A non local approximation of the Gaussian perimeter: Gamma convergence and Isoperimetric properties

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Abstract

We study a non local approximation of the Gaussian perimeter, proving the Gamma convergence to the local one. Surprisingly, in contrast with the local setting, the halfspace turns out to be a volume constrained stationary point if and only if the boundary hyperplane passes through the origin. In particular, this implies that Ehrhard symmetrization can in general increase the non local Gaussian perimeter taken into consideration.

1 Introduction

The Gaussian isoperimetric inequality says that the halfspace has the smallest Gaussian perimeter among all sets with prescribed Gaussian measure, [4]. In the Euclidean setting, an increasing interest has been devoted to the study of non local approximations of the perimeter and their isoperimetric shapes, since the pioneering work of Caffarelli, Roquejoffre and Savin, [6].

The aim of this paper is to provide an analogous non local approximation of the Gaussian perimeter, showing the Gamma convergence to the local one. Moreover, we study the isoperimetric properties of this non local functional and observe that, in contrast with the local setting, an halfspace is a volume constrained critical point if and only if it has Gaussian measure $\frac{1}{2}$. In particular, we deduce that Ehrhard symmetrization can in general increase the non local Gaussian perimeter taken into consideration.

We remark that the non local approximation of the Gaussian perimeter we study is different from the one recently proposed in [22]. The non local functional we introduce has the advantage of having a more explicit formulation, while it has the drawback that the isoperimetric shapes and the Ehrhard symmetrization are not preserved.

For an extensive description of the main differences between the local and the non local framework, we refer the interested reader to [14]. A discussion about recent Γ-convergence results in the non local setting has been presented in [15].

Inspired by [1], for a measurable set $E \subset \mathbb{R}^n$, $n \geq 1$, $0 < s < 1$, and a connected, open set $\Omega \Subset \mathbb{R}^n$ with Lipschitz boundary (or simply $\Omega = (a, b) \Subset \mathbb{R}$ if $n = 1$), we define the Gaussian, non local functional

$$\mathcal{J}_s^\gamma(E, \Omega) := \mathcal{J}_s^{1,\gamma}(E, \Omega) + \mathcal{J}_s^{2,\gamma}(E, \Omega),$$

where

$$\mathcal{J}_s^{1,\gamma}(E, \Omega) := \int_{E\cap\Omega} \int_{E^c\cap\Omega} \frac{\gamma(x,\, y)}{|x-y|^{n+s}}\, dxdy,$$
$$\mathcal{J}_s^{2,\gamma}(E, \Omega) := \int_{E\cap\Omega} \int_{E^c\cap\Omega^c} \frac{\gamma(x,\, y)}{|x-y|^{n+s}}\, dxdy + \int_{E\cap\Omega^c} \int_{E^c\cap\Omega} \frac{\gamma(x,\, y)}{|x-y|^{n+s}}\, dxdy,$$

and

$$\gamma\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^+,\ \gamma(x,\, y) = \exp\left(-\frac{1}{4}\left(|x|^2 + |y|^2\right)\right).$$

When Ω coincides with the whole space, we just write $\mathcal{J}_s^\gamma(E)$.

In [1], Amborsio, De Philippis and Martinazzi have studied the Euclidean version of it, namely $\mathcal{J}_s = \mathcal{J}_s^1 + \mathcal{J}_s^2$, where

$$\mathcal{J}_s^1(E,\Omega) := \int_{E\cap\Omega}\int_{E^c\cap\Omega} \frac{1}{|x-y|^{n+s}}\,dxdy,$$
$$\mathcal{J}_s^2(E,\Omega) := \int_{E\cap\Omega}\int_{E^c\cap\Omega^c} \frac{1}{|x-y|^{n+s}}\,dxdy + \int_{E\cap\Omega^c}\int_{E^c\cap\Omega} \frac{1}{|x-y|^{n+s}}\,dxdy.$$

The authors point out that $\mathcal{J}_s(E,\Omega)$ can be thought of as a fractional perimeter of E in Ω, and they show the Γ-convergence of $(1-s)\mathcal{J}_s(\cdot,\Omega)$ to $\omega_{n-1}P(\cdot,\Omega)$ as $s\to 1^-$, where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$, $P(E,\,\Omega) := \mathcal{H}^{n-1}(\mathcal{F}E\cap\Omega)$ is the Euclidean perimeter, $\mathcal{H}^\alpha$ denotes the classical α-Hausdorff measure and $\mathcal{F}E$ the reduced boundary of E. Moreover, they prove the convergence of any sequence $\{E_i\}$ of local minimizers for $\mathcal{J}_{s_i}(\cdot,\Omega)$ to a local minimizer for $P(\cdot,\Omega)$, see [1, Theorem 3].

The first aim of this paper is to generalize [1, Theorem 3] to the Gaussian case, thus building a relation between the functional $\mathcal{J}_s^\gamma$ and the Gaussian perimeter

$$P^\gamma(E,\,\Omega) := \int_{\mathcal{F}E\cap\Omega} e^{-\frac{1}{2}|x|^2}\,d\mathcal{H}^{n-1}(x).$$

The second goal is to investigate whether the halfspaces are volume constrained critical points of $\mathcal{J}_s^\gamma$. This turns out to be true if and only if the boundary hyperplane passes through the origin.

The paper is divided in four Sections. In Section 2 we prove the $\Gamma-$convergence of the functional $\mathcal{J}_s^\gamma$ to P^γ. In Section 3 we compute the first and second variation of $\mathcal{J}_s^\gamma$ (for the local framework see [3] or [19]). In Section 4 we prove that halfspaces are volume constrained stationary points for $\mathcal{J}_s^\gamma$ if and only if their Gaussian volume is $\frac{1}{2}$.

2 The Gamma-convergence

In this section we extend [1, Theorem 3] to the Gaussian case. Namely, we show:

Theorem 1 (Convergence of local minimizers) *Assume that $s_i \uparrow 1$, E_i are local minimizers of $\mathcal{J}_{s_i}^\gamma(\cdot,\Omega)$, and $\chi_{E_i}\to\chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\limsup_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}^\gamma(E_i,\Omega') < +\infty \qquad \forall \Omega' \Subset \Omega, \tag{1}$$

E is a local minimizer of $P^\gamma(\cdot,\Omega)$ and $(1-s_i)\mathcal{J}_{s_i}^\gamma(E_i,\Omega')\to\omega_{n-1}P(E,\Omega')$ whenever $\Omega'\Subset\Omega$ and $P(E,\partial\Omega')=0$.

The proof of Theorem 1 is almost identical to the Euclidean one for [1, Theorem 3]. We limit our study to the parts which differ from it. In particular we will prove the following two propositions. Let ω_k denote the volume of the unit ball in $\mathbb{R}^k$ for $k\geq 1$, and set $\omega_0 := 1$.

Proposition 2 *For every measurable set $E \subset \mathbb{R}^n$ we have*

$$\Gamma - \liminf_{s\uparrow 1}(1-s)\mathcal{J}_s^{1,\gamma}(E,\Omega) \geq \omega_{n-1}P^\gamma(E,\Omega) \tag{2}$$

w.r.t. the L^1_{loc} convergence of the corresponding characteristic functions in $\mathbb{R}^n$, i.e.

$$\liminf_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}^{1,\gamma}(E_i,\Omega) \geq \omega_{n-1}P^\gamma(E,\Omega) \qquad \textit{whenever } \chi_{E_i}\to\chi_E \textit{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n),\ s_i\uparrow 1.$$

Proposition 3 *For every measurable set $E \subset \mathbb{R}^n$ we have*

$$\Gamma - \limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^{\gamma}(E,\Omega) \leq \omega_{n-1}P^\gamma(E,\Omega) \tag{3}$$

w.r.t. the L^1_{loc} convergence of the corresponding characteristic functions in $\mathbb{R}^n$. Inequality (3) *means that for every measurable set E and sequence $s_i\uparrow 1$ there exists a sequence E_i with $\chi_{E_i}\to\chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$\limsup_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}^{\gamma}(E_i,\Omega) \leq \omega_{n-1}P^\gamma(E,\Omega).$$

In these two propositions lurk the main differences between the Gaussian case and the Euclidean case. Once we have proved Proposition 2 and Proposition 3, we are done: the proof of Theorem 1 is completely identical to the proof of [1, Theorem 3], with the only forethought of adding a γ-superscript in every considered functional, and remembering the simple inequality $\gamma(x,\,y)\leq 1$.

