SIAM J. MATH. ANAL.
Vol. 52, No. 3, pp. 3114–3130

ONE-DIMENSIONAL SCALAR CONSERVATION LAWS WITH REGULATED DATA*

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Abstract. We show that the Kružkov solution of the Cauchy problem for a scalar conservation law in one spatial dimension propagates regulated initial data into regulated solutions at later times. The proof is based on standard front-tracking and an extension, due to Fraňková, of Helly's selection principle.

Key words. scalar conservation law, regulated initial data, one dimensional

AMS subject classifications. 35L65, 35B65

DOI. 10.1137/19M1267957

1. Introduction.

It is well known that, in any space dimension, the spatial variation of the unique admissible Kružkov solution of a scalar conservation law is nonincreasing with time [3, 7]. Furthermore, in the one-dimensional case, the use of the space of functions with bounded variation (BV) as data space is particularly convenient: it is large enough to include discontinuous functions, a sine qua non for any discussion of global existence, while at the same time being equipped with a user-friendly criterion for strong compactness (Helly's selection principle); the method of front-tracking offers a direct, self-contained existence proof for BV solutions; and the existence of one-sided traces along shock waves is guaranteed due to the propagation of BV-regularity.

The goal of the present work is to extend the approach via front-tracking to the larger class of *regulated* L^∞ initial data for one-dimensional scalar conservation laws. By definition, a function $v : \mathbb{R} \to \mathbb{R}$ is regulated provided it possesses one-sided limits at every point, i.e.,

$$v(x+) := \lim_{y\downarrow x} v(y) \qquad \text{and} \qquad v(x-) := \lim_{y\uparrow x} v(y) \tag{1.1}$$

exist as finite numbers for all $x \in \mathbb{R}$. We note that regulated solutions provide the natural setting for the study of generalized characteristics for conservation laws; see [2, 3].

Recall that, strictly speaking, the Kružkov solution (i.e., the unique admissible solution) of a scalar conservation law is an equivalence class of almost everywhere equal functions. We shall show that, whenever the initial data $\bar{u}$ are regulated, there is a version $u(t, x)$ of the Kružkov solution to

$$\partial_t u(t,x) + \partial_x[f(u(t,x))] = 0, \qquad t > 0,\ x \in \mathbb{R} \tag{1.2}$$

$$u(0,x) = \bar{u}(x), \tag{1.3}$$

*Received by the editors June 12, 2019; accepted for publication (in revised form) May 22, 2020; published electronically June 30, 2020.
https://doi.org/10.1137/19M1267957
Funding: The work of the first author was partially supported by the National Science Foundation grant DMS-1813283.
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with the property that $u(t,x)$ is a regulated function of x at all times $t > 0$. The proof makes use of an extension, due to Fraňková [5], of Helly's selection principle to the space of regulated functions.

The key ingredient, introduced by Fraňková [5], is the notion of *ε-variation* and its role in providing a criterion for relative compactness in the space of regulated functions. (The ε-variation of a bounded function v on a compact interval $[a,b]$ is defined as the infimum of variations of BV functions uniformly ε-close to v; see Definition 2.3 below.) Fraňková [5] shows that a function v defined on a compact interval is regulated if and only if its ε-variation is finite for every $\varepsilon > 0$. Our main result generalizes the property of nonincrease of variation along BV solutions to (1.2) by providing the existence of a version of the Kružkov solution whose ε-variation is nonincreasing in time whenever the data are regulated and of compact support.

The analysis naturally divides into two parts. In the first part we verify that the ε-variation of a solution does not increase in time when the initial data belong to BV. This is done by monitoring the ε-variation in front-tracking approximations that converge to the Kružkov solution. Apart from the ε-variation aspect, the convergence argument is standard; see Oleĭnik [8]. We stress that it uses Helly's selection principle twice: first to define the solution candidate at rational times, and then again to define its extension to all times $t > 0$. This last step requires some care: the solution candidate is defined separately at each time as a pointwise everywhere limit, and it is necessary to verify that the resulting function is jointly measurable in (t,x). (For details see pages 122–123 in [8] or the proof of Theorem 2.4 in [1].) The standard argument depends on the fact that BV regularity implies that the solution operator $t \mapsto u(t,\cdot)$ is uniformly Lipschitz continuous as a map from $\mathbb{R}^+_{0,t}$ into $L^1_{loc}(\mathbb{R}_x)$.

The only new aspect in our treatment of the BV case is to verify that the ε-variation is nonincreasing along front-tracking approximations in this case. Since the solution is given as a pointwise everywhere limit of the front-tracking approximations, a result of Fraňková (Proposition 2.7 below) implies that the same conclusion holds for the exact solution as well.

The second part of the analysis is to argue that nonincrease of ε-variation continues to hold when the initial data are merely regulated. For this we are not able to simply repeat the argument outlined above: BV regularity was there used in an essential manner to obtain Lipschitz continuity of the solution operator. Instead we shall exploit the following two facts. First, the Kružkov solution operator has the L^1-contraction property, and second, the Kružkov solution for general L^∞ data belongs to $C^0(\mathbb{R}^+_0; L^1_{loc}(\mathbb{R}))$, a result which is a consequence of the existence theory based, e.g., on vanishing viscosity approximations (see sections 6.2 and 6.3 in [3]).

This last aspect of the argument renders our approach not fully self-contained, i.e., it is not entirely based on front-tracking. On the other hand, other methods of constructing the solution do not easily provide the type of "fine" pointwise information about the Kružkov solution which is required for monitoring its ε-variation. Instead, we build a version u of the Kružkov solution "by hand" via front-tracking, essentially retracing the steps in the standard proof of existence for BV data. The differences are that we now apply Fraňková's extension of Helly's selection principle, rather than the latter, to obtain convergence and that we exploit L^1-contractivity and mere continuity of the solution operator to show that the regulated solution candidate we build is indeed a version of the Kružkov solution. This last step again requires an argument for joint measurability of our solution candidate. By insisting on right-continuity of the solution candidate at each time, we exploit the fact that it may be realized as the limit of spatial Steklov averages.

The rest of the paper is organized as follows. In section 2 we introduce regulated functions and the notion of ε-variation. We also record some known properties, state two key results from Fraňková's work [5], and formulate our main result. Section 3 establishes a series of results on step functions, their ε-variation, and how it changes when values are added or removed. These results are then used in section 4 to show that the ε-variation does not increase along a suitable version of the Kružkov solution when the initial data belong to BV. In section 5 we apply Fraňková's extension of Helly's selection principle to build a solution candidate with nonincreasing ε-variation. Finally, it is shown that this is a version of the Kružkov solution, thereby establishing the main result.

2. Regulated functions, ε-variation, and main result.

For one-sided limits we use the notations in (1.1) above.

