho_i}\exp\Big\{A_2\int_{-r}^0\mu(ds)\int_s^0 e^{\gamma(u-s)}h\big(\boldsymbol{\varphi}(u)\big)du\Big\}$
$V_{\mathbf{0}}(\boldsymbol{\varphi})$	$:=\Big(1+\mathbf{c}^\top\mathbf{x}\Big)\exp\Big\{A_2\int_{-r}^0\mu(ds)\int_s^0 e^{\gamma(u-s)}h\big(\boldsymbol{\varphi}(u)\big)du\Big\}$
$\mathcal{C}_{V,M}$	$:=\{\boldsymbol{\varphi}\in\mathcal{C}_+:A_2\gamma\int_{-r}^0\mu(ds)\int_s^0 e^{\gamma(u-s)}h\big(\boldsymbol{\varphi}(u)\big)du\le A_0 \text{ and } \|\mathbf{x}\|\le M\}$
n^*	constant satisfying $\gamma_0(n^*-1)-A_0>0$
p_1	constant satisfying condition (3.1) and $p_1>p_0$
$\mathcal{C}_V(\widehat{\delta})$	$:=\{\boldsymbol{\varphi}\in\mathcal{C}_+^\circ\cap\mathcal{C}_{V,M} \text{ and } \|\varphi_i(0)\|\le\widehat{\delta} \text{ for some } i\}$
$\widehat{\alpha}_i, \alpha_*, \kappa_e$	constants satisfying (4.2)
T_e, δ_e	constants determined in Lemma 4.2

In this paper, we treat a stochastic delay Kolmogorov system of equations

$$\begin{cases} dX_i(t)=X_i(t)f_i(\mathbf{X}_t)dt+X_i(t)g_i(\mathbf{X}_t)dE_i(t), \quad i=1,\dots,n,\\ \mathbf{X}_0=\boldsymbol{\phi}\in\mathcal{C}_+. \end{cases} \tag{2.1}$$

Denote the solution of (2.1) by $\mathbf{X}^{\boldsymbol{\phi}}(t)$. For convenience, we often suppress the superscript "$\boldsymbol{\phi}$" and use $\mathbb{P}_{\boldsymbol{\phi}}$ and $\mathbb{E}_{\boldsymbol{\phi}}$ to denote the probability and expectation given the initial value $\boldsymbol{\phi}$, respectively. We also assume that the initial value is non-random. Denote by $\{\mathcal{F}_t\}_{t\ge0}$ the filtration satisfying the usual conditions and assume that the n-dimensional Brownian motion $\mathbf{B}(t)$ is adapted to $\{\mathcal{F}_t\}_{t\ge 0}$. Note that a segment process is also referred to as a memory segment function. We assume the following assumptions hold in the rest of the paper.

Assumption 2.1. The coefficients of (2.1) satisfy

(1) $\operatorname{diag}(g_1(\boldsymbol{\varphi}),\dots,g_n(\boldsymbol{\varphi}))\Gamma^\top\Gamma\operatorname{diag}(g_1(\boldsymbol{\varphi}),\dots,g_n(\boldsymbol{\varphi}))=(g_i(\boldsymbol{\varphi})g_j(\boldsymbol{\varphi})\sigma_{ij})_{n\times n}$ is a positive definite matrix for any $\boldsymbol{\varphi}\in\mathcal{C}_+$.
(2) $f_i(\cdot), g_i(\cdot):\mathcal{C}_+\to\mathbb{R}$ are Lipschitz continuous in each bounded set of $\mathcal{C}_+$ for any $i=1,\dots,n$.
(3) There exist $\mathbf{c}=(c_1,\dots,c_n)\in\mathbb{R}^n$, $c_i>0, \forall i$ and $\gamma_b,\gamma_0>0$, $A_0>0$, $A_1>A_2>0$, $M>0$, a continuous function $h:\mathbb{R}^n\to\mathbb{R}_+$ and a probability measure μ concentrated on $[-r,0]$ such that for any $\boldsymbol{\varphi}\in\mathcal{C}_+$

$$\begin{aligned}
&\frac{\sum_{i=1}^n c_i x_i f_i(\boldsymbol{\varphi})}{1+\mathbf{c}^\top\mathbf{x}}-\frac{1}{2}\frac{\sum_{i,j=1}^n \sigma_{ij}c_ic_jx_ix_jg_i(\boldsymbol{\varphi})g_j(\boldsymbol{\varphi})}{(1+\mathbf{c}^\top\mathbf{x})^2}\\
&\qquad+\gamma_b\sum_{i=1}^n\Big(|f_i(\boldsymbol{\varphi})|+g_i^2(\boldsymbol{\varphi})\Big)\\
&\qquad\le A_0\mathbf{1}_{\{|\mathbf{x}|<M\}}-\gamma_0-A_1h(\mathbf{x})+A_2\int_{-r}^0 h\big(\boldsymbol{\varphi}(s)\big)\mu(ds),
\end{aligned}\tag{2.2}$$

where $\mathbf{x}:=\boldsymbol{\varphi}(0)$. We assume without loss of generality that $h:\mathbb{R}^n\to[1,\infty)$, otherwise, we can always change γ_0 and A_1, A_2 to fulfill this requirement.

Assumption 2.2. One of following conditions holds:

(a) There is a constant $\widetilde{K}$ such that for any $\boldsymbol{\varphi}\in\mathcal{C}_+$, $\mathbf{x}=\boldsymbol{\varphi}(0)$,

$$\sum_{i=1}^n|f_i(\boldsymbol{\varphi})|+\sum_{i=1}^n g_i^2(\boldsymbol{\varphi})\le\widetilde{K}\Big[h(\mathbf{x})+\int_{-r}^0 h(\boldsymbol{\varphi}(s))\mu(ds)\Big].\tag{2.3}$$

(b) There exist $b_1,b_2>0$, a function $h_1:\mathbb{R}^n\to[1,\infty]$, and a probability measure μ_1 on $[-r,0]$ such that for any $\boldsymbol{\varphi}\in\mathcal{C}_+$, $\mathbf{x}=\boldsymbol{\varphi}(0)$

$$b_1h_1(\mathbf{x})\le\sum_{i=1}^n|f_i(\boldsymbol{\varphi})|+\sum_{i=1}^n g_i^2(\boldsymbol{\varphi})\le b_2\Big[h_1(\mathbf{x})+\int_{-r}^0 h_1(\boldsymbol{\varphi}(s))\mu_1(ds)\Big].\tag{2.4}$$

Remark 1.

- The assumptions above and additional assumptions to follow are not restrictive and are easily verifiable. Such conditions are common in population dynamics in the literature; see Section 5.

- Parts (2) and (3) of Assumption 2.1 guarantee the existence and uniqueness of a strong solution to (2.1). We need part (1) of Assumption 2.1 to ensure that the solution to (2.1) is a non-degenerate diffusion. Moreover, as will be shown later that (3) implies the tightness of the family of transition probabilities associated with the solution to (2.1). One difficulty stems from the positive term $A_2 \int_{-r}^{0} h\big(\boldsymbol{\varphi}(s)\big)\mu(ds)$ on the right-hand side of (2.2), which cannot be relaxed in practice.
- Assumption 2.2 plays an important role in ensuring the π-uniform integrability of $\sum_i \big(|f_i(\cdot)| + g_i^2(\cdot)\big)$, for any invariant measure π.

As was mentioned, persistence and extinction are concepts of vital importance in biology and ecology. It turns out that such concepts are features in all stochastic functional Kolmogorov systems. The termination of a species in biology is referred to as extinction, the moment of extinction is generally considered to be the death of the last individual of the species. In contrast to extinction, we have the persistence of a species. We first define persistence and extinction similar to [25,56,57].

Definition 2.1. Let $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))^\top$ be the solution of (2.1). The process $\mathbf{X}$ is strongly stochastically persistent if for any $\varepsilon > 0$, there exists an $R > 0$ such that for any $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$

$$\liminf_{t\to\infty} \mathbb{P}_{\boldsymbol{\phi}} \left\{ R^{-1} \le |X_i(t)| \le R \right\} \ge 1 - \varepsilon \text{ for all } i = 1, \dots, n. \tag{2.5}$$

Definition 2.2. With $\mathbf{X}(t)$ given in Definition 2.1, for $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$ and some $i \in \{1, \dots, n\}$, we say X_i goes extinct with probability $p_{\boldsymbol{\phi}} > 0$ if

$$\mathbb{P}_{\boldsymbol{\phi}} \left\{ \lim_{t\to\infty} X_i(t) = 0 \right\} = p_{\boldsymbol{\phi}}.$$

Let $\mathcal{M}$ be the set of ergodic invariant probability measures of $\mathbf{X}_t$ supported on the boundary $\partial\mathcal{C}_+$. Letting $\boldsymbol{\delta}^*$ be the Dirac measure concentrated at $\mathbf{0}$, then $\boldsymbol{\delta}^* \in \mathcal{M}$ so that $\mathcal{M} \neq \emptyset$. For a subset $\widetilde{\mathcal{M}} \subset \mathcal{M}$, denote by $\mathrm{Conv}(\widetilde{\mathcal{M}})$ the convex hull of $\widetilde{\mathcal{M}}$ (the set of probability measures π of the form $\pi(\cdot) = \sum_{\nu\in\widetilde{\mathcal{M}}} p_\nu \nu(\cdot)$ with $p_\nu \ge 0$ and $\sum_{\nu\in\widetilde{\mathcal{M}}} p_\nu = 1$). For any $\pi \in \mathrm{Conv}(\mathcal{M})$, we define

$$\lambda_i(\pi) := \int_{\partial\mathcal{C}_+} \left(f_i(\boldsymbol{\varphi}) - \frac{\sigma_{ii} g_i^2(\boldsymbol{\varphi})}{2} \right) \pi(d\boldsymbol{\varphi}).$$

For a subset I of $\{1, \dots, n\}$, denote $I^c := \{1, \dots, n\} \setminus I$,

$$\mathcal{C}_+^I := \left\{ \boldsymbol{\varphi} \in \mathcal{C}_+ : \|\varphi_i\| = 0 \text{ if } i \in I^c \right\},$$
$$\mathcal{C}_+^{I,\circ} := \left\{ \boldsymbol{\varphi} \in \mathcal{C}_+ : \|\varphi_i\| = 0 \text{ if } i \in I^c \text{ and } \varphi_i(s) > 0 \text{ for all } s \in [-r, 0] \text{ if } i \in I \right\},$$

and

$$\partial\mathcal{C}_+^I := \left\{ \boldsymbol{\varphi} \in \mathcal{C}_+ : \|\varphi_i\| = 0 \text{ if } i \in I^c \text{ and } \|\varphi_i\| = 0 \text{ for some } i \in I \right\}.$$

In case $I = \emptyset$, $\mathcal{C}_+^I = \mathcal{C}_+^{I,\circ} = \{\mathbf{0}\}$. Denote by $\mathcal{M}^I$, $\mathcal{M}^{I,\circ}$, ∂M^I the sets of ergodic probability measures on $\mathcal{C}_+^I$, $\mathcal{C}_+^{I,\circ}$ and ∂C_+^I, respectively.

Consider $\pi \in \mathcal{M} \setminus \{\boldsymbol{\delta}^*\}$. Since the diffusion $\mathbf{X}_t$ is non-degenerate in each subspace, there exists a subset of $\{1, \ldots, n\}$, denoted by I_π such that $\pi(\mathcal{C}_+^{I_\pi,\circ}) = 1$. The following conditions will imply persistence cannot happen.

Assumption 2.3. There exists a subset $I \subset \{1, \ldots, n\}$ such that

$$\max_{i \in I_\pi^c,\, \pi \in \mathcal{M}^{I,\circ}} \{\lambda_i(\pi)\} < 0. \tag{2.6}$$

If $I \neq \emptyset$, we assume further that

$$\max_{i \in I} \{\lambda_i(\nu)\} > 0, \tag{2.7}$$

for any $\nu \in \mathrm{Conv}(\partial \mathcal{M}^I)$.

Assumption 2.4. The inverse of the matrix $(x_i x_j \sigma_{ij} g_i(\boldsymbol{\varphi}) g_j(\boldsymbol{\varphi}))_{n \times n}$ is uniformly bounded in $D_{\varepsilon,R}$ for each ε, $R > 0$, where

$$D_{\varepsilon,R} := \{\boldsymbol{\varphi} \in \mathcal{C}_+ : \|\boldsymbol{\varphi}\| \le R, x_i \ge \varepsilon \ \forall i; \mathbf{x} := \boldsymbol{\varphi}(0)\}.$$

Theorem 2.1. *Assume Assumptions 2.1, 2.2, 2.3, and 2.4 hold. For any $p < p_0$ with p_0 being a sufficiently small constant (as given in Lemma 3.1), and any initial value $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$, we have*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}_{\boldsymbol{\phi}} \bigwedge_{i=1}^n \|X_{i,t}\|^p \, dt = 0, \tag{2.8}$$

where $\bigwedge_{i=1}^n x_i := \min_{i=1,\ldots,n}\{x_i\}$ and $\mathbf{X}_t =: (X_{1,t}, \ldots, X_{n,t})$.

With additional technical conditions, we can determine which species goes extinct, and which persists. First, we define random normalized occupation measures

$$\widetilde{\Pi}_t(\cdot) := \frac{1}{t} \int_0^t \mathbf{1}_{\{\mathbf{X}_s \in \cdot\}} ds, \ t > 0. \tag{2.9}$$

For any initial condition $\boldsymbol{\phi} \in \mathcal{C}_+$, denote the weak*-limit set of the family $\left\{\widetilde{\Pi}_t(\cdot), t \ge 1\right\}$ by $\mathcal{U} = \mathcal{U}(\omega)$.

Assumption 2.5. Assume one of the following conditions holds.

- Assumption 2.2(a) holds and there exist constants $p_2 > 0$ and $B_1 > B_2 > 0$, $B_0 > 0$, and $B_3 > 0$ such that for any $\boldsymbol{\varphi} \in \mathcal{C}_+$, $\mathbf{x} := \boldsymbol{\varphi}(0)$

$$
\begin{aligned}
(1+\mathbf{c}^\top\mathbf{x})^{p_2}&\left(\frac{\sum_{i=1}^n c_i x_i f_i(\boldsymbol{\varphi})}{1+\mathbf{c}^\top\mathbf{x}}-\frac{1}{2}\frac{\sum_{i,j=1}^n \sigma_{ij}c_ic_jx_ix_jg_i(\boldsymbol{\varphi})g_j(\boldsymbol{\varphi})}{(1+\mathbf{c}^\top\mathbf{x})^2}\right)\\
&\le B_0-B_1(1+\mathbf{c}^\top\mathbf{x})^{p_2}h(\mathbf{x})\\
&+B_2\int_{-r}^{0}(1+\mathbf{c}^\top\boldsymbol{\varphi}(s))^{p_2}h\big(\boldsymbol{\varphi}(s)\big)\mu(ds),
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
(1+\mathbf{c}^\top\mathbf{x})^{2p_2}\sum_{i=1}^n g_i^2(\boldsymbol{\varphi})\le& B_3(1+\mathbf{c}^\top\mathbf{x})^{p_2}h(\mathbf{x})\\
&+B_3\int_{-r}^{0}(1+\mathbf{c}^\top\boldsymbol{\varphi}(s))^{p_2}h\big(\boldsymbol{\varphi}(s)\big)\mu(ds).
\end{aligned}
\tag{2.11}
$$

- Assumption 2.2(b) is satisfied, (2.10), and (2.11) hold with h, μ replaced by h_1, μ_1.

