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Tverberg-Type Theorems with Altered Intersection Patterns (Nerves)

Jesús A. De Loera[1] · Thomas A. Hogan[1] · Deborah Oliveros[2] · Dominic Yang[1]

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Abstract

Tverberg's theorem says that a set with sufficiently many points in $\mathbb{R}^d$ can always be partitioned into m parts so that the $(m-1)$-simplex is the (nerve) intersection pattern of the convex hulls of the parts. The main results of our paper demonstrate that Tverberg's theorem is just a special case of a more general situation, where other simplicial complexes must always arise as nerve complexes, as soon as the number of points is large enough. We prove that, given a set with sufficiently many points, all trees and all cycles can also be induced by at least one partition of the point set. We also discuss how some simplicial complexes can never be achieved this way, even for arbitrarily large sets of points.

Keywords Tverberg's theorem · Nerve complexes · Geometric Ramsey theory · Combinatorial convexity · Intersection graphs · Clustering

Mathematics Subject Classification 52A35 · 52A01 · 52C40 · 05C55

Editor in Charge: János Pach

Jesús A. De Loera
deloera@math.ucdavis.edu

Thomas A. Hogan
tahogan@math.ucdavis.edu

Deborah Oliveros
dolivero@matem.unam.mx

Dominic Yang
dyang@ucdavis.edu

1 Department of Mathematics, University of California, Davis, USA

2 Instituto de Matemáticas, Universidad Nacional Autónoma de México, Mexico City, Mexico

1 Introduction

The celebrated theorem of Helge Tverberg states (see [2] and references therein):

Tverberg's Theorem ([27]) *Every set S with at least* $(d+1)(m-1)+1$ *points in Euclidean d-space* $\mathbb{R}^d$ *can be partitioned into m parts* $\mathcal{P} = S_1, \dots, S_m$ *such that all the convex hulls of these parts have nonempty intersection. The special case of a bi-partition,* $m = 2$*, is called* ***Radon's lemma****.*

The nerve (intersection pattern) of the convex hulls in Tverberg's theorem is very specific, a simplex; our paper investigates other possible nerves. Informally, the main results of our paper demonstrate that, given sufficiently many points, other kinds of nerves can always be induced by a suitable partition of the point set. In particular, we show that any tree or cycle can be induced as the nerve.

Our geometric results are naturally motivated from two independent research directions. First of all, there is Ramsey theory (see [11]) where one studies how every sufficiently large system must contain a large well-organized subsystem. Here "sufficiently large" is governed by Ramsey numbers. In geometry a classical example of a Ramsey-type theorem is Erdős–Szekeres' theorem saying that every sufficiently large point set in the plane must contain a sub-configuration forming a convex k-gon (in this case the constant is hard to compute; see [24] and references there). We stress that Tverberg's theorem is also Ramsey-type, although in this case the constant is explicit and easy. Our paper proves new Ramsey–Tverberg-type results, where nerve structures are shown to arise once we have sufficiently many points. Our results are also a kind of universality result, in the spirit of Pór [19]. We will see our results depend on some universal Ramsey-like constants too and we use Ramsey numbers of hypergraphs for our geometric estimates.

A second motivation comes from clustering and data classification [4,7]. Clustering algorithms aim to "color" or "label" data points by groups that share common characteristics (see [10,13] and references there). Classification is then a partitioning of the data set. Two sets of points will intersect if they share members with both characteristics. When doing a classification researchers face the question, is the proposed partition of data showing intrinsic data properties, unique to the particular input data points, or is this a mere artifact appearing in *all* data sets after having a sufficiently large data set? Our Tverberg-type theorems with altered nerves will be relevant to data science when analyzing the statistical significance of a proposed classification in large-scale data sets.

To state our results precisely we begin with some terminology and notation typical of geometric topological combinatorics (see [15,26] for details, especially on simplicial complexes discussed here). Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of convex sets in $\mathbb{R}^d$. The *nerve* $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ is the simplicial complex with vertex set $[m] := \{1, 2, \dots, m\}$ whose faces are $I \subset [m]$ such that $\bigcap_{i \in I} F_i \neq \emptyset$.

Given a collection of points $S \subset \mathbb{R}^d$ and an n-partition into n color classes $\mathcal{P} = S_1, \dots, S_n$ of S, we define *the nerve of the partition*, $\mathcal{N}(\mathcal{P})$, to be the nerve complex $\mathcal{N}(\{\mathrm{conv}(S_1), \dots, \mathrm{conv}(S_n)\})$, where $\mathrm{conv}(S_i)$ is the convex hull of the elements in the color class i. Similarly, given a partition $\mathcal{P}$, we define the *intersection graph of the partition*, denoted $\mathcal{N}^1(\mathcal{P})$, as the 1-skeleton of the nerve of $\mathcal{P}$.

Given a simplicial complex K, and a finite set of points S in $\mathbb{R}^d$, we say that K is *partition induced on* S if there exists a partition $\mathcal{P}$ of S such that the nerve of the partition is isomorphic to K. We say that K is *d-partition induced* if there exists at least one set of points $S \subset \mathbb{R}^d$ such that K is partition induced on S. These types of complexes are special cases of the *convex set representable complexes*, i.e., those complexes which are nerves of families of convex sets (see [25] for details). It was shown by Perelman [17] that every d-dimensional simplicial complex is $(2d+1)$-partition induced on some point set. This result is in fact optimal, because the barycentric subdivision of the d-skeleton of a $(2d+2)$-dimensional simplex is not $2d$-partition induced, see [28] and [25] for details.

Motivated by Tverberg's theorem, we introduce another property of simplicial complexes which is much stronger than being d-partition induced because it has to hold, not in one, but in *all* point sets with sufficiently many points.

Definition 1.1 A simplicial complex K is *d-Tverberg* if there exists a constant $\mathrm{Tv}(K,d)$ such that K is partition induced on all point sets $S \subset \mathbb{R}^d$ in general position with $|S| > \mathrm{Tv}(K,d)$. The minimal such constant $\mathrm{Tv}(K,d)$ is called the *Tverberg number for K in dimension d*.

Let us briefly examine the definition of d-Tverberg complexes. First of all, note one can re-state the classical Tverberg's theorem as follows:

Tverberg's Theorem rephrased *The* $(m-1)$*-simplex is a d-Tverberg complex for all* $d \geq 1$*, with Tverberg number* $(d+1)(m-1)+1$.

Definition 1.1 can be compared with earlier work by Reay and others [18,20,22], who asked what happens when we only demand that each k of the convex hulls intersect. They looked for the smallest number n of points sufficient so that some partition induces a nerve which contains the $(k-1)$-skeleton of a simplex. In fact, Reay's conjecture says, for every $n \leq (d+1)(m-1)$ there exists an n point set $X \subset \mathbb{R}^d$ such that no partition of X induces the complete graph K_m as its intersection graph. In contrast, we are looking for an exact nerve of general kind.

Definition 1.1 is most interesting for sets $S \subset \mathbb{R}^d$ in general position. The reason is that for collinear points the only type of nerve complexes possible are those whose graphs are *interval graphs*. Interval graphs have been classified [14] and in particular are *chordal*. With Definition 1.1 the 4-cycle graph is not 1-Tverberg, because it is not chordal, but we will show later that it is d-Tverberg for all $d \geq 2$. Similarly, while every d-Tverberg complex K is clearly d-partition induced, the converse is not true. The complex in Fig. 1 is a graph that is partition induced on some planar point sets, but not for points in convex position, regardless of how many points we use. Thus it is not a 2-Tverberg complex. Details are presented in Appendix A.

The key contribution of our paper is generalization of the classical Tverberg theorem, with other simplicial complexes—not just simplices—being d-Tverberg complexes.

Before stating our first result, recall that the complete k-hypergraph with s vertices is the hypergraph whose hyperedges are all the k-subsets of $\{1,2,\ldots,s\}$. When $k=2$ this is exactly the complete graph K_s. The *k-hypergraph Ramsey number* $R_k(m)$ is the smallest integer N such that for every 2-coloring of the hyperedges of the

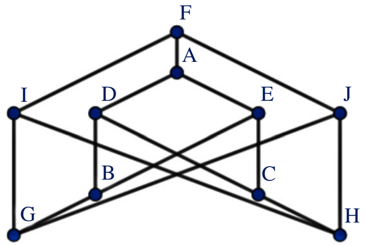


Fig. 1 A 2-partition induced one-dimensional complex that is not 2-Tverberg

complete k-hypergraph with N vertices, say with colors red and blue, it contains either a red complete k-hypergraph with m vertices or a blue complete k-hypergraph with m vertices. Here being a red (respectively blue) complete k-hypergraph means all k-tuples are colored red (respectively blue). See more on Ramsey numbers in [5] and references therein.

Theorem 1.2 *All trees and cycles are d-Tverberg complexes for all $d \geq 2$.*

(A) *Every tree T_n on n nodes is a d-Tverberg complex for $d \geq 2$. The Tverberg number* $\mathrm{Tv}(T_n, d)$ *exists and it is at most* $R_{d+1}((d+1)(n-1)+1)$. *More strongly,* $\mathrm{Tv}(T_n, 2)$ *is at most* $\binom{4n-4}{2n-2} + 1$.
(B) *Every n-cycle C_n with $n \geq 4$ is a d-Tverberg complex for $d \geq 2$. The Tverberg number exists and* $\mathrm{Tv}(C_n, d)$ *is at most* $nd + n + 4d$.