DEFINITION 2.1. *Let I be any interval in $\mathbb{R}$ (i.e., I may be open, closed, half open, finite, or infinite). A function $u : I \to \mathbb{R}$ is* regulated *on I provided its right and left limits exist (as finite numbers) at all points in the interior of I and it has a finite right limit at the left endpoint and a finite left limit at the right endpoint, whether or not these endpoints are finite or belong to I. The class of regulated functions on I is denoted $\mathcal{R}(I)$; we write $\mathcal{R}$ for $\mathcal{R}(\mathbb{R})$ and $\mathcal{R}[a,b]$ for $\mathcal{R}([a,b])$.*

DEFINITION 2.2. *With I as above, a function $u : I \to \mathbb{R}$ is of* bounded (pointwise) variation *provided*

$$\operatorname{var} u := \sup \sum_{i=1}^{k} |u(x_i) - u(x_{i-1})| < \infty, \tag{2.1}$$

where the sup *is over all $k \in \mathbb{N}$ and all finite selections of points $x_0 < x_1 < \cdots < x_k$ in I. We denote the set of such functions by* $\mathrm{BV}(I)$ *(writing* $\mathrm{BV}[a,b]$ *for* $\mathrm{BV}([a,b])$*).*

In what follows a and b always denote finite numbers with $a < b$. Also, throughout, $\|\cdot\|$ denotes sup-norm on $[a,b]$:

$$\|u\| := \sup_{a \le x \le b} |u(x)|.$$

DEFINITION 2.3. *For any function $u : [a,b] \to \mathbb{R}$ and any $\varepsilon > 0$, its* ε-variation *on $[a,b]$ is given by*

$$\varepsilon\text{-}\operatorname{var} u := \inf\{\operatorname{var} v \,:\, v \in \mathcal{V}(u;\varepsilon)\}, \tag{2.2}$$

where

$$\mathcal{V}(u;\varepsilon) := \{v \in \mathrm{BV}[a,b] \,:\, \|v - u\| \le \varepsilon\}. \tag{2.3}$$

If need be, we use the notations $\varepsilon\text{-}\operatorname{var}_a^b u$ and $\mathcal{V}(u;\varepsilon;a,b)$ to emphasize the dependence on the interval $[a,b]$.

We note that if $u \in \mathrm{BV}[a,b]$, then

$$\varepsilon\text{-}\operatorname{var}_a^b u \le \operatorname{var}_a^b u, \tag{2.4}$$

and that the inequality is typically strict. E.g., for the one-step function

$$u(x) = \begin{cases} u_L & x < 0, \\ u_R & x \ge 0, \end{cases}$$

we have

$$\varepsilon\text{-var}\, u = \max(|u_L - u_R| - 2\varepsilon, 0).$$

DEFINITION 2.4. *A set $\mathcal{A} \subset \mathcal{R}[a,b]$ has* uniformly bounded ε-variations *provided that for every $\varepsilon > 0$ there is a finite number K_ε such that ε-var $u \le K_\varepsilon$ for all $u \in \mathcal{A}$.*

For $u : [a,b] \to \mathbb{R}$ we let $J(u)$ denote the jump set of u, i.e.,

$$J(u) := \{x \in [a,b] \,|\, \text{at least one of } u(x+) \text{ or } u(x-) \text{ differs from } u(x)\}.$$

A function $u : [a,b] \to \mathbb{R}$ is a *step function* provided there is a finite, increasing sequence of points $x_0 = a < x_1 < \cdots < x_{N-1} < x_N = b$ such that u is constant on each of the open intervals (x_i, x_{i+1}), $i = 0, \dots, N$. The following results about regulated functions are known (see [5, 4]):

(R1) if $u \in \mathcal{R}$, then $J(u)$ is a countable set;
(R2) a function $u : [a,b] \to \mathbb{R}$ is regulated if and only if $\varepsilon\text{-var}_a^b\, u < \infty$ for all $\varepsilon > 0$;
(R3) a function $u : [a,b] \to \mathbb{R}$ is regulated if and only if it is the uniform limit of a sequence of step functions on $[a,b]$.

Remark 2.5. It is immediate to verify that if $u : [a,b] \to \mathbb{R}$ is regulated and right-continuous (i.e., $u(x) = u(x+)$ for all $x \in [a,b)$), then the step functions in (R3) may be chosen as right-continuous as well.

Fraňková's extension of Helly's selection principle takes the following form.

THEOREM 2.6 (see [5, Theorem 3.8]). *Assume that the sequence (v_n) in $\mathcal{R}[a,b]$ has uniformly bounded ε-variations and $(|v_n(a)|)$ is a bounded sequence. Then there are a subsequence (v_{n_k}) of (v_n) and a function $v_0 \in \mathcal{R}[a,b]$ such that $v_{n_k}(x) \to v_0(x)$ for every $x \in [a,b]$.*

We shall also make use of the following result from [5].

PROPOSITION 2.7 (see [5, Proposition 3.6]). *Assume that the sequence (v_n) in $\mathcal{R}[a,b]$ has uniformly bounded ε-variations. If $v_n(x) \to v(x)$ for each $x \in [a,b]$, then $v(x) \in \mathcal{R}[a,b]$ and*

$$\varepsilon\text{-var}_a^b\, v \le \liminf_n\, \varepsilon\text{-var}_a^b\, v_n \qquad \textit{for every } \varepsilon > 0.$$

The main result we establish is the following.

THEOREM 2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function, and assume $\bar{u} \in \mathcal{R}$ has compact support contained in (a,b). Let L denote the Lipschitz constant of $f|_{range(\bar{u})}$. Then there is a version $u = u(t,x)$ of the Kružkov solution of the Cauchy problem* (1.2)–(1.3) *(i.e., an element in its $L^1_{loc}(\mathbb{R}_t^+ \times \mathbb{R}_x)$-equivalence class) which satisfies*

$$\varepsilon\text{-var}_{a-Lt}^{b+Lt}\, u(t,\cdot) \le \varepsilon\text{-var}_a^b\, \bar{u} \qquad \textit{for all } \epsilon > 0 \textit{ and all } t \ge 0.$$

We note that the version $u(t,x)$ we construct in section 5 of the Kružkov solution is right-continuous at all times: $u(t,x+) = u(t,x)$ for all $x \in \mathbb{R}$, $t \ge 0$.

3. Preliminary results. Let $u \in \mathcal{R}$; it follows from Definition 2.1 that the function

$$u_r(x) := u(x+) \equiv \lim_{y \downarrow x} u(y) \tag{3.1}$$

is well defined for all $x \in \mathbb{R}$. According to (R1), $u_r(x) = u(x)$ except for at most a countable set of x-values.

LEMMA 3.1. *Assume $u \in \mathcal{R}$, and define the function $u_r(x)$ as in* (3.1). *Then the following hold.*

(a) *For any compact interval $[a,b]$ where b is a point of continuity of u, and for any $\varepsilon > 0$, we have $\varepsilon\text{-var}_a^b\, u_r \le \varepsilon\text{-var}_a^b\, u$. In particular, according to* (R2), *$u_r \in \mathcal{R}[a,b]$.*

(b) *u_r can be realized as the limit of right Steklov averages of u, i.e.,*

$$u_r(x) = \lim_{\delta \downarrow 0} u^\delta(x),$$

where

$$u^\delta(x) := \frac{1}{\delta}\int_x^{x+\delta} u(\xi)\, d\xi.$$

(c) *u_r is right-continuous at all points $x \in \mathbb{R}$.*

Proof. For (a) we first note that u is continuous at $x = b$ by assumption, so that $u_r(b) = u(b)$. Now, for $\varepsilon > 0$ fixed, we choose any $v \in \mathcal{V}(u;\varepsilon;a,b)$ and define the function $\tilde{v} : [a,b] \to \mathbb{R}$ by

$$\tilde{v}(x) := \begin{cases} v(x+) & x \in [a,b), \\ v(b) & x = b. \end{cases}$$

