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Stability of Stochastic Functional Differential Equations with Regime-Switching: Analysis Using Dupire's Functional Itô Formula

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Abstract

This work focuses on almost sure and L^p stability of stochastic functional differential equations by using Lyapunov functionals with the help of the recently developed Dupire's functional Itô formula. Novel conditions for stability, which are different from those in the existing literature, are given in terms of Lyapunov functionals. It is demonstrated that the conditions are useful for stochastic stabilization. It is also shown that adding a diffusion term can stabilize an unstable system of deterministic differential equations with Markov switching. Furthermore, a robustness result is obtained, which states that the stability of stochastic differential equations with regime-switching is preserved under delayed perturbations when the delay is small enough.

Keywords Switching diffusion · Functional stochastic differential equation with switching · Stability

1 Introduction

Functional differential equations (FDEs) arise from a wide range of applications. As observed that in real world applications, including queueing systems, biological and ecological systems, finance and economics, control engineering, networked systems, wired and wireless communications, and other related fields, delays are often unavoidable. Dealing with such systems, one of the main ingredients is that the underlying dynamics of the systems have memory and include the past dependence; see e.g., [1, 10, 24, 30]. It has also

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been well recognized that the systems under consideration are often corrupted by noise. Thus it is necessary to take into account of random perturbations. Because of the pressing needs, stochastic functional differential equations (SFDEs) and applications have been studied extensively in the past decades; see [7, 11, 16, 22] and references therein. Several types of stability for SFDEs including moment stability, almost sure stability, and stability in probability have been considered using Razumikhin methods and Lyapunov functionals in [3, 8, 11, 13, 15, 23, 29] and references therein.

There are several main difficulties in handling stochastic differential delay equations and stochastic functional differential equations. To begin, the so-called segment process associated to a stochastic functional equation, belongs to an infinite dimensional space even if the differential equations and solutions live in a finite dimensional space. For instance, one considers a stochastic differential equation. Rather than the running time t, suppose that one can only observe the system at discrete epoch or sampling time $\lfloor t/t_0 \rfloor t_0$ for a constant $t_0 > 0$, where $\lfloor t/t_0 \rfloor$ denotes the integer part of t/t_0. Then one immediately faces systems with delays.

Second, in the study of stability of stochastic differential equations (diffusions and switching diffusions) with an equilibrium point 0 (under Lipschitz condition), an important observation is: If the solutions do not start at 0, they will never reach 0 in finite time. Thus we can conveniently construct needed Lyapunov functions. This is no longer true if we consider SFDEs. As a results, unlike for SDEs or switching diffusions in which we can use Lyapunov functions of the form $V = |x|^p$ for $p \in (0, 1)$ (e.g., [9, 13]) practical Lyapunov functions (or functionals) for treating SFDEs are often of quadratic forms, which prevent the use of a relatively large class of Lyapunov functions (or functionals) to prove L^p or almost surely stability; see also the classical results on stochastic differential equations [26].

In addition, for stochastic functional differential equations, the solution processes are no longer Markovian. Although there were many excellent works on stochastic delay equations, because of the solution processes being non-Markovian due to delay, there had been virtually no bona fide operators and functional Itô formulas except some general setup in a Banach space such as [16] before 2009. The setup in a Banach space, though general, is not suitable to be used in analysis involving functional stochastic differential equations. Recently, in [6], Dupire generalized the Itô formula to a functional setting by using pathwise functional derivatives. The Itô formula developed has substantially eased the difficulties and encouraged development with a wide range of applications. Subsequently, his work was developed further by [4, 5]. The functional Itô formula enables us to obtain a bona fide operator for SFDEs and facilitate the use of Lyapunov functionals to a larger class of stochastic systems with delays, including stochastic functional differential equations with regime-switching. It is known switching functional stochastic differential equations can describe complex systems that cannot be modeled with continuous states alone. A distinct feature of stochastic functional differential equations with regime-switching is that both continuous dynamics and discrete events are influencing the systems.

In this work, we demonstrate that Dupire's Functional Itô formula is useful for carrying out stability analysis. With the help of the functional Itô formula, we obtain sufficient conditions for almost sure and L^p stability of SFDEs with regime switching by using Lyapunov functionals that are different from the existing literature. For some of the recent works on switching diffusions, we refer the reader to [2, 15, 17–20, 27, 28] and the references therein. We further show that the stability result can be used for stabilizing Markovian switching ordinary differential equations. This is done by adding a diffusion term to an unstable ordinary differential equation with Markovian switching, which opens doors for further consideration of stabilization of a wide variety of systems.

As a bi-product of the stability, we derive certain robustness results. We demonstrate that if a stochastic differential equation is stable, then an associate stochastic functional differential equation is also stable provided the delay is small enough. Similar results have been given in [8, 25] by estimating the difference between a SFDE and its SDE counterpart in each finite interval. In contrast, using suitable Lyapunov functionals, we can obtain similar results with weaker conditions and simpler proofs.

The rest of the paper is organized as follows. Section 2 recalls the notion of stochastic functional differential equations with regime-switching and introduces the functional Itô formula. We also use a functional Lyapunov function to prove the existence and uniqueness of solutions. Section 3 is devoted to new conditions for almost sure stability and L^p stability of SFDEs with regime switching. Section 4 concentrates on the robustness. We treat almost sure stability of SFDEs with regime switching when the delayed time is small.

2 SFDEs with Regime Switching and Functional Itô Formula

Let r be a fixed positive number. Denote by $\mathcal{C}([a,b],\mathbb{R}^{n_0})$ the set of $\mathbb{R}^{n_0}$-valued continuous functions defined on $[a,b]$. In what follows, we mainly work with $\mathcal{C}([-r,0],\mathbb{R}^{n_0})$, and simply denote it by $\mathcal{C} := \mathcal{C}([-r,0],\mathbb{R}^{n_0})$. For each $\phi \in \mathcal{C}$, we use the sup norm metric $\|\phi\| = \sup\{|\phi(t)| : t \in [-r,0]\}$; for $t \ge 0$, we use y_t to denote the segment function or memory segment function $y_t = \{y(t+s) : -r \le s \le 0\}$. Denote by $|x|$ the Euclidean norm of x for $x \in \mathbb{R}^{n_0}$. For an $m \times n$ matrix A, we use the operator norm

$$|A| = \sup\{\|Ax\| : x \in \mathbb{R}^n, |x| = 1\}.$$

We work with $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$, a complete filtered probability space with the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual condition, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $W(t)$ be an $\mathcal{F}_t$-adapted and $\mathbb{R}^d$-valued Brownian motion, and $b(\cdot,\cdot) : \mathbb{R}^{n_0} \times \mathcal{S} \to \mathbb{R}^{n_0}$, where $\mathcal{S} := \{1,\dots,m_0\}$. Let $\alpha(t)$ be a homogeneous Markov chain taking value in $\mathcal{S}$, and assume that $\alpha(t)$ is independent of the Brownian motion $W(t)$. Suppose that the generator of $\alpha(t)$ is $Q = (q_{ij})_{m_0\times m_0}$ so that for sufficiently small $\Delta > 0$,

$$\begin{aligned} &\mathbb{P}\{\alpha(t+\Delta) = j|\alpha(t) = i\} = q_{ij}\Delta + o(\Delta) \text{ if } i \ne j \text{ and} \\ &\mathbb{P}\{\alpha(t+\Delta) = i|\alpha(t) = i\} = 1 - q_{ii}\Delta + o(\Delta). \end{aligned} \tag{2.1}$$

Assume that $\alpha(t)$ is an irreducible Markov chain with a unique invariant probability measure $\boldsymbol{\nu} = (\nu_1,\dots,\nu_{m_0})$. Suppose that a continuous state component $X(t)$ satisfies

$$dX(t) = f(X_t,\alpha(t))dt + g(X_t,\alpha(t))dW(t). \tag{2.2}$$

Recall that a strong solution to Eq. 2.2 on $[0,T]$ with initial data (ξ, i_0) with ξ being a $\mathcal{C}$-valued $\mathcal{F}_0$-measurable random variable and $i_0 \in \mathcal{S}$, is an $\mathcal{F}_t$-adapted process $X(t)$ such that

- $X(t)$ is continuous with probability 1 (w.p.1).
- $X(t) = \xi(t)$ for $t \in [-r,0]$ and $\alpha(0) = i_0$
- $X(t)$ satisfies (2.2) for all $t \in [0,T]$ w.p.1.

