# Uniqueness of critical points of the anisotropic isoperimetric problem for finite perimeter sets

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**Abstract**

Given an elliptic integrand of class $\mathscr{C}^{2,\alpha}$, we prove that finite unions of disjoint open Wulff shapes with equal radii are the only volume-constrained critical points of the anisotropic surface energy among all sets with finite perimeter and reduced boundary almost equal to its closure.

## 1 Introduction

### Overview

The classical anisotropic isoperimetric problem (or Wulff problem) amounts to minimizing the anisotropic boundary energy among all sets of finite perimeter with prescribed volume. For all positive (continuous) integrands the solution is uniquely characterized up to translation by the Wulff shape, as proved by Taylor in [42] and [43]. Alternative proofs can be found in [19, 30, 5]. This isoperimetric shape was constructed by Wulff in [44] and plays a central role in crystallography.

Instead of considering minima, a more subtle question asks to characterize *critical points* of the anisotropic boundary energy with prescribed volume. For integrands of class $\mathscr{C}^1$, this is equivalent to characterize sets of finite perimeter whose anisotropic mean curvature in the sense of varifolds is constant. For all convex integrands in $\mathbf{R}^2$, Morgan proved in [32] that Wulff shapes are the only critical points among all planar regions with boundary given by a closed and connected rectifiable curve. To the best of our knowledge, the characterization in every dimension for smooth boundaries has been conjectured for the first time by Giga in [20] and Morgan in [32]. For smooth elliptic integrands, this has been positively answered in [21] for dimension 3, and in [22] for every dimension. In particular, He, Li, Ma and Ge proved in [22] that if $F$ is a smooth elliptic integrand and $\Omega$ is a set with smooth boundary and constant $F$-mean curvature (more generally constant higher order $F$-mean curvature), then $\Omega$ is a Wulff shape. This is the anisotropic counterpart of the celebrated Alexandrov's theorem [1]. Moreover, quantitative stability versions of this rigidity theorem have been showed in [6, 15, 14]. Related results for immersed smooth hypersurfaces are [33, 21, 26]; for piecewise smooth hypersurfaces see [34, 25].

In the non-smooth setting, Maggi has conjectured in [28, Conjecture] the characterization of the Wulff shapes among *sets of finite perimeter*:

**Conjecture** ([28])**.** *For a positive convex integrand, Wulff shapes are the unique sets of finite perimeter and finite volume that are critical points of the anisotropic boundary energy at fixed volume.*

Since the integrand is assumed to be convex, but may fail to be $\mathscr{C}^1$, the notion of first variation and critical points are suitably defined using the convexity in time of the functional along any prescribed variational flow (see [28, p. 35-36]). Maggi specifies in [28] the significant interest from the physical viewpoint for crystalline integrands. Moreover he points out that this conjecture is open even for smooth elliptic anisotropic energies and among sets with Lipschitz boundaries, [28, p. 36].

In the full generality of sets of finite perimeter (and dimension greater than 2) the conjecture has been solved so far only for the *area functional* by Delgadino and Maggi in [8]. In this important paper [8], the authors prove that among sets of finite perimeter, finite unions of balls with equal radii are the unique volume-constrained critical points of the isotropic surface area. They obtain this result generalizing the argument of Montiel and Ros in [31] to sets of finite perimeter by means of the strong maximum principle of Schätzle [40]. They also prove a weak version of the Heintze-Karcher inequality for all sets of finite perimeter and, in a preliminary version of [8] (see [7]), they treat the special case of *local minimizers* of anisotropic smooth energies, using an anisotropic strong maximum principle proved in [12]. However, the reliance on the strong maximum principles of [40] and [12] prevents the extension of their method to treat critical points of anisotropic energies.

In the present paper we provide a positive answer to the aforementioned conjecture for elliptic integrands of class $\mathscr{C}^{2,\alpha}$ among finite perimeter sets with reduced boundary almost equal to their closure, see Corollary 6.9. Our Theorem 6.5 actually proves a stronger result, namely the following anisotropic Heintze-Karcher inequality for sets of finite perimeter (we refer to Section 2 for the notation) and the characterization of finite unions of disjoint open Wulff shapes as the unique configurations realizing the equality case.

**1.1 Theorem.** *Suppose $\alpha \in (0,1)$, $F$ is an elliptic integrand of class $\mathscr{C}^{2,\alpha}$ (see 2.12), $E \subseteq \mathbf{R}^{n+1}$ is a set of finite perimeter such that*

$$\mathscr{H}^n(\operatorname{Clos}(\partial^* E) \sim \partial^* E) = 0, \tag{1}$$

*the unit-density varifold associated with $\partial^* E$ has first variation absolutely continuous with respect to $\mathscr{H}^n \llcorner \partial^* E$ and its distributional mean curvature $H$ with respect to the inner measure-theoretic normal of $E$ is bounded and positive. Furthermore we suppose that $H$ is locally of class $\mathscr{C}^{0,\alpha}$ on the $\mathscr{C}^{1,\alpha}$ regular part of $\partial^* E$.*

*Then*

$$\mathscr{L}^{n+1}(E) \leq \frac{n}{n+1} \int_{\partial^* E} \frac{F(\mathbf{n}(E,x))}{H(x)} \, \mathrm{d}\mathscr{H}^n(x)\,. \tag{2}$$

*Equality holds if and only if $E$ coincides up to a set of $\mathscr{L}^{n+1}$ measure zero with a finite union of disjoint open Wulff shapes with radii not smaller than $n/\|H\|_{L^\infty}$.*

This theorem extends to the anisotropic setting an analogous result in the isotropic framework given in [38]. The proof of the inequality (2) is based on a suitable generalization of the Montiel-Ros argument, that is different from the generalization given in [8]. In fact the core of our proof is based on certain geometric properties of the generalized unit normal bundle (see 2.8) of the support of a varifold with bounded distributional mean curvature, whose proof does not depend on the Schätzle's maximum principle. It is also important to remark that, while for smooth varieties the analysis of the equality case in (2) easily follows from the well known rigidity of smooth (anisotropic) umbilical surfaces in the Euclidean space (see [31] and [22]), in a singular setting such analysis is a delicate issue.

The characterization of finite unions of Wulff shapes as the only volume-constrained critical points of the anisotropic perimeter among sets with reduced boundary almost equal to its closure, see Corollary 6.9, follows as a simple corollary of our main theorem. Here we denote by $\mathcal{P}_F$ *the $F$-perimeter functional*, i.e.

$$\mathcal{P}_F(E) = \int_{\partial^* E} F(\mathbf{n}(E,x)) \, \mathrm{d}\mathscr{H}^n(x)$$

for every $E \subseteq \mathbf{R}^{n+1}$ with finite perimeter.

**1.2 Corollary** (cf. 6.9)**.** *Suppose $\alpha \in (0,1)$, $F$ is an elliptic integrand of class $\mathscr{C}^{2,\alpha}$ and $E \subseteq \mathbf{R}^{n+1}$ is a finite perimeter set with finite volume such that*

$$\mathscr{H}^n(\operatorname{Clos}(\partial^* E) \sim \partial^* E) = 0.$$

*If $E$ is a volume-constrained critical point of $\mathcal{P}_F$, then $E$ is equivalent to a finite union of disjoint open Wulff shapes.*

It is worth to comment on the condition (1). By standard measure-theoretic results, see [18, 2.10.19(4)], such condition always holds if $E$ is a set of finite perimeter such that

$$\boldsymbol{\Theta}^{*n}(\mathscr{H}^n \llcorner \partial^* E, x) > 0 \quad \text{for every } x \in \operatorname{Clos}(\partial^* E)\,. \tag{3}$$

The density condition (3) is always satisfied if $E$ is a domain with Lipschitz boundary or if $E$ is either a local minimiser or an almost minimiser as considered in [7]. On the other hand it is not known if this density condition holds for every set of finite perimeter $E$ such that the unit-density varifold $\mathbf{v}(\partial^* E, 1)$ associated to the reduced boundary $\partial^* E$ has constant *anisotropic* distributional mean curvature. In the *isotropic* setting, this follows from the well known monotonicity formula. However, no monotonicity formula is known for anisotropic energies. In view of these facts, condition (1) is a natural hypothesis in the context of anisotropic energies.

Finally we remark that we consider here $\mathscr{C}^{2,\alpha}$ (for $0 < \alpha < 1$) integrands to ensure that the *regular part* of $\partial^* E$ is a classical smooth hypersurface. In fact, employing the Allard's regularity theory for codimension 1 integral varifolds with bounded anisotropic mean curvature [2], one gets that the regular part of $\partial^* E$ is a $\mathscr{C}^{1,\alpha}$-hypersurface. Assuming, as we did in our main theorem, that the distributional mean curvature is $\mathcal{C}^{0,\alpha}$, it follows from the classical regularity theory of elliptic PDEs that the regular part of $\partial^* E$ is $\mathscr{C}^{2,\alpha}$. This is one of the several parts where the method employed here conceptually differs from the method in [38], which instead is independent of Allard's regularity theory. This is due to the fact that some steps in [38] are based on the theory of curvature developed in [36] and on the locality theorem of Schätzle in [41] and these items are currently not available in the anisotropic setting. It is interesting to ask if they might be extended to the anisotropic setting, in which case one expects that the results of the present paper might be generalized to all elliptic integrands of class $\mathscr{C}^2$.

## Method of proof

Suppose $F$ is an elliptic integrand of class $\mathcal{C}^2$, $F^*$ is the conjugate of $F$ (see 2.29) and $B^{F^*}(a,r) = \{x : F^*(x-a) \leq r\}$. Let $A \subseteq \mathbf{R}^n$ be a closed set and we define the anisotorpic distance function by

$$\boldsymbol{\delta}^F_A(x) = \inf\{F^*(a-x) : a \in A\} \quad \text{for every } x \in \mathbf{R}^{n+1}.$$

The generalized anisotropic normal bundle of $A$ is given by

$$N^F(A) = (A \times \partial \mathbf{B}^{F^*}(0,1)) \cap \{(a,u) : \boldsymbol{\delta}^F_A(a+su) = s \text{ for some } s > 0\}.$$

The pillar of our proof is the following key geometric property of the generalized normal bundle of the support of a varifold with bounded anisotropic mean curvature.

**1.3 Theorem** (cf. 4.10 and 4.11)**.** *Suppose $V$ is an $n$-dimensional varifold in $\mathbf{R}^{n+1}$ such that the $F$-anisotropic first variation is absolutely continuous with respect to the weight measure $\|V\|$ associated to $V$ and the $F$-anisotropic distributional mean curvature vector $\mathbf{h}_F(V,\cdot)$ is bounded in length.*

*Then the following Lusin (N) condition holds:*

$$\mathscr{H}^n(N(\operatorname{spt}\|V\|) \cap \{(a,u) : a \in S\}) = 0$$

*whenever $S \subseteq \operatorname{spt}\|V\|$ with $\mathscr{H}^n(S) = 0$.*

This theorem provides a fundamental control on the singular part of $\operatorname{spt}\|V\|$, that allows to get integral inequalities. This will be the key to extend the Montiel-Ros argument to a singular setting.

We now explain the strategy to prove our main theorem (from now on the integrand $F$ is assumed to be of class $\mathscr{C}^{2,\alpha}$ elliptic). Let $V = \mathbf{v}(\partial^* E, 1)$ be the unit-density varifold associated to the essential boundary of $E$. Firstly, we use 6.2 to replace the set $E$ with an open set $\Omega$ such that $\mathscr{L}^{n+1}((\Omega \sim E) \cup (E \sim \Omega)) = 0$ and $\mathscr{H}^n(\partial\Omega \sim \partial^*\Omega) = 0$. Using Allard's regularity theory for codimension 1 varifolds with bounded anisotropic mean curvature [2], we deduce that $\mathscr{H}^n$ almost all of $\partial^*\Omega$ is $\mathscr{C}^{2,\alpha}$ regular. On the $\mathscr{C}^{2,\alpha}$ regular part we can express the distributional anisotropic mean curvature vector $\mathbf{h}_F(V,\cdot)$ as the trace of the anisotropic second fundamental form as in 2.21. The main difficulty to obtain the inequality (2) is to prove that one can perform the anisotropic version of the integral estimates of the Montiel-Ros argument only on the regular part of $\partial^* E$ to get the conclusion. This very delicate issue is resolved using Theorem 1.3. We deal now with the equality case. Firstly we notice that the principal curvatures of $\partial\Omega$ at $z$ must all be equal to $-n/H(z)$, for $z$ in the regular part of $\partial\Omega$. If $\partial\Omega$ was a regular hypersurface, then we could immediately conclude that $\partial\Omega$ is equal to a Wulff shape, because of the well known rigidity of umbilical surfaces. However there is no way to deduce from the regularity theory that $\partial\Omega$ has no singular part. Consequently, the fact that the regular part of $\partial\Omega$ is umbilical only implies that $\partial\Omega$ is made of a countable collection of pieces of Wulff shapes and of a singular set of $\mathscr{H}^n$ measure zero. Therefore the global shape of $\partial\Omega$ might be a priori arbitrarily complicated. To solve this issue, we first provide a general criterion to prove that a closed set has positive anisotropic reach (see 5.5), by means of an anisotropic Steiner formula, see 5.9. This result generalizes to the anisotropic setting a theorem of Heveling-Hug-Last in [23] and it is of independent interest. Then we prove that a set $E$ realizing the equality case in (2) satisfies the Steiner formula in (16). To this aim, we employ Theorem 1.3 and consequently we deduce that the anisotropic reach of $C$ satisfies $\operatorname{reach}^F C > n/\|H\|_{L^\infty}$. This is a crucial information, since it implies that the level-sets of the anisotropic distance function $S^F(C,r)$ are closed $\mathscr{C}^{1,1}$-hypersurfaces for every $0 < r < n/\|H\|_{L^\infty}$. Using once again 1.3, we also obtain that $S^F(C,r)$ are umbilical. Hence, by 3.2, we can conclude that $S^F(C,r)$ are finite unions of boundaries of Wulff shapes of radii not smaller than $(n - r\|H\|_{L^\infty})/\|H\|_{L^\infty}$. We conclude that each connected component of $\Omega$ must be a Wulff shape of radius at least $n/\|H\|_{L^\infty}$. Moreover, since the perimeter of $\Omega$ is finite, we also get that there are at most finitely many connected components of $\Omega$.

### Structure of the paper

In Section 2, after having recalled some background material, we provide some classical facts about Wulff shapes and we study some basic properties of the anisotropic nearest point projection onto an arbitrary closed set. In Section 3 we prove that the only totally umbilical closed and connected hypersurface of class $\mathscr{C}^{1,1}$ is the Wulff shape. In Section 4 we prove Theorem 1.3. Actually we prove this result in the more general class of anisotropic $(n,h)$-sets introduced in [11], thus extending an analogous result for isotropic $(n,h)$ sets obtained in [39, 3.8-3.9]. In Section 5 we establish the general criterion to prove that a closed set has positive reach by means of an anisotropic Steiner formula. To conclude, in Section 6 we prove our main theorem and its corollary.

## 2 Preliminaries

### Notation

The natural number $n \geq 1$ shall be fixed for the whole paper.

In principle, but with some exceptions explained below, we shall follow the notation of Federer (see [18, pp. 669 – 671]). The domain and the image of a map $f$ are denoted by $\operatorname{dmn} f$ and $\operatorname{im} f$. The set-theoretic difference between two sets $A$ and $B$ is denoted by $A \sim B$. Whenever $A \subseteq \mathbf{R}^{n+1}$ we denote by $\operatorname{Int} A$ and $\operatorname{Clos} A$ the interior and the closure of $A$ in $\mathbf{R}^{n+1}$. If $T \in \mathbf{G}(n+1,k)$, then we write $T_\natural$ for the linear orthogonal projection of $\mathbf{R}^{n+1}$ onto $T$. The symbol $\mathbb{N}$ stands for the set of non-negative integers. We use standard abbreviations for intervals $(a,b) = \mathbf{R} \cap \{t : a < t < b\}$ and $[a,b] = \mathbf{R} \cap \{t : a \leq t \leq b\}$. We also employ the terminology introduced in [18, 3.2.14] when dealing with rectifiable sets. Moreover, given a measure $\phi$ and a positive integer $m$ the notions of $(\phi,m)$ approximate tangent cone $\operatorname{Tan}^m(\phi,\cdot)$, $(\phi,m)$ approximate differentiability and $(\phi,m)$ approximate differential are used in agreement with [18, 3.2.16]. The $m$-dimensional density of a measure $\phi$ is denoted by $\boldsymbol{\Theta}^m(\phi,\cdot)$, see [18, 2.10.19]. We also introduce the symbol $\mathbf{S}^n$ for the unit $n$-dimensional sphere in $\mathbf{R}^{n+1}$. If $X \subseteq Y$ are sets, we write $\mathbf{1}_X : Y \to \mathbf{R}$ for the characteristic function of the set $X$ and $\operatorname{id}_Y : Y \to Y$ for the identity function on $Y$.

Concerning varifolds and submanifolds of $\mathbf{R}^{n+1}$ we use the notation introduced in [3]. The space of all $m$-dimensional varifolds on an open subset $U$ of $\mathbf{R}^{n+1}$ is denoted by $\mathbf{V}_m(U)$. If $M$ is a submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^1$, we write $\mathscr{X}(M)$ for compactly supported tangent vectorfields on $M$ of class $\mathscr{C}^1$; cf. [3, 2.5]. We say that $M$ is a *closed submanifold* of $\mathbf{R}^{n+1}$ if it is a submanifold of $\mathbf{R}^{n+1}$ and a closed (but not necessarily compact) subset of $\mathbf{R}^{n+1}$; in particular, $\partial M \sim M = \varnothing$.

We also use the following convention. Whenever $X$, $Y$ are normed vectorspaces, $A \subseteq X$, and $f : A \to Y$ we write $\mathrm{D}f$ for the derivative of $f$ that is a $\operatorname{Hom}(X,Y)$ valued function whose domain is the set of points of differentiability of $f$. If $Y = \mathbf{R}$ and $X$ is equipped with a scalar product, then we write $\operatorname{grad} f$ for the $X$ valued function characterised by

$$\langle u, \mathrm{D}f(x)\rangle = \operatorname{grad} f(x) \bullet u \quad \text{for } x \in \operatorname{dmn} \mathrm{D}f \text{ and } u \in X\,.$$

### The unit normal bundle of a closed set

Let $A \subseteq \mathbf{R}^{n+1}$ be a closed set.

**2.1 Definition.** Given $A \subseteq \mathbf{R}^{n+1}$ we define *the distance function to $A$* as

$$\boldsymbol{\delta}_A(x) = \inf\{|x-a| : a \in A\} \quad \text{for every } x \in \mathbf{R}^{n+1}.$$

Moreover,

$$S(A,r) = \{x : \boldsymbol{\delta}_A(x) = r\} \quad \text{for } r > 0.$$

*2.2 Remark* (cf. [36, 2.13]). If $r > 0$ then $\mathscr{H}^n(S(A,r) \cap K) < \infty$ whenever $K \subseteq \mathbf{R}^n$ is compact and $S(A,r)$ is countably $(\mathscr{H}^n, n)$ rectifiable of class 2.

**2.3 Definition** (cf. [36, 3.1]). If $U$ is the set of all $x \in \mathbf{R}^{n+1}$ such that there exists a unique $a \in A$ with $|x-a| = \boldsymbol{\delta}_A(x)$, we define the *nearest point projection onto* $A$ as the map $\boldsymbol{\xi}_A$ characterised by the requirement

$$|x - \boldsymbol{\xi}_A(x)| = \boldsymbol{\delta}_A(x) \quad \text{for } x \in U.$$

We set $U(A) = \operatorname{dmn} \boldsymbol{\xi}_A \sim A$. The functions $\boldsymbol{\nu}_A$ and $\boldsymbol{\psi}_A$ are defined by

$$\boldsymbol{\nu}_A(z) = \boldsymbol{\delta}_A(z)^{-1}(z - \boldsymbol{\xi}_A(z)) \quad \text{and} \quad \boldsymbol{\psi}_A(z) = (\boldsymbol{\xi}_A(z), \boldsymbol{\nu}_A(z)),$$

whenever $z \in U(A)$.