Then, for $x \in [a,b)$ we have

$$|u_r(x) - \tilde{v}(x)| = \lim_{y \downarrow x} |u(y) - v(y)| \le \varepsilon,$$

while for $x = b$ we have

$$|u_r(b) - \tilde{v}(b)| = |u(b) - v(b)| \le \varepsilon.$$

Next, fix any partition $a = x_0 < \cdots < x_{N-1} < x_N = b$ and any $\mu > 0$. Consider any other partition $a < x'_0 < \cdots < x'_{N-1} < x'_N := b$. We have

$$\sum_{i=1}^N |\tilde{v}(x_i) - \tilde{v}(x_{i-1})| \le \sum_{i=1}^N \big(|\tilde{v}(x_i) - v(x'_i)| + |v(x'_i) - v(x'_{i-1})| + |v(x'_{i-1}) - \tilde{v}(x_{i-1})|\big) \tag{3.2}$$

$$\le \text{var}_a^b\, v + \sum_{i=1}^{N-1} |\tilde{v}(x_i) - v(x'_i)| + \sum_{i=1}^{N} |v(x'_{i-1}) - \tilde{v}(x_{i-1})|, \tag{3.3}$$

where we have used that $|\tilde{v}(x_N) - v(x'_N)| = |\tilde{v}(b) - v(b)| = 0$. Since $\tilde{v}(x) = v(x+)$ on $[a,b)$, by choosing each x'_i, $i = 0,\dots,N-1$, sufficiently close to and strictly larger than each x_i, we can arrange that

$$\sum_{i=1}^{N-1} |\tilde{v}(x_i) - v(x'_i)| + \sum_{i=1}^{N} |v(x'_{i-1}) - \tilde{v}(x_{i-1})| < \mu.$$

As $\mu > 0$ is arbitrary it follows from (3.3) that

$$\sum_{i=1}^N |\tilde{v}(x_i) - \tilde{v}(x_{i-1})| \le \text{var}_a^b\, v$$

for any partition $\{x_i\}_{i=0}^N$ of $[a,b]$ (with $x_0 = a$ and $x_N = b$), and thus $\operatorname{var}\tilde{v} \le \operatorname{var} v$. This shows that $\tilde{v} \in \mathrm{BV}[a,b]$ so that $\tilde{v} \in \mathcal{V}(u_r;\varepsilon;a,b)$. Thus, for each $v \in \mathcal{V}(u;\varepsilon;a,b)$, there is a $\tilde{v} \in \mathcal{V}(u_r;\varepsilon;a,b)$ with $\operatorname{var}\tilde{v} \le \operatorname{var} v$, showing that $\varepsilon\text{-var}_a^b\, u_r \le \varepsilon\text{-var}_a^b\, u$, establishing part (a).

For (b), fix any point $x \in \mathbb{R}$ and any $\delta > 0$. Since $u \in \mathcal{R}$, there is an $h > 0$ with the property that

$$|u(\xi) - u_r(x)| < \delta \qquad \text{whenever } \xi \in (x, x+h). \tag{3.4}$$

Thus,

$$|u^\delta(x) - u_r(x)| \le \frac{1}{\delta}\int_x^{x+\delta} |u(\xi) - u_r(x)|\, d\xi \le \delta$$

whenever $0 < \delta < h$.

Finally, for (c), fix any $x \in \mathbb{R}$ and any $\delta > 0$. Then there is an $h > 0$ such that (3.4) holds. We proceed to show that $|u_r(x+y) - u_r(x)| < 2\delta$ whenever $0 < y < h$. Fix any such y; as $u \in \mathcal{R}$, there is a $k > 0$ with the property that

$$|u_r(x+y) - u(\eta)| < \delta \qquad \text{whenever } \eta \in (x+y, x+y+k). \tag{3.5}$$

Set $\mu := \min\{h, y+k\}$ such that $y < \mu \le h$. We then have $(x+y, x+\mu) \subset (x, x+h) \cap (x+y, x+y+k)$. Thus, using any $\eta \in (x+y, x+\mu)$, we have from (3.4)–(3.5) that

$$|u_r(x+y) - u_r(x)| \le |u_r(x+y) - u(\eta)| + |u(\eta) - u_r(x)| < 2\delta$$

whenever $0 < y < h$, showing that u_r is right-continuous at x. ◻

In preparation for monitoring the ε-variation of piecewise constant approximations to scalar conservation laws, we consider ε-variation of step functions.

Lemma 3.2. *Let $u : [a,b] \to \mathbb{R}$ be a right-continuous step function with* $\operatorname{supp} u \subset (a,b)$*. Then, for any $\varepsilon > 0$, we have that*

$$\varepsilon\text{-var}\, u = \inf\{\operatorname{var} z \mid z \in \mathcal{V}_s(u;\varepsilon)\}, \tag{3.6}$$

where

$$\begin{aligned} \mathcal{V}_s(u;\varepsilon) \equiv \mathcal{V}_s(u;\varepsilon;a,b) :=& \{z : [a,b] \to \mathbb{R} \mid z \text{ is a right-continuous step function,} \\ & J(z) \subset J(u), \text{ and } \|z - u\| \le \varepsilon\}. \end{aligned} \tag{3.7}$$

Furthermore, the infimum in (3.6) *is attained.*

Proof. Since $\mathcal{V}(u;\varepsilon) \supset \mathcal{V}_s(u;\varepsilon)$, it is immediate that $\le$ holds in (3.6). To establish the opposite inequality, we shall show that for each $v \in \mathcal{V}(u;\varepsilon)$ there is a $z \in \mathcal{V}_s(u;\varepsilon)$ with $\operatorname{var} v \ge \operatorname{var} z$; equality in (3.6) then follows from definition (2.2) of $\varepsilon\text{-var}\, u$. For this let $x_1 < \cdots < x_N$ be the jump loci of u. (If $u \equiv 0$ the result is obvious.) Since u is right-continuous with $\operatorname{supp} u \subset (a,b)$, we have $x_0 := a < x_1$, $x_N < x_{N+1} := b$ and that $u \equiv 0$ on $[x_N, b]$. Define the points

$$\xi_i := \frac{1}{2}(x_i + x_{i+1}) \qquad \text{for } i = 0, \dots, N.$$

Given $v \in \mathcal{V}(u;\varepsilon)$ we define $z : [a,b] \to \mathbb{R}$ to be the right-continuous step function which takes the value $v(\xi_i)$ on $[x_i, x_{i+1})$ for each $i = 0, \dots, N-1$ and which takes the value $v(\xi_N)$ on $[x_N, b]$, i.e.,

$$z(x) = \sum_{i=0}^{N-1} z_i \mathbf{1}_{[x_i,x_{i+1})}(x) + z_N \mathbf{1}_{[x_N,b]}(x)\,, \quad \text{where } z_i = v(\xi_i) \text{ for } i = 0,\dots,N.$$

It follows that $z \in \mathcal{V}_s(u;\varepsilon)$ and that $\operatorname{var} v \geq \operatorname{var} z$.