The proof of Theorem 1.2 relies on several powerful non-constructive tools such as the Ham-Sandwich theorem (see [15, Sect. 1.3]), a characterization of oriented matroids of cyclic polytopes [6], and the multi-dimensional version of Erdős–Szekeres theorem (this is due to Grünbaum [12] and Cordovil and Duchet [6], see also [3, Chapter 9], and the survey [16]). These tools are enough to show the existence of a Tverberg number $\mathrm{Tv}(T_n, d)$, but the bounds are far from tight. Details are presented in Sect. 2.

We can prove the following general lower bound for the Tverberg numbers (see Appendix A for the argument).

Lemma 1.3 *For any connected simplicial complex K with $n \geq 2$ vertices, if it exists,* $\mathrm{Tv}(K, d) \geq 2n$.

In addition to this general lower bound, we show that the upper bounds of Theorem 1.2 can indeed be improved by giving better bounds on the Tverberg numbers of *caterpillar trees*. Caterpillar trees are those in which all the vertices are within distance one of a central path; these include paths and stars. See Sect. 3.

Theorem 1.4 *If a tree T_n is a caterpillar tree with n nodes, then T_n is a d-Tverberg complex for all $d > 0$, and its d-Tverberg number* $\mathrm{Tv}(T_n, d)$ *is no more than* $(d + 1)(n - 1) + 1$.

In terms of intersection properties caterpillar graphs have been shown by Eckhoff [8] to be precisely the trees that are also interval graphs. In other words, the previous theorem implies that a tree T_n is also 1-Tverberg if and only if T_n is a caterpillar tree. Furthermore, in dimension two we can give some info on Tverberg numbers for trees:

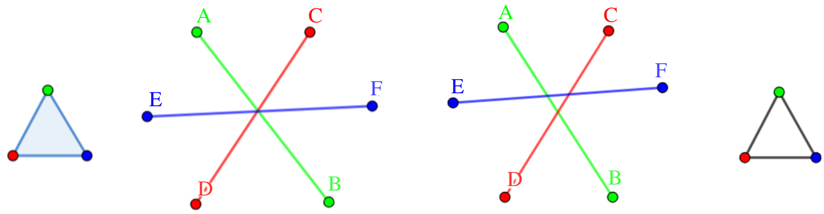


Fig. 2 Only the 1-skeleton of the nerve is preserved by order-preserving bijection

Theorem 1.5

(A) *The* 2*-Tverberg number* $\mathrm{Tv}(S_n, 2)$ *for a star tree with n nodes equals* $2n$.
(B) *The* 2*-Tverberg numbers of the path and cycle with four nodes are* $\mathrm{Tv}(P_4, 2) = 9$ *and* $11 \le \mathrm{Tv}(C_4, 2) \le 13$.

The proof of Theorem 1.5 (B) requires exhaustive computer enumeration of all possible partitions, over all possible order types of point sets with fewer than ten points. Luckily, these order types were classified in [1].

Recall that for an ordered set of points $S = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n) \in \mathbb{R}^d$, the *order type* (see [15, 9.3]) of S is defined as the mapping assigning to each $(d+1)$-tuple $(i_1, i_2, \ldots, i_{d+1})$ of indices, $1 < i_1 < i_2 < \ldots < i_{d+1} \le n$, the orientation of the $(d+1)$-tuple $(\boldsymbol{p}_{i_1}, \boldsymbol{p}_{i_2}, \ldots, \boldsymbol{p}_{i_{d+1}})$ (i.e., the sign of the determinant of the corresponding matrix). The order type of S is encoded by the *chirotope* of S which is the sequence of resulting $\binom{n}{d+1}$ signs of possible determinants. This is a vector of $+1$'s and -1's, with $\binom{n}{d+1}$ entries.

The proof of Theorem 1.5 (B) also uses the following lemma to ensure that it suffices to check one representative configuration of points from each order type, reducing calculations to finitely many cases. See details in Appendix A.

Lemma 1.6 *Suppose* S_1 *and* S_2 *are two point sets in* $\mathbb{R}^d$ *with the same order type, and let* σ *be a bijection from* S_1 *to* S_2 *that preserves the orientation of any* $(d+1)$*-tuple in* S_1. *Then any partition* $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ *of* S_1 *and the corresponding partition of* S_2 *via* σ, *denoted* $\sigma\mathcal{P} = \{\sigma(P_1), \sigma(P_2), \ldots, \sigma(P_n)\}$, *have the same intersection graph* $\mathcal{N}^1(\mathcal{P})$.

Lemma 1.6 cannot be extended to arbitrary nerve complexes as we see in the example of Fig. 2. Despite the fact that the chirotope-preserving bijections do not preserve the higher-dimensional skeleton of the nerve of a partition we can still make use of Lemma 1.6 throughout our paper because our results are only about *triangle-free* simplicial complexes, thus their nerve complexes equal their 1-skeleton.

2 A Tverberg Theorem for Trees and Cycles

2.1 Proof of Theorem 1.2 (A) in the Plane

Because the case of dimension two exemplifies the key ideas very well and because we can provide a better bound, we first give the proof of Theorem 1.2(A) in the plane. To summarize the proof, first, we show in Theorem 2.1 that the result holds if the points are arranged as the vertices of a convex polygon. Second, given any set $\bar{S}$ with at least $\binom{4n-4}{2n-2}+1$ points in the plane, we apply the Erdős–Szekeres theorem to deduce that $\bar{S}$ has a sub-configuration S of $2n$ points in convex position. Then we apply Theorem 2.1 to obtain a partition of S whose nerve is the tree T_n, and finally, in Lemma 2.3, we prove we can extend the partition of S to the rest of $\bar{S}$ while preserving the nerve. Later in Sect. 2.2 we present the general case in $\mathbb{R}^d$ following a similar strategy, but some of the key steps are different.

Theorem 2.1 *Let T_n be a tree with n nodes, and let $S \subset \mathbb{R}^2$ be any $2n$ point set in convex position. Then S admits a partition $\mathcal{P}$ such that its nerve $\mathcal{N}(\mathcal{P})$ is isomorphic to T_n.*

Proof The proof is by induction on n, the number of vertices in T_n. For an example of the construction see Fig. 3. For $n=1$, the tree consists of a single node and S is a set of two points in $\mathbb{R}^2$. Coloring both points with color 1 will trivially satisfy the theorem. When $n=2$, the only tree with two vertices is K_2. Any set of four points in S, say $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4$ in counterclockwise order, can be partitioned with intersection graph K_2. Note that in this case, coloring the points in $S = S_1 \cup S_2$ with two alternating colors $\boldsymbol{s}_1 = 1, \boldsymbol{s}_2 = 2, \boldsymbol{s}_3 = 1, \boldsymbol{s}_4 = 2$ will yield the required partition.

For performing the induction step, we can assume T_n was obtained from a tree T_{n-1} by adding the leaf node v_n to a node $v_r \in T_{n-1}$ such that $\{v_n, v_r\}$ is an edge of T_n. Note that in our labeling of the n nodes, r may not be $n-1$, but all trees are constructed by a sequence of leaf additions.

By the induction hypothesis, for any set S' with exactly $2n-2$ points in convex position in $\mathbb{R}^2$, there exists a partition $\mathcal{P}'$ of S' into $n-1$ color classes, where each color $i \in \{1, 2, \ldots, n-1\}$ is used twice, such that $T_{n-1} = \mathcal{N}(\mathcal{P}')$. Thus, we may assume that there exists a two-to-one "coloring function" $\mathcal{C}\colon S' \to [n-1]$ that associates two points in S' with a color i (the color of node v_i).

Let S be a set of $2n$ points in convex position in $\mathbb{R}^2$, ordered in a clockwise manner, say $S = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_{2n}\}$, and assume without loss of generality that $\boldsymbol{s}_1$ is at twelve o'clock. Next, consider the set $S' := S \setminus \{\boldsymbol{s}_2, \boldsymbol{s}_{2n}\}$. To this set S' we can apply the induction hypothesis, it is properly colored and gives T_{n-1}. Now we show how to add color n to the remaining points in S to give T_n. There are two cases.