For a Lyapunov function $V : \mathbb{R}^{n_0} \times \mathcal{S} \mapsto \mathbb{R}_+$, which is twice continuously differentiable with respect to the first variable, one often works with a map from $\mathcal{C} \times \mathcal{S} \mapsto \mathbb{R}$. With a slight abuse of notation, we write it as

$$\mathcal{L}V(\phi, i) = \mathcal{L}_i V(\phi) + \sum_{j\in\mathcal{S}} q_{ij}\big[V(\phi(0), j) - V(\phi(0), i)\big]$$

where

$$\mathcal{L}_i U(\phi) = U_x(\phi(0)) f(\phi, i) + \frac{1}{2}\,\mathrm{tr}\,(U_{xx}(\phi(0))A(\phi, i))$$

with $U : \mathbb{R}^{n_0} \mapsto \mathbb{R}_+$ being twice continuously differentiable with respect to its variable and $A(\phi, i) = g^\top(\phi, i)g(\phi, i)$.

Remark 2.1 The notation needs some explanation. Note that the variables ϕ and (ϕ, i) in $\mathcal{L}_i U(\phi)$ and $\mathcal{L}V(\phi, i)$ represent the variables appearing in $\mathcal{L}_i U$ and $\mathcal{L}V$, respectively, because the term X_t appears in the coefficient of the system in Eq. 2.2. They do not represent the variables in functions U and V. The dependence of the variable x (corresponding the solution of Eq. 2.2) in $\mathcal{L}_i U(\phi)$ and $\mathcal{L}V(\phi, i)$ is indicated by $\phi(0)$.

Now we state the functional Itô formula for our process (see [5] for more details). Let $\mathbb{D}$ be the space of cadlag functions $f : [-r, 0] \mapsto \mathbb{R}^{n_0}$. For $\phi \in \mathbb{D}$, with $h \geq 0$ and $y \in \mathbb{R}^n$, we define horizontal and vertical perturbations as

$$\phi_h(s) = \begin{cases} \phi(s+h) \ \text{ if } \ s \in [-r, -h], \\ \phi(0) \ \text{ if } \ s \in [-h, -0], \end{cases}$$

and

$$\phi^y(s) = \begin{cases} \phi(s) \ \text{ if } \ s \in [-r, 0), \\ \phi(0) + y, \ \text{ if } \ s = 0 \end{cases}$$

respectively. Let $\mathcal{V} : \mathbb{D} \times \mathcal{S} \mapsto \mathbb{R}$. The horizontal and vertical partial derivatives of V at (ϕ, i) are defined as

$$\mathcal{V}_t(\phi, i) = \lim_{h\to 0} \frac{\mathcal{V}(\phi_h, i) - \mathcal{V}(\phi, i)}{h}, \tag{2.3}$$

and

$$\partial_i \mathcal{V}(\phi, k) = \lim_{h\to 0} \frac{\mathcal{V}(\phi^{he_k}, i) - \mathcal{V}(\phi, i)}{h}, \tag{2.4}$$

respectively, if these limits exist. In Eq. 2.4, e_k is the standard unit vector in $\mathbb{R}^{n_0}$ whose k-th component is 1 and other components are 0. Let $\mathbb{F}$ be the family of function $\mathcal{V}(\cdot, \cdot) : \mathbb{D} \times \mathcal{S} \times \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying that

- $\mathcal{V}$ is continuous, that is, for any $\varepsilon > 0$, $(\phi, i) \in \mathbb{D} \times \mathcal{S}$, there is a $\delta > 0$ such that $|\mathcal{V}(\phi, i) - \mathcal{V}(\phi', i)| < \varepsilon$ as long as $\|\phi - \phi'\| < \delta$.
- The derivatives $\mathcal{V}_t$, $\mathcal{V}_x = (\partial_k \mathcal{V})$, and $\mathcal{V}_{xx} = (\partial_{kl} \mathcal{V})$ exist and are continuous.
- $\mathcal{V}$, $\mathcal{V}_t$, $\mathcal{V}_x = (\partial_k \mathcal{V})$ and $\mathcal{V}_{xx} = (\partial_{kl} \mathcal{V})$ are bounded in each $B_R := \{(\phi, i) : \|\phi\| \leq R, i \leq R\}$, $R > 0$.

Let $\mathcal{V}(\cdot, \cdot) \in \mathbb{F}$, we define the operator

$$\mathbb{L}\mathcal{V}(\phi, i) = \mathbb{L}_i \mathcal{V}(\phi) + \sum_{j\in\mathcal{S}} q_{ij}\big[\mathcal{V}(\phi, j) - \mathcal{V}(\phi, i)\big] \tag{2.5}$$

where

$$
\begin{aligned}
\mathbb{L}_i\mathcal{V}(\phi) =&\mathcal{V}_t(\phi,i)+\mathcal{V}_x(\phi,i)f(\phi(0,i)+\frac{1}{2}\operatorname{tr}\Big(\mathcal{V}_{xx}(\phi,i)A(\phi,i)\Big)\\
=&\mathcal{V}_t(\phi,i)+\sum_{k=1}^{n_0}f_k(\phi,i)\mathcal{V}_k(\phi,i)+\frac{1}{2}\sum_{k,l=1}^{n_0}a_{kl}(\phi,i)\mathcal{V}_{kl}(\phi,i).
\end{aligned}
\tag{2.6}
$$

Again, the variable ϕ in $\mathbb{L}_i\mathcal{V}(\phi)$ and (ϕ,i) in $\mathbb{L}\mathcal{V}(\phi,i)$ are variables of $\mathbb{L}_i\mathcal{V}$ and $\mathbb{L}\mathcal{V}$ respectively. We have the functional Itô formula (see [4, 5])

$$
d\mathcal{V}(X_t,\alpha(t))=\Big(\mathbb{L}\mathcal{V}(X_t,\alpha(t))\Big)dt+\mathcal{V}_x(X_t,\alpha(t))g(X_t,\alpha(t))dW(t) \tag{2.7}
$$

Remark 2.2 The recently developed functional Itô formula in [6] encouraged subsequent advances; for example, [4, 5]. Such development proved to be very useful for a wide range of applications.

To proceed, we compute the Dupire derivatives of some functionals in certain forms.

- Consider

  $$V_0(\phi,i)=f_1(\phi(0),i)$$

  where $f(\cdot,\cdot):\mathbb{R}^n\times\mathcal{S}\mapsto\mathbb{R}$ is a function that is twice continuously differentiable in the first variable. Then

  $$
  V_{0t}(\phi,i)=0,\qquad \partial_k V_0(\phi,i)=\frac{\partial f}{\partial x_k}(\phi(0),i),\qquad \partial_{kl}V_0(\phi,i)=\frac{\partial^2 f}{\partial x_k\partial x_l}(\phi(0),i). \tag{2.8}
  $$

- If

  $$\mathcal{V}_1(\phi,i)=\int_s^0 g(u,i)f_1(\phi(u),i)du$$

  where $s\in[-r,0]$ is a fixed number, $f_1(\cdot,\cdot):\mathbb{R}^n\times\mathcal{S}\mapsto\mathbb{R}$ is a function that is twice continuously differentiable in the first variable and $g(\cdot,\cdot):\mathbb{R}_+\times\mathcal{S}\mapsto\mathbb{R}$ be a continuously differentiable function in the first variable. Then at $(\phi,i)\in\mathcal{C}\times\mathcal{S}$ we have (see [21] for the detailed computations)

  $$
  \begin{aligned}
  &\mathcal{V}_{1t}(\phi,i)=g(0,i)f_1(\phi(0),i)-g(s,i)f_1(\phi(s),i)-\int_s^0 f_1(\phi(u),i)dg(u,i),\\
  &\partial_k\mathcal{V}_1(\phi,i)=0,\qquad \partial_{kl}\mathcal{V}_1(\phi,i)=0.
  \end{aligned}
  \tag{2.9}
  $$

- If

  $$\mathcal{V}_2(\phi,i)=\int_{-r}^0 g_2(s,i)\mu(ds)\int_s^0 g_1(u,i)f_2(\phi(u),i)du$$

  where $f_2(\cdot,\cdot):\mathbb{R}^n\times\mathcal{S}\mapsto\mathbb{R}$ is a function that is twice continuously differentiable in the first variable and g_1 be a continuously differentiable function in the first variable and g_2 is continuous. Then applying (2.9) and Fubini's theorem, we can easily obtain

  $$
  \begin{aligned}
  \mathcal{V}_{2t}(\phi,i)=&\,g_1(0,i)f_2(\phi(0),i)\int_{-r}^0 g_2(s,i)\mu(ds)-\int_{-r}^0 g_1(s,i)g_2(s,i)f_2(\phi(s),i)\mu(ds)\\
  &-\int_{-r}^0 g_2(s,i)\mu(ds)\int_s^0 f_2(\phi(t),i)dg_1(t,i),\\
  \partial_k\mathcal{V}_2(\phi,i)\,=&\,0,\qquad \partial_{kl}\mathcal{V}(\phi,i)=0.
  \end{aligned}
  \tag{2.10}
  $$

Assumption 2.1 *Suppose that f and g are locally Lipschitz. That is, for any $n > 0$, there exists $K_n > 0$ such that*

$$|f(\phi, i) - f(\psi, i)| + |g(\phi, i) - g(\psi, i| \le K_n \|\phi - \psi\|$$

given that $\phi, \psi \in \mathcal{C}$, $\|\phi\| \le n$, $\|\psi\| \le n$.