We will use the following notation: we write $x\in\mathbb{R}^n$ as (x',x_n) with $x'\in\mathbb{R}^{n-1}$ and $x_n\in\mathbb{R}$; we denote by H the halfspace $\{x:\ x_n\leq 0\}$ and by $Q=(-1/2,1/2)^n$ the canonical unit cube; we denote by $B_r(x)$ the ball of radius r centered at x and, unless otherwise specified, $B_r := B_r(0)$; for every $h\in\mathbb{R}^n$ and function u defined on $U\subset\mathbb{R}^n$ we set $\tau_h u(x) := u(x+h)$ for all $x\in U-h$. For the definition and basic properties of the perimeter $P(E,\Omega)$ in the sense of De Giorgi, we refer to [2, 18].

2.1 Proof of proposition 2

We denote by $\mathcal{C}$ the family of all n-cubes in $\mathbb{R}^n$

$$\mathcal{C} := \{R(x+rQ):\ x\in\mathbb{R}^n,\ r>0,\ R\in SO(n)\}.$$

Let $s_i\uparrow 1$ and sets $E_i\subset\mathbb{R}^n$ with $\chi_{E_i}\to\chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $i\to\infty$ be given. We need to show the inequality

$$\liminf_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}^{1,\gamma}(E_i,\Omega) \geq \omega_{n-1}P^\gamma(E,\Omega). \tag{4}$$

We can assume that the left-hand side of (4) is finite, otherwise the inequality is trivial. We choose an arbitrary $\Omega'\Subset\Omega$, and find a positive constant $c_0=c_0(\Omega')$ so that $c_0\leq\gamma(x,\,y),\ \forall\, x,\,y\in\Omega'$. Then we easily obtain the inequality

$$c_0\limsup_i \mathcal{J}_{s_i}^1(1-s_i)(E_i,\,\Omega_i') \leq \lim_i (1-s_i)\mathcal{J}_{s_i}^{1,\gamma}(E_i,\,\Omega') < +\infty.$$

By [1, Theorem 1] and the arbitrariness of Ω', we conclude that E has locally finite perimeter. We shall denote by μ its perimeter measure, i.e. $\mu(A)=|D\chi_E|(A)$ for any Borel set $A\subset\Omega$, and we shall use the following property of sets of finite perimeter: for μ-a.e. $x_0\in\Omega$ there exists $R_{x_0}\in SO(n)$ such that $(E-x_0)/r$ locally converge in measure to $R_{x_0}H$ as $r\to 0$. In addition,

$$\lim_{r\to 0}\frac{\mu(x_0+rR_{x_0}Q)}{r^{n-1}} = 1, \quad \text{for } \mu\text{-a.e. } x_0. \tag{5}$$

Indeed this property holds for every $x_0 \in \mathcal{F}E$, see [2, Theorem 3.59(b)].

Now, given a cube $C \in \mathcal{C}$ contained in Ω, we set

$$\alpha_i(C) := (1-s_i)\mathcal{J}^{1,\gamma}_{s_i}(E_i, C), \qquad \text{and} \qquad \alpha(C) := \liminf_{i\to\infty} \alpha_i(C).$$

Moreover, we define $C_r(x_0) := x_0 + rR_{x_0}Q$, where R_{x_0} is as in (5), and the measure

$$\nu(E) = \int_E e^{-\frac{1}{2}|x|^2}\, d\mu(x), \qquad \text{for every } E \text{ Borel set.}$$

We claim that for μ-a.e. $x_0 \in \mathbb{R}^n$ it holds

$$\omega_{n-1} \le \liminf_{r\to 0} \frac{\alpha(C_r(x_0))}{\nu(C_r(x_0))}. \tag{6}$$

If the claim is true, then we observe that for all $\varepsilon > 0$ the family

$$\mathcal{A} := \Big\{ C_r(x_0) \subset \Omega \ :\ \omega_{n-1}\nu(C_r(x_0)) \le (1+\varepsilon)\alpha(C_r(x_0)) \Big\}$$

is a fine covering of μ-almost all of Ω. By a suitable variant of Vitali's theorem (see [21]), we can extract a countable subfamily of disjoint cubes $\{C_j \subset \Omega : j \in J\}$ such that $\nu\big(\Omega \setminus \bigcup_{j\in J} C_j\big) = 0$, whence

$$\begin{aligned}\omega_{n-1}P^\gamma(E,\Omega) &= \omega_{n-1}\nu\Big(\bigcup_{j\in J} C_j\Big) = \omega_{n-1}\sum_{j\in J}\nu(C_j) \le (1+\varepsilon)\sum_{j\in J}\alpha(C_j) \le (1+\varepsilon)\liminf_{i\to\infty}\sum_{j\in J}\alpha_i(C_j)\\ &\le (1+\varepsilon)\liminf_{i\to\infty}(1-s_i)\mathcal{J}^{1,\gamma}_{s_i}(E_i,\Omega).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get the $\Gamma - \liminf$ estimate.

We now prove the inequality in (6) at any point x_0 such that $(E-x_0)/r$ converges locally in measure as $r \to 0$ to $R_{x_0}H$ and (5) holds. The continuity of the exponential ensures that

$$\lim_{r\to 0} \fint_{C_r(x_0)} e^{-\frac{1}{2}|x|^2}\, d\mu(x) = e^{-\frac{1}{2}|x_0|^2}.$$

Thus, thanks to (5), we just need to show the inequality

$$\liminf_{r\to 0} \frac{\alpha(C_r(x_0))}{r^{n-1}} \ge \omega_{n-1} e^{-\frac{1}{2}|x_0|^2}. \tag{7}$$

Since from now on x_0 is fixed, we may assume $R_{x_0} = I$, so that the limit hyperplane is H and the cubes $C_r(x_0)$ are the standard ones $x_0 + rQ$. Let us choose a sequence $r_k \to 0$ such that

$$\liminf_{r\to 0} \frac{\alpha(C_r(x_0))}{r^{n-1}} = \lim_{k\to\infty} \frac{\alpha(C_{r_k}(x_0))}{r_k^{n-1}}.$$

For $k > 0$ we can choose $i(k)$ large enough that the following conditions hold:

$$\begin{cases} \alpha_{i(k)}(C_{r_k}(x_0)) \le \alpha(C_{r_k}(x_0)) + r_k^n, \\ r_k^{1-s_{i(k)}} \ge 1 - \dfrac{1}{k}, \\ \displaystyle\fint_{C_{r_k}(x_0)} |\chi_{E_{i(k)}} - \chi_E| dx < \frac{1}{k}. \end{cases}$$

We observe that, although $\mathcal{J}_s^{1,\gamma}$ does not enjoy the nice scaling properties of $\mathcal{J}_s^1$, it still satisfies the equality

$$\mathcal{J}_s^{1,\gamma}(E,\, C_r(x_0)) = r^{n-s}\mathcal{J}_s^{1,\,\gamma_{x_0,\,r}}((E-x_0)/r,\, Q),$$

where we have set

$$\gamma_{x_0,\,r}(x,\, y) = \exp\left(-\frac{1}{4}\left(|x_0+rx|^2+|x_0+ry|^2\right)\right).$$