Assumption 2.6. Let S be a family of subsets I satisfying Assumption 2.3. We assume either that $S^c := 2^{\{1,\dots,n\}} \setminus S$ is empty, where $2^{\{1,\dots,n\}}$ denotes the family of all subsets of $\{1,\dots,n\}$, or

$$
\max_{i=1,\dots,n}\{\lambda_i(\nu)\}>0 \text{ for any } \nu\in\operatorname{Conv}(\cup_{J\notin S}\mathcal{M}^{J,\circ}).
$$

Theorem 2.2. *Suppose that Assumptions 2.1, 2.3, 2.4, 2.5, and 2.6 are satisfied. Then for any $\boldsymbol{\phi}\in\mathcal{C}_+^\circ$,*

$$
\sum_{I\in S}P_{\boldsymbol{\phi}}^I=1,\quad P_{\boldsymbol{\phi}}^I>0, \tag{2.12}
$$

where for $\boldsymbol{\phi}\in\mathcal{C}_+^\circ$,

$$
P_{\boldsymbol{\phi}}^I:=\mathbb{P}_{\boldsymbol{\phi}}\left\{\mathcal{U}(\omega)\subset\operatorname{Conv}(\mathcal{M}^{I,\circ}) \text{ and } \lim_{t\to\infty}\frac{\ln X_i(t)}{t}\subset\left\{\lambda_i(\pi):\pi\in\operatorname{Conv}(\mathcal{M}^{I,\circ})\right\}, i\in I^c\right\}.
$$

In the above, $\lim_{t\to\infty}x(t)$ can be understood as the set of limit points of $x(\cdot)$ as $t\to\infty$.

Remark 2. From a dynamic system point of view, we have the following observations; see also [25].

- Assumption 2.3 states the existence of an attracting subspace on the boundary which normally results in extinction.
- Assumption 2.6 is a technical condition ensuring that the interior of the attracting subspace in Assumption 2.3 is an attractor in that subspace.
- Assumption 2.5 is a condition to control the volatility of the diffusion part while the nondegenaracy of the diffusion part due to Assumption 2.4 leads to the accessibility to the boundary from any interior point.

3. Technical results

3.1. Well-posedness of the problem

The well-posedness of the problem (2.1) and some basic properties have been studied in the first part [46]. We restate some results, while the proofs are referred to [46]. The following series of results provide the estimating of the infinitesimal operator $\mathcal{L}V_{\boldsymbol{\rho}}$, the well-posedness of the problem, the "local" compactness of the solution, the regularity of the solution and the continuity on the initial data, respectively. The formula of the infinitesimal operator and the functional Itô formula are referred to the first part [46].

Lemma 3.1. *For any $\gamma < \gamma_b$ and $p_0 > 0$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ satisfying*

$$|\boldsymbol{\rho}| < \min\left\{\frac{\gamma_b}{2}, \frac{1}{n}, \frac{\gamma_b}{4\sigma^*}\right\} \text{ and } p_0 < \min\left\{1, \frac{\gamma_b}{8n\sigma^*}\right\}, \tag{3.1}$$

where $\sigma^* := \max\{\sigma_{ij} : 1 \le i, j \le n\}$, let

$$V_{\boldsymbol{\rho}}(\boldsymbol{\varphi}) := \Big(1 + \mathbf{c}^\top \mathbf{x}\Big) \prod_{i=1}^n x_i^{\rho_i} \exp\left\{A_2 \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\gamma(u-s)} h\big(\boldsymbol{\varphi}(u)\big) du\right\}.$$

Then, we have

$$\begin{aligned}\mathcal{L}V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\varphi}) \le p_0 V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\varphi}) \Bigg[& A_0 \mathbf{1}_{\{|\mathbf{x}|<M\}} - \gamma_0 - A h(\mathbf{x}) \\ & - A_2 \gamma \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\gamma(u-s)} h\big(\boldsymbol{\varphi}(u)\big) du - \frac{\gamma_b}{2} \sum_{i=1}^n \Big(|f_i(\boldsymbol{\varphi})| + g_i^2(\boldsymbol{\varphi}) \Big) \Bigg],\end{aligned} \tag{3.2}$$

where $\mathbf{x} := \boldsymbol{\varphi}(0)$ and A is a positive number satisfying $A < A_1 - A_2 \int_{-r}^0 e^{-\gamma s} \mu(ds)$. Recall that $\mathbf{c}$, M, A_0, A_1, A_2, γ_0, γ_b, $h(\cdot)$, $\mu(\cdot)$ are defined in Assumption 2.1(3).

Theorem 3.1. *For any initial condition $\boldsymbol{\phi} \in \mathcal{C}_+$, there exists a unique global solution of (2.1). It remains in $\mathcal{C}_+$ (resp. $\mathcal{C}_+^\circ$), provided $\boldsymbol{\phi} \in \mathcal{C}_+$ (resp. $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$). Moreover, for any $p_0, \boldsymbol{\rho}$ satisfying condition (3.1), we have*

$$\mathbb{E}_{\boldsymbol{\phi}} V_{\boldsymbol{\rho}}^{p_0}(\mathbf{X}_t) \le V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\phi}) e^{A_0 p_0 t}. \tag{3.3}$$

In addition, if $\rho_i \ge 0, \forall i$, then

$$\mathbb{E}_{\boldsymbol{\phi}} V_{\boldsymbol{\rho}}^{p_0}(\mathbf{X}_t) \le V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\phi}) e^{-\gamma_0 p_0 t} + \overline{M}_{p_0,\boldsymbol{\rho}}, \tag{3.4}$$

where

$$\overline{M}_{p_0,\boldsymbol{\rho}} := \frac{A_0}{\gamma_0} \sup_{\boldsymbol{\varphi}\in\mathcal{C}_{V,M}} V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\varphi}) < \infty \text{ provided } \rho_i \ge 0\,\forall i,$$

and $\mathcal{C}_{V,M} = \{\boldsymbol{\varphi} \in \mathcal{C}_+ : A_2\gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h\big(\boldsymbol{\varphi}(u)\big)du \le A_0 \text{ and } |\mathbf{x}| \le M\}$.

Lemma 3.2. *For any* $R_1 > 0$, $T > r$, $\varepsilon > 0$, *there exists an* $R_2 > 0$ *such that*

$$\mathbb{P}_{\boldsymbol{\phi}}\Big\{ \|\mathbf{X}_t\| \le R_2,\ \forall t \in [r, T]\Big\} > 1 - \varepsilon,$$

for any initial point $\boldsymbol{\phi}$ *satisfying* $V_{\mathbf{0}}(\boldsymbol{\phi}) < R_1$, *where* $V_{\mathbf{0}}$ *is defined as in Lemma 3.1 corresponding to* $\boldsymbol{\rho} = \mathbf{0} = (0, \dots, 0)$.

Lemma 3.3. *There is a sufficiently small* $\alpha > 0$ *such that for any* $R > 0$ *and* $\varepsilon > 0$, *there exists* $R_3 = R_3(R, \varepsilon) > 0$ *satisfying*

$$\text{if } \|\boldsymbol{\phi}\| \le R \text{ then } \mathbb{P}_{\boldsymbol{\phi}}\{\|\mathbf{X}_t\|_{2\alpha} \le R_3\ \forall t \in [r, 3r]\} \ge 1 - \frac{\varepsilon}{2}. \tag{3.5}$$

As a consequence, for any $R > 0$ *and* $\varepsilon > 0$, *there exists an* $R_4 = R_4(\varepsilon, R) > 0$ *satisfying that*

$$\text{if } V_{\mathbf{0}}(\boldsymbol{\phi}) \le R \text{ then } \mathbb{P}_{\boldsymbol{\phi}}\{\|\mathbf{X}_t\|_{2\alpha} \le R_4\ \forall t \in [2r, 3r]\} \ge 1 - \varepsilon. \tag{3.6}$$

Proposition 3.1. *The following results hold.*

(i) *Let* $\rho_1^{(3)}$ *be a fixed constant satisfying* $0 < \rho_1^{(3)} < \min\left\{\frac{\gamma_b}{2}, \frac{1}{n}, \frac{\gamma_b}{4\sigma^*}\right\}$. *For any* $T > r$ *and* $m > 0$ *there exists a finite constant* $K_{m,T}$ *such that*

$$\mathbb{E}_{\boldsymbol{\phi}} \left\|X_{i,t}\right\|^{p_0\rho_1^{(3)}} \le K_{m,T}\phi_i^{p_0\rho_1^{(3)}}(0),\ \forall t \in [r, T], i = 1, \dots, n,$$

given that

$$|\boldsymbol{\phi}(0)| + \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\gamma(u-s)} h\big(\boldsymbol{\phi}(u)\big)du < m,$$

where $\mathbf{X}_t =: (X_{1,t}, \dots, X_{n,t})$ *and* $\boldsymbol{\phi} =: (\phi_1, \dots, \phi_n)$ *is the initial value.*

(ii) *For any* $T > r$, $\varepsilon > 0$, $R > 0$, *there exists an* $\varepsilon_1 > 0$ *such that*

$$\mathbb{P}\left\{\left\|\mathbf{X}_T^{\boldsymbol{\phi}_1} - \mathbf{X}_T^{\boldsymbol{\phi}_2}\right\| \le \varepsilon\right\} \ge 1 - \varepsilon \text{ whenever } V_{\mathbf{0}}(\boldsymbol{\phi}_i) < R, \left\|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\right\| \le \varepsilon_1. \tag{3.7}$$

Moreover, the solution $(\mathbf{X}_t)$ *has the Feller property in* $\mathcal{C}_+$.

3.2. Random occupation measures: tightness and convergence

Next, we deal with the tightness and uniform integrability of the random normalized occupation measures, which are defined by

$$\widetilde{\Pi}_t(\cdot) := \frac{1}{t}\int_0^t \mathbf{1}_{\{\mathbf{X}_s\in\cdot\}}ds,\ t>0.$$

Lemma 3.4. *Suppose that Assumptions 2.1 and 2.5 are satisfied. Then the following results hold*

- *There is a $\widetilde{G}>0$ such that for all $\boldsymbol{\phi}\in\mathcal{C}_+$*

$$\mathbb{P}_{\boldsymbol{\phi}}\Bigg\{\limsup_{T\to\infty}\frac{1}{T}\int_0^T\Bigg[\Big(1+\sum_i c_iX_i(t)\Big)^{p_2}h(\mathbf{X}(t)) + \int_{-r}^0\Big(1+\sum_i c_iX_i(t+s)\Big)^{p_2}h(\mathbf{X}(t+s))\mu(ds)\Bigg]dt\le\widetilde{G}\Bigg\}=1,$$

  where p_2 is as in Assumption 2.5.
- *Suppose that we have a sample path of $\mathbf{X}$ satisfying*

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\Big[\Big(1+\sum_i c_iX_i(t)\Big)^{p_2}h(\mathbf{X}(t)) + \int_{-r}^0\Big(1+\sum_i c_iX_i(t+s)\Big)^{p_2}h(\mathbf{X}(t+s))\mu(ds)\Big]dt\le\widetilde{G},$$

  and that there is a sequence $(T_k)_{k\in\mathbb{N}}\subset\mathbb{R}^n_+$ such that $\lim_{k\to\infty}T_k=\infty$ and $\big\{\widetilde{\Pi}_{T_k}(\cdot)\big\}_{k\in\mathbb{N}}$ converges weakly to an invariant probability measure π of $\mathbf{X}$ when $k\to\infty$. Then for this sample path, we have

$$\int_{\mathcal{C}_+}K(\boldsymbol{\varphi})\widetilde{\Pi}_{T_k}(d\boldsymbol{\varphi})\to\int_{\mathcal{C}_+}K(\boldsymbol{\varphi})\pi(d\boldsymbol{\varphi}),$$

  for any continuous function $K:\mathcal{C}_+\to\mathbb{R}$ satisfying $\forall\boldsymbol{\varphi}\in\mathcal{C}_+, 0<p<p_2$,

$$|K(\boldsymbol{\varphi})|<C_K\left[\Big(1+\mathbf{c}^\top\mathbf{x}\Big)^p h(\mathbf{x})+\int_{-r}^0\Big(1+\mathbf{c}^\top\boldsymbol{\varphi}(s)\Big)^p h(\boldsymbol{\varphi}(s))\mu(ds)\right],$$

  with C_K being a positive constant.

- *There is a constant* $\widehat{K}_1 > 1$ *such that*

$$\mathbb{P}_{\boldsymbol{\phi}}\left\{\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{\|\mathbf{X}_s\|\le\widehat{K}_1\}}ds \ge \frac{1}{2}\right\} = 1,\ \boldsymbol{\phi}\in\mathcal{C}_+. \tag{3.8}$$

Moreover, for any ε_1 *and* $\varepsilon_2 > 0$*, there is a* $\beta > 0$ *such that for each* $i = 1, \dots, n$*,*

$$\mathbb{P}_{\boldsymbol{\phi}}\{X_i(t) > \beta, \forall t\in[0, n^*T_e]\} > 1-\varepsilon_1 \text{ if } \boldsymbol{\phi}\in\mathcal{C}_+,\ V_{\mathbf{0}}(\boldsymbol{\phi})\le\widehat{K}_1,\ \phi_i(0) > \varepsilon_2, \tag{3.9}$$

where n^* *and* T_e *are as in Lemma 4.2.*

Proof. Consider the first assertion. We obtain from the functional Itô formula and (2.10) that

$$\begin{aligned}
&\frac{\left(1+\mathbf{c}^\top\mathbf{X}(t)\right)^{p_2} - \left(1+\mathbf{c}^\top\mathbf{X}(0)\right)^{p_2}}{t}\\
&\le B_0 - \frac{B_1}{t}\int_0^t (1+\mathbf{c}^\top\mathbf{X}(s))^{p_2}h(\mathbf{X}(s))ds\\
&\quad + \frac{B_2}{t}\int_0^t ds\int_{-r}^0 (1+\mathbf{c}^\top\mathbf{X}(u+s))^{p_2}h(\mathbf{X}(u+s))\mu(du) + \frac{\overline{L}(t)}{t}\\
&\le B_0 + \frac{B_2}{t}\int_0^r ds\int_{-r}^0 (1+\mathbf{c}^\top\mathbf{X}(u+s))^{p_2}h(\mathbf{X}(u+s))\mu(du)\\
&\quad - \frac{B_1-B_2}{t}\int_0^t (1+\mathbf{c}^\top\mathbf{X}(s))^{p_2}h(\mathbf{X}(s))ds + \frac{\overline{L}(t)}{t},
\end{aligned} \tag{3.10}$$

where $\overline{L}(t)$ is the diffusion part of $\left(1+\mathbf{c}^\top\mathbf{X}(t)\right)^{p_2}$. Due to (2.11), we have the following estimate for the quadratic variation $\langle\overline{L}_\cdot, \overline{L}_\cdot\rangle_t$ of $\overline{L}(t)$

$$\begin{aligned}\langle \overline{L}_\cdot, \overline{L}_\cdot\rangle_t \leq & B_3 \int_0^t (1+\mathbf{c}^\top \mathbf{X}(s))^{p_2} h(\mathbf{X}(s))ds \\ & + B_3 \int_0^t ds \int_{-r}^0 (1+\mathbf{c}^\top \mathbf{X}(u+s))^{p_2} h(\mathbf{X}(u+s))\mu(du) \\ \leq & B_3 \int_0^r ds \int_{-r}^0 (1+\mathbf{c}^\top \mathbf{X}(u+s))^{p_2} h(\mathbf{X}(u+s))\mu(du) \\ & + 2B_3 \int_0^t (1+\mathbf{c}^\top \mathbf{X}(s))^{p_2} h(\mathbf{X}(s))ds.\end{aligned}$$

It follows from the strong law of large numbers for local martingales that

$$\limsup_{t\to\infty} \left(-\frac{B_1 - B_2}{2t} \int_0^t (1+\mathbf{c}^\top \mathbf{X}(s))^{p_2} h(\mathbf{X}(s))ds + \frac{\overline{L}(t)}{t} \right) \leq 0 \text{ a.s.} \tag{3.11}$$

Applying (3.11) and noting $\liminf_{t\to\infty} \dfrac{(1+\mathbf{c}^\top \mathbf{X}(t))^{p_2} - (1+\mathbf{c}^\top \mathbf{X}(0))^{p_2}}{t} \geq 0$ to (3.10), we have

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t (1+\mathbf{c}^\top \mathbf{X}(s))^{p_2} h(\mathbf{X}(s))ds \leq \frac{2B_0}{B_1 - B_2} \text{ a.s.}$$

Therefore, the first part of Lemma 3.4 is proved.

The proof of the second part is the same as that of [46, Lemma 3.5] and is omitted. The proof of the third part can be found in [25, Proof of Lemma 5.5]. □

Compared to [25, Lemma 5.7], it is much more difficult to prove the tightness and characterize the limit of the normalized occupation measures in this setting because $\mathcal{C}$ is an infinite dimensional space.