Theorem 2.1 *Suppose there exists a probability measure μ on $[-r, 0]$ and a function $V(x)$ satisfying $c_1|x|^2 \le V(x) \le c_2|x^2|$ and*

$$\mathcal{L}_i V(\phi) \le aV(\phi(0)) + b\int_{-r}^{0} V(\phi(s))\mu(ds).$$

Then there exists a unique solution to Eq. 2.2 *for any initial value (ϕ, i)*

Proof The existence and uniqueness of local solutions can be seen in [15] due to the local Lipschitz continuity of the coefficient. Let $\mathcal{V}(\phi) = V(\phi(0)) + b\int_{-r}^{0}\mu(ds)\int_{s}^{0} V(\phi(u))du$

$$\begin{aligned}\mathbb{L}\mathcal{V}(\phi, i) =&\mathcal{L}_i V(\phi) + bV(\phi(0)) - b\int_{-r}^{0}\mu(ds)V(\phi(s))\\ \le& aV(\phi(0)) + b\int_{-r}^{0} V(\phi(s))\mu(ds) + bV(\phi(0)) - b\int_{-r}^{0}\mu(ds)V(\phi(s))\\ \le&(a+b)V(\phi(0)) \le (a+b)\mathcal{V}(\phi).\end{aligned}$$

Let $\tau_n = \inf\{t \ge 0 : \mathcal{V}(X_t) \ge n\}$, by Itô's formula, we have

$$\begin{aligned}\mathbb{E}_{\phi,i}\mathcal{V}(X_{t\wedge\tau_n}) =&\mathcal{V}(\phi) + \mathbb{E}_{\phi,i}\int_0^{t\wedge\tau_n}\mathbb{L}\mathcal{V}(X_s, \alpha(s))ds\\ \le&\mathcal{V}(\phi) + (a+b)\mathbb{E}_{\phi,i}\int_0^{t\wedge\tau_n}\mathcal{V}(X_s)ds\\ \le&\mathcal{V}(\phi) + (a+b)\int_0^t \mathbb{E}_{\phi,i}\mathcal{V}(X_{s\wedge\tau_n})ds.\end{aligned}$$

By Gronwall's inequality, we have

$$\mathbb{E}_{\phi,i}\mathcal{V}(X_{t\wedge\tau_n}) \le \mathcal{V}(\phi)e^{(a+b)t}.$$

As a result,

$$\mathbb{P}_{\phi,i}\{\mathcal{V}(X_t) \le n\} \le \frac{\mathcal{V}(\phi)e^{(a+b)t}}{n} \to 0 \text{ as } n \to \infty,$$

which implies that $\tau_\infty > t$ a.s. for any $t > 0$. The existence of global solutions is proved. □

3 Almost Sure and L^p Stability Using Functional Itô Formula and Lyapunov Functionals

We start by recall a Razumikin's type theorem in [11, 29] with the use of Lyapunov functionals.

Theorem 3.1 (A Razumikhin's Type Theorem) *Let $V : \mathbb{R}^{n_0} \times \mathcal{S} \mapsto \mathbb{R}_+$ be a C^2 function satisfying*

$$c_1|x|^2 \le V(x, i) \le c_2|x|^2,\ \forall x \in D \text{ for some } c_1, c_2 > 0. \tag{3.1}$$

Suppose that there exists $\lambda_1 > \lambda_2 > 0$ *such that*

$$\mathcal{L}V(\phi, i) \leq -\lambda_1|\phi(0)|^2 + \lambda_2 \int_{-r}^{0} |\phi(t)|^2 \mu(dt)$$

for some probability measure μ *in* $[-r, 0]$. *Then,*

$$\lim_{t\to\infty} \frac{\ln \mathbb{E}_{\phi,i}|X(t)|^2}{t} \leq -\frac{\lambda}{1 \vee c_1} \tag{3.2}$$

for any $\lambda > 0$ *satisfying that* $\lambda_2 \int_{-r}^{0} e^{-\lambda s}\mu(ds) + \lambda \leq \lambda_1$. *Since* $\lambda_1 > \lambda_2$, *such* λ *always exists.*

Proof Let

$$\mathcal{V}(\phi, i) = V(\phi(0), i) + \lambda_2 \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\lambda(u-s)}|\phi(u)|^2(du)$$

Then, Eq. 2.10 implies

$$\begin{aligned}
\mathbb{L}\mathcal{V}(\phi, i) &= \mathcal{L}V(\phi, i) + \lambda_2|\phi(0)|^2 \int_{-r}^{0} e^{-\lambda s}\mu(ds) - \lambda_2 \int_{-r}^{0} |\phi(s)|^2\mu(ds) - \lambda\lambda_2 \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\lambda(u-s)}|\phi(u)|^2(du) \\
&\leq -\left(\lambda_1 - \lambda_2 \int_{-r}^{0} e^{-\lambda s}\mu(ds)\right)|\phi(0)|^2 - \lambda\lambda_2 \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\lambda(u-s)}|\phi(u)|^2(du) \\
&\leq -\frac{\lambda}{c_1} V(\phi(0)) - \lambda\lambda_2 \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\lambda(u-s)}|\phi(u)|^2(du) \\
&\leq -\frac{\lambda}{1 \vee c_1} \mathcal{V}(\phi, i).
\end{aligned}$$

Then, standard arguments of Lyapunov methods show that

$$\lim_{t\to\infty} \frac{\ln \mathbb{E}_{\phi,i} \mathcal{V}(X_t, \alpha(t))}{t} \leq -\frac{\lambda}{1 \vee c_1},$$

which together with Eq. 3.1 leads to Eq. 3.2. □

Razumikhin's methods for SFDEs with regime-switching usually require uniform (in switching states) estimates for $\mathcal{L}V(\phi, i)$, that seems to be very restrictive (see e.g., [11, 14]). The following theorem allows us to relax this condition so as to have different estimates in different switching states.

Theorem 3.2 *Let* $V : \mathbb{R}^{n_0} \mapsto \mathbb{R}_+$ *be a* C^2 *function satisfying*

$$c_1|x|^2 \leq V(x) \leq c_2|x|^2, \ \forall x \in \mathbb{R}^{n_0} \text{ for some } c_1, c_2 > 0. \tag{3.3}$$

Suppose that there exist $\gamma > 0$, $b \geq 0$, $a(i) \in \mathbb{R}$, $p_0 \in (0, \frac{1}{2})$, *and a probability measure* μ *on* $[-r, 0]$ *such that*

$$\mathcal{L}_i V(\phi) - \frac{\left(\frac{1}{2} - p_0\right)|V_x(\phi(0))g(\phi, i)|^2}{V(\phi(0)) + b\int_{-r}^{0} \mu(ds)\int_{s}^{0} e^{\gamma(u-s)}V(\phi(u))(du)} \leq a(i)V(\phi(0)) + b\int_{-r}^{0} V(\phi(t))\mu(dt), \tag{3.4}$$

for $\phi \not\equiv 0$ and that

$$-\Lambda := \sum_{i\in\mathcal{M}} \left((-\gamma) \vee \left(a(i) + b\int_{-r}^{0} e^{-\gamma t}\mu(dt) \right) \right) \nu_i < 0. \tag{3.5}$$

Then

$$\mathbb{P}_{\phi,i} \left\{ \limsup_{t\to\infty} \frac{\ln |X(t)|}{t} \le -\frac{\Lambda}{2} \right\} = 1,\ (\phi, i) \in \mathcal{C} \times \mathcal{S}, \tag{3.6}$$

and for sufficiently small $p > 0$,

$$\limsup_{t\to\infty} \frac{\mathbb{E}_{\phi,i} V^p(X(t))}{t} \le -\frac{p\Lambda}{2},\ (\phi, i) \in \mathcal{C} \times \mathcal{S}. \tag{3.7}$$

Remark 3.1 Another distinctive feature of Theorem 3.2 is the appearance of the negative term $-\left(\frac{1}{2} - p_0\right) \frac{|V_x(\phi(0))g(\phi,i)|^2}{V(\phi(0))+b\int_{-r}^{0}\mu(ds)\int_s^0 e^{\gamma(s-u)}V(\phi(u))(du)}$ in Eq. 3.4, which is normally not shown in the traditional stability analysis. Not only does it improve existing stability conditions but it also indicates that the diffusion term can stabilize the system which, in practice, is not shown using Lyapunov functions and Razumikhin's methods. This fact will be illustrated in Example 3.3.