Case 1 If $\mathcal{C}(\boldsymbol{s}_1) = r$, then extend $\mathcal{P}'$ to a partition $\mathcal{P}$ of S by assigning color n to the points $\boldsymbol{s}_2$ and $\boldsymbol{s}_{2n}$. Thus $\mathcal{P} = \mathcal{P}' \cup \{\boldsymbol{s}_2, \boldsymbol{s}_{2n}\}$. Let L_n be the line through $\boldsymbol{s}_2$ and $\boldsymbol{s}_{2n}$. Observe that on one side of L_n, say L_n^+, there is only $\boldsymbol{s}_1$. Then the other points in S' are contained in the other open half plane L_n^-. In particular, one point, say $\boldsymbol{s}_j$, is such that $\mathcal{C}(\boldsymbol{s}_j) = r$. Thus $\boldsymbol{s}_1$ and $\boldsymbol{s}_j$ have color r. Then $\operatorname{conv}(\boldsymbol{s}_2, \boldsymbol{s}_{2n})$ and $\operatorname{conv}(\boldsymbol{s}_1, \boldsymbol{s}_j)$ intersect so $\mathcal{N}(\mathcal{P})$ contains the edge (r, n). Furthermore, for any $i \neq n, r$, we have that $\mathcal{N}(\mathcal{P})$

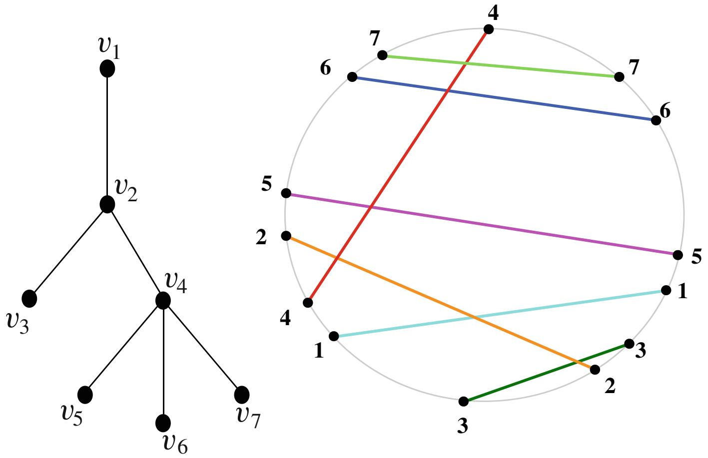


Fig. 3 Example of a tree with seven nodes shown as partition induced on a set S of 14 points in convex position

does not contain the edge (i, n), since the points with color i are contained in L_n^- and so their convex hull cannot intersect $\operatorname{conv}(\boldsymbol{s}_2, \boldsymbol{s}_{2n})$. Thus the nerve of $\mathcal{P}$ is T_n.

Before starting Case 2 consider the relabeling of $S' := S \setminus \{\boldsymbol{s}_2, \boldsymbol{s}_{2n}\} = \{\boldsymbol{x}_1 = \boldsymbol{s}_1, \boldsymbol{x}_2 = \boldsymbol{s}_3, \ldots, \boldsymbol{x}_{2n-2} = \boldsymbol{s}_{2n-1}\}$.

Case 2 If $\mathcal{C}(\boldsymbol{s}_1) \neq r$, then we know that on one side of the line L_n (through $\boldsymbol{s}_2$ and $\boldsymbol{s}_{2n}$) there are two points in S', say $\boldsymbol{x}_i, \boldsymbol{x}_{i+k}$ (as above) such that $\mathcal{C}(\boldsymbol{x}_i) = \mathcal{C}(\boldsymbol{x}_{i+k}) = r$ for $i \geq 3$ and $1 < k \leq (2n-2) - i$. Apply to S' the following new coloring $\bar{\mathcal{C}}\colon S' \to [n-1]$ defined as $\bar{\mathcal{C}}(\boldsymbol{x}_j) = \mathcal{C}(\boldsymbol{x}_{j+2n-i-1}) \bmod 2n-2$; that is, the rotation that sends the corresponding color in $\boldsymbol{x}_i$ to $\boldsymbol{x}_1$. Observe that this rotation preserves all the intersection patterns that existed before (by Lemma 1.6), and thus $\mathcal{N}(\mathcal{P}')$ is T_{n-1}. Lastly, we are now in the position to apply Case 1 again, so the theorem follows.

This completes the proof that any set S of $2n$ points in convex position in the plane has a partition whose nerve is isomorphic to any given tree T_n. □

To extend our result to the case that the points are in general position, we will use a famous theorem in combinatorial geometry, the Erdős–Szekeres theorem. This theorem says that every sufficiently large set of points in general position contains a subset of k points in convex position. The fact that this number $N = N(k, 2)$ exists for every k was first established in a seminal paper of Erdős and Szekeres [9], who proved the following bounds on $N(k, 2)$:

$$2^{k-2} + 1 \leq N(k, 2) \leq \binom{2k-4}{k-2} + 1.$$

A handful of recent papers have improved the upper bound (see for instance [16] for an excellent survey and a very recent paper by Suk [24] showing that $N(k, 2) = 2^{k+o(k)}$).

By the Erdős–Szekeres theorem we know that $\binom{4n-4}{2n-2} + 1$ points always contain a $2n$-gon. Then, we can use Theorem 2.1. Finally we explain how to extend the partition (or coloring) given by Theorem 2.1 to the rest of the points in $\bar{S}$.

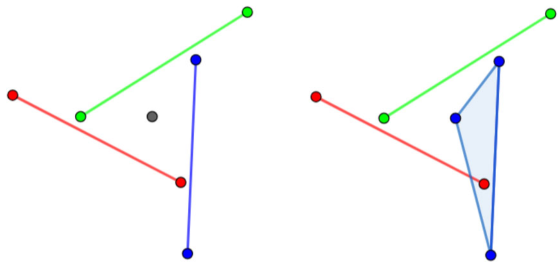


Fig. 4 From left to right: Illustration of the construction at steps 1 and 2, and the resulting partition

Definition 2.2 Let S be a set of points in $\mathbb{R}^d$ and let $\mathcal{P} = S_1, \ldots, S_n$ be an n-partition of S into n color classes that yields a specific nerve $\mathcal{N}(\mathcal{P})$. We say that $\mathcal{P}$ is *extendable* if for all $\bar{S}$ containing S, there is a partition $\bar{\mathcal{P}} = \bar{S}_1 \ldots \bar{S}_n$ of $\bar{S}$ extending $\mathcal{P}$ (meaning $S_i \subset \bar{S}_i$ for all i) such that $\mathcal{N}(\bar{\mathcal{P}})$ is isomorphic to $\mathcal{N}(\mathcal{P})$.

Observe that in general, such an extension is not necessarily possible; for example, Fig. 4 shows a set of six vertices, and a partition in three color classes (see left side of the figure), that is not extendable. Note that any extension that includes the midpoint will change the intersection pattern (see right side of the figure). Surprisingly, in the case of the nerves of the partitions obtained in Theorem 2.1 (and Theorem 2.4 in the next subsection), this extension is possible.

Lemma 2.3 *Let T_n be a given tree on n nodes and let S be a set of $2n$ points in convex position in the plane. Then the partition $\mathcal{P}$ obtained in the proof of Theorem 2.1 is extendable.*

Proof Let $\bar{S}$ be an arbitrary finite set of points in $\mathbb{R}^d$ such that $S \subset \bar{S}$. We begin by assuming that the "color partition function" $\mathcal{C}\colon S \to [n]$ is the one given in Theorem 2.1. It yields a partition $\mathcal{P}$ of S with nerve $\mathcal{N}(\mathcal{P})$ isomorphic to T_n, and n is the last color added. Recall that we denoted by v_r the node in T_{n-1} such that $\{v_n, v_r\}$ is the leaf of T_n in which we added v_n.

The extension of $\mathcal{P}$ will be given through induction on n, by a "color partition function" $\bar{\mathcal{C}_n}\colon \bar{S} \to [n]$ as follows.

- For $n = 1$, let $\bar{\mathcal{C}}_1(\boldsymbol{x}) = 1$ for every point in $\bar{S}$.
- For the induction step, the extension $\bar{\mathcal{C}}_{n-1}\colon \bar{S} \to [n-1]$ exists by induction hypothesis. Here is how we obtain the extension $\bar{\mathcal{C}}_n$: Let S_j denote the set of points in S of color v_j, or j for simplicity. Consider the line L_n through S_n (it is given by points $\mathcal{S}_2$ and $\mathcal{S}_{2n}$ in Theorem 2.1), and recall that this line leaves only one element of S_r on one side, say L_n^+, and the rest of the points of S on the other side L_n^-. We define $\bar{\mathcal{C}}\colon \bar{S} \to [n]$ as follows: $\bar{\mathcal{C}}_n(\boldsymbol{x}) = \bar{\mathcal{C}}_{n-1}(\boldsymbol{x})$ when $\boldsymbol{x} \in L_n^-$, $\bar{\mathcal{C}}_n(\boldsymbol{x}) = r$ when $\boldsymbol{x} \in \operatorname{conv}(S_r)$, and, finally, $\bar{\mathcal{C}}_n(\boldsymbol{x}) = n$ when $\boldsymbol{x} \in \operatorname{closure}(L_n^+)$ but $\boldsymbol{x} \notin \operatorname{conv}(S_r)$. Here $\operatorname{closure}(L_n^+)$ denotes the closed half-plane on the right of L_n.