Since $\|D_x(e^{-x^2})\|_{L^\infty} \le 1$, for every $x, y \in Q$ and $r>0$ the following inequality holds:

$$\left|\, \gamma_{x_0,\,r}(x,\, y) - e^{-\frac{1}{2}|x_0|^2} \,\right| \le r.$$

Then we infer

$$\begin{aligned}
\frac{\alpha(C_{r_k}(x_0))}{r_k^{n-1}} &\ge \frac{\alpha_{i(k)}(C_{r_k}(x_0))}{r_k^{n-1}} - r_k = \frac{(1-s_{i(k)})\mathcal{J}_{s_{i(k)}}^{1,\,\gamma_{x_0,\,r_k}}((E_{i(k)}-x_0)/r_k, Q) r_k^{n-s_{i(k)}}}{r_k^{n-1}} - r_k \\
&\ge \Big(1-\frac{1}{k}\Big)(1-s_{i(k)})\mathcal{J}_{s_{i(k)}}^{1,\,\gamma_{x_0,\,r_k}}((E_{i(k)}-x_0)/r_k, Q) - r_k \\
&\ge \Big(1-\frac{1}{k}\Big)(1-s_{i(k)})\mathcal{J}_{s_{i(k)}}^{1}((E_{i(k)}-x_0)/r_k, Q)(e^{-\frac{1}{2}|x_0^2|} - r_k) - r_k,
\end{aligned}$$

i.e.

$$\lim_{k\to\infty}\frac{\alpha(C_{r_k}(x_0))}{r_k^{n-1}} \ge e^{-\frac{1}{2}|x_0|^2}\liminf_{k\to\infty}(1-s_{i(k)})\mathcal{J}_{s_{i(k)}}^{1}((E_{i(k)}-x_0)/r_k, Q).$$

Since we have

$$\lim_{k\to\infty}\int_Q |\chi_{(E_{i(k)}-x_0)/r_k} - \chi_{(E-x_0)/r_k}|dx = 0,$$

and

$$\lim_{k\to\infty}\int_Q |\chi_{(E-x_0)/r_k} - \chi_H|dx = 0,$$

it follows that $(E_{i(k)}-x_0)/r_k \to H$ in $L^1(Q)$. If we define

$$\Gamma_n := \inf\Big\{ \liminf_{s\uparrow 1}(1-s)\mathcal{J}_s^1(E_s, Q) \;\Big|\; \chi_{E_s}\to\chi_H \text{ in } L^1(Q)\Big\}, \tag{8}$$

it has been proved in [1, Lemmata 7, 11, 12] that $\Gamma_n = \omega_{n-1}$. Hence we conclude the claimed inequality (7).

2.2 Proof of proposition 3

As in [1], it is enough to prove the $\Gamma-\limsup$ inequality for the collection $\mathcal{B}$ of polyhedra Π of finite perimeter which satisfy $P(\Pi, \partial\Omega) = 0$. $\mathcal{B}$ is dense in energy, i.e. such that for every set E of finite perimeter there exists $E_k \in \mathcal{B}$ with $\chi_{E_k}\to\chi_E$ in $L^1_{\rm loc}(\mathbb{R}^n)$ as $k\to\infty$ and $\limsup_k P^\gamma(E_k,\Omega) = P^\gamma(E,\Omega)$. We recall that a polyhedron Π is in the class $\mathcal{B}$ if and only if

$$\lim_{\delta\to 0} P(\Pi, \Omega_\delta^+\cup\Omega_\delta^-) = 0, \quad \text{or equivalently} \quad \lim_{\delta\to 0} P^\gamma(\Pi, \Omega_\delta^+\cup\Omega_\delta^-) = 0,$$

where we set

$$\Omega_\delta^+ := \{x\in\Omega^c \mid d(x,\Omega)<\delta\}, \qquad \Omega_\delta^- := \{x\in\Omega \mid d(x,\Omega^c)<\delta\}. \tag{9}$$

We are going to prove that for a polyhedron $\Pi \subset \mathbb{R}^n$ there holds

$$\limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^\gamma(\Pi,\Omega) \le \Gamma_n^* P^\gamma(\Pi,\Omega) + 2\Gamma_n^* \lim_{\delta\to 0} P^\gamma(\Pi,\Omega_\delta^+ \cup \Omega_\delta^-), \tag{10}$$

where

$$\Gamma_n^* := \limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^1(H,Q). \tag{11}$$

Again, as in [1, Lemmata 7, 11, 12] we have the equality $\Gamma_n^* = \omega_{n-1}$. We shall divide the proof into two main steps.

Step 1. We first estimate $\mathcal{J}_s^{1,\gamma}(\Pi,\Omega)$. For a fixed $\varepsilon > 0$ set

$$(\partial\Pi)_\varepsilon := \{x \in \Omega \mid d(x,\partial\Pi) < \varepsilon\}, \quad (\partial\Pi)_\varepsilon^- := (\partial\Pi)_\varepsilon \cap \Pi.$$

We can find N_ε disjoint cubes $Q_i^\varepsilon \subset \Omega$, $1 \le i \le N_\varepsilon$, of side length ε satisfying the following properties:

(i) if $\tilde{Q}_i^\varepsilon$ denotes the dilation of Q_i^ε by a factor $(1+\varepsilon)$, then each cube $\tilde{Q}_i^\varepsilon$ intersects exactly one face Σ of $\partial\Pi$, its barycenter belongs to Σ and each of its sides is either parallel or orthogonal to Σ;

(ii) $\mathcal{H}^{n-1}\left(((\partial\Pi)\cap\Omega) \setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon\right) = |P(\Pi,\Omega) - N_\varepsilon \varepsilon^{n-1}| \to 0$ as $\varepsilon \to 0$.

Property (ii), combined with the continuity of the exponential and the property of measures, easily implies

$$\left| P^\gamma(\Pi,\Omega) - \varepsilon^{n-1} \sum_{i=1}^{N_\varepsilon} e^{-\frac{1}{2}|x_i^\varepsilon|^2} \right| \to 0 \text{ as } \varepsilon \to 0, \tag{12}$$

where we have set by x_i^ε the center of the cubes Q_i^ϵ. For $x \in \mathbb{R}^n$ set

$$I_s(x) := \int_{\Pi^c\cap\Omega} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}}\, dy.$$

We consider several cases.

Case 1: $x \in (\Pi\cap\Omega) \setminus (\partial\Pi)_\varepsilon^-$. Then for $y \in \Pi^c \cap \Omega$ we have $|x-y| \ge \varepsilon$, hence

$$I_s(x) \le \int_{(B_\varepsilon(x))^c} \frac{1}{|x-y|^{n+s}} dy = n\omega_n \int_\varepsilon^\infty \frac{1}{\rho^{s+1}} d\rho = \frac{n\omega_n}{s\varepsilon^s},$$

since $n\omega_n = \mathcal{H}^{n-1}(S^{n-1})$. Therefore

$$\int_{(\Pi\cap\Omega)\setminus(\partial\Pi)_\varepsilon^-} I_s(x) e^{-\frac{1}{4}|x|^2}\, dx \le \frac{n\omega_n}{s\varepsilon^s} \int_{\Pi\cap\Omega} e^{-\frac{1}{4}|x|^2}\, dx. \tag{13}$$

Case 2: $x \in (\partial\Pi)_\varepsilon^- \setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon$. Then

$$I_s(x) \le \int_{(B_{d(x,\Pi^c\cap\Omega)}(x))^c} \frac{1}{|x-y|^{n+s}} dy = n\omega_n \int_{d(x,\Pi^c\cap\Omega)}^\infty \frac{1}{\rho^{s-1}} d\rho = \frac{n\omega_n}{s[d(x,\Pi^c\cap\Omega)]^s}. \tag{14}$$