Proof Let

$$\mathcal{V}(\phi) = V(\phi(0)) + b\int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\gamma(u-s)} V(\phi(u))(du).$$

By the functional Itô formula, we have for $X_t \not\equiv 0$ that

$$\begin{aligned} d\mathcal{V}(X_t) = &\Big(\mathcal{L}_{\alpha(t)} V(X(t)) - \gamma b \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\gamma(u-s)} V(X_t(u))(du) \\ &+ b\int_{-r}^{0} e^{-\gamma s} \mu(ds) V(X_t(s)) - b\int_{-r}^{0} V(X_t(s)) d\mu(s) \Big) dt \\ &+ V_x(X(t)) g(X_t, \alpha(t)) dW(t), \end{aligned}$$

which leads to

$$\begin{aligned} d\ln\mathcal{V}(X_t) = &\frac{1}{\mathcal{V}(X_t)} \Big(\mathcal{L}_{\alpha(t)} V(X_t) + \gamma b \int_{-r}^{0} \mu(ds) \int_{s}^{0} e^{\gamma(u-s)} V(X_t(u))(du) \\ &+ b\int_{-r}^{0} e^{-\gamma s} \mu(ds) V(X(t)) - b\int_{-r}^{0} V(X_t(s)) d\mu(s) \Big) dt \\ &- \frac{|V_x(X(t))g(X_t,\alpha(t))|^2}{2\mathcal{V}^2(X_t)} dt + \frac{V_x(X(t))g(X_t,\alpha(t))dW(t)}{\mathcal{V}(X_t)} \\ \le &\left[\left(a(\alpha(t)) + b\int_{-r}^{0} e^{-\gamma t}\mu(dt) \right) \vee (-\gamma) \right] dt \\ &- p_0 \frac{|V_x(X(t))g(X_t,\alpha(t))|^2}{\mathcal{V}^2(X_t)} dt + \frac{V_x(X(t))g(X_t,\alpha(t))dW(t)}{\mathcal{V}(X_t)}. \end{aligned} \tag{3.8}$$

Let $\tau^* = \inf\{t : X_t \equiv 0\}$. By the exponential martingale inequality (see e.g., [12]), we have

$$\mathbb{P}\left\{\int_0^t \frac{V_x(X(t))g(X_s,\alpha(t))dW(s)}{\mathcal{V}(X_s)} - p_0\int_0^t \frac{|V_x(X(t))g(X_s,\alpha(s))|^2}{2\mathcal{V}^2(X_s)}ds \geq \frac{\ln n}{p_0},\ \text{for some } t\in[0,\tau^*\wedge n)\right\} \leq n^{-2}.$$

Since $\sum_{n=1}^{\infty} n^{-2} < \infty$, it follows from the Borel-Cantelli lemma that with probability 1, there exists a random integer $n^* = n^*(\omega)$ such that for any $k > n^*$,

$$\int_0^t \frac{V_x(X(s))g(X_s,\alpha(s))dW(s)}{\mathcal{V}(X_s)} - p_0\int_0^t \frac{|V_x(X(s))g(X_s))|^2}{2\mathcal{V}^2(X_s)}ds < \frac{\ln k}{p_0} \text{ for all } t\in[0,k\wedge\tau^*).$$

As a result, for almost every $\omega \in \{\tau^* = \infty\}$, we have

$$\int_0^t \frac{V_x(X(s))g(X_s,\alpha(s))dW(s)}{\mathcal{V}(X(s))} - p_0\int_0^t \frac{|V_x(X(s))g(X_s)|^2}{2\mathcal{V}^2(X_s)}ds < \frac{\ln k}{p_0}\ \forall t\in[k-1,k), k>n^*. \tag{3.9}$$

Since $\lim_{k\to\infty}\frac{\ln k}{k-1} = 0$, we deduce from Eq. 3.8 that

$$\limsup_{t\to\infty}\frac{1}{t}\left[\int_0^t \frac{V_x(X(s))g(X_s,\alpha(s))dW(s)}{\mathcal{V}(X_s)} - p_0\int_0^t \frac{|V_x(X(s))g(X_s,\alpha(s))|^2}{2\mathcal{V}^2(X_s)}ds\right] \leq 0 \tag{3.10}$$

for almost every $\omega \in \{\tau^* = \infty\}$. Combining (3.8) and (3.10) yields

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln\mathcal{V}(X(t))}{t} &\leq \limsup_{t\to\infty}\frac{1}{t}\int_0^t(-\gamma)\vee\left(a(\alpha(s)) + b\int_{-r}^0 e^{-\gamma u}\mu(du)\right)ds \\ &\leq \sum_{i\in\mathcal{S}}\widetilde{a}(i)\nu_i \text{ for almost every } \omega\in\{\tau^*=\infty\},\end{aligned} \tag{3.11}$$

where

$$\widetilde{a}(i) = (-\gamma)\vee\left(a(i) + b\int_{-r}^0 e^{-\gamma u}\mu(du)\right).$$

Because of the uniqueness of the solution, $X(t) = 0$, $t \geq \tau^*$ for almost all $\omega \in \{\tau^* < \infty\}$. This together with Eq. 3.11 proves (3.6).

Similar to Eq. 3.8, we can show for $\mathcal{V}^p(\phi) = (\mathcal{V}(\phi))^p$ that

$$\mathbb{L}_i\mathcal{V}^p(\phi) \leq \widetilde{a}(i)p\mathcal{V}^p(\phi) \text{ for } \phi\neq 0,\ p\in(0,p_0).$$

Since $\sum_{i\in\mathcal{S}}(\Lambda + \widetilde{a}(i))\,\nu_i = 0$ (due to Eq. 3.5), an application of the Fredholm alternative (see e.g., [9]) is the existence of γ_i such that

$$\sum_{j\in\mathcal{S}} q_{ij}\gamma_j = -\left(\Lambda + \widetilde{a}(i)\right).$$

Let $p \in (0, p_0)$ be sufficiently small such that

$$2p\gamma_i < 1,\ \text{and } p\gamma_j\widetilde{a}(i) < \frac{\Lambda}{2},\ \text{ for } i\in\mathcal{S}.$$

We have

$$\sum_{j\in\mathcal{S}} q_{ij}(1-p\gamma_j) = -p\sum_{j\in\mathcal{S}} q_{ij}\gamma_j = -p\widetilde{a}(i),\ i\in\mathcal{S}.$$

Let $\mathcal{U}(\phi, i) = (1 - p\gamma_i)\mathcal{V}^p(\phi)$. We have that

$$\begin{aligned}\mathbb{L}\mathcal{U}(\phi, i) =&(1 - p\gamma_j)\mathbb{L}_i\mathcal{V}^p(\phi) + \mathcal{V}^p(\phi)\sum_{j\in\mathcal{S}} q_{ij}(1 - p\gamma_j)\\ \le& p(1 - p\gamma_j)\widetilde{a}(i)\mathbb{L}_i\mathcal{V}^p(\phi) + \mathcal{V}^p(\phi)\sum_{j\in\mathcal{S}} q_{ij}(1 - p\gamma_j)\\ \le& - p\Lambda\mathcal{V}^p(\phi) - p^2\gamma_j\widetilde{a}(i)\mathcal{V}^p(\phi)\\ \le& - \frac{p\Lambda}{2}\mathcal{V}^p(\phi) \le -\frac{p\Lambda}{2}\mathcal{U}^p(\phi), \phi \ne 0.\end{aligned}$$