To show that the infimum in (3.6) is attained, let

$$u(x) = \sum_{i=0}^{N} u_i \mathbf{1}_{[x_i,x_{i+1})}(x), \qquad x \in [a,b],$$

and observe that the expression on the right-hand side of (3.6) agrees with the infimum of the continuous function

$$\mathbb{R}^{N+1} \ni z = (z_0,\dots,z_N) \mapsto \sum_{i=0}^{N-1} |z_{i+1} - z_i|$$

when restricted to the compact set

$$\prod_{i=0}^{N} [u_i - \varepsilon, u_i + \varepsilon].$$

By continuity and compactness this infimum is attained. ∎

Remark 3.3. Assume u and $\tilde{u}$ are two right-continuous step functions that take the same values in the same order but possibly on different intervals, i.e.,

$$u(x) = \sum_{i=0}^{N} u_i \mathbf{1}_{[x_i,x_{i+1})}(x) \quad \text{and} \quad \tilde{u}(x) = \sum_{i=0}^{N} u_i \mathbf{1}_{[\tilde{x}_i,\tilde{x}_{i+1})}(x)\,, \quad x \in [a,b]\,,$$

with $a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$ and $a = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_N < \tilde{x}_{N+1} = b$. Then

$$\varepsilon\text{-var}\, u(x) = \varepsilon\text{-var}\, \tilde{u}(x)\,.$$

This follows from (3.6), as for any $z \in \mathcal{V}_s(u;\epsilon)$, there is $\tilde{z} \in \mathcal{V}_s(\tilde{u};\epsilon)$ such that

$$\operatorname{var} z = \sum_{i=1}^{N} |z(x_i+) - z(x_i-)| = \sum_{i=1}^{N} |\tilde{z}(\tilde{x}_i+) - \tilde{z}_i(\tilde{x}_i-)| = \operatorname{var} \tilde{z}\,.$$

The next result shows that in approximating a regulated function u with a step function that takes values of u, the ε-variation will not increase.

LEMMA 3.4. *Let $u \in \mathcal{R}[a,b]$ with $\operatorname{supp} u \subset (a,b)$, and fix any sequence $x_1 < \cdots < x_N$ in (a,b) with the property that $\operatorname{supp} u \subset (a,x_N)$. Let $\mathcal{S}[u] : [a,b] \to \mathbb{R}$ be the right-continuous step function which coincides with u at the left endpoints of each subinterval, i.e.,*

$$\mathcal{S}[u](x) := \begin{cases} u(x_i) & x \in [x_{i-1},x_i),\ i = 1,\dots,N, \\ u(x_N) & x \in [x_N,b], \end{cases} \tag{3.8}$$

where $x_0 := a$. (Note that $u(x_N) = 0$ according to the assumptions on u.) Then, for any $\epsilon > 0$ we have

$$\varepsilon\text{-var}\, \mathcal{S}[u] \leq \varepsilon\text{-var}\, u. \tag{3.9}$$

Proof. Fix the sequence $x_1 < \cdots < x_N$ and $\varepsilon > 0$. According to definition (2.2) and Lemma 3.2, (3.9) amounts to

$$\inf\{\operatorname{var} z \mid z \in \mathcal{V}_s(\mathcal{S}[u]; \varepsilon; a, b)\} \leq \inf\{\operatorname{var} v \mid v \in \mathcal{V}(u; \varepsilon; a, b)\}. \tag{3.10}$$

To establish this inequality we fix any $v \in \mathcal{V}(u; \varepsilon; a, b)$ and proceed to generate a $z \in \mathcal{V}_s(\mathcal{S}[u]; \varepsilon; a, b)$ with $\operatorname{var} z \leq \operatorname{var} v$. For this, consider the function $\mathcal{S}[v]$, defined as in (3.8) with u replaced by v. It is immediate to verify that $\mathcal{S}[v] \in \mathcal{V}_s(\mathcal{S}[u]; \varepsilon; a, b)$ (since $\mathcal{S}[v]$ and $\mathcal{S}[u]$ are both constant on each subinterval, their values there are the values of v and u, respectively, at the left endpoints of the subintervals, and $\|v - u\| \leq \varepsilon$) and that $\operatorname{var} \mathcal{S}[v] \leq \operatorname{var} v$ (since $\mathcal{S}[v]$ takes on values of v). It follows that

$$\inf\{\operatorname{var} z \mid z \in \mathcal{V}_s(\mathcal{S}[u]; \varepsilon; a, b)\} \leq \operatorname{var} \mathcal{S}[v] \leq \operatorname{var} v.$$

As $v \in \mathcal{V}(u; \varepsilon; a, b)$ is arbitrary, (3.10) follows. ∎

We conclude this section by showing that the ε-variation of a step function does not increase when some of its values are omitted and that it remains the same if values are added in a monotone manner. This is the key property which will be used below to show that approximate solutions of (1.2)–(1.3) generated via front-tracking do not increase in ε-variation as time increases.

LEMMA 3.5. *Assume $u : [a, b] \to \mathbb{R}$ is a right-continuous step function with $\operatorname{supp} u \subset (a, b)$. Let $x_1 < \cdots < x_N$ be the jump loci of u, and set $x_0 := a$ and $x_{N+1} := b$ such that*

$$u(x) = \sum_{i=0}^{N} u_i \mathbf{1}_{[x_i, x_{i+1})}(x), \qquad x \in [a, b], \tag{3.11}$$

for constants u_i (in particular $u_0 = u_N = 0$). Consider any sequence of $N + 1$ points $\tilde{x}_1 < \cdots < \tilde{x}_j < \hat{x} < \tilde{x}_{j+1} < \cdots < \tilde{x}_N$ in (a, b) and any value $\hat{u} \in \mathbb{R}$. Then the right-continuous, simple function $\tilde{u} : [a, b] \to \mathbb{R}$ defined by

$$\tilde{u}(x) := \begin{cases} u_i & x \in [\tilde{x}_i, \tilde{x}_{i+1}), \quad i = 0, \dots, j-1, j+1, \dots, N, \\ u_j & x \in [\tilde{x}_j, \hat{x}), \\ \hat{u} & x \in [\hat{x}, \tilde{x}_{j+1}), \end{cases}$$

where $\tilde{x}_0 := a$ and $\tilde{x}_{N+1} := b$, has the following properties: for any $\varepsilon > 0$,

1. $\varepsilon\text{-}\operatorname{var} u \leq \varepsilon\text{-}\operatorname{var} \tilde{u}$;
2. *if $u_j \leq \hat{u} \leq u_{j+1}$ or $u_{j+1} \leq \hat{u} \leq u_j$, then $\varepsilon\text{-}\operatorname{var} u = \varepsilon\text{-}\operatorname{var} \tilde{u}$.*

Proof. Applying Lemma 3.2 to $\tilde{u}$, we obtain a right-continuous, simple function $\tilde{z} : [a, b] \to \mathbb{R}$ with the same jump set as $\tilde{u}$ and satisfying $\|\tilde{z} - \tilde{u}\| \leq \varepsilon$ and $\operatorname{var} \tilde{z} = \varepsilon\text{-}\operatorname{var} \tilde{u}$. Let the $N + 2$ consecutive values (from left to right) taken by $\tilde{z}$ be $z_0, \dots, z_j, \hat{z}, z_{j+1}, \dots, z_N$. We then define the right-continuous, simple function z to have the same jump set as the given function u and taking the $N + 1$ consecutive values $z_0, \dots, z_j, z_{j+1}, \dots, z_N$. Then $\|z - u\| \leq \varepsilon$ such that

$$\begin{aligned} \varepsilon\text{-}\operatorname{var} u &\equiv \inf\{\operatorname{var} v \,:\, v \in \mathrm{BV}[a, b] \,:\, \|v - u\| \leq \varepsilon\} \\ &\leq \operatorname{var} z = \sum_{i=1}^{j} |z_i - z_{i-1}| + |z_{j+1} - z_j| + \sum_{i=j+2}^{N} |z_i - z_{i-1}| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{j} |z_i - z_{i-1}| + |\hat{z} - z_j| + |z_{j+1} - \hat{z}| + \sum_{i=j+2}^{N} |z_i - z_{i-1}| \\
&= \operatorname{var} \tilde{z} = \varepsilon\text{-var}\, \tilde{u},
\end{aligned}
$$

establishing part (1).

