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A FAST SOLVER FOR THE FRACTIONAL HELMHOLTZ EQUATION*

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Abstract. The purpose of this paper is to study a Helmholtz problem with a spectral fractional Laplacian, instead of the standard Laplacian. Recently, it has been established that such a fractional Helmholtz problem better captures the underlying behavior in geophysical electromagnetics. We establish the well-posedness and regularity of this problem. We introduce a hybrid spectral-finite element approach to discretize it and show well-posedness of the discrete system. In addition, we derive a priori discretization error estimates. Finally, we introduce an efficient solver that scales as well as the best possible solver for the classical integer-order Helmholtz equation. We conclude with several illustrative examples that confirm our theoretical findings.

Key words. fractional Helmholtz equation, spectral fractional Laplacian, error analysis

AMS subject classifications. 65N12, 65N22, 65N30, 65N38, 65N55

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1. Introduction.

Recently, starting from the Maxwell's equations, the article [26] derived the scalar fractional Helmholtz equation. It also established existence of fractional (anomalous) behavior for the Magnetotelluric Problem in geophysical electromagnetics by showing a direct qualitative match between numerical tests and actual data. Motivated by these results, the goal of this paper is to take a step towards a rigorous mathematical foundation of the fractional Helmholtz equation. In particular, we show its well-posedness, introduce a new hybrid (spectral-finite element) approach for its discretization, establish a priori error estimates, and introduce an efficient solver that scales as well as the best solver in the classical (integer-order) case.

Let Ω be a bounded open domain in $\mathbb{R}^d$. We consider the fractional-order Helmholtz

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problem

$$\text{(fH)}\qquad \begin{cases} (-\Delta)^s u(\boldsymbol{x}) - k^{2s} u(\boldsymbol{x}) &= f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) &= 0, \qquad \boldsymbol{x} \in \partial\Omega, \end{cases}$$

with a given wave number $k \in \mathbb{C}$ and right-hand side data f. We restrict ourselves to the case of homogeneous Dirichlet boundary conditions. Nonhomogeneous conditions can be incorporated by solving an auxiliary local problem with a homogeneous right-hand side; see, for instance, [3, 4]. For $s \in (0,1)$, $(-\Delta)^s$ denotes the fractional powers of the realization in $L^2(\Omega)$ of the classical Laplacian $(-\Delta)$ supplemented with zero Dirichlet boundary conditions. For a rigorous definition of $(-\Delta)^s$, see section 2. For completeness, we mention that the spectral Laplacian in (fH) is not the only choice for fractional Laplacian; another popular choice is the so-called *integral fractional Laplacian*. The two definitions coincide when $\Omega = \mathbb{R}^d$ but are different when Ω is bounded [22]. Towards this end, we emphasize that our choice to use the spectral fractional Laplacian is directly motivated by the fact that the article [26] has shown a direct comparison between the numerical simulations using spectral fractional Laplacian and real life data from magnetotelluric measurements. The fractional-order model was found to match the subsurface response more accurately than the classical, integer-order equation. In addition, [26] provided a derivation of the fractional Helmholtz equation under the assumption of Ohm's constitutive law in terms of a fractional space derivative.

The article [26] solved the nonlocal operator $(-\Delta)^s$ using the so-called Kato formula [14]; the use of this formula in the context of the fractional Poisson equation was first proposed in [7], where it is referred to as the Balakrishnan formula. However, in this work we use the so-called extension approach that stems from probability literature [18], but has been pioneered by Caffarelli and Silvestre [9] and Stinga and Torrea [24]. The extension approach says that $(-\Delta)^s$ is the Dirichlet-to-Neumann map for a harmonic extension of the solution. The key advantage of this is the fact that the extension problem is local, albeit it is posed on a semi-infinite domain, $\Omega \times (0,\infty) \subset \mathbb{R}^{d+1}$, with one additional space dimension. This fact introduces computational challenges. In order to create finite element based numerical approximation, the article [19], in the case of the Poisson equation, introduced a truncation approach so that the resulting domain is bounded. On the other hand, [1] introduced a different approach where no such truncation is needed. Our hybrid spectral-finite element discretization of (fH) is motivated by the latter.

The remainder of this work is structured as follows: Section 2 introduces the necessary notation and spaces. In section 3, we show well-posedness of (fH). In section 4 we introduce the extension problem and derive properties of its eigenfunctions. Section 5 deals with the hybrid spectral-finite element discretization of the problem and a priori error estimates. In section 6 we discuss the solver of the resulting linear system. We conclude by showing numerical examples in section 7.

Remark 1.1. The choice of the coefficient $-k^{2s}$ in (fH) might appear nonintuitive at first. In [26], the fractional Helmholtz problem is stated as

$$\begin{cases} (-\Delta)^s u(\boldsymbol{x}) - \kappa^2 u(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\Omega, \end{cases}$$

with wave number $\kappa \in \mathbb{C}$. Clearly, this is just a matter of notation, and $k = \kappa^{1/s}$. We prefer the coefficient $-k^{2s}$, because this choice leads to weaker restrictions on the mesh size h when solving the fractional Helmholtz problem for fixed wave number k and different values of the fractional order s. We also note that with the proposed formulation, we need to solve the classical, integer-order, Helmholtz problem with wave number k. Nevertheless, everything that follows also holds if we use $\kappa^{1/s}$ instead of k.

2. Notation. The purpose of this section is to introduce relevant notation and preliminary results. The content of this section is well-known. Unless otherwise stated, Ω will be a bounded Lipschitz domain in $\mathbb{R}^d$. To this end, we define the fractional-order Sobolev (Hilbert) space as

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \mid \|u\|_{H^s(\Omega)} < \infty \right\}, \tag{2.1}$$

equipped with the norm

$$\|u\|_{H^s(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \int_\Omega d\boldsymbol{x} \int_\Omega d\boldsymbol{y} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{d+2s}} \right)^{\frac{1}{2}}.$$

An equivalent norm is defined by

$$\|u\|^2_{H^s(\Omega)} = \min_{U|_\Omega = u, U \in H^s(\mathbb{R}^d)} \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \left| \widehat{U}(\xi) \right|^2 \, d\xi,$$

where $\widehat{U}$ is the Fourier transform of U.

Next, we define the spectral fractional Laplacian $(-\Delta)^s$. Let $0 < \lambda_0 \leq \lambda_1 \leq \cdots$, and let $\phi_0, \phi_1, \ldots$ be the eigenvalues and eigenfunctions of the standard Laplacian, i.e.,

$$\begin{cases} -\Delta \phi_m(\boldsymbol{x}) = \lambda_m \phi_m(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \phi_m(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \partial\Omega, \end{cases} \tag{Eig}$$

normalized so that $\|\phi_m\|_{L^2(\Omega)} = 1$. Then $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(\boldsymbol{x}) = \sum_{m=0}^{\infty} u_m \lambda_m^s \phi_m(\boldsymbol{x}) \quad \text{with} \quad u_m = (u, \phi_m)_{L^2}.$$

Notice that the eigenfunctions $\{\phi_m\}_{m=0}^{\infty}$ form a complete orthonormal basis of $L^2(\Omega)$.

Using the spectrum of the Laplacian $\{(\lambda_m, \phi_m)\}_{m \in \mathbb{N}}$, we define yet another fractional-order Sobolev space [25], [8, Appendix B]

$$\widetilde{H}^s(\Omega) = \left\{ u \in L^2(\Omega) \mid |u|_{\widetilde{H}^s(\Omega)} < \infty \right\}, \tag{2.2}$$

where the norm is given by

$$|u|_{\widetilde{H}^s(\Omega)} = \left(\sum_{m=0}^{\infty} |u_m|^2 \lambda_m^s \right)^{\frac{1}{2}},$$

and $u_m = (u, \phi_m)_{L^2(\Omega)}$. The two spaces in (2.1) and (2.2) are related to each other. Indeed, for $s > 1/2$, $\widetilde{H}^s(\Omega)$ coincides with the space $H_0^s(\Omega)$ defined to be the closure of $C_0^\infty(\Omega)$ with respect to the $H^s(\Omega)$-norm, while for $s < 1/2$, $\widetilde{H}^s(\Omega)$ is identical to

$H^s(\Omega)$. In the critical case $s = 1/2$, $\widetilde{H}^s(\Omega) \subset H_0^s(\Omega)$, and the inclusion is strict, $\widetilde{H}^{\frac{1}{2}}(\Omega)$ is known as the Lions–Magenes space (see, for example, [15, Chapter 3]). These relationships between $\widetilde{H}^s$ and H^s imply that the boundary conditions in (fH) are understood in the classical trace sense only when $s > 1/2$. We denote the dual space of $\widetilde{H}^s(\Omega)$ by $\widetilde{H}^{-s}(\Omega)$ and use $\langle \cdot, \cdot \rangle_{\widetilde{H}^s(\Omega), \widetilde{H}^{-s}(\Omega)}$ to denote the duality pairings. For simplicity we drop the subscripts from the duality pairings when it is clear from the context.

The spaces $\widetilde{H}^s(\Omega)$ are useful to describe the properties of the spectral fractional Laplacian. For instance, suppose $f \in \widetilde{H}^r(\Omega)$, $r \geq -s$, and $f = \sum_{m=0}^{\infty} f_m \phi_m(\boldsymbol{x})$ with $f_m = \langle f, \phi_m \rangle$; then, the solution u to the fractional Poisson problem of order s with right-hand side f,

$$\text{(fP)} \qquad \begin{cases} (-\Delta)^s u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \partial\Omega, \end{cases}$$

is given by

$$u = \sum_{m=0}^{\infty} u_m \phi_m(\boldsymbol{x}), \qquad u_m = f_m \lambda_m^{-s}, \tag{2.3}$$

and hence $u \in \widetilde{H}^{r+2s}(\Omega)$. Notice that no additional smoothness on the domain Ω is needed to get this higher regularity. Nevertheless, to establish an equivalence between $\widetilde{H}^{r+2s}$, when $r + 2s > 1$, with higher order Sobolev spaces additional smoothness on the domain Ω is needed, as shown by the following result.

LEMMA 2.1. *Let $r \in (\ell - 1, \ell]$ for $\ell \geq 2$, and assume that $\partial\Omega \in C^\ell$. Then $\widetilde{H}^r(\Omega) \subset H^r(\Omega)$.*

Proof. By bootstrapping classical regularity results for the Poisson equation [11, Theorem 3.10] and using that $\partial\Omega \in C^\ell$, the eigenfunctions of problem (Eig) satisfy $\phi_m \in H^\ell(\Omega)$. Now, let $f \in \widetilde{H}^\ell(\Omega)$, and expand with respect to the eigenfunctions as $f = \sum_{m=0}^{\infty} f_m \phi_m(\boldsymbol{x})$ with $f_m = \langle f, \phi_m \rangle$. Since $\partial\Omega$ is Lipschitz, ϕ_m can be extended by zero to the whole space [15]. Then,

$$\begin{aligned} \|f\|_{H^\ell(\Omega)}^2 &\leq \int_{\mathbb{R}^d} (1 + |\xi|^2)^\ell \left|\widehat{f}(\xi)\right|^2 \, d\xi \\ &= \sum_{m,n} f_m \overline{f_n} \int_{\mathbb{R}^d} (1 + |\xi|^2)^\ell \widehat{\phi_m}(\xi) \widehat{\phi_n}(\xi) \, d\xi. \end{aligned}$$

But, since

$$\int_{\mathbb{R}^d} |\xi|^{2j} \, \widehat{\phi_m}(\xi) \widehat{\phi_n}(\xi) \, d\xi = \lambda_m^j \delta_{mn} \|\phi_m\|_{L^2(\Omega)}^2 = \lambda_m^j \delta_{mn}$$

for $0 \leq j \leq \ell$ due to Plancherel's theorem, we have that

$$\|f\|_{H^\ell(\Omega)}^2 = \sum_m |f_m|^2 (1 + \lambda_m)^\ell \leq C \|f\|_{\widetilde{H}^\ell(\Omega)}^2.$$

The result then follows by interpolation of spaces between $\widetilde{H}^{\ell-1}(\Omega)$ and $\widetilde{H}^\ell(\Omega)$. ∎

Similar results to the above lemma can also be obtained under different sets of assumptions on Ω that give rise to a regularity lifting for the classical Poisson problem. For example, for Ω being a convex polyhedral domain [11, Theorem 3.12], the above result holds true for $\ell = 2$. A more detailed regularity theory for spectral Poisson problems can be found in [12].

In principle one could use the expression (2.3) to compute u. However, the cost of precomputing the unknown eigenvalues *and* eigenfunctions makes this an expensive task. To overcome this hurdle, as mentioned in the introduction, we follow the approach of Stinga and Torrea [24].