Then, standard arguments show that

$$\limsup_{t\to\infty}\frac{\ln \mathbb{E}_{\phi,i}\mathcal{U}(X_t, \alpha(t))}{t} \le -\frac{p\Lambda}{2}.$$

Consequently, Eq. 3.7 follows. □

Example 3.1 Consider a scalar switching system

$$dX(t) = \left(a(\alpha(t))X(t) + \sum_{j=1}^{m} bX(t - r_j)\right)dt, \text{ for } r_j \le r.$$

Let $V(x) = x^2$. Then

$$\begin{aligned}\mathcal{L}_i V(\phi) &= 2a(i)(\phi(0))^2 + 2a(i)b\phi(0)\sum_{j=1}^{m}\phi(-r_j)\\ &\le (2a(i) + a^2(i))(\phi(0))^2 + b^2m\sum_{j=1}^{m}(\phi(-r_j))^2\\ &\le (2a(i) + a^2(i))(\phi(0))^2 + b^2m^2\int_{-r}^{0}(\phi(s))^2\mu(ds),\end{aligned} \tag{3.12}$$

where $\mu(\cdot) = \frac{1}{m}\sum_{j=1}^{m}\boldsymbol{\delta}_j(\cdot)$ and $\boldsymbol{\delta}_j$ is the Dirac measure with mass at $-r_j$. As a result, a condition for almost sure stability is $-\gamma = \inf\{2a(i) + a^2(i) + b^2m^2 : i \in \mathcal{S}\} < 0$ and

$$\sum_{i\in\mathcal{S}}(2a(i) + a^2(i))\nu_i + b^2m^2\int_{-r}^{0}e^{-\gamma u}\mu(du) = \sum_{i\in\mathcal{S}}(2a(i) + a^2(i))\nu_i + b^2m\sum_{j=1}^{m}e^{\gamma r_j} < 0.$$

Example 3.2 Consider a linear stochastic delay differential equation with regime-switching of the form:

$$dX(t) = A(\alpha(t))X(t)dt + \sum_{j=1}^{d}C_j(\alpha(t))X(t - r)dW_j. \tag{3.13}$$

Let $\Sigma(i) = \sum_{j=1}^{d}C_j^\top(i)C_j(i)$. For a symmetric matrix $D \in \mathbb{R}^{n_0\times n_0}$, let

$$\Lambda^M(D) = \sup\{x^\top Dx : x \in \mathbb{R}^{n_0}, |x| = 1\}.$$

Let

$$a = \sum_{i\in\mathcal{S}}\left(\Lambda^M(A^\top(i) + A(i))\right)\nu(i),\ \ b = \sum_{i\in\mathcal{S}}\left(\Lambda^M(\Sigma(i))\right)^2\nu(i),$$

and $\gamma = -\inf\{\Lambda^M(A^\top(i) + A(i)) : i \in \mathcal{S}\}$. We show that, the condition for almost sure stability and L^p-stability with a small p of the system is

$$a < 0, \text{ and } a^2 > be^{\gamma r}. \tag{3.14}$$

Indeed, under that condition, $\gamma > 0$ (since $a < 0$) and we can find $c > 0$ such that $c^2 + ac + \dfrac{be^{\gamma r}}{4} < 0$ or equivalently,

$$a + \frac{be^{\gamma r}}{4c} \le -c. \tag{3.15}$$

We have the estimate that

$$\begin{aligned} x^\top \Sigma(i) y \le& \frac{\Lambda^M(\Sigma(i))}{4c} x^\top \Sigma(i) x + \frac{c}{\Lambda^M(\Sigma(i))} y^\top \Sigma(i) y \\ \le& \frac{\left(\Lambda^M(\Sigma(i))\right)^2}{4c} |x|^2 + c|y|^2 \end{aligned}$$

Let $V(x) = |x|^2$. Then we have

$$\begin{aligned} \mathcal{L}_i V(\phi) =& (\phi(0))^\top (A^\top(i) + A(i))\phi(0) + \phi(-r)\Sigma(i)(\phi(-r))^\top \\ \le& (\phi(0))^\top (A^\top(i) + A(i))\phi(0) + \frac{\left(\Lambda^M(\Sigma(i))\right)^2}{4c} |\phi(0)|^2 + c|\phi(-r)|^2 \\ \le& \left(\Lambda^M(A^\top(i) + A(i)) + \frac{\left(\Lambda^M(\Sigma(i))\right)^2}{4c}\right) |\phi(0)|^2 + c|\phi(-r)|^2 \end{aligned}$$

In view of Theorem 3.2 and Eqs. 3.15 and 3.13 is exponentially stable almost surely under (3.14).

Example 3.3 Let $r < \frac{1}{4}$ and $0 < \delta < \frac{1}{2}$ and consider the scalar equation without switching.

$$dX(t) = X(t)dt + 2\left(X^2(t) + \delta \int_{-r}^0 |X(t+s)|^2 ds\right)^{-\frac{1}{2}} dW(t).$$

Consider $V(x) = x^2$. We have $\mathcal{L}V(\phi) = 2\phi^2(0) + 4(\phi^2(0) + \delta \int_{-r}^0 \phi^2(s)ds)$. It is easy to show that there are no $\lambda_1 > \lambda_2 \ge 0$ and an probability measure μ such that

$$\mathcal{L}V(\phi) \le -\lambda_1 \phi^2(0) + \lambda_2 \int_{-r}^0 \phi(s)\mu(ds).$$

Thus, Razumikhin's method does not work for the Lyapunov function $V(x) = x^2$ in this example.

On the other hand, choose $\gamma > 0$ to be determined. We have

$$\begin{aligned} \frac{\phi^2(0)\left(\phi^2(0) + \delta \int_{-r}^0 \phi^2(s)ds\right)}{\phi^2(0) + 4\delta \int_{-r}^0 ds \int_s^0 e^{\gamma(s-u)}|\phi(u)|^2 du} =& \phi^2(0) \frac{\phi^2(0) + \delta \int_{-r}^0 \phi^2(s)ds}{\phi^2(0) + 4\delta \int_{-r}^0 ds \int_s^0 e^{\gamma(u-s)}\phi^2(u)du} \\ \ge& \phi^2(0) \frac{\phi^2(0) + \delta \int_{-r}^0 \phi^2(s)ds}{\phi^2(0) + 4r\delta e^{\gamma r} \int_{-r}^0 |\phi(s)|^2 ds} \\ \ge& \phi^2(0) \text{ if } 4re^{\gamma r} < 1. \end{aligned}$$

As a result,

$$\begin{aligned}\mathcal{L}V(\phi)-(\frac{1}{2}-p_0)&\frac{16\phi^2(0)\left(4(\phi^2(0)+\delta\int_{-r}^0\phi^2(s)ds)\right)}{\phi^2(0)+4\delta\int_{-r}^0 ds\int_s^0 e^{\gamma(s-u)}|\phi(u)|^2du}\\ &\le -(8-2p_0)\phi^2(0)+6\phi^2(0)+4\delta\int_{-r}^0\phi^2(s)ds\\ &\le -(2-2p_0)\phi^2(0)+4\delta\int_{-r}^0\phi^2(s)ds \ \text{ if } 4re^{\gamma r}<1.\end{aligned}$$

Since $4\delta < 2$, and $\lim_{\gamma\to 0}\frac{e^{\gamma r}-1}{\gamma}=1$, we can choose γ and p_0 sufficiently small that

$$2-2p_0-4\delta\int_{-r}^0 e^{-\gamma s}ds = 2-2p_0-4\delta\frac{e^{\gamma r}-1}{\gamma}<0 \text{ and } 4re^{\gamma r}<1.$$

With such p_0 and γ, the system is exponentially stable with probability 1 because Eqs. 3.4 and 3.5 are satisfied with $b=4\delta$ and $a=2-2p_0$. Note that without the diffusion term, the system is unstable. Thus, our theorem shows that the diffusion term can stabilize the system, which seems to be impossible to obtain using the Lyapunov function approach.