**2.4 Definition** (cf. [36, 3.6, 3.8, 3.13]). We define the function $\rho(A,\cdot)$ setting

$$\rho(A,x) = \sup\bigl\{t : \boldsymbol{\delta}_A(\boldsymbol{\xi}_A(x) + t(x - \boldsymbol{\xi}_A(x))) = t\boldsymbol{\delta}_A(x)\bigr\} \quad \text{for } x \in U(A)\,,$$

and we say that $x \in U(A)$ is a *regular point of* $\boldsymbol{\xi}_A$ if and only if $\boldsymbol{\xi}_A$ is approximately differentiable at $x$ with symmetric approximate differential and $\operatorname{ap}\lim_{y\to x} \rho(A,y) = \rho(A,x) > 1$. The set of regular points of $\boldsymbol{\xi}_A$ is denoted by $R(A)$.

For $\tau \geq 1$ we define

$$A_\tau = U(A) \cap \{x : \rho(A,x) \geq \tau\}\,.$$

*2.5 Remark* (cf. [36, 3.7]). The function $\rho(A,\cdot)$ is upper semicontinuous and its image is contained $[1,\infty]$.

**2.6 Definition** (cf. [36, 4.9]). Suppose $x \in R(A)$. Then $\chi_{A,1}(x) \leq \ldots \leq \chi_{A,n}(x)$ denote the eigenvalues of the symmetric linear map $\operatorname{ap} \mathrm{D}\boldsymbol{\nu}_A(x)|\{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$.

*2.7 Remark.* Notice that $\mathscr{H}^n(S(A,r) \sim R(A)) = 0$ for $\mathscr{L}^1$ a.e. $r > 0$ (cf. [36, 3.15]) and

$$\operatorname{Tan}^n(\mathscr{H}^n \llcorner S(A,r), x) = \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$$

for $\mathscr{H}^n$ a.e. $x \in S(A,r)$ and for $\mathscr{L}^1$ a.e. $r > 0$, cf. [36, 3.12].

The functions $\chi_{A,i}$ are the approximate principal curvatures of $S(A,r)$ in the direction of $\boldsymbol{\nu}_A(x)$. In fact, as proved in [36, 3.12], they coincide with the eigenvalues of the approximate second-order differential $\operatorname{ap} \mathrm{D}^2 S(A,r)$ of $S(A,r)$; cf. [37] for the general theory of higher order approximate differentiability for sets.

**2.8 Definition** (cf. [36, 4.1], [24, §2.1]). *The generalized unit normal bundle of* $A$ is defined as

$$N(A) = (A \times \mathbf{S}^n) \cap \{(a,u) : \boldsymbol{\delta}_A(a + su) = s \text{ for some } s > 0\}$$

and $N(A,a) = \{v : (a,v) \in N(A)\}$ for $a \in A$.

*2.9 Remark* (cf. [36, 4.3]). The set $N(A)$ is a countably $n$ rectifiable Borel subset of $\mathbf{R}^{n+1} \times \mathbf{S}^n$.

## Anisotropic mean curvature

Here we recall the notion of ellipticity for an integrand $F$ and the associated concept of $F$-mean curvature vector both for varifolds and smooth varieties.

**2.10 Definition.** Let $k \in \mathbb{N}$, $\alpha \in [0,1]$. By a *(convex) integrand of class* $\mathscr{C}^{k,\alpha}$ we mean a non-negative (convex) function $F : \mathbf{R}^{n+1} \to \mathbf{R}$ such that $F|\mathbf{R}^{n+1} \sim \{0\}$ is of class $\mathscr{C}^{k,\alpha}$ and

$$F(\lambda\nu) = |\lambda| F(\nu) \quad \text{for } \nu \in \mathbf{R}^{n+1} \text{ and } \lambda \in \mathbf{R}$$

By an *integrand* we mean an integrand of class $\mathscr{C}^0$. Moreover, we say that a convex integrand is *strictly convex* if

$$F(x+y) < F(x) + F(y) \quad \text{for all linearly independent } x, y \in \mathbf{R}^{n+1}\,.$$

*2.11 Remark.* Evidently strictly convex integrands correspond to strictly convex norms on $\mathbf{R}^{n+1}$.

**2.12 Definition** (cf. [18, 5.1.2] and [2, 3.1(4)])**.** We say that an integrand $F$ is *elliptic* if there exists a number $\gamma > 0$ such that the map $\mathbf{R}^{n+1} \ni u \mapsto F(u) - \gamma|u|$ is convex. We call $\gamma$ the *ellipticity constant of* $F$.

*2.13 Remark* (cf. [18, 5.1.3]). Assume $F$ is an integrand of class $\mathscr{C}^{1,1}$. Then ellipticity of $F$ with ellipticity constant $\gamma > 0$ is equivalent to the condition

$$\left\langle (v,v),\, \mathrm{D}^2F(u) \right\rangle \geq \gamma \frac{|u \wedge v|^2}{|u|^3} = \gamma \frac{|v|^2 - (v \bullet u/|u|)^2}{|u|} \quad \text{for } u \in \operatorname{dmn} \mathrm{D}^2F,\, u \neq 0,\, v \in \mathbf{R}^{n+1}\,. \tag{4}$$

In particular, if $F$ is elliptic, $u \in \operatorname{dmn} \mathrm{D}^2F$, $|u| = 1$, and $v \in \operatorname{span}\{u\}^{\perp}$, then

$$\left\langle (v,v),\, \mathrm{D}^2F(u) \right\rangle \geq \gamma |v|^2\,,$$

which shows that $F$ is uniformly elliptic in the sense of [11, §2]. The interested reader can find a more exhaustive discussion about ellipticity conditions (in general codimension) in [9, 16].

**2.14 Definition.** Assume $F$ is an elliptic integrand with ellipticity constant $\gamma > 0$. We define

$$C(F) = \sup\left( \left\{ \gamma^{-1},\, \sup F[\mathbf{S}^n] / \inf F[\mathbf{S}^n] \right\} \cup \left\{ \|\mathrm{D}^2F(\nu)\| : \nu \in \mathbf{S}^n \cap \operatorname{dmn} \mathrm{D}^2F \right\} \right).$$

*2.15 Remark.* Let $U \subseteq \mathbf{R}^{n+1}$ be open. For any $T \in \mathbf{G}(n+1,n)$ we choose arbitrarily $\nu(T) \in T^{\perp}$ such that $|\nu(T)| = 1$. In the sequel we shall tacitly identify any $V \in \mathbf{V}_n(U)$ with a Radon measure $\bar{V}$ over $U \times \mathbf{R}^{n+1}$ such that

$$\bar{V}(\alpha) = \frac{1}{2} \int \alpha(x, \nu(T)) + \alpha(x, -\nu(T))\, \mathrm{d}V(x,T) \quad \text{for } \alpha \in C^0_c(U, \mathbf{R})\,.$$

Clearly, this definition does not depend on the choice of $\nu(T)$.

**2.16 Definition.** Let $U \subseteq \mathbf{R}^{n+1}$ be open, $F$ be an integrand of class $\mathscr{C}^1$, $V \in \mathbf{V}_n(U)$. We define the *first variation of* $V$ *with respect to* $F$ by the formula

$$\delta_F V(g) = \int \mathrm{D}g(x) \bullet B_F(\nu)\, \mathrm{d}V(x,\nu) \quad \text{for } g \in \mathscr{X}(U)\,,$$

where $B_F(\nu) \in \operatorname{Hom}(\mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ is given by

$$B_F(\nu)u = F(\nu)u - \nu \cdot \langle u,\, \mathrm{D}F(\nu) \rangle \quad \text{for } \nu, u \in \mathbf{R}^{n+1},\, \nu \neq 0\,.$$

*2.17 Remark* (cf. [2],[9, Appendix A],[13],[10]). If $\varphi : \mathbf{R} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is smooth, $\varphi(0,x) = x$ for $x \in \mathbf{R}^{n+1}$, and $g = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \varphi(t,\cdot) \in \mathscr{X}(\mathbf{R}^{n+1})$, then

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \Phi_F(\varphi_{t\#}V) = \delta_F V(g)\,,$$

where the functional $\Phi_F : \mathbf{V}_n(U) \to [0,\infty]$ is defined as

$$\Phi_F(V) = \int F(\nu)\, \mathrm{d}V(x,\nu)\,.$$

**2.18 Definition** (cf. [11, §2])**.** Let $\Omega \subseteq \mathbf{R}^{n+1}$ be open, $V \in \mathbf{V}_n(\Omega)$, $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be an integrand of class $\mathscr{C}^1$. Assume that $\|\delta_F V\|$ is a Radon measure. Then

$$\delta_F V(g) = -\int \overline{\mathbf{h}}_F(V,x) \bullet g(x)\, \mathrm{d}\|V\|(x) + \int \boldsymbol{\eta}_F(V,x) \bullet g(x)\, \mathrm{d}\|\delta_F V\|_{\mathrm{sing}}(x) \quad \text{for } g \in \mathscr{X}(\Omega)\,,$$

where $\|\delta_F V\|_{\mathrm{sing}}$ is the singular part of $\|\delta_F V\|$ with respect to $\|V\|$, $\overline{\mathbf{h}}_F(V,\cdot)$ is an $\mathbf{R}^{n+1}$ valued $\|V\|$-integrable function, and $\boldsymbol{\eta}_F(V,\cdot)$ is an $\mathbf{S}^n$ valued $\|\delta_F V\|$-integrable function.

For $\|V\|$-a.e. $x$ we define the *$F$-mean curvature vector of $V$ at $x$*, denoted $\mathbf{h}_F(V,x)$, by the formula

$$\mathbf{h}_F(V,x) = \frac{\overline{\mathbf{h}}_F(V,x)}{\int F(\nu)\, \mathrm{d}V^{(x)}(\nu)}\,,$$

where $V^{(x)}$ is the probability measure on $\mathbf{S}^n$ coming from disintegration of $V$; see [3, §3.3].

**2.19 Definition.** Define $\boldsymbol{\Xi} : \bigodot^2 \mathbf{R}^{n+1} \to \operatorname{Hom}(\mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ to be the linear map[1] characterised by

$$\langle u, \boldsymbol{\Xi}(A)\rangle \bullet v = A(u, v) \quad \text{for } A \in \textstyle\bigodot^2 \mathbf{R}^{n+1} \text{ and } u, v \in \mathbf{R}^{n+1}.$$

*2.20 Remark.* In particular, if $f : \mathbf{R}^{n+1} \to \mathbf{R}$ is twice differentiable at $x \in \mathbf{R}^{n+1}$, then

$$\boldsymbol{\Xi}(\mathrm{D}^2 f(x)) = \mathrm{D}(\operatorname{grad} f)(x) \in \operatorname{Hom}(\mathbf{R}^{n+1}, \mathbf{R}^{n+1}).$$

*2.21 Remark.* Let $G \subseteq \mathbf{R}^{n+1}$ be open, $v_1, \dots, v_{n+1}$ be an orthonormal basis of $\mathbf{R}^{n+1}$, $M$ be a submanifold of $G$ of dimension $n$ of class $\mathscr{C}^2$, $V = \mathbf{v}_n(M) \in \mathbf{V}_n(G)$, $x \in M$, $\nu : G \to \mathbf{R}^{n+1}$ be of class $\mathscr{C}^1$ and satisfy

$$|\nu(y)| = 1, \quad \nu(y) \in \operatorname{Nor}(M, y), \quad \text{and} \quad \langle \nu(y), \mathrm{D}\nu(y)\rangle = 0 \quad \text{for } y \in M.$$

In [11, Proposition 2.1] the authors show that if $F$ is an elliptic integrand of class $\mathscr{C}^2$ then

$$-F(\nu(x))\mathbf{h}_F(V, x) = \nu(x) \operatorname{tr}\big(\mathrm{D}(\operatorname{grad} F \circ \nu)(x)\big) = \nu(x) \sum_{j=1}^{n+1} \big\langle (\mathrm{D}\nu(x)v_j, v_j),\, \mathrm{D}^2 F(\nu(x))\big\rangle.$$

**2.22 Definition** (cf. [18, 4.5.5])**.** Let $A \subseteq \mathbf{R}^{n+1}$ and $b \in \mathbf{R}^{n+1}$. We say that $u$ is an *exterior normal of $A$ at $b$* if $u \in \mathbf{R}^{n+1}$, $|u| = 1$,

$$\begin{gathered} \boldsymbol{\Theta}^{n+1}(\mathscr{L}^{n+1} \llcorner \{x : (x - b) \bullet u > 0\} \cap A, b) = 0, \\ \text{and} \quad \boldsymbol{\Theta}^{n+1}(\mathscr{L}^{n+1} \llcorner \{x : (x - b) \bullet u < 0\} \sim A, b) = 0. \end{gathered}$$

We also set $\mathbf{n}(A, b) = u$ if $u$ is the exterior normal of $A$ at $b$ and $\mathbf{n}(A, b) = 0$ if there exists no exterior normal of $A$ at $b$.

**2.23 Definition** (cf. [4, Def. 3.54])**.** Let $A \subseteq \mathbf{R}^{n+1}$ be a set of finite perimeter and $V = \mathbf{v}_{n+1}(A) \in \mathbf{V}_{n+1}(\mathbf{R}^{n+1})$. Then $\|\delta V\|$ is a Radon measure (cf. [3, 4.7]) and there exists $\|\delta V\|$ measurable function $\boldsymbol{\eta}(V, \cdot)$ with values in $\mathbf{S}^n$ as in [3, 4.3]. We define the *reduced boundary of $A$*, denoted $\partial^* A$, as the set of points $x \in \operatorname{dmn} \boldsymbol{\eta}(V, \cdot)$ for which

$$\|\delta V\| \, \mathbf{B}(x, r) > 0 \quad \text{for } r > 0 \text{ and} \quad \lim_{r \downarrow 0} \frac{1}{\|\delta V\| \, \mathbf{B}(x, r)} \int_{\mathbf{B}(x,r)} \boldsymbol{\eta}(V, \cdot) \, \mathrm{d}\|\delta V\| = \boldsymbol{\eta}(V, x).$$

**2.24 Definition.** Let $E \subseteq \mathbf{R}^{n+1}$ and $x \in \mathbf{R}^{n+1}$. We define

$$\mathbf{n}^F(E, x) = \operatorname{grad} F(\mathbf{n}(E, x)) \quad \text{if } \mathbf{n}(E, x) \neq 0 \quad \text{and} \quad \mathbf{n}^F(E, x) = 0 \quad \text{if } \mathbf{n}(E, x) = 0.$$

*2.25 Remark.* Assume $X$ is a Hilbert space, $\dim X = k \in \mathbb{N}$, $A, B \in \operatorname{Hom}(X, X)$ are self-adjoint automorphisms of $X$, and $A$ is positive definite. With the help of the (tiny) spectral theorem [27, Chap. VIII, Thm. 4.3] we find a self-adjoint and positive definite map $C \in \operatorname{Hom}(X, X)$ such that $A = C \circ C$. Next, we observe that $E = C^{-1} \circ A \circ B \circ C = C \circ B \circ C$ is self-adjoint. Employing again the (tiny) spectral theorem we find an orthonormal basis $v_1, \dots, v_k \in X$ and real numbers $\lambda_1, \dots, \lambda_k$ such that $Ev_i = \lambda_i v_i$ for $i \in \{1, 2, \dots, k\}$. We obtain

$$A \circ B(Cv_i) = C \circ Ev_i = \lambda_i C v_i \quad \text{for } i \in \{1, 2, \dots, k\}$$

and we see that $Cv_1, \dots, Cv_k$ is a basis of eigenvectors of $A \circ B$ with eigenvalues $\lambda_1, \dots, \lambda_k$.

In particular, if $G$, $M$, $x$, and $\nu$ are as in 2.21, $F$ is an elliptic integrand, $u = \nu(x) \in \operatorname{dmn} \mathrm{D}^2 F$, and $X = \operatorname{Tan}(M, x)$, then the maps $A = \boldsymbol{\Xi}(\mathrm{D}^2 F(\nu(x)) | X \times X)$ and $B = \mathrm{D}\nu(x) | X \in \operatorname{Hom}(X, X)$ are self-adjoint and $A$ is positive definite; hence, $A \circ B$ has exactly $n$ real eigenvalues.

Observe also that since $F$ is positively 1-homogeneous, $\operatorname{grad} F$ is positively 0-homogeneous, i.e., $\operatorname{grad} F(\lambda v) = \operatorname{grad} F(v)$ for $\lambda \in (0, \infty)$ and $v \in \operatorname{dmn} \operatorname{grad} F$; hence,

$$v \in \ker \mathrm{D}(\operatorname{grad} F)(v) \quad \text{for } v \in \operatorname{dmn} \mathrm{D}^2 F. \tag{5}$$

Since $\mathrm{D}^2 F(\nu(x)) \in \bigodot^2 \mathbf{R}^{n+1}$ is symmetric it follows that $\mathrm{D}(\operatorname{grad} F)(\nu(x)) \in \operatorname{Hom}(\mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ is self-adjoint and we have

$$\operatorname{im} \mathrm{D}(\operatorname{grad} F)(\nu(x)) = \big(\ker \mathrm{D}(\operatorname{grad} F)(\nu(x))\big)^{\perp}$$

so that $\mathrm{D}(\operatorname{grad} F)(\nu(x)) | X \in \operatorname{Hom}(X, X)$ by (5). Seeing that also $\mathrm{D}\nu(x) | X \in \operatorname{Hom}(X, X)$ we conclude

$$\mathrm{D}(\operatorname{grad} F \circ \nu)(x) | X \in \operatorname{Hom}(X, X).$$

[1] As in [18, 1.10] the symbol $\bigodot^2 X$ denotes the vectorspace of bilinear maps of the type $X \times X \to \mathbf{R}$.

**2.26 Definition.** Let $F$ be an elliptic integrand of class $\mathscr{C}^{1,1}$, $G \subseteq \mathbf{R}^{n+1}$ be open, $M$ be a submanifold of $G$ of dimension $n$ of class $\mathscr{C}^{1,1}$, $\nu : G \to \mathbf{R}^{n+1}$ be Lipschitz continuous and such that $|\nu(z)| = 1$ and $\nu(z) \in \operatorname{Nor}(M, z)$ for $z \in M$, $x \in \operatorname{dmn} \mathrm{D}\nu$, and $u = \operatorname{grad} F(\nu(x))$. We define the *$F$-principal curvatures of $M$ at $(x, u)$*

$$\kappa^F_{M,1}(x, u) \leq \ldots \leq \kappa^F_{M,n}(x, u)$$

to be the eigenvalues of the map $\mathrm{D}(\operatorname{grad} F \circ \nu)(x)| \operatorname{Tan}(M, x) \in \operatorname{Hom}(\operatorname{Tan}(M, x), \operatorname{Tan}(M, x))$; cf. 2.25.

*2.27 Remark.* If $V = \mathbf{v}_n(M) \in \mathbf{V}_n(G)$, then one may check using 2.21 that

$$F(\nu(x))\mathbf{h}_F(V, x) = \overline{\mathbf{h}}_F(V, x) = -\nu(x) \textstyle\sum_{i=1}^n \kappa^F_{M,i}(x, u)\,.$$

## Wulff shapes

Here we collect basic facts on Wulff shapes for readers convenience.