For part (2), assume for concreteness that $u_{j+1} \leq \hat{u} \leq u_j$. (The proof is entirely similar when $u_j \leq \hat{u} \leq u_{j+1}$.) Applying Lemma 3.2 to u, we obtain a right-continuous, simple function $z : [a, b] \to \mathbb{R}$ with the same jump set as u and satisfying $\|z - u\| \leq \varepsilon$ and $\operatorname{var} z = \varepsilon\text{-var}\, u$. Let the $N + 1$ consecutive values (from left to right) taken by z be $z_0, \ldots, z_N$.

Claim: we can choose a value $\hat{z}$ such that

(i) $\hat{z}$ is located between z_j and z_{j+1} (possibly being equal to one of these), and

(ii) $|\hat{z} - \hat{u}| \leq \varepsilon$.

Assuming for now that this claim is valid, we define the right-continuous, simple function $\tilde{z}$ as follows:

$$
\tilde{z}(x) := \begin{cases} z_i & x \in [\tilde{x}_i, \tilde{x}_{i+1}), \quad i = 0, \ldots, j-1, j+1, \ldots, N, \\ z_j & x \in [\tilde{x}_j, \hat{x}), \\ \hat{z} & x \in [\hat{x}, \tilde{x}_{j+1}). \end{cases}
$$

In particular, $\tilde{z}$ has the same jump set as $\tilde{u}$. As $\|z - u\| \leq \varepsilon$, and by part (ii) of the claim above, we have $\|\tilde{z} - \tilde{u}\| \leq \varepsilon$. Also, according to part (i) of the claim we get

$$
\begin{aligned}
\varepsilon\text{-var}\, \tilde{u} &= \inf\{\operatorname{var} v \,:\, v \in \mathrm{BV}[a, b],\ \|v - \tilde{u}\| \leq \varepsilon\} \\
&\leq \operatorname{var} \tilde{z} = \sum_{i=0}^{j-1} |z_{i+1} - v_i| + |\hat{z} - z_j| + |z_{j+1} - \hat{z}| + \sum_{i=j+1}^{N-1} |z_{i+1} - v_i| \\
&= \sum_{i=0}^{j-1} |z_{i+1} - v_i| + |z_{j+1} - z_j| + \sum_{i=j+1}^{N-1} |z_{i+1} - v_i| = \operatorname{var} z = \varepsilon\text{-var}\, u.
\end{aligned}
$$

Together with part (1) of the lemma, we obtain part (2), modulo the claim above. For this we observe that if $\hat{u}$ lies between z_j and z_{j+1}, then $\hat{z} := \hat{u}$ obviously satisfies both parts of the claim. For the case that $\hat{u}$ does not lie between z_j and z_{j+1} we consider separately the cases $z_j \gtrless z_{j+1}$, in each case considering the two possible locations of $\hat{u}$. This gives the following four subcases:

- if $\hat{u} \geq z_j \geq z_{j+1}$, we set $\hat{z} := z_j$;
- if $z_j \geq z_{j+1} \geq \hat{u}$, we set $\hat{z} := z_{j+1}$;
- if $\hat{u} \geq z_{j+1} \geq z_j$, we set $\hat{z} := z_{j+1}$;
- if $z_{j+1} \geq z_j \geq \hat{u}$, we set $\hat{z} := z_j$.

I.e., in each case we choose $\hat{z}$ to the one of z_j and z_{j+1} closer to $\hat{u}$. It is straightforward to verify that this choice of $\hat{z}$ satisfies both parts of the claim. ∎

4. ε-variation of BV-solutions.

For completeness and later reference we provide a brief description of front-tracking as it applies to the scalar conservation law (1.2) with a locally Lipschitz continuous flux $f(u)$ and BV initial data $\bar{u}$. (See [1, 3, 6] for full details.)

Let $(\bar{u}_n)$ be a sequence of step functions that converge uniformly to $\bar{u}$. For each n let $f_n(u)$ be the continuous and piecewise affine function with break points at the elements of the set

$$\mathcal{B}_n := \frac{1}{n}\mathbb{Z} \cup \operatorname{range}(\bar{u}_n),$$

at which points $f_n(u)$ coincides with $f(u)$. An analysis shows that the unique Kružkov solution $u_n(t, x)$ to the "approximate" Cauchy problem

$$\partial_t u(t,x) + \partial_x[f_n(u(t,x))] = 0 \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \tag{4.1}$$

$$u(0,x) = \bar{u}_n(x), \tag{4.2}$$

can be obtained in the following manner. At time $t = 0$ the Riemann problems at the jumps locations $\{\bar{x}_i\}_{i=1}^{N_n}$ of $\bar{u}_n$ are resolved. The result is a fan of straight lines (fronts) emanating from each jump location and separating constant values of u_n, each of which belongs to the set $\mathcal{B}_n$. The speed of any front in u_n is bounded by the Lipschitz constant of $f|_{\operatorname{range}(\bar{u})}$. In any Riemann problem these outgoing states form a monotone sequence of u-values with the values of the Riemann data as extreme values. The fronts are propagated until two or more of them meet (interact), at which point a new Riemann problem is defined. This is then resolved into a new fan of propagating fronts that separate constant u-values, again forming a finite monotone sequence from the set $\mathcal{B}_n$.

Assume an interaction occurs at time $\bar{t}$. It will be convenient to consider the resolution of the interaction as a two-step process where, first, all states between the two extreme incoming fronts are removed (giving the solution precisely at time $\bar{t}$), followed by the resolution of the Riemann problem defined by the extreme incoming states, $u_\pm$, say.

The works [1, 3, 6] show that this algorithm is well defined: each solution $u_n(t, x)$ contains only finitely many interactions and fronts, globally in time. This is a consequence of the fact that the spatial variation $\operatorname{var} u_n(t, \cdot)$ is nonincreasing in time. More precisely, an analysis shows that the variation remains constant across any interaction where the incoming states form a monotone sequence, while the variation decreases strictly in all other interactions.