Lemma 3.5. *Let Assumptions 2.1 and 2.5 be satisfied. For any initial condition $\boldsymbol{\phi} \in \mathcal{C}_+$, the family $\left\{\widetilde{\Pi}_t(\cdot), t \geq 1\right\}$ is tight in $\mathcal{C}_+$, and its weak*-limit set, denoted by $\mathcal{U} = \mathcal{U}(\omega)$, is a family of invariant probability measures of $\mathbf{X}_t$ with probability 1.*

Proof. For simplicity of notation, denote $\widetilde{h}(\mathbf{x}) = (1 + \mathbf{c}^\top \mathbf{x})^{p_2} h(\mathbf{x})$. We have

$$\begin{aligned}\int_0^T \left[\int_{-r}^0 \widetilde{h}(\mathbf{X}_t(u))du\right] dt &= \int_0^T \left[\int_{-r}^0 \widetilde{h}(\mathbf{X}(t+u))du\right] dt \\ &\leq \int_{-r}^0 \left[\int_{-r}^T \widetilde{h}(\mathbf{X}(t))dt\right] du \leq r\int_0^T \widetilde{h}(\mathbf{X}(t))dt + r\int_{-r}^0 \widetilde{h}(\mathbf{X}(u))du.\end{aligned} \tag{3.12}$$

Hence, by Lemma 3.4 and (3.12), we deduce that

$$\lim_{T\to\infty} \int_0^T \left[\widetilde{h}(\mathbf{X}(t)) + \int_{-r}^0 \widetilde{h}(\mathbf{X}_t(u))du\right] dt \leq \widetilde{G}(1+r). \tag{3.13}$$

Define

$$\widetilde{C}_R := \left\{\boldsymbol{\varphi} \in \mathcal{C}_+^\circ : \widetilde{h}(\mathbf{x}) + \int_{-r}^0 \widetilde{h}(\boldsymbol{\varphi}(u))du < R, \mathbf{x} = \boldsymbol{\varphi}(0)\right\}.$$

A consequence of (3.13) is that for any $\varepsilon > 0$, there is an $R > 0$ such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{\{\mathbf{X}_t \in \widetilde{C}_R\}} dt > 1 - \varepsilon. \tag{3.14}$$

It is easily seen that

$$\sup_{\boldsymbol{\varphi}\in\widetilde{C}_R} V_{\mathbf{0}}^{p_0}(\boldsymbol{\varphi}) < \infty \ \forall R > 0.$$

By Lemma 3.3, there is a compact set $\mathcal{K} := \{\boldsymbol{\varphi} : \|\boldsymbol{\varphi}\|_{2\alpha} < R_4\}$ of $\mathcal{C}^\alpha$, for some $\alpha > 0$ and $R_4 = R_4(\varepsilon, R)$ such that

$$\mathbb{P}_{\boldsymbol{\phi}}\{\mathbf{X}_t \in \mathcal{K} \ \forall t \in [2r, 3r]\} \geq 1 - \varepsilon, \quad \boldsymbol{\phi} \in \widetilde{C}_R. \tag{3.15}$$

Let $Y_k := \mathbf{1}_{\{\mathbf{X}_{kr}\in\mathcal{K}\}}$ for $k \in \mathbb{N}$, then $\sum_{l=1}^k Y_l = A_k + M_k$ with

$$A_k := \sum_{l=1}^k \mathbb{E}\left[Y_l|\mathcal{F}_{(l-1)r}\right] \ ; \ M_k := Y_0 + \sum_{l=1}^k \left(Y_l - \mathbb{E}\left[Y_l|\mathcal{F}_{(l-1)r}\right]\right).$$

By strong law of large number for martingales, it is easily seen that

$$\lim_{k\to\infty} \frac{M_k}{k} = 0 \text{ a.s.} \tag{3.16}$$

To proceed, we estimate $\mathbb{E}\left[Y_l|\mathcal{F}_{(l-1)r}\right]$ for $l \geq 2$. In the event

$$\left\{\frac{1}{r}\int\limits_{(l-2)r}^{(l-1)r} \mathbf{1}_{\{\mathbf{X}_t\in\widetilde{C}_R\}}dt > 0\right\},$$

$\mathbf{X}_t \in \widetilde{C}_R$ for some $t \in [(l-2)r, (l-1)r]$ and then, by the strong Markov property of $\mathbf{X}_t$ and (3.15), we have

$$\mathbb{E}\left[Y_l\Bigg|\frac{1}{r}\int\limits_{(l-2)r}^{(l-1)r} \mathbf{1}_{\{\mathbf{X}_t\in\widetilde{C}_R\}}dt > 0\right] \geq 1-\varepsilon.$$

Thus, owing to $\frac{1}{r}\int_{(l-2)r}^{(l-1)r} \mathbf{1}_{\{\mathbf{X}_t\in\widetilde{C}_R\}}dt \leq 1$, we can write

$$\mathbb{E}\left[Y_l|\mathcal{F}_{(l-1)r}\right] \geq \frac{1-\varepsilon}{r}\int\limits_{(l-2)r}^{(l-1)r} \mathbf{1}_{\{\mathbf{X}_t\in\widetilde{C}_R\}}dt \ \text{ a.s.}$$

As a result,

$$\frac{A_k}{k} \geq \frac{1-\varepsilon}{kr}\int\limits_{0}^{(k-1)r} \mathbf{1}_{\{\mathbf{X}_t\in\widetilde{C}_R\}}dt \text{ a.s. },$$

which together with (3.14) implies that

$$\liminf_{k\to\infty}\frac{A_k}{k} \geq (1-\varepsilon)^2 \text{ a.s.} \tag{3.17}$$

We deduce from (3.16) and (3.17) that

$$\liminf_{k\to\infty}\frac{\sum_{l=1}^k Y_l}{k} \geq 1-2\varepsilon \text{ a.s.} \tag{3.18}$$

We have the following estimate

$$1-\mathbf{1}_{\{\mathbf{X}_{lr}\in\mathcal{K} \text{ and } \mathbf{X}_{(l+1)r}\in\mathcal{K}\}} \leq (1-Y_l)+(1-Y_{l+1}). \tag{3.19}$$

Due to (3.18) and (3.19), we get

$$\lim_{k\to\infty}\frac{\sum_{l=1}^k \mathbf{1}_{\{\mathbf{X}_{lr}\in\mathcal{K} \text{ and } \mathbf{X}_{(l+1)r}\in\mathcal{K}\}}}{k} \geq 1-4\varepsilon \text{ a.s.} \tag{3.20}$$

By Lemma 3.3, it is easy to show that there is a compact set $\widetilde{\mathcal{K}}$ of $\mathcal{C}$ such that

$$\mathbf{X}_t \in \widetilde{\mathcal{K}}, \forall t \in [lr, (l+1)r] \text{ if } \mathbf{X}_{lr} \in \mathcal{K} \text{ and } \mathbf{X}_{(l+1)r} \in \mathcal{K}. \tag{3.21}$$

A consequence of (3.20) and (3.21) is that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{\{\mathbf{X}_t \in \widetilde{\mathcal{K}}\}} \geq 1 - 4\varepsilon \text{ a.s.} \tag{3.22}$$

As a result, the tightness of $\{\widetilde{\Pi}_t(\cdot)\}$ is obtained.

Moreover, from (3.22), we obtain a family of compact sets $\{\mathcal{K}_\varepsilon \subset \mathcal{C}_+, \varepsilon \in (0,1)\}$ such that for any $\varepsilon > 0$, $\mathbb{P}_{\boldsymbol{\phi}} \{\mathcal{U}(\omega) \in \mathcal{U}_{\mathcal{K}}\} = 1$, where $\mathcal{U}_{\mathcal{K}}$ is the set of probability measures on $\mathcal{C}$ satisfying $\pi(\mathcal{K}_\varepsilon) > 1 - \varepsilon$ for all $\varepsilon \in (0,1)$. From the definition of $\mathcal{U}_{\mathcal{K}}$, there exists a countable family $\{v_k\}$ of bounded and continuous functions from $\mathcal{C}$ to $\mathbb{R}$ such that for any bounded and continuous function v and measure $\pi \in \mathcal{U}_{\mathcal{K}}$, we have

$$\int v(\boldsymbol{\varphi})\pi(d\boldsymbol{\varphi}) = \lim_{k_n\to\infty} \int v_{k_n}(\boldsymbol{\varphi})\pi(d\boldsymbol{\varphi}). \tag{3.23}$$

Using the standard arguments in [20, Proof of Theorem 4.2], we can show that outside a null set, any weak limit of $\{\widetilde{\Pi}_t(\cdot)\}$, denoted by $\widetilde{\pi}$, satisfies

$$\int_{\mathcal{C}_+} \widetilde{\pi}(d\boldsymbol{\varphi}) \int P(t, \boldsymbol{\varphi}, \boldsymbol{\psi}) v_k(\boldsymbol{\psi}) = \int v_k(\boldsymbol{\varphi})\widetilde{\pi}(d\boldsymbol{\varphi}). \tag{3.24}$$

From (3.23) and (3.24), we have that outside a null set, for any bounded and continuous function v,

$$\int_{\mathcal{C}_+} \widetilde{\pi}(d\boldsymbol{\varphi}) \int P(t, \boldsymbol{\varphi}, \boldsymbol{\psi}) v(\boldsymbol{\psi}) = \int v(\boldsymbol{\varphi})\widetilde{\pi}(d\boldsymbol{\varphi}). \tag{3.25}$$

The lemma is thus proved. $\square$

4. Extinction

Following the development in the last section, this section focuses on obtaining the criteria of extinction. To start, we have the following Lemma, whose proof is easily obtained by modifying the proof of [25, Lemma 5.1].

Lemma 4.1. *For any $\pi \in \mathcal{M}$ and $i \in I_\pi$, we have $\lambda_i(\pi) = 0$.*

The intuition behind Lemma 4.1 is clear. If the process evolves in the interior of the support of an ergodic invariant measure μ, it will eventually approach the "stationary" state with probability measure μ and it cannot grow or decay exponentially fast.

It is shown in [57, Lemma 4], by the min-max principle, that condition (2.7) is equivalent to the existence of $0 < \widehat{\alpha}_i < p_0, i \in I$ such that

$$\inf_{\nu\in\partial\mathcal{M}^I}\sum_{i\in I}\widehat{\alpha}_i\lambda_i(\nu)>0.$$

Thus, there is an $\alpha_*>0$ sufficiently small such that

$$\inf_{\nu\in\partial\mathcal{M}^I}\sum_{i\in I}\widehat{\alpha}_i\lambda_i(\nu)-\alpha_*\max_{i\in I^c}\{\lambda_i(\nu)\}>0. \tag{4.1}$$

In view of (4.1), (2.6), and Lemma 4.1, there is a $\kappa_e>0$ such that for any $\nu\in\mathcal{M}^I$,

$$\sum_{i\in I}\widehat{\alpha}_i\lambda_i(\nu)-\alpha_*\max_{i\in I^c}\{\lambda_i(\nu)\}>3\kappa_e. \tag{4.2}$$

Now, denote

$$\begin{aligned}Q_{\mathbf{0}}(\boldsymbol{\varphi})=&A_2h(\mathbf{x})\int_{-r}^{0}e^{-\gamma s}\mu(ds)-A_2\int_{-r}^{0}h\big(\boldsymbol{\varphi}(s)\big)\mu(ds)\\&-A_2\gamma\int_{-r}^{0}\mu(ds)\int_{s}^{0}e^{\gamma(u-s)}h\big(\boldsymbol{\varphi}(u)\big)du\\&+\frac{\sum_{i=1}^n c_ix_if_i(\boldsymbol{\varphi})}{1+\mathbf{c}^\top\mathbf{x}}-\frac{1}{2}\sum_{i,j=1}^n\frac{c_ic_j\sigma_{ij}x_ix_jg_i(\boldsymbol{\varphi})g_j(\boldsymbol{\varphi})}{\big(1+\mathbf{c}^\top\mathbf{x}\big)^2},\end{aligned}$$

and let n^* be a sufficient large integer such that

$$\gamma_0(n^*-1)-A_0>0. \tag{4.3}$$

Lemma 4.2. *Assume that Assumptions 2.1, 2.2, and 2.3 hold. Let $I\subset\{1,\dots,n\}$ satisfy Assumption 2.3. Then there are $T_e\ge 0$ and $\delta_e>0$ such that for any $T\in[T_e,n^*T_e]$, $\boldsymbol{\phi}\in\mathcal{C}_+^\circ\cap\mathcal{C}_{V,M}$, $\phi_i(0)<\delta_e,\forall i\in I^c$, we have*

$$\begin{aligned}&\frac{1}{T}\int_0^T\mathbb{E}_{\boldsymbol{\phi}}Q_{\mathbf{0}}(\mathbf{X}_t)dt-\sum_{i\in I}\widehat{\alpha}_i\frac{1}{T}\int_0^T\mathbb{E}_{\boldsymbol{\phi}}\left(f_i(\mathbf{X}_t)-\frac{\sigma_{ii}g_i^2(\mathbf{X}(t))}{2}\right)dt\\&\qquad+\alpha_*\max_{i\in I^c}\left\{\frac{1}{T}\int_0^T\mathbb{E}_{\boldsymbol{\phi}}\left(f_i(\mathbf{X}_t)-\frac{\sigma_{ii}g_i^2(\mathbf{X}_t)}{2}\right)dt\right\}\le-\kappa_e,\end{aligned} \tag{4.4}$$

Proof. The proof is similar to [46, Lemma 4.2]. First, using (4.2), we can prove that for any compact set $\mathcal{K}$, there exists a $T_{\mathcal{K}}>0$ such that for any $T>T_{\mathcal{K}}$, $\boldsymbol{\phi}\in\mathcal{C}_+^\circ\cap\mathcal{K}$, we have

$$
\begin{aligned}
&\frac{1}{T}\int_0^T \mathbb{E}_{\boldsymbol\phi} Q_{\mathbf{0}}(\mathbf{X}_t)dt - \sum_{i\in I}\widehat{\alpha}_i \frac{1}{T}\int_0^T \mathbb{E}_{\boldsymbol\phi}\left(f_i(\mathbf{X}_t) - \frac{\sigma_{ii} g_i^2(\mathbf{X}_t)}{2}\right)dt \\
&\qquad + \alpha_* \max_{i\in I^c}\left\{\frac{1}{T}\int_0^T \mathbb{E}_{\boldsymbol\phi}\left(f_i(\mathbf{X}_t) - \frac{\sigma_{ii} g_i^2(\mathbf{X}_t)}{2}\right)dt\right\} \le -2\kappa_e.
\end{aligned}
\tag{4.5}
$$

Then, although the Feller property of $(\mathbf{X}_t)$ is not directly applied here because a bounded set in an infinite dimensional space is not necessarily pre-compact, we can overcome the difficulty by using Lemma 3.3 and Proposition 3.1(ii). The detailed calculations are analogous to that of [46, Lemma 4.2] and are omitted. □

Lemma 4.3. *Suppose that Assumptions 2.1, 2.2, and 2.3 hold, and $\widehat{\alpha}_i$, α_*, δ_e, n^*, T_e are as in Lemma 4.2. Then there is a $\theta \in (0, p_0)$ such that for any $T \in [T_e, n^* T_e]$ and $\boldsymbol\phi \in \mathcal{C}_+^\circ \cap \mathcal{C}_{V,M}$ satisfying $\phi_i(0) < \delta_e$, $\forall i \in I^c$ one has*

$$
\mathbb{E}_{\boldsymbol\phi}\widehat{U}_\theta(\mathbf{X}_T) \le \exp\left(-\frac{1}{4}\theta\kappa_e T\right)\widehat{U}_\theta(\boldsymbol\phi),
$$

where

$$
\begin{aligned}
\widehat{U}_\theta(\boldsymbol\varphi) &:= \sum_{i\in I^c} V^\theta_{\boldsymbol\rho^{i,e}}(\boldsymbol\varphi) \\
&= \sum_{i\in I^c}\left[(1+\mathbf{c}^\top\mathbf{x})\frac{x_i^{\alpha_*}}{\prod_{j\in I} x_j^{\widehat{\alpha}_j}}\exp\left\{A_2\int_{-r}^0 \mu(ds)\int_s^0 e^{\gamma(u-s)}h\big(\boldsymbol\varphi(u)\big)du\right\}\right]^\theta,
\end{aligned}
$$

and $\boldsymbol\rho^{i,e} = (\rho_1^{i,e}, \dots, \rho_n^{i,e})$ and

$$
\rho_j^{i,e} = \alpha_* \text{ if } j = i,\ \ \rho_j^{i,e} = -\widehat{\alpha}_j \text{ if } j \ne i,\ j\in I \text{ and otherwise, } \rho_j^{i,e} = 0.
$$