**2.28 Definition.** Let $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be a norm, $x \in \mathbf{R}^{n+1}$ and $r > 0$. We define

$$\mathbf{U}^F(x, r) = \{y : F(y - x) < r\} \quad \text{and} \quad \mathbf{B}^F(x, r) = \{y : F(y - x) \leq r\}\,.$$

**2.29 Definition.** Let $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be a norm. Define the *conjugate norm* $F^*$ on $\mathbf{R}^{n+1}$ by setting

$$F^*(w) = \sup\{w \bullet u : u \in \mathbf{R}^{n+1},\ F(u) \leq 1\}.$$

By a *Wulff shape (of $F$)* we mean any open ball with respect to the $F^*$ norm.

*2.30 Remark.* If $F$ is a norm on $\mathbf{R}^{n+1}$ then $F^*$ is a norm on $\mathbf{R}^{n+1}$ and $F = F^{**}$. Moreover,

$$F^*(w) = \sup\{w \bullet u : u \in \mathbf{R}^{n+1},\ F(u) = 1\}.$$

*2.31 Remark.* If $f : \mathbf{R}^{n+1} \to \mathbf{R}$ is a convex function then, following [35, pag. 104], we define *the conjugate of $f$* to be the convex function $g : \mathbf{R}^{n+1} \to \mathbf{R}$ given by

$$g(x) = \sup\bigl\{(x \bullet u) - f(u) : u \in \mathbf{R}^{n+1}\bigr\} \quad \text{for } x \in \mathbf{R}^{n+1}\,.$$

Let $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be a 1-homogeneous function such that $F^{-1}\{0\} = \{0\}$ and set $f = \frac{1}{2}F^2$. Fix $x \in \mathbf{R}^{n+1} \sim \{0\}$. Whenever $u \in \mathbf{R}^{n+1}$ satisfies $F(u) = 1$ define

$$\varphi_u(\lambda) = x \bullet (\lambda u) - f(\lambda u) = \lambda(x \bullet u) - \tfrac{1}{2}\lambda^2 \quad \text{for } \lambda \in \mathbf{R}\,.$$

Note that $\varphi_u$ is a quadratic polynomial having exactly one maximum

$$\sup\{\varphi_u(\lambda) : \lambda \in \mathbf{R}\} = \varphi_u(x \bullet u) = \tfrac{1}{2}(x \bullet u)^2\,.$$

Maximizing with respect to $u \in \mathbf{R}^{n+1}$ satisfying $F(u) = 1$ we see that

$$\sup\bigl\{(x \bullet u) - f(u) : u \in \mathbf{R}^{n+1}\bigr\} = \tfrac{1}{2} \sup\bigl\{(x \bullet u)^2 : u \in \mathbf{R}^{n+1}, F(u) = 1\bigr\}\,.$$

Therefore, if $F$ is a norm, then the conjugate of $\frac{1}{2}F^2$ equals $\frac{1}{2}(F^*)^2$.

In the next lemma we summarize few facts about $F$ and $F^*$.

**2.32 Lemma.** *Let $F$ be a strictly convex norm, differentiable at each point of $\mathbf{R}^{n+1} \sim \{0\}$. Define $W = \mathbf{U}^F(0, 1)$, $W^* = \mathbf{U}^{F^*}(0, 1)$, $G = \operatorname{grad} F$ and $G^* = \operatorname{grad} F^*$.*

*Then the following statements hold.*

(a) *$F$ is continuously differentiable on $\mathbf{R}^{n+1} \sim \{0\}$.*

(b) *$F^*$ is strictly convex norm and continuously differentiable on $\mathbf{R}^{n+1} \sim \{0\}$.*

(c) *$F(x) = x \bullet G(x)$ and $G(\lambda x) = (\lambda/|\lambda|)G(x)$ for $\lambda \neq 0$ and $x \in \mathbf{R}^{n+1} \sim \{0\}$. The same statement holds if we replace $F$ and $G$ with $F^*$ and $G^*$.*

(d) *For every $w \in \mathbf{R}^{n+1} \sim \{0\}$ there exists a unique $u \in \partial W$ such that $F^*(w) = u \bullet w$ and $G(u)F^*(w) = w$. The same statement holds if we replace $W$, $F^*$ and $G$ with $W^*$, $F$ and $G^*$.*

(e) *$G[\mathbf{R}^{n+1} \sim \{0\}] = \partial W^*$ and $G^*[\mathbf{R}^{n+1} \sim \{0\}] = \partial W$.*

(f) *$G|\partial W$ is an injective map onto $\partial W^*$ with $(G|\partial W)^{-1} = G^*|\partial W^*$.*

(g) *$G|\mathbf{S}^n$ is an injective map onto $\partial W^*$ with $(G|\mathbf{S}^n)^{-1} = \mathbf{n}(W^*, \cdot)$; $G^*|\mathbf{S}^n$ is an injective map onto $\partial W$ with $(G^*|\mathbf{S}^n)^{-1} = \mathbf{n}(W, \cdot)$.*

(h) *$\mathbf{n}(W^*, x) = F(\mathbf{n}(W^*, x))G^*(x)$ for $x \in \partial W^*$ and $\mathbf{n}(W, x) = F^*(\mathbf{n}(W, x))G(x)$ for $x \in \partial W$.*

(i) *If $F$ is an elliptic integrand of class $\mathscr{C}^{1,1}$, then $F^*$ is an elliptic integrand of class $\mathscr{C}^{1,1}$.*

*Proof.* The assertion in (a) follows from [35, 25.5]. Noting 2.31, one sees that (b) is a consequence of [35, 26.3]. Then the assertion in (c) directly follows from the positive 1-homogeneity of $F$ and $F^*$.

Claim 1: *$G|\partial W$ and $G^*|\partial W^*$ are injective maps.* Assume that $G|\partial W$ is not injective, i.e., that there exist $a, b \in \partial W$ such that $a \neq b$ and $G(a) = G(b)$. Then it follows from (c) that $a$ and $b$ are linearly independent. Set $u = b - a$ and define the map $f : [0, 1] \to \mathbf{R}^{n+1}$ by the formula $f(t) = F(a+tu)$, which is a strictly convex function on $[0, 1]$ because $a$ and $b$ are linearly independent. Then $f'(0) = G(a) \bullet u$ and $f'(1) = G(b) \bullet u$, so $f'(0) = f'(1)$ which contradicts strict convexity of $f$. Therefore $G|\partial W$ is injective. In view of (b) and (c) we also have that $G^*|\partial W^*$ is injective.

For any $w \in \mathbf{R}^{n+1}$ define $g_w : \mathbf{R}^{n+1} \to \mathbf{R}$ by the formula $g_w(u) = u \bullet w$ for $u \in \mathbf{R}^{n+1}$. Let $w \in \mathbf{R}^{n+1} \sim \{0\}$ and select $u \in \partial W$ such that $g_w(u) = \sup\{g_w(v) : v \in \partial W\} = F^*(w)$. Then $w = \operatorname{grad} g_w(u) \in \operatorname{Nor}(\partial W, u)$, whence we deduce that $w = \lambda G(u)$ for some $\lambda \in \mathbf{R}$. Using (c) we infer that

$$F^*(w) = u \bullet w = \lambda u \bullet G(u) = \lambda F(u) = \lambda \,.$$

Henceforth, $G(u)F^*(w) = w$ and the uniqueness asserted in (d) is a consequence of the injectivity of $G|\partial W$ proved in Claim 1. Noting that $F^{**} = F$ and (b), the second part of (d) follows from its first part by duality.

Let $y \in \mathbf{R}^{n+1} \sim \{0\}$. It follows from (d) that there exists a unique $u \in \partial W$ such that $G(u) = \frac{y}{F^*(y)}$ and there exists a unique $v \in \partial W^*$ such that $F(u) = u \bullet v$ and $G^*(v) = \frac{u}{F(u)} = u$. Since $F(u) = u \bullet G(u)$ by (c) and both $G(u)$ and $v$ belong to $\partial W^*$, we conclude from the aforementioned uniqueness that $\frac{y}{F^*(y)} = G(u) = v$. Therefore we get from the 0-homogeneity of $G^*$ that

$$\frac{y}{F^*(y)} = G(G^*(v)) = G(G^*(y/F^*(y))) = G(G^*(y)). \tag{6}$$

By duality we also obtain

$$\frac{y}{F(y)} = G^*(G(y)). \tag{7}$$

Now, noting that the inclusions $\partial W^* \subseteq G[\partial W]$ and $\partial W \subseteq G^*[\partial W^*]$ follow immediately from (d), one may infer from the equalities in (6) and (7) that $\partial W^* = G[\partial W]$ and $\partial W = G^*[\partial W^*]$. Henceforth, (f) is proved and the statement in (e) follows from the 0-homogeneity of $G$ claimed in (c).

Let $x \in \partial W^*$. Since $G^*(x)$ is an exterior normal to $W^*$ at $x$, then $G^*(x) = \lambda \mathbf{n}(W^*, x)$ for some $\lambda > 0$ and we employ (c), (e) and (f) to compute

$$x = G(G^*(x)) = G(\lambda \mathbf{n}(W^*, x)) = G(\mathbf{n}(W^*, x)),$$

$$1 = F(G^*(x)) = \lambda F(\mathbf{n}(W^*, x)).$$

Noting that $\operatorname{im}(\mathbf{n}(W^*, \cdot)) = \mathbf{S}^n$, we readily obtain the first parts of the statements in (g) and (h), while the second parts follow, as usual, by duality.

We assume now that $F$ is an elliptic integrand of class $\mathscr{C}^{1,1}$. It follows that $G|\mathbf{S}^n$ is a bi-lipschitzian homeomorphism. Henceforth, by (g) and (h), $\mathbf{n}(W^*, \cdot)$ and $G^*|\partial W^*$ are Lipschitz maps. It follows that $F^*$ is of class $\mathscr{C}^{1,1}$. Moreover, since $\mathbf{n}(W, \cdot)$ is a Lipschitz map, we infer from (g) that $(G^*|\mathbf{S}^n)^{-1}$ is Lipschitz and, consequently, $F^*$ is elliptic.

□

**2.33 Corollary.** *Assume $F$ is an elliptic integrand of class $\mathscr{C}^{1,1}$, $r \in \mathbf{R}$ is positive, $W = \mathbf{U}^{F^*}(0, r)$, $\eta : \partial W \to \mathbf{R}^{n+1}$ is given by $\eta(z) = \operatorname{grad} F(\mathbf{n}(W, z))$ for $z \in \partial W$. We have $\eta(z) = z/r$ for $z \in \partial W$ so $\mathrm{D}\eta(y)v = v/r$ for $v \in \operatorname{Tan}(\partial W, y)$ and $y \in \partial W$; hence, recalling 2.26 and 2.27 we see that*

$$\kappa^F_{\partial W, i}(y, \eta(y)) = 1/r \quad \text{for } y \in \partial W \text{ and } i = 1, \dots, n \,.$$

Now we prove a basic one-sided estimate for the anisotropic principal curvatures of a smooth submanifold at the touching points with Wulff shapes.

**2.34 Lemma.** *Assume $F$ is an elliptic integrand of class $\mathscr{C}^2$, $\Omega \subseteq \mathbf{R}^{n+1}$ is open, $\partial\Omega$ is a manifold of class $\mathscr{C}^2$, $a \in \Omega$, $r > 0$, $W = \mathbf{R}^{n+1} \cap \{x : F^*(x-a) < r\}$, $W \subseteq \Omega$, $b \in \partial\Omega \cap \partial W$, $\nu_\Omega : \partial\Omega \to \mathbf{R}^{n+1}$, $\nu_\Omega(z) = \mathbf{n}(\Omega, z)$ for $z \in \partial\Omega$.*

*Then $\kappa^F_{\partial\Omega,i}(b, \operatorname{grad} F(\nu_\Omega(b))) \leq 1/r$ for $i = 1, \dots, n$.*

*Proof.* Define $\nu_W : \partial W \to \mathbf{R}^{n+1}$ by $\nu_W(z) = \mathbf{n}(W, z)$ for $z \in \partial W$ and set $T = \operatorname{Tan}(\partial\Omega, b)$ and $\nu = \nu_\Omega(b)$. Since $W \subseteq \Omega$ we have $T = \operatorname{Tan}(\partial W, b)$ and $\nu = \nu_W(b)$; cf. 2.38(h).

Assume $b = 0$. If $U \subseteq \mathbf{R}^{n+1}$ is an open neighbourhood of $b$ and $f_\Omega : T \to \mathbf{R}$ and $f_W : T \to \mathbf{R}$ are functions of class $\mathscr{C}^2$ such that

$$U \cap \{z : f_\Omega(T_\natural z) < -\nu \bullet z\} = U \cap \Omega$$

$$U \cap \{z : f_W(T_\natural z) < -\nu \bullet z\} = U \cap W,$$

then the inclusion $W \subseteq \Omega$ implies $f_\Omega(T_\natural z) \leq f_W(T_\natural z)$ for $z \in U$ and $\mathrm{D}^2 f_\Omega(0)(u,u) \leq \mathrm{D}^2 f_W(0)(u,u)$ for all $u \in T$. We conclude that

$$\mathrm{D}\nu_\Omega(0)u \bullet u = \mathrm{D}^2 f_\Omega(0)(u,u) \leq \mathrm{D}^2 f_W(0)(u,u) = \mathrm{D}\nu_W(0)u \bullet u \quad \text{for } u \in T. \tag{8}$$

Define the linear maps $A, B, E \in \operatorname{Hom}(T,T)$ by the formulas

$$A = \mathrm{D}(\operatorname{grad} F)(\nu)|T, \quad B = \mathrm{D}\nu_\Omega(0), \quad E = \mathrm{D}\nu_W(0).$$

Since $A = A^*$ and $Au \bullet u > 0$ for all nonzero $u \in T$, recalling 2.25, we find a self-adjoint and positive map $C \in \operatorname{Hom}(T,T)$ such that $A = C \circ C$. Noting from 2.33 that $A \circ Eu = \mathrm{D}(\operatorname{grad} F \circ \nu_W)(b)u = r^{-1}u$ for $u \in T$, it follows that

$$C \circ E \circ Cu = C^{-1} \circ A \circ E \circ Cu = \frac{u}{r} \quad \text{for } u \in T.$$

Next, applying (8) with $Cu$ in place of $u$, we get

$$C \circ B \circ Cu \bullet u = B(Cu) \bullet (Cu) \leq E(Cu) \bullet (Cu) = C \circ E \circ Cu \bullet u = \frac{|u|^2}{r} \quad \text{for } u \in T.$$

Finally, noting that $C \circ B \circ C$ is a self-adjoint map with the same eigenvalues of the map $A \circ B$ and $A \circ B = \mathrm{D}(\operatorname{grad} F \circ \nu_\Omega)(b)|T$, we get the conclusion. □

*2.35 Remark.* The idea of the proof of Lemma 2.34 is taken from [7, pp. 25-26].

## Anisotropic nearest point projection

Here we introduce the anisotropic nearest point projection onto an arbitrary closed set and we prove some basic properties.

**2.36 Definition.** Let $F$ be a norm on $\mathbf{R}^{n+1}$. Given $A \subseteq \mathbf{R}^{n+1}$, we define *the anisotropic distance function to $A$* as

$$\boldsymbol{\delta}^F_A(x) = \inf\{F^*(a-x) : a \in A\} \quad \text{for every } x \in \mathbf{R}^{n+1}.$$

Moreover,

$$S^F(A, r) = \{x : \boldsymbol{\delta}^F_A(x) = r\} \quad \text{for } r > 0.$$

**2.37 Definition.** Suppose $A \subseteq \mathbf{R}^n$ is closed and $W$ is the set of all $x \in \mathbf{R}^n$ such that there exists a unique $a \in A$ with $F^*(x-a) = \boldsymbol{\delta}^F_A(x)$. The *anisotropic nearest point projection onto $A$* is the map $\boldsymbol{\xi}^F_A : W \to A$ characterised by the requirement

$$F^*(x - \boldsymbol{\xi}^F_A(x)) = \boldsymbol{\delta}^F_A(x) \quad \text{for } x \in W.$$

We also define $\boldsymbol{\nu}^F_A : W \sim A \to \partial\mathbf{B}^{F^*}(0,1)$ and $\boldsymbol{\psi}^F_A : W \sim A \to A \times \partial\mathbf{B}^{F^*}(0,1)$ by the formulas

$$\boldsymbol{\nu}^F_A(z) = \boldsymbol{\delta}^F_A(z)^{-1}(z - \boldsymbol{\xi}^F_A(z)) \quad \text{and} \quad \boldsymbol{\psi}^F_A(z) = (\boldsymbol{\xi}^F_A(z), \boldsymbol{\nu}^F_A(z)) \quad \text{for } z \in W \sim A.$$

**2.38 Lemma.** *Let $F : \mathbf{R}^{n+1} \to \mathbf{R}$ be norm over $\mathbf{R}^{n+1}$, $G = \operatorname{grad} F$, $G^* = \operatorname{grad} F^*$, $A \subseteq \mathbf{R}^{n+1}$ be closed. Then*

(a) $|\boldsymbol{\delta}^F_A(y) - \boldsymbol{\delta}^F_A(z)| \le F^*(y-z)$ *for* $y, z \in \mathbf{R}^{n+1}$.

(b) $\boldsymbol{\xi}^F_A$ *is continuous.*

(c) *Suppose* $x \in \mathbf{R}^{n+1} \sim A$ *and* $a \in A$ *are such that* $\boldsymbol{\delta}^F_A(x) = F^*(x-a)$. *Then*

$$\boldsymbol{\delta}^F_A(a + t(x-a)) = tF^*(x-a) = t\boldsymbol{\delta}^F_A(x) \quad \text{for } 0 < t \le 1 .$$

*In case $F$ is a strictly convex norm of class $\mathscr{C}^1$ then the following additional statements hold.*

(d) *Suppose* $x \in \mathbf{R}^{n+1} \sim A$ *such that* $\mathrm{D}\boldsymbol{\delta}^F_A(x)$ *exists. Then* $x \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ *and*

$$\operatorname{grad} \boldsymbol{\delta}^F_A(x) = G^*\Big(\frac{x - \boldsymbol{\xi}^F_A(x)}{\boldsymbol{\delta}^F_A(x)}\Big), \quad G(\operatorname{grad} \boldsymbol{\delta}^F_A(x)) = \frac{x - \boldsymbol{\xi}^F_A(x)}{\boldsymbol{\delta}^F_A(x)}.$$

(e) *The maps* $\boldsymbol{\delta}^F_A | \operatorname{Int}\big(\operatorname{dmn} \boldsymbol{\xi}^F_A \sim A\big)$ *and* $(\boldsymbol{\delta}^F_A)^2 | \operatorname{Int}\big(\operatorname{dmn} \boldsymbol{\xi}^F_A\big)$ *are continuously differentiable and*

$$\big\langle u, \mathrm{D}(\boldsymbol{\delta}^F_A)^2(y) \big\rangle = \big\langle u, \mathrm{D}(F^*)^2(y - \boldsymbol{\xi}^F_A(y)) \big\rangle \quad \text{for } y \in \operatorname{Int}\big(\operatorname{dmn} \boldsymbol{\xi}^F_A\big) \text{ and } u \in \mathbf{R}^{n+1} .$$

(f) $\mathscr{L}^{n+1}(\mathbf{R}^{n+1} \sim \operatorname{dmn} \boldsymbol{\xi}^F_A) = 0$.

(g) *Assume* $a \in A$, $u \in \partial \mathbf{B}^{F^*}(0,1)$, $t > 0$, *and* $\boldsymbol{\delta}^F_A(a + tu) = t$. *Then* $a + su \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ *and* $\boldsymbol{\xi}^F_A(a + su) = a$ *for all* $0 < s < t$. *In particular,*

$$\{s : \boldsymbol{\xi}^F_A(a+su) = a\} \subseteq \{s : \boldsymbol{\delta}^F_A(a+su) = s\} = \operatorname{Clos}\{s : \boldsymbol{\xi}^F_A(a+su) = a\} .$$

(h) *Assume* $a \in A$, $x \in \mathbf{R}^{n+1}$, *and* $\boldsymbol{\delta}^F_A(x) = F^*(x-a)$. *Then*

$$x - a \in G(\operatorname{Nor}(A, a)) .$$

*In particular, if* $\mathbf{n}(A,a) \neq 0$, *then*

$$\mathbf{n}^F(A,a) = \boldsymbol{\nu}^F_A(x) = \frac{x-a}{F^*(x-a)} .$$

*Proof.* We mimic parts of the proof of [17, 4.8].