We define the weighted norms on a generic domain $\mathcal{D}$ for a nonnegative weight function ω by

$$\|u\|_{L^2_\omega(\mathcal{D})} = \left(\int_{\mathcal{D}} \omega(\boldsymbol{x}) \left|u(\boldsymbol{x})\right|^2 \, d\boldsymbol{x}\right)^{\frac{1}{2}}, \qquad |u|_{H^1_\omega(\mathcal{D})} = \left(\int_{\mathcal{D}} \omega(\boldsymbol{x}) \left|\nabla u(\boldsymbol{x})\right|^2 \, d\boldsymbol{x}\right)^{\frac{1}{2}},$$
$$\|u\|_{H^1_\omega(\mathcal{D})} = \left(\|u\|^2_{L^2_\omega(\mathcal{D})} + |u|^2_{H^1_\omega(\mathcal{D})}\right)^{\frac{1}{2}},$$

along with the associated weighted spaces

$$L^2_\omega\left(\mathcal{D}\right) = \left\{u \text{ measurable } \mid \|u\|_{L^2_\omega} < \infty\right\}, \quad H^1_\omega\left(\mathcal{D}\right) = \left\{u \in L^2_\omega\left(\mathcal{D}\right) \mid \|u\|_{H^1_\omega} < \infty\right\}.$$

We use C to denote a generic constant that could change from line to line but is independent of the mesh size h and the wave number k. We will also drop the differential in the integrand when the integration variable is clear from the context.

3. Well-posedness and regularity of fractional Helmholtz equation. The main goal of this section is to establish existence and uniqueness of the solution to the fractional Helmholtz equation (fH).

In order for problem (fH) to be well-posed, we require in what follows that

$$C_{\text{reg}} \left|k\right|^{s+\alpha} \inf_{m\in\mathbb{N}} \left|1 - \left(\frac{k^2}{\lambda_m}\right)^s\right| \geq 1, \tag{3.1}$$

where $\lambda_m > 0$ are the eigenvalues of the standard Laplacian with zero Dirichlet boundary conditions (see (Eig)) and where $C_{\text{reg}} > 0$ and $\alpha \geq -s$ are constants that are independent of k. In particular, this means that $\lambda_m \neq k^2$ for all $m \in \mathbb{N}$. Condition (3.1) will, in particular, be used in the proof of the regularity result in Proposition 3.5. We further comment on the particular form of the condition in Remark 3.6.

We first state the notion of weak solutions.

Definition 3.1. *Given $f \in \widetilde{H}^{-s}(\Omega)$ we say that $u \in \widetilde{H}^s(\Omega)$ is a weak solution to* (fH) *if*

$$a(u, v) = \langle f, v\rangle \quad \forall\, v \in \widetilde{H}^s(\Omega), \tag{3.2}$$

where

$$a(u, v) := \sum_{m=0}^{\infty} \left(\lambda_m^s - k^{2s}\right) \int_\Omega u\phi_m \int_\Omega \overline{v}\phi_m. \tag{3.3}$$

Next, we shall establish the uniqueness of the solution to (3.2). We operate under the condition that $k \in \mathbb{C}$ is a constant.

LEMMA 3.2. *Let $f \in \widetilde{H}^{-s}(\Omega)$ be given, and let $k \in \mathbb{C}$ be a given constant. Assume that* (3.1) *holds. Then every $u \in \widetilde{H}^s(\Omega)$ solving* (fH) *in the weak sense is unique.*

Proof. It is sufficient to show that when the data $f \equiv 0$, then u solving (3.2) is identically zero. By setting $v = u_\ell \phi_\ell$ (where $\ell \in \mathbb{N}$ is arbitrary) in (3.2), using the orthonormality of $\{\phi_m\}_{m=0}^\infty$, and the assumption that k is a constant, we obtain that

$$a(u, u_\ell \phi_\ell) = \sum_{m=0}^{\infty} \left(\lambda_m^s - k^{2s}\right) \int_\Omega u\phi_m \int_\Omega \overline{u}_\ell \phi_\ell \phi_m = \left(\lambda_\ell^s - k^{2s}\right) |u_\ell|^2 = 0.$$

Since according to (3.1) $\lambda_\ell^s \neq k^{2s}$, we obtain that $u_\ell = 0$, i.e., $\int_\Omega u\phi_\ell = 0$. Since ℓ was arbitrary, we obtain that $u = 0$ a.e. in Ω. This completes the proof. ◻

LEMMA 3.3 (Gårding's (in)equality). *Let $u \in \widetilde{H}^s(\Omega)$ solve* (fH) *in the weak sense, and let $k \in \mathbb{C}$ be a given constant. Then*

$$a(u, u) + k^{2s}\|u\|_{L^2(\Omega)}^2 = \|u\|_{\widetilde{H}^s(\Omega)}^2.$$

Proof. From the definition of $a(\cdot, \cdot)$ in (3.3) we obtain that

$$a(u, u) = \sum_{m=0}^{\infty} \left(\lambda_m^s - k^{2s}\right) |u_m|^2 = \|u\|_{\widetilde{H}^s(\Omega)}^2 - k^{2s}\|u\|_{L^2(\Omega)}^2.$$

By rearranging terms in the above equality, we obtain the desired result. ◻

THEOREM 3.4. *Let $f \in \widetilde{H}^{-s}(\Omega)$ be given, and let $k \in \mathbb{C}$ be a given constant. Assume that* (3.1) *holds. Then there exists a unique $u \in \widetilde{H}^s(\Omega)$ solving* (fH) *in the weak sense.*

Proof. Lemmas 3.2 and 3.3, in conjunction with the Fredholm alternative, give the asserted result. We refer to [6, Theorem 3.3] for similar arguments in the case of the standard Laplacian. ◻

The next result establishes regularity of solutions of the fractional Helmholtz equation.

PROPOSITION 3.5 (regularity). *Assume that* (3.1) *holds. If $f \in \widetilde{H}^r(\Omega)$, $r \geq -s$, then the weak solution to the fractional Helmholtz problem $u \in \widetilde{H}^{r+2s}(\Omega)$, and*

$$\|u\|_{\widetilde{H}^{r+2s}(\Omega)} \leq C_{\text{reg}} |k|^{\alpha+s} \|f\|_{\widetilde{H}^r(\Omega)}.$$

Proof. Assume that $u \in \widetilde{H}^\beta(\Omega)$ for some $\beta \geq s$. Then $(-\Delta)^s u = f + k^{2s}u \in \widetilde{H}^{\min\{r,\beta\}}$. By the regularity result for the fractional Poisson problem, we obtain that $u \in \widetilde{H}^{\min\{r,\beta\}+2s}$. Since $u \in \widetilde{H}^s(\Omega)$, we obtain the desired result by iteration. Now, if we expand $f = \sum_{m=0}^\infty f_m\phi_m$, then $u = \sum_{m=0}^\infty f_m(\lambda_m^s - k^{2s})^{-1}\phi_m$, and

$$\begin{aligned}
\|u\|_{\widetilde{H}^{r+2s}(\Omega)}^2 &= \sum_{m=0}^{\infty} \frac{\lambda_m^{r+2s}}{|\lambda_m^s - k^{2s}|^2} |f_m|^2 \leq \sup_{m\in\mathbb{N}} \frac{\lambda_m^{2s}}{|\lambda_m^s - k^{2s}|^2} \sum_{m=0}^{\infty} \lambda_m^r |f_m|^2 \\
&= \frac{1}{\inf_{m\in\mathbb{N}} |1 - (k^2/\lambda_m)^s|^2} \|f\|_{\widetilde{H}^r(\Omega)}^2 \leq C_{\text{reg}}^2 |k|^{2\alpha+2s} \|f\|_{\widetilde{H}^r(\Omega)}^2,
\end{aligned}$$

where we have used assumption (3.1). ◻

Remark 3.6. Consider the pair of solution and right-hand side given by $f = \phi_m$, $u = (\lambda_m^s - k^{2s})^{-1}\phi_m$, $m \in \mathbb{N}$. Then $\|f\|_{\widetilde{H}^r(\Omega)} = \lambda_m^{r/2}$ and $\|u\|_{\widetilde{H}^{r+2s}(\Omega)} = \lambda_m^{r/2} / \left|1-(k^2/\lambda_m)^s\right|$.

By choosing k^2 arbitrarily close to λ_m, the ratio

$$\|u\|_{\widetilde{H}^{r+2s}(\Omega)} \,/\, \|f\|_{\widetilde{H}^r(\Omega)} = 1/\left|1-(k^2/\lambda_m)^s\right|$$

can be made arbitrarily large, which motivates the restriction on k of (3.1). Assumption (3.1) might appear arbitrary at first, but it allows us to analyze the fractional Helmholtz problem with homogeneous Dirichlet conditions much in the same way as its classical equivalent with Robin condition. Although not the topic of this work, we expect that the presented analysis will extend to more general boundary conditions. The comparison with the classical Robin case also motivates the inclusion of the factor $|k|^{s+\alpha}$ in assumption (3.1). The resulting regularity result of Proposition 3.5 mirrors estimates available in the integer-order case; see, e.g., [17, Assumption 4.8]. In the integer-order case, the value of α depends on smoothness properties of the domain. We note that it can be difficult to determine whether a given fractional Helmholtz problem satisfies assumption (3.1), since the eigenvalues of the Laplacian are generally not known in closed form. It should be observed though that this is in fact the same assumption that needs to hold in the classical, integer-order case for $s = 1$.

4. The extension problem.

By using [9, 24] we can equivalently cast the fractional Helmholtz problem (fH) as a problem over the extruded domain $\mathcal{C} = \Omega \times (0,\infty)$:

$$\text{(Ext)} \qquad \begin{cases} -\nabla \cdot \omega(y)\nabla U(\boldsymbol{x},y) = 0, & (\boldsymbol{x},y) \in \mathcal{C}, \\ U(\boldsymbol{x},y) = 0, & (\boldsymbol{x},y) \in \partial_L\mathcal{C} := \partial\Omega \times [0,\infty), \\ \dfrac{\partial U}{\partial \nu^\omega}(\boldsymbol{x}) - k^{2s}U(\boldsymbol{x},0) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases}$$

where $\omega(y) = y^\alpha/d_s$, $\alpha = 1-2s$, $d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}$, and

$$\frac{\partial U}{\partial \nu^\omega}(\boldsymbol{x}) = -\lim_{y\to 0^+}\omega(y)\frac{\partial U}{\partial y}(\boldsymbol{x},y) = (-\Delta)^s U(\boldsymbol{x},0). \tag{4.1}$$

The solution to (fH) is then recovered by taking the trace of U on Ω, i.e., $u = \operatorname{tr}_\Omega U$.

We define the solution space $\mathcal{H}^1_\omega$ on the semi-infinite cylinder $\mathcal{C}$ as

$$\mathcal{H}^1_\omega = \left\{ V \in H^1_\omega(\mathcal{C}) \mid V = 0 \text{ on } \partial_L\mathcal{C} \right\}$$

and we denote its dual by $(\mathcal{H}^1_\omega)^*$. Notice that

$$\operatorname{tr}_\Omega \mathcal{H}^1_\omega \equiv \widetilde{H}^s(\Omega),$$

where tr_Ω denotes the Ω-trace operator. Moreover, due to the Poincaré inequality in the weighted Sobolev spaces, we have that the seminorm $|\cdot|_{H^1_\omega}$ is a norm on $\mathcal{H}^1_\omega$, and we write $\|\cdot\|_{\mathcal{H}^1_\omega} := |\cdot|_{H^1_\omega}$. We refer to [10] for details.