4 Stability of Systems with Small Delay

This section addresses the following equation. Suppose we have a SFDE with regime switching

$$dX(t)=\widehat{f}(X_t(t),\alpha(t))dt+\widehat{g}(X_t,\alpha(t))dW(t). \tag{4.1}$$

Suppose further that the system

$$dX(t)=\widehat{f}(\phi_c(X(t)),\alpha(t))dt+\widehat{g}(\phi_c(X_t),\alpha(t))dW(t).$$

is stable where $\phi_c(X(t))$ satisfies $\phi_c(X(t))(s)\equiv X(t)$, $s\in[-r,0]$. An immediate question comes up. Is Eq. 4.1 stable when r is sufficiently small? For instance, if the scalar equation

$$dX(t)=\Big(a(\alpha(t))-b(\alpha(t)\Big)X(t)dt+\sigma(\alpha(t))X(t)dW(t)$$

is almost surely stable, we expect that when τ is sufficiently small,

$$dX(t)=(a(\alpha(t))X(t)-b(\alpha(t)X(t-r))dt+\sigma(\alpha(t))X(t)dW(t)$$

is almost surely stable.

We rewrite Eq. 4.1 in the form

$$dX(t)=\left(f(X(t),\alpha(t))+\widetilde{f}(X_t,\alpha(t))\right)dt+\left(g(X(t),\alpha(t))+\widetilde{g}(X_t,\alpha(t))\right)dW(t), \tag{4.2}$$

where $\widetilde{f}$ and $\widetilde{g}$ satisfy

$$|\widetilde{f}(\phi)|^2\le c_{\widetilde{f}}^2\int_{-r}^0\mu(ds)|\phi(0)-\phi(u)|^2du \quad\text{and}\quad |\widetilde{g}(\phi)|^2\le c_{\widetilde{g}}^2\int_{-r}^0\mu(ds)|\phi(0)-\phi(u)|^2du, \tag{4.3}$$

for some probability measure $\mu(\cdot)$.

Suppose that $f(x,i)$ and $g(x,i)$ are locally Lipschitz and

$$|f^\top(x,i)|\le c_f|x|, \ \text{ and } |g(x,i)|\le c_g|x|. \tag{4.4}$$

$$a_i=\sup_{x\ne 0}\left\{\frac{2x^\top f(x,i)+\text{tr}(g^\top(x,i)g(x,i))-2|x^\top g|^2}{|x|^2}\right\}<\infty \tag{4.5}$$

We impose the condition $\sum_{i\in\mathcal{S}} a_i\nu_i < 0$ for almost sure stability of the SDE with regime switching

$$dX(t) = f(X(t), \alpha(t))dt + g(X(t), \alpha(t))dW(t). \tag{4.6}$$

It can be shown in the proof of Theorem 4.1 (or see [9, 20]) that if $\sum_{i\in\mathcal{S}} a_i\nu_i < 0$, then Eq. 4.6 is exponentially stable almost surely.

Theorem 4.1 *Suppose that* Eqs. 4.3, 4.4, and 4.5 *hold. If* $-\Lambda := \sum_{i\in\mathcal{S}} a_i\nu_i < 0$, *then for any* $\widetilde{\Lambda} < \Lambda$, *there exists an* $r^* > 0$ *such that any solutions of* Eq. 4.2 *satisfy*

$$\mathbb{P}_{\phi,i}\left\{\frac{\ln|X(t)|^2}{t} \le -\widetilde{\Lambda}\right\} = 1$$

if $r \le r^*$.

Proof Let c_{tr} be a universal constant such that $\operatorname{tr} G^\top\widetilde{G} \le c_{\mathrm{tr}}|G||G|$ for any $G, \widetilde{G} \in \mathbb{R}^{n_0\times d}$. Let $\overline{a} = -\min_{i\in\mathcal{S}}\{a_i\} > 0$. Let $\varepsilon_0 = \Lambda - \widetilde{\Lambda}$ and $M > 0$, $\lambda > 0$ given by

$$\frac{8c_{\widetilde{f}}^2 + 8c_{\mathrm{tr}}c_g c_{\widetilde{g}}}{\varepsilon_0} + 2c_{\mathrm{tr}}c_{\widetilde{g}}^2 + c_f^2 + 6c_{\widetilde{g}}^2 + \overline{a} + \frac{\varepsilon_0}{16} = M; \quad \lambda = \overline{a} + \frac{\varepsilon_0}{16} + \frac{\varepsilon_0}{16}2c_g^2 + 2. \tag{4.7}$$

Consider the Lyapunov functional

$$U(\phi) = M\int_{-r}^0 \mu(ds)\int_s^0 e^{\lambda(u-s)}|\phi(0) - \phi(u)|^2 du.$$

Computing either directly from the definition or by expanding $|\phi(0) - \phi(u)|^2$ and then using Eq. 2.10 and Itô's formula for a product, we can obtain the Dupire derivatives of $U(\phi)$, which are given as follows

$$U_t(\phi) = -M\int_{-r}^0 |\phi(0) - \phi(s)|^2\mu(ds) - \lambda U(\phi).$$

$$U_{x_i}(\phi) = 2\int_{-r}^0 \mu(ds)\int_s^0 e^{\lambda(u-s)}(\phi_i(0) - \phi_i(u))du,\ i = 1, \dots, n_0.$$

$$U_{x_ix_i}(\phi) = 2\phi_i(0)\int_{-r}^0 \mu(ds)\int_s^0 e^{\lambda(u-s)}du = 2c_{\lambda,r}\phi_i(0); \quad U_{x_ix_j}(\phi) = 0 \text{ for } i \ne j,$$

where

$$c_{\lambda,r} := \int_{-r}^0 \mu(ds)\int_s^0 e^{\lambda(u-s)}du \le \frac{\exp(\lambda r) - 1}{\lambda} \to 0 \text{ as } r \to 0.$$

By the Cauchy-Schwarz inequality, we obtain the estimate

$$\begin{aligned}
|U_x|^2 =& 4M^2\sum_{i=1}^{n_0}\left(\int_{-r}^0 \mu(ds)\int_s^0 e^{\lambda(u-s)}(\phi_i(0) - \phi_i(u))du\right)^2 \\
\le& 4M^2\int_{-r}^0 \mu(ds)\sum_{i=1}^{n_0}\left(\int_s^0 e^{\lambda(u-s)}(\phi_i(0) - \phi_i(u))du\right)^2 \\
\le& 4M^2\int_{-r}^0 \mu(ds)\int_s^0 e^{\lambda(u-s)}du\int_s^0 e^{\lambda(u-s)}\sum_{i=1}^{n_0}(\phi_i(0) - \phi_i(u))^2 du \\
\le& 4Mc_{\lambda,r}U(\phi).
\end{aligned} \tag{4.8}$$

Letting $\mathcal{V}(\phi) = |\phi(0)|^2 + U(\phi)$, we have

$$\begin{aligned}
\mathbb{L}_i\mathcal{V} =& \frac{2x^\top(f+\widetilde{f})+\text{tr}[(g+\widetilde{g})^\top(g+\widetilde{g})-MQ-\lambda U+U_x(f+\widetilde{f})+Mc_{\lambda,r}\,\text{tr}[(g+\widetilde{g})^\top(g+\widetilde{g})]}{|x|^2+U}\\
&-\frac{|(2x^\top+U_x)(g+\widetilde{g})|^2}{2(|x|^2+U)^2}\\
=& \frac{2x^\top f+\text{tr}(g^\top g)-\overline{a}Q-\overline{a}U+\varepsilon_0|x|^2}{|x|^2+U}-\frac{|2x^\top g|^2}{2|x|^2(|x|^2+U)}+\frac{|2x^\top g|^2}{2|x|^2(|x|^2+U)}\\
&-\frac{|(2x^\top+U_x)(g+\widetilde{g})|^2}{2(|x|^2+U)^2}+\frac{-\varepsilon_0|x|^2-(M-\overline{a})Q-(\lambda-\overline{a})U}{|x|^2+U}\\
&+\frac{2x^\top\widetilde{f}+Mc_{\lambda,r}\,\text{tr}[g^\top g]+(1+Mc_{\lambda,r})(\text{tr}[2g^\top\widetilde{g}+\widetilde{g}^\top\widetilde{g}])+U_x(f+\widetilde{f})}{|x|^2+U}.
\end{aligned}$$