Let $y, z \in \mathbf{R}^{n+1}$, then

$$\boldsymbol{\delta}^F_A(y) \le \boldsymbol{\delta}^F_A(z) + F^*(y-z) \quad \text{and} \quad \boldsymbol{\delta}^F_A(z) \le \boldsymbol{\delta}^F_A(y) + F^*(y-z) ;$$

hence, claim (a) follows.

Assume that (b) does not hold. Then there are $y_i \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ for $i \in \mathbb{N}$ and $\varepsilon > 0$ such that $\lim_{i\to\infty} y_i = y \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ but $F^*(\boldsymbol{\xi}^F_A(y_i) - \boldsymbol{\xi}^F_A(y)) \ge \varepsilon$. Using (a) we get

$$F^*(\boldsymbol{\xi}^F_A(y_i) - y) \le \boldsymbol{\delta}^F_A(y) + 2F^*(y_i - y) \quad \text{for } i \in \mathbb{N} ;$$

hence, the set $\{\boldsymbol{\xi}^F_A(y_i) : i \in \mathbb{N}\}$ is a bounded subset of the closed set $A$ and we may assume that $\lim_{i\to\infty} \boldsymbol{\xi}^F_A(y_i) = z \in A$. Then

$$\boldsymbol{\delta}^F_A(y) = \lim_{i\to\infty} \boldsymbol{\delta}^F_A(y_i) = \lim_{i\to\infty} F^*(\boldsymbol{\xi}^F_A(y_i) - y_i) = F^*(z-y) ;$$

hence, $\boldsymbol{\xi}^F_A(y) = z$ which is incompatible with

$$F^*(z - \boldsymbol{\xi}^F_A(y)) = \lim_{i\to\infty} F^*(\boldsymbol{\xi}^F_A(y_i) - \boldsymbol{\xi}^F_A(y)) \ge \varepsilon .$$

Assume (c) does not hold. Then there are $0 < t < 1$ and $b \in A$ such that setting $y = a + t(x-a)$ we get $F^*(y-b) < F^*(y-a)$ and

$$F^*(x-a) \le F^*(x-b) \le F^*(x-y) + F^*(y-b) < F^*(x-y) + F^*(y-a) = F^*(x-a)$$

a contradiction.

Now we prove (d). Let $a \in A$ be such that $\boldsymbol{\delta}_A^F(x) = F^*(x-a)$. By (c) we have

$$\boldsymbol{\delta}_A^F(x + t(a-x)) = \boldsymbol{\delta}_A^F(x) - t\boldsymbol{\delta}_A^F(x) \quad \text{for } 0 < t < 1\,,$$

which implies

$$\operatorname{grad} \boldsymbol{\delta}_A^F(x) \bullet \frac{x-a}{\boldsymbol{\delta}_A^F(x)} = \frac{\mathrm{D}\boldsymbol{\delta}_A^F(x)(a-x)}{-\boldsymbol{\delta}_A^F(x)} = 1. \tag{9}$$

Noting that (9) and (a) imply that

$$\begin{aligned} 1 = \sup\{\mathrm{D}\boldsymbol{\delta}_A^F(x)u : u \in \mathbf{R}^{n+1},\ F^*(u) \le 1\} \\ = \sup\{\operatorname{grad} \boldsymbol{\delta}_A^F(x) \bullet u : u \in \mathbf{R}^{n+1},\ F^*(u) \le 1\} \\ = F^{**}(\operatorname{grad} \boldsymbol{\delta}_A^F(x)) = F(\operatorname{grad} \boldsymbol{\delta}_A^F(x)), \end{aligned}$$

we employ 2.32(d)(f) to conclude that

$$G(\operatorname{grad} \boldsymbol{\delta}_A^F(x)) = \frac{x-a}{\boldsymbol{\delta}_A^F(x)}, \quad \operatorname{grad} \boldsymbol{\delta}_A^F(x) = G^*\Big(\frac{x - \boldsymbol{\xi}_A^F(x)}{\boldsymbol{\delta}_A^F(x)}\Big).$$

The formula for $\mathrm{D}(\boldsymbol{\delta}_A^F)^2$ postulated in (e) follows from (d) arguing exactly as in [17, 4.8(5)] and noting that

$$\langle u,\, \mathrm{D}(F^*)^2(y)\rangle = 2F^*(y)G^*(y) \bullet u \quad \text{for } u \in \mathbf{R}^{n+1}.$$

Continuity of the derivatives of $\boldsymbol{\delta}_A^F | \mathbf{R}^{n+1} \sim A$ and $(\boldsymbol{\delta}_A^F)^2$ follows from the formulas and a reasoning completely analogous to the proof of [17, 4.8(5)].

Item (f) is a consequence of (d) and the Rademacher theorem [18, 3.1.6].

For the proof of (g), firstly we notice that $\boldsymbol{\delta}_A^F(a+su) = s$ for $0 < s \le t$ by (c). Now assume to the contrary, that there exist $0 < s < t$ and $b \in A$, $b \neq a$ such that $s = F^*(a+su-a) = F^*(a+su-b) = \boldsymbol{\delta}_A^F(a+su)$. Set $p = a+su$ and $q = a+tu$. Clearly $b \neq p+su$ since otherwise $t = \boldsymbol{\delta}_A^F(q) \le F^*(q-b) = F^*(a+tu-(a+2su)) = t-2s < t$ which is impossible. Therefore, $q-a$ and $q-b$ are linearly independent and, since $F^*$ is strictly convex by 2.32(b), we obtain the contradictory estimate

$$t \le F^*(q-b) < F^*(q-p) + F^*(p-b) = t - s + s = t\,.$$

To prove (h) we observe that

$$\mathbf{U}^{F^*}(x, F^*(x-a)) \cap A = \varnothing\,; \quad \text{hence,} \quad -\mathbf{n}(\mathbf{B}^{F^*}(x, F^*(x-a)), a) \in \operatorname{Nor}(A,a)\,.$$

Indeed, otherwise there would exist $v \in \operatorname{Tan}(A,a)$, $|v| = 1$, such that $v \bullet \mathbf{n}(\mathbf{B}^{F^*}(x, F^*(x-a)), a) < 0$ so there would be points $y_i \in A$ such that $|y_i - a| \to 0$ and $(y_i - a)/|y_i - a| \to v$ as $i \to \infty$ and then, since $F^*$ is of class $\mathscr{C}^1$, we could find $i \in \mathbb{N}$ for which $y_i \in \mathbf{U}^{F^*}(x, F^*(x-a)) \cap A$ and this cannot happen. Henceforth, employing 2.32(c)(g) we see that

$$\frac{x-a}{F^*(x-a)} = G(-\mathbf{n}(\mathbf{B}^{F^*}(x, F^*(x-a)), a)) \in G(\operatorname{Nor}(A,a))\,. \qquad \square$$

## 3 Totally umbilical hypersurfaces

In 2.33 we proved that $\partial \mathbf{B}^{F^*}(0,r)$ has all $F$-principal curvatures equal to $1/r$. In this section we show that this condition actually characterises the manifold $\partial \mathbf{B}^{F^*}(0,r)$.

**3.1 Lemma.** *Suppose $M$ is a connected submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ of dimensions $n$, $\eta : M \to \mathbf{R}^{n+1}$ is Lipschitz, and $\kappa : M \to \mathbf{R}$ is such that*

$$\mathrm{D}\eta(z)(u) = \kappa(z)u \quad \textit{for } \mathscr{H}^n \textit{ almost all } z \in M \textit{ and all } u \in \operatorname{Tan}(M,z)\,.$$

*Then $\kappa$ is a constant function.*

*Proof.* Since $M$ is connected it suffices to show the claim only locally. Let $a \in M$. We represent $M$ near $a$ as the graph of some $\mathscr{C}^{1,1}$ function $f$, i.e., we find $p \in \mathbf{O}^*(n+1,n)$, $q \in \mathbf{O}^*(n+1,1)$, $U \subseteq \mathbf{R}^n$ an open ball centred at $p(a)$, and $f : U \to \mathbf{R}$ of class $\mathscr{C}^{1,1}$ such that[2], setting $L = p^* + q^* \circ f$, there holds

$$a \in L[U] \subseteq M \quad \text{and} \quad q \circ p^* = 0\,.$$

For each $v \in \mathbf{R}^n$ we define

$$\gamma_v : U \to \mathbf{R} \quad \text{by} \quad \gamma_v(x) = \eta(L(x)) \bullet v\,.$$

Then

$$\begin{aligned} \mathrm{D}\gamma_v(x)u &= \mathrm{D}\eta(L(x))(\mathrm{D}L(x)u) \bullet v = \kappa(L(x))(\mathrm{D}L(x)u) \bullet v \\ &= \kappa(L(x))(p^*(u) + q^*(\mathrm{D}f(x)u)) \bullet v = \kappa(L(x))(u \bullet p(v) + \mathrm{D}f(x)u \bullet q(v)) \\ &\qquad \text{for } \mathscr{L}^n \text{ almost all } x \in U,\ u \in \mathbf{R}^n,\ v \in \mathbf{R}^{n+1}\,. \end{aligned}$$

Now, choose an orthonormal basis $e_1, \dots, e_n$ of $\mathbf{R}^n$ and set $\gamma_i = \gamma_{p^*(e_i)}$ for $i = 1, 2, \dots, n$. Since $q \circ p^* = 0$ and $p \circ p^* = \mathbf{1}_{\mathbf{R}^n}$, we obtain

$$\text{(10)} \quad \mathrm{D}\gamma_i(x)e_j = \kappa(L(x))(e_i \bullet e_j) = 0 \quad \text{and} \quad \mathrm{D}\gamma_i(x)e_i = \kappa(L(x)) \\ \text{for } \mathscr{L}^n \text{ almost all } x \in U,\ i,j \in \{1,2,\dots,n\}, \text{ and } i \neq j\,.$$

Recall that $U$ is an open ball centred at $p(a)$. Define $J = \{(x - p(a)) \bullet e_1 : x \in U\}$. Since $\eta$ is Lipschitz we see that $\gamma_1, \dots, \gamma_n$ are absolutely continuous and deduce from (10) that there exist Lipschitz functions $a_1, \dots, a_n : J \to \mathbf{R}$ such that

$$\begin{aligned} &\gamma_i(x) = a_i((x - p(a)) \bullet e_i) \\ &\qquad \text{and} \quad a_i'((x - p(a)) \bullet e_i) = a_j'((x - p(a)) \bullet e_j) = \kappa(L(x)) \\ &\qquad\qquad \text{for } \mathscr{L}^n \text{ almost all } x \in U,\ i,j \in \{1,2,\dots,n\}\,. \end{aligned}$$

It follows that $a_i'$ is a constant function for $i = 1, 2, \dots, n$; hence, $\kappa$ is also constant. □

**3.2 Lemma.** *Suppose $F$ is an elliptic integrand of class $\mathscr{C}^{1,1}$, $M$ is a connected $n$-dimensional submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ satisfying* $\operatorname{Clos} M \sim M = \varnothing$, *$\nu : M \to \mathbf{R}^{n+1}$ is Lipschitz and such that $\nu(z) \in \operatorname{Nor}(M,z)$ and $|\nu(z)| = 1$, $\eta : M \to \mathbf{R}^{n+1}$ is defined by $\eta(y) = \operatorname{grad} F(\nu(y))$, and there exists a scalar function $\kappa : M \to \mathbf{R}$ such that*

$$\mathrm{D}\eta(y)u = \kappa(y)u \quad \textit{for } \mathscr{H}^n \textit{ almost all } y \in M \textit{ and all } u \in \operatorname{Tan}(M,y)\,.$$

*Then there exists $\lambda \in \mathbf{R}$ such that $\kappa(y) = \lambda$ for $y \in M$ and either $\lambda = 0$ and $M$ is a hyperplane in $\mathbf{R}^{n+1}$ or $\lambda \neq 0$ and $M = \partial \mathbf{B}^F(a, |\lambda|^{-1})$ for some $a \in \mathbf{R}^{n+1}$.*

*Proof.* In view of 3.1 we obtain $\lambda \in \mathbf{R}$ such that

$$\mathrm{D}\eta(z)u = \lambda u \quad \text{for all } \mathscr{H}^n \text{ almost all } z \in M \text{ and } u \in \operatorname{Tan}(M,z)\,.$$

Therefore, $\mathrm{D}(\eta - \lambda \mathrm{id}_{\mathbf{R}^n}) = 0$ and we obtain $c \in \mathbf{R}^n$ such that

$$\eta(z) - \lambda z = c \quad \text{for all } z \in M\,.$$

If $\lambda = 0$, then $\eta$ is constant and $M$ must be a hyperplane because $\operatorname{Clos} M \sim M = \varnothing$. In case $\lambda \neq 0$ we set $a = -c\lambda^{-1}$ and $\rho = |\lambda|^{-1}$. Then

$$F^*(z - a) = \rho F^*(\eta(z)) = \rho F^*(\operatorname{grad} F(\nu(z))) = \rho \quad \text{for all } z \in M\,,$$

by 2.32(e). Hence, $M = \partial \mathbf{B}^{F^*}(a, \rho)$ because $\operatorname{Clos} M \sim M = \varnothing$. □

[2] As in [18, 1.7.4] we write $\mathbf{O}^*(n,k)$ for the set of $\alpha \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^k)$ such that $\alpha^* \circ \alpha = (\operatorname{im} \alpha^*)_\natural$ and $\alpha \circ \alpha^* = \mathrm{id}_{\mathbf{R}^k}$.

# 4 The Lusin property for anisotropic (n,h)-sets

In this section $F$ is an elliptic integrand of class $\mathscr{C}^2$ and $\Omega \subseteq \mathbf{R}^{n+1}$ is open.

**4.1 Definition** (cf. [11, Definition 3.1])**.** We say that $Z \subseteq \Omega$ is an *(n,h)-set with respect to* $F$ if $Z$ is relatively closed in $\Omega$ and for any open set $N \subseteq \Omega$ such that $\partial N \cap \Omega$ is smooth and $Z \subseteq \operatorname{Clos} N$ there holds

$$F(\mathbf{n}(N,p))\mathbf{h}_F(\mathbf{v}_n(\partial N),p) \bullet \mathbf{n}(N,p) \geq -h \quad \text{for } p \in Z \cap \partial N \cap \Omega \,.$$

In order to prove the main result of this section we need to prove a weak maximum principle for $(n,h)$ sets (see 4.8), where the barrier is not in general a smooth hypersurface but only a graph of function that is a twice differentiable at the touching point with the $(n,h)$ set. For this purpose, it seems convenient to introduce a concept of *pointwise anisotropic mean curvature* for those sets which are twice differentiable in the sense of the recent work [29].

**4.2 Definition** (cf. [29, §2.7])**.** Let $k \in \mathbb{N}$, $X$, $Y$ be normed vectorspaces, $A \subseteq X$, $f : A \to Y$, and $a \in X$. Then $f$ is called *pointwise differentiable of order $k$ at $a$* if there exists an open set $U \subseteq X$ and a function $g : U \to Y$ of class $k$ such that

$$a \in U \subseteq A \,, \quad f(a) = g(a) \,, \quad \text{and} \quad \lim_{x \to a} \frac{|f(x) - g(x)|}{|x-a|^k} = 0 \,.$$

Whenever this is satisfied one defines also the *pointwise differential of order $i$ of $f$ at $a$* by

$$\operatorname{pt} \mathrm{D}^i f(a) = \mathrm{D}^i g(a) \quad \text{for } i \in \{0, 1, \ldots, k\} \,.$$

**4.3 Definition** (cf. [29, §3.3])**.** Suppose $k, n \in \mathbb{N}$ and $A \subseteq \mathbf{R}^{n+1}$. Then $A$ is called *pointwise differentiable of order $k$ at $a$* if there exists a submanifold $B$ of $\mathbf{R}^{n+1}$ of class $k$ such that $a \in B$,

$$\begin{aligned} &\lim_{r \downarrow 0} r^{-1} \sup |\operatorname{distance}(\cdot, A) - \operatorname{distance}(\cdot, B)| [\mathbf{B}(a,r)] = 0 \,, \\ &\quad \text{and} \quad \lim_{r \downarrow 0} r^{-k} \sup \operatorname{distance}(\cdot, B) [A \cap \mathbf{B}(a,r)] = 0 \,. \end{aligned}$$

**4.4 Definition** (cf. [29, §3.12])**.** Suppose $n, k \in \mathbb{N}$ and $A \subseteq \mathbf{R}^{n+1}$. Then $\operatorname{pt} \mathrm{D}^k A$ is the function whose domain consists of pairs $(a, S)$ such that $a \in \operatorname{Clos} A$, $A$ is pointwise differentiable of order $k$ at $a$, $S \in \mathbf{G}(n+1, \dim \operatorname{Tan}(A,a))$, and $S^\perp \cap \operatorname{Tan}(A,a) = \{0\}$ and whose value at $(a,S)$ equals the unique $\phi \in \bigodot^k(\mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ such that whenever $f : S \to S^\perp$ is of class $k$ and satisfies

$$\begin{aligned} &\lim_{r \downarrow 0} r^{-1} \sup |\operatorname{distance}(\cdot, A) - \operatorname{distance}(\cdot, B)| [\mathbf{B}(a,r)] = 0 \,, \\ &\quad \text{and} \quad \lim_{r \downarrow 0} r^{-k} \sup \operatorname{distance}(\cdot, B) [A \cap \mathbf{B}(a,r)] = 0 \,, \end{aligned}$$

where $B = \{x + f(x) : x \in S\}$, then $\phi = \mathrm{D}^k(f \circ S_\natural)(a)$.

*4.5 Remark* (cf. [29, §§3.14, 3.15]). Assume $n, d, k \in \mathbb{N}$, $S \in \mathbf{G}(n+1, d)$, $U \subseteq S$ is open, $f : U \to S^\perp$ is continuous, $x \in U$, $A = \{\chi + f(\chi) : \chi \in S\}$. Then $A$ is pointwise differentiable of order $k$ at $a = x + f(x)$ if and only if $f$ is pointwise differentiable of order $k$ at $x$. Moreover, $\operatorname{pt} \mathrm{D}^i A(a, S) = \operatorname{pt} \mathrm{D}^i (f \circ S_\natural)(x)$ for $i \in \{0, 1, \ldots, k\}$.

**4.6 Definition.** Assume $M \subseteq \mathbf{R}^{n+1}$ is pointwise differentiable of order 2 at $a \in \operatorname{Clos} M$, $T \in \mathbf{G}(n+1, n)$, $f : T \to T^\perp$ is pointwise differentiable of order 2 at 0, $f(0) = 0$, $\operatorname{pt} \mathrm{D} f(0) = 0$, $B = \mathbf{R}^{n+1} \cap \{a + x + f(x) : x \in T\}$, $\nu \in T^\perp$, $|\nu| = 1$, and

$$\begin{aligned} &\lim_{r \downarrow 0} r^{-1} \sup |\operatorname{distance}(\cdot, M) - \operatorname{distance}(\cdot, B)| [\mathbf{B}(a,r)] = 0 \,, \\ &\quad \text{and} \quad \lim_{r \downarrow 0} r^{-2} \sup \operatorname{distance}(\cdot, B) [M \cap \mathbf{B}(a,r)] = 0 \,. \end{aligned}$$

We define the *pointwise $F$-mean curvature vector of $M$ at $a$*, denoted $\operatorname{pt} \mathbf{h}_F(M,a)$, by the formula

$$-F(\nu) \operatorname{pt} \mathbf{h}_F(M,a) = \nu \operatorname{tr}\big(\boldsymbol{\Xi}(\mathrm{D}^2 F(\nu)) \circ \boldsymbol{\Xi}(\operatorname{pt} \mathrm{D}^2 (f \circ T_\natural)(0) \bullet \nu)\big) \,.$$

*4.7 Remark.* Note that the above definition does not depend on the choice of $\nu$ and $f$.