The weak formulation of the extension problem (Ext) consists of seeking $U \in \mathcal{H}^1_\omega$ such that

$$\mathcal{A}(U,V) = \langle f, V\rangle_\Omega \quad \forall V \in \mathcal{H}^1_\omega, \tag{wExt}$$

where we have

$$\mathcal{A}(U,V) = \int_{\mathcal{C}} \omega \nabla U \cdot \nabla \overline{V} - k^{2s} \int_{\Omega} U\overline{V} \quad \text{and} \quad \langle f, V\rangle_{\Omega} := \left\langle f, \operatorname{tr}_{\Omega} \overline{V}\right\rangle.$$

We will also frequently use the shorthand

$$\|V\|_{L^2(\Omega)} := \|\operatorname{tr}_{\Omega} V\|_{L^2(\Omega)}.$$

We seek a solution of the extension problem using classical separation of variables: $U(\boldsymbol{x},y) = \Phi(\boldsymbol{x})\Psi(y)$. Then

$$\frac{-\Delta_{\boldsymbol{x}}\Phi}{\Phi} = \frac{\partial_y\left(\omega(y)\partial_y\Psi\right)}{\omega(y)\Psi} = A,$$

where A is a constant that is independent of $\boldsymbol{x}$ and y. Thanks to (Eig), the boundary condition on the lateral face of the cylinder $\mathcal{C}$ shows that $\Phi = \phi_m$ and $A = \lambda_m$ for $m \in \mathbb{N}$. The associated solution $\Psi = \psi_m$ in the extension direction must therefore satisfy

$$\partial_y\left(\omega(y)\partial_y\psi_m\right) = \lambda_m \omega(y)\psi_m. \tag{4.2}$$

Notice that $\psi_m(0) = 1$. Moreover, using (4.1) we obtain that

$$\frac{\partial \psi_m}{\partial \nu^{\omega}} = \lambda_m^s. \tag{4.3}$$

By applying integration by parts to (4.2) and using (4.3) we obtain that

$$\lambda_m \int_0^{\infty} \omega\psi_m\psi_n + \int_0^{\infty} \omega\psi_m'\psi_n' = \lambda_m^s, \tag{4.4}$$

which is uniquely solvable when we impose $\psi_m(+\infty) = 0$. Subtracting the same identity with indices m and n interchanged results in

$$\int_0^{\infty} \omega\psi_m\psi_n = \begin{cases} \frac{\lambda_m^s - \lambda_n^s}{\lambda_m - \lambda_n} & \text{if } m \neq n, \\ s\lambda_m^{s-1} & \text{if } m = n, \end{cases} \tag{4.5}$$

and by substituting in (4.4)

$$\int_0^{\infty} \omega\psi_m'\psi_n' = \begin{cases} \frac{\lambda_m\lambda_n^s - \lambda_n\lambda_m^s}{\lambda_m - \lambda_n} & \text{if } m \neq n, \\ (1-s)\lambda_m^s & \text{if } m = n, \end{cases} \tag{4.6}$$

where the identities for $m = n$ are obtained by taking the limit as $\lambda_n \to \lambda_m$. The solution to the extension problem (Ext) is then given by

$$U(\boldsymbol{x},y) = \sum_{m=0}^{\infty} u_m\phi_m(\boldsymbol{x})\psi_m(y), \quad \text{where } u_m = \left(\lambda_m^s - k^{2s}\right)^{-1} f_m, \tag{4.7}$$

while $u(\boldsymbol{x}) = \sum_{m=0}^{\infty} u_m\phi_m(\boldsymbol{x})$ as in (2.3). The separable solution (4.7) is the basis for our choice of discretization of the extension problem to be described in the next section. The main advantage of this approach is that the extension problem involves only integer-order derivatives but comes at the price of having to deal with a degenerate weight $\omega(y)$.

We conclude this section with the following well-posedness result for (wExt).

Proposition 4.1. *Let Ω be a bounded Lipschitz domain, and let $f \in \widetilde{H}^{-s}(\Omega)$. Then there exists $U \in \mathcal{H}^1_\omega$ solving* (wExt), *and such a solution depends continuously on the data*

$$\|U\|_{\mathcal{H}^1_\omega} \leq C_d(k) \|f\|_{\widetilde{H}^{-s}(\Omega)}$$

with $C_d(k) = C_{reg} |k|^{s+\alpha}$.

Proof. The proof follows along the lines of Theorem 3.4, i.e., we need to show uniqueness of U and prove Gårding's inequality. Then, the result will follow from the Fredholm alternative. Construction of a unique solution using separation of variables is given above. Gårding's inequality can be shown as follows. We have

$$\begin{aligned}\|U\|^2_{\mathcal{H}^1_\omega} &= \mathcal{A}(U,U) + k^{2s} (\operatorname{tr}_\Omega U, \operatorname{tr}_\Omega U)_{L^2(\Omega)} \\ &= \mathcal{A}(U,U) + k^{2s} \langle \mathcal{T} \operatorname{tr}_\Omega U, \operatorname{tr}_\Omega U\rangle_{\widetilde{H}^{-s}(\Omega),\widetilde{H}^{s}(\Omega)} \\ &= \mathcal{A}(U,U) + k^{2s} \langle (\operatorname{tr}^\star_\Omega \mathcal{T} \operatorname{tr}_\Omega) U, U\rangle_{(\mathcal{H}^1_\omega)^*,\mathcal{H}^1_\omega}, \end{aligned} \tag{4.8}$$

where in the second equality we have used the existence of a compact operator $\mathcal{T} : \widetilde{H}^s(\Omega) \to \widetilde{H}^{-s}(\Omega)$. Moreover, in the last equality we have used that the trace operator $\operatorname{tr}_\Omega : \mathcal{H}^1_\omega \to \widetilde{H}^s(\Omega)$ is bounded linear and thus its adjoint $\operatorname{tr}^\star_\Omega : \widetilde{H}^{-s}(\Omega) \to (\mathcal{H}^1_\omega)^*$ is well-defined. Notice that the operator $\operatorname{tr}^\star_\Omega \mathcal{T} \operatorname{tr}_\Omega : \mathcal{H}^1_\omega \to (\mathcal{H}^1_\omega)^*$ is compact (composition of bounded and compact operators); thus we have shown Gårding's (in)equality [20, Remark 2.1.58]. Now, just as in (4.7), we can expand $f = \sum_{m=0}^\infty f_m \phi_m$, so that $U = \sum_{m=0}^\infty f_m (\lambda_m^s - k^{2s})^{-1} \phi_m \psi_m$. Since ϕ_m are orthonormal in $L^2(\Omega)$, we find

$$\begin{aligned}\|U\|^2_{\mathcal{H}^1_\omega} &= \int_{\mathcal{C}} \omega |\nabla U|^2 \\ &= \sum_{m=0}^\infty |f_m|^2 \frac{1}{(\lambda_m^s - k^{2s})^2} \left(\int_\Omega |\nabla \phi_m|^2 \int_0^\infty \omega (\psi_m)^2 + \int_\Omega \phi_m^2 \int_0^\infty \omega (\psi_m')^2 \right) \\ &= \sum_{m=0}^\infty |f_m|^2 \frac{\lambda_m^s}{(\lambda_m^s - k^{2s})^2} \\ &\leq \sup_{m\in\mathbb{N}} \frac{\lambda_m^{2s}}{(\lambda_m^s - k^{2s})^2} \sum_{m=0}^\infty \lambda_m^{-s} |f_m|^2 \\ &\leq C^2_{\text{reg}} |k|^{2(s+\alpha)} \|f\|^2_{\widetilde{H}^{-s}(\Omega)}. \end{aligned}$$

This completes the proof. ∎

5. Discretization of the extension problem and a priori error bounds. For the remainder of this paper, we assume that Ω is sufficiently smooth so that $\widetilde{H}^{r+2s}(\Omega)$ can be associated with the classical fractional-order Sobolev space $H^{r+2s}(\Omega)$ (see Lemma 2.1). We expect that domains of limited smoothness will reduce the achievable rate of convergence. We propose approximating the variational problem (wExt) using a Galerkin scheme with the subspace consisting of standard low-order nodal finite elements of order $p \geq 1$ in the $\boldsymbol{x}$-variable and a spectral method in the y-direction. To this end, we let $\mathcal{T}_h$ be a shape regular, globally quasi-uniform triangulation of Ω, and let

$$S_h = \left\{ v_h \in C^0\left(\overline{\Omega}\right) \mid v_h\big|_K \in \mathbb{P}_p(K) \ \forall K \in \mathcal{T}_h \right\}.$$

In the y-direction, ideally, we would like to use y-basis functions $\{\psi_m\}$ given in the previous section. Unfortunately, this requires knowledge of the true eigenvalues λ_m of $(-\Delta)$ over Ω. Therefore, we use approximations $\widetilde{\lambda}_m \approx \lambda_m$ in place of the true eigenvalues in (4.5) and (4.6).

The Galerkin subspace for the extension problem is then taken to be

$$\mathcal{V}_h = \left\{ V_h = \sum_{m=0}^{M-1} v_{h,m}(\boldsymbol{x})\, \widetilde{\psi}_m(y) \mid v_{h,m} \in S_h \text{ and } \widetilde{\psi}_m \text{ solves (4.4) with } \widetilde{\lambda}_m \right\} \subset \mathcal{H}^1_\omega.$$

Notice that we do not need an analytic expression for the basis functions $\{\widetilde{\psi}_m\}$, and it is sufficient to know mass and stiffness matrices (4.5) and (4.6). The spectral expansion order M will depend on s, h, and the regularity of the solution. This dependency will be made more explicit in Assumption 5.3. The efficient approach to find approximations $\widetilde{\lambda}_m$ is discussed in [1] and is based on using the asymptotic law for the eigenvalues of the integer-order Laplacian as well as coarse finite-element discretizations. We further emphasize that $\mathcal{O}(|\log h|)$ eigenvalue approximations are sufficient to get "good approximation" properties.

The Galerkin approximation of (wExt) seeks $U_h \in \mathcal{V}_h$ such that

$$\text{(wExt}_\text{h}\text{)} \qquad \mathcal{A}(U_h, V_h) = \langle f, V_h \rangle_\Omega \quad \forall V_h \in \mathcal{V}_h,$$

with the approximation of the fractional Helmholtz problem given by

$$u_h := \operatorname{tr}_\Omega U_h.$$

Having introduced the discrete problem, our next goal is to obtain an estimate for the error $u - u_h$. The trace inequality in [10, Proposition 2.1] (see also [19]) implies that

$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \le C \|U - U_h\|_{\mathcal{H}^1_\omega},$$

where the constant is independent of p, M, and h. We also refer to [5, Theorem 2.3] for a more general trace inequality. Hence, in order to bound $u - u_h$, it suffices to bound $\|U - U_h\|_{\mathcal{H}^1_\omega}$, the discretization error of the extension problem (wExt$_\text{h}$).

Define the norm

$$\|V\|^2 := \|V\|^2_{\mathcal{H}^1_\omega} + |k|^{2s} \|V\|^2_{L^2(\Omega)}, \qquad V \in \mathcal{H}^1_\omega.$$

Using the trace inequality in [10, Proposition 2.1] (see also [19]), we find that $\mathcal{A}$ is continuous

$$|\mathcal{A}(U,V)| \le \|U\|_{\mathcal{H}^1_\omega} \|V\|_{\mathcal{H}^1_\omega} + |k|^{2s} \|U\|_{L^2(\Omega)} \|V\|_{L^2(\Omega)} \le C \|U\| \, \|V\| \quad \forall U, V \in \mathcal{H}^1_\omega$$

and satisfies the Gårding type (in)equality

$$\mathcal{A}(U,U) + k^{2s} \|U\|^2_{L^2(\Omega)} = \|U\|^2_{\mathcal{H}^1_\omega} \qquad \forall U \in \mathcal{H}^1_\omega.$$

Define the solution operator $\mathcal{S}_k : \widetilde{H}^{-s}(\Omega) \to \mathcal{H}^1_\omega$ via

$$\mathcal{A}(\mathcal{S}_k f, V) = \langle f, V \rangle_\Omega \qquad \forall V \in \mathcal{H}^1_\omega$$

and the adjoint solution operator $\mathcal{S}_k^* : \widetilde{H}^{-s}(\Omega) \to \mathcal{H}_\omega^1$ via

$$\mathcal{A}(W, \mathcal{S}_k^* f) = \langle f, W \rangle_\Omega \qquad \forall\, W \in \mathcal{H}_\omega^1.$$

The two operators can be expressed in terms of each other as

$$\mathcal{S}_k^* = \mathcal{S}_{\overline{k}}.$$

Moreover, let

$$\eta := \sup_{f \in L^2(\Omega) \setminus \{0\}} \inf_{V_h \in \mathcal{V}_h} \frac{\|\!|\mathcal{S}_k^* f - V_h|\!\|}{\|f\|_{L^2(\Omega)}}.$$

η measures how well solutions of the adjoint problem can be approximated using functions in $\mathcal{V}_h$. Naturally, we expect η to decrease as h decreases. The following two results, Theorems 5.1 and 5.2, closely mimic the ideas developed in [17, 16] for the integer-order case. We refer to the supplementary material file (supp.pdf [local/web 276KB]) for their respective proofs. Similar to the integer-order Helmholtz equation, the mesh size needs to be small enough to resolve the wave number k.