Proof. The argument to prove this proposition is similar to that of [46, Proposition 4.1]. For each $i \in I^c$, by making use of Lemma 4.2, there exists a $\theta > 0$ such that for $T \in [T_e, n^* T_e]$, $\boldsymbol\phi \in \mathcal{C}_+^\circ \cap \mathcal{C}_{V,M}$ with $\phi_i(0) < \delta_e$, we have

$$
\mathbb{E}_{\boldsymbol\phi} V^\theta_{\boldsymbol\rho^{i,e}}(\mathbf{X}_T) \le \exp\left(-\frac{1}{4}\theta\kappa_e T\right) V^\theta_{\boldsymbol\rho^{i,e}}(\boldsymbol\phi).
$$

Therefore, the proposition follows from the definition of $\widehat{U}_\theta$. □

Proposition 4.1. *Under Assumptions 2.1, 2.2, and 2.3. For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$
\mathbb{P}_{\boldsymbol\phi}\left\{\lim_{t\to\infty}\widehat{U}_\theta(\mathbf{X}_t) = 0\right\} \ge 1-\varepsilon, \text{ for all } \boldsymbol\phi \text{ satisfying } \widehat{U}_\theta(\boldsymbol\phi) < \delta. \tag{4.6}
$$

Proof. Let

$$C_0 := \sup_{\boldsymbol{\varphi}\in\mathcal{C}_+^\circ}\left\{\frac{\prod_{i\in I} x_i^{\widehat{\alpha}_i}}{(1+\mathbf{c}^\top\mathbf{x})\exp\left\{A_2\int_{-r}^0 \mu(ds)\int_s^0 e^{\gamma(u-s)}h\big(\boldsymbol{\varphi}(u)\big)du\right\}} : \mathbf{x}=\boldsymbol{\varphi}(0)\right\} < \infty,$$

and

$$d(\delta_e) := \frac{(\delta_e)^{\theta\alpha_*}}{C_0^\theta}, \tag{4.7}$$

where δ_e is as in Lemma 4.2. By (3.2), we have

$$\mathcal{L}\widehat{U}_\theta(\boldsymbol{\varphi}) \le -\theta\gamma_0\widehat{U}_\theta(\boldsymbol{\varphi}) \text{ if } \boldsymbol{\phi}\in\mathcal{C}_+^\circ, \boldsymbol{\varphi}\notin\mathcal{C}_{V,M}. \tag{4.8}$$

Because of (4.8), (3.3), and Proposition 4.1, by the same procedure as [46, Theorem 4.1], we obtain that

$$\text{if } \widehat{U}_\theta(\boldsymbol{\phi}) \le d(\delta_\varepsilon) \text{ then } \mathbb{E}_{\boldsymbol{\phi}} Z(1) \le q_1 Z(0), \text{ for some } q_1\in(0,1), \tag{4.9}$$

where $Z(k) := d(\delta_e)\wedge\widehat{U}_\theta(\mathbf{X}_{kn^*T_e}), k\in\mathbb{N}$. The reader can also see [25, Proof of Theorem 5.1] for detailed calculations of this argument.

For each $m < d(\delta_e)$, define the stopping time

$$\beta_m := \inf\{k\in\mathbb{N} : Z(k)\ge m\}.$$

By (4.9),

$$\mathbb{E}_{\boldsymbol{\phi}}\mathbf{1}_{\{\beta_m>k\}}Z(k) \le q_1^k Z(0). \tag{4.10}$$

In view of (4.10), we have

$$\mathbb{P}_{\boldsymbol{\phi}}\{\beta_m>k\}\ge 1-\varepsilon, \forall k\in\mathbb{N} \text{ if } \widehat{U}_\theta(\boldsymbol{\phi}) < m\varepsilon.$$

Hence, letting $k\to\infty$ yields

$$\mathbb{P}_{\boldsymbol{\phi}}\{\beta_m=\infty\}\ge 1-\varepsilon \text{ if } \widehat{U}_\theta(\boldsymbol{\phi}) < m\varepsilon. \tag{4.11}$$

On the other hand, using (3.2) again and by the definition of $\widehat{U}_\theta$, we have

$$\mathcal{L}\widehat{U}_\theta(\boldsymbol{\varphi}) \le A_0\theta\widehat{U}_\theta(\boldsymbol{\varphi}).$$

Hence, by a standard argument as in [46, Proof of Theorem 3.1], we get

$$\mathbb{E}_{\boldsymbol{\phi}} e^{-A_0\theta(t\wedge\zeta_u)}\widehat{U}_\theta(X_{t\wedge\zeta_u}) \le \widehat{U}_\theta(\boldsymbol{\phi}),$$

where for each $u > 0$

$$\zeta_u := \inf\{t \geq 0 : \widehat{U}_\theta(\mathbf{X}_t) > u\widehat{U}_\theta(\boldsymbol{\phi})\},$$

which implies that

$$ue^{-A_0\theta t}\widehat{U}_\theta(\boldsymbol{\phi})\mathbb{P}_{\boldsymbol{\phi}}\{\zeta_u > t\} \leq \widehat{U}_\theta(\boldsymbol{\phi}).$$

Thus, for any $u > 0$,

$$\mathbb{P}_{\boldsymbol{\phi}}\{\zeta_u > n^*T_e\} \leq \frac{e^{A_0\theta n^*T_e}}{u}. \tag{4.12}$$

Let $q_2, q_3 \in (0, q_1)$, $q_2 < q_3$, where q_1 is as in (4.9). We obtain from (4.10) that

$$\mathbb{P}_{\boldsymbol{\phi}}\left\{\mathbf{1}_{\{\beta_m > k\}}Z(k) \leq Z(0)q_2^k\right\} \geq 1 - \left(\frac{q_1}{q_2}\right)^k. \tag{4.13}$$

We deduce from (4.12), the Markov property of $\mathbf{X}_t$, and (4.13) that

$$\begin{aligned}
&\mathbb{P}_{\boldsymbol{\phi}}\left\{\mathbf{1}_{\{\beta_m > k\}}\sup_{s\in[kn^*T_e,(k+1)n^*T_e]}\widehat{U}_\theta(\mathbf{X}_s) \leq Z(0)q_3^k\right\}\\
&\geq \left(1 - \frac{e^{A_0\theta n^*T_e}}{\left(\frac{q_3}{q_2}\right)^k}\right)\mathbb{P}_{\boldsymbol{\phi}}\left\{\mathbf{1}_{\{\beta_m > k\}}Z(k) \leq Z(0)q_2^k\right\}\\
&\geq \left(1 - \frac{e^{A_0\theta n^*T_e}}{\left(\frac{q_3}{q_2}\right)^k}\right)\cdot\left(1 - \left(\frac{q_1}{q_2}\right)^k\right)\\
&\geq 1 - \left(e^{A_0\theta n^*T_e}\left(\frac{q_2}{q_3}\right)^k + \left(\frac{q_1}{q_2}\right)^k\right),
\end{aligned} \tag{4.14}$$

for any $k > k_0$, where k_0 satisfies

$$e^{A_0\theta n^*T_e} < \left(\frac{q_3}{q_2}\right)^{k_0}.$$

Since

$$\sum_{k=k_0}^{\infty}\left(e^{A_0\theta n^*T_e}\left(\frac{q_2}{q_3}\right)^k + \left(\frac{q_1}{q_2}\right)^k\right) < \infty,$$

by the Borel-Cantelli Lemma, we deduce from (4.14) that

$$\mathbb{P}_{\boldsymbol{\phi}}\left\{\lim_{k\to\infty}\mathbf{1}_{\{\beta_m>k\}}\sup_{s\in[kn^*T_e,(k+1)n^*T_e]}\widehat{U}_\theta(\mathbf{X}_s)=0\right\}=1. \tag{4.15}$$

Combining (4.15) and (4.11) implies that

$$\mathbb{P}_{\boldsymbol{\phi}}\left\{\lim_{s\to\infty}\widehat{U}_\theta(\mathbf{X}_s)=0\right\}\geq 1-\varepsilon \text{ if } \widehat{U}_\theta(\boldsymbol{\phi})<m\varepsilon.$$

Then the proposition is proved. □

Lemma 4.4. *Assume Assumption 2.1 and 2.4 hold. For any $\varepsilon, R>0$, $\mathbf{y}^*=(y_1^*,\dots,y_n^*)\in\mathbb{R}_+^{n,\circ}:=\{\mathbf{y}\in\mathbb{R}^n: y_i>0\,\forall i\}$, $\delta>0$, $t^*\geq 2r$*

$$\inf_{\boldsymbol{\phi}\in D_{\varepsilon,R}}\mathbb{P}_{\boldsymbol{\phi}}\left\{\mathbf{X}_{t^*}\in B_{\mathbf{y}^*,\delta}\right\}>0, \tag{4.16}$$

where

$$B_{\mathbf{y}^*,\delta}:=\left\{\boldsymbol{\varphi}\in\mathcal{C}_+^\circ:\left|\boldsymbol{\varphi}(s)-\mathbf{y}^*\right|<\delta;\forall s\in[-r,0]\right\}.$$

Proof. To prove (4.16), we modify slightly the proof of [24, Lemma 3.8] as follows. Let $\delta_1\in(0,\frac{\delta}{2})$ be sufficiently small such that

$$\min\left\{\min_{i=1,\dots,n}y_i^*-\delta_1,\varepsilon-\delta_1\right\}=:2\delta_2>2\delta_1.$$

Define

$$D(t):=|\mathbf{X}(t)-\mathbf{k}(t)|^2-(\delta_1/2)^2,$$

where $\mathbf{k}:[0,t^*]\to\mathbb{R}^n$ is continuously differentiable with Lipschitz constant at most $\frac{2(R+|\mathbf{y}^*|+\delta)}{r}$ satisfying

$$k_i(t)\geq 2\delta_2\ \forall t\in[0,t^*]\ \forall i,\ \text{ and } \mathbf{k}(0)=\boldsymbol{\phi}(0)-(\delta_1/2,0,\dots,0);\ \mathbf{k}(t)=\mathbf{y}^*, t\in[r,t^*].$$

It is noted that

$$X_i(t)\geq\delta_2\ \forall i \text{ and } |\mathbf{X}(t)|<2(R+|\mathbf{y}^*|+\delta), t\in[0,t^*] \text{ whenever } |D(t)|\leq(\delta_1/4)^2. \tag{4.17}$$

Hence, under Assumption 2.4 for the diffusion coefficients and (4.17), we can mimic the remaining of proof in [24, Lemma 3.8] (with $\mathbf{k}$ in place of $\mathbf{h}$) and obtain that

$$\inf_{\boldsymbol{\phi}\in D_{\varepsilon,R}}\mathbb{P}_{\boldsymbol{\phi}}\left\{\mathbf{X}_{t^*}\in B_{\mathbf{y}^*,\delta}\right\}>0.\quad\square$$

Theorem 4.1. *Assume that Assumptions 2.1, 2.2, 2.3, and 2.4 hold. For any $p < p_0$ and any $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$, we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_{\boldsymbol{\phi}} \bigwedge_{i=1}^n \|X_{i,t}\|^p\, dt = 0, \tag{4.18}$$

where $\bigwedge_{i=1}^n x_i = \min_{i=1,\dots,n}\{x_i\}$ and $\mathbf{X}_t = (X_{1,t},\dots,X_{n,t})$.

Proof. It is clear that if $\lim_{t\to\infty} \widehat{U}_\theta(\mathbf{X}_t) = 0$ then $\mathbf{X}_t$ tends to the boundary of $\mathcal{C}_+$ as $t\to\infty$. Moreover, we can choose suitable y^* and δ_1 such that $\forall \boldsymbol{\varphi} \in B_{y^*,\delta_1}$, $\widehat{U}_\theta(\boldsymbol{\varphi})$ is small enough. Therefore, in view of Proposition 4.1 and Lemma 4.4, the probability that $\mathbf{X}_t$ tends to the boundary is positive for any initial data. As a consequence, there is no invariant probability measure in $\mathcal{C}_+^\circ$. Therefore, we can deduce that the weak*-limit of $\Pi_t^{\boldsymbol{\phi}}(\cdot)$ is a probability measure concentrated on $\partial\mathcal{C}_+$. By noting that the function $\left(\bigwedge_{i=1}^n \varphi_i^p(0)\right)$, $p < p_0$ of variable $\boldsymbol{\varphi}$ satisfies condition [46, (3.41)], the theorem follows from [46, Lemma 3.5]. □

Lemma 4.5. *Suppose that Assumption 2.1, 2.3, and 2.5 are satisfied and let I be the subset of $\{1,\dots,n\}$ in Assumption 2.3. Then the following results hold:*

- *For any $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$,*

$$\mathbb{P}_{\boldsymbol{\phi}}\left\{\mathcal{U}(\omega) \subset \operatorname{Conv}(\mathcal{M}^I)\right\} = \mathbb{P}_{\boldsymbol{\phi}}\left\{\mathcal{U}(\omega) \subset \operatorname{Conv}(\mathcal{M}^{I,\circ})\right\}.$$

- *For any $m > 0$, $\delta > 0$, and $\varepsilon > 0$, there is a $R > 0$ such that*

$$\begin{aligned}\mathbb{P}_{\boldsymbol{\phi}}\Bigg\{&\mathcal{U}(\omega) \subset \operatorname{Conv}(\mathcal{M}^{I,\circ}) \text{ and} \\ &\quad \lim_{t\to\infty} \frac{\ln X_i(t)}{t} \subset \left\{\lambda_i(\pi), \pi \in \operatorname{Conv}(\mathcal{M}^{I,\circ})\right\}, i \in I^c\Bigg\} \\ &> 1-\varepsilon, \quad \text{for all } \boldsymbol{\phi} \in \Delta_I^{m,\delta,R},\end{aligned}$$

where

$$\begin{aligned}\Delta_I^{m,\delta,R} := \Big\{&\boldsymbol{\varphi} \in \mathcal{C}_+^\circ : m \le x_i \text{ for } i \in I, x_i < \delta \text{ for } i \in I^c \text{ and} \\ &V_{\mathbf{0}}(\boldsymbol{\varphi}) < R, \mathbf{x} := \boldsymbol{\varphi}(0)\Big\}.\end{aligned}$$

Proof. The proof of the first part is similar to [25, Proof of Lemma 5.8].