Now write $(\partial\Pi)\cap\Omega = \bigcup_{j=1}^J \Sigma_j$, where each Σ_j is the intersection of a face of $\partial\Pi$ with Ω, and define

$$(\partial\Pi)_{\varepsilon,j}^- := \{x \in (\partial\Pi)_\varepsilon^- : \operatorname{dist}(x,\Pi^c\cap\Omega) = \operatorname{dist}(x,\Sigma_j)\}.$$

Clearly $(\partial\Pi)^-_\varepsilon = \bigcup_{j=1}^J (\partial\Pi)^-_{\varepsilon,j}$. Moreover we have

$$(\partial\Pi)^-_{\varepsilon,j} \subset \{x + t\nu : x \in \Sigma_{\varepsilon,j},\, t \in (0,\varepsilon),\, \nu \text{ is the interior unit normal to } \Sigma_{\varepsilon,j}\},$$

and $\Sigma_{\varepsilon,j}$ is the set of points x belonging to the same hyperplane as Σ_j and with $\operatorname{dist}(x,\Sigma_j) \le \varepsilon$. Clearly $\mathcal{H}^{n-1}(\Sigma_{\varepsilon,j}) \le \mathcal{H}^{n-1}(\Sigma_j) + C\varepsilon$ as $\varepsilon \to 0$. Then from (14) we infer

$$\begin{aligned}
\int_{(\partial\Pi)^-_\varepsilon \setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} I_s(x) e^{-\frac{1}{4}|x|^2}\,dx &\le \frac{n\omega_n}{s} \sum_{j=1}^J \int_{(\partial\Pi)^-_{\varepsilon,j} \setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} \frac{1}{[d(x,\Pi^c)]^s}\,dx \\
&\le \frac{n\omega_n}{s} \sum_{j=1}^J \int_{(\partial\Pi)^-_{\varepsilon,j} \setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} \frac{1}{[d(x,\Sigma_{\varepsilon,j})]^s}\,dx \\
&\le \frac{n\omega_n}{s} \sum_{j=1}^J \int_{(\Sigma_{\varepsilon,j}) \setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} \left(\int_0^\varepsilon \frac{dt}{t^s} \right) d\mathcal{H}^{n-1} \\
&= \frac{n\omega_n \varepsilon^{1-s}}{s(1-s)} \mathcal{H}^{n-1} \left(\left(\bigcup_{j=1}^J \Sigma_{\varepsilon,j} \right) \setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon \right) = \frac{\varepsilon^{1-s} o(1)}{s(1-s)},
\end{aligned} \tag{15}$$

with error $o(1) \to 0$ as $\varepsilon \to 0$ and independent of s.

Case 3: $x \in \Pi \cap \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon$. In this case we write

$$\begin{aligned}
I_s(x) &= \int_{(\Pi^c \cap \Omega) \cap \{y : |x-y| \ge \varepsilon^2\}} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}}\,dy + \int_{(\Pi^c \cap \Omega) \cap \{y : |x-y| < \varepsilon^2\}} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}}\,dy \\
&=: I_s^1(x) + I_s^2(x).
\end{aligned}$$

Then, similar to the case 1,

$$I_s^1(x) \le n\omega_n \int_{\varepsilon^2}^\infty \frac{1}{\rho^{s+1}}\,d\rho = \frac{n\omega_n}{s\varepsilon^{2s}},$$

hence (since all cubes are contained in Ω)

$$\int_{\Pi \cap \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} I_s^1(x) e^{-\frac{1}{4}|x|^2}\,dx \le \frac{n\omega_n}{s\varepsilon^{2s}} \int_\Omega e^{-\frac{1}{4}|x|^2}\,dx. \tag{16}$$

As for $I_s^2(x)$ observe that if $x \in Q_i^\varepsilon$ and $|x-y| \le \varepsilon^2$, then $y \in \tilde{Q}_i^\varepsilon$, where $\tilde{Q}_i^\varepsilon$ is the cube obtained by dilating Q_i^ε by a factor $1+\varepsilon$ (hence the side length of $\tilde{Q}_i^\varepsilon$ is $\varepsilon + \varepsilon^2$). Then

$$\begin{aligned}
\int_{\Pi \cap \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} I_s^2(x) e^{-\frac{1}{4}|x|^2}\,dx &\le \sum_{i=1}^{N_\varepsilon} \int_{\Pi \cap Q_i^\varepsilon} \int_{\Pi^c \cap \tilde{Q}_i^\varepsilon} \frac{e^{-\frac{1}{4}(|x|^2+|y|^2)}}{|x-y|^{n+s}}\,dy dx \\
&\le \sum_{i=1}^{N_\varepsilon} \int_{\Pi \cap \tilde{Q}_i^\varepsilon} \int_{\Pi^c \cap \tilde{Q}_i^\varepsilon} \frac{e^{-\frac{1}{4}(|x|^2+|y|^2)}}{|x-y|^{n+s}}\,dy dx \\
&\le \left(\sum_{i=1}^{N_\varepsilon} e^{-\frac{1}{2}|x_i^\varepsilon|^2} \right) \mathcal{J}_s^1(H, (\varepsilon+\varepsilon^2)Q)(1+\varepsilon^2) \\
&= \underbrace{\left(\varepsilon^{n-1} \sum_{i=1}^{N_\varepsilon} e^{-\frac{1}{2}|x_i^\varepsilon|^2} \right)}_{=P^\gamma(\Pi,\Omega)+o(1)} \varepsilon^{1-s}(1+\varepsilon)^{n-s} \mathcal{J}_s^1(H,Q)(1+\varepsilon^2),
\end{aligned} \tag{17}$$

where in the last identity we used the scaling property

$$\mathcal{J}_s^i(\lambda E, \lambda\Omega) = \lambda^{n-s}\mathcal{J}_s^i(E,\Omega) \qquad \text{for } \lambda > 0,\ i = 1, 2. \tag{18}$$

Keeping $\varepsilon > 0$ fixed, letting s go to 1 and putting (13)-(17) together, we infer

$$\limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^{1,\gamma}(\Pi,\Omega) \le o(1) + \Gamma_n^* P^\gamma(\Pi,\Omega) = o(1) + \omega_{n-1}P^\gamma(\Pi,\,\Omega),$$

with error $o(1) \to 0$ as $\varepsilon \to 0$ uniformly in s. Since $\varepsilon > 0$ is arbitrary, we conclude

$$\limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^{1,\gamma}(\Pi,\Omega) \le \omega_{n-1}P^\gamma(\Pi,\Omega). \tag{19}$$

Step 2. It now remains to estimate $\mathcal{J}_s^{2,\gamma}$. Let us start by considering the term

$$\int_{\Pi\cap\Omega}\int_{\Pi^c\cap\Omega^c} \frac{e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}}{|x-y|^{n+s}}\,dydx.$$

Case 1: $x \in \Pi \cap (\Omega \setminus \Omega_\delta^-)$. Then for $y \in \Pi^c \cap \Omega^c$ we have $|x-y| \ge \delta$, whence

$$I(x) := \int_{\Pi^c\cap\Omega^c} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}}\,dy \le n\omega_n\int_\delta^\infty \frac{d\rho}{\rho^{1+s}} = \frac{n\omega_n}{s\delta^s}.$$

Case 2: $x \in \Pi \cap \Omega_\delta^-$. In this case, using the same argument of case 1 for $y \in \Pi^c \cap (\Omega^c \setminus \Omega_\delta^+)$, we have

$$I(x) = \int_{\Pi^c\cap\Omega_\delta^+} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}}\,dy + \int_{\Pi^c\cap(\Omega^c\setminus\Omega_\delta^+)} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}}\,dy \le \int_{\Pi^c\cap\Omega_\delta^+} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}}\,dy + \frac{n\omega_n}{s\delta^s}.$$