THEOREM 5.1. *Assume that*

$$\eta\,|k|^s \le \gamma \tag{5.1}$$

for small enough constant γ that is independent of h and k. Then $\mathcal{A}$ satisfies the discrete inf-sup condition

$$\inf_{U_h \in \mathcal{V}_h} \sup_{V_h \in \mathcal{V}_h} \frac{|\mathcal{A}(U_h, V_h)|}{\|\!|U_h|\!\|\,\|\!|V_h|\!\|} \ge \frac{1 - C\gamma}{1 + 2(C_d(k) + \eta)\,|k|^s}.$$

THEOREM 5.2. *Let $U \in \mathcal{H}_\omega^1$ be the solution of* (wExt), *and let $U_h \in \mathcal{V}_h$ be the solution of* (wExt$_\mathrm{h}$). *Assume that* (5.1) *holds for small enough constant γ that is independent of h and k. Then*

$$\|\!|U - U_h|\!\| \le C \inf_{V_h \in \mathcal{V}_h} \|\!|U - V_h|\!\|,$$
$$\|U - U_h\|_{L^2(\Omega)} \le C\eta\,\|\!|U - U_h|\!\|,$$

where the constants are independent of h and k.

Before we turn our attention to the approximation results, we state the required assumptions on the approximation space $\mathcal{V}_h$ and the eigenvalue approximations $\{\widetilde{\lambda}_m\}$, parameterized by a parameter t that will be linked to the solution regularity.

ASSUMPTION 5.3. *Given $t \ge s$, assume that the following hold:*

- *M is large enough such that $\lambda_M^{(s-t)/2} \sim h^{\min\{p,t-s\}}$, where p is the polynomial degree.*
- *For $0 \le m \le M-1$ it holds that*

$$\left(\frac{\widetilde{\lambda}_m}{\lambda_m}\right)^s, \left(\frac{\lambda_m}{\widetilde{\lambda}_m}\right)^{1-s} \le c_\sigma^2 \tag{5.2}$$

with a positive constant c_σ that is independent of h.

- *For* $0 \le m \le M-1$ *it holds that*

$$g\left(s, \widetilde{\lambda}_m/\lambda_m\right) \le \lambda_m^{t-s} h^{2\min\{p,t-s\}}, \tag{5.3}$$

where

$$g(s,\rho) = 1 - \frac{1}{(1-s)\rho^s + s\rho^{s-1}}. \tag{5.4}$$

We refer the reader to [1] for a discussion on how these requirements can be achieved in practice using the asymptotic behavior of the eigenvalues and by finite element discretization. The assumptions (5.6) and (5.4) in the following theorem are also discussed in [1] in more detail. In what follows, let π_h denote the Scott–Zhang interpolant [21].

THEOREM 5.4. *Let* $s \le t \le p+1$ *and assume that Assumption* 5.3 *holds for* t. *Moreover, assume that there exist positive constants* C_0, C_1 *independent of* h *such that the following two inequalities hold for any* $\boldsymbol{y} \in \mathbb{R}^M$:

$$\begin{aligned}\sum_{m,n=0}^{M-1} y_m y_n \int_\Omega (\phi_m - \pi_h\phi_m)(\phi_n - \pi_h\phi_n) \\ \le C_0 |\log \lambda_M| \sum_{m=0}^{M-1} y_m^2 \|\phi_m - \pi_h\phi_m\|_{L^2(\Omega)}^2,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\sum_{m,n=0}^{M-1} y_m y_n \int_\Omega \nabla(\phi_m - \pi_h\phi_m)\cdot\nabla(\phi_n - \pi_h\phi_n) \\ \le C_1 |\log \lambda_M| \sum_{m=0}^{M-1} y_m^2 \|\nabla(\phi_m - \pi_h\phi_m)\|_{L^2(\Omega)}^2,\end{aligned} \tag{5.6}$$

where π_h *is the Scott–Zhang interpolant.*

Let U *satisfy the variational equality* (wExt), *and assume that* $u = \mathrm{tr}_\Omega U \in \widetilde{H}^q(\Omega)$ *for* $s \le q \le t$. *Then*

$$\begin{aligned}&\inf_{V_h\in\mathcal{V}_h} \|U - V_h\| \\ &\le C\, |u|_{\widetilde{H}^q(\Omega)} \sqrt{|\log h|} \left\{ h^{\min\{p,t-s\}\frac{q-s}{t-s}} + |k|^s\, h^{\min\{p,t-s\}\min\{q,2t-2s\}/(t-s)} \right\},\end{aligned}$$

where C *is independent of* h *and* k.

Given the length of the proof of the above theorem, we record it in the supplementary material file (supp.pdf [local/web 276KB]).

Remark 5.5. Given a right-hand side function $f \in \widetilde{H}^r(\Omega)$, the regularity result in Proposition 3.5 gives that the solution to the fractional Helmholtz problem has regularity of order $r+2s$. Using elements of order p, we want to select M and eigenvalue approximations $\widetilde{\lambda}_m$ to satisfy Assumption 5.3 for $t = \max\{0, \min\{r, p-s\}\} + 2s$. Satisfying the conditions for larger values of t will not lead to any improvements in the approximation result. This also shows that the method cannot take advantage of right-hand side regularity $r \ge p - s$.

The following stable splitting is inspired by the classical, integer-order case (see, e.g., [16]).

PROPOSITION 5.6. *Assume that assumption* (3.1) *holds. Then, for* $g \in L^2(\Omega)$, $\mathcal{S}_k^* g$ *can be split as*

$$\mathcal{S}_k^* g = U_{\mathcal{E}}(g) + U_{\mathcal{A}}(g),$$

where $U_{\mathcal{E}}(g)$, $U_{\mathcal{A}}(g) \in \mathcal{H}^1_\omega$, *and their respective traces* $u_{\mathcal{E}}(g) := \operatorname{tr}_\Omega U_{\mathcal{E}}(g)$ *and* $u_{\mathcal{A}}(g) := \operatorname{tr}_\Omega U_{\mathcal{A}}(g)$ *satisfy*

$$\begin{aligned} u_{\mathcal{E}}(g) &\in \widetilde{H}^{2s}(\Omega), & |u_{\mathcal{E}}(g)|_{\widetilde{H}^{2s}(\Omega)} &\le C \|g\|_{L^2(\Omega)}, \\ u_{\mathcal{A}}(g) &\in \widetilde{H}^{t}(\Omega), & |u_{\mathcal{A}}(g)|_{\widetilde{H}^{t}(\Omega)} &\le C |k|^{\alpha+(t-s)} \|g\|_{L^2(\Omega)} \quad \forall t, \end{aligned}$$

for constants that are independent of k.

Proof. Set $\mathcal{I}_k := \left\{ m \in \mathbb{N} \mid \lambda_m \le 2|k|^2 \right\}$. Expand g with respect to $\{\phi_m\}$ and write, using $g_m = (g, \phi_m)_{L^2}$,

$$\begin{aligned} g &= \sum_{m=0}^{\infty} g_m \phi_m = \sum_{m \in \mathcal{I}_k} g_m \phi_m + \sum_{m \notin \mathcal{I}_k} g_m \phi_m =: g_{\mathcal{A}} + g_{\mathcal{E}}, \\ \mathcal{S}_k^* g &= \mathcal{S}_k^* g_{\mathcal{A}} + \mathcal{S}_k^* g_{\mathcal{E}} =: U_{\mathcal{A}}(g) + U_{\mathcal{E}}(g). \end{aligned}$$

Since for $m \notin \mathcal{I}_k$ we have that $\left|\lambda_m^s - k^{2s}\right| = \lambda_m^s - k^{2s} > (1 - 1/2^s)\lambda_m^s$, we find

$$|u_{\mathcal{E}}(g)|^2_{\widetilde{H}^{2s}(\Omega)} = \sum_{m \notin \mathcal{I}_k} \frac{\lambda_m^{2s}}{|\lambda_m^s - k^{2s}|^2} |g_m|^2 \le C \sum_{m \notin \mathcal{I}_k} |g_m|^2 = C \|g_{\mathcal{A}}\|^2_{L^2(\Omega)} \le C \|g\|^2_{L^2(\Omega)}.$$

On the other hand, we have that

$$\begin{aligned} |u_{\mathcal{A}}(g)|^2_{\widetilde{H}^{t}(\Omega)} &= \sum_{m \in \mathcal{I}_k} \frac{\lambda_m^{t}}{|\lambda_m^s - k^{2s}|^2} |g_m|^2 \\ &\le C |k|^{2(t-2s)} \sum_{m \in \mathcal{I}_k} \frac{\lambda_m^{2s}}{|\lambda_m^s - k^{2s}|^2} |g_m|^2 \\ &\le C |k|^{2(t-2s)} \sup_{m \in \mathcal{I}_k} \frac{\lambda_m^{2s}}{|\lambda_m^s - k^{2s}|^2} \|g_{\mathcal{A}}\|^2_{L^2(\Omega)} \\ &= C |k|^{2(t-2s)} \frac{1}{\inf_{m \in \mathcal{I}_k} |1 - (k^2/\lambda_m)^s|^2} \|g_{\mathcal{A}}\|^2_{L^2(\Omega)} \\ &\le C C_{\text{reg}}^2 |k|^{2(t-2s)} |k|^{2s+2\alpha} \|g\|^2_{L^2(\Omega)}, \end{aligned}$$

where we have used assumption (3.1). ∎

THEOREM 5.7. *Let* $f \in \widetilde{H}^r(\Omega)$, $r \ge -s$. *Assume that*

$$\left\{ 1 + (h|k|)^{\max\{0, \min\{r, p-s\}\}} |k|^{\alpha+s} \right\} \sqrt{|\log h|} \left[(h|k|)^s + (h|k|)^{2s} \right] \le \gamma \tag{5.7}$$

for small enough constant γ *that is independent of* h *and* k, *and that Assumption* 5.3 *is satisfied for* $t = \max\{0, \min\{r, p-s\}\} + 2s$ *and that the conditions of Theorem* 5.4 *hold. Let* $U \in \mathcal{H}^1_\omega$ *be the solution of* (wExt), *and let* $U_h \in \mathcal{V}_h$ *be the solution of* (wExt_h) *and* u *and* u_h *be their respective traces on* Ω. *Then*

$\|u - u_h\|_{\widetilde{H}^s(\Omega)}$

$$\leq C \|U - U_h\|_{\mathcal{H}^1_\omega} \leq C \|\!|U - U_h|\!\| \leq C |k|^{\alpha+s} h^{\min\{p,r+s\}} \sqrt{|\log h|}\, |f|_{\widetilde{H}^r(\Omega)},$$

$$\|u - u_h\|_{L^2(\Omega)}$$
$$\leq C |k|^{\alpha+s} \left\{1 + (h|k|)^{\max\{0,\min\{r,p-s\}\}} |k|^{\alpha+s}\right\} h^{\min\{p+s,r+2s\}} |\log h|\, |f|_{\widetilde{H}^r(\Omega)},$$

where the constants are independent of h and k.

Remark 5.8. Condition (5.7) is the fractional-order equivalent of the usual conditions on the relation between wave number k and mesh size h; see, e.g., [16, 27]. Assumption 5.3 guarantees good approximation properties and is, loosely speaking, satisfied by choosing enough eigenvalue approximations $\widetilde{\lambda}_m$ given a mesh of mesh size h, and reduces to the classical, integer-order case (compare [16]) up to the logarithmic factor. The conditions (5.6) and (5.4) of Theorem 5.4 are shown to hold in practice; see [1].