We proceed to prove the second part. By the third part of Lemma 3.4, there is a $k_0 > 0$ such that

$$\mathbb{P}_{\boldsymbol{\phi}}\left\{\max_{i\in I^c}\{X_i(t)\} > k_0, \forall t \in [0, n^* T_e]\right\} > \frac{1}{2}, \ \boldsymbol{\phi} \in \mathcal{A}_I, \tag{4.19}$$

where

$$\mathcal{A}_I = \{\boldsymbol{\varphi} \in \mathcal{C}_+ : V_{\mathbf{0}}(\boldsymbol{\varphi}) \le \widehat{K}_1, \max_{i\in I^c}\{x_i\} \ge 1, \mathbf{x} := \boldsymbol{\varphi}(0)\}.$$

It can be seen that

$$\nu(\mathcal{A}_I) > 0 \text{ for } \nu \in \mathcal{M} \setminus \mathcal{M}^I. \tag{4.20}$$

As in Proposition 4.1, consider $U_e(\boldsymbol{\varphi}) := d(\delta_e) \wedge \widehat{U}_\theta(\boldsymbol{\varphi})$, where δ_e is defined in (4.7). By the definition of $U_e(\cdot)$, there is a $\delta > 0$ sufficiently small such that

$$\sup_{\boldsymbol{\varphi}\in\Delta_I^{m,\delta,R}} \{U_e(\boldsymbol{\varphi})\} \le \frac{\varepsilon}{2} \inf_{\boldsymbol{\varphi}\in\mathcal{C}_+^\circ, x_i\ge k_0, \text{ for some } i\in I^c} \{U_e(\boldsymbol{\varphi})\}. \tag{4.21}$$

In view of (4.11), we obtain if $\boldsymbol{\phi} \in \Delta_I^{m,\delta,R}$

$$\mathbb{P}_{\boldsymbol{\phi}} \left\{ U_e(\mathbf{X}(kn^*T_e)) < \inf_{\boldsymbol{\varphi}\in\mathcal{C}_+^\circ, x_i\ge k_0, \text{ for some } i\in I^c} \{U_e(\boldsymbol{\varphi})\}, \ \forall k \in \mathbb{N} \right\} > 1 - \frac{\varepsilon}{2}.$$

Thus

$$\mathbb{P}_{\boldsymbol{\phi}} \left\{ \max_{i\in I^c}\{X_i(kn^*T_e)\} < k_0 \text{ for all } k \in \mathbb{N} \right\} > 1 - \frac{\varepsilon}{2} \text{ if } \boldsymbol{\phi} \in \Delta_I^{m,\delta,R}. \tag{4.22}$$

Now, we prove

$$\mathbb{P}_{\boldsymbol{\phi}} \left\{ \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{\mathbf{X}_s\in\mathcal{A}_I\}} ds = 0 \right\} > 1 - \varepsilon, \ \boldsymbol{\phi} \in \Delta_I^{m,\delta,R}, \tag{4.23}$$

by a contradiction argument. Assume that there is a $\boldsymbol{\phi} \in \Delta_I^{m,\delta,R}$ satisfying

$$\mathbb{P}_{\boldsymbol{\phi}} \left\{ \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{\mathbf{X}_s\in\mathcal{A}_I\}} ds > 0 \right\} > \varepsilon. \tag{4.24}$$

Then

$$\mathbb{P}_{\boldsymbol{\phi}}\{\tau_{\mathcal{A}_I} < \infty\} > \varepsilon, \tag{4.25}$$

where $\tau_{\mathcal{A}_I} = \inf\{t > 0 : \mathbf{X}_t \in \mathcal{A}_I\}$. By the strong Markov property of $\{\mathbf{X}_t\}$, it follows from (4.19) and (4.25) that

$$\mathbb{P}_{\boldsymbol{\phi}} \left(\{\tau_{\mathcal{A}_I} < \infty\} \bigcap \left\{ \max_{i\in I^c}\{X_i(t)\} \ge k_0 \text{ for } t \in [\tau_{\mathcal{A}_I}, \tau_{\mathcal{A}_I} + n^*T_e] \right\} \right) > \frac{1}{2}\varepsilon,$$

which contradicts (4.22) and hence, (4.23) holds.

We observe that if for an $\omega \in \Omega$ and a sequence $\{t_j\}$ with $\lim_{j\to\infty} t_j = \infty$, $\widetilde{\Pi}_{t_j}(\cdot)$ converges weakly to an invariant probability of the form $\pi_0 = (1-q)\pi_1 + q\pi_2$ with $q \in [0,1]$, $\pi_1 \in \text{Conv}(\mathcal{M}^I)$, and $\pi_2 \in \text{Conv}(\mathcal{M} \setminus \mathcal{M}^I)$, then by (4.20)

$$\limsup_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} \mathbf{1}_{\{\mathbf{X}_s \in \mathcal{A}_I\}} ds \geq \pi_0(\mathcal{A}_I) \geq q\pi_2(\mathcal{A}_I).$$

This inequality combined with Lemma 3.5, (4.20), and (4.23) implies that $q = 0$ and

$$\mathbb{P}_{\boldsymbol{\phi}} \left\{ \mathcal{U}(\omega) \subset \text{Conv}(\mathcal{M}^I) \right\} > 1 - \varepsilon,\ \boldsymbol{\phi} \in \Delta_I^{m,\delta,R}.$$

The first claim of Lemma 4.5 and the above estimates lead to

$$\mathbb{P}_{\boldsymbol{\phi}} \left\{ \mathcal{U}(\omega) = \{\text{Conv}(\mathcal{M}^{I,\circ})\} \right\} > 1 - \varepsilon, \boldsymbol{\phi} \in \Delta_I^{m,\delta,R}. \tag{4.26}$$

In view of Lemma 3.4 and (4.26), we have for $\boldsymbol{\phi} \in \Delta_I^{m,\delta,R}$ and for each $i = 1, \ldots, n$ that

$$\mathbb{P}_{\boldsymbol{\phi}} \left\{ \lim_{t\to\infty} \frac{1}{t} \int_0^t \left(f_i(\mathbf{X}_s) - \frac{\sigma_{ii} g_i^2(\mathbf{X}_s)}{2} \right) ds \subset \left\{ \lambda_i(\pi) : \pi \in \text{Conv}(\mathcal{M}^{I,\circ}) \right\} \right\} > 1 - \varepsilon. \tag{4.27}$$

On the other hand, it is easy to see

$$\mathbb{P}_{\boldsymbol{\phi}} \left\{ \lim_{t\to\infty} \frac{1}{t} \int_0^t g_i(\mathbf{X}_s) dE_i(s) = 0,\ i = 1, \ldots, n \right\} = 1. \tag{4.28}$$

The second claim of Lemma 4.5 follows from (4.27), (4.28), and an application of the functional Itô formula. □

With the above Lemmas in hand, we can modify slightly the proof of [25, Theorem 5.2] to obtain the following Theorem.

Theorem 4.2. *Suppose that Assumptions 2.1, 2.3, 2.4, 2.5, and 2.6 are satisfied. Then for any* $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$,

$$\sum_{I \in S} P_{\boldsymbol{\phi}}^I = 1, \quad P_{\boldsymbol{\phi}}^I > 0, \tag{4.29}$$

where for $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$,

$$P_{\boldsymbol{\phi}}^I := \mathbb{P}_{\boldsymbol{\phi}} \Bigg\{ \mathcal{U}(\omega) \subset \text{Conv}(\mathcal{M}^{I,\circ}) \text{ and } \lim_{t\to\infty} \frac{\ln X_i(t)}{t} \in \left\{ \lambda_i(\pi), \pi \in \text{Conv}(\mathcal{M}^{I,\circ}) \right\}, i \in I^c \Bigg\}.$$

5. Applications

This section presents a number of applications of our main results Theorems 2.1 and 2.2. We provide sufficient conditions for the extinction of several popular biological and ecological systems. These results are complements of the permanence characterization in [46] in that excluding the critical cases, if the system is not permanent, the extinction will happen and vice versa.

5.1. Stochastic delay Lotka-Volterra competitive model

Introduced in [39,64] by Lotka and Volterra in 1926, the Lotka-Volterra model is one of the most important models in mathematical biology and has been studied widely in the literature. The Lotka-Volterra competitive models are used to describe the dynamics of the species when they live in proximity, share the same basic resources, and compete for food, habitat, territory, etc. Because of the influences of many complex properties in real life, other terms (white noises, Markov switching, delayed time, etc.) are added to the original system to reflect better the phenomena. Stochastic delay Lotka-Volterra competitive models have also been widely studied; see, for example, [2,35] and references therein. However, there is no unified general framework to handle that except the work [46, Section 5.1], which provided criteria for persistence.

For the case of two-dimensional competitive stochastic delay system, this kind model takes the form

$$\begin{cases} dX_1(t) = X_1(t)\big(a_1 - b_{11}X_1(t) - b_{12}X_2(t) - \widehat{b}_{11}X_1(t-r) - \widehat{b}_{12}X_2(t-r)\big)\,dt \\ \qquad\qquad +X_1(t)dE_1(t), \\ dX_2(t) = X_2(t)\big(a_2 - b_{21}X_1(t) - b_{22}X_2(t) - \widehat{b}_{21}X_1(t-r) - \widehat{b}_{22}X_2(t-r)\big)\,dt \\ \qquad\qquad +X_2(t)dE_2(t), \end{cases} \tag{5.1}$$

where $X_i(t)$ denotes the size of the species i at time t; $a_i > 0$ represents the growth rate of the species i; $b_{ii} > 0$ stands for the intra-specific competition of the i^{th} species; $b_{ij} \geq 0, (i \neq j)$ is the inter-specific competition; $\widehat{b}_{ij} > -b_{ii}$ $(i, j = 1, 2)$ (i.e., $\widehat{b}_{ij}$ can be negative); r is the delay time; $(E_1(t), E_2(t))^\top = \Gamma^\top \mathbf{B}(t)$ with $\mathbf{B}(t) = (B_1(t), B_2(t))^\top$ being a vector of independent standard Brownian motions and Γ being a 2×2 matrix such that $\Gamma^\top\Gamma = (\sigma_{ij})_{2\times 2}$ is a positive definite matrix.

As a complement of [46, Section 5.1] that provides the conditions for the persistence, we characterize the extinction to complete the long-time characterization in this paper. Applying our Theorems in Section 2, we have that $\lambda_i(\boldsymbol{\delta}^*) = a_i - \dfrac{\sigma_{ii}}{2}, i = 1, 2$. In view of Theorem 2.1, if $\lambda_1(\boldsymbol{\delta}^*) < 0$, (resp. $\lambda_2(\boldsymbol{\delta}^*) < 0$) there is no invariant probability measure on $\mathcal{C}^\circ_{1+} := \{(\varphi_1, 0) \in \mathcal{C}_+ : \varphi_1(s) > 0\ \forall s \in [-r, 0]\}$ (resp. $\mathcal{C}^\circ_{2+} := \{(0, \varphi_2) \in \mathcal{C}_+ : \varphi_2(s) > 0\ \forall s \in [-r, 0]\}$). By Lemma 4.1, we have

$$\lambda_i(\pi_i) = a_i - \frac{\sigma_{ii}}{2} - \int\limits_{\mathcal{C}^\circ_{i+}} \big(b_{ii}\varphi_i(0) + \widehat{b}_{ii}\varphi_i(-r)\big)\,\pi_i(d\boldsymbol{\varphi}) = 0, \text{ where } \boldsymbol{\varphi} = (\varphi_1, \varphi_2),$$

which implies

$$\int\limits_{\mathcal{C}^{\circ}_{i+}} \left(b_{ii}\varphi_i(0)+\widehat{b}_{ii}\varphi_i(-r)\right)\pi_i(d\boldsymbol{\varphi}) = a_i - \frac{\sigma_{ii}}{2}. \tag{5.2}$$

Since π_i is an invariant probability measure of $\{\mathbf{X}_t\}$, it is easy to see that

$$\int\limits_{\mathcal{C}^{\circ}_{i+}} \varphi_i(0)\pi_i(d\boldsymbol{\varphi}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T X_{i,t}(0)dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T X_i(t)dt, \tag{5.3}$$

where $(X_{1,t}, X_{2,t}) = \mathbf{X}_t$. Similarly,

$$\int\limits_{\mathcal{C}^{\circ}_{i+}} \varphi_i(-r)\pi_i(d\boldsymbol{\varphi}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T X_i(t-r)dt. \tag{5.4}$$

By virtue of (5.3) and (5.4), we can prove that

$$\int\limits_{\mathcal{C}^{\circ}_{i+}} \varphi_i(0)\pi_i(d\boldsymbol{\varphi}) = \int\limits_{\mathcal{C}^{\circ}_{i+}} \varphi_i(-r)\pi_i(d\boldsymbol{\varphi}). \tag{5.5}$$

Combining (5.2) and (5.5) yields that

$$\int\limits_{\mathcal{C}^{\circ}_{i+}} \varphi_i(0)\pi_i(d\boldsymbol{\varphi}) = \int\limits_{\mathcal{C}^{\circ}_{i+}} \varphi_i(-r)\pi_i(d\boldsymbol{\varphi}) = \frac{a_i - \frac{\sigma_{ii}}{2}}{b_{ii}+\widehat{b}_{ii}}.$$

Therefore, we have

$$\begin{aligned}\lambda_2(\pi_1) &= \int\limits_{\mathcal{C}^{\circ}_{1+}} \left[a_2 - \frac{\sigma_{22}}{2} - b_{21}\varphi_1(0) - \widehat{b}_{21}\varphi_1(-r)\right]\pi_1(d\boldsymbol{\varphi}) \\ &= a_2 - \frac{\sigma_{22}}{2} - \left(a_1 - \frac{\sigma_{11}}{2}\right)\cdot\frac{b_{21}+\widehat{b}_{21}}{b_{11}+\widehat{b}_{11}},\end{aligned}$$

and

$$\begin{aligned}\lambda_1(\pi_2) &= \int\limits_{\mathcal{C}^{\circ}_{2+}} \left[a_1 - \frac{\sigma_{11}}{2} - b_{12}\varphi_2(0) - \widehat{b}_{12}\varphi_2(-r)\right]\pi_2(d\boldsymbol{\varphi}) \\ &= a_1 - \frac{\sigma_{11}}{2} - \left(a_2 - \frac{\sigma_{22}}{2}\right)\cdot\frac{b_{12}+\widehat{b}_{12}}{b_{22}+\widehat{b}_{22}}.\end{aligned}$$

By applying Theorem 2.2 to characterize the extinction together with the characterization of persistence in [46, Section 4.1], we have the following results.

- If $\lambda_i(\boldsymbol{\delta}^*) < 0, i = 1, 2$ then $X_i(t)$ converges to 0 almost surely with the exponential rate $\lambda_i(\boldsymbol{\delta}^*)$ for any initial condition $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{C}_+^\circ$.
- If $\lambda_i(\boldsymbol{\delta}^*) > 0, \lambda_j(\boldsymbol{\delta}^*) < 0$ for one $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, then $\lambda_j(\pi_i) < 0$ and $X_j(t)$ converges to 0 almost surely with the exponential rate $\lambda_j(\pi_i)$ for any initial condition $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{C}_+^\circ$ and the randomized occupation measure converges weakly to π_i almost surely.
- If $\lambda_i(\boldsymbol{\delta}^*) > 0, i \in \{1, 2\}$ and $\lambda_1(\pi_2) < 0, \lambda_2(\pi_1) < 0$ then $P_i^{\boldsymbol{\phi}} > 0, i = 1, 2$ and $P_1^{\boldsymbol{\phi}} + P_2^{\boldsymbol{\phi}} = 1$ where

$$P_i^{\boldsymbol{\phi}} = \mathbb{P}_{\boldsymbol{\phi}} \left\{ \mathcal{U}(\omega) = \{\pi_i\} \text{ and } \lim_{t \to \infty} \frac{\ln X_j(t)}{t} = \lambda_j(\pi_i), j \in \{1, 2\} \setminus \{i\} \right\}.$$

- If $\lambda_1(\boldsymbol{\delta}^*) > 0, \lambda_2(\boldsymbol{\delta}^*) > 0$, $\lambda_j(\pi_i) < 0, \lambda_i(\pi_j) > 0$ for $i, j \in \{1, 2\}, i \neq j$ then $X_j(t)$ converges to 0 almost surely with the exponential rate $\lambda_j(\pi_i)$ and the randomized occupation measure converges weakly to π_i almost surely for any initial condition $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{C}_+^\circ$.
- If $\lambda_1(\boldsymbol{\delta}^*) > 0, \lambda_2(\boldsymbol{\delta}^*) > 0$ and $\lambda_1(\pi_2) > 0, \lambda_2(\pi_1) > 0$, any invariant probability measure in $\partial\mathcal{C}_+$ has the form $\pi = q_0\boldsymbol{\delta}^* + q_1\pi_1 + q_2\pi_2$ with $0 \le q_0, q_1, q_2$ and $q_0 + q_1 + q_2 = 1$. Then, one has $\max_{i=1,2}\{\lambda_i(\pi)\} > 0$ for any π having the form above. Therefore, there is a unique invariant probability measure π^* on $\mathcal{C}_+^\circ$.

The above characterization generalizes the results of long-term properties in [35].