Observe that, by the induction process, the intersection patterns of $\bar{S}_1, \ldots, \bar{S}_{n-1}$ are the same in L_n^- by construction. Furthermore, $\operatorname{closure}(L_n^+)$ does not intersect any other element in the partition, so no new intersections occur. □

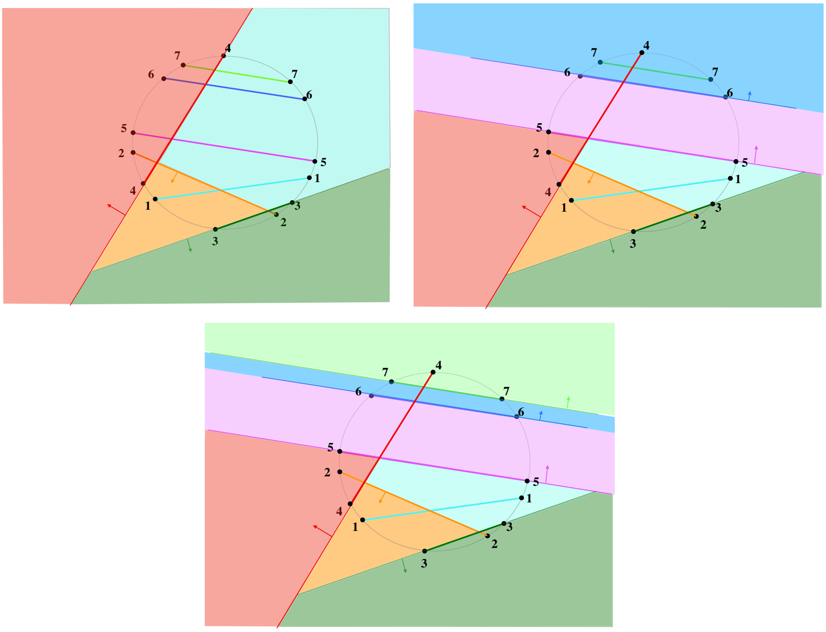


Fig. 5 The extension of the partition in Fig. 3. The left figure is the extension up to $n = 4$, the central figure is the extension up to $n = 6$, and the right figure is up to $n = 7$

2.2 Proof of Theorem 1.2 (A) in $\mathbb{R}^d$

Next, we will show a general dimension version of Theorem 2.1. The pattern of the proof is very similar to the planar case, but we will need to use properties of cyclic polytopes and their oriented matroids. A parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^d$ is a *d-order curve* (sometimes called *alternating*) when no affine hyperplane H in $\mathbb{R}^d$ meets the curve in more than d points. An example is the famous *moment curve*. See [3,6,23].

In what follows we will use *ordered cyclic d-polytopes* $C_m(d)$ which are obtained as the convex hull of m vertices $S := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\}$ along a d-order curve in $\mathbb{R}^d$, and thus we may order the vertices of this polytope in an increasing sequential manner, say $\alpha(t_1) = \boldsymbol{x}_1 < \alpha(t_2) = \boldsymbol{x}_2 < \cdots < \alpha(t_m) = \boldsymbol{x}_m$. Ordered cyclic polytopes are very special because every subpolytope is also cyclic with respect to the same vertex order, i.e., the corresponding oriented matroid is *alternating*. Alternating means the chirotope has all positive signs. See Section 9.4 in the book [3].

Theorem 2.4 *Let T_n be any tree with n nodes, and let S be the vertices of an ordered cyclic d-polytope $C_m(d)$ with $m = (n-1)(d+1)+1$ vertices in $\mathbb{R}^d$. Then, there exists a partition $\mathcal{P}$ of S such that the nerve $\mathcal{N}(\mathcal{P})$ is isomorphic to T_n.*

Proof As we mentioned before, for dimension two, we relied on Erdős–Szekeres to build a convex polygon. For the general case in $\mathbb{R}^d$, we need a multi-dimensional version of the Erdős–Szekeres theorem which follows from an application of the hypergraph Ramsey theorem [5]. The theorem we need was first given by Grünbaum

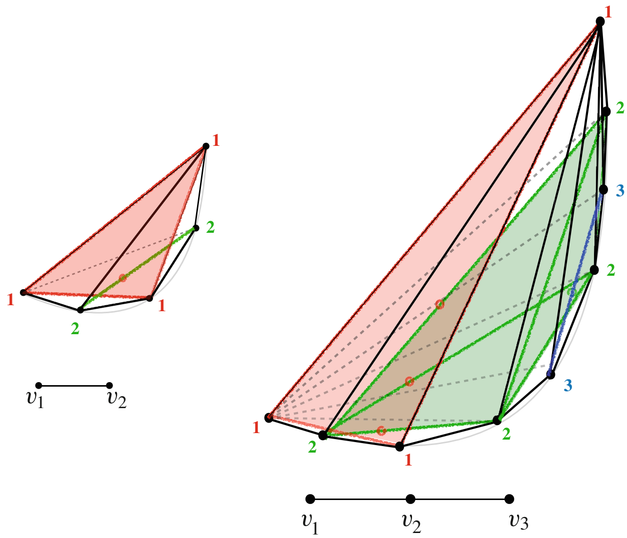


Fig. 6 Left: a tree on two nodes shown as a partition in a set S of five vertices of the cyclic polytope $C_5(3)$. Right: a tree on three nodes as a partition of the nine vertices of another cyclic polytope in $\mathbb{R}^3$, this time $C_9(3)$

[12, Exercise 7.3.6] and Cordovil and Duchet [6] using oriented matroid methods. See [3, Proposition 9.4.7] for a short proof. The theorem shows the existence of a number $N = N(k, d)$ such that every set of N points in general position in $\mathbb{R}^d$ contains the vertices of an ordered cyclic d-polytope. N is bounded from above by the hypergraph Ramsey number $R_{d+1}(m)$ (see the introduction) ensuring the existence of an alternating oriented matroid (hence an ordered cyclic polytope with m vertices).

According to [23], when an oriented matroid is alternating, then its cyclic d-polytope is on a d-order curve in $\mathbb{R}^d$ and every subpolytope of it is also cyclic. This is quite a useful fortuity, since it is well known, that in odd dimensions there exist combinatorial cyclic polytopes containing subpolytopes which are not cyclic (see page 104 of the same paper). By these facts, we know that if $\bar{S}$ is a set of points in general position in $\mathbb{R}^d$ with at least $R_{d+1}((n-1)(d+1)+1)$ points, then $\bar{S}$ contains a set S consisting of the $m = (n-1)(d+1)+1$ vertices of an ordered cyclic d-polytope $C_m(d)$.

Let $C_m(d)$ denote an ordered cyclic d-polytope, with m vertices and assume as before $S := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\}$ along the curve. As in the case of the plane, the proof will be given by induction on n, the number of nodes of the tree T_n (Fig. 5).

If $n = 1$, again there is nothing to prove. If $n = 2$, the only tree with two vertices is K_2. Then by Radon's theorem, any set of $d+2$ points in S can be partitioned into $S = S_1 \cup S_2$ with $2 \le |S_i| \le d$ for $i \in \{1, 2\}$, and intersection graph K_2, see Fig. 6, left.

For the induction step, suppose T_n was obtained from T_{n-1} by adding the node v_n to a node $v_r \in T_{n-1}$ such that $\{v_n, v_r\}$ is a leaf of T_n, and assume that T_{n-1} is the nerve of some set $\mathcal{N}(\mathcal{P}')$ where the set S' are the vertices of the ordered cyclic polytope with exactly $(n-1)(d+1)-d$ vertices in $\mathbb{R}^d$ and $\mathcal{P}' = \{S'_1, S'_2, \ldots, S'_{n-1}\}$ are the color classes with colors $1, 2, \ldots, n-1$, respectively, via a "coloring function" $\mathcal{C}' : S' \to [n-1]$.

Let k be the maximum number in $[n]$ such that $\boldsymbol{x}_k$ is in S'_r. Next, in $C_m(d)$ consider the subpolytope Q of $(n-1)(d+1)-d$ vertices, obtained as the convex hull conv $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k, \boldsymbol{x}_{k+d+2}, \ldots, \boldsymbol{x}_m)$, and let R be the polytope consisting of the convex hull of the complement of Q and $\boldsymbol{x}_k$, thus $R = \text{conv}(\{\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_{k+d+1}\})$. Note both Q and R are ordered cyclic polytopes and $Q \cap R = \{\boldsymbol{x}_k\}$. Thus, by the induction hypothesis there exists a partition of the vertices of Q into $n-1$ color classes whose nerve is isomorphic to T_{n-1} as before. Next, by Radon's lemma there exists a partition into two color classes A and B of the $d+2$ vertices of R.

Say $\boldsymbol{x}_k \in A$, then define a "coloring function" $\mathcal{C} : S \to [n]$ in the following way: $\mathcal{C}(\boldsymbol{x}) = \mathcal{C}'(\boldsymbol{x})$ if $\boldsymbol{x}$ is a vertex of Q, $\mathcal{C}(\boldsymbol{x}) = r$ if $\boldsymbol{x} \in A$, and, finally, $\mathcal{C}(\boldsymbol{x}) = n$ if $\boldsymbol{x} \in B$. That is, $S_n \cap S_r \neq \emptyset$.