Therefore

$$\begin{aligned}\int_{\Pi\cap\Omega}\int_{\Pi^c\cap\Omega^c} \frac{e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}}{|x-y|^{n+s}}\,dydx &\le \frac{2n\omega_n}{s\delta^s}\int_\Omega e^{-\frac{1}{4}|x|^2}\,dx + \int_{\Pi\cap\Omega_\delta^-}\int_{\Pi^c\cap\Omega_\delta^+} \frac{e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}}{|x-y|^{n+s}}\,dydx \\ &\le \frac{2n\omega_n}{s\delta^s}\int_\Omega e^{-\frac{1}{4}|x|^2}\,dx + \int_{\Pi\cap(\Omega_\delta^-\cup\Omega_\delta^+)}\int_{\Pi^c\cap(\Omega_\delta^-\cup\Omega_\delta^+)} \frac{e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}}{|x-y|^{n+s}}\,dydx.\end{aligned}$$

An obvious similar estimate can be obtained by swapping Π and Π^c, finally yielding

$$\begin{aligned}\mathcal{J}_s^{2,\gamma}(\Pi,\Omega) &\le \frac{4n\omega_n}{s\delta^s}\int_\Omega e^{-\frac{1}{4}|x|^2}\,dx + 2\int_{\Pi\cap(\Omega_\delta^-\cup\Omega_\delta^+)}\int_{\Pi^c\cap(\Omega_\delta^-\cup\Omega_\delta^+)} \frac{e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}}{|x-y|^{n+s}}\,dydx \\ &= \frac{4n\omega_n}{s\delta^s}\int_\Omega e^{-\frac{1}{4}|x|^2}\,dx + 2\mathcal{J}_s^{1,\gamma}(\Pi,\Omega_\delta^-\cup\Omega_\delta^+).\end{aligned}$$

Using inequality (19) applied with the open set $\Omega_\delta^- \cup \Omega_\delta^+$, we get

$$\limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^{2,\gamma}(\Pi,\Omega) \le 2\omega_{n-1}P^\gamma(\Pi,\Omega_\delta^-\cup\Omega_\delta^+).$$

Since $\delta > 0$ is arbitrary, letting δ go to zero, we conclude the proof of the Proposition.

3 First and second variation

In this section we calculate the first and second variation of $\mathcal{J}_s^\gamma(E)$. A similar analysis has been done in [17] in order to prove the local minimality of the ball for a functional involving non local terms.

First, we fix some notation. Given a vector field $X \in C_c^2(\mathbb{R}^n, \mathbb{R}^n)$, the *associated flow* is defined as the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(x,t) = X(\Phi(x,t)) \\ \Phi(x,0) = x. \end{cases} \tag{20}$$

In the following, we shall always write Φ_t to denote the map $\Phi(\cdot, t)$. Note that for any given X there exists $\delta > 0$ such that, for $t \in [-\delta, \delta]$, the map Φ_t is a diffeomorphism coinciding with the identity map outside a compact set.

If $E \subset \mathbb{R}^n$ is measurable, we set $E_t := \Phi_t(E)$. Denoting by $J\Phi_t$ the n-dimensional Jacobian of Φ_t, the first and second derivatives of $J\Phi_t$ are given by

$$\frac{\partial}{\partial t}|_{t=0} J\Phi_t = \mathrm{div} X, \qquad \frac{\partial^2}{\partial t^2}|_{t=0} J\Phi_t = \mathrm{div}((\mathrm{div} X)X). \tag{21}$$

Finally, given a sufficiently smooth bounded open set $E \subset \mathbb{R}^n$ and a vector field X, we recall that the *first variation of* $\mathcal{J}_s^\gamma(E)$ along the vector field X is defined by

$$\delta \mathcal{J}_s^\gamma(E)[X] := \frac{d}{dt}|_{t=0} \mathcal{J}_s^\gamma(E_t),$$

where Φ_t is the flow associated with X. The *second variation of* $\mathcal{J}_s^\gamma(E)$ along the vector field X is defined by

$$\delta^2 \mathcal{J}_s^\gamma(E)[X] = \frac{d^2}{dt^2}|_{t=0} \mathcal{J}_s^\gamma(E_t).$$

If X is a vector field such that $X := \phi \nu_E$ on ∂E, where ν_E denotes the exterior normal to E, using the area formula and the divergence theorem, the first variation of the Gaussian volume can be computed as

$$\begin{aligned} \frac{d}{dt}|_{t=0} \gamma(E_t) = \frac{d}{dt}|_{t=0} \int_E J\Phi_t(x) e^{-\frac{|\Phi_t|^2}{2}} dx &= \int_E \left(\mathrm{div}\, X - \langle X, x\rangle\right) e^{-\frac{|x|^2}{2}} dx \\ &= \int_E \mathrm{div}\left(X e^{-\frac{|x|^2}{2}}\right) dx = \int_{\partial E} \phi(x) e^{-\frac{|x|^2}{2}} d\mathcal{H}_x^{n-1}. \end{aligned} \tag{22}$$

If E is a set of class C^2, given a smooth function $\phi : \partial E \to \mathbb{R}$, it can be extended in a neighborhood U of ∂E so that

$$\frac{\partial}{\partial \nu}\phi + \phi(H - \langle x, \nu_E\rangle) = 0 \quad \text{on } \partial E. \tag{23}$$

The second variation of the Gaussian volume along the vector field X such that $X = \phi \nu_E$ on ∂E and ϕ satisfies (23), can be calculated using the divergence theorem and reads as

$$\frac{d^2}{dt^2}|_{t=0} \gamma(E_t) = \int_E \mathrm{div}\left(\mathrm{div}\left(X e^{-\frac{|x|^2}{2}}\right) X\right) dx = \int_{\partial E} \phi \left(\frac{\partial}{\partial \nu}\phi + \phi(H - \langle x, \nu_E\rangle)\right) e^{-\frac{|x|^2}{2}} d\mathcal{H}_x^{n-1} = 0$$

Thus, we say that a vector field preserves the Gaussian volume of E if it satisfies

$$\int_{\partial E} \phi(x) e^{-\frac{|x|^2}{2}} d\mathcal{H}_x^{n-1} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial \nu} + \phi(H - \langle x, \nu_E\rangle) = 0 \ \text{ for } \ x \in \partial E. \tag{24}$$

We note that without these assumptions the expression of the second variation of the Gaussian perimeter even in the local framework is quite complicated, see [3, Eq. (17)].

In order to compute the first and second variation of $\mathcal{J}_s^\gamma$, due to the singularity of the Kernel in the integrand, we need to pass through approximations. Thus, given $\delta \in [0, 1/2)$, let $\eta_\delta \in C_c^\infty([0,+\infty),[0,1])$ be such that $\eta_\delta = 1$ on $[0,\delta]\cup[1/\delta,\infty]$, $\eta_\delta = 0$ on $[2\delta, 1/(2\delta)]$, $|\eta'| \le 2/\delta$ on $[0,\infty)$, and $\eta_\delta \downarrow 0$ as $\delta \to 0^+$. Then we define

$$K_\delta(z) := (1-\eta_\delta(|z|))\frac{1}{|z|^{n+s}}.$$

Moreover, we now introduce the two quantities

$$H^*_{\partial E,\delta}(x) = \int_{\mathbb{R}^n} (\chi_{E^c}(y) - \chi_E(y))\, K_\delta(x-y) e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)} dy$$

and

$$H^*_{\partial E}(x) = \int_{\mathbb{R}^n} (\chi_{E^c}(y) - \chi_E(y))\, K(x-y) e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)} dy$$

corresponding respectively to the fractional mean curvature with respect to the regularized non local Gaussian perimeter and to the fractional mean curvature with respect to the non local Gaussian perimeter. Bearing these definitions in mind we now show the following theorem.