In the formula, $x=\phi(0)$, $Q=\int_{-r}^0|\phi(0)-\phi(s)|^2\mu(ds)$, the variables $(x,i)=(\phi(0,i)$ in f, g and the variables (ϕ,i) in $\widetilde{f}$, $\widetilde{g}$, U_x, U are dropped for sake of notational simplicity. If $x=0$ and $U\neq 0$, then $f(x,i)=0$ and $g(x,i)=0$. We have

$$\begin{aligned}
\mathbb{L}_i\mathcal{V} =& \frac{(1+Mc_{\lambda,r})\,\text{tr}(\widetilde{g}^\top g)-MQ-\lambda U}{U}-\frac{|U_x\widetilde{g}|^2}{2U^2}\\
\le& \frac{(1+Mc_{\lambda,r})c_{\text{tr}}c_{\widetilde{g}}^2Q-MQ-\lambda U}{U}-\frac{|U_x\widetilde{g}|^2}{2U^2}\\
\le& -\overline{a}-\frac{|U_x\widetilde{g}|^2}{2U^2},
\end{aligned}\tag{4.9}$$

if

$$Mc_{\lambda,r}<1.\tag{4.10}$$

If $x\neq 0$, we have

$$\begin{aligned}
\mathbb{L}_i\mathcal{V} =& \frac{2x^\top f+\text{tr}[g^\top g)-(\overline{a}+\frac{\varepsilon_0}{16})Q-(\overline{a}+\frac{\varepsilon_0}{16})U+\varepsilon_0|x|^2}{|x|^2+U}-\frac{|2x^\top g|^2}{2|x|^2(|x|^2+U)}\\
&+\frac{|2x^\top g|^2}{2|x|^2(|x|^2+U)}-\frac{|(2x^\top+U_x)(g+\widetilde{g})|^2}{2(|x|^2+U)^2}\\
&-\frac{\frac{\varepsilon_0}{16}(|x|^2+Q+U)}{|x|^2+U}-\frac{\frac{15\varepsilon_0}{16}|x|^2+(M-\overline{a})Q+(\lambda-\overline{a})U}{|x|^2+U}\\
&+\frac{2x^\top\widetilde{f}+Mc_{\lambda,r}\,\text{tr}[g^\top g]+(1+Mc_{\lambda,r})(\text{tr}[2g^\top\widetilde{g}+\widetilde{g}^\top\widetilde{g}])+U_x(f+\widetilde{f})}{|x|^2+U}.
\end{aligned}\tag{4.11}$$

We have the following estimates:

$$
\begin{aligned}
&2|x^\top \widetilde{f}| \le \varepsilon_1 |x|^2 + \varepsilon_1^{-1}|\widetilde{f}|^2 \le \frac{\varepsilon_0}{8}|x|^2 + \frac{8c_{\widetilde{f}}^2}{\varepsilon_0}|Q|,\\
&Mc_{\lambda,r}\,\mathrm{tr}[g^\top g] \le Mc_{\mathrm{tr}}c_{\lambda,r}c_g^2|x|^2 \le \frac{\varepsilon_0}{16}|x|^2,\\
&(1+Mc_{\lambda,r})(\mathrm{tr}[2g^\top \widetilde{g}) \le 4\,\mathrm{tr}[g^\top \widetilde{g}] \le 4c_{\mathrm{tr}}|g||\widetilde{g}| \le \frac{\varepsilon_0}{8}|x|^2 + \frac{8c_{\mathrm{tr}}c_g c_{\widetilde{g}}}{\varepsilon_0}Q,\\
&(1+Mc_{\lambda,r})\,\mathrm{tr}[\widetilde{g}^\top \widetilde{g}] \le 2c_{\mathrm{tr}}c_{\widetilde{g}}^2 Q,\\
&U_x f \le \frac{\varepsilon_0}{8}|x|^2 + \frac{2c_f^2|U_x|^2}{\varepsilon_0} \le \frac{\varepsilon_0}{8}|x|^2 + \frac{8Mc_f^2 c_{\lambda,r}U}{\varepsilon_0},\\
&U_x \widetilde{f} \le |\widetilde{f}| + \frac{|U_x|^2}{4} \le c_{\widetilde{f}}^2 Q + Mc_{\lambda,r}U,
\end{aligned}
\tag{4.12}
$$

if

$$
Mc_{\mathrm{tr}}c_{\lambda,r}c_g^2 \le \frac{\varepsilon_0}{16}, \text{ and } Mc_{\lambda,r} \le 1. \tag{4.13}
$$

Then Eq. 4.12 and 4.7 lead to

$$
\begin{aligned}
&\frac{-\varepsilon_0|x|^2 - (M-\overline{a})Q - (\lambda-\overline{a})U}{|x|^2+U}\\
&\qquad + \frac{2x^\top \widetilde{f} + Mc_{\lambda,r}\,\mathrm{tr}[g^\top g] + (1+Mc_{\lambda,r})(\mathrm{tr}[2g^\top \widetilde{g} + \widetilde{g}^\top \widetilde{g}]) + U_x(f+\widetilde{f})}{|x|^2+U}\\
&\quad \le \frac{-0.5\varepsilon_0|x|^2 - 6c_{\widetilde{g}}^2 Q - (2c_g^2+1)U}{|x|^2+U}
\end{aligned}
\tag{4.14}
$$

if

$$
\frac{8Mc_f^2 c_{\lambda,r}}{\varepsilon_0} + c_{\lambda,r} \le 1, \text{ and } 8Mc_{\mathrm{tr}}c_{\lambda,r}c_g^2 \le \varepsilon_0. \tag{4.15}
$$

As a result,

$$
\begin{aligned}
&\frac{|2x^\top g|^2}{2|x|^2(|x|^2+U)} - \frac{|(2x^\top + U_x)(g+\widetilde{g})|^2}{2(|x|^2+U)^2}\\
&\quad = \frac{|2x^\top g|^2 - |(2x^\top + U_x)(g+\widetilde{g})|^2}{2(|x|^2+U)^2} + \frac{|2x^\top g|^2 U}{2|x|^2(|x|^2+U)^2}\\
&\quad \le \frac{|2x^\top \widetilde{g} + U_x(g+\widetilde{g})|^2}{2(|x|^2+U)^2} + \frac{|2x^\top g|^2 U}{2|x|^2(|x|^2+U)^2}\\
&\quad \le \frac{3|2x^\top \widetilde{g}|^2 + 3|U_x g|^2 + 3|U_x \widetilde{g}|^2}{2(|x|^2+U)^2} + \frac{4c_g^2 U}{2(|x|^2+U)^2}\\
&\quad \le \frac{6c_{\widetilde{g}}^2|x|^2 Q + 6Mc_{\lambda,r}c_g^2|x|^2 U + 6Mc_{\lambda,r}c_{\widetilde{g}}^2 QU}{(|x|^2+U)^2} + \frac{4c_g^2 U}{2(|x|^2+U)^2}\\
&\quad \le \frac{6c_{\widetilde{g}}^2|x|^2 Q + 6Mc_{\lambda,r}c_g^2|x|^2 U + 6Mc_{\lambda,r}c_{\widetilde{g}}^2 QU}{(|x|^2+U)^2} + \frac{2c_g^2 U}{(|x|^2+U)},
\end{aligned}
\tag{4.16}
$$

which implies

$$\begin{aligned}
&\frac{-0.5\varepsilon_0|x|^2-6c_g^2Q-(2c_g^2+1)U}{|x|^2+U}+\frac{|2x^\top g|^2}{2|x|^2(|x|^2+U)}-\frac{|(2x^\top+U_x)(g+\widetilde{g})|^2}{2(|x|^2+U)^2}\\
&\quad\le\frac{-\Big(0.5\varepsilon_0|x|^2+6c_{\widetilde{g}}^2Q+U\Big)(|x|^2+U)+\Big(6c_{\widetilde{g}}^2|x|^2Q+6Mc_{\lambda,r}c_g^2|x|^2U+6Mc_{\lambda,r}c_{\widetilde{g}}^2QU\Big)}{(|x|^2+U)^2}\\
&\quad\le\frac{-\Big(0.5\varepsilon_0-6Mc_g^2c_{\lambda,r}\Big)|x|^2U-6(c_{\widetilde{g}}^2-Mc_{\lambda,r}c_{\widetilde{g}}^2)QU}{(|x|^2+U)^2}\\
&\quad\le 0,
\end{aligned}\tag{4.17}$$

if

$$0.5\varepsilon_0\ge 6Mc_g^2c_{\lambda,r}\ \text{ and }\ 1\ge Mc_{\lambda,r}.\tag{4.18}$$

Applying Eqs. 4.14 and 4.17) to Eq. 4.11 and then combining with Eq. 4.9, we have

$$\mathbb{L}_i\ln\mathcal{V}\le(a_i-\varepsilon_0)-\frac{\varepsilon_0(|x|^2+Q+U)}{16(|x|^2+U)}.$$