Finally, an application of Helly's selection principle, together with uniform Lipschitz continuity of the solution maps $\mathbb{R}^+ \ni t \mapsto u_n(t) \in L^1_{loc}(\mathbb{R}_x)$, show that the solutions u_n tend to a limit function $u(t, x)$ in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$ as $n \to \infty$. (As mentioned in the introduction, the argument that the limit u is jointly measurable in (t, x) requires some care.) Due to locally uniform convergence of f_n to f and strong L^1-convergence of the u_n to u, it follows that u is the unique Kružkov solution of (1.2). More precisely, if the Kružkov solution is viewed as an L^1-equivalence class of almost everywhere defined functions, the function u generated by front-tracking is one *version* (representative) of the Kružkov solution.

Remark 4.1. To fix a unique version $u(t, x)$ of the Kružkov solution constructed via front-tracking, we impose right-continuity of $u(t, \cdot)$ at each fixed time t. Passing from $u(t, x)$ to $u(t, x+)$ does not increase the variation or the ε-variation of the solutions we consider (cf. part (a) of Lemma 3.1).

Our goal in the present work is to extend this approach to the case of regulated data $\bar{u} \in \mathcal{R}$. As noted in the introduction, the key property allowing such an extension is the fact that not only the variation but also the ε-variation of a front-tracking approximation is nonincreasing in time. This is a consequence of Lemma 3.5. We start by considering the situation at the initial time.

LEMMA 4.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, piecewise affine function. Consider any right-continuous step function $\bar{u}(x)$ with compact support contained in (a, b) and* *with values among the break points of f. Let L be the Lipschitz constant of $f|_{range(\bar{u})}$, and assume $t^* > 0$ is the first time of interaction between fronts in the front-tracking solution $u(t,x)$ of (1.2)–(1.3). Then*

$$\varepsilon\text{-var}_{a-Lt}^{b+Lt}\, u(t,\cdot) = \varepsilon\text{-var}_a^b\, \bar{u} \qquad \textit{for all } t \in [0,t^*).$$

Proof. Fix any $t \in [0,t^*)$. As outlined above, the function $u(t,\cdot)$ consists of a finite number of wave fans emanating from the points where $\bar{u}$ jumps. Due to finite speed of propagation, $\text{supp}(u(t,\cdot)) \subset (a - Lt, b + Lt)$. Each wave fan consists of a monotone sequence of u-values separated by fronts. We can now let the initial data $\bar{u}(x)$ play the role of the step function $u(x)$ in Lemma 3.5 and then repeatedly apply part 2 of that lemma. For each fixed wave fan connecting the states u_i to u_{i+1} (see (3.11)), say, we successively insert the values of the Riemann solution (which play the role of $\hat{u}$ in the statement of Lemma 3.5). Since these values form a monotone sequence with u_i and u_{i+1} as extreme members, part 2 of Lemma 3.5 implies that the ε-variation is unchanged in each step, and the conclusion follows. ∎

Next we consider the behavior of the ε-variation at later times. First, from Remark 3.3 it is clear that the ε-variation of a front-tracking solution does not change during the open time intervals between interactions. Next, recall that the resolution of an interaction in the front-tracking solution amounts to the removal of at least one value present in the solution before the interaction, followed by the insertion of a monotone sequence of values whose extreme values were present before the interaction. We can therefore apply Lemma 3.5 and deduce that the ε-variation is nonincreasing along any front-tracking solution. More precisely, we have the following.

LEMMA 4.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, piecewise affine function. Consider any right-continuous step function $\bar{u}(x)$ with compact support in (a,b) and values among the break points of f. Let L denote the Lipschitz constant of $f|_{range(\bar{u})}$. Then the front-tracking solution $u(t,x)$ of (1.2)–(1.3) satisfies*

$$\varepsilon\text{-var}_{a-Lt}^{b+Lt}\, u(t,\cdot) \le \varepsilon\text{-var}_a^b\, \bar{u} \tag{4.3}$$

for all $t \ge 0$ and all $\varepsilon > 0$.

Proof. Let $\bar{t} > 0$ be any time at which two or more fronts meet in the solution $u(t,x)$. Without loss of generality we can assume that no other interactions occur at different locations at time $\bar{t}$. (There can be at most finitely many of them, and we may treat each of them in turn in the same manner as explained in the following.) Let $u_0,\dots,u_m$ denote the values taken by $u(t,\cdot)$ at times t just prior to $\bar{t}$, i.e.,

$$u(t,x) = \sum_{i=0}^{m} u_i \mathbf{1}_{[x_i(t),x_{i+1}(t))}(x) \qquad \text{for } t \in (\bar{t}-\delta,\bar{t}),$$

where $\delta > 0$ is so small that no other interaction occurs during $[\bar{t}-\delta,\bar{t}+\delta]$ and $x_0(t) < x_1(t) < \cdots < x_m(t) < x_{m+1}(t)$ are affine functions on $(\bar{t}-\delta,\bar{t})$. Setting

$$\mathcal{I} = \{i \,|\, x_i(\bar{t}) = x_{i+1}(\bar{t})\},$$

we have that $u(t,\cdot)$ for $t > \bar{t}$ is the solution of the conservation law $u_t + f(u)_x = 0$ with initial data

$$u(\bar{t},x) = \sum_{\substack{i=0\\ i\notin\mathcal{I}}}^{m} u_i \mathbf{1}_{[x_i(\bar{t}),x_{i+1}(\bar{t}))}(x)$$

at time $\bar{t}$, i.e., the states $\{u_i\}_{i\in\mathcal{I}}$ are removed in the interaction. We can now apply Lemma 3.5 several times to compare the ε-variations of $u(\bar{t}-\delta,x)$ and $u(\bar{t},x)$: letting $u(\bar{t}-\delta,x)$ play the role of $\tilde{u}(x)$ in the lemma and then removing the values $\{u_i\}_{i\in\mathcal{I}}$ successively until we obtain $u(\bar{t},x)$, part 1 of Lemma 3.5 yields

$$\varepsilon\text{-var}\, u(\bar{t},\cdot) \le \varepsilon\text{-var}\, u(\bar{t}-\delta,\cdot).$$

According to Lemma 4.2, applied with $\bar{t}$ as the initial time, we also have

$$\varepsilon\text{-var}\, u(\bar{t}+\delta,\cdot) = \varepsilon\text{-var}\, u(\bar{t},\cdot).$$

We conclude that $\varepsilon\text{-var}\, u(\bar{t}+\delta,\cdot) \le \varepsilon\text{-var}\, u(\bar{t}-\delta,\cdot)$. As the ε-variation does not change between interactions, the conclusion follows. ∎

Before considering regulated data it remains to establish, still in the setting of BV data, that the ε-variation is nonincreasing also for general conservation laws (1.2) with Lipschitz continuous flux function.