Although we only provide the explicit computations for 2-dimension cases, our results (in both this paper and [46]) can be applied to characterize the long-time behavior of solutions for stochastic delay Lotka-Volterra competitive models with n-species,

5.2. *Stochastic delay Lotka-Volterra predator-prey model*

This section is devoted to the application of our results to stochastic delay Lotka-Volterra predator-prey models. In contrast to Lotka-Volterra competitive model in which two species compete for food, habitat, territory, etc, the Lotka-Volterra predator-prey models are frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other one as prey. The Lotka-Volterra predator-prey system with one prey and two competing predators is given as follows

$$\begin{cases} dX_1(t) = X_1(t)\Big\{a_1 - b_{11}X_1(t) - b_{12}X_2(t) - b_{13}X_3(t) \\ \qquad\qquad -\widehat{b}_{11}X_1(t-r) - \widehat{b}_{12}X_2(t-r) - \widehat{b}_{13}X_3(t-r)\Big\}dt + X_1(t)dE_1(t), \\ dX_2(t) = X_2(t)\Big\{-a_2 + b_{21}X_1(t) - b_{22}X_2(t) - b_{23}X_3(t) \\ \qquad\qquad -\widehat{b}_{21}X_1(t-r) - \widehat{b}_{22}X_2(t-r) - \widehat{b}_{23}X_3(t-r)\Big\}dt + X_2(t)dE_2(t), \\ dX_3(t) = X_3(t)\Big\{-a_3 + b_{31}X_1(t) - b_{32}X_2(t) - b_{33}X_3(t) \\ \qquad\qquad -\widehat{b}_{31}X_1(t-r) - \widehat{b}_{32}X_2(t-r) - \widehat{b}_{33}X_3(t-r)\Big\}dt + X_3(t)dE_3(t), \end{cases} \tag{5.6}$$

where $X_1(t)$, $X_2(t)$, and $X_3(t)$ denote the densities at time t of the prey, and two predators, respectively; $a_1 > 0$ denotes the growth rate; $a_2, a_3 > 0$ represent the death rate of X_2, X_3;

$b_{ii} > 0, i = 1, 2, 3$ are the intra-specific competition coefficient of X_i; $b_{ij} \geq 0, i \neq j = 1, 2, 3$, in which b_{12}, b_{13} represent the capture rates, b_{21}, b_{31} represent the growth from food, and b_{23} and b_{32} signify the competitions between predators (species 2 and 3); $\widehat{b}_{ij}$ is either positive or in $(-b_{ii}, 0]$; r is the time delay for each $i, j \in \{1, 2, 3\}$; $(E_1(t), E_2(t), E_3(t))^\top = \Gamma^\top \mathbf{B}(t)$ with $\mathbf{B}(t) = (B_1(t), B_2(t), B_3(t))^\top$ being a vector of independent standard Brownian motions and Γ being a 3×3 matrix such that $\Gamma^\top \Gamma = (\sigma_{ij})_{3\times 3}$ is a positive definite matrix. It is worth noting that system (5.6) is the (stochastic delay) Lotka-Volterra model with two predators competing for one prey, which was considered in [33]. If we switch the sign of a_i or $b_{ij}, i \neq j$, we can obtain a stochastic time-delay Lotka-Volterra system with the prey and the mesopredator or intermediate predator. The case involving a superpredator or top predator, was studied in [34,67], and the stochastic time-delay Lotka-Volterra system with one predator and two preys was investigated in [23].

Our assumptions are verified for (5.6) in the first part [46, Section 5.2]. To characterize the extinction, first, let us consider the equation on the boundary $\{(0, \varphi_2, \varphi_3) \in \mathcal{C}_+ : \varphi_2(s), \varphi_3(s) \geq 0\ \forall s \in [-r, 0]\}$. Since $\lambda_i(\boldsymbol{\delta}^*) = -a_i - \frac{\sigma_{ii}}{2} < 0, i = 2, 3$, by applying Theorem 2.1 for the space $\{(0, \varphi_2, \varphi_3) : \varphi_2(s), \varphi_3(s) \geq 0\ \forall s \in [-r, 0]\}$, we obtain that there is only one invariant probability measure on $\{(0, \varphi_2, \varphi_3) : \varphi_2(s), \varphi_3(s) \geq 0\ \forall s \in [-r, 0]\}$, which is $\boldsymbol{\delta}^*$. It indicates that without the prey, both predators die out.

Second, we consider the equation on the boundaries $\mathcal{C}_{12+} := \{(\varphi_1, \varphi_2, 0) \in \mathcal{C}_+ : \varphi_1(s), \varphi_2(s) \geq 0\ \forall s \in [-r, 0]\}$ and $\mathcal{C}_{13+} := \{(\varphi_1, 0, \varphi_3) \in \mathcal{C}_+ : \varphi_1(s), \varphi_3(s) \geq 0,\ \forall s \in [-r, 0]\}$. If $\lambda_1(\boldsymbol{\delta}^*) = a_1 - \frac{\sigma_{11}}{2} < 0$, an application of Theorem 2.1 implies that $\boldsymbol{\delta}^*$ is the unique invariant probability measure on $\mathcal{C}_+$. If $\lambda_1(\boldsymbol{\delta}^*) > 0$, there is an invariant probability measure π_1 on $\mathcal{C}^\circ_{1+} := \{(\varphi_1, 0, 0) \in \mathcal{C}_+ : \varphi_1(s) > 0\ \forall s \in [-r, 0]\}$.

In view of Lemma 4.1, we obtain

$$\int\limits_{\mathcal{C}^\circ_{1+}} \big(b_{11}\varphi_1(0) + \widehat{b}_{11}\varphi_1(-r)\big)\,\pi_1(d\boldsymbol{\varphi}) = a_1 - \frac{\sigma_{11}}{2}. \tag{5.7}$$

Similar to the process of getting (5.5), we obtain from (5.7) that

$$\int\limits_{\mathcal{C}^\circ_{1+}} \varphi_1(0)\pi_1(d\boldsymbol{\varphi}) = \int\limits_{\mathcal{C}^\circ_{1+}} \varphi_1(-r)\pi_1(d\boldsymbol{\varphi}) = \frac{a_1 - \frac{\sigma_{11}}{2}}{b_{11} + \widehat{b}_{11}}.$$

Therefore,

$$\begin{aligned}\lambda_i(\pi_1) &= \int\limits_{\mathcal{C}^\circ_{1+}} \Big[-a_i - \frac{\sigma_{ii}}{2} + b_{i1}\varphi_1(0) - \widehat{b}_{i1}\varphi_1(-r)\Big]\pi_1(d\boldsymbol{\varphi}) \\ &= -a_i - \frac{\sigma_{ii}}{2} + \Big(a_1 - \frac{\sigma_{11}}{2}\Big)\cdot \frac{b_{i1} - \widehat{b}_{i1}}{b_{11} + \widehat{b}_{11}},\ i = 2, 3.\end{aligned}$$

If $\lambda_1(\boldsymbol{\delta}^*) > 0$ and $\lambda_i(\pi_1) < 0, i = 2, 3$, in view of Theorem 2.1, there is no invariant probability measure on $\mathcal{C}^\circ_{1i+}$.

By Theorem 2.1 and Theorem 2.2, we have the following classification for extinction.

- If $\lambda_1(\boldsymbol{\delta}^*) < 0$ then for any initial condition $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$, $X_1(t)$, $X_2(t)$, $X_3(t)$, converge to 0 almost surely with the exponential rates $\lambda_i(\boldsymbol{\delta}^*)$, $i = 1, 2, 3$, respectively.
- If $\lambda_1(\boldsymbol{\delta}^*) > 0$, $\lambda_i(\pi_1) < 0$, $i = 2, 3$ then $X_i(t)$, $i = 2, 3$ converge to 0 almost surely with the exponential rate $\lambda_i(\pi_1)$, $i = 2, 3$, respectively, and the occupation measure converges almost surely for any initial condition $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$ to π_1.
- If $\lambda_1(\boldsymbol{\delta}^*) > 0$, $\lambda_i(\pi_1) > 0$, $\lambda_j(\pi_{1i}) < 0$, and $\lambda_j(\pi_1) < 0$ for $i, j \in \{2, 3\}$ and $i \neq j$, then $X_j(t)$ converges to 0 almost surely with the exponential rate $\lambda_j(\pi_{1i})$ and the occupation measure converges almost surely for any initial condition $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$ to π_{1i}.
- If $\lambda_1(\boldsymbol{\delta}^*) > 0$, $\lambda_2(\pi_1) > 0$, $\lambda_3(\pi_1) > 0$, $\lambda_j(\pi_{1i}) < 0$, $\lambda_i(\pi_{1j}) > 0$ for $i, j \in \{2, 3\}$ and $i \neq j$, then $X_j(t)$ converges to 0 almost surely with the exponential rate $\lambda_j(\pi_{1i})$ and the occupation measure converges almost surely for any initial condition $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$ to π_{1i}.
- If $\lambda_1(\boldsymbol{\delta}^*) > 0$, $\lambda_2(\pi_1) > 0$, $\lambda_3(\pi_1) > 0$, $\lambda_2(\pi_{13}) < 0$, $\lambda_3(\pi_{12}) < 0$, then $p_i^{\boldsymbol{\phi}} > 0, i = 2, 3$ and $p_2^{\boldsymbol{\phi}} + p_3^{\boldsymbol{\phi}} = 1$, where

$$p_i^{\boldsymbol{\phi}} = \mathbb{P}_{\boldsymbol{\phi}} \left\{ \mathcal{U}(\omega) = \{\pi_{1i}\} \text{ and } \lim_{t\to\infty} \frac{\ln X_i(t)}{t} = \lambda_i(\pi_{1j}), i \in \{2, 3\} \setminus \{j\} \right\}.$$

The above assertions generalize the results in [33]. Moreover, if we switch the sign of a_i or $b_{ij}, i \neq j$ as we mentioned at the beginning of this section and modify slightly the above characterization, we improve the results in [23,34,67].

Confining our analysis and setting to $\mathcal{C}_{12+}$, which describes the evolution of one predator and its prey, we obtain

$$\begin{cases} dX_1(t) = X_1(t)\Big[a_1 - b_{11}X_1(t) - \widehat{b}_{11}X_1(t-r) - b_{12}X_2(t) - \widehat{b}_{12}X_2(t-r)\Big]dt \\ \qquad\qquad + X_1(t)dE_1(t), \\ dX_2(t) = X_2(t)\Big[-a_2 + b_{21}X_1(t) + \widehat{b}_{21}X_1(t-r) - b_{22}X_2(t) - \widehat{b}_{22}X_2(t-r)\Big]dt \\ \qquad\qquad + X_2(t)dE_2(t). \end{cases} \tag{5.8}$$

The above characterization can be specialized as:

- If $\lambda_1(\boldsymbol{\delta}^*) < 0$ then $X_1(t)$, $X_2(t)$ converge to 0 almost surely with the exponential rates $\lambda_1(\boldsymbol{\delta}^*)$ and $\lambda_2(\boldsymbol{\delta}^*)$, respectively.
- If $\lambda_1(\boldsymbol{\delta}^*) > 0$ and $\lambda_2(\pi_1) < 0$ then $X_2(t)$ converges to 0 almost surely with the exponential rate $\lambda_2(\pi_1)$ and the occupation measure converges to π_1.

This result generalizes that of [32].

5.3. Stochastic delay replicator equation

The replicator equation, which is a deterministic monotone, non-linear, and non-innovative game dynamic system plays a popular and important role in evolutionary game theory. Such an equation was introduced in 1978 by Taylor and Jonker in [60]. Since then significant contributions have been made in biology [26,50], economics [65], and optimization and control for a variety of systems [7,51,53,61]. To capture the random factors in nature, the deterministic system has been generalized to stochastic systems. This section is devoted to applying our main results to stochastic delay replicator equation. The replicator dynamics for a game with n strategies, involving social-type time delay (see, e.g., [1] for details of such delays) and white noise perturbation is given by

$$\begin{cases} dx_i(t) = x_i(t)\Big(f_i(\mathbf{x}(t-r)) - \dfrac{1}{X}\displaystyle\sum_{j=1}^{n} x_j(t) f_j(\mathbf{x}(t-r))\Big)dt \\ \qquad\qquad +x_i(t)\Big(\sigma_i dB_i(t) - \dfrac{1}{X}\displaystyle\sum_{j=1}^{n} \sigma_j x_j dB_j(t)\Big);\ i = 1, \dots, n, \\ \mathbf{x}(s) = \mathbf{x}_0(s);\ t \in [-r, 0], \end{cases} \tag{5.9}$$

where X is the size of the populations; $x_i(t)$ is the portion of population that has selected the i^{th} strategy and the distribution of the whole population among the strategy; the fitness functions $f_i(\cdot) : \mathbb{R}^n_+ \to \mathbb{R}$, $i = 1, \dots, n$ are the payoffs obtained by the individuals playing the i^{th} strategy; r is the time delay; and $\mathbf{x}_0(s) \in \Delta_X := \{\mathbf{x} \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = X\}$ for all $s \in [-r, 0]$ is the initial value. Some special cases of (5.9) have been studied in literature. For instance, [27,28] considered equation (5.9) without time delay in the case $f_i(\cdot) : \mathbb{R}^n_+ \to \mathbb{R}$, $i = 1, \dots, n$ being linear mappings; and [1,52] studied the deterministic version of equation (5.9).

Recall that by a similar argument as in [52,65], we can show that Δ_X remains invariant a.s. As a consequence, our assumptions are verified. Hence, our results in this paper (Theorem 2.1 and Theorem 2.2) hold for (5.9); see [46, Section 4.3]. We first apply our results to characterize the extinction for some low-dimensional systems. Consider equation (5.9) for two-dimensional systems. Define

$$\begin{aligned} \mathcal{C}^X_+ &:= \{(\varphi_1, \varphi_2) : \varphi_1(s) + \varphi_2(s) = X \text{ and } \varphi_1(s), \varphi_2(s) \ge 0 \text{ for all } s \in [-r, 0]\}, \\ \partial\mathcal{C}^X_+ &:= \{(\varphi_1, \varphi_2) \in \mathcal{C}^X_+ : \|\varphi_1\| = 0 \text{ or } \|\varphi_2\| = 0\}, \\ \mathcal{C}^{X,\circ}_+ &:= \{(\varphi_1, \varphi_2) \in \mathcal{C}^X_+ : \varphi_1(s), \varphi_2(s) > 0 \text{ for all } s \in [-r, 0]\}. \end{aligned}$$

In this case, it is clear that there are two invariant probability measures on the boundary $\partial\mathcal{C}^X_+$, which are $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ concentrating on $(X, 0)$ and $(0, X)$, respectively, where 0, X are understood to be constant functions. We have

$$\begin{aligned} \lambda_1(\boldsymbol{\delta}_2) &= f_1((0, X)) - f_2((0, X)) - \frac{\sigma_1^2 + \sigma_2^2}{2}, \\ \lambda_2(\boldsymbol{\delta}_1) &= f_2((X, 0)) - f_1((X, 0)) - \frac{\sigma_1^2 + \sigma_2^2}{2}. \end{aligned}$$

Using Theorem 2.1, and Theorem 2.2, we have the following classification for the extinction of (5.9): If $\lambda_1(\boldsymbol{\delta}_2) < 0$ (resp., $\lambda_2(\boldsymbol{\delta}_1) < 0$), there is no invariant probability measure on $\mathcal{C}^{X,\circ}_+$. Moreover, $x_1(t)$ tends to 0 (resp., $x_2(t)$) almost surely.