As a result, for $t<\tau_*:=\inf\{s\ge0:\mathcal{V}(X_s)=0\}$, we have

$$\begin{aligned}
\ln\mathcal{V}(X_t)&=\ln\mathcal{V}(X_0)+\int_0^t\mathbb{L}_{\alpha(s)}\ln V(X_s,\alpha(s))ds+H(t)\\
&\le\int_0^t(a(\alpha(s))+\varepsilon_0)ds-\int_0^t\frac{\varepsilon_0(|X(s)|^2+Q(X_s)+U(X_s))}{16(|X(s)|^2+U(X_s))}ds+H(t),
\end{aligned}$$

where

$$H(t):=\int_0^t\frac{(2X(s)^\top+U_x(X_s))(g(X_s,i)+\widetilde{g}(X_s,i))dW(s)}{|X(s)|^2+U(X_s)}.$$

By virtue of Eqs. 4.3, 4.4, and 4.8, we have the following inequality for the quadratic variation of $H(t)$

$$\int_0^t\frac{|(2X(s)^\top+U_x(X_s))(g(X_s,i)+\widetilde{g}(X_s,i))|^2}{(|X(s)|^2+U(X_s))^2}\le C\int_0^t\frac{|X(s)|^2+Q(X_s)+U(X_s)}{|X(s)|^2+U(X_s)}ds,\ t<\tau_*$$

for some constant C. As a result, we can apply the exponential martingale inequality and proceed in the same manner as in the proof of Theorem 3.2 to show that

$$\lim_{t\to\infty}\frac{H(t)}{t}-\frac{1}{t}\int_0^t\frac{\varepsilon_0|X(s)|^2+6c_{\widetilde{g}}^2Q(X_s)+(2c_g^2+1)U(X_s)}{16(|X(s)|^2+U(X_s))}ds=0$$

for almost every $\omega\in\{\tau_*=\infty\}$, which implies that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t(a(\alpha(s))+\varepsilon_0)ds=\sum(a(i)+\varepsilon_0)\nu_i=-\widetilde{\Lambda}\text{for almost every }\omega\in\{\tau_*=\infty\}.$$

Thus we have for almost every $\omega\in\{\tau_*=\infty\}$ that

$$\limsup_{t\to\infty}\frac{\ln\mathcal{V}(X(t))}{t}\le\sum(a(i)+\varepsilon_0)\nu_i=-\widetilde{\Lambda}.$$

if Eqs. 4.10, 4.13, 4.15, and 4.18 are satisfied. Let

$$K = \min\left\{\frac{1}{M}, \frac{\varepsilon_0}{16Mc_{\mathrm{tr}}c_g^2}, \frac{\varepsilon}{8Mc_f^2+\varepsilon}, \frac{\varepsilon_0}{8Mc_{\mathrm{tr}}c_g^2}\right\}.$$

Since $c_\lambda \le \dfrac{e^{\lambda r}-1}{\lambda}$, Eqs. 4.10, 4.13, 4.15, and 4.18 are satisfied if $r \le r^*$ where $r^* = \dfrac{\ln(K\lambda+1)}{\lambda}$. The proof is complete by noting that $X(t) = 0, t \ge \tau_*$ almost surely in the event $\{\tau_* < \infty\}$. □

Example 4.1 Consider the system

$$\begin{aligned} dX(t) &= (AX(t) + BX(t-r))\,dt + \sum_{j=1}^{d}\big(C_jX(t) + D_jX(t-r)\big)\,dW_j(t) \\ &= \big(\overline{A}X(t) + B(X(t-r) - X(t))\big)\,dt + \sum_{j=1}^{d}\big(\overline{C}_jX(t) + D_j(X(t-r) - X(t))\big)\,dW_j(t) \end{aligned} \tag{4.19}$$

where $\overline{A} = A + B$, $\overline{C}_j = C_j + D_j$. Let

$$a_i := \inf_{|x|=1}\left\{x^\top\overline{A}x + \sum_{j=1}^{d}\left(\frac{1}{2}(x^\top\sigma_j^\top(i)\sigma_j(i)x) - (x^\top\sigma_j^\top(i)x)\right)\right\}.$$

If $\sum a_i\nu_i < 0$, then the system (4.19) is exponentially stable when r is small enough.

As a consequence, if $a - b - 0.5\sigma^2 < 0$, the scalar system

$$dX(t) = (aX(t) - b(X(t-r)))dt + \sigma X(t-r)dW(t),$$

is exponentially almost surely stable. Thus, the system can be stable even that the leading coefficient a is positive, which seems practically impossible to obtain using Razumikhin's method.

Now, consider the case when the perturbations $\widetilde{f}$ and $\widetilde{g}$ satisfy

$$|\widetilde{f}(\phi)|^2 \le c_{\widetilde{f}}^2\int_{-r}^{0}|\phi(u)|^2du \quad\text{and}\quad |\widetilde{g}(\phi)|^2 \le c_{\widetilde{g}}^2\int_{-r}^{0}|\phi(u)|^2du. \tag{4.20}$$

Theorem 4.2 *Suppose that* Eqs. 4.20, 4.4, and 4.5 *hold. If* $-\Lambda := \sum a_i\nu_i < 0$, *then for any* $\widetilde{\Lambda} < \Lambda$, *there exists an* $r_* > 0$ *such that any solutions of* Eq. 4.2 *satisfy*

$$\mathbb{P}_{\phi,i}\left\{\frac{\ln|X(t)|^2}{t} \le -\widetilde{\Lambda}\right\} = 1$$

if $r \le r_*$.

Proof Let $\varepsilon_0 = \Lambda - \widetilde{\Lambda}$. Define

$$U(\phi) = \frac{\varepsilon_0}{4}\int_{-r}^{0} e^{\overline{a}(s+r)}|\phi(s)|^2ds.$$

Then

$$U_t(\phi) = -\overline{a}U(\phi) + \frac{\varepsilon_0}{4}|\phi(0)|^2e^{-\overline{a}r} - \frac{\varepsilon_0}{4}|\phi(-r)|^2$$

and $U_x(\phi, t) = 0$. Consider the function

$$\mathcal{V}(\phi) = |\phi(0)|^2 + U(\phi).$$

Similar to the proof of Theorem 4.1, when r is sufficiently small, $t < \tau^*$,

$$\ln \mathcal{V}(X_t) \le \ln \mathcal{V}(\phi) + \int_0^t (a_{\alpha(s)} - \varepsilon_0))ds - \int_0^t \frac{0.5\varepsilon_0(|X(s)|^2 + U(X_s))}{|X(s)|^2 + U(X_s)}$$
$$+ \int_0^t \frac{X(s)^\top (g(X(s), \alpha(s)) + g(X_s, \alpha(s)))dW(s)}{|X(s)|^2 + U(X_s)}.$$

Proceeding in the same manner as in the proof of Theorem 4.1, we can obtain the desired results. □

Example 4.2 Consider the scalar equation

$$dX(t) = \left(f(X(t), \alpha(t)) + \int_{-r}^0 \Phi_1(s)X(t+s)ds \right) dt$$
$$+ \left(g(X(t), \alpha(t)) + \int_{-r}^0 \Phi_2(s)X(s+t)ds) \right) dW(t). \quad (4.21)$$

Suppose that $\|\Phi_1\| \wedge \|\Phi_2\| \le K$. then an application of the Cauchy-Schwarz inequality yields

$$\left| \int_{-r}^0 \Phi_m(s)\phi(s)ds \right|^2 \le Kr \int_{-r}^0 |\phi(s)|^2 ds, m = 1, 2.$$

As a result, if we have $\sum_{i\in\mathcal{S}} a_i \nu_i < 0$, where a_i is defined by Eq. 4.5, system (4.21) is stable when r is sufficiently small.

Remark 4.1 Although our conditions for perturbed functions $\widetilde{f}$ and $\widetilde{g}$ are more restrictive than Assumption 2.2 in [25] since we require $\widetilde{f}$ and $\widetilde{g}$ to have some structures in Eqs. 4.3 or 4.20, we do not need the global Lipschitz conditions for f and g like [25, Assumption 2.1]. The conditions (4.5) and $\sum a_i \nu_i < 0$ are also more relaxed than the conditions

$$\sum_{i\in\mathcal{S}} (\alpha_i + 05\rho_i^2 - \sigma_i^2)\nu_i < 0$$

and

$$x^\top f(x, i) \le \alpha_i |x|^2, |g(x, i)| \le \rho_i |x|, |x^\top g(x, i)| \ge \sigma_i |x^2|$$

used in [25].

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