**4.8 Lemma.** *Suppose $T \in \mathbf{G}(n+1,n)$, $\eta \in T^\perp$, $|\eta| = 1$, $f : T \to T^\perp$ is pointwise differentiable of order 2 at 0 and satisfies $f(0) = 0$ and $\operatorname{pt} \mathrm{D} f(0) = 0$, $\Sigma = \{x + f(x) : x \in T\}$, $h \geq 0$, and $\Gamma$ is an $(n,h)$ subset of $\Omega$ with respect to $F$ such that $0 \in \Gamma$ and*

$$\Gamma \cap V \subseteq \big\{ z : z \bullet \eta \leq f \circ T_\natural(z) \bullet \eta \big\}$$

*for some open neighbourhood $V$ of 0. Then*

$$F(\eta) \operatorname{pt} \mathbf{h}_F(\Sigma, 0) \bullet \eta \geq -h.$$

*Proof.* We mimic the proof of [39, 3.4]. Fix $\varepsilon > 0$, define $P, \psi : T \to T^\perp$ by

$$\begin{aligned} P(x) &= \tfrac{1}{2} \langle (x,x), \operatorname{pt} \mathrm{D}^2 f(0) \rangle \quad \text{for } x \in T\,, \\ \psi(x) &= \big( P(x) \bullet \eta + \varepsilon |x|^2 \big) \eta \quad \text{for } x \in T\,, \\ \text{and set} \quad M &= \mathbf{R}^n \cap \big\{ x + \psi(x) : x \in T \big\}\,. \end{aligned}$$

Note that since $f$ is pointwise differentiable of order 2 at 0, it follows that

$$\lim_{x \to 0} \frac{|f(x) - P(x)|}{|x|^2} = 0\,.$$

Hence, we choose $r > 0$ such that $f(x) \bullet \eta \leq \psi(x) \bullet \eta$ for $x \in \mathbf{U}(0,r) \cap T$. Since $\Gamma$ is an $(n,h)$ subset of $\Omega$, $M$ is smooth and touches $\Gamma$ at 0, and $\Gamma \cap \mathbf{U}(0,r) \subseteq \mathbf{R}^{n+1} \{x : x \bullet \eta \leq \psi(x) \bullet \eta\}$, we may use the barrier principle [11, Proposition 3.1(iii)] to derive the estimate

$$F(\eta) \mathbf{h}_F(M, 0) \bullet \eta \geq -h\,.$$

Recall 4.6 to see that

$$-F(\eta) \operatorname{pt} \mathbf{h}_F(M,0) = \eta \operatorname{tr}\big( \mathbf{\Xi}(\mathrm{D}^2 F(\eta)) \circ \mathbf{\Xi}(\mathrm{D}^2 (\psi \circ T_\natural)(0) \bullet \eta) \big)\,.$$

Since

$$\mathrm{D}^2 (\psi \circ T_\natural)(0)(u,v) \bullet \eta = \operatorname{pt} \mathrm{D}^2 (f \circ T_\natural)(0)(u,v) \bullet \eta + 2\varepsilon u \bullet T_\natural v \quad \text{for } u, v \in \mathbf{R}^n$$

we see that

$$-F(0,\eta) \operatorname{pt} \mathbf{h}_F(\Sigma,0) = -F(\eta) \operatorname{pt} \mathbf{h}_F(M,0) - 2\varepsilon\eta \operatorname{tr}\big( \mathbf{\Xi}(\mathrm{D}^2 F(\eta)) \big)\,.$$

Passing to the limit $\varepsilon \downarrow 0$ we obtain the claim. ☐

**4.9 Definition.** Suppose $A \subseteq \mathbf{R}^{n+1}$ is a closed set. We say that $N(A)$ satisfies the $n$ *dimensional Lusin (N) condition in* $\Omega$ if and only if

$$S \subseteq A \cap \Omega \quad \text{and} \quad \mathscr{H}^n(S) = 0 \quad \text{implies that} \quad \mathscr{H}^n(N(A)|S) = 0\,.$$

**4.10 Theorem.** *Suppose $0 \leq h < \infty$, $A$ is an $(n,h)$ subset of $\Omega$ with respect to $F$.*
*Then $N(A)$ satisfies the $n$ dimensional Lusin (N) condition in $\Omega$.*

*Proof.* We modify the proof of [39, 3.8]. Let $\tau > \lambda = 2C(F)^2(n-1) + 1$, where $C(F) > 0$ is defined in 2.14.

Claim 1: *Assume $r \in \mathbf{R}$ satisfies $0 \leq h < \frac{1}{2C(F)r}$, and $x \in S(A,r) \cap R(A) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(A)$ (see 2.4) is such that $\mathbf{\Theta}^n(\mathscr{H}^n \llcorner S(A,r) \sim A_\tau, x) = 0$, and the conclusions of [39, 2.9] are satisfied. Consider an orthonormal basis $v_1, \ldots, v_{n+1}$ in which the matrix of $\operatorname{ap} \mathrm{D} \boldsymbol{\nu}_A(x)$ is diagonal and $v_{n+1} = \boldsymbol{\nu}_A(x)$. We introduce abbreviations*

$$\partial_{ij} F(\nu) = \langle (v_i, v_j), \mathrm{D}^2 F(\nu) \rangle \quad \textit{for } i,j \in \{1, 2, \ldots, n+1\}\,.$$

*Then we have*

$$\sum_{i=1}^n \partial_{ii} F(\boldsymbol{\nu}_A(x)) \chi_{A,i}(x) \leq h \quad \textit{and} \quad \big\| {\textstyle\bigwedge_n} \big( (\mathscr{H}^n \llcorner S(A,r), n) \operatorname{ap} \mathrm{D} \boldsymbol{\xi}_A(x) \big) \big\| > 0\,.$$

Noting that $\boldsymbol{\xi}_A | A_\lambda$ is approximately differentiable at $x$ (since $x \in R(A)$), we employ [36, 3.7, 3.10(3)(6)] and [18, 3.2.16] to conclude that

$$\chi_{A,j}(x) \geq -(\lambda-1)^{-1} r^{-1} \quad \text{for } j = 1, \ldots, n\,, \tag{11}$$

$$\operatorname{ap} \mathrm{D}\boldsymbol{\xi}_A(x) | \operatorname{Tan}(\mathscr{H}^n \llcorner S(A,r), x) = (\mathscr{H}^n \llcorner S(A,r), n) \operatorname{ap} \mathrm{D}\boldsymbol{\xi}_A(x). \tag{12}$$

We choose $f$, $V$ and $T$ as in [39, 2.9] and $0 < s < r/2$ such that $\mathbf{U}(x,s) \subseteq V$. We assume $\boldsymbol{\xi}_A(x) = 0 \in \Gamma$ and we notice that $T_\natural(x) = 0$ and $\boldsymbol{\nu}_A(x) = r^{-1}x$. Then we define $g(\zeta) = f(\zeta) - x$ for $\zeta \in T$,

$$U = T_\natural\big(\mathbf{U}(x,s) \cap \{\chi + f(\chi) : \chi \in T\}\big), \quad W = \{y - x : y \in T_\natural^{-1}(U) \cap \mathbf{U}(x,s)\}\,.$$

It follows that $W$ is an open neighbourhood of $0$ and

$$W \cap A \subseteq \{z : z \bullet \boldsymbol{\nu}_A(x) \leq g(T_\natural(z)) \bullet \boldsymbol{\nu}_A(x)\}\,. \tag{13}$$

Indeed, if (13) did not hold, then there would be $y \in \mathbf{U}(x,s) \cap T_\natural^{-1}[U]$ such that $y - x \in A$ and $y \bullet \boldsymbol{\nu}_A(x) > f(T_\natural(y)) \bullet \boldsymbol{\nu}_A(x)$; noting that

$$T_\natural(y) + f(T_\natural(y)) \in \mathbf{U}(x,s) \cap S(A,r) \quad \text{and} \quad |T_\natural(y) + f(T_\natural(y)) - y| < r\,,$$

we would conclude

$$|T_\natural(y) + f(T_\natural(y)) - (y-x)| = r - (y - f(T_\natural(y))) \bullet \boldsymbol{\nu}_A(x) < r = \boldsymbol{\delta}_A(T_\natural(y) + f(T_\natural(y)))$$

which is a contradiction.

Since $-\chi_{A,1}(x), \ldots, -\chi_{A,n}(x)$ are the eigenvalues of $\operatorname{pt} \mathrm{D}^2 g(0) \bullet \boldsymbol{\nu}_A(x)$ and $0 \in A$, we may apply 4.8 to infer that

$$\partial_{11} F(\boldsymbol{\nu}_A(x)) \chi_{A,1}(x) + \ldots + \partial_{nn} F(\boldsymbol{\nu}_A(x)) \chi_{A,n}(x) \leq h \tag{14}$$

and combining (4), 2.14, (11), and (14) we get that for every $j = 1, \ldots, n$

$$\begin{aligned}\chi_{A,j}(x) \leq C(F)\partial_{jj} F(\boldsymbol{\nu}_A(x))\chi_{A,j}(x) &\leq C(F)h - C(F) \sum_{k \neq j, k=1}^{n} \partial_{kk} F(\boldsymbol{\nu}_A(x)) \chi_{A,k}(x) \\ &\leq C(F)h - \frac{C(F)^2(n-1)}{(\lambda-1)r} < \frac{1}{r}\,.\end{aligned}$$

From (12) and [36, 3.5] follows that $1 - r\chi_{A,j}(x)$ are the eigenvalues of $(\mathscr{H}^n \llcorner S(A,r), n) \operatorname{ap} \mathrm{D}\boldsymbol{\xi}_A(x)$ for $j = 1, \ldots, n$; hence, we obtain

$$\big\|\textstyle\bigwedge_n\big((\mathscr{H}^n \llcorner S(A,r), n) \operatorname{ap} \mathrm{D}\boldsymbol{\xi}_A(x)\big)\big\| \geq \displaystyle\prod_{i=1}^{n} \big(1 - \chi_{A,i}(x) r\big) > 0\,.$$

Claim 2: *For $\mathscr{H}^n$ a.e. $x \in S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(A)$ and for $\mathscr{L}^1$ a.e. $0 < r < \frac{1}{2C(F)h}$ the conclusion of Claim 1 holds.*

This is immediate since

$$\boldsymbol{\Theta}^n(\mathscr{H}^n \llcorner S(A,r) \sim A_\tau, x) = 0$$

for $\mathscr{H}^n$ a.e. $x \in S(A,r) \cap A_\tau$ and for every $r > 0$ by [36, 2.13(1)] and [18, 2.10.19(4)], and $\mathscr{H}^n(S(A,r) \sim R(A)) = 0$ for $\mathscr{L}^1$ a.e. $r > 0$ by [36, 3.15].

Claim 3: *$N(A)$ satisfies the $n$ dimensional Lusin (N) condition in $\Omega$.*

Let $R \subseteq A$ be such that $\mathscr{H}^n(R) = 0$. For $r > 0$ it follows from [36, 3.16, 3.17(1), 4.3] that $\boldsymbol{\psi}_A | A_\tau \cap S(A,r)$ is a bilipschitz homeomorphism and

$$\boldsymbol{\psi}_A(\boldsymbol{\xi}_A^{-1}\{x\} \cap A_\tau \cap S(A,r)) \subseteq N(A,x) \quad \text{for } x \in A\,.$$

Noting Claim 2 and [36, 3.10(1)], we can apply [39, 3.5, 3.6] with $W$, $S$, and $f$ replaced by $S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(A)$, $R$, and $\boldsymbol{\xi}_A | S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(A)$ to infer that

$$\mathscr{H}^n(\boldsymbol{\xi}_A^{-1}(R) \cap S(A,r) \cap A_\tau) = 0 \quad \text{for } \mathscr{L}^1 \text{ a.e. } 0 < r < \frac{1}{2C(F)} h^{-1}.$$

We notice that $N(A)|R = \bigcup_{r>0} \boldsymbol{\psi}_A(S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(R))$ by [36, 4.3] and $\boldsymbol{\psi}_A(S(A,r) \cap A_\tau) \subseteq \boldsymbol{\psi}_A(S(A,s) \cap A_\tau)$ if $s < r$ by [36, 3.17(2)]. Henceforth, it follows that

$$\mathscr{H}^n(N(A)|R) = 0\,. \qquad \square$$

The following weak maximum principle is a simple consequence of [11, Theorem 3.4].

**4.11 Lemma.** *Assume*

$$V \in \mathbf{V}_n(\Omega), \quad F(\mathbf{h}_F(V,x)) \le h \quad \textit{for } \|V\| \textit{ almost all } x, \quad \|\delta_F V\|_{\mathrm{sing}} = 0.$$

*Then* $\operatorname{spt}\|V\|$ *is an* $(n,h)$ *subset of* $\Omega$ *with respect to* $F$*.*

*Proof.* For every $k \in \mathbb{N}$ let $V_k = k \cdot V$. Note that

$$u \bullet v = \frac{u}{F(u)} \bullet \frac{v}{F^*(v)} F(u)F^*(v) \le F(u)F^*(v) \quad \text{whenever } u, v \in \mathbf{R}^{n+1};$$

thus, for $k \in \mathbb{N}$ and $g \in \mathscr{X}(\Omega)$, we compute

$$\begin{aligned} \delta_F V_k(g) &= -\int \overline{\mathbf{h}}_F(V,x) \bullet g(x)\, \mathrm{d}\|V_k\|(x) = -\int \mathbf{h}_F(V,x) \bullet g(x) F(\nu)\, \mathrm{d}V_k(x,\nu) \\ &\le h \int F^*(g(x))F(\nu)\, \mathrm{d}V_k(x,\nu). \end{aligned}$$

Moreover, the area blowup set

$$Z = \big\{x \in \Omega : \limsup_{k\to\infty} \|V_k\|(\mathbf{B}(x,r)) = +\infty \text{ for every } r > 0\big\} \tag{15}$$

coincides with $\operatorname{spt}\|V\|$; hence, [11, Theorem 3.4] yields that $\operatorname{spt}\|V\| = Z$ is an $(n,h)$ set. □

*4.12 Remark.* Although the area blowup set $Z$ is defined in [11, Theorem 3.4] as

$$Z = \big\{x \in \operatorname{Clos}\Omega : \limsup_{k\to\infty} \|V_k\|(\mathbf{B}(x,r)) = +\infty \text{ for every } r > 0\},$$

the correct definition should be the one used in (15), in order to be consistent with [11, Definition 3.1] (requiring an $(n,h)$ set being a relatively closed subset of $\Omega$)

# 5 The anisotropic unit normal bundle

In this section we will need to work with a suitable anisotropic variant of the normal bundle for closed sets. Let us introduce some definitions.

**5.1 Definition.** Suppose $F$ is an elliptic integrand and $A \subseteq \mathbf{R}^{n+1}$ is closed. The *generalized anisotropic unit normal bundle of* $A$ is defined as

$$N^F(A) = (A \times \partial\mathbf{B}^{F^*}(0,1)) \cap \{(a,u) : \boldsymbol{\delta}^F_A(a+su) = s \text{ for some } s > 0\}.$$

**5.2 Lemma.** *Suppose* $F$ *is an elliptic integrand of class* $\mathscr{C}^{1,1}$ *and* $A \subseteq \mathbf{R}^{n+1}$ *is closed. Then*

$$N^F(A) = (\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F)[N(A)] = \big\{(a, \operatorname{grad} F(u)) : (a,u) \in N(A)\big\}.$$

*In particular,* $N^F(A)$ *is a countably* $n$ *rectifiable Borel subset of* $\mathbf{R}^{n+1} \times \partial\mathbf{B}^{F^*}(0,1)$*.*

*Proof.* Given $(a,u) \in N^F(A)$, there exists $s > 0$ such that

$$a \in A \cap \partial\mathbf{U}^{F^*}(a+su,s) \quad \text{and} \quad \mathbf{U}^{F^*}(a+su,s) \cap A = \varnothing.$$

Since $\partial\mathbf{U}^{F^*}(a+su,s)$ is submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ (see 2.32(i)), there exists $r > 0$ and $x \in \mathbf{R}^{n+1}$ such that $\mathbf{U}(x,r) \subseteq \mathbf{U}^{F^*}(a+su,s)$ and $a \in \partial\mathbf{U}(x,r)$. It follows that

$$\mathbf{n}(\mathbf{U}(x,r),a) = \mathbf{n}(\mathbf{U}^{F^*}(a+su,s),a) \quad \text{and} \quad (a, -\mathbf{n}(\mathbf{U}^{F^*}(a+su,s),a)) \in N(A).$$

Since $\operatorname{grad} F(\mathbf{n}(\mathbf{U}^{F^*}(0,1),z)) = z$ for every $z \in \partial\mathbf{U}^{F^*}(0,1)$ (see 2.32(g)), it follows that

$$\operatorname{grad} F\big(-\mathbf{n}(\mathbf{U}^{F^*}(a+su,s),a)\big) = -\operatorname{grad} F\big(\mathbf{n}(\mathbf{U}^{F^*}(a+su,s),a)\big) = -\frac{a-(a+su)}{s} = u,$$

i.e. $(a,u) \in (\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F)(N(A))$.

The proof of the reverse inclusion $(\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F)(N(A)) \subseteq N^F(A)$ is completely analogous and the postscript follows from [36, 4.3]. □

**5.3 Definition.** Suppose $\Omega \subseteq \mathbf{R}^{n+1}$ is open, $F$ is an elliptic integrand, and $A \subseteq \mathbf{R}^{n+1}$ is closed. We say that $N^F(A)$ satisfies the *$n$ dimensional Lusin (N) condition in $\Omega$* if and only if the following implication holds,

$$S \subseteq A \cap \Omega, \quad \mathscr{H}^n(S) = 0 \quad \implies \quad \mathscr{H}^n(N^F(A)|S) = 0\,.$$

**5.4 Lemma.** *Assume $F$ is an elliptic integrand of class $\mathscr{C}^{1,1}$, $\Omega \subseteq \mathbf{R}^{n+1}$ is open, and $A \subseteq \mathbf{R}^{n+1}$ is closed. Then $N(A)$ satisfies the $n$ dimensional Lusin (N) condition in $\Omega$ if and only if $N^F(A)$ satisfies the $n$ dimensional Lusin (N) condition in $\Omega$.*

*Proof.* Let $S \subseteq A \cap \Omega$ be such that $\mathscr{H}^n(S) = 0$. Assume that either $\mathscr{H}^n(N^F(A)|S) = 0$ or $\mathscr{H}^n(N(A)|S) = 0$. Since the map $\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F$ is a bilipschitz homeomorphism (see 2.32(i)), we deduce that $\mathscr{H}^n(N^F(A)|S) = \mathscr{H}^n((\mathrm{id}_{\mathbf{R}^{n+1}} \times \operatorname{grad} F)(N(A)|S)) = \mathscr{H}^n(N(A)|S) = 0$ as desired. □

**5.5 Definition.** Let $F$ be an elliptic integrand and $A \subseteq \mathbf{R}^{n+1}$ be closed. The *anisotropic reach function* $r^F_A : N^F(A) \to [0, \infty]$ is defined by

$$r^F_A(a, u) = \sup\{s : \boldsymbol{\delta}^F_A(a + su) = s\} \quad \text{for } (a, u) \in N^F(A)\,.$$

The *anisotropic reach of $A$* is defined by

$$\operatorname{reach}^F(A) = \inf\big\{\sup\{r : \mathbf{U}^{F^*}(a, r) \subseteq \operatorname{dmn} \boldsymbol{\xi}^F_A\} : a \in A\big\} = \sup\big\{r : \{x : \boldsymbol{\delta}^F_A(x) < r\} \subseteq \operatorname{dmn} \boldsymbol{\xi}^F_A\big\}\,.$$

*5.6 Remark.* Since $\boldsymbol{\delta}^F_A$ is Lipschitz continuous (see 2.32(a)), the function $f_s : N^F(A) \to \mathbf{R}$ given by $f_s(a, u) = \min\{\boldsymbol{\delta}^F_A(a+su)\,, s\}$ is also Lipschitz for any $s \in \mathbf{R}$ and $r^F_A(a, u) = \sup\{f_s(a, u) : s \in (0, \infty)\}$. Therefore $r^F_A$ is lower-semicontinuous. In particular, $r^F_A$ is a Borel function.