Observe that there exists a facet $\mathcal{F}_n$ of Q, and therefore a $(d-1)$-hyperplane $H_n = \langle \mathcal{F}_n \rangle$, containing $\boldsymbol{x}_k, \boldsymbol{x}_{k+d+2}$, and some other vertices of Q, that leaves Q completely contained in the closure of one of the sides of this hyperplane, say H_n^-, and leaving points $\{\boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_{k+d+1}\}$ on the other side. (For instance, if the vertices of $\mathcal{F}_n$ are $V(\mathcal{F}_n) = \{\boldsymbol{x}_k, \boldsymbol{x}_{k+d+2}, y_1, y_2, \ldots, y_{d-2}\}$, according to the Gale evenness condition this could be defined in the following way: If d is even, $y_1, y_2, \ldots, y_{d-2}$ are consecutive vertices before $\boldsymbol{x}_k$ if $k \geq m/2$, or after $\boldsymbol{x}_{k+d+2}$ if $k \leq m/2$. If d is odd, $k \geq m/2$ and if $k \neq m$ then $y_{d-2} = \boldsymbol{x}_m$ and $y_1, y_2, \ldots, y_{d-3}$ are consecutive vertices before $\boldsymbol{x}_k$, and if $k = m$ then $y_1 = \boldsymbol{x}_1$ and $y_2, \ldots, y_{d-2}$ are consecutive vertices before $\boldsymbol{x}_k$. Similarly, if $k \leq m/2$ and $k \neq 1$ then $y_1 = \boldsymbol{x}_1$ and $y_2, \ldots, y_{d-2}$ are consecutive vertices after $\boldsymbol{x}_{k+d+2}$, and if $k = 1$ then $y_{d-2} = \boldsymbol{x}_m$ and $y_1, y_2, \ldots, y_{d-3}$ are consecutive vertices after $\boldsymbol{x}_{k+d+2}$). Therefore H_n strictly separates every point of color different than r and m from points of color n and therefore no further intersections occur. By the construction, the parts of $\mathcal{P}$ consist of the n color classes determined by the coloring $\mathcal{C}$. The nerve $\mathcal{N}(\mathcal{P})$ is isomorphic to T_n. □

To finish the proof we just need to "extend", as we did in the case of the plane, the partition given in Theorem 2.4 (for the vertices of $C_m(d)$) to a partition $\bar{P}$ of $\bar{S}$ in such a way that the nerve $\mathcal{N}(\bar{\mathcal{P}})$ is preserved. Lemma 2.5 below guarantees that this is always possible, finishing the proof of Theorem 1.2 (A).

Lemma 2.5 *Let T_n be a given tree and let S be the vertices of an ordered cyclic polytope with $m = (n-1)(d+1)+1$ vertices in $\mathbb{R}^d$. Then the specific partition $\mathcal{P}$ of S obtained in Theorem 2.4 is extendable to any set $\bar{S}$ containing S with the same nerve complex.*

Proof Let $\bar{S}$ be an arbitrary finite set of points in $\mathbb{R}^d$ such that $S \subset \bar{S}$. Let S_j denote the set of points in S of color v_j, or j for simplicity. Let us begin by assuming that the "color partition function" $\mathcal{C} : S \to [n]$, given in Theorem 2.4, yields a partition $\mathcal{P}$ of S with nerve $\mathcal{N}(\mathcal{P})$ isomorphic to T_n. The extension of $\mathcal{P}$ of S will be given by induction on the number of nodes n.

- In the case $n = 1$ assign $\bar{\mathcal{C}}_1(\boldsymbol{x}) = 1$ for every point $\boldsymbol{x}$ in $\bar{S}$.
- For the induction step note that the induction hypothesis guarantees the extension $\bar{\mathcal{C}}_{n-1} : \bar{S} \to [n-1]$ exists.

To begin observe that polytopes Q and R defined in Theorem 2.4 satisfy that $Q \cap R = \{\boldsymbol{x}_k\}$ so $(Q \setminus \{\boldsymbol{x}_k\}) \cap (R \setminus \{\boldsymbol{x}_k\}) = \emptyset$. Recall that there exists a $(d-1)$-hyperplane H_n that leaves points of color r on both sides of the hyperplane, and strictly separates points of color n and all those of other colors. Furthermore, Q is completely contained in the closure of one of the sides of this hyperplane, say H_n^- (see the proof of Theorem 2.4). Then "color partition function" $\bar{\mathcal{C}}_n : \bar{S} \to [n]$ is given as follows: $\bar{\mathcal{C}}_n(\boldsymbol{x}) = \bar{\mathcal{C}}_{n-1}(\boldsymbol{x})$ if $\boldsymbol{x} \in H_n^-$, $\bar{\mathcal{C}}_n(\boldsymbol{x}) = r$ if $\boldsymbol{x} \in S_r$, and $\bar{\mathcal{C}}_n(\boldsymbol{x}) = n$ if $\boldsymbol{x} \in \text{closure}(H_n^+)$ and $\boldsymbol{x} \notin S_n$.

As before, $\text{closure}(H_n^+)$ is the closed half-hyperplane containing only points in R of colors n and r. Observe that, by the induction process, the intersection patterns of $\bar{S}_1, \dots \bar{S}_{n-1}$ are the same in H_n^- by construction, and $\text{closure}(\mathrm{H}_\mathrm{n}^+) \cap S_r \neq \emptyset$ yields the leaf $\{v_r, v_n\}$. Furthermore, $S_n \subset \text{closure}(H_n^+)$ does not intersect any other elements in the partition since they are contained in H_n^-, so no further intersections occur. □

2.3 Proof of Theorem 1.2 (B)

Suppose that $\bar{S}$ is a set of at least $nd + n + 4d$ points in general position in $\mathbb{R}^d$. We start by projecting the points onto a generic 2-plane H where we can assume, without loss of generality, that the points of $\bar{S}$ have distinct projections onto it. Let S be the projection of $\bar{S}$, now planar points.

Lemma 2.6 *There exists a disk D containing all these projected planar points in S, and a subdivision D' of D into n sectors such that:*

(i) *Each sector contains at least $d + 1$ points.*
(ii) *No two adjacent sectors form a combined angle of more than π radians.*

Proof We start by picking a line L_1 with at least $\lfloor (nd + n + 4d)/2 \rfloor$ points on both sides of L_1. Denote by L_1^- and L_1^+, respectively, the open half-spaces defined by L_1 and by M_1^+, M_1^- the points of S on the two half-spaces of L_1. Applying the Ham Sandwich Theorem (see [15, Sect. 1.3]) to the sets M_1^- and M_1^+, we can find a line L_2 so that L_1 and L_2 together separate the plane into four regions, say R_1, R_2, R_3, and R_4 with at least $\lfloor (nd + n + 4d)/4 \rfloor$ projected points in each region. Note that $\lfloor (nd + n + 4d)/4 \rfloor \geq d + 1$ points.

Denote by $\boldsymbol{p}$ the point in the plane where L_1 and L_2 intersect, and let D be a disk centered at $\boldsymbol{p}$ that contains all the projected points. Now we choose arcs emanating from $\boldsymbol{p}$ to subdivide each of the four regions R_1, R_2, R_3, and R_4 into as many subregions, containing at least $d + 1$ points (note that each of the R_i has at least $d + 1$ points in them by construction). This can be done as follows.

If R_i' has at least $2d + 2$ points, then take a line emanating from $\boldsymbol{p}$ and rotate it until it divides R_i into two regions, one with $d + 1$ points, denoted R_{i1}, and the other with the remaining (at least $d + 1$) points in R_i, denoted R_i'. Otherwise do nothing. Repeating this process as many times as possible, we will obtain a subdivision of each R_i into subregions, all but one of which have exactly $d + 1$ points, and none of which has more than $2d + 1$ points. We call the final regions of this recursive process sectors.

Since the original four regions R_1, R_2, R_3, R_4 satisfy property (ii) of the lemma, and the process of subdivision is made to show (i) holds after subdividing the four regions, all we have left to do is to check there are n sectors. For this, let k_1, k_2, k_3, and k_4 denote the respective number of sectors formed from each of the four regions, and j_1, j_2, j_3, and j_4 denote the number of points in each region. It suffices to show that $k_1 + k_2 + k_3 + k_4 \geq n$ because we can always merge adjacent sectors within the same region R_i, while preserving claims (i) and (ii).

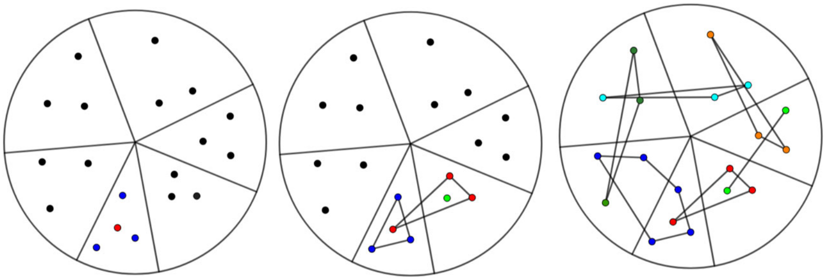
Our procedure for generating subdivisions guarantees that $j_i \leq k_i(d+1) + d$ for all $i = 1, 2, 3, 4$. Summing these inequalities up we get $j_1 + j_2 + j_3 + j_4 \leq (k_1 + k_2 + k_3 + k_4)(d+1) + 4d$, so

$$nd + n + 4d \leq (k_1 + k_2 + k_3 + k_4)(d+1) + 4d,$$

which implies that $k_1 + k_2 + k_3 + k_4 \geq n$. This completes the proof of the lemma. □

Now we will use the the subdivision D', whose existence is guaranteed by Lemma 2.6, to find our desired partition of the data points whose partition nerve is an n-cycle.

We construct a partition one sector at a time. In the first step, we notice that one of the n sectors, say Q_1, has at least $d+2$ points by the pigeonhole principle. Use Radon's lemma to partition the points in Q_1 into two sets S_1 and S_2 so that the convex hulls of S_1 and S_2 intersect.