Proof. Without loss of generality, assume that $r+s \leq p$. (Otherwise, we can take $r = p-s$.) Let $g \in L^2(\Omega)$. According to Proposition 5.6 we split $\mathcal{S}_k^* g = U_\mathcal{E}(g) + U_\mathcal{A}(g)$, with $u_\mathcal{E}(g) = \operatorname{tr}_\Omega U_\mathcal{E}(g)$ having regularity $q_\mathcal{E} = 2s$ and $u_\mathcal{A}(g) = \operatorname{tr}_\Omega U_\mathcal{A}(g)$ having regularity $q_\mathcal{E} = t$. Since $q_\mathcal{E} = 2s \leq \max\{0,r\} + 2s = t = q_\mathcal{A} \leq 2t - 2s$, we have that

$$\min\{p, t-s\} \frac{q_\mathcal{E} - s}{t-s} = q_\mathcal{E} - s = s,$$
$$\min\{p, t-s\} \min\{q_\mathcal{E}, 2t-2s\}/(t-s) = \min\{q_\mathcal{E}, 2t-2s\} = 2s,$$
$$\min\{p, t-s\} \frac{q_\mathcal{A} - s}{t-s} = q_\mathcal{A} - s = t - s,$$
$$\min\{p, t-s\} \min\{q_\mathcal{A}, 2t-2s\}/(t-s) = \min\{q_\mathcal{A}, 2t-2s\} = t.$$

Hence, by applying Theorem 5.4 and Proposition 5.6, we have that

$$\inf_{V_h \in \mathcal{V}_h} \|\!|\mathcal{S}_k^* g - V_h|\!\|$$
$$\leq \inf_{V_h \in \mathcal{V}_h} \|\!|U_\mathcal{E}(g) - V_h|\!\| + \inf_{V_h \in \mathcal{V}_h} \|\!|U_\mathcal{A}(g) - V_h|\!\|$$
$$\leq C \left\{ |u_\mathcal{E}(g)|_{\widetilde{H}^{2s}(\Omega)} + h^{t-2s} |u_\mathcal{A}(g)|_{\widetilde{H}^t(\Omega)} \right\} \sqrt{|\log h|} h^s \left[1 + (h|k|)^s\right]$$
$$\leq C \left\{ 1 + h^{\max\{0,\min\{r,p-s\}\}} |k|^{\alpha+(t-s)} \right\} \sqrt{|\log h|} h^s \left[1 + (h|k|)^s\right] \|g\|_{L^2(\Omega)}$$
$$\leq C \left\{ 1 + (h|k|)^{\max\{0,\min\{r,p-s\}\}} |k|^{\alpha+s} \right\} \sqrt{|\log h|} h^s \left[1 + (h|k|)^s\right] \|g\|_{L^2(\Omega)}.$$

Therefore, we find

$$\eta \leq C \left\{ 1 + (h|k|)^{\max\{0,\min\{r,p-s\}\}} |k|^{\alpha+s} \right\} \sqrt{|\log h|} h^s \left[1 + (h|k|)^s\right].$$

Combining the latter with Theorem 5.1 and (5.7), we obtain that the discrete inf-sup condition holds.

Now, let $f \in \widetilde{H}^r(\Omega)$. Then the solution of the fractional Helmholtz problem is in $\widetilde{H}^{r+2s}(\Omega)$ and hence, applying Theorem 5.4 with $q = r + 2s \leq t$ and the regularity estimate from Proposition 3.5, we obtain that

$$\inf_{V_h \in \mathcal{V}_h} \|\!|U - V_h|\!\| \leq C |u|_{\widetilde{H}^{r+2s}(\Omega)} \sqrt{|\log h|} h^{r+s} \left[1 + (h|k|)^s\right]$$
$$= C C_{\text{reg}} |f|_{\widetilde{H}^r(\Omega)} |k|^{\alpha+s} \sqrt{|\log h|} h^{r+s} \left[1 + (h|k|)^s\right].$$

Combining this with Theorem 5.2 and (5.7), we obtain the estimates

$$\begin{aligned}
&\left\| U - U_h \right\| \\
&\qquad \leq C \left|k\right|^{\alpha+s} h^{\min\{p,r+s\}} \sqrt{\left|\log h\right|} \left|f\right|_{\widetilde{H}^r(\Omega)}, \\
&\left\| U - U_h \right\|_{L^2(\Omega)} \\
&\qquad \leq C \left|k\right|^{\alpha+s} \left\{ 1 + \left(h\left|k\right|\right)^{\max\{0,\min\{r,p-s\}\}} \left|k\right|^{\alpha+s} \right\} h^{\min\{p+s,r+2s\}} \left|\log h\right| \left|f\right|_{\widetilde{H}^r(\Omega)}.
\end{aligned}$$

We conclude by noting that due to the trace inequality,

$$\left\| u - u_h \right\|_{\widetilde{H}^s(\Omega)} \leq C \left\| U - U_h \right\|_{\mathcal{H}^1_\omega} \leq C \left\| U - U_h \right\|. \qquad \square$$

6. Solution of the linear system. Let $\{\Phi_i\}_{i=1}^n$ denote the nodal basis functions of the finite element solution space S_h; then the solution of the discretized fractional Helmholtz problem can be written as $u_h(\boldsymbol{x}) = \sum_{i=1}^n u_{h;i} \Phi_i(\boldsymbol{x}) = \boldsymbol{u}_h \cdot \boldsymbol{\Phi}(\boldsymbol{x})$.

Here, for ease of notation, we have assumed that the eigenvalue approximations $\widetilde{\lambda}_m$, $m = 0, \ldots, \widetilde{M} - 1$, are all distinct. To select $\widetilde{M}$ distinct eigenvalue approximations, $\widetilde{M} \leq M$, we use the same procedure that is detailed in [1]. In a nutshell, we approximate the lower part of the spectrum using coarse finite element discretizations and multigrid preconditioned LOBPCG, and the upper part of the spectrum using Weyl's asymptotic law. Finally, a decimation procedure is applied. This works since Assumption 5.3 does not require the same level of accuracy for all eigenvalues. The larger the eigenvalue, the more freedom we have in approximating it. We refer to [1] for all the details.

By expanding the finite element functions as linear combinations with respect to the basis functions Φ_i, the solution $U_h(\boldsymbol{x}, y)$ of the extension problem ($\mathrm{wExt_h}$) can be written in the form

$$U_h(\boldsymbol{x}, y) = \sum_{m=0}^{\widetilde{M}-1} \sum_{i=1}^{n} c_{i,m} \Phi_i(\boldsymbol{x}) \widetilde{\psi}_m(y) \in \mathcal{V}_h$$

with the coefficients $(c_{i,m}) = \boldsymbol{U}_h$ obtained by solving the linear system

$$\left[M_{FE} \otimes \left(S_\sigma - k^{2s} B_\sigma\right) + S_{FE} \otimes M_\sigma\right] \boldsymbol{U}_h = \boldsymbol{F}_h, \tag{6.1}$$

where

$$\begin{aligned}
M_{FE} &= \left(\int_\Omega \Phi_i \Phi_j\right), & S_{FE} &= \left(\int_\Omega \nabla\Phi_i \nabla\Phi_j\right), \\
M_\sigma &= \left(\int_0^\infty \omega \widetilde{\psi}_m \widetilde{\psi}_n\right), & S_\sigma &= \left(\int_0^\infty \omega \widetilde{\psi}'_m \widetilde{\psi}'_n\right), \\
B_\sigma &= \mathbf{1}_{\widetilde{M}} \otimes \mathbf{1}_{\widetilde{M}}^T, & & \\
\boldsymbol{F}_h &= \boldsymbol{f}_h \otimes \mathbf{1}_{\widetilde{M}}, & \boldsymbol{f}_h &= \left(\langle f, \Phi_i \rangle\right).
\end{aligned}$$

Here, $\mathbf{1}_{\widetilde{M}}$ is the vector of ones of length $\widetilde{M}$. The approximation to the solution of the fractional Helmholtz problem is then obtained by taking the trace of U_h on Ω:

$$u_h = \operatorname{tr}_\Omega U_h = \sum_{i=1}^n \left(\sum_{m=0}^{\widetilde{M}-1} c_{i,m} \right) \Phi_i(\boldsymbol{x}), \tag{6.2}$$

where we recall the normalization $\widetilde{\psi}_m(0) = 1$. In matrix form, the trace operator is given by $I \otimes \mathbf{1}_{\widetilde{M}}^T \in \mathbb{R}^{n \times \mathcal{N}}$, so that $\boldsymbol{u}_h = \left[I \otimes \mathbf{1}_{\widetilde{M}}^T\right] \boldsymbol{U}_h$.

PROPOSITION 6.1. *There exist weights w_m and shift coefficients μ_m such that*

$$\boldsymbol{u}_h = \sum_{m=0}^{\widetilde{M}-1} w_m \left[M_{FE}\mu_m + S_{FE}\right]^{-1} \boldsymbol{f}_h. \tag{6.3}$$

When k^{2s} is real, all μ_m are real and at most one coefficient μ_m is negative.

Proof. We consider the following generalized eigenvalue problem:

$$\left(S_\sigma - k^{2s} B_\sigma\right) Q = M_\sigma Q \mu \tag{6.4}$$

with the normalization $Q^H M_\sigma Q = I$ and μ a diagonal matrix with entries μ_m. If k^{2s} is real, then all μ_m are real since S_σ, B_σ, and M_σ are real-valued and symmetric matrices. Then

$$\left(I \otimes Q^H\right) \left[M_{FE} \otimes \left(S_\sigma - k^{2s} B_\sigma\right) + S_{FE} \otimes M_\sigma\right] (I \otimes Q) = M_{FE} \otimes \mu + S_{FE} \otimes I.$$

Hence, we have

$$\left[M_{FE} \otimes \left(S_\sigma - k^{2s} B_\sigma\right) + S_{FE} \otimes M_\sigma\right]^{-1} = (I \otimes Q) \left[M_{FE} \otimes \mu + S_{FE} \otimes I\right]^{-1} \left(I \otimes Q^H\right).$$

Since $\boldsymbol{F}_h = \boldsymbol{f}_h \otimes \mathbf{1}_{\widetilde{M}}$ and $\boldsymbol{u}_h = \left[I \otimes \mathbf{1}_{\widetilde{M}}^T\right] \boldsymbol{U}_h$, we obtain

$$\boldsymbol{u}_h = \sum_{m=0}^{\widetilde{M}-1} \underbrace{\left(Q^H \mathbf{1}_{\widetilde{M}}\right)_m^2}_{=:w_m} \left(M_{FE}\mu_m + S_{FE}\right)^{-1} f_h.$$

Both the spectral mass matrix M_σ and the spectral stiffness matrix S_σ are real-valued, symmetric, and nonnegative, and so we know that the eigenvalues $\mu_m^{(0)}$ of the related problem

$$S_\sigma Q_0 = M_\sigma Q_0 \mu^0, \qquad Q_0^T M_\sigma Q_0 = I$$

are all real and nonnegative. Here, the entries of the diagonal matrix μ^0 are $\mu_m^{(0)}$. Without loss of generality, we assume that $0 \leq \mu_0^{(0)} \leq \mu_1^{(0)} \leq \cdots \leq \mu_{\widetilde{M}-1}^{(0)}$. The eigenvalues μ_m, in turn, satisfy the characteristic equation

$$\begin{aligned} 0 &= \det\left(S_\sigma - k^{2s} B_\sigma - \mu M_\sigma\right) \\ &= \det\left(Q_0\right)^{-2} \det\left(\mu^0 - k^{2s} \boldsymbol{z} \otimes \boldsymbol{z}^T - \mu I\right), \end{aligned}$$

where $\boldsymbol{z} = Q_0^T \mathbf{1}_{\widetilde{M}}$. Here, we have exploited the tensor product structure of B_σ. This means that we are interested in the impact on the spectrum of a rank one perturbation of a diagonal matrix. The eigenvalues of the rank one perturbation are $-k^{2s}\|z\|^2$ with multiplicity one and 0 with multiplicity $\widetilde{M}-1$. If k^{2s} is real, then $\mu^0 - k^{2s} \boldsymbol{z} \otimes \boldsymbol{z}^T$ is Hermitian and all μ_m are real. We assume without loss of generality that $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_{\widetilde{M}-1}$. Applying Weyl's theorem [13, Theorem 4.3.7], one can then show that

$$\mu_0^{(0)} - k^{2s}\|z\|^2 \leq \mu_0 \leq \mu_0^{(0)},$$

$$\mu_{m-1}^{(0)} \leq \mu_m \leq \mu_m^{(0)} \qquad \text{for } m \geq 1.$$

Since all $\mu_m^{(0)}$ are nonnegative, we can conclude that at most one eigenvalue μ_m is negative. ∎

The above proposition shows that we need to solve a sequence of systems with matrix of the form

$$M_{FE}\mu + S_{FE}, \quad \mu \in \mathbb{C}.$$

Depending on μ, we use different solver strategies.

- μ is real and nonnegative (this corresponds to a classical, integer-order reaction-diffusion problem):
  We employ a conjugate gradient solver with a standard geometric multigrid preconditioner.
- μ is real and negative (this corresponds to an integer-order Helmholtz problem):
  We use GMRES preconditioned by a geometric multigrid which has been constructed from the shifted system $S_{FE} + (1 + i\beta)\mu M_{FE}$ with $\beta = 0.5$.
- μ is complex, $\operatorname{Re}\mu$ is nonnegative:
  We use GMRES preconditioned by standard geometric multigrid.
- μ is complex, $\operatorname{Re}\mu$ is negative:
  Let $\mu =: -\nu(1+i\alpha)$ with $\nu \in \mathbb{R}_{\geq 0}$ and $\alpha \in \mathbb{R}$. We use GMRES preconditioned by a geometric multigrid which has been constructed from the shifted system $S_{FE} - \nu\,(1 + i\beta)\,M_{FE}$ with $\beta = 0.5$.