Theorem 4 *Let E be an open set of class C^2 and $X \in C_c^2(\mathbb{R}^n,\mathbb{R}^n)$ a vector field such that $X = \phi\nu_E$ on ∂E with $\phi \in C^2(\partial E)$. Then the first variation of $\mathcal{J}_s^\gamma(E)$ along a vector field X is given by*

$$\partial\mathcal{J}_s^\gamma(E)[X] = \int_{\partial E} H^*_{\partial E}(x)\langle X(x),\nu_E(x)\rangle d\mathcal{H}^{n-1}_x, \tag{25}$$

while the second variation reads as

$$\begin{aligned}\partial^2\mathcal{J}_s^\gamma(E)[X] = &\int_{\partial E}\int_{\partial E} \frac{e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}}{|x-y|^{n+s}}\left(|\phi(x)-\phi(y)|^2 - \phi^2|\nu_E(x)-\nu_E(y)|\right) d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y \\ &+ \int_{\partial E} H^*_{\partial E}\left(\phi\left(H_{\partial E} - \langle x,\nu_E\rangle\right) + \frac{\partial\phi}{\partial\nu}\right)\phi\, d\mathcal{H}^{n-1}_x \\ &- \int_{\partial E}\phi^2(x)\int_{\mathbb{R}^n}\frac{\langle y-x,\nu_E\rangle}{2}\frac{e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}}{|x-y|^{n+s}} dy\, d\mathcal{H}^{n-1}_x,\end{aligned} \tag{26}$$

where $H_{\partial E}(x)$ stands for the mean curvature of ∂E at the point x. Moreover, if X is volume preserving, then

$$\begin{aligned}\partial^2\mathcal{J}_s^\gamma(E)[X] = &\int_{\partial E}\int_{\partial E} \frac{e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}}{|x-y|^{n+s}}\left(|\phi(x)-\phi(y)|^2 - \phi^2|\nu_E(x)-\nu_E(y)|\right) d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y \\ &- \int_{\partial E}\phi^2(x)\int_{\mathbb{R}^n}\frac{\langle y-x,\nu_E(x)\rangle}{2}\frac{e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}}{|x-y|^{n+s}} dy\, d\mathcal{H}^{n-1}_x.\end{aligned} \tag{27}$$

Proof. Let us call $\mathcal{J}_s^\delta$ the integral associated to the regularized kernel, namely

$$\mathcal{J}_s^\delta(E) = \int_{E^c}\int_E K_\delta(x-y) e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)} dxdy.$$

By the definition of Φ_t, the implicit function theorem gives the existence of $\varepsilon > 0$ such that the map Φ_t is a diffeomorphism for all $t \in [-\delta, \delta]$. Using the area formula, we compute

$$\mathcal{J}_s^\delta(E_t) = \int_{E^c}\int_E K_\delta(\Phi(x,t)-\Phi(y,t))e^{-\frac{1}{4}\left(|\Phi(x,t)|^2+|\Phi(y,t)|^2\right)}J\Phi(x,t)J\Phi(y,t)dxdy$$

We use the first equation in (21), the divergence theorem and the symmetry of the kernel K_δ to compute the first variation of $\mathcal{J}_s^\delta$

$$\begin{aligned}
\frac{d}{dt}_{|t=0}\mathcal{J}_s^\delta(E_t) &= \int_{E^c}\int_E e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}D_z\left(K_\delta(x-y)\right)(X(x)-X(y))dxdy \\
&\quad + \int_{E^c}\int_E K_\delta(x-y)\left(D_x(e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)})X(x) + D_y(e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)})X(y)\right)dxdy \\
&\quad + \int_{E^c}\int_E K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\left(\operatorname{div}X(x)+\operatorname{div}X(y)\right)dxdy \\
&= \int_{E^c}\int_E \operatorname{div}_x\left(e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}K_\delta(x-y)X(x)\right)dxdy \\
&\quad + \int_{E^c}\int_E \operatorname{div}_y\left(e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}K_\delta(x-y)X(y)\right)dxdy \\
&= \int_{\partial E}\int_{\mathbb{R}^n}\left(\chi_{E^c}(y)-\chi_E(y)\right)K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\langle X(x),\nu_E(x)\rangle dyd\mathcal{H}_x^{n-1}.
\end{aligned}$$

Now we compute the second variation of $\mathcal{J}_s^\delta$.

$$\begin{aligned}
\delta^2\mathcal{J}_s^\delta[X] &= \frac{d^2}{dt^2}_{|t=0}\int_{E^c}\int_E K_\delta(\Phi(x,t)-\Phi(y,t))e^{-\frac{1}{4}\left(|\Phi(x,t)|^2+|\Phi(y,t)|^2\right)}\operatorname{div}X(x)\operatorname{div}X(y)dxdy \\
&= \int_{E^c}\int_E D^2_{xx}\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right)[X(x),X(x)]dxdy \\
&\quad + \int_{E^c}\int_E\left\langle D_x\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right), DX(x)X(x)\right\rangle dxdy \\
&\quad + 2\int_{E^c}\int_E D^2_{yx}\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right)[X(x),X(y)]dxdy \\
&\quad + \int_{E^c}\int_E D^2_{yy}\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right)[X(y),X(y)]dxdy \\
&\quad + \int_{E^c}\int_E\left\langle D_y\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right), DX(y)X(y)\right\rangle dxdy \\
&\quad + 2\int_{E^c}\int_E\left\langle D_x\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right), X(x)\right\rangle\left(\operatorname{div}_x X(x)+\operatorname{div}_y X(y)\right)dxdy \\
&\quad + 2\int_{E^c}\int_E\left\langle D_y\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right), X(y)\right\rangle\left(\operatorname{div}_x X(x)+\operatorname{div}_y X(y)\right)dxdy \\
&\quad + \int_{E^c}\int_E K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\big(\operatorname{div}[X(x)\operatorname{div}(X(x))]+\operatorname{div}[X(y)\operatorname{div}(X(y))]\big)dxdy \\
&\quad + 2\int_{E^c}\int_E K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\operatorname{div}_x X(x)\operatorname{div}_y X(y)dxdy
\end{aligned}$$

We now use the divergence theorem and exploit the symmetry of K_δ in order to simplify the above

expression as follows

$$\begin{aligned}\delta^2\mathcal{J}_s^\delta &= \int_{E^c}\int_E \operatorname{div}_x\left[X(x)\operatorname{div}_x\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(x)\right)\right]dxdy\\ &+\int_{E^c}\int_E \operatorname{div}_y\left[X(y)\operatorname{div}_y\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(y)\right)\right]dxdy\\ &+\int_{E^c}\int_E \operatorname{div}_x\left[X(x)\operatorname{div}_y\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(y)\right)\right]dxdy\\ &+\int_{E^c}\int_E \operatorname{div}_y\left[X(y)\operatorname{div}_x\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(x)\right)\right]dxdy\\ &= I_1+I_2+I_3+I_4.\end{aligned} \tag{28}$$

Using Fubini and the divergence theorems we have

$$I_1=\int_{\partial E}\langle X(x),\nu_E(x)\rangle\int_{E^c}\operatorname{div}_x\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(x)\right)dyd\mathcal{H}_x^{n-1}$$

and

$$\begin{aligned}I_3&=\int_{\partial E}\langle X(x),\nu_E(x)\rangle\int_{E^c}\operatorname{div}_x\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(y)\right)dyd\mathcal{H}_x^{n-1}\\ &=-\int_{\partial E}\int_{\partial E}(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\langle X(y),\nu_E(y)\rangle\langle X(x),\nu_E(x)\rangle d\mathcal{H}_x^{n-1}d\mathcal{H}_y^{n-1}.\end{aligned} \tag{29}$$