**5.7 Lemma.** *Suppose $F$ is an elliptic integrand of class $\mathscr{C}^{1,1}$ and $A$ is a closed submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^1$ such that $\operatorname{reach}^F A > 0$. Then $\operatorname{reach} A > 0$ and $A$ is a submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$.*

*Proof.* Set $W = \mathbf{B}^{F^*}(0, 1)$. First observe that $\partial W$ is a submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ by 2.32(i). Therefore, there exists $\rho \in (0, 1)$ such that for each $x \in \partial W$ we have

$$\mathbf{B}(x + \rho \mathbf{n}(W, x), \rho) \subseteq W\,.$$

Assume $\operatorname{reach}^F A = s > 0$. Let $z \in \mathbf{R}^{n+1}$ be such that $\boldsymbol{\delta}_A(z) = r < \rho s$ and find $x \in A$ with $|z - x| = \boldsymbol{\delta}_A(z)$. Set $B = \mathbf{B}(z, r)$, $u = -\mathbf{n}(B, x)$, and $w = x + r \operatorname{grad} F(u)/\rho$. Note that $u \in \operatorname{Tan}(A, x)^\perp$. We have $\boldsymbol{\delta}^F_A(w) = r/\rho < s$ so $w \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ and $\mathbf{B}^{F^*}(w, r/\rho) \cap A = \{x\}$ and $\mathbf{B}(z, r) \subseteq \mathbf{B}^{F^*}(w, r/\rho)$; hence, $z \in \operatorname{dmn} \boldsymbol{\xi}_A$.

Since $z$ was arbitrary we see that $\{x : \boldsymbol{\delta}_A(x) < \rho s\} \subseteq \operatorname{dmn} \boldsymbol{\xi}_A$ which shows that $\operatorname{reach} A \geq \rho s$. The second part of the conclusion readily follows from [17, 4.20]. □

**5.8 Corollary.** *Suppose $A \subseteq \mathbf{R}^{n+1}$ is closed and $\operatorname{reach}^F A > 0$. Then $S^F(A, r)$ is a submanifold of $\mathbf{R}^{n+1}$ of class $\mathscr{C}^{1,1}$ of dimension $n$ for every $0 < r < \operatorname{reach}^F A$.*

*Proof.* Since $R = \operatorname{reach}^F A > 0$, we have that $\mathbf{R}^{n+1} \cap \{y : \boldsymbol{\delta}^F_A(y) < R\} \subseteq \operatorname{dmn} \boldsymbol{\xi}^F_A$. Therefore, from 2.38(e)(d) it follows that $\boldsymbol{\delta}^F_A | \mathbf{R}^{n+1} \cap \{y : 0 < \boldsymbol{\delta}^F_A(y) < R\}$ is of class $\mathscr{C}^1$ and

$$\operatorname{grad} \boldsymbol{\delta}^F_A(y) = \operatorname{grad} F^*\Big(\frac{x - \boldsymbol{\xi}^F_A(y)}{\boldsymbol{\delta}^F_A(y)}\Big) \neq 0 \quad \text{for } y \in \mathbf{R}^{n+1} \text{ with } 0 < \boldsymbol{\delta}^F_A(y) < R\,.$$

Consequently, for every $0 < r < R$ we see that $S^F(A, r) = (\boldsymbol{\delta}^F_A)^{-1}\{r\}$ is a closed submanifold of $\Omega$ of class $\mathscr{C}^1$ of dimension $n$. Moreover, we have $\operatorname{reach}^F S^F(A, r) \geq \min\{R - r, r\} > 0$ so the conclusion follows from 5.7. □

We prove now the anisotropic version of [23, Theorem 3], whose proof is essentially along the same lines.

**5.9 Theorem.** *Assume $F$ is an elliptic integrand of class $\mathscr{C}^{1,1}$ and $A \subseteq \mathbf{R}^{n+1}$ is closed. Let $r > 0$ and suppose that for every $\mathscr{H}^n$ measurable bounded function $f : \mathbf{R}^{n+1} \times \partial \mathbf{U}^{F^*}(0, 1) \to \mathbf{R}$ with compact support there are numbers $c_1(f), \ldots, c_{n+1}(f) \in \mathbf{R}$ such that*

$$\int_{\mathbf{R}^{n+1} \sim A} f \circ \boldsymbol{\psi}^F_A \cdot \mathbf{1}_{\{x : \boldsymbol{\delta}^F_A(x) \leq t\}} \, \mathrm{d}\mathscr{L}^{n+1} = \sum_{j=1}^{n+1} c_j(f) t^j \quad \textit{for } 0 < t < r\,. \tag{16}$$

*Then* $\operatorname{reach}^F(A) \geq r$.

*Proof.* Let $S = \{(x,u,t) : (x,u) \in N^F(A),\ r^F_A(x,u) > t\}$ and define $\phi : N^F(A) \times (0,\infty) \to \mathbf{R}^{n+1}$

$$\phi(x,u,t) = x + tu \quad \text{for } (x,u,t) \in N^F(A) \times (0,\infty).$$

Claim 1: $\mathscr{L}^{n+1}(\operatorname{dmn} \boldsymbol{\xi}^F_A \sim (A \cup \phi(S))) = 0$; *hence,*

$$\mathscr{L}^{n+1}(\mathbf{R}^{n+1} \sim (A \cup \phi(S))) = 0\,.$$

Recalling 2.32(g) we see that

$$\operatorname{dmn} \boldsymbol{\xi}^F_A \sim (A \cup \phi(S)) = \phi(\{(x,u,t) : (x,u) \in N^F(A),\ t = r^F_A(x,u) > 0\})\,.$$

Since $\phi$ is a locally Lipschitz map, it suffices to prove that

$$\mathscr{H}^{n+1}(\{(x,u,t) : (x,u) \in K,\ M > t = r^F_A(x,u) > 0\}) = 0 \tag{17}$$

for all $M \in \mathbb{N}$ and $K \subseteq N^F(A)$ bounded. By 5.2 and [18, 3.2.29] we know that $N^F(A)$ is countably $n$ rectifiable. Hence, it suffices to prove (17) for all $M \in \mathbb{N}$ and $K \subseteq A$ being $n$ rectifiable. Assume $K$ and $M$ are such. Employing [18, 3.2.23] we get

$$\mathscr{H}^{n+1}(K \times (0, M+1)) = (M+1)\mathscr{H}^n(K) < \infty\,. \tag{18}$$

Recall 5.6. For $q \in \mathbf{R}$ define the Borel set

$$V_q = \{(x,u,t+q) : (x,u) \in K,\ M > t = r^F_A(x,u) > 0\}$$

and observe that

$$V_q \cap V_p = \varnothing \quad \text{whenever } p \neq q\,, \quad V_q \subseteq K \times (0, M+1) \quad \text{for } 0 < q < 1\,,$$
$$\text{and} \quad \mathscr{H}^{n+1}(V_q) = \mathscr{H}^{n+1}(V_0) \quad \text{for any } q \in \mathbf{R}\,.$$

Therefore, if $\mathscr{H}^{n+1}(V_0) > 0$, then $\mathscr{H}^{n+1}(\bigcup\{V_q : 0 < q < 1,\ q \text{ rational}\}) = \infty$ which contradicts (18).

Claim 2:

$$\mathscr{L}^{n+1}\big(\{z : 0 < \boldsymbol{\delta}^F_A(z) \le r,\ r^F_A(\boldsymbol{\psi}^F_A(z)) < r\}\big) = 0\,. \tag{19}$$

In the following sequence of estimates we have to deal with the problem that $N^F(A)$ might not have locally finite measure so $\mu = \mathscr{H}^n \llcorner N^F(A)$ might not be Radon and $(\mu, n)$ approximate Jacobian of $\phi$ might not be well defined.

Recalling 2.32(g) one readily infers that $\phi|S$ is injective. Since $N^F(A)$ is Borel and countably $n$ rectifiable (see 5.2) we may find a partition

$$N^F(A) = \textstyle\bigcup_{i=1}^\infty N_i$$

such that each $N_i$ is a Borel $n$ rectifiable set (in particular, $\mathscr{H}^n(N_i) < \infty$) and the family $\{N_i : i \in \mathbb{N}\}$ is disjointed; cf. [18, 2.1.6]. For $i \in \mathbb{N}$ w define

$$\mu_i = \mathscr{H}^n \llcorner N_i\,, \quad S_i = S \cap \big(N_i \times (0,\infty)\big)\,, \quad \text{and} \quad J = \sum_{i=1}^\infty \big\|\textstyle\bigwedge_n[(\mu_i, n) \operatorname{ap} \mathrm{D}\phi]\big\| \mathbf{1}_{S_i}\,.$$

We apply Claim 1 and the coarea formula [18, 3.2.22] to find that

$$\begin{aligned}
&\int_{\mathbf{R}^{n+1} \sim A} g \,\mathrm{d}\mathscr{L}^{n+1} = \int_{\phi(S)} g \,\mathrm{d}\mathscr{L}^{n+1} = \sum_{i=1}^\infty \int_{\phi(S_i)} g \,\mathrm{d}\mathscr{L}^{n+1} \\
&= \int_0^\infty \sum_{i=1}^\infty \int_{N_i} \big\|\textstyle\bigwedge_n[(\mu_i, n) \operatorname{ap} \mathrm{D}\phi(x,u,t)]\big\| g(x+tu) \mathbf{1}_{\{(w,v) : r^F_A(w,v) > t\}}(x,u) \,\mathrm{d}\mathscr{H}^n(x,u) \,\mathrm{d}t \\
&= \int_0^\infty J(x,u,t) g(x+tu) \mathbf{1}_{\{(w,v) : r^F_A(w,v) > t\}}(x,u) \,\mathrm{d}\mathscr{H}^n(x,u) \,\mathrm{d}t
\end{aligned} \tag{20}$$

whenever $g : \mathbf{R}^{n+1} \to \mathbf{R}$ is a non-negative Borel function with compact support.

Let $B \subseteq \mathbf{R}^{n+1}$ be compact, $0 < \tau < r$ and $\tau < t < r$. We define

$$N_{\tau,B} = N^F(A) \cap \{(x,u) : r^F_A(x,u) \leq \tau,\ x \in B\}\,,$$

and we apply (16) to the function $\mathbf{1}_{N_{\tau,B}}$ and (20) to the function $g = (\mathbf{1}_{N_{\tau,B}} \circ \boldsymbol{\psi}^F_A) \cdot \mathbf{1}_{\{w : \boldsymbol{\delta}^F_A(w) \leq t\}}$ to compute

$$(21)\quad \sum_{j=1}^{n+1} c_j(f) t^j \overset{(16)}{=} \int_{\mathbf{R}^{n+1}\sim A} \mathbf{1}_{N_{\tau,B}}(\boldsymbol{\psi}^F_A(z)) \mathbf{1}_{\{w:\boldsymbol{\delta}^F_A(w)\leq t\}}(z)\, \mathrm{d}\mathscr{L}^{n+1} z$$
$$\overset{(20)}{=} \int_0^\infty \int_{N^F(A)} J(x,u,s) \mathbf{1}_{\{w:\boldsymbol{\delta}^F_A(w)\leq t\}}(x+su) \mathbf{1}_{\{(w,v): r^F_A(w,v) > s\}}(x,u) \mathbf{1}_{N_{\tau,B}}(\boldsymbol{\psi}^F_A(x+su))\, \mathrm{d}\mathscr{H}^n(x,u)\, \mathrm{d}s$$
$$= \int_0^\infty \int_{N^F(A)} J(x,u,s) \mathbf{1}_{\{w:\boldsymbol{\delta}^F_A(w)\leq t\}}(x+su) \mathbf{1}_{\{(w,v): r^F_A(w,v) > s\}}(x,u) \mathbf{1}_{N_{\tau,B}}(x,u)\, \mathrm{d}\mathscr{H}^n(x,u)\, \mathrm{d}s$$
$$= \int_0^\infty \int_{N^F(A)} J(x,u,s) \mathbf{1}_{\{w:\boldsymbol{\delta}^F_A(w)\leq t\}}(x+su) \mathbf{1}_{\{(w,v): s< r^F_A(w,v) \leq \tau\}}(x,u) \mathbf{1}_{B}(x)\, \mathrm{d}\mathscr{H}^n(x,u)\, \mathrm{d}s$$
$$= \int_0^\infty \int_{N^F(A)} J(x,u,s) \mathbf{1}_{\{(w,v): s< r^F_A(w,v) \leq \tau\}}(x,u) \mathbf{1}_{B}(x)\, \mathrm{d}\mathscr{H}^n(x,u)\, \mathrm{d}s\,,$$

where the last equality follows because $\boldsymbol{\delta}^F_A(x+su) = s < r^F_A(x,u) \leq \tau < t$, for every $\tau < t < r$. Whence, we deduce that $\sum_{j=1}^{n+1} c_j(f) t^j$ is independent of $t$, for every $\tau < t < r$. Therefore, this polynomial is identically zero, a condition that implies, by the first equality in (21),

$$\mathscr{L}^{n+1}\big(\{z : 0 < \boldsymbol{\delta}^F_A(z) \leq r,\ \boldsymbol{\psi}^F_A(z) \in N_{\tau,B}\}\big) = 0\,.$$

Since the last equation holds for every $0 < \tau < r$ and for every compact set $B \subseteq \mathbf{R}^{n+1}$, we conclude that (19) holds.

Claim 3: $\operatorname{reach}^F(A) \geq r$.

Let $z \in \mathbf{R}^{n+1} \sim A$ satisfy $0 < \boldsymbol{\delta}^F_A(z) < r$. Then there exists a sequence $\{z_i : i \in \mathbb{N}\} \subseteq \operatorname{dmn} \boldsymbol{\xi}^F_A$ which converges to $z$ and such that

$$0 < \boldsymbol{\delta}^F_A(z_i) \leq r \quad \text{and} \quad r^F_A(\boldsymbol{\psi}^F_A(z_i)) \geq r\,.$$

Noting that $(\boldsymbol{\xi}^F_A(z_i))$ is a bounded sequence, and passing to a subsequence if necessary, we find $p \in A$ and $u \in \partial \mathbf{U}^{F^*}(0,1)$ such that

$$\boldsymbol{\xi}^F_A(z_i) \to p, \qquad \boldsymbol{\nu}^F_A(z_i) \to u\,.$$

In particular, $z = p + \boldsymbol{\delta}^F_A(z) u$. We find $t \in \mathbf{R}$ such that $\boldsymbol{\delta}^F_A(z) < t < r$, and notice that

$$\mathbf{U}^{F^*}(\boldsymbol{\xi}^F_A(z_i) + t\boldsymbol{\nu}^F_A(z_i), t) \cap A = \varnothing \quad \text{for } i \geq 1\,; \quad \text{hence,} \quad \mathbf{U}^{F^*}(p + tu, t) \cap A = \varnothing\,.$$

This shows that $\boldsymbol{\delta}^F_A(p+tu) = t > \boldsymbol{\delta}^F_A(z)$; hence, 2.32(g) yields $z \in \operatorname{dmn} \boldsymbol{\xi}^F_A$ and $\boldsymbol{\xi}^F_A(z) = p$. □

*5.10 Remark.* We point out that Claim 1 proves that the set of centers of maximal $F$-balls contained in the complement of $A$ has $\mathscr{L}^{n+1}$ measure zero. This set, in turn, contains the set of non-differentiability points of the distance function $\boldsymbol{\delta}^F_A$.

# 6 Heintze Karcher inequality

Here we prove our main theorem 6.5. Firstly we need the following basic facts on sets of finite perimeter.

*6.1 Remark.* Suppose $E \subseteq \mathbf{R}^{n+1}$ is of finite perimeter. We recall that the reduced boundary (see 2.23) and the essential boundary (cf. [18, 4.5.12] and [4, Def. 3.60]) of $E$ are $\mathscr{H}^n$ almost the same (see [4, Thm. 3.61]). Recalling [3, 4.7] we deduce that $\mathbf{n}(E,\cdot)|\partial^* E : \partial^* E \to \mathbf{R}^{n+1}$ equals the negative of the *generalised inner normal to $E$* defined in [4, Def. 3.54].

**6.2 Lemma.** *Let $E$ be a set of finite perimeter in $\mathbf{R}^{n+1}$ such that*

$$\mathscr{H}^n(\operatorname{Clos}\partial^* E \sim \partial^* E) = 0\,.$$

*Then there exists an open set $P \subseteq \mathbf{R}^{n+1}$ such that*

$$\mathscr{L}^{n+1}((P \sim E) \cup (E \sim P)) = 0 \quad \textit{and} \quad \mathscr{H}^n(\partial P \sim \partial^* P) = 0\,.$$

*Proof.* We define

$$\begin{aligned} P &= \mathbf{R}^{n+1} \cap \{x : \mathscr{L}^{n+1}(\mathbf{U}(x,\rho) \sim E) = 0 \text{ for some } \rho > 0\}\,, \\ Q &= \mathbf{R}^{n+1} \cap \{x : \mathscr{L}^{n+1}(\mathbf{U}(x,\rho) \cap E) = 0 \text{ for some } \rho > 0\}\,, \end{aligned}$$

and we notice that they are open subsets of $\mathbf{R}^{n+1}$. It follows from [18, 4.5.3] that

$$\operatorname{spt} \mathscr{H}^n \llcorner \partial^* E = \mathbf{R}^{n+1} \sim (P \cup Q)\,. \tag{22}$$

Simple use of a Vitali covering lemma [18, 2.8.18] yields

$$\begin{gathered} \mathscr{L}^{n+1}(P \sim E) = 0\,, \qquad \mathscr{L}^{n+1}(E \cap Q) = 0\,; \\ \mathscr{L}^{n+1}(E \sim P) = \mathscr{L}^{n+1}(E \cap Q) + \mathscr{L}^{n+1}(\operatorname{spt} \mathscr{H}^n \llcorner \partial^* E) = 0\,. \end{gathered}$$

From 6.1 and [18, 4.5.11] we deduce that $\partial^* P = \partial^* E$ and, since $\partial P \subseteq \operatorname{spt} \mathscr{H}^n \llcorner \partial^* E$ by (22), we conclude

$$\mathscr{H}^n(\partial P \sim \partial^* P) = 0\,. \qquad \square$$

*6.3 Remark.* Let $F$ be an elliptic integrand. Recalling [18, 5.1.1] we define $\Phi : \mathbf{R}^{n+1} \times \bigwedge_n \mathbf{R}^{n+1} \to \mathbf{R}$, a parametric integrand of degree $n$ on $\mathbf{R}^{n+1}$, by setting

$$\Phi(z,\xi) = F(*\xi) \quad \text{for } z \in \mathbf{R}^{n+1} \text{ and } \xi \in \textstyle\bigwedge_n \mathbf{R}^{n+1}\,,$$

where $*$ denotes the Hodge star operator associated with the standard scalar product and orientation on $\mathbf{R}^{n+1}$; see [18, 1.7.8]. By 2.12 and [18, 5.1.2] we see that $\Phi$ is elliptic in the sense of [18, 5.1.2]. Moreover, if $\Phi^\S$ is the nonparametric integrand associated with $\Phi$ (see [18, 5.1.9]) and $\Phi^\S_z(\xi) = \Phi^\S(z,\xi)$ for $(z,\xi) \in \mathbf{R}^{n+1} \times \bigwedge_n \mathbf{R}^{n+1}$, then $\mathrm{D}^2\Phi^\S_z(\xi)$ is strongly elliptic in the sense of [18, 5.2.3] for all $(z,\xi) \in \mathbf{R}^{n+1} \times \bigwedge_n \mathbf{R}^{n+1}$ by [18, 5.2.17].