In the second step, we denote the slice on the left to Q_1 as Q_2. By Radon's lemma, any point $\boldsymbol{x}_2$ in S_2 from step one, combined with the (at least) $d+1$ points in Q_2 can be partitioned into two sets S_2' and S_3 so that the convex hulls of S_2 and S_3 intersect. Without loss of generality we can assume that $\boldsymbol{x}_2 \in S_2'$, and then set $S_2 = S_2 \cup S_2'$. In step k, where $3 \leq k \leq n-1$, we continue in the same way. We denote the slice on the left to Q_{k-1} as Q_k. By Radon's lemma, any point $\boldsymbol{x}_k$ in S_k from step $k-1$, combined with the (at least) $d+1$ points in Q_{k+1} can be partitioned into two sets S_k' and S_{k+1} so that the convex hulls of S_k' and S_{k+1} intersect. Without loss of generality we can assume that $\boldsymbol{x}_{k-1} \in S_{k-1}'$, and then set $S_k = S_k \cup S_k'$. Finally, in step $n-1$ we set $S_1 = S_1 \cup S_n$.

We claim that the nerve of the resulting partition $\mathcal{P} = \{S_1, S_2, \ldots, S_n\}$ is the n-cycle. This is a consequence of two facts: We used Radon's lemma to guarantee that any two subsets appearing in the same sector have intersecting convex hulls. Each subset appears in at most two sectors, and since two adjacent sectors have a combined angle of at most π radians, there is a line separating any two subsets that do not appear in the same sector. Thus we have that $\operatorname{conv}(S_i) \cap \operatorname{conv}(S_j) \neq \emptyset$ if and only if there is some sector containing points from both S_i and S_j. If we let $\boldsymbol{v}_i$ denote the vertex of the nerve corresponding to subset S_i, we see that the edges of $\mathcal{N}(\mathcal{P})$ consist precisely of $(\boldsymbol{v}_n, \boldsymbol{v}_1)$ and $(\boldsymbol{v}_i, \boldsymbol{v}_{i+1})$ where $i \in [n-1]$. This completes the proof.

3 Improved Tverberg Numbers of Special Trees and in Low Dimensions

3.1 Proof of Theorem 1.4: Better Bounds for Tverberg Numbers of Caterpillar Trees

To make the notation easier, we adopt the following convention throughout the proof of Theorem 1.4: All point sets $S \subset \mathbb{R}^d$ are indexed in increasing order with respect to their first coordinate. That is, if $S = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$, with $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{id})$, then we assume that $x_{11} \leq x_{21} \leq \ldots \leq x_{n1}$. Furthermore, by rotating the axes, we can assume that no two points have the same first coordinate and that the previous inequalities are strict.

We first prove the special case of stars in Theorem 1.4 as a lemma. A caterpillar is a sequence of stars, thus we can later use induction again.

Lemma 3.1 *For any $(d+1)(n-1)+1$ points in $\mathbb{R}^d$, we can find a partition of those points with nerve St_n, the star tree on n vertices (i.e., with $n-1$ spokes).*

Proof We prove this by induction on n. For $n = 1$, the partition of one point to get St_1 is obvious. Now assume the result is true for some n. We need to show that any $(d+1)n+1$ points can be partitioned with partition nerve St_{n+1}. Let $M = (n-1)(d+1)+1$. By induction hypothesis, the subset $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M\} \subset S$ admits a partition $\mathcal{P} = \{A_1, \ldots, A_n\}$ with $\mathcal{N}(\mathcal{P}) \simeq \mathrm{St}_n$. Without loss of generality, assume that A_1 is the central vertex of the star graph. Let $\boldsymbol{x} \in S$ be some point in A_1. By Radon's lemma, there is a way to partition the $d+2$ points $\boldsymbol{x}, \boldsymbol{x}_M, \boldsymbol{x}_{M+1}, \ldots, \boldsymbol{x}_{M+d+1}$ into two sets X, Y with $\mathrm{conv}(X) \cap \mathrm{conv}(Y) \neq \emptyset$, and we can assume that $\boldsymbol{x} \in X$. The set $\mathrm{conv}(Y)$ intersects $\mathrm{conv}(A_1 \cup X)$ but does not intersect any of $\mathrm{conv}(A_i)$, $2 \leq i \leq n$, because every point in Y has larger first coordinate than any point in A_i. Then we see $\{A_1 \cup X, A_2, \ldots, A_n, Y\}$ is a partition which will induce the star graph St_n. □

Proof of Theorem 1.4 Now we prove that for every caterpillar tree T_n with at most n nodes, every set S with at least $(d+1)(n-1)+1$ points in $\mathbb{R}^d$ admits a partition $\mathcal{P}$ with $\mathcal{N}(\mathcal{P}) \simeq T_n$. An illustration of the partition constructed in the proof is given in Fig. 7. The proof is by induction on the length of the central path in T_n, which we will denote by m. The induction hypothesis says that for every $m \in \mathbb{N}$ and any caterpillar tree T_n with n vertices and a central path of length m the following two statements hold:

(i) Every set S of $(d+1)(n-1)+1$ points in $\mathbb{R}^d$ admits a partition $\mathcal{P}$ with $\mathcal{N}(\mathcal{P}) \simeq T_n$.
(ii) Denote by v the last vertex of the central path, and denote by St_{k+1} the star subgraph induced by v and its k neighbors. Then the subsets in $\mathcal{P}$ corresponding to vertices in St_{k+1} are comprised of the $(d+1)k+1$ points in S with largest first coordinate.

If the length of the central path is one, both parts of the induction hypothesis follow by applying Lemma 3.1. Assume the result holds if the central path is of length m. We consider caterpillar graphs that have central paths of length $m+1$. Let G be such a graph with n vertices. We consider the endpoint of the path v_{m+1} and the vertex prior v_m. If we consider the subgraph of G consisting of the path $v_1, \ldots, v_m$ and all vertices adjacent to it except v_{m+1}, this is a caterpillar graph with a path of length m. Let p

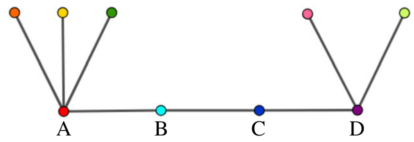


Fig. 7 An example of caterpillar graph G with nine vertices

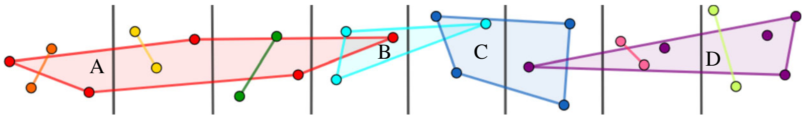


Fig. 8 An example of how a set of points can be partitioned with nerve G. The vertical lines indicate how we start with a Radon partition of the leftmost $d+2$ points, then partition the points from left to right, considering $d+1$ more points at each step. Notice there are extra points on the right, which are added to the subset corresponding to the last vertex on the central path

denote the number of vertices of this graph. By induction hypothesis, we can represent this graph using the $(d+1)(p-1)+1$ points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{(d+1)(p-1)+1}$. We will have the partition $\{A_1, \ldots, A_p\}$ where we take A_1 to be the set corresponding to v_m. Then take a point $\boldsymbol{x} \in A_1$ and the next $d+1$ points $\boldsymbol{x}_{(d+1)(p_1)+2}, \ldots, \boldsymbol{x}_{(d+1)(p_1)+d+2}$ to have a Radon partition X, Y with $\boldsymbol{x} \in X$. Our new partition will be $\{A_1 \cup X, A_2, \ldots, A_p, Y\}$. Y will correspond to the vertex v_{m+1} and will not intersect any of the other sets due to having larger first coordinate. In addition, $A_1 \cup X$ will not intersect any new sets by how we have arranged the points due to the induction hypothesis. Now, as in the proof of the lemma, we can add new sets by considering $d+1$ points in iteration for each of the other vertices adjacent to v_{m+1}. Since there were $n-p$ vertices and we used $d+1$ points for each, in total we used $(d+1)(n-1)+1+(d+1)(n-p) = (d+1)(n-1)+1$ points. This is the desired number (Fig. 8).

Thus we have proven the induction hypothesis. To complete the proof of the theorem, we note that if we have more than $(d+1)(n-1)+1$ points, we can apply the induction hypothesis to find the desired partition of the $(d+1)(n-1)+1$ points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_{(d+1)(n-1)+1}$, then add any remaining points to the subset corresponding to the endpoint of the central path in the caterpillar graph. □

3.2 Proof of Theorem 1.5: Tverberg Numbers of Trees in Dimension Two

Now we focus on the situation in two dimensions.