We remark that I_1 (resp. I_3) has the same expression of I_2 (resp. I_4) exchanging x and y. Using this observation and the symmetry of K_δ, we compute

$$I_1+I_2=\int_{\partial E}\langle X(x),\nu_E(x)\rangle\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))\operatorname{div}_x\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(x)\right)dyd\mathcal{H}_x^{n-1},$$

and

$$I_3+I_4=-2\int_{\partial E}\int_{\partial E}(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\langle X(y),\nu_E(y)\rangle\langle X(x),\nu_E(x)\rangle d\mathcal{H}_x^{n-1}d\mathcal{H}_y^{n-1}.$$

Next we write $\operatorname{div}_x X(x)=\operatorname{div}_{\nu(x)}X(x)+\operatorname{div}_{\tau(x)}X(x)$, where $\operatorname{div}_{\nu(x)}X(x):=\langle DX[\nu_E(x)],\nu_E(x)\rangle$. Using Fubini's theorem and the divergence theorem on manifolds, we get

$$\begin{aligned}I_1+I_2&=\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))\int_{\partial E}\langle X(x),\nu_E(x)\rangle\operatorname{div}_{\tau(x)}\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(x)\right)d\mathcal{H}_x^{n-1}dy\\ &+\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))\int_{\partial E}\langle X(x),\nu_E(x)\rangle\operatorname{div}_{\nu(x)}\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(x)\right)d\mathcal{H}_x^{n-1}dy\\ &=\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))\int_{\partial E}H_{\partial E}\langle X(x),\nu_E(x)\rangle^2\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right)d\mathcal{H}_x^{n-1}dy\\ &+\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))\int_{\partial E}\langle X(x),\nu_E(x)\rangle\operatorname{div}_{\nu(x)}\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}X(x)\right)d\mathcal{H}_x^{n-1}dy\\ &=\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))\int_{\partial E}H_{\partial E}(x)\phi^2(x)\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right)d\mathcal{H}_x^{n-1}dy\\ &+\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))\int_{\partial E}\phi^2(x)\frac{\partial}{\partial\nu(x)}\left(K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}\right)d\mathcal{H}_x^{n-1}dy\\ &+\int_{\mathbb{R}^n}(\chi_{E^c}(y)-\chi_E(y))\int_{\partial E}\phi(x)\frac{\partial\phi}{\partial\nu(x)}K_\delta(x-y)e^{-\frac{1}{4}\left(|x|^2+|y|^2\right)}d\mathcal{H}_x^{n-1}dy,\end{aligned} \tag{30}$$

where we used that $X = \phi\nu_E$ and then $\langle D_\tau f, X\rangle = 0$ for every $f \in C^1(\partial E)$. We recall that $\phi \in C^2(\partial E)$, hence the integrals are well defined. Regarding the second addend of the expression above, using again Fubini's theorem and the fact that $D_x K_\delta = -D_y K_\delta$, we get

$$\begin{aligned}
&\int_E \frac{\partial}{\partial \nu(x)} \left(K_\delta(x-y) e^{-\frac{1}{4}(|x|^2+|y|^2)} \right) dy \\
&= -\int_E \left[\langle D_y K_\delta(x-y), \nu_E(x)\rangle + \frac{\langle x, \nu_E(x)\rangle}{2} K_\delta(x-y) \right] e^{-\frac{1}{4}(|x|^2+|y|^2)} dy \\
&= -\int_{\partial E} K_\delta(x-y) e^{-\frac{1}{4}(|x|^2+|y|^2)} \langle \nu_E(x), \nu_E(y)\rangle \, d\mathcal{H}^{n-1}_y - \int_E \frac{\langle x+y, \nu_E(x)\rangle}{2} K_\delta(x-y) e^{-\frac{1}{4}(|x|^2+|y|^2)} dy,
\end{aligned} \tag{31}$$

Finally, thanks to the identity $|\nu_E(x) - \nu_E(y)|^2 = 2 - 2\langle \nu_E(x), \nu_E(y)\rangle$, after some elementary calculations we deduce

$$\begin{aligned}
\partial^2 \mathcal{J}^\delta_s(E)[X] = &\int_{\partial E}\int_{\partial E} e^{-\frac{1}{4}(|x|^2+|y|^2)} K_\delta(x-y) \left(|\phi(x) - \phi(y)|^2 - \phi^2 |\nu_E(x) - \nu_E(y)| \right) d\mathcal{H}^{n-1} d\mathcal{H}^{n-1} \\
&+ \int_{\partial E} H^*_{\partial E,\delta} \left(\phi \left(H_{\partial E} - \langle x, \nu_E\rangle \right) + \frac{\partial \phi}{\partial \nu} \right) \phi \, d\mathcal{H}^{n-1} \\
&- \int_{\partial E} \phi^2(x) \int_{\mathbb{R}^n} \left(\chi_{E^c}(y) - \chi_E(y) \right) \frac{\langle y-x, \nu_E(x)\rangle}{2} K_\delta(x-y) e^{-\frac{1}{4}(|x|^2+|y|^2)} dy d\mathcal{H}^{n-1}_x.
\end{aligned} \tag{32}$$

At this point we just need to show that the first and second variation of $\mathcal{J}^\delta_s$ converge, respectively, to the first and second variation of $\mathcal{J}^\gamma_s$ as δ goes to 0. The proof of this fact is exactly the same as in [17]. □

Note that, since

$$\frac{d}{dt}\Big|_{t=0} \int_{E_t} e^{-\frac{x^2}{2}} dx = \int_{\partial E} e^{-\frac{|x|^2}{2}} \langle X(x), \nu_E(x)\rangle d\mathcal{H}^{n-1},$$

we have that the flow Φ associated to X preserves the Gaussian volume if

$$\int_{\partial E} e^{-\frac{|x|^2}{2}} \langle X(x), \nu_E(x)\rangle d\mathcal{H}^{n-1} = 0.$$

Thus, the Euler-Lagrange equation for the problem

$$\min_{|E|=m} \mathcal{J}^\gamma_s(E) \tag{33}$$

is

$$\int_{\partial E} \phi(x) \int_{\mathbb{R}^n} \left(\chi_{E^c}(y) - \chi_E(y) \right) \frac{e^{-\frac{1}{4}(|x|^2+|y|^2)}}{|x-y|^{n+s}} dy d\mathcal{H}^{n-1}_x = \lambda \int_{\partial E} \phi(x) e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}_x.$$

Moreover, if E is a set of class C^2, then thanks to the fundamental lemma of the calculus of variations the above equation can be rewritten as

$$\int_{\mathbb{R}^n} \left(\chi_{E^c}(y) - \chi_E(y) \right) \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}} dy = \lambda e^{-\frac{|x|^2}{4}}, \qquad \forall x \in \partial E. \tag{34}$$

E is said to be stationary with respect to the non local Gaussian isoperimetric problem, or equivalently a volume constrained critical point, if it satisfies equation (34).

4 Volume constrained stationary shapes

In this section we prove that, as opposed to the local setting, the only halfspaces which are stationary with respect to the non local Gaussian isoperimetric problem are the ones generated by hyperplanes passing through the origin.