We note that this solution approach exposes a significant amount of parallelism. The solution of the $\widetilde{M}$ decoupled problems is embarrassingly parallel, and each of the integer-order problems can be performed in parallel. We also note that these solvers merely reuse existing solver technologies. In particular, this implies that any improvements that can be made for the (potentially costly) solution of the integer-order Helmholtz equation will benefit the solution of its fractional equivalent. This is why we can limit ourselves to a comparatively simple solution strategy. More advanced methods are available in the literature. It should be noted that the iterative solution of (integer-order) Helmholtz problems is notoriously difficult, particularly in the case of large wave numbers. It is therefore no surprise that the fractional-order case is faced with the same issues.

In Figure 6.1, we plot the distance between the shift coefficients and $-k^2$ for $\Omega = [0,1]^2$, $s \in \{0.6, 0.9\}$, and different wave numbers k. It can be observed that as $h \to 0$, the distance decays towards zero. While we do not explore this property any further from a theoretical standpoint, this shows that one of the subsystems recovers asymptotically the equivalent integer-order Helmholtz system.

6.1. Comparison with the integer-order case. When k^{2s} is real, only a single μ_m is negative according to Proposition 6.1. The above observation entails that the single integer-order Helmholtz problem that needs to be solved has wave number (very close to) k. This permits a comparison of the solution complexity of the classical integer-order Helmholtz problem to the fractional case. The fractional-order case differs in that we need to

- compute eigenvalue approximations $\widetilde{\lambda}_m$, $m = 0, \ldots, \widetilde{M}$,
- solve a generalized eigenvalue problem to obtain shifts μ_m and weights w_m, $m = 0, \ldots, \widetilde{M}$, and

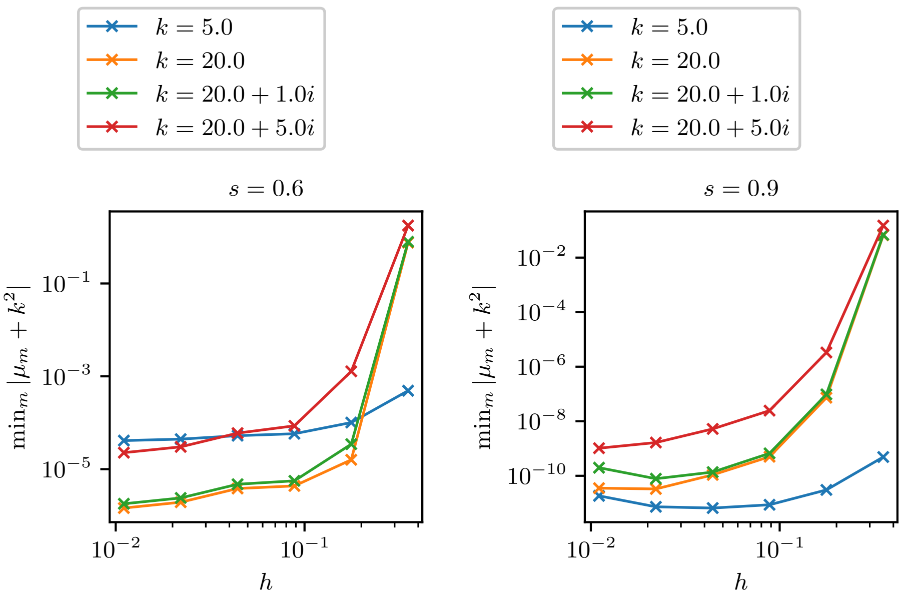

FIG. 6.1. *Convergence of the shift coefficient* μ *closest to* $-k^2$ *for* $\Omega = [0,1]^2$ *and several wave numbers* k.

- solve $\widetilde{M} - 1$ reaction-diffusion type systems.

The generalized eigenvalue problem (6.4) can be solved in $\mathcal{O}(\widetilde{M}^3)$ operations, as outlined in [1], and the weights w_m can be computed in $\mathcal{O}(\widetilde{M}^2)$ operations. Since finding the eigenvalue approximations is also an inexpensive operation (cf. [1]), the computations are entirely dominated by the linear solves. Solving an integer-order Helmholtz problem can be significantly more costly than solving reaction-diffusion problems, especially when the wave number k is large. Therefore, we expect that the overall cost of solving the fractional Helmholtz problem is comparable to the classical integer-order case.

6.2. Solving sequences of problems with variable fractional order. If the eigenvalue approximations are chosen such that they satisfy Assumption 5.3 for a range of fractional orders $[s_{\min}, s_{\max}] \subset (0,1)$, the resulting solver can be used to solve fractional Helmholtz problems of order $s_{\min} \leq s \leq s_{\max}$ without rediscretization. This is quite beneficial since the exact value of the fractional exponent s is generally unknown and needs to be determined through repeated linear solves with varying s in the framework of an inverse problem. See, for instance, [23, 2, 5], where the exponent s is spatially dependent. We do not explore the property further in the context of the present work.

7. Numerical examples. Let $\Omega = [0,1]^d$. We solve the fractional-order Helmholtz equation

$$\begin{cases} (-\Delta)^s u(\boldsymbol{x}) - k^{2s} u &= f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) &= 0, \qquad \boldsymbol{x} \in \partial\Omega. \end{cases}$$

In order to evaluate the error convergence rates, we consider the manufactured analytic solution $u = C \prod_{i=1}^d [x_i(1-x_i)]^{r+2s-1/2}$ with a given right-hand side regularity of index r and C chosen to normalize u. We obtain an approximation of the corresponding right-hand side function f via the discrete sine transform. We observe that

$$f \in \widetilde{H}^r(\Omega), \qquad u \in \begin{cases} C^\infty\left(\overline{\Omega}\right) & \text{for } r + 2s \in \mathbb{N} + 3/2, \\ \widetilde{H}^{r+2s}(\Omega) & \text{else.} \end{cases}$$

We have to resort to this approach since we are not aware of any nontrivial analytic pairs of right-hand side and solution for the fractional Helmholtz problem that reflect the regularity properties of the equation. We also note that prescribing the solution, and finding an approximation to the right-hand side function f, instead of the other way around, permits us to compute the L^2-error as follows:

$$\|u - u_h\|^2_{L^2(\Omega)} = \|u\|^2_{L^2(\Omega)} - 2\operatorname{Re}(u, u_h)_{L^2(\Omega)} + \|u_h\|^2_{L^2(\Omega)}.$$

Here, the first term can be evaluated analytically and the second and third term can be evaluated using quadrature of sufficiently high order. The $\mathcal{H}^1_\omega$-error is given by

$$\|U - U_h\|^2_{\mathcal{H}^1_\omega} = (f, u)_{L^2(\Omega)} - 2\operatorname{Re}(f, u_h)_{L^2(\Omega)} + (f, u_h)_{L^2(\Omega)} + k^{2s}\|u - u_h\|^2_{L^2(\Omega)} + 2i\operatorname{Im}(k^{2s})(u_h, u)_{L^2(\Omega)}.$$

Here, we have used the variational formulation given in (wExt). Since the expansion coefficients u_m and the exact eigenvalues λ_m are known, we can use the expansion

$$(f, u)_{L^2(\Omega)} = \sum_{m=0}^{\infty} \left(\lambda_m^s - k^{2s}\right)^{-1} |u|^2_m$$

to approximate the true value by truncating the sum to a sufficient number of terms, as long as we make sure that the truncation error is dominated by the discretization error. As stated in Theorem 5.7 the $\widetilde{H}^s$-error is bounded by the $\mathcal{H}^1_\omega$-error.

It is important to note that the fact that the domains Ω are hypercubes is exploited only to obtain approximations for pairs of solutions and right-hand sides in order to compute error norms. The discretization of the problem as well as the solver are entirely oblivious to this fact and do not take advantage of it.

In what follows, we solve the above problem for the d-hypercube, $d = 2$, fractional order $s \in \{0.6, \dots, 0.9\}$, and

I. a low regularity case ($r = 1/2$), using piecewise linear ($p = 1$) elements and real-valued wave number $k \in \{5, 20\}$,

II. a low regularity case ($r = 1/2$), using piecewise quadratic ($p = 2$) elements and real-valued wave number $k \in \{5, 20\}$,

III. a high regularity case ($r = 2$), using piecewise quadratic ($p = 2$) elements and real-valued wave number $k \in \{5, 20\}$, and

IV. a low regularity case ($r = 1/2$), using piecewise linear ($p = 1$) elements and complex-valued wave number $k = 20 + 5i$.

In all settings, we use quasi-uniform meshes.

In Tables 7.1 and 7.2, we display the solution errors measured in the $\mathcal{H}^1_\omega$- and the $L^2(\Omega)$-norm for the first two test cases, I and II. In Tables 7.3 and 7.4, the convergence results of the latter two test cases, III and IV, are shown. As predicted by Theorem 5.7, order $h^{\min\{p, r+s\}}$ convergence is observed in the $\mathcal{H}^1_\omega$-norm. For the L^2-error, we often observe convergence of order $h^{\min\{p+1, r+2s\}}$, which is better than the theoretical rate of $h^{\min\{p+s, r+2s\}}$ predicted by Theorem 5.7.

TABLE 7.1
L^2-errors and $\mathcal{H}^1_\omega$-errors for the solution of the fractional Helmholtz problem on the unit square for test case I*: wave number $k = 5$ (*left*) and $k = 20$ (*right*), fractional orders $s \in \{0.6, 0.7, 0.8, 0.9\}$, and piecewise linear ($p = 1$) finite elements for a right-hand side $f \in \widetilde{H}^r(\Omega)$, $r = 1/2$.*

	$s = 0.6,\ k = 5.0$				$s = 0.6,\ k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.177	8.99e-02		3.12e-01		3.36e-02		2.78e-01	
0.0884	2.00e-02	2.17	1.11e-01	1.49	6.99e-03	2.27	1.04e-01	1.41
0.0442	4.91e-03	2.03	5.07e-02	1.13	7.29e-03	-0.06	6.53e-02	0.67
0.0221	1.23e-03	2.00	2.49e-02	1.03	1.25e-02	-0.78	7.93e-02	-0.28
0.011	3.12e-04	1.98	1.24e-02	1.00	1.07e-03	3.54	1.40e-02	2.50
0.00552	7.95e-05	1.97	6.24e-03	0.99	2.31e-04	2.21	6.39e-03	1.13
0.00276	2.05e-05	1.96	3.13e-03	1.00	5.61e-05	2.04	3.15e-03	1.02
0.00138	5.34e-06	1.94	1.57e-03	1.00	1.40e-05	2.00	1.57e-03	1.00
Theoretical		1.60		1.00		1.60		1.00
	$s = 0.7,\ k = 5.0$				$s = 0.7,\ k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.177	8.39e-02		3.67e-01		4.68e-02		4.57e-01	
0.0884	1.89e-02	2.15	1.41e-01	1.38	1.88e-02	1.31	1.97e-01	1.21
0.0442	4.65e-03	2.02	6.62e-02	1.09	5.77e-03	1.70	7.92e-02	1.32
0.0221	1.17e-03	2.00	3.28e-02	1.01	1.00e-02	-0.80	8.79e-02	-0.15
0.011	2.94e-04	1.99	1.64e-02	1.00	8.76e-04	3.52	1.79e-02	2.30
0.00552	7.42e-05	1.99	8.25e-03	0.99	1.90e-04	2.21	8.39e-03	1.09
0.00276	1.88e-05	1.98	4.13e-03	1.00	4.61e-05	2.04	4.15e-03	1.02
0.00138	4.76e-06	1.98	2.06e-03	1.00	1.14e-05	2.01	2.07e-03	1.01
Theoretical		1.70		1.00		1.70		1.00
	$s = 0.8,\ k = 5.0$				$s = 0.8,\ k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.177	7.89e-02		4.42e-01		5.33e-02		6.79e-01	
0.0884	1.79e-02	2.14	1.81e-01	1.29	4.06e-02	0.39	4.76e-01	0.51
0.0442	4.41e-03	2.02	8.68e-02	1.06	4.28e-03	3.24	9.66e-02	2.30
0.0221	1.10e-03	2.00	4.31e-02	1.01	7.37e-03	-0.78	9.17e-02	0.07
0.011	2.76e-04	2.00	2.16e-02	1.00	6.62e-04	3.48	2.27e-02	2.01
0.00552	6.91e-05	2.00	1.08e-02	1.00	1.44e-04	2.20	1.09e-02	1.06
0.00276	1.73e-05	2.00	5.40e-03	1.00	3.50e-05	2.04	5.41e-03	1.01
0.00138	4.34e-06	2.00	2.70e-03	1.00	8.70e-06	2.01	2.70e-03	1.00
Theoretical		1.80		1.00		1.80		1.00
	$s = 0.9,\ k = 5.0$				$s = 0.9,\ k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.177	7.46e-02		5.41e-01		5.44e-02		9.32e-01	
0.0884	1.71e-02	2.12	2.33e-01	1.21	6.07e-02	-0.16	9.28e-01	0.01
0.0442	4.21e-03	2.02	1.13e-01	1.04	3.00e-03	4.34	1.20e-01	2.96
0.0221	1.05e-03	2.00	5.62e-02	1.01	4.79e-03	-0.67	9.05e-02	0.40
0.011	2.62e-04	2.00	2.81e-02	1.00	4.52e-04	3.41	2.88e-02	1.65
0.00552	6.56e-05	2.00	1.40e-02	1.00	9.99e-05	2.18	1.41e-02	1.03
0.00276	1.64e-05	2.00	7.02e-03	1.00	2.43e-05	2.04	7.03e-03	1.01
0.00138	4.09e-06	2.00	3.51e-03	1.00	6.04e-06	2.01	3.51e-03	1.00
Theoretical		1.90		1.00		1.90		1.00