PROPOSITION 4.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a locally Lipschitz continuous function, and assume $\bar{u}\in BV(\mathbb{R})$ has compact support contained in (a,b). Let L denote the Lipschitz constant of $f|_{range(\bar{u})}$. Then there is a right-continuous version $u=u(t,x)$ of the Kružkov solution of the Cauchy problem* (1.2)–(1.3) *(i.e., an element in its $L^1_{loc}(\mathbb{R}^+_{0,t}\times\mathbb{R}_x)$-equivalence class) which satisfies*

$$\varepsilon\text{-var}_{a-Lt}^{b+Lt}\, u(t,\cdot) \le \varepsilon\text{-var}_a^b\, \bar{u} \qquad \textit{for all } \epsilon>0.$$

Proof. Step 1. Without loss of generality we may assume that $\bar{u}$ is right-continuous. (If not, replace $\bar{u}$ by its right-continuous version $\bar{u}_r$ defined in section 3; as $\bar{u}$ and $\bar{u}_r$ agree almost everywhere, they generate the same Kružkov solution.) We start by constructing a sequence $(\bar{u}_n)$ of right-continuous step functions which coincide with $\bar{u}$ at the left endpoints of each interval of constancy and that converge uniformly to $\bar{u}$ as $n\to\infty$. To do this, first observe that as $\mathrm{supp}(\bar{u})$ is compact, property (R3) in section 2 (applied to $\bar{u}|_{[a,b]}\in\mathcal{R}[a,b]$) yields a sequence of right-continuous step functions (v_k) that converge uniformly to $\bar{u}$ (cf. Remark 2.5). We then define the sequence $(\bar{u}_n)$ as follows. For each n, let $k(n)$ be such that

$$\|v_{k(n)}-\bar{u}\| \le \frac{1}{2n}\,. \tag{4.4}$$

Let $\{x_{k,i}\}_{i=1}^{N_k}$ denote the jump set of v_k such that

$$v_k(x)\equiv v_k(x_{k,i}) \qquad \text{for } x\in[x_{k,i},x_{k,i+1}),\ i=1,\dots,N_k-1, \tag{4.5}$$

and $v_k(x)$ vanishes for $x<x_{k,1}$ as well as for $x\ge x_{k,N_k}$. We then define

$$\bar{u}_n(x) := \bar{u}(x_{k(n),i}) \qquad \text{for } x\in[x_{k(n),i},x_{k(n),i+1}),\ i=1,\dots,N_{k(n)}-1,$$

and $\bar{u}_n(x):=0$ everywhere else. As a consequence of (4.4) and (4.5), given any $x\in[a,b]$, $x\in[x_{k(n),i},x_{k(n),i+1})$, say, we have

$$\begin{aligned}
|\bar{u}_n(x)-\bar{u}(x)| &= |\bar{u}(x_{k(n),i})-\bar{u}(x)| \\
&\le |\bar{u}(x_{k(n),i})-v_{k(n)}(x_{k(n),i})| + |v_{k(n)}(x_{k(n),i})-v_{k(n)}(x)| \\
&\quad + |v_{k(n)}(x)-\bar{u}(x)| \\
&\le \frac{1}{2n}+0+\frac{1}{2n}=\frac{1}{n}.
\end{aligned}$$

As $\bar{u}_n(x)=\bar{u}(x)=0$ for all $x\notin[a,b]$, this shows that $\bar{u}_n$ converges uniformly to $\bar{u}$.

Step 2. Next we execute the front-tracking algorithm as outlined at the beginning of the present section. Let f_n denote the piecewise affine approximation of the flux f that coincides with f at the points $\frac{1}{n}\mathbb{Z}\cup\mathrm{range}(\bar{u}_n)$, and let $u_n(t,x)$ denote the Kružkov solution of the conservation law $u_t + f_n(u)_x = 0$ with initial data $\bar{u}_n$. According to Lemma 4.3 and Lemma 3.4 we have

$$\varepsilon\text{-var}_{a-Lt}^{b+Lt}\, u_n(t,\cdot) \le \varepsilon\text{-var}_a^b\, \bar{u}_n \le \varepsilon\text{-var}_a^b\, \bar{u}. \tag{4.6}$$

As detailed in [1, 3, 6], Helly's selection principle and a diagonal argument give a subsequence (u_{n_k}) and a function $u(t,x)$ with the property that $u(t,\cdot) \in \mathrm{BV}$ for all $t \in \mathbb{Q}^+$ and such that

$$u_{n_k}(t,x) \to u(t,x) \quad \text{for all } t \in \mathbb{Q}^+,\ x \in \mathbb{R}.$$

Applying Proposition 2.7 together with (4.6) gives

$$\varepsilon\text{-var}_{a-Lt}^{b+Lt}\, u(t,\cdot) \le \liminf_k\ \varepsilon\text{-var}_{a-Lt}^{b+Lt}\, u_{n_k}(t,\cdot) \le \varepsilon\text{-var}_a^b\, \bar{u} \qquad \text{for all } t \in \mathbb{Q}^+. \tag{4.7}$$

Finally, one extends u to irrational times as follows. For a given $t \in \mathbb{R}^+ \setminus \mathbb{Q}^+$, fix a sequence $(t_\ell) \subset \mathbb{Q}^+$ with $t_l \to t$, and apply Helly's selection principle to obtain a subsequence (t_{l_m}) for which $(u(t_{l_m},x))_m$ converges for all $x \in \mathbb{R}$. We then define

$$u(t,x) := \lim_m u(t_{l_m},x) \quad \text{for all } x \in \mathbb{R}.$$

Therefore, Proposition 3.6 in [5] together with (4.6) show that (4.7) holds also at irrational times. According to the analysis in [1], it is known that $u(t,x)$ is a version the Kružkov entropy solution of (1.2) with initial data (1.3). Finally, by passing to the right-continuous version $u(t,x+)$ and applying part (a) of Lemma 3.1, we obtain the conclusion. ∎

5. Proof of main result. We finally consider the original Cauchy problem, repeated here for convenience:

$$\partial_t u(t,x) + \partial_x[f(u(t,x))] = 0, \qquad t > 0,\ x \in \mathbb{R}, \tag{5.1}$$

$$u(0,x) = \bar{u}(x), \tag{5.2}$$

where the initial data $\bar{u}$ are now any regulated function with compact support in (a,b). As before let L denote the Lipschitz constant of $f|_{\mathrm{range}(\bar{u})}$.

It is known that the Kružkov solution of (5.1)–(5.2) is unique within the space $L^1(\mathbb{R}_0^+ \times \mathbb{R}; dt \otimes dx)$, whose elements are equivalence classes of functions agreeing almost everywhere on $\mathbb{R}_0^+ \times \mathbb{R}$; see Chapter 6 in [3]. To prove Theorem 2.8 we need to establish the existence of a version $u(t,x)$ of the Kružkov solution of (5.1)–(5.2) with the regularity property that

$$\varepsilon\text{-var}_{a-Lt}^{b+Lt}\, u(t,\cdot) \le \varepsilon\text{-var}_a^b\, \bar{u} \qquad \text{for all } t \ge 0 \text{ and all } \varepsilon > 0. \tag{5.3}$$

Consider first any version $\tilde{u}(t,x)$ of the unique Kružkov solution of (5.1)–(5.2). The map $(t,x) \mapsto \tilde{u}(t,x)$ is then jointly measurable, and it is known that the solution defines a continuous map into $L^1_{loc}(\mathbb{R}_x)$ (see Theorem 6.2.2. in [3]):

$$\tilde{u} \in C^0(\mathbb{R}_0^+; L^1_{loc}(\mathbb{R})). \tag{5.4}$$

On the other hand, merely as an element of $L^1(\mathbb{R}_0^+ \times \mathbb{R}; dt \otimes dx) \cap C^0(\mathbb{R}_0^+; L^1_{loc}(\mathbb{R}))$, we do not have enough information to conclude that $\tilde{u}(t,x)$ satisfies (5.3), or even is a regulated function, at times $t > 0$. While it is possible that other constructive approaches (e.g., vanishing viscosity) can be used, we shall base our argument on front-tracking and approximation via BV solutions.