Theorem 5 *We fix $a \in \mathbb{R}$ and $\omega \in \mathbb{S}^{n-1}$. If $H_{\omega,a} := \{x \in \mathbb{R}^n : \langle x, \omega\rangle < a\}$ is stationary with respect to the non local Gaussian isoperimetric problem, then $a = 0$, or equivalently,*

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{H_{\omega,a}} e^{-\frac{|x|^2}{2}} dx = \frac{1}{2}.$$

Proof. Up to rotation, we can assume $\omega = e_n$ and to shorten the notations we write H_a instead of $H_{e_n,a}$. We start observing that, for every $x \in \partial H_a$, it holds $\langle x, e_n\rangle = a$. This implies that, with the change of coordinate $z = y - x$, if $\langle y, e_n\rangle < a$, then $\langle z, e_n\rangle < 0$ and then we can write

$$\begin{aligned}\int_{H_a} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}} dy &= \int_{\{y_n<a\}} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}} dy = \int_{\{z_n<0\}} \frac{e^{-\frac{1}{4}(|z|^2+|x|^2+2\langle z,x\rangle)}}{|z|^{n+s}} dy \\ &= \int_{\{z_n>0\}} \frac{e^{-\frac{1}{4}(|z|^2+|x|^2-2\langle z,x\rangle)}}{|z|^{n+s}} dy, \qquad \forall x \in \partial H_a.\end{aligned} \tag{35}$$

Analogously, we compute

$$\int_{H_a^c} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}} dy = \int_{\{y_n>a\}} \frac{e^{-\frac{1}{4}|y|^2}}{|x-y|^{n+s}} dy = \int_{\{z_n>0\}} \frac{e^{-\frac{1}{4}(|z|^2+|x|^2+2\langle z,x\rangle)}}{|z|^{n+s}} dz \qquad \forall x \in \partial H_a. \tag{36}$$

Plugging equations (35), (36) in equation (34), we get

$$\int_{\{z_n>0\}} \frac{e^{-\frac{1}{4}(|z|^2+|x|^2)}}{|z|^{n+s}} \left(e^{\frac{\langle z,x\rangle}{2}} - e^{-\frac{\langle z,x\rangle}{2}}\right) dz = \lambda e^{-\frac{|x|^2}{4}}, \qquad \forall x \in \partial H_a, \tag{37}$$

which in turn reads

$$2\int_{\{z_n>0\}} \frac{e^{-\frac{1}{4}|z|^2}}{|z|^{n+s}} \sinh\left(\frac{\langle z,x\rangle}{2}\right) dz = \lambda, \qquad \forall x \in \partial H_a. \tag{38}$$

We remark that the integral in (38) is well defined, since $\lim_{x\to 0} \frac{\sinh(x)}{x} = 1$.

We split $x = (x', x_n)$ and we observe that

$$\begin{aligned}\sinh\left(\frac{\langle z,x\rangle}{2}\right) &= \sinh\left(\frac{\langle z',x'\rangle + z_n x_n}{2}\right) \\ &= \sinh\left(\frac{\langle z',x'\rangle}{2}\right)\cosh\left(\frac{z_n x_n}{2}\right) + \cosh\left(\frac{\langle z',x'\rangle}{2}\right)\sinh\left(\frac{z_n x_n}{2}\right).\end{aligned} \tag{39}$$

Plugging (39) in (38), we deduce the following equation for every $x \in \partial E$

$$\begin{aligned}A + B := &\int_{\{z_n>0\}} \frac{e^{-\frac{1}{4}|z|^2}}{|z|^{n+s}} \sinh\left(\frac{\langle z',x'\rangle}{2}\right)\cosh\left(\frac{z_n a}{2}\right) dz \\ &+ \int_{\{z_n>0\}} \frac{e^{-\frac{1}{4}|z|^2}}{|z|^{n+s}} \cosh\left(\frac{\langle z',x'\rangle}{2}\right)\sinh\left(\frac{z_n a}{2}\right) dz = \frac{\lambda}{2}.\end{aligned} \tag{40}$$

Since $\frac{e^{-\frac{1}{4}(|z'|^2+|z_n|^2)}}{(|z'|^2+|z_n|^2)^{\frac{n+s}{2}}}\sinh\left(\frac{\langle z',x'\rangle}{2}\right)\cosh\left(\frac{z_n a}{2}\right)$ is odd in z', we deduce

$$A=\int_0^\infty\int_{\mathbb{R}^{n-1}}\frac{e^{-\frac{1}{4}(|z'|^2+|z_n|^2)}}{\left(|z'|^2+|z_n|^2\right)^{\frac{n+s}{2}}}\sinh\left(\frac{\langle z',x'\rangle}{2}\right)\cosh\left(\frac{z_n a}{2}\right)dz'dz_n=0.$$

Plugging this information in (40) and taking the partial derivative in x_j, for every $j=1,\dots,n-1$, we deduce

$$\int_{\{z_n>0\}}\frac{e^{-\frac{1}{4}|z|^2}}{|z|^{n+s}}\frac{\partial}{\partial x_j}\left(\cosh\left(\frac{\langle z',x'\rangle}{2}\right)\right)\sinh\left(\frac{z_n a}{2}\right)dz=0.$$

Assuming without loss of generality that $j=n-1$ and denoting $x'=(x'',x_{n-1})$, we obtain

$$\begin{aligned}C+D:=&\int_{\{z_n>0\}}\frac{e^{-\frac{1}{4}|z|^2}}{|z|^{n+s}}\frac{\partial}{\partial x_j}\left(\cosh\left(\frac{\langle z'',x''\rangle}{2}\right)\cosh\left(\frac{z_{n-1}x_{n-1}}{2}\right)\right)\sinh\left(\frac{z_n a}{2}\right)dz\\&+\int_{\{z_n>0\}}\frac{e^{-\frac{1}{4}|z|^2}}{|z|^{n+s}}\frac{\partial}{\partial x_j}\left(\sinh\left(\frac{\langle z'',x''\rangle}{2}\right)\sinh\left(\frac{z_{n-1}x_{n-1}}{2}\right)\right)\sinh\left(\frac{z_n a}{2}\right)dz=0.\end{aligned}\tag{41}$$

Since $\frac{e^{-\frac{1}{4}|z|^2}}{|z|^{n+s}}\cosh\left(\frac{z_{n-1}x_{n-1}}{2}\right)z_{n-1}$ is odd in z_{n-1}, we deduce

$$D=\int_0^\infty\int_{\mathbb{R}^{n-2}}\int_{\mathbb{R}}\frac{e^{-\frac{1}{4}|z|^2}}{|z|^{n+s}}\sinh\left(\frac{\langle z'',x''\rangle}{2}\right)\cosh\left(\frac{z_{n-1}x_{n-1}}{2}\right)z_{n-1}\sinh\left(\frac{z_n a}{2}\right)dz=0.$$

Plugging this information in (41), we get that for every $x\in\partial E$ it holds

$$\int_{\{z_n>0\}}\frac{e^{-\frac{1}{4}|z|^2}}{|z|^{n+s}}\cosh\left(\frac{\langle z'',x''\rangle}{2}\right)\sinh\left(\frac{z_{n-1}x_{n-1}}{2}\right)\sinh\left(\frac{z_n a}{2}\right)z_{n-1}dz=0.\tag{42}$$

We denote

$$C(z_n,x):=\int_{\mathbb{R}^{n-1}}\frac{e^{-\frac{1}{4}|(z',z_n)|^2}}{|(z',z_n)|^{n+s}}\cosh\left(\frac{\langle z'',x''\rangle}{2}\right)\sinh\left(\frac{z_{n-1}x_{n-1}}{2}\right)z_{n-1}dz''dz_{n-1},$$

and we observe that if $x_{n-1}\neq 0$, then $C(z_n,x)\neq 0$ since the integrand is even in the variables z'' and z_{n-1}. Equation (42) then reads

$$\int_0^\infty C(z_n,x)\sinh\left(\frac{z_n a}{2}\right)dz_n=0,\qquad\forall x\in\partial E,\tag{43}$$

and since for every $z_n>0$

$$\sinh\left(\frac{z_n a}{2}\right)\begin{cases}>0 & \text{if } a>0\\ =0 & \text{if } a=0\\ <0 & \text{if } a<0\end{cases},$$

equation (43) can hold if and only if $a=0$. □

Remark 6 *It remains an interesting open question to characterize the critical points of the isoperimetric problem. We refer the interested reader to [11, 12, 13] for the characterization of critical points of the isoperimetric porblem in other settings.*

5 Acknowledgements

The first author has been partially supported by the NSF DMS Grant No. 1906451. The second author was partially supported by the Academy of Finland grant 314227.

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