Let $W \subseteq \mathbf{R}^n$ be open and bounded, $V \in \mathbf{V}_n(W \times \mathbf{R})$, $\mathbf{p} : \mathbf{R}^{n+1} \to \mathbf{R}^n$ and $\mathbf{q} : \mathbf{R}^{n+1} \to \mathbf{R}$ be given by $\mathbf{p}(z_1,\ldots,z_{n+1}) = (z_1,\ldots,z_n)$ and $\mathbf{q}(z_1,\ldots,z_{n+1}) = z_{n+1}$ for $(z_1,\ldots,z_{n+1}) \in \mathbf{R}^{n+1}$. Assume $f : \mathbf{R}^n \to \mathbf{R}$ is of class $\mathscr{C}^1$, and $V$ is the unit density varifold associated to the graph of $f$, i.e., $V = \mathbf{v}_n(\operatorname{im}(\mathbf{p}^* + \mathbf{q}^* \circ f))$. Recalling [18, 5.1.9] we see that for any $\theta : W \to \mathbf{R}$ of class $\mathscr{C}^1$ with compact support there holds

$$\delta_F V(\mathbf{q}^* \circ \theta \circ \mathbf{p}) = \int \big\langle (0, \theta(x), \mathrm{D}\theta(x)),\, \mathrm{D}\Phi^\S(x, f(x), \mathrm{D}f(x)) \big\rangle \,\mathrm{d}\mathscr{L}^{n+1}(x)\,.$$

Suppose $\alpha \in (0,1)$, $F$ is of class $\mathscr{C}^{2,\alpha}$, $f$ is of class $\mathscr{C}^{1,\alpha}$, $\|\delta_F V\|$ is a Radon measure, $\|\delta_F V\|_{\mathrm{sing}} = 0$, and $\mathbf{h}_F(V,\cdot) : \operatorname{spt}\|V\| \to \mathbf{R}^{n+1}$ is of class $\mathscr{C}^{0,\alpha}$. Define $\eta : W \to \mathbf{R}^{n+1}$ and $H : W \to \mathbf{R}$ by the formulas

$$\begin{aligned} \eta(x) &= (\mathbf{q}^*(1) - \mathbf{p}^*(\operatorname{grad} f(x))) \cdot (1 + |\operatorname{grad} f(x)|^2)^{-1/2} \\ \text{and} \quad H(x) &= -F(\eta(x)) \cdot \mathbf{q} \circ \mathbf{h}_F(V, (\mathbf{p}^* + \mathbf{q}^* \circ f)(x)) \cdot \sqrt{1 + |\operatorname{grad} f|^2} \end{aligned}$$

for $x \in W$. Note that $\eta(x)$ is the unit normal vector to the graph of $f$ at $(\mathbf{p}^* + \mathbf{q}^* \circ f)(x)$ for $x \in W$. Employing the area formula [18, 3.2.3] we get

$$\delta_F V(\mathbf{q}^* \circ \theta \circ \mathbf{p}) = -\int_{\operatorname{spt}\|V\|} \theta(\mathbf{p}(z)) \cdot \mathbf{q}(\mathbf{h}(V,z)) \cdot F(\eta(\mathbf{p}(z))\,\mathrm{d}\mathscr{H}^n(z) = \int_W \theta(x) \cdot H(x)\,\mathrm{d}\mathscr{L}^n(x)$$

so that

$$\int_W \big\langle (0, \theta(x), \mathrm{D}\theta(x)),\, \mathrm{D}\Phi^\S(x, f(x), \mathrm{D}f(x)) \big\rangle \,\mathrm{d}\mathscr{L}^{n+1}(x) = \int_W \theta(x) \cdot H(x)\,\mathrm{d}\mathscr{L}^n(x) \tag{23}$$

$$\text{for any } \theta \in \mathscr{D}(W, \mathbf{R})\,.$$

Since $H$ is of class $\mathscr{C}^{0,\alpha}$ and $F$ of class $\mathscr{C}^{2,\alpha}$ a slight modification of the proof of [18, 5.2.15] shows that $f$ is actually of class $\mathscr{C}^{2,\alpha^2}$.

To support the last claim recall the proof of [18, 5.2.15] with 2, $n+1$, $n$, $\alpha^2$, $W$, $\Phi^\S$ in place of $q$, $n$, $m$, $\delta$, $U$, $G$. Using all the symbols defined in [18, 5.2.15], for any integer $\nu$ such that $\nu > 1/d$, define $R_\nu : \mathbf{B}(b, \rho - d) \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R})$ so that

$$\sigma \bullet R_\nu(x) = \int_0^1 \sigma(e_i) \cdot H(x + te_i/\nu) \, \mathrm{d}\mathscr{L}^1(t) \quad \text{for } \sigma \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}) \text{ and } x \in \mathbf{B}(b, \rho - d) .$$

We remark that we intentionally refer to [18, 5.2.15] with $\alpha^2$ in place of $\delta$, since in this case the estimates for the $\delta$-Hölder constants of $Q_\nu$ and $A_\nu$ from page 556 can be adjusted by replacing the supremum norm of $\mathrm{D}^3G$ with the $\alpha$-Hölder norm of $\mathrm{D}^2G$. Moreover, since in our case $f$ satisfies (23) rather than [18, 5.2.15(4)] the displayed equation in the middle of page 556 of [18], i.e.,

$$\int_{\mathbf{U}(b,\rho-d)} \langle \mathrm{D}f_\nu(x) \odot \mathrm{D}\theta(x),\, A_\nu(x) \rangle \, \mathrm{d}\mathscr{L}^n(x) = (P_\nu - Q_\nu,\, \mathrm{D}\theta)_{b,\rho-d}$$

turns into

$$\int_{\mathbf{U}(b,\rho-d)} \langle \mathrm{D}f_\nu(x) \odot \mathrm{D}\theta(x),\, A_\nu(x) \rangle \, \mathrm{d}\mathscr{L}^n(x) = (P_\nu - Q_\nu - R_\nu,\, \mathrm{D}\theta)_{b,\rho-d} .$$

Clearly $R_\nu$ is $\alpha$-Hölder continuous with Hölder constant independent of $\nu$ so all the estimates from the upper half of page 557 of [18] hold in the modified case with an additional term coming from $R_\nu$. Thus, one can still use [18, 5.2.2] to conclude that $\mathrm{D}_i f$ is of class $\mathscr{C}^{1,\alpha}$; hence, $f$ is of class $\mathscr{C}^{2,\alpha}$.

**6.4 Definition.** Let $A \subseteq \mathbf{R}^{n+1}$, $k \in \mathbb{N}$, $\alpha \in [0,1]$. We say that $x \in A$ is a *$\mathscr{C}^{k,\alpha}$-regular point of $A$* if there exists an open set $W \subseteq \mathbf{R}^{n+1}$ such that $x \in W$ and $A \cap W$ is an $n$-dimensional submanifold of class $\mathscr{C}^{k,\alpha}$ of $\mathbf{R}^{n+1}$. The set of all $\mathscr{C}^{k,\alpha}$ regular points of $A$ shall be called the *$\mathscr{C}^{k,\alpha}$ regular part of $A$*. If $\alpha = 0$ we omit it in the notation.

**6.5 Theorem.** *Suppose*

$$\begin{gathered} n \geq 2 , \quad c \in (0, \infty) , \quad \alpha \in (0,1) , \quad F \textit{ is an elliptic integrand of class } \mathscr{C}^{2,\alpha} , \\ E \subseteq \mathbf{R}^{n+1} \textit{ is a set of finite perimeter} , \quad \mathscr{H}^n \big( \operatorname{Clos}(\partial^* E) \sim \partial^* E \big) = 0 , \\ V = \mathbf{v}_n(\partial^* E) \in \mathbf{RV}_n(\mathbf{R}^{n+1}) , \quad \| \delta_F V \|_{\mathrm{sing}} = 0 , \\ \mathbf{h}_F(V, \cdot)|K \textit{ is of class } \mathscr{C}^{0,\alpha} \textit{ for each compact subset } K \textit{ of the } \mathscr{C}^{1,\alpha} \textit{ regular part of } \operatorname{spt} \|V\| , \\ 0 < -\overline{\mathbf{h}}_F(V,x) \bullet \mathbf{n}(E,x) \leq c \quad \textit{for } \|V\| \textit{ almost all } x . \end{gathered}$$

*Then*

$$\mathscr{L}^{n+1}(E) \leq \frac{n}{n+1} \int_{\partial^* E} \frac{1}{|\mathbf{h}_F(V,x)|} \, \mathrm{d}\mathscr{H}^n(x) \tag{24}$$

*and equality holds if and only if here there exists a finite union $\Omega$ of disjoint open Wulff shapes with radii not smaller than $n/c$ such that $\mathscr{L}^{n+1}\big( (\Omega \sim E) \cup (E \sim \Omega) \big) = 0$.*

*Proof.* First we employ 6.2 to obtain an open set $\Omega \subseteq \mathbf{R}^{n+1}$ such that

$$\mathscr{L}^{n+1}\big( (\Omega \sim E) \cup (E \sim \Omega) \big) = 0 \quad \text{and} \quad \mathscr{H}^n(\partial\Omega \sim \partial^*\Omega) = 0 .$$

Directly from the definition (see [18, 4.5.12, 4.5.11]) it follows that the essential boundaries of $\Omega$ and $E$ coincide; hence, recalling 6.1, we obtain $V = \mathbf{v}_n(\partial^*\Omega)$. We shall consider $\Omega$ instead of $E$ in the sequel. Let us define

$$\begin{gathered} H : \operatorname{spt} \|V\| \to [0, c] \quad \text{so that} \quad H(x) = -\overline{\mathbf{h}}_F(V,x) \bullet \mathbf{n}(E,x) \quad \text{for } \|V\| \text{ almost all } x , \\ C = \mathbf{R}^{n+1} \sim \Omega , \quad Q = \partial C \cap \big\{ x : x \text{ is a } \mathscr{C}^2\text{-regular point of } \partial C \big\} . \end{gathered}$$

Note that $\partial^* C = \partial^* \Omega$, $\mathbf{n}^F(C, \cdot) = -\mathbf{n}^F(\Omega, \cdot)$, and $H(x) = F(\mathbf{n}(E,x)) |\mathbf{h}_F(V,x)|$ for $\|V\|$ almost all $x$.

Claim 1: *If $x \in Q$, $y \in \Omega$, and $\boldsymbol{\xi}_C^F(y) = x$ (in other words: $y \in \Omega \cap (\boldsymbol{\xi}_C^F)^{-1}(Q)$), then*

$$0 \leq \frac{1}{n} H(x) \leq -\kappa_{Q,1}^F(\boldsymbol{\psi}_C^F(y)) \leq \boldsymbol{\delta}_C^F(y)^{-1} .$$

We clearly have

$$\mathbf{U}^{F^*}(y, \boldsymbol{\delta}_C^F(y)) \cap C = \varnothing \quad \text{and} \quad \partial \mathbf{U}^{F^*}(y, \boldsymbol{\delta}_C^F(y)) \cap C = \{x\}\,;$$

hence, recalling 2.33, 2.27, 2.34 and that $x$ is a $\mathscr{C}^2$-regular point of $\partial C$, wee see that

$$\frac{1}{n} H(x) \leq -\kappa^F_{Q,1}(\boldsymbol{\psi}_C^F(y)) \leq -\kappa^F_{\partial \mathbf{U}^{F^*}(y, \boldsymbol{\delta}_C^F(y)),1}(\boldsymbol{\psi}_C^F(y)) = \boldsymbol{\delta}_C^F(y)^{-1}$$

and the claim is proven.

Claim 2: $\mathscr{L}^{n+1}(\Omega \sim (\boldsymbol{\xi}_C^F)^{-1}(Q)) = 0$.

Note that $F(\overline{\mathbf{h}}_F(V,x)) = H(x)F(\mathbf{n}(\Omega,x))$ for $\|V\|$ almost all $x$ so applying Lemma 4.11 we conclude that $\partial\Omega$ is an $(n, c\,C(F))$ subset of $\mathbf{R}^{n+1}$. It follows by Theorem 4.10 that $\mathscr{H}^n(N(\partial\Omega)|S) = 0$ whenever $S \subseteq \mathbf{R}^{n+1}$ satisfies $\mathscr{H}^n(S) = 0$. Combining this with Lemma 5.4, we deduce that $\mathscr{H}^n(N^F(\partial\Omega)|S) = 0$ whenever $S \subseteq \mathbf{R}^{n+1}$ satisfies $\mathscr{H}^n(S) = 0$. Since $N^F(C) \subseteq N^F(\partial\Omega)$, one readily infers that $\mathscr{H}^n(N^F(C)|S) = 0$ whenever $S \subseteq \mathbf{R}^{n+1}$ satisfies $\mathscr{H}^n(S) = 0$. We also observe that for $\|V\|$ almost all $z$ there exists a radius $r > 0$ such that $V$ satisfies all the assumption of [2, The Regularity Theorem, pp. 27-28] inside $\mathbf{U}(z,r)$. This implies that for $\mathscr{H}^n$ almost all $z \in \partial C$ there exists an open set $G \subset \mathbf{R}^{n+1}$ with $z \in G$ and such that $\partial C \cap G$ coincides with a rotated graph of some function $f : \mathbf{R}^n \to \mathbf{R}$ of class $\mathscr{C}^{1,\alpha}$. However, employing 6.3, we see that $f$ is actually of class $\mathscr{C}^{2,\alpha^2}$. Therefore,

$$\mathscr{H}^n(\partial C \sim Q) = 0 \quad \text{and} \quad \mathscr{H}^n(N^F(C)|(\partial C \sim Q)) = 0\,. \tag{25}$$

Since $\boldsymbol{\psi}_C^{\boldsymbol{F}}\big(S^F(C,r) \cap (\operatorname{dmn} \boldsymbol{\xi}_C^F) \sim (\boldsymbol{\xi}_C^F)^{-1}(Q)\big) \subseteq N(C)|(\partial C \sim Q)$ for every $r > 0$, we get

$$\mathscr{H}^n\big(\boldsymbol{\psi}_C^{\boldsymbol{F}}\big(S^F(C,r) \cap (\operatorname{dmn} \boldsymbol{\xi}_C^F) \sim (\boldsymbol{\xi}_C^F)^{-1}(Q)\big)\big) = 0 \quad \text{for every } r > 0\,.$$

Moreover, we have $\big(\boldsymbol{\psi}_C^{\boldsymbol{F}}|(S^F(C,r) \cap \operatorname{dmn} \boldsymbol{\xi}_C^F \sim C)\big)^{-1} \in \mathscr{C}^1$ and we deduce that

$$\mathscr{H}^n\big(S^F(C,r) \cap (\operatorname{dmn} \boldsymbol{\xi}_C^F) \sim (\boldsymbol{\xi}_C^F)^{-1}(Q)\big) = 0 \quad \text{for every } r > 0.$$

Combining 2.38(f)(a)(d) with the coarea formula [18, 3.2.22], we get

$$\mathscr{H}^n(S^F(C,r) \sim \operatorname{dmn} \boldsymbol{\xi}_C^F) = 0 \quad \text{for } \mathscr{L}^1 \text{ almost all } r > 0\,.$$

From 2.38(d) it follows that $F(\operatorname{grad} \boldsymbol{\delta}_C^F(x)) = 1$; hence, recalling 2.14, we obtain $|\operatorname{grad} \boldsymbol{\delta}_C^F(x)| \geq \frac{1}{C(F)}$. Using the coarea formula, we compute

$$\begin{aligned} &\frac{1}{C(F)} \mathscr{L}^{n+1}(\Omega \sim (\boldsymbol{\xi}_C^F)^{-1}(Q)) \\ &\qquad \leq \int_{\Omega \sim (\boldsymbol{\xi}_C^F)^{-1}(Q)} |\operatorname{grad} \boldsymbol{\delta}_C^F(x)| dx = \int_0^\infty \mathscr{H}^n(S^F(C,r) \sim (\boldsymbol{\xi}_C^F)^{-1}(Q))\, \mathrm{d}r = 0\,. \end{aligned}$$

In particular we get that $\mathscr{L}^{n+1}(\Omega \sim (\boldsymbol{\xi}_C^F)^{-1}(Q)) = 0$, which settles Claim 2.

We define

$$\begin{gathered} Z = (Q \times \mathbf{R}) \cap \big\{(x,t) : 0 < t \leq -\kappa^F_{Q,1}(x, \mathbf{n}^F(C,x))^{-1}\big\}\,, \\ \zeta : Z \to \mathbf{R}^{n+1}\,, \quad \zeta(x,t) = x + t\mathbf{n}^F(C,x)\,. \end{gathered}$$

For brevity of the notation we also set

$$J_{n+1}\zeta(x,t) = \|{\textstyle\bigwedge}_{n+1}(\mathscr{H}^{n+1} \llcorner Z, n+1)\, \mathrm{ap}\, \mathrm{D}\zeta(x,t)\| \quad \text{whenever } (x,t) \in Z\,.$$

Claim 3: *There holds*

$$J_{n+1}\zeta(x,t) = F(\mathbf{n}(C,x)) \prod_{i=1}^n \big(1 + t\, \kappa^F_{Q,i}(x, \mathbf{n}^F(C,x))\big) \quad \textit{for } (x,t) \in Z\,.$$

Let $(x,t) \in Z$ and $u = \mathbf{n}^F(C,x)$. Recalling 2.25 we find a basis $\tau_1(x), \ldots, \tau_n(x)$ of $\operatorname{Tan}(Q,x)$ consisting of eigenvectors of $\mathrm{D}(\mathbf{n}^F(C,\cdot))(x)$ and such that

$$\langle \tau_i(x), \mathrm{D}\mathbf{n}^F(C,\cdot)(x)\rangle = \kappa^F_{Q,i}(x,u)\,\tau_i(x) \quad \text{for } i \in \{1,2,\ldots,n\}\,,$$
$$|\tau_1(x) \wedge \cdots \wedge \tau_n(x)| = 1\,.$$

Noting that $\operatorname{Tan}(Z,(x,t)) = \operatorname{Tan}(Q,x) \times \mathbf{R}$,

$$\langle (0,1), \mathrm{D}\zeta(x,t)\rangle = \mathbf{n}^F(C,x) = \operatorname{grad} F(\mathbf{n}(C,x))\,,$$
$$\langle (\tau_i(x),0), \mathrm{D}\zeta(x,t)\rangle = (1 + t\kappa^F_{Q,i}(x,u))\,\tau_i(x) \quad \text{for } i \in \{1,\ldots,n\}\,,$$

we compute

$$\begin{aligned} J_{n+1}\zeta(x,t) &= \prod_{i=1}^n (1 + t\kappa^F_{Q,i}(x,u))\,|\mathbf{n}^F(C,x) \wedge \tau_1(x) \wedge \cdots \wedge \tau_n(x)| \\ &= \operatorname{grad} F(\mathbf{n}(C,x)) \bullet \mathbf{n}(C,x) \prod_{i=1}^n (1 + t\kappa^F_{Q,i}(x,u))\,|\mathbf{n}(C,x) \wedge \tau_1(x) \wedge \cdots \wedge \tau_n(x)| \end{aligned}$$

and Claim 3 follows from 2.32(c) and [18, 1.7.5].