Lemma 3.2 *Let $S \in \mathbb{R}^2$ be a set of points in the plane. Denote by $L_{\boldsymbol{p}_1\boldsymbol{p}_2}$ the line segment between points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Suppose that there exists $\boldsymbol{p}_1, \boldsymbol{p}_2 \in S$ such that $L_{\boldsymbol{p}_1\boldsymbol{p}_2}$ separates the remaining points of S into two sets A, B and such that for any $\boldsymbol{a} \in A$, $\boldsymbol{b} \in B$, we have that $L_{\boldsymbol{ab}}$ intersects $L_{\boldsymbol{p}_1\boldsymbol{p}_2}$. Then it is possible to pair off elements $\boldsymbol{a}_i \in A$, $\boldsymbol{b}_i \in B$, so that for $i, j = 1, \ldots, \min(|A|, |B|)$, $i \neq j$, $L_{\boldsymbol{a}_i\boldsymbol{b}_i}$ does not intersect $L_{\boldsymbol{a}_j\boldsymbol{b}_j}$. When both A, B have equal size, the pairing process yields a perfect matching of the points in A, B.*

Proof Suppose we have points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as hypothesized and partition the remaining points into A and B. Let L be the line between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. To pair off the points,

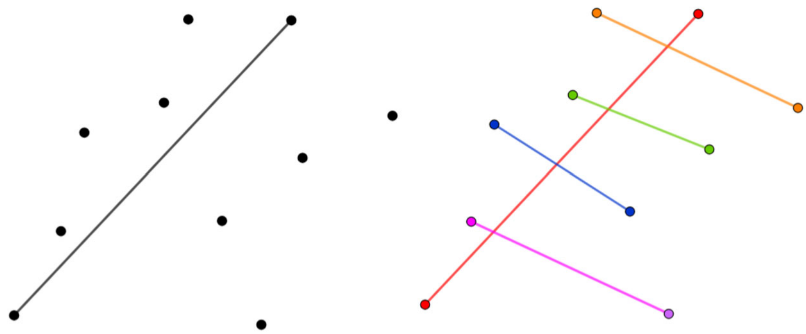


Fig. 9 In the first case, there is a partition which divides the remaining points into two sets of equal size. Then we can pair off points so that the segment connecting them intersects the dividing line, but no other segment

we consider the vertices of $\operatorname{conv}(A \cup B)$. Since L separates the points of A and B, we must have that there is a pair of adjacent vertices of $\operatorname{conv}(A \cup B)$ such that one, $\boldsymbol{a}_1$, is a member of A and the other, $\boldsymbol{b}_1$, a member of B. The segment between this pair cannot intersect the segment between any other pair of points as this segment forms the boundary of the convex hull. We pair off these two points and then consider $\operatorname{conv}(A \setminus \{\boldsymbol{a}_1\} \cup B \setminus \{\boldsymbol{b}_1\})$. We see that L separates $A \setminus \{\boldsymbol{a}_1\}$ and $B \setminus \{\boldsymbol{b}_1\}$, so we can repeat this argument to pair off $\boldsymbol{a}_2$ and $\boldsymbol{b}_2$. Continuing in this fashion until we have paired off all the elements (perfect matching situation) or we ran out of points in either A or B, we will have a pairing $(\boldsymbol{a}_1, \boldsymbol{b}_1), (\boldsymbol{a}_2, \boldsymbol{b}_2), \dots, (\boldsymbol{a}_n, \boldsymbol{b}_n)$ where $L_{\boldsymbol{a}_i \boldsymbol{b}_i}$ does not intersect $L_{\boldsymbol{a}_j \boldsymbol{b}_j}$ for $i \neq j$. □

Proof of Theorem 1.5 (A) Let $A \in \mathbb{R}^2$ be a collection of $2n$ points in general position in the plane. Our initial goal will be to find two points which can separate the remaining points into two sets of equal size $n-1$ so we can apply Lemma 3.2. This will not always be possible, so we will try to make the size of the two sets as close as possible by separating instead with a triangle-bounded set. In both cases the separating set (either a segment, or a triangle-bounded set), will be the center of the star tree.

To do this, we will consider the vertices of the convex hull of A. We pick arbitrarily a vertex $\boldsymbol{p}_1$ of $\operatorname{conv}(A)$ and order the remaining vertices $\boldsymbol{p}_2, \dots, \boldsymbol{p}_k$ in counterclockwise order where k is the number of vertices. For each $i = 2, \dots, k$, we divide the remaining points of A into two sets B_i, C_i where B_i is the set of points in A to the left of $L_{\boldsymbol{p}_1 \boldsymbol{p}_i}$ and C_i is the set of points to the right of $L_{\boldsymbol{p}_1 \boldsymbol{p}_i}$. We note that the size of B_i decreases from $2n-2$ to 0 as i increases and the size of C_i increases from 0 to $2n-2$.

We consider two cases. The first case is that there exists i such that $|B_i| = |C_i| = n-1$ and then we can apply the above lemma as the line segment between every pair of points in $B_i \times C_i$ intersects $L_{\boldsymbol{p}_1 \boldsymbol{p}_i}$ since $L_{\boldsymbol{p}_1 \boldsymbol{p}_i}$ separates B_i and C_i and $\boldsymbol{p}_1, \boldsymbol{p}_i$ are vertices of $\operatorname{conv}(A)$. Then we have a (perfect) pairing $(\boldsymbol{b}_1, \boldsymbol{c}_1), \dots, (\boldsymbol{b}_{n-1}, \boldsymbol{c}_{n-1})$ where for any two pairs the segments do not intersect, but each intersects $L_{\boldsymbol{p}_1 \boldsymbol{p}_i}$. Then the partition $\{\{\boldsymbol{b}_1, \boldsymbol{c}_1\}, \dots, \{\boldsymbol{b}_{n-1}, \boldsymbol{c}_{n-1}\}, \{\boldsymbol{p}_1, \boldsymbol{p}_i\}\}$ is a partition which induces the star graph St_n. For an example of this case and how to partition the points, see Fig. 9.

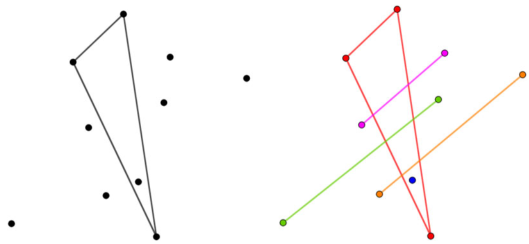


Fig. 10 In the second case, we find a central dividing triangle of a given point configuration. Then we pair off as many points on opposite sides of the triangle as possible using Lemma 3.2, and make points in the interior of the triangle singletons until we have n subsets. Any extra points are added to the subset containing the central dividing triangle

The second case is that there does not exist such a vertex $\boldsymbol{p}_i$. In this case, we find an index i such that $|B_i| > |C_i|$ and $|B_{i+1}| < |C_{i+1}|$. Set $D = \{\boldsymbol{p}_1, \boldsymbol{p}_i, \boldsymbol{p}_{i+1}\}$ and notice that $\operatorname{conv}(D)$ must contain at least one point of A in its interior. D will form the center vertex of our star graph. See Fig. 10 for a depiction of this central triangle.

To construct the remaining subsets of our partition, we first count the number of points in each of the B_i and C_i. If we let I denote the interior of $B_i \setminus B_{i+1}$, then we see that $|B_{i+1}| = |B_i| - |I| - 1$ and $|C_{i+1}| = |C_i| + |I| + 1$. This is because at each i, we are moving the points from the interior of $\operatorname{conv}(\{\boldsymbol{p}_1, \boldsymbol{p}_i, \boldsymbol{p}_{i+1}\})$ and the point $\boldsymbol{p}_i$ from B_i to C_{i+1}.

Now we will assemble pairs of points, one from B_{i+1} and one from C_i, to form disjoint segments which will intersect $\operatorname{conv}(D)$ using Lemma 3.2 with separating line segment $L_{\boldsymbol{p}_1 \boldsymbol{p}_{i+1}}$. If we let $m = \min(|B_{i+1}|, |C_i|)$, we pair off the points, $\{\{b_1, c_1\}, \ldots, \{b_m, c_m\}\}$, so that the segments between any two points do not intersect but each intersects the central triangle. Note that this set may be empty as one of $|B_{i+1}|$ and $|C_i|$ may be zero. Then to fill out the remaining subsets, we add singleton sets $\{x_1\}, \ldots, \{x_{n-1-m}\}$ from the interior I. Each x_i is a singleton set which intersects $\operatorname{conv}(D)$, but will not intersect any of the segments since the points are in general position. After constructing these sets, there may be remaining points (left over unmatched points in either B_{i+1} or C_i), Y, which are still unassigned. These can be added to the set D without introducing new intersections. Then our final partition is $\{D \cup Y, \{b_1, c_1\}, \ldots, \{b_m, c_m\}, \{x_1\}, \ldots, \{x_{n-1-m}\}\}$.

This procedure will work provided that there are enough points in the interior I and on either side of the triangle $\operatorname{conv}(\{p_1, p_i, p_{i+1}\})$ to have $n-1$ sets whose convex hulls intersect $\operatorname{conv}(D)$. Therefore, if we can show that $m+|I| = \min(|B_{i+1}|, |C_i|)+|I| \geq n-1$, we will be done. To see this note that for any i, we have $|B_i|+|C_i|+2 = 2n$ just by counting the points in each set. Then since $|B_{i+1}| = |B_i| - |I| - 1$ and $|B_i| > |C_i|$, we can write

$$|B_{i+1}| > |C_i| - |I| - 1 = 2n - 3 - |B_i| - |I|. \tag{1}$$

Substituting $|B_i| = |B_{i+1}| + |I| + 1$ in (1) and rearranging, we get $|B_{i+1}| > n - 2 - |I| \geq n - 1 - |I|$. Similarly, using that $|C_{i+1}| > |B_{i+1}|$, $|B_{i+1}| = |B_i| - |I| - 1$, and $|C_{i+1}| = |C_i| + |I| + 1$, we get $|C_i| > |B_i| - 2|I| - 2$. Since $|B_i| + |C_i| + 2 = 2n$, we can finally write $|C_i| > n - 2 - |I| \geq n - 1 - |I|$. Then we have that the number of intersections we can have equals $\min(|B_{i+1}, |C_i|) + |I| \geq n - 1$ which is exactly enough to form a nerve complex equal to the star graph on n vertices. □

Proof of Theorem 1.5 (B) As a consequence of Lemma 1.6, when enumerating partition induced graphs it is enough to consider the partitions of point sets of combinatorial types. We can check whether a given simplex complex is 2-partition induced on a representative for each order type.