All computations are performed on a dual socket Intel Xeon E5-2650V3, 2.30GHz, 20 core workstation. In Figures 7.1 and 7.2 we display timings for the solution of the linear problems of test cases I–IV. We display both the total solve time as well as the cumulative time for all reaction-diffusion type solves. We observe that, as expected, the integer-order Helmholtz solve dominates the overall solution time for larger wave number k. This is essentially due to an increase in the number of iterations. For example, solving the integer-order Helmholtz problem for test case I on the finest mesh takes 13 iterations for $k = 5$, but 41 for $k = 20$. The reaction-diffusion type

Table 7.2

L^2-errors and $\mathcal{H}^1_\omega$-errors for the solution of the fractional Helmholtz problem on the unit square for test case II*: wave number $k = 5$ (*left*) and $k = 20$ (*right*), fractional orders $s \in \{0.6, 0.7, 0.8, 0.9\}$, and piecewise quadratic ($p = 2$) finite elements for a right-hand side $f \in \widetilde{H}^r(\Omega)$, $r = 1/2$.*

	$s = 0.6$, $k = 5.0$				$s = 0.6$, $k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	1.07e-02		7.63e-02		1.02e-02		8.53e-02	
0.177	2.00e-03	2.41	2.53e-02	1.59	3.89e-03	1.40	3.23e-02	1.40
0.0884	5.42e-04	1.89	6.79e-03	1.90	1.15e-03	1.76	9.13e-03	1.82
0.0442	1.59e-04	1.77	4.48e-03	0.60	1.67e-04	2.78	4.49e-03	1.02
0.0221	4.80e-05	1.73	2.08e-03	1.11	4.81e-05	1.79	2.08e-03	1.11
0.011	1.46e-05	1.71	9.61e-04	1.11	1.46e-05	1.72	9.61e-04	1.11
Theoretical		1.70		1.10		1.70		1.10
	$s = 0.7$, $k = 5.0$				$s = 0.7$, $k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	1.31e-02		1.22e-01		1.50e-02		1.51e-01	
0.177	2.37e-03	2.47	4.19e-02	1.54	4.68e-03	1.68	5.32e-02	1.51
0.0884	5.50e-04	2.10	1.12e-02	1.90	1.36e-03	1.78	1.51e-02	1.82
0.0442	1.36e-04	2.02	6.69e-03	0.74	1.49e-04	3.19	6.71e-03	1.17
0.0221	3.51e-05	1.95	2.90e-03	1.21	3.54e-05	2.07	2.90e-03	1.21
0.011	9.26e-06	1.92	1.25e-03	1.22	9.26e-06	1.93	1.25e-03	1.22
Theoretical		1.90		1.20		1.90		1.20
	$s = 0.8$, $k = 5.0$				$s = 0.8$, $k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	1.43e-02		1.75e-01		1.85e-02		2.39e-01	
0.177	2.19e-03	2.71	5.62e-02	1.64	4.29e-03	2.11	6.93e-02	1.79
0.0884	4.23e-04	2.38	1.15e-02	2.29	1.24e-03	1.79	1.73e-02	2.00
0.0442	8.35e-05	2.34	7.29e-03	0.66	1.01e-04	3.62	7.32e-03	1.24
0.0221	1.81e-05	2.21	2.92e-03	1.32	1.84e-05	2.46	2.92e-03	1.33
0.011	4.06e-06	2.15	1.16e-03	1.33	4.07e-06	2.18	1.16e-03	1.33
Theoretical		2.10		1.30		2.10		1.30
	$s = 0.9$, $k = 5.0$				$s = 0.9$, $k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	1.48e-02		2.36e-01		2.14e-02		3.60e-01	
0.177	1.88e-03	2.98	6.85e-02	1.78	3.34e-03	2.68	8.02e-02	2.16
0.0884	3.02e-04	2.63	4.94e-03	3.79	9.65e-04	1.79	1.46e-02	2.45
0.0442	4.29e-05	2.82	6.69e-03	-0.44	6.23e-05	3.95	6.72e-03	1.12
0.0221	7.25e-06	2.57	2.41e-03	1.47	7.78e-06	3.00	2.41e-03	1.48
0.011	1.31e-06	2.47	8.76e-04	1.46	1.32e-06	2.56	8.62e-04	1.48
Theoretical		2.30		1.40		2.30		1.40

subproblems do not display such behavior. This shows that for high wave number k, solution of the fractional problem and its integer-order equivalent have very comparable cost. We also observe that for fixed k, the solution time scales roughly linearly with the number of degrees of freedom $n = \dim S_h$ of the finite element space.

In a final example, we solve the fractional Helmholtz problem on the Fichera cube $\Omega = [0, 2]^3 \setminus [1, 2]^3$ for wave number $k = 5$ and $f = 1$. Since no analytic solution is known, errors are computed with respect to a solution on a fine mesh (mesh size $h \approx 0.0135$, $n \approx 14.5 \times 10^6$ unknowns). While we did not use available closed-form expressions for the eigenvalue before, this example further illustrates that such expressions are not needed. In Table 7.5, we display the solution errors measured in the $\mathcal{H}^1_\omega$- and the $L^2(\Omega)$-norm. Again, we observe that the L^2-error converges faster than predicted by Theorem 5.7, but that order $h^{\min\{p,r+s\}}$ convergence is observed in the $\mathcal{H}^1_\omega$-norm. The apparent speed-up of convergence for finer meshes is due to the fact that we are using the solution on a fine mesh to compute errors.

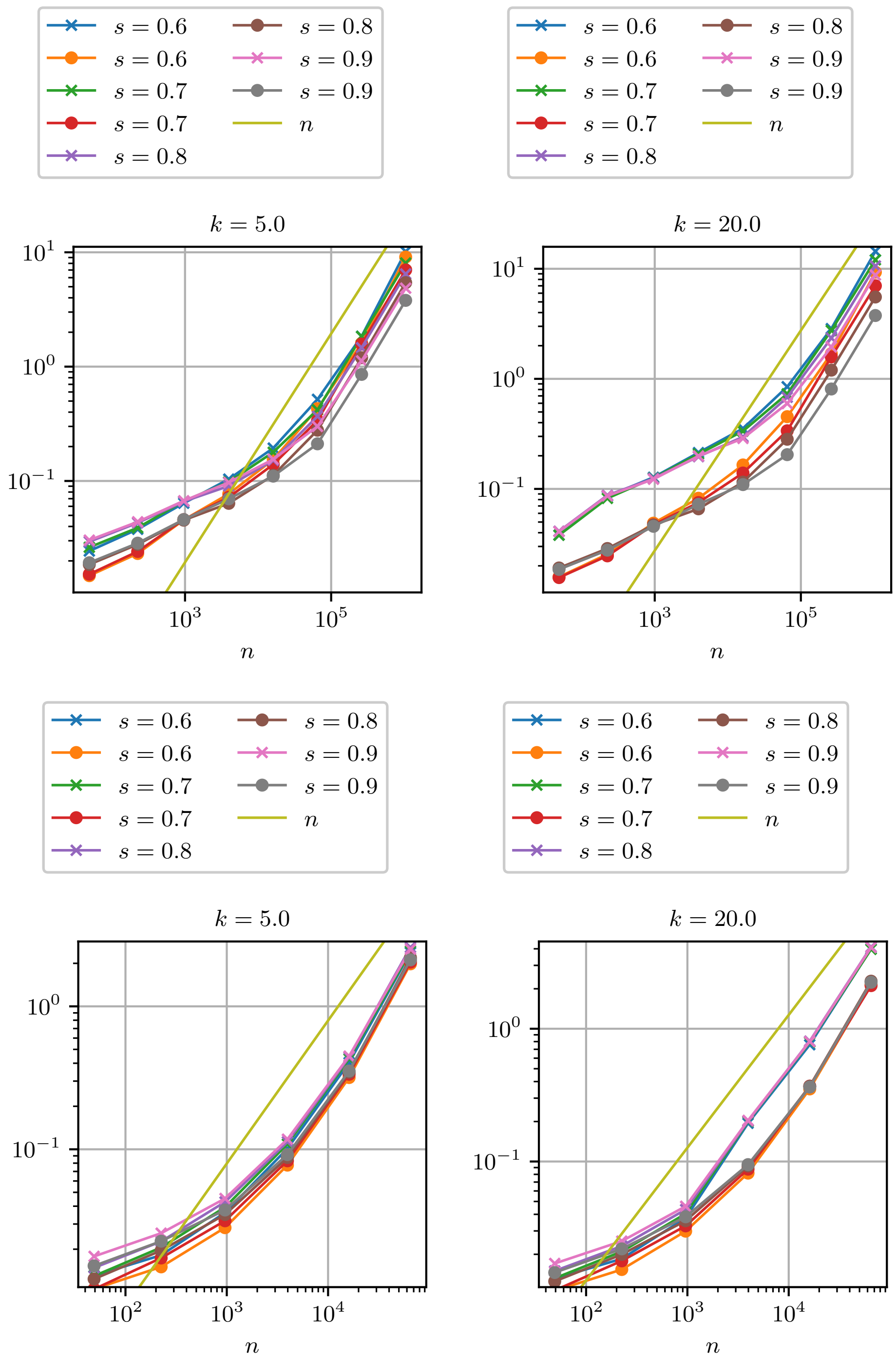

FIG. 7.1. *Solution times for the fractional Helmholtz problem on the unit square for test cases* I *(*top*) and* II *(*bottom*). Also, the total time (×) and time for all reaction-diffusion type subproblems (•).*