Claim 4: *Inequality* (24) *holds.*

Employing Claim 1 and Claim 2 we see that $\mathscr{L}^{n+1}(\Omega \sim \zeta(Z)) = 0$. Hence, using the area formula and then Claim 3, we get

$$\begin{aligned} (26)\quad \mathscr{L}^{n+1}(\Omega) \le \mathscr{L}^{n+1}(\zeta(Z)) &\le \int_{\zeta(Z)} \mathscr{H}^0(\zeta^{-1}(y))\,\mathrm{d}\mathscr{L}^{n+1}(y) = \int_Z J_{n+1}\zeta\,\mathrm{d}\mathscr{H}^{n+1} \\ &= \int_Q F(\mathbf{n}(C,x)) \int_0^{-1/\kappa^F_{Q,1}(x,\mathbf{n}^F(C,x))} \prod_{i=1}^n \big(1 + t\kappa^F_{Q,i}(x,\mathbf{n}^F(C,x))\big)\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^n(x)\,. \end{aligned}$$

Using again Claim 1, then the standard inequality between the arithmetic and the geometric mean, and finally 2.27, we obtain

$$\begin{aligned} (27)\quad \mathscr{L}^{n+1}(\Omega) &\le \int_Q F(\mathbf{n}(C,x)) \int_0^{-1/\kappa^F_{Q,1}(x,\mathbf{n}^F(C,x))} \Big(\frac{1}{n}\sum_{i=1}^n \big(1 + t\kappa^F_{Q,i}(x,\mathbf{n}^F(C,x))\big)\Big)^n \mathrm{d}t\,\mathrm{d}\mathscr{H}^n(x) \\ &\le \int_Q F(\mathbf{n}(C,x)) \int_0^{n/H(x)} \Big(1 - t\frac{H(x)}{n}\Big)^n \mathrm{d}t\,\mathrm{d}\mathscr{H}^n(x) \\ &= \frac{n}{n+1} \int_{\partial\Omega} \frac{F(\mathbf{n}(C,x))}{H(x)}\,\mathrm{d}\mathscr{H}^n(x)\,, \end{aligned}$$

which implies (24) by 2.18.

We assume now that equality holds in (24). Since the chains of inequalities (26) and (27) become chains of equalities, we deduce that

$$(28)\qquad \mathscr{L}^{n+1}(\zeta(Z) \sim \Omega) = 0\,,$$

$$(29)\qquad \mathscr{H}^0(\zeta^{-1}(y)) = 1 \quad \text{for } \mathscr{L}^{n+1} \text{ almost all } y \in \zeta(Z)\,,$$

$$(30)\qquad -\kappa^F_{Q,j}(z,\mathbf{n}^F(C,z))^{-1} = \frac{n}{H(z)} \quad \text{for } \mathscr{H}^n \text{ almost all } z \in Q \text{ and all } j = 1,\ldots,n\,.$$

Our goal is to prove that $\Omega$ is a finite union of disjoint open Wulff shapes. We need two preliminary claims, whence the conclusion will be easily deduced.

Claim 5: $\operatorname{reach}^F C \ge n/c$.

Recall that $H(z) \le c$ for $\mathscr{H}^n$ almost all $z \in \partial C$. Let $0 < \rho < n/c$ and

$$Q_\rho = Q \cap \{z : \rho < -\kappa^F_{Q,1}(z,\mathbf{n}^F(C,z))^{-1}\}.$$

It follows from (25), (30), and the fact that $\partial C$ is an $(n, c\,C(F))$ subset of $\mathbf{R}^{n+1}$, that

$$\mathscr{H}^n(\partial C \sim Q_\rho) = 0 \quad \text{and} \quad \mathscr{H}^n(N(C)|\partial C \sim Q_\rho) = 0\,;$$

hence, we argue as in Claim 2 to conclude that $\mathscr{L}^{n+1}(\Omega \sim \boldsymbol{\xi}_C^{-1}(Q_\rho)) = 0$. We define

$$C_\rho^F = \{z : \boldsymbol{\delta}_C^F(z) \leq \rho\} \quad \text{and} \quad Z_\rho = Q_\rho \times \{t : 0 < t \leq \rho\}$$

and we notice that

$$\boldsymbol{\xi}_C^{-1}(Q_\rho) \cap \Omega \cap C_\rho^F \subseteq \zeta(Z_\rho) \subseteq C_\rho^F\,, \quad \mathscr{L}^{n+1}(\Omega \cap C_\rho^F \sim \zeta(Z_\rho)) = 0\,.$$

Let $f : \mathbf{R}^{n+1} \times \partial \mathbf{U}^{F^*}(0,1) \to \mathbf{R}$ be a Borel measurable function with compact support. Then we use Claim 1, (28), (29), (30), and [36, 5.4] to compute

$$\begin{aligned}
\int_{\Omega \cap C_\rho^F} f(\boldsymbol{\psi}_C^F(y))\,\mathrm{d}\mathscr{L}^{n+1}(y) &= \int_{\Omega \cap \zeta(Z_\rho)} f(\boldsymbol{\psi}_C^F(y))\,\mathrm{d}\mathscr{L}^{n+1}(y) \\
&= \int_{\Omega \cap \zeta(Z_\rho)} \int_{\zeta^{-1}(y)} f(z, \mathbf{n}^F(C,z))\,\mathrm{d}\mathscr{H}^0(z)\,\mathrm{d}\mathscr{L}^{n+1}(y) \\
&= \int_{\zeta(Z_\rho)} \int_{\zeta^{-1}(y)} f(z, \mathbf{n}^F(C,z))\,\mathrm{d}\mathscr{H}^0(z)\,\mathrm{d}\mathscr{L}^{n+1}(y) \\
&= \int_{Z_\rho} J_{n+1}\zeta(z,t)\, f(z, \mathbf{n}^F(C,z))\,\mathrm{d}\mathscr{H}^{n+1}(z,t) \\
&= \int_{Q_\rho} f(z, \mathbf{n}^F(C,z)) F(\mathbf{n}(C,z)) \int_0^\rho \Big(1 - t\frac{H(z)}{n}\Big)^n\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^n(z) \\
&= \int_{\partial C} f(z, \mathbf{n}^F(C,z)) F(\mathbf{n}(C,z)) \int_0^\rho \Big(1 - t\frac{H(z)}{n}\Big)^n\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^n(z) \\
&= \sum_{i=1}^{n+1} c_i(f)\rho^i,
\end{aligned}$$

where, for $i = 1, \ldots, n+1$,

$$c_i(f) = \Big(-\frac{1}{n}\Big)^{i-1} \frac{n!}{i!(n-i+1)!} \int_{\partial C} f(z, \mathbf{n}^F(C,z)) F(\mathbf{n}(C,z)) H(z)^{i-1}\,\mathrm{d}\mathscr{H}^n(z)\,.$$

Therefore, $\operatorname{reach}^F C \geq n/c$ by Theorem 5.9.

Claim 6: *Let $0 < r < n/c \leq \operatorname{reach}^F C$. Then $S^F(C,r)$ is a finite union of Wulff shapes of radii not smaller than $c^{-1}(n - rc)$.*

Since $\operatorname{reach}^F C \geq n/c$ we employ 5.8 to find that $S^F(C,r)$ is a submanifold of $\mathbf{R}^{n+1}$ of dimension $n$ of class $\mathscr{C}^{1,1}$. We define

$$C_r = \mathbf{R}^{n+1} \cap \{z : \boldsymbol{\delta}_C^F(z) < r\}\,.$$

Noting that $\mathbf{n}^F(C_r, \cdot)|S^F(C,r) = \operatorname{grad} F \circ \mathbf{n}(C_r, \cdot)|S^F(C,r)$ and $\operatorname{grad} F$ is a $\mathscr{C}^1$ function, we deduce that $\mathbf{n}^F(C_r, \cdot)|S^F(C,r)$ is a Lipschitzian vector field. We define

$$T = Q \cap \big\{z : \kappa_{Q,j}^F(z) = -H(z)/n \ \text{ for } j = 1, \ldots, n\big\}\,,$$

and we notice that $\mathscr{H}^n(\partial C \sim T) = 0$ by (25) and (30); then the Lusin (N) condition implies

$$\mathscr{H}^n(S^F(C,r) \sim (\boldsymbol{\xi}_C^F)^{-1}(T)) = 0\,. \tag{31}$$

Recalling 2.38(h) we see that

$$\mathbf{n}^F(C_r, z) = \frac{z - \boldsymbol{\xi}_C^F(z)}{r} = \operatorname{grad} F\big(\mathbf{n}(C, \xi(z))\big) = \mathbf{n}^F(C, \cdot) \circ \boldsymbol{\xi}_C^F(z) \quad \text{whenever } z \in S^F(C,r)\,.$$

Let us set

$$\sigma = \boldsymbol{\xi}_C^F | S^F(C,r) \cap (\boldsymbol{\xi}_C^F)^{-1}(T) \quad \text{and} \quad \varphi = \zeta | T \times \{r\}\,.$$

Observe that if $x \in T$, then $z = x + r\mathbf{n}^F(C,x) \in S^F(C,r)$, $\boldsymbol{\xi}_C^F(z) = x$, and $\operatorname{Tan}(S^F(C,r), z) = \operatorname{Tan}(T,x)$; hence, $\sigma = \varphi^{-1}$ and we get

$$\begin{gathered}
\big\langle u, \mathrm{D}\varphi(x)\big\rangle = (1 - rH(x)/n)u \quad \text{for } x \in T \text{ and } u \in \operatorname{Tan}(T,x)\,, \\
\big\langle u, \mathrm{D}\sigma(z)\big\rangle = (1 - rH(\boldsymbol{\xi}_C^F(z))/n)^{-1}u \quad \text{for } z \in \operatorname{dmn}\sigma \text{ and } u \in \operatorname{Tan}(T, \boldsymbol{\xi}_C^F(z))\,, \\
\mathrm{D}\mathbf{n}^F(C_r, \cdot)(z)u = \frac{-H(\boldsymbol{\xi}_C^F(z))}{n - rH(\boldsymbol{\xi}_C^F(z))}u \quad \text{for } \mathscr{H}^n \text{ a.a. } z \in S^F(C,r) \text{ and } u \in \operatorname{Tan}(T, \boldsymbol{\xi}_C^F(z))\,.
\end{gathered} \tag{32}$$

Employing 3.2 we conclude that $S^F(C,r)$ is a union of at most countably many boundaries of Wulff shapes with radii not smaller than $c^{-1}(n-rc)$. Since $E$ has finite perimeter we have $\mathscr{H}^n(\partial\Omega) < \infty$ so using (32) and (31) we conclude that $\mathscr{H}^n(S^F(C,r)) < \mathscr{H}^n(\partial^*\Omega) < \infty$ and Claim 6 follows.

We are now ready to conclude the proof. We notice from [17, 4.20] that

$$\partial C = \{x : \dim \operatorname{Nor}(C,x) \geq 1\}$$

and by Lemma 5.2, we also get that

$$\partial C = \{x : \dim \operatorname{Nor}^F(C,x) \geq 1\}\,.$$

We claim that

$$\boldsymbol{\xi}_C^F(S^F(C,r)) = \partial C \quad \text{ for } 0 < r < n/c\,.$$

Indeed, since $0 < r < \operatorname{reach}^F C$, for every $x \in \partial C$ there exists $\nu \in \operatorname{Nor}^F(C,x)$ such that $x + r\nu \in S^F(C,r) \cap \operatorname{dmn} \boldsymbol{\xi}_C^F$ and consequently $\boldsymbol{\xi}_C^F(x+r\nu) = x$. We deduce that $\partial C \subseteq \boldsymbol{\xi}_C^F(S^F(C,r))$. The reverse inclusion is trivial.

Consider a connected component $S_1$ of $S^F(C,r)$. By Claim 6 we obtain $s \geq n/c - r$ and $z \in \mathbf{R}^{n+1}$ such that $S_1 = \partial \mathbf{B}^{F^*}(z,s)$. Observe that

$$S^F(\mathbf{R}^{n+1} \sim \mathbf{B}^{F^*}(z, s+r), r) = S_1\,;$$

hence,

$$\partial \mathbf{B}^{F^*}(z, s+r) = \boldsymbol{\xi}_C^F(S_1) \subseteq \partial C$$

and, using, e.g., the constancy theorem [18, 4.1.7], we deduce that $\mathbf{U}^{F^*}(z, s+r)$ is a connected component of $\Omega$. Since $S_1$ was chosen arbitrarily we see that $\Omega$ must be a finite union of open disjoint Wulff shapes of radii at least $n/c$. ☐

*6.6 Remark.* This theorem extends to sets of finite perimeter the analogous result for smooth boundaries in [22, Theorem 4].

We use now Theorem 6.5 to study the critical points of the anisotropic surface area for a given volume.

**6.7 Definition** (cf. [3, 4.1])**.** A smooth function $h : (-\epsilon, \epsilon) \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is called *local variation* if and only if

(a) $h(0,x) = x$ for every $x \in \mathbf{R}^{n+1}$,

(b) $h(t,\cdot) : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is a diffeomorphism for every $t \in (-\epsilon, \epsilon)$,

(c) the set $\{x : h(t,x) \neq x \text{ for some } t \in (-\epsilon,\epsilon)\}$ has compact closure in $\mathbf{R}^{n+1}$.

We set $h_t = h(t,\cdot)$ and $\dot{h}_t(x) = \lim_{u\to 0} u^{-1}(h_{t+u}(x) - h_t(x))$ for every $(t,x) \in (-\epsilon,\epsilon) \times \mathbf{R}^{n+1}$.

Given an integrand $F$ we define *the $F$-perimeter functional* as

$$\mathcal{P}_F(E) = \int_{\partial^* E} F(\mathbf{n}(E,x))\, \mathrm{d}\mathscr{H}^n x$$

for every $E \subseteq \mathbf{R}^{n+1}$ with finite perimeter, and *the $F$-isoperimetric functional* as

$$\mathcal{I}_F(E) = \frac{\mathcal{P}_F(E)^{n+1}}{\mathscr{L}^{n+1}(E)^n}$$

for every $E \subseteq \mathbf{R}^{n+1}$ with finite perimeter and finite volume.

**6.8 Corollary.** *Let $E \subseteq \mathbf{R}^{n+1}$ be a set of finite perimeter and finite volume such that*

$$\mathscr{H}^n(\operatorname{Clos}(\partial^* E) \sim \partial^* E) = 0\,.$$

*If $\alpha \in (0,1)$, $F$ is an elliptic integrand of class $\mathscr{C}^{2,\alpha}$ and for every local variation $h$ it holds that*

$$\frac{d}{dt} \mathcal{I}_F(h_t(E))\Big|_{t=0} = 0\,,$$

*then there exists a finite union $\Omega$ of disjoint open Wulff shapes with equal radii such that*

$$\mathscr{L}^{n+1}((\Omega \sim E) \cup (E \sim \Omega)) = 0\,.$$

*Proof.* Let $h$ be a local variation and $V = \mathbf{v}(\partial^* E)$. Define $p(t) = \mathcal{P}_F(h_t(E))$ and $v(t) = \mathscr{L}^{n+1}(h_t(E))$ for $-\epsilon < t < \epsilon$. We observe that

$$p'(0) = \delta_F V(\dot{h}_0)\,,$$
$$v'(0) = \int_E \operatorname{div} \dot{h}_0 \, \mathrm{d}\mathscr{L}^{n+1} = \int_{\partial^* E} \dot{h}_0(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x)\,.$$

Noting that the derivative of the function $\frac{p^{n+1}}{v^n}$ equals

$$\Big(\frac{p(t)}{v(t)}\Big)^n \Big[(n+1)p'(t) - n\frac{p(t)}{v(t)}v'(t)\Big] \quad \text{for } -\epsilon < t < \epsilon\,,$$

it follows that

$$(n+1)p'(0) - n\frac{p(0)}{v(0)}v'(0) = 0$$

and the arbitrariness of $h$ implies that

$$\|\delta_F V\|_{\mathrm{sing}} = 0 \quad \text{and} \quad \overline{\mathbf{h}}_F(V,x) = -\frac{n}{n+1}\frac{\mathcal{P}_F(E)}{\mathscr{L}^{n+1}(E)}\mathbf{n}(E,x)\,.$$

It follows that the hypothesis of Theorem 6.5 and the equality in (24) are realized. Henceforth, the conclusion follows from Theorem 6.5. $\square$

**6.9 Corollary.** *Let $E \subseteq \mathbf{R}^{n+1}$ be a set of finite perimeter and finite volume such that*

$$\mathscr{H}^n(\mathrm{Clos}(\partial^* E) \sim \partial^* E) = 0\,.$$

*If $\alpha \in (0,1)$, $F$ is an elliptic integrand of class $\mathscr{C}^{2,\alpha}$, and for every local variation $h$*

$$\int_{\partial^* E} \dot{h}_0(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x) = 0 \quad \textit{implies} \quad \frac{d}{dt}\mathcal{P}_F(h_t(E))\Big|_{t=0} = 0\,, \tag{33}$$

*then there exists a finite union $\Omega$ of disjoint open Wulff shapes with equal radii such that*

$$\mathscr{L}^{n+1}((\Omega \sim E) \cup (E \sim \Omega)) = 0\,.$$

*Proof.* Let $V = \mathbf{v}(\partial^* E)$. Given $g \in \mathscr{X}(\mathbf{R}^{n+1})$ such that $\int_{\partial^* E} g(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x) = 0$ and $\varepsilon \in (0,1)$ define $h : (-\varepsilon, \varepsilon) \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ by the formula $h(t,x) = x + tg(x)$ and observe that $h$ is a local variation provided $\varepsilon$ is small enough. Moreover, it satisfies $\int_{\partial^* E} \dot{h}_0(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x) = 0$; hence, by (33), we deduce that

$$\delta_F V(g) = 0 \quad \text{whenever} \quad g \in \mathscr{X}(\mathbf{R}^{n+1}) \quad \text{and} \quad \int_{\partial^* E} g(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x) = 0\,. \tag{34}$$

Given $g_1, g_2 \in \mathscr{X}(\mathbf{R}^{n+1})$ such that $\int_{\partial^* E} g_i(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x) \neq 0$ for $i \in \{1,2\}$, we define

$$g_3 = g_1 - \frac{\int_{\partial^* E} g_1(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x)}{\int_{\partial^* E} g_2(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x)} g_2$$

and observe that

$$\int_{\partial^* E} g_3(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x) = 0\,;$$

hence, by (34), we get $\delta_F V(g_3) = 0$, which in turn reads

$$\frac{\int \mathrm{D}g_1(x) \bullet B_F(\nu) \, \mathrm{d}V(x,\nu)}{\int_{\partial^* E} g_1(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x)} = \frac{\int \mathrm{D}g_2(x) \bullet B_F(\nu) \, \mathrm{d}V(x,\nu)}{\int_{\partial^* E} g_2(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x)}\,.$$

In particular, there exists $\lambda \in \mathbf{R}$ such that

$$\frac{\int \mathrm{D}g(x) \bullet B_F(\nu) \, \mathrm{d}V(x,\nu)}{\int_{\partial^* E} g(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x)} = \lambda \quad \text{for} \quad g \in \mathscr{X}(\mathbf{R}^{n+1}) \quad \text{with} \quad \int_{\partial^* E} g(x) \bullet \mathbf{n}(E,x) \, \mathrm{d}\mathscr{H}^n(x) \neq 0\,.$$

Hence, recalling Definition 2.18, the hypothesis of Theorem 6.5 and the equality in (24) are realized as in the proof of 6.8. Theorem 6.5 provides the desired conclusion. $\square$

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