To complete part (B) we rely on an explicit computer enumeration of all order types on small point set provided by [1]. It turns out, that the point configuration displayed in Fig. 11 is the only point configuration for which it is impossible to generate P_4. Its specific coordinatization is $A(222, 243)$, $B(238, 13)$, $C(131, 50)$, $D(154, 105)$, $E(166, 145)$, $F(134, 106)$, $G(174, 188)$, $H(18, 51)$. For every other point configuration of eight or less vertices, we found a partition which induced the path graph P_4. From this we assert that $\mathrm{Tv}(P_4, 2) \geq 9$. Since we also found a partition inducing P_4 for every single order type on nine points, we are sure that $\mathrm{Tv}(P_4, 2) = 9$ because in the case of 10 or more points, we can use the weaker bound given in the proof of the second part of Theorem 1.4.

Similarly for the cycle C_4. We have the configuration with coordinates $A(0, 0)$, $B(8, 5)$, $C(18, 3)$, $D(7, 4)$, $E(14, 5)$, $F(10, 8)$, $G(11, 7)$, $H(14, 17)$, $I(11, 6)$, $J(12, 12)$, which gives the desired lower bound. The upper bound is given by following the proof of Theorem 1.2 (B), except starting with any set of 13 points (the bound given in the theorem is higher since it accounts for divisibility issues that can occur in certain cases). □

4 Final Remarks and Open Problems

In this paper we generalized Tverberg's theorem by showing that many simplicial complexes, which we call Tverberg complexes, are always induced by the nerve of some partition of any sufficiently large set of points in a fixed dimension. The study of Tverberg complexes is part of the study of simplicial complexes that are nerves of convex sets. We conclude by listing a few open questions:

1. What is the exact value of $\mathrm{Tv}(T_n, d)$ where T_n is a tree with n nodes? Is $(d+1)(n-1) + 1$ the correct value? What about the case $d = 2$?
2. What is the computational complexity of determining if a point configuration can partition induce a given simplicial complex?
3. What is the computational complexity of computing the Tverberg numbers of a given Tverberg complex, such as a tree?
4. Are there topological versions of Tverberg theorems for other simplicial complexes?
5. Is there a graph G that is not 3-Tverberg?
6. Is there a simplicial complex K that is not d-Tverberg for any d?

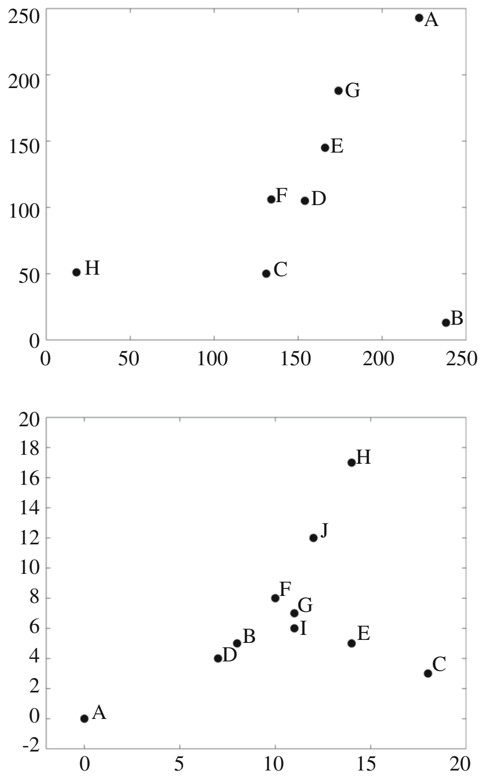


Fig. 11 Two point configuration which cannot be partitioned to induce, respectively, P_4 (top; on eight points) and C_4 (bottom; on ten points)

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Appendix A: Proofs of Auxiliary Lemmas

We include proofs of some supplementary lemmas mentioned in the introduction.

Proof of Lemma 1.3 Suppose by contradiction that $\mathrm{Tv}(K, d) < 2n$. Let $S \subset \mathbb{R}^d$ be a set of points in convex position with $|S| = \mathrm{Tv}(K, d)$. By the pigeonhole principle, if we partition S into n disjoint subsets, there must be at least one subset that is a singleton $\{\boldsymbol{x}\}$. Since K is connected, the node corresponding to the singleton $\{\boldsymbol{x}\}$ is connected, by an edge, to at least one other node, implying that $\{\boldsymbol{x}\}$ is in the convex hull of another subset. However, this is a contradiction as the points are in convex position. $\square$

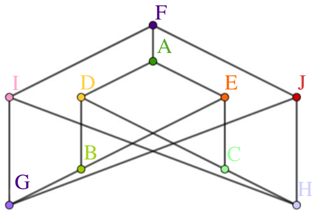


Fig. 12 Graph K

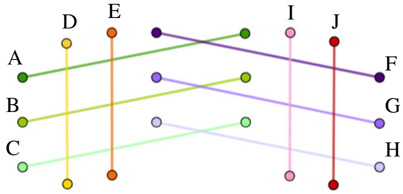
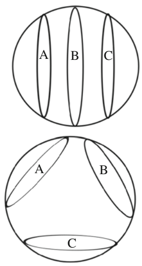


Fig. 13 Partitioned point set with nerve K

Proof of Lemma 1.6 To show that $\mathcal{N}^1(\mathcal{P}) = \mathcal{N}^1(\sigma(\mathcal{P}))$ it suffices to show that $\mathrm{conv}(P_{i_1}) \cap \mathrm{conv}(P_{i_2}) \neq \emptyset$ if and only if $\mathrm{conv}(\sigma(P_{i_1})) \cap \mathrm{conv}(\sigma(P_{i_2})) \neq \emptyset$ for all $i_1, i_2 \in [n]$. Suppose $\mathrm{conv}(P_{i_1}) \cap \mathrm{conv}(P_{i_2}) \neq \emptyset$. Then they contain respectively P'_{i_1} and P'_{i_2}, which are an inclusion minimal Radon partition of S_1. Since σ is an order-preserving bijection, σ is an isomorphism between oriented matroids (see, for instance, [21]) determined by S_1 and S_2. The minimal Radon partitions in S_1 correspond to the circuits of the oriented matroids and therefore are preserved under σ. Thus $\mathrm{conv}(\sigma(P'_{i_1})) \cap \mathrm{conv}(\sigma(P'_{i_2})) \neq \emptyset$. The reverse implication is shown by the reasoning applied to σ^{-1}. □

As we mentioned in the introduction, the graph K in Fig. 12 is 2-partition induced (in particular, by the partitioned point set in Fig. 13), but not 2-Tverberg, as implied by the following lemma:

Lemma A.1 *Suppose S is any set of points in convex position in $\mathbb{R}^2$. Then the graph K in Fig. 12 is not partition induced on S.*



Proof We note that since K is a triangle free graph, it suffices to show that it is not the intersection graph of any partition of points in convex position. We argue by contradiction. Suppose that there is a set of points in convex position partitioned so that the graph above is their intersection graph. By Lemma 1.6 we may assume the points are arranged on the boundary of a disc $\mathcal{D}$. Call the convex hull of the points corresponding to each node i by region i. In the rest of the proof of Lemma A.1, we will rely on the following.

Claim *Consider the independent set of nodes* $\{A, B, C\}$ *in Fig. 12. Up to exchanging their labels (note that the graph is symmetric about* A, B, C*), there are two possible arrangements of the regions A, B, and C, pictured in Fig. 5.*

Proof of the claim The region $\mathcal{M} - B$ has two connected components. If regions A and C lie in different connected components of $\mathcal{M} - B$, then regions A, B, and C must be arranged as in Fig. 5. Otherwise, A and C lie in the same connected component, say $\mathcal{N}$, of $\mathcal{M} - B$. If we walk clockwise around the boundary of $\mathcal{N}$, we can only alternate twice between being in regions A and C, reducing to the two possibilities shown. □

By the claim, we see that A, B, and C must be arranged (up to symmetry) as in one of the two cases pictured above. If they are arranged as in Fig. 5, note that regions E and F both intersect regions A, B, and C. In that case it is easy to see that regions E and F must intersect, which is a contradiction.

If the regions are arranged as in Fig. 5, consider those regions D, F, G, and H. Note that the region D intersects A, B, C. Also, region F is disjoint from all the regions B through H, while is intersecting A. Similarly, region G is disjoint from all the regions A through H except B. Also region H is disjoint from all the regions A through H except C. Considering the two cases: F, G, H lie in the same connected component of $\mathcal{M} - D$, or F, G, H lie in different connected components of $\mathcal{M} - D$, it is easy to see that, in both cases, F, G, and H must be arranged as A, B, and C are in Fig. 5. Then I, J are disjoint but both intersect F, G, and H, which is a contradiction by the argument above. Thus K cannot be the nerve of a set of points in convex position. □

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