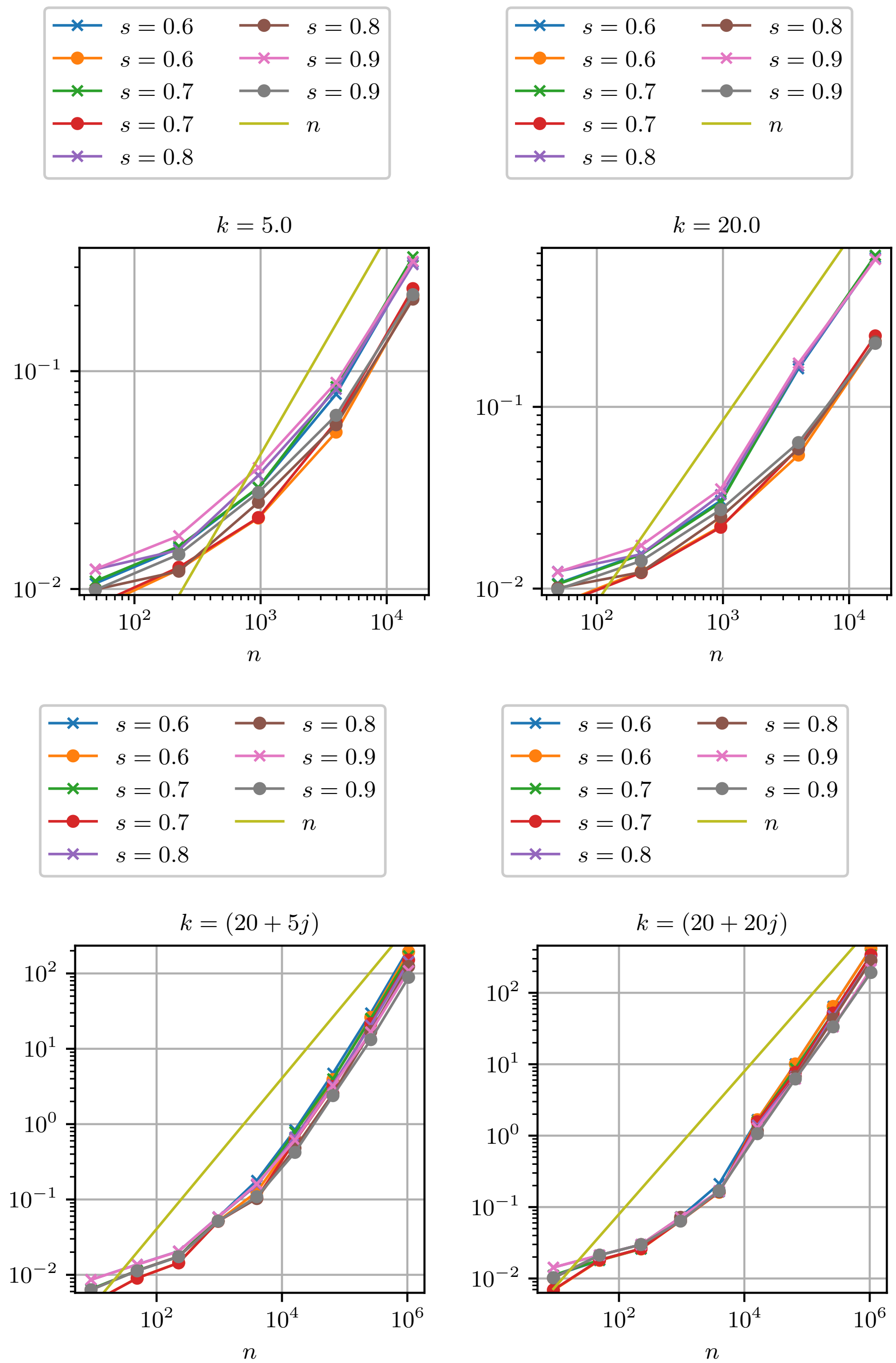

FIG. 7.2. *Solution times for the fractional Helmholtz problem on the unit square for test cases* III (top) *and* IV (bottom). *Also, the total time* ($\times$) *and time for all reaction-diffusion type subproblems* ($\bullet$).

Table 7.3

L^2-errors and $\mathcal{H}^1_\omega$-errors for the solution of the fractional Helmholtz problem on the unit square for test case III: *wave number $k = 5$ (*left*) and $k = 20$ (*right*), fractional orders $s \in \{0.6, 0.7, 0.8, 0.9\}$, and piecewise quadratic ($p = 2$) finite elements for a right-hand side $f \in \widetilde{H}^r(\Omega)$, $r = 2$.*

	$s = 0.6,\ k = 5.0$				$s = 0.6,\ k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	1.17e-02		9.35e-02		3.26e-02		2.12e-01	
0.177	1.51e-03	2.96	2.69e-02	1.80	4.30e-03	2.93	3.64e-02	2.54
0.0884	2.04e-04	2.89	7.19e-03	1.90	1.07e-03	2.01	9.60e-03	1.92
0.0442	2.69e-05	2.92	1.84e-03	1.96	5.74e-05	4.22	1.87e-03	2.36
0.0221	3.49e-06	2.95	4.70e-04	1.97	4.71e-06	3.61	4.68e-04	2.00
Theoretical		2.60		2.00		2.60		2.00
	$s = 0.7,\ k = 5.0$				$s = 0.7,\ k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	1.22e-02		1.28e-01		3.31e-02		2.92e-01	
0.177	1.62e-03	2.91	3.77e-02	1.76	4.57e-03	2.86	5.16e-02	2.50
0.0884	2.12e-04	2.93	9.99e-03	1.92	1.17e-03	1.97	1.37e-02	1.91
0.0442	2.72e-05	2.97	2.54e-03	1.98	6.19e-05	4.24	2.58e-03	2.41
0.0221	3.43e-06	2.99	6.44e-04	1.98	4.88e-06	3.66	6.55e-04	1.98
Theoretical		2.70		2.00		2.70		2.00
	$s = 0.8,\ k = 5.0$				$s = 0.8,\ k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	1.29e-02		1.76e-01		3.29e-02		3.96e-01	
0.177	1.73e-03	2.90	5.26e-02	1.75	4.65e-03	2.82	7.13e-02	2.47
0.0884	2.19e-04	2.98	1.38e-02	1.93	1.22e-03	1.94	1.91e-02	1.90
0.0442	2.74e-05	3.00	3.48e-03	1.98	6.43e-05	4.24	3.54e-03	2.43
0.0221	3.42e-06	3.00	8.82e-04	1.98	5.00e-06	3.69	8.82e-04	2.00
Theoretical		2.80		2.00		2.80		2.00
	$s = 0.9,\ k = 5.0$				$s = 0.9,\ k = 20.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	1.39e-02		2.44e-01		3.23e-02		5.35e-01	
0.177	1.83e-03	2.92	7.28e-02	1.74	4.57e-03	2.82	9.66e-02	2.47
0.0884	2.25e-04	3.02	1.89e-02	1.95	1.22e-03	1.91	2.60e-02	1.89
0.0442	2.79e-05	3.01	4.77e-03	1.99	6.50e-05	4.23	4.85e-03	2.43
0.0221	3.48e-06	3.00	1.20e-03	1.98	5.06e-06	3.68	1.21e-03	2.01
Theoretical		2.90		2.00		2.90		2.00

TABLE 7.4

L^2-errors and $\mathcal{H}^1_\omega$-errors for the solution of the fractional Helmholtz problem on the unit square for test case IV*: wave number $k = 20 + 5i$ (*left*) and $k = 20 + 20i$ (*right*), fractional orders $s \in \{0.6, 0.7, 0.8, 0.9\}$, and piecewise linear ($p = 1$) finite elements for a right-hand side $f \in \widetilde{H}^r(\Omega)$, $r = 1/2$.*

	$s=0.6,\ k=20+5i$				$s=0.6,\ k=20+20i$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	3.80e-02		3.94e-01		3.50e-02		2.75e-01	
0.177	1.02e-02	1.90	1.90e-01	1.05	8.86e-03	1.98	1.37e-01	1.01
0.0884	3.33e-03	1.61	9.42e-02	1.01	2.70e-03	1.72	6.84e-02	1.00
0.0442	1.02e-03	1.71	4.72e-02	1.00	8.69e-04	1.63	3.44e-02	0.99
0.0221	3.09e-04	1.72	2.38e-02	0.99	2.77e-04	1.65	1.73e-02	0.99
0.011	9.36e-05	1.72	1.20e-02	0.99	8.70e-05	1.67	8.71e-03	0.99
0.00552	2.84e-05	1.72	6.01e-03	0.99	2.71e-05	1.68	4.39e-03	0.99
0.00276	8.66e-06	1.72	3.02e-03	0.99	8.38e-06	1.69	2.21e-03	0.99
0.00138	2.64e-06	1.71	1.51e-03	1.00	2.59e-06	1.70	1.11e-03	0.99
Theoretical		1.60		1.00		1.60		1.00
	$s=0.7,\ k=20+5i$				$s=0.7,\ k=20+20i$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	4.25e-02		4.94e-01		3.97e-02		3.04e-01	
0.177	1.26e-02	1.75	2.43e-01	1.03	1.14e-02	1.80	1.55e-01	0.97
0.0884	4.29e-03	1.56	1.22e-01	1.00	3.42e-03	1.74	7.68e-02	1.01
0.0442	1.24e-03	1.79	6.09e-02	1.00	1.03e-03	1.74	3.87e-02	0.99
0.0221	3.48e-04	1.83	3.07e-02	0.99	2.99e-04	1.78	1.96e-02	0.98
0.011	9.61e-05	1.86	1.54e-02	0.99	8.50e-05	1.81	9.92e-03	0.98
0.00552	2.64e-05	1.87	7.75e-03	0.99	2.38e-05	1.83	5.01e-03	0.99
0.00276	7.18e-06	1.88	3.89e-03	1.00	6.60e-06	1.85	2.52e-03	0.99
0.00138	1.95e-06	1.88	1.95e-03	1.00	1.81e-06	1.86	1.26e-03	1.00
Theoretical		1.70		1.00		1.70		1.00
	$s=0.8,\ k=20+5i$				$s=0.8,\ k=20+20i$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	4.95e-02		6.45e-01		4.69e-02		3.57e-01	
0.177	1.41e-02	1.81	3.17e-01	1.03	1.28e-02	1.87	1.77e-01	1.02
0.0884	4.51e-03	1.65	1.57e-01	1.01	3.50e-03	1.88	8.49e-02	1.06
0.0442	1.23e-03	1.88	7.81e-02	1.01	9.59e-04	1.87	4.27e-02	0.99
0.0221	3.22e-04	1.93	3.91e-02	1.00	2.58e-04	1.89	2.16e-02	0.98
0.011	8.36e-05	1.95	1.96e-02	1.00	6.81e-05	1.92	1.09e-02	0.98
0.00552	2.15e-05	1.96	9.83e-03	1.00	1.78e-05	1.94	5.51e-03	0.99
0.00276	5.50e-06	1.97	4.92e-03	1.00	4.58e-06	1.96	2.76e-03	1.00
0.00138	1.40e-06	1.98	2.46e-03	1.00	1.17e-06	1.97	1.38e-03	1.00
Theoretical		1.80		1.00		1.80		1.00
	$s=0.9,\ k=20+5i$				$s=0.9,\ k=20+20i$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.354	5.56e-02		8.52e-01		5.31e-02		4.70e-01	
0.177	1.46e-02	1.93	4.14e-01	1.04	1.32e-02	2.01	2.12e-01	1.15
0.0884	4.33e-03	1.75	2.02e-01	1.03	3.28e-03	2.01	9.90e-02	1.10
0.0442	1.12e-03	1.95	9.94e-02	1.02	8.39e-04	1.97	4.94e-02	1.00
0.0221	2.85e-04	1.98	4.95e-02	1.00	2.14e-04	1.97	2.48e-02	0.99
0.011	7.19e-05	1.99	2.48e-02	1.00	5.42e-05	1.98	1.25e-02	0.99
0.00552	1.81e-05	1.99	1.24e-02	1.00	1.37e-05	1.99	6.26e-03	0.99
0.00276	4.53e-06	2.00	6.20e-03	1.00	3.43e-06	1.99	3.13e-03	1.00
0.00138	1.13e-06	2.00	3.10e-03	1.00	8.58e-07	2.00	1.56e-03	1.00
Theoretical		1.90		1.00		1.90		1.00

TABLE 7.5

L^2-errors and $\mathcal{H}^1_\omega$-errors for the solution of the fractional Helmholtz problem on the Fichera cube: fractional orders $s \in \{0.6, 0.7, 0.8, 0.9\}$ and piecewise linear ($p = 1$) finite elements for a right-hand side $f \in \widetilde{H}^r(\Omega)$, $r = 1/2$.

	$s = 0.6$, $k = 5.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.433	1.11e+01		2.93e+01	
0.217	8.21e-01	3.76	2.27e+00	3.69
0.108	1.13e-01	2.86	4.91e-01	2.21
0.0541	3.21e-02	1.82	2.25e-01	1.13
0.0271	7.66e-03	2.07	1.04e-01	1.12
Theoretical		1.60		1.00
	$s = 0.7$, $k = 5.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.433	6.90e+00		2.14e+01	
0.217	5.10e-01	3.76	1.67e+00	3.68
0.108	7.04e-02	2.86	3.76e-01	2.15
0.0541	1.99e-02	1.82	1.75e-01	1.10
0.0271	4.75e-03	2.07	8.13e-02	1.11
Theoretical		1.70		1.00
	$s = 0.8$, $k = 5.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.433	4.37e+00		1.59e+01	
0.217	3.23e-01	3.76	1.25e+00	3.67
0.108	4.46e-02	2.86	2.92e-01	2.10
0.0541	1.26e-02	1.82	1.38e-01	1.08
0.0271	3.01e-03	2.07	6.45e-02	1.10
Theoretical		1.80		1.00
	$s = 0.9$, $k = 5.0$			
h	L^2	RoC	$\mathcal{H}^1_\omega$	RoC
0.433	2.82e+00		1.21e+01	
0.217	2.08e-01	3.76	9.56e-01	3.66
0.108	2.87e-02	2.86	2.30e-01	2.06
0.0541	8.14e-03	1.82	1.10e-01	1.07
0.0271	1.95e-03	2.06	5.17e-02	1.09
Theoretical		1.90		1.00

8. Conclusion. In this work, we have presented a fractional-order Helmholtz problem. We have discussed well-posedness and convergence of a hybrid spectral-finite element discretization. An efficient solver has been proposed that scales as well as the best possible solver for the integer-order Helmholtz equation, making the more appropriate fractional equation a preferable alternative for geophysical electromagnetics modeling. Numerical examples have been used to illustrate the obtained theoretical results.

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