To proceed, we consider (5.9) for three-dimensional systems. We define the following sets

$$\begin{aligned} \mathcal{C}^X_+ := \Big\{&(\varphi_1, \varphi_2, \varphi_3) : \varphi_1(s) + \varphi_2(s) + \varphi_3(s) = X \text{ and} \\ &\qquad \varphi_1(s), \varphi_2(s), \varphi_3(s) \ge 0 \text{ for all } s \in [-r, 0]\Big\}, \end{aligned}$$

$$\partial\mathcal{C}_+^X := \mathcal{C}_{12+}^X \cup \mathcal{C}_{23+}^X \cup \mathcal{C}_{13+}^X,$$

$$\mathcal{C}_{ij+}^X := \{(\varphi_1,\varphi_2,\varphi_3) \in \mathcal{C}_+^X : \|\varphi_k\| = 0, k \neq i, j\}, \text{ for } i \neq j \in \{1,2,3\},$$

$$\mathcal{C}_+^{X,\circ} := \{(\varphi_1,\varphi_2,\varphi_3) \in \mathcal{C}_+^X : \varphi_1(s), \varphi_2(s), \varphi_3(s) > 0 \text{ for all } s \in [-r,0]\}.$$

Denote by $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\delta}_3$ the invariant probability measures on the boundary $\partial\mathcal{C}_+^X$ of (5.9), concentrating on $(X,0,0)$, $(0,X,0)$, and $(0,0,X)$, respectively. We have

$$\lambda_i(\boldsymbol{\delta}_1) = f_i((X,0,0)) - f_1((X,0,0)) - \frac{\sigma_1^2 + \sigma_i^2}{2}, i = 2,3,$$

$$\lambda_j(\boldsymbol{\delta}_2) = f_j((0,X,0)) - f_2((0,X,0)) - \frac{\sigma_2^2 + \sigma_j^2}{2}, j = 1,3,$$

and

$$\lambda_k(\boldsymbol{\delta}_3) = f_k((0,0,X)) - f_3((0,0,X)) - \frac{\sigma_3^2 + \sigma_k^2}{2}, k = 1,2.$$

If $\max_{j=1,3} \lambda_j(\boldsymbol{\delta}_2) > 0$ and $\max_{k=1,2} \lambda_k(\boldsymbol{\delta}_3) > 0$, there is a unique invariant probability measure on $\mathcal{C}_{23+}^X$, denoted by π_{23}. If $\max_{j=1,3} \lambda_j(\boldsymbol{\delta}_2) < 0$ or $\max_{k=1,2} \lambda_k(\boldsymbol{\delta}_3) < 0$, the invariant probability measure on $\mathcal{C}_{23+}^X$ does not exist. If π_{23} exists, we have

$$\begin{aligned}\lambda_1(\pi_{23}) = -\frac{\sigma_1^2}{2} + \int\limits_{\mathcal{C}_{23+}^X} \Big(f_1(\boldsymbol{\varphi}) - \frac{2X\varphi_2(0) f_2(\boldsymbol{\varphi}) + \sigma_2^2\varphi_2^2(0)}{X^2} \\ - \frac{2X\varphi_3(0) f_3(\boldsymbol{\varphi}) + \sigma_3^2\varphi_3^2(0)}{X^2}\Big)\pi_{23}(d\boldsymbol{\varphi}).\end{aligned}$$

By Lemma 4.1 and $\lambda_2(\pi_{23}) = \lambda_3(\pi_{23}) = 0$, we have

$$\begin{aligned}&\int\limits_{\mathcal{C}_{23+}^X} \left(\frac{2X\varphi_2(0) f_2(\boldsymbol{\varphi}) + \sigma_2^2\varphi_2^2(0)}{2X^2} + \frac{2X\varphi_3(0) f_3(\boldsymbol{\varphi}) + \sigma_3^2\varphi_3^2(0)}{2X^2} \right) \pi_{23}(d\boldsymbol{\varphi}) \\ &= \frac{\sigma_2^2}{2} + \int\limits_{\mathcal{C}_{23+}^X} f_2(\boldsymbol{\varphi})\pi_{23}(d\boldsymbol{\varphi}) \\ &= \frac{\sigma_3^2}{2} + \int\limits_{\mathcal{C}_{23+}^X} f_3(\boldsymbol{\varphi})\pi_{23}(d\boldsymbol{\varphi}).\end{aligned}$$

As a result, one has

$$
\begin{aligned}
\lambda_1(\pi_{23}) &= -\frac{\sigma_1^2+\sigma_2^2}{2} + \int\limits_{\mathcal{C}_{23+}^X} (f_1(\boldsymbol{\varphi}) - f_2(\boldsymbol{\varphi}))\,\pi_{23}(d\boldsymbol{\varphi}) \\
&= -\frac{\sigma_1^2+\sigma_3^2}{2} + \int\limits_{\mathcal{C}_{23+}^X} (f_1(\boldsymbol{\varphi}) - f_3(\boldsymbol{\varphi}))\,\pi_{23}(d\boldsymbol{\varphi}).
\end{aligned}
$$

The conditions to guarantee the existence of the unique invariant probability measure π_{12}, π_{13} on the boundary $\mathcal{C}_{12+}^X, \mathcal{C}_{13+}^X$ are similarly obtained and $\lambda_2(\pi_{13}), \lambda_3(\pi_{12})$ can be computed similar to $\lambda_1(\pi_{23})$. Therefore, we have the following classification for the extinction of solution of (5.9). For $l \in \{1,2,3\}$, $X_l(t)$ tends to 0 almost surely exponentially fast if one of following conditions holds:

- $\max_{i\neq l} \lambda_i(\boldsymbol{\delta}_l) < 0$,
- $\max_{i=2,3} \lambda_i(\boldsymbol{\delta}_1) > 0$, $\max_{j=1,3} \lambda_j(\boldsymbol{\delta}_2) > 0$, $\max_{k=1,2} \lambda_k(\boldsymbol{\delta}_3) > 0$ and $\lambda_l(\pi_{ij}) < 0$, $\{i,j,l\} = \{1,2,3\}$.

The explicit characterization of (5.9) in n-dimensional systems is more complex. However, our results (Theorem 2.1, and Theorem 2.2) as well as that of [46] are applicable under suitable conditions. Finally, it is worth noting that if $r = 0$ (i.e., there is no time delay) and $f_i(\cdot), i = 1,\dots,n$ are linear, the characterization of the long-term behavior of (5.9) in this section is equivalent to the results in [27,28].

5.4. Stochastic delay epidemic SIR model

The epidemic SIR model is one of the basic building blocks of compartmental models, from which many infectious disease models are derived; and was first introduced by Kermack and McKendrick in [29,30], and are deemed effective to depict the spread of many common diseases with permanent immunity such as rubella, whooping cough, measles, and smallpox. The model consists of three compartments, S (the number of susceptible), I (the number of infectious), and R (the number of recovered (or immune)). Much effort has been devoted to studying the behavior of the SIR epidemic systems and its variants; see [13–16] and the references therein. In this subsection, we investigate the stochastic epidemic SIR model with time delay. First, we consider the equation with linear incidence rate of the following form

$$
\begin{cases}
dS(t) = (a - b_1S(t) - c_1I(t)S(t) - c_2I(t)S(t-r))\,dt + S(t)dE_1(t), \\
dI(t) = (-b_2I(t) + c_1I(t)S(t) + c_2I(t)S(t-r))\,dt + I(t)dE_2(t),
\end{cases}
\tag{5.10}
$$

where $S(t)$ is the density of susceptible individuals, $I(t)$ is the density of infected individuals, $a > 0$ is the recruitment rate of the population, $b_i > 0$, $i = 1,2$ are the death rates, $c_i > 0$, $i = 1,2$ are the incidence rates, r is the delayed time, $(E_1(t), E_2(t))^\top = \Gamma^\top \mathbf{B}(t)$ with $\mathbf{B}(t) = (B_1(t), B_2(t))^\top$ being a vector of independent standard Brownian motions, and Γ being a 2×2 matrix such that $\Gamma^\top\Gamma = (\sigma_{ij})_{2\times 2}$ is a positive definite matrix. It is well-known that the dynamics of recovered individuals have no effect on the disease transmission dynamics and that is why we only consider the dynamics of $S(t), I(t)$ in (5.10).

While the conditions for persistence of (5.10) were given in [46, Section 5.4], we develop the conditions for extinction here. First, we consider the equation on the boundary $\{(\varphi_1, 0) : \varphi_1(s) \geq 0\ \forall s \in [-r, 0]\}$ and let $\widehat{S}(t)$ be the solution of the equation on this boundary as following

$$d\widehat{S}(t) = \big(a - b_1\widehat{S}(t)\big)\,dt + \widehat{S}(t)dE_1(t). \tag{5.11}$$

Since the drift coefficient of this equation is negative if $\widehat{S}(t)$ is sufficiently large and positive, if $\widehat{S}(t)$ is sufficiently small, we can show that there is a unique invariant probability measure π of (5.10) on $\mathcal{C}^\circ_{1+} := \{(\varphi_1, 0) : \varphi_1(s) > 0\ \forall s \in [-r, 0]\}$. On the other hand, since $\lambda_2(\boldsymbol{\delta}^*) = -b_2 - \dfrac{\sigma_{22}}{2} < 0$, there is no invariant probability measure in $\mathcal{C}^\circ_{2+} := \{(0, \varphi_2) : \varphi_2(s) > 0; \forall s \in [-r, 0]\}$. We define the following threshold

$$\lambda(\pi) = -b_2 - \frac{\sigma_{22}}{2} + \int\limits_{\mathcal{C}^\circ_{1+}} (c_1\varphi_1(0) + c_2\varphi_1(-r))\,\pi(d\boldsymbol{\varphi}), \tag{5.12}$$

whose sign will be able to characterize the permanence and extinction. As an application of Lemma 4.1, we get

$$\int\limits_{\mathcal{C}^\circ_{1+}} \varphi_1(0)\pi(d\boldsymbol{\varphi}) = \frac{a}{b_1}. \tag{5.13}$$

By (5.5), we have that

$$\int\limits_{\mathcal{C}^\circ_{1+}} \varphi_1(-r)\pi(d\boldsymbol{\varphi}) = \int\limits_{\mathcal{C}^\circ_{1+}} \varphi_1(0)\pi(d\boldsymbol{\varphi}) = \frac{a}{b_1}.$$

Therefore, under this condition, we obtain from (5.12) and (5.13) that

$$\lambda(\pi) = -b_2 - \frac{\sigma_{22}}{2} + \frac{a(c_1 + c_2)}{b_1}.$$

Using the same idea and technique, it is possible to obtain similar results of Theorem 2.1, and Theorem 2.2. Therefore, we have the following classifications:

- If $\lambda(\pi) < 0$, $I(t)$ converges to 0 almost surely with exponential rate $\lambda(\pi)$ while $S(t)$ tends to $\widehat{S}(t)$.
- If $\lambda(\pi) > 0$, (5.10) has a unique invariant probability measure in $\mathcal{C}^\circ_+$ (follows the first part [46, Section 5.4]).

This characterization is equivalent to the result in [36,37].

In the above, we consider the linear incidence to make our computations be more explicit. The characterizations still hold for the following stochastic delay SIR epidemic model with more general incidence rate

$$\begin{cases} dS(t) = \big(a - b_1 S(t) - I(t) f_1\big(S(t), S(t-r), I(t), I(t-r)\big)\big)\,dt + S(t)dE_1(t), \\ dI(t) = \big(-b_2 I(t) + I(t) f_2\big(S(t), S(t-r), I(t), I(t-r)\big)\big)\,dt + I(t)dE_2(t), \end{cases} \tag{5.14}$$

where $f_i : \mathbb{R}^4 \to \mathbb{R}, i = 1, 2$ are the incidence functions satisfying

- $f_1(0, 0, i_1, i_2) = f_2(0, 0, i_1, i_2) = 0$.
- there exists some $\kappa \in (0, \infty)$ such that for all $\boldsymbol{\varphi} \in \mathcal{C}_+$

$$\begin{aligned} f_2\big(\varphi_1(0), \varphi_1(-r), \varphi_2(0), \varphi_2(-r)\big) &\le \kappa f_1\big(\varphi_1(0), \varphi_1(-r), \varphi_2(0), \varphi_2(-r)\big) \\ &\le \kappa^2 \left(1 + |\boldsymbol{\varphi}(0)| + |\boldsymbol{\varphi}(-r)|\right). \end{aligned}$$

- $f_2(s_1, s_2, i_1, i_2)$ is non-decreasing in s_1, s_2 and is non-increasing in i_1, i_2.

It is important to mention that our conditions are verified by almost all incidence functions used in the literature, including linear functional response, Holling type II functional response, Beddington-DeAngelis functional response, etc. In the general case, the long-run behavior is almost completely characterized the same as the case of linear incidence rate by the threshold $\lambda(\pi)$ given by

$$\lambda(\pi) = -b_2 - \frac{\sigma_{22}}{2} + \int_{\mathcal{C}_{1+}^\circ} f_2\big(\varphi_1(0), \varphi_1(-r), \varphi_2(0), \varphi_2(-r)\big)\pi(d\boldsymbol{\varphi}),$$

where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ and π is the invariant probability measure of (5.11). These results significantly generalize and improve that of [8,38,41].

5.5. Stochastic delay chemostat model

Chemostat models, introduced by Novick and Szilard in [49], play an important role in microbiology, biotechnology, and population biology. A chemostat is a bio-reactor, in which fresh medium is continuously added, and culture liquid containing left-over nutrients, metabolic end products, and microorganisms are continuously removed at the same rate to keep a constant culture volume.

This section studies a model of n-microbial populations competing for a single nutrient in a chemostat, in which we take both the delayed times and the white noises into consideration. Consider the following system of stochastic functional differential equations

$$\begin{cases} dS(t) = \left(1 - S(t) + aS(t-r) - \displaystyle\sum_{i=1}^{n} x_i(t) p_i(S(t))\right)dt + S(t)dE_0(t), \\ dx_i(t) = x_i(t)\left(p_i(S(t-r)) - 1\right)dt + x_i(t)dE_i(t),\ i = 1, \dots, n, \end{cases} \tag{5.15}$$

where $S(t)$ is the concentration of nutrient at time t; $0 \le a < 1$ is a constant; $x_i(t), i = 1, \dots, n$ are the concentrations of the competing microbial populations; $p_i(S), i = 1, \dots, n$ are the density-dependent uptakes of nutrient by population x_i; r is the delayed time; and $(E_0(t), \dots, E_n(t))^\top = \Gamma^\top \mathbf{B}(t)$ with $\mathbf{B}(t) = (B_0(t), \dots, B_n(t))^\top$ being a vector of independent standard Brownian motions and Γ being a $(n+1) \times (n+1)$ matrix such that $\Gamma^\top \Gamma =$

$(\sigma_{ij})_{(n+1)\times(n+1)}$ is a positive definite matrix. Moreover, $\mathcal{C} := \mathcal{C}([-r,0],\mathbb{R}^{n+1})$ instead of $\mathcal{C}([-r,0],\mathbb{R}^{n})$. While the deterministic version of (5.15) was studied with the long-time behavior characterized in [19,22,66], recent attention on the stochastic counterpart can be found in [58,59,68].

Similar to Section 5.4 as well as [46, Section 5.5], if we assume that $p_i : \mathbb{R} \to \mathbb{R}, i = 1,\dots,n$ satisfying non-decreasing and bounded properties and $p_i(0) = 0$, then our Assumptions hold. Therefore, our results in this paper can be applied to (5.15). Before considering the higher dimensional systems, we consider $n = 1$ and 2. If $n = 1$, there is only one population x_1 together with the nutrient $S(t)$. Similar to Section 5.4, there is no invariant probability measure of (S_t, x_{1t}) in $\mathcal{C}^\circ_{1+} := \{(0,\varphi_1) \in \mathcal{C}_+ : \varphi_1(s) > 0, \forall s \in [-r,0]\}$, where x_{1t} is the memory segment function of $x_1(t)$. Moreover, there is a unique invariant probability measure π_0 in $\mathcal{C}^\circ_{0+} := \{(\varphi_0, 0) \in \mathcal{C}_+ : \varphi_0(s) > 0, \forall s \in [-r,0]\}$. Hence, it is easy to see that for any invariant probability measure π in $\partial\mathcal{C}_+$, we have

$$\lambda_1(\pi) = \lambda_1(\pi_0) = -1 - \frac{\sigma_{11}}{2} + \int\limits_{\mathcal{C}^\circ_{0+}} p_1(\varphi_0(-r))\pi_0(d\boldsymbol{\varphi}).$$

Therefore, our results yield the following classification.

- If $\lambda_1(\pi_0) > 0$ then (S_t, x_{1t}) admits a unique invariant probability measure in $\mathcal{C}^\circ_+$; followed by [46, Section 5.5].
- If $\lambda_1(\pi_0) < 0$ then $x_1(t)$ tends to 0 almost surely with exponential rate $\lambda_1(\pi_0)$ while $S(t)$ tends to $\widehat{S}(t)$, where $\widehat{S}(t)$ is the solution of

$$d\widehat{S}(t) = \big(1 - \widehat{S}(t) + a\widehat{S}(t-r)\big)\,dt + \widehat{S}(t)dE_0(t).$$

To proceed, we study the characterization of the longtime behavior in the case $n = 2$. Similar to the case of $n = 1$, there is no invariant probability measure in $\mathcal{C}^\circ_{i+} := \{(0,\varphi_1,\varphi_2) \in \mathcal{C}_+ : \|\varphi_j\| = 0, j \neq i$ and $\varphi_i(s) > 0, \forall s \in [-r,0]\}$, and there is a unique measure π_0 in $\mathcal{C}^\circ_{0+} := \{(\varphi_0, 0, 0) \in \mathcal{C}_+ : \varphi_0(s) > 0, \forall s \in [-r,0]\}$. If $\lambda_i(\pi_0) > 0$, where

$$\lambda_i(\pi_0) = -1 - \frac{\sigma_{ii}}{2} + \int\limits_{\mathcal{C}^\circ_{0+}} p_i(\varphi_0(-r))\pi_0(d\boldsymbol{\varphi}),\ i = 1,2,$$

then there is a unique invariant probability measure π_{0i} in $\mathcal{C}^\circ_{0i+} := \{(\varphi_0,\varphi_1,\varphi_2) \in \mathcal{C}_+ : \|\varphi_j\| = 0, j \neq i$ and $\varphi_0(s), \varphi_i(s) > 0, \forall s \in [-r,0]\}$. Hence, let

$$\lambda_j(\pi_{0i}) = -1 - \frac{\sigma_{jj}}{2} + \int\limits_{\mathcal{C}^\circ_{0+}} p_j(\varphi_0(-r))\pi_{0i}(d\boldsymbol{\varphi}),\ j \neq i.$$

The extinction is classified as follows.

- If $\lambda_1(\pi_0) < 0$, $\lambda_2(\pi_0) < 0$ then $x_1(t), x_2(t)$ tend to 0 almost surely with exponential rate $\lambda_1(\pi_0)$, $\lambda_2(\pi_0)$, respectively, while $S(t)$ tends to $\widehat{S}(t)$, where $\widehat{S}(t)$ is the solution of

$$d\widehat{S}(t) = \big(1 - \widehat{S}(t) + a\widehat{S}(t-r)\big)\,dt + \widehat{S}(t)dE_0(t).$$

- If $\lambda_i(\pi_0) > 0,\ \lambda_j(\pi_0) < 0,\ \lambda_j(\pi_{0i}) < 0,\ i \neq j \in \{1,2\}$ then $x_j(t)$ converges to 0 almost surely with exponential rate $\lambda_j(\pi_{0i})$ and the random occupation measure converges to π_{0i}.
- If $\lambda_1(\pi_0) > 0,\ \lambda_2(\pi_0) > 0,\ \lambda_i(\pi_{0j}) > 0,\ \lambda_j(\pi_{0i}) < 0,\ i \neq j \in \{1,2\}$ then $x_j(t)$ converges to 0 almost surely with exponential rate $\lambda_j(\pi_{0i})$ and the random occupation measure converges to π_{0i}.
- If $\lambda_1(\pi_0) > 0,\ \lambda_2(\pi_0) > 0,\ \lambda_1(\pi_{02}) < 0,\ \lambda_2(\pi_{01}) < 0$ then $q_i > 0, i = 1,2$ and $q_1 + q_2 = 1$ where

$$q_i = \mathbb{P}_{\boldsymbol{\phi}}\left\{\mathcal{U}(\omega) = \{\pi_{0i}\} \text{ and } \lim_{t\to\infty}\frac{\ln X_i(t)}{t} = \lambda_i(\pi_{1j}), i \in \{1,2\}\setminus\{j\}\right\}.$$

On the other hand, combining this section and [46, Section 5.5] leads to that (S_t, x_{1t}, x_{2t}) admits a unique invariant probability measure in $\mathcal{C}_+^\circ$ if one of following conditions holds

- $\lambda_1(\pi_0) > 0,\ \lambda_2(\pi_0) < 0,\ \lambda_2(\pi_{01}) > 0.$
- $\lambda_1(\pi_0) < 0,\ \lambda_2(\pi_0) > 0,\ \lambda_1(\pi_{02}) > 0.$
- $\lambda_1(\pi_0) > 0,\ \lambda_2(\pi_0) > 0,\ \lambda_1(\pi_{02}) > 0,\ \lambda_2(\pi_{01}) > 0.$

For higher dimensional systems, although, it is somewhat difficult to show concretely in case of general functions $p_i(\cdot)$, it is computable in certain cases. These classifications improve the results in [58,68].

Remark 3. In Sections 5.1-5.5, to present the main ideas without notation complication we used a single delay. However, the results for models with multi-delays or distributed delays can be obtained similarly. On the other hand, if $r = 0$, i.e., there is no time delay, the above results are consistent with and/or even improve the existing results in the literature.

References

[1] J. Alboszta, J. Miekisz, Stability of evolutionarily stable strategies in discrete replicator dynamics with time delay, J. Theor. Biol. 231 (2004) 175–179.
[2] A. Bahar, X. Mao, Stochastic delay Lotka-Volterra model, J. Math. Anal. Appl. 292 (2004) 364–380.
[3] J. Bao, G. Yin, C. Yuan, Asymptotic Analysis for Functional Stochastic Differential Equations, SpringerBriefs Math., Springer, 2016.
[4] M. Benaïm, Stochastic persistence, preprint, 2018.
[5] M. Benaïm, C. Lobry, Lotka-Volterra with randomly fluctuating environments or “how switching between beneficial environments can make survival harder”, Ann. Appl. Probab. 26 (2016) 3754–3785.
[6] M. Benaïm, E. Strickler, Random switching between vector fields having a common zero, Ann. Appl. Probab. 29 (2019) 326–375.
[7] I. Bomze, M. Pelillo, V. Stix, Approximating the maximum weight clique using replicator dynamics, IEEE Trans. Neural Netw. 11 (2000) 1228–1241.
[8] G. Chen, T. Li, Stability of stochastic delayed SIR model, Stoch. Dyn. 9 (2009) 231–252.
[9] P.L. Chesson, S. Ellner, Invasibility and stochastic boundedness in monotonic competition models, J. Math. Biol. 27 (1989) 117–138.
[10] R. Cont, D.-A. Fournié, Change of variable formulas for non-anticipative functionals on path space, J. Funct. Anal. 259 (2010) 1043–1072.
[11] R. Cont, D.-A. Fournié, Functional Itô calculus and stochastic integral representation of martingales, Ann. Probab. 41 (2013) 109–133.

[12] J.M. Cushing, Integrodifferential Equations and Delay Models in Population Dynamics, vol. 20, Springer Science & Business Media, 2013.

[13] N.T. Dieu, D.H. Nguyen, N.H. Du, G. Yin, Classification of asymptotic behavior in a stochastic SIR model, SIAM J. Appl. Dyn. Syst. 15 (2016) 1062–1084.

[14] N.H. Du, N.T. Dieu, N.N. Nguyen, Conditions for permanence and ergodicity of certain SIR epidemic models, Acta Appl. Math. 160 (2019) 81–99.

[15] N.H. Du, N.N. Nguyen, Permanence and extinction of certain stochastic SIR models perturbed by a complex type of noises, Appl. Math. Lett. 64 (2017) 223–230.

[16] N.H. Du, N.N. Nguyen, Permanence and extinction for the stochastic SIR epidemic model, J. Differ. Equ. 269 (2020) 9619–9652.

[17] N.H. Du, D.H. Nguyen, G. Yin, Conditions for permanence and ergodicity of certain stochastic predator-prey models, J. Appl. Probab. 53 (2016) 187–202.

[18] B. Dupire, Functional Itô Calculus, Bloomberg Portfolio Research Paper No. 2009-04-FRONTIERS, Available at SSRN: http://ssrn.com/abstract=1435551, 2009, or https://doi.org/10.2139/ssrn.1435551.

[19] S.F. Ellermeyer, Competition in the chemostat: global asymptotic behavior of a model with delayed response in growth, SIAM J. Appl. Math. 54 (1994) 456–465.

[20] S.N. Evans, A. Henning, S.J. Schreiber, Protected polymorphisms and evolutionary stability of patch-selection strategies in stochastic environments, J. Math. Biol. 71 (2015) 325–359.

[21] S.N. Eithier, T.G. Kurtz, Markov Processes: Characterization and Convergence, Wiley, New York, 1986.

[22] H.I. Freedman, J.W.-H. So, P. Waltman, Coexistence in a model of competition in the chemostat incorporating discrete delays, SIAM J. Appl. Math. 49 (1989) 859–870.

[23] J. Geng, M. Liu, Y. Zhang, Stability of a stochastic one-predator-two-prey population model with time delays, Commun. Nonlinear Sci. Numer. Simul. 53 (2017) 65–82.

[24] M. Hairer, J.C. Mattingly, M. Scheutzow, Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations, Probab. Theory Relat. Fields 149 (2011) 223–259.

[25] A. Hening, D.H. Nguyen, Coexistence and extinction for stochastic Kolmogorov systems, Ann. Appl. Probab. 28 (2018) 1893–1942.

[26] J. Hofbauer, K. Sigmund, Evolutionary Games and Population Dynamics, Cambridge Univ. Press, 1998.

[27] J. Hofbauer, L.A. Imhof, Time averages, recurrence and transience in the stochastic replicator dynamics, Ann. Appl. Probab. 19 (2009) 1347–1368.

[28] L.A. Imhof, The long-run behavior of the stochastic replicator dynamics, Ann. Appl. Probab. 15 (2005) 1019–1045.

[29] W.O. Kermack, A.G. McKendrick, Contributions to the mathematical theory of epidemics–I, Proc. R. Soc. Lond. Ser. A 115 (1927) 700–721.

[30] W.O. Kermack, A.G. McKendrick, Contributions to the mathematical theory of epidemics–II, Proc. R. Soc. Lond. Ser. A 138 (1932) 55–83.

[31] R. Khasminskii, Stochastic Stability of Differential Equations, 2nd ed., Springer, Heidelberg, 2012.

[32] M. Liu, H. Qiu, K. Wang, A remark on a stochastic predator-prey system with time delays, Appl. Math. Lett. 26 (2013) 318–323.

[33] M. Liu, C. Bai, Y. Jin, Population dynamical behavior of a two-predator one-prey stochastic model with time delay, Discrete Contin. Dyn. Syst. 37 (2017) 2513–2538.

[34] M. Liu, M. Fan, Stability in distribution of a three-species stochastic cascade predator-prey system with time delays, IMA J. Appl. Math. 82 (2017) 396–423.

[35] M. Liu, C. Bai, Optimal harvesting of a stochastic delay competitive model, Discrete Contin. Dyn. Syst., Ser. B 22 (2017) 1493–1508.

[36] Q. Liu, Q. Chen, D. Jiang, The threshold of a stochastic delayed SIR epidemic model with temporary immunity, Physica A 450 (2016) 115–125.

[37] Q. Liu, D. Jiang, The threshold of a stochastic delayed SIR epidemic model with vaccination, Physica A 461 (2016) 140–147.

[38] Q. Liu, D. Jiang, N. Shi, T. Hayat, A. Alsaedi, Asymptotic behaviors of a stochastic delayed SIR epidemic model with nonlinear incidence, Commun. Nonlinear Sci. Numer. Simul. 40 (2016) 89–99.

[39] A.J. Lotka, Elements of Physical Biology, William and Wilkins, Baltimore, 1925.

[40] Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, J. Differ. Equ. 223 (2) (2006) 400–426.

[41] M. Mahrouf, J. Adnani, N. Yousfi, Stability analysis of a stochastic delayed SIR epidemic model with general incidence rate, Appl. Anal. 97 (2018) 2113–2121.

[42] X. Mao, Stochastic Differential Equations and Their Applications, Horwood Publishing Limited, Chichester, 1997.

[43] X. Mao, C. Yuan, Stochastic Differential Equations with Markovian Switching, Imperial College Press, London, 2006.
[44] S.-E.A. Mohammed, Stochastic Functional Differential Equations, Longman Scientific and Technical, 1986.
[45] D.H. Nguyen, N.N. Nguyen, G. Yin, General nonlinear stochastic systems motivated by chemostat models: complete characterization of long-time behavior, optimal controls, and applications to wastewater treatment, Stoch. Process. Appl. 130 (2020) 4608–4642.
[46] D.H. Nguyen, N.N. Nguyen, G. Yin, Stochastic functional Kolmogorov equations I: persistence, under revision for, Stoch. Process. Appl. (2021), arXiv:2105.03551.
[47] D.H. Nguyen, G. Yin, Stability of stochastic functional differential equations with regime switching: analysis using Dupire's functional Itô formula, Potential Anal. 53 (2020) 247–265.
[48] D.H. Nguyen, G. Yin, C. Zhu, Long-term analysis of a stochastic SIRS model with general incidence rates, SIAM J. Appl. Math. 80 (2020) 814–838.
[49] A. Novick, L. Szilard, Description of the chemostat, Science 112 (1950) 715–716.
[50] M.A. Nowak, K. Sigmund, Evolutionary dynamics of biological games, Science 303 (2004) 793–799.
[51] G. Obando, A. Pantoja, N. Quijano, Building temperature control based on population dynamics, IEEE Trans. Control Syst. Technol. 22 (2014) 404–412.
[52] G. Obando, J.I. Poveda, N. Quijano, Replicator dynamics under perturbations and time delays, Math. Control Signals Syst. 28 (2016) 20.
[53] E. Ramírrez-Llanos, N. Quijano, A population dynamics approach for the water distribution problem, Int. J. Control 83 (2010) 1947–1964.
[54] M. Reiß, M. Riedle, O. van Gaans, Delay differential equations driven by Levy processes: stationarity and Feller properties, Stoch. Process. Appl. 116 (2006) 1409–1432.
[55] G. Roth, S.J. Schreiber, Persistence in fluctuating environments for interacting structured populations, J. Math. Biol. 69 (2014) 1267–1317.
[56] S.J. Schreiber, Persistence for stochastic difference equations: a mini-review, J. Differ. Equ. Appl. 18 (2012) 1381–1403.
[57] S.J. Schreiber, M. Benaïm, K.A.S. Atchadé, Persistence in fluctuating environments, J. Math. Biol. 62 (2011) 655–683.
[58] S. Sun, X. Zhang, Asymptotic behavior of a stochastic delayed chemostat model with nutrient storage, J. Biol. Syst. 26 (2018) 225–246.
[59] S. Sun, X. Zhang, Asymptotic behavior of a stochastic delayed chemostat model with nonmonotone uptake function, Physica A 512 (2018) 38–56.
[60] P.D. Taylor, L.B. Jonker, Evolutionary stable strategies and game dynamics, Math. Biosci. 40 (1978) 145–156.
[61] H. Tembine, E. Altman, R. El-Azouzi, Y. Hayel, Evolutionary games in wireless networks, IEEE Trans. Syst. Man Cybern., Part B 40 (2010) 634–646.
[62] T.D. Tuong, N.N. Nguyen, G. Yin, Longtime behavior of a class of stochastic tumor-immune systems, Syst. Control Lett. 146 (2020) 104806.
[63] P.F. Verhulst, Notice sur la loi que la population poursuit dans son accroissement, Correspondance Math. Phys. 10 (1838) 113–121.
[64] V. Volterra, Variazioni e fluttuazioni del numero d'individui in specie d'animali conviventi, Mem. Acad. Lincei 2 (1926) 31–113.
[65] J. Weibull, Evolutionary Game Theory, MIT Press, Cambridge, MA, USA, 1997.
[66] G.S.K. Wolkowicz, H. Xia, Global asymptotic behavior of a chemostat model with discrete delays, SIAM J. Appl. Math. 57 (1997) 1019–1043.
[67] J. Wu, Analysis of a three-species stochastic delay predator-prey system with imprecise parameters, Methodol. Comput. Appl. Probab. 21 (2019) 43–67.
[68] X. Zhang, R. Yuan, The existence of stationary distribution of a stochastic delayed chemostat model, Appl. Math. Lett. 93 (2019) 15–21.