Without loss of generality we assume that $\bar{u}$ is right-continuous. As in Step 1 of the proof of Proposition 4.4, we construct a sequence $(\bar{u}_n)$ of right-continuous step functions which coincide with $\bar{u}$ at the left endpoints of each interval of constancy and such that

$$\bar{u}_n \to \bar{u} \qquad \text{uniformly on } [a,b]. \tag{5.5}$$

Let $u_n(t,x)$ denote the version of the Kružkov solution of (5.1) with initial data $\bar{u}_n$ given by Proposition 4.4. According to Proposition 4.4 and Lemma 3.4 we have

$$\varepsilon\text{-var}_{a-Lt}^{b+Lt}\, u_n(t,\cdot) \le \varepsilon\text{-var}_a^b\, \bar{u}_n \le \varepsilon\text{-var}_a^b\, \bar{u} \qquad \text{for all } t \ge 0 \text{ and all } \varepsilon > 0. \tag{5.6}$$

We also record the fact that the L^1-contraction property (see section 6.2 of [3]), together with (5.5), yield

$$u_n(t,\cdot) \to \tilde{u}(t,\cdot) \qquad \text{in } L^1_{loc}(\mathbb{R}) \text{ at all times } t \ge 0. \tag{5.7}$$

However, this is not sufficient to conclude that $\tilde{u}(t,\cdot)$, or a version of it, is regulated. Instead we shall construct, via Fraňková's theorem (Theorem 2.6 above), an alternative version $u(t,x)$ of the Kružkov solution—a version for which we *can* monitor the ε-variation. Note that the issue at this point is not existence of the Kružkov solution. Rather, the goal is to exploit its known properties and use Fraňková's extension of Helly's selection principle to identify a "good" version of it.

So, with $\bar{u}_n$ and $u_n(t,x)$ as above, consider the set of rational times $s \in \mathbb{Q}_0^+$. Applying Theorem 2.6, a standard diagonal argument, (5.6), and Proposition 2.7, we obtain a subsequence (u_{n_k}) of (u_n) and a function

$$v : \mathbb{Q}_0^+ \times \mathbb{R} \to \mathbb{R} \tag{5.8}$$

(at this stage only defined at rational times) with the following properties:

(A) $u_{n_k}(s,x) \to v(s,x)$ for all $s \in \mathbb{Q}_0^+$ and all $x \in \mathbb{R}$;

(B) $v(s,\cdot)$ is regulated for all $s \in \mathbb{Q}_0^+$ with

$$\varepsilon\text{-var}_{a-Ls}^{b+Ls}\, v(s,\cdot) \le \varepsilon\text{-var}_a^b\, \bar{u} \qquad \text{for all } s \in \mathbb{Q}_0^+. \tag{5.9}$$

It follows from property (A) and (5.7) that

$$v(s,\cdot) = \tilde{u}(s,\cdot) \qquad \text{as } L^1(\mathbb{R}_x)\text{-functions at each time } s \in \mathbb{Q}_0^+. \tag{5.10}$$

We now want to extend $v(s,x)$ to all of $\mathbb{R}_0^+ \times \mathbb{R}$ in such a manner that (5.9) and (5.10) continue to hold for all $t \ge 0$. To this end, fix any $t \in \mathbb{R}_0^+ \setminus \mathbb{Q}_0^+$, choose a sequence (s_m) of rational times such that $s_m \to t$, and consider the sequence $(v(s_m,\cdot))$. Thanks to property (B) above we can apply Theorem 2.6 and extract a subsequence $(v(s_{m_l},\cdot))$ (depending on t) with the property that $(v(s_{m_l},x))$ converges for all $x \in \mathbb{R}$; we define

$$v(t,x) := \lim_l v(s_{m_l},x). \tag{5.11}$$

Together with (A) above this defines $v(t,x)$ at all points in the upper half-plane. According to (5.9) and Proposition 2.7, we then have

$$\varepsilon\text{-var}_{a-Lt}^{b+Lt}\, v(t,\cdot) \leq \varepsilon\text{-var}_a^b\, \bar{u} \qquad \text{for all } t \in \mathbb{R}_0^+ \text{ and all } \varepsilon > 0. \tag{5.12}$$

In particular, as a consequence of (R2) in section 2, it follows that

$$v(t,\cdot) \text{ is a regulated function at all times } t \geq 0. \tag{5.13}$$

Furthermore, by exploiting the continuity property (5.4) of the Kružkov solution, together with (5.10) and the pointwise convergence in (5.11), we deduce that the difference

$$\|\tilde{u}(t,\cdot)-v(t,\cdot)\|_1 \leq \|\tilde{u}(t,\cdot)-\tilde{u}(s_{m_l},\cdot)\|_1 + \overbrace{\|\tilde{u}(s_{m_l},\cdot)-v(s_{m_l},\cdot)\|_1}^{0 \text{ by } (5.10)} + \|v(s_{m_l},\cdot)-v(t,\cdot)\|_1 \tag{5.14}$$

can be made arbitrarily small by choosing l sufficiently large. It follows that

$$v(t,\cdot) = \tilde{u}(t,\cdot) \qquad \text{as } L^1(\mathbb{R}_x)\text{-functions for all times } t \in \mathbb{R}_0^+. \tag{5.15}$$

We finally define the function $u(t,x)$ by setting

$$u(t,x) := v(t,x+), \tag{5.16}$$

which is well defined according to (5.13) and part (a) of Lemma 3.1. We record that (5.13) and property (R1) in section 2 imply that

$$u(t,x) := v(t,x) \qquad \text{for all but countably many values of } x \tag{5.17}$$

at every time $t \geq 0$. This u is our candidate for the "good" version of the Kružkov solution. Indeed, (5.3) is an immediate consequence of (5.12) and part (a) of Lemma 3.1.

The only remaining issue is to verify that $u(t,x)$ is a version of the Kružkov solution. I.e., we need to argue that u and $\tilde{u}$ agree almost everywhere with respect to the product measure $dt \otimes dx$ on $\mathbb{R}_0^+ \times \mathbb{R}$. This will follow from (5.15), (5.17), and Fubini's theorem once we verify that $u(t,x)$ is jointly measurable with respect to (t,x). This is not immediate since u is defined separately at each time t; in particular, it is not a consequence of the statement in (5.15). We follow [1] and exploit part (b) of Lemma 3.1, according to which u can be realized as the limit of Steklov averages of $v(t,\cdot)$:

$$u(t,x) = \lim_{\delta \downarrow 0} v^\delta(t,x), \tag{5.18}$$

where

$$v^\delta(t,x) := \frac{1}{\delta}\int_x^{x+\delta} v(t,\xi)\,d\xi.$$

We proceed to show that each $v^\delta(t,x)$ jointly continuous in (t,x) for each fixed δ. First note that, thanks to (5.15),

$$v^\delta(t,x) = \frac{1}{\delta}\int_x^{x+\delta} \tilde{u}(t,\xi)\,d\xi. \tag{5.19}$$

We then consider continuity with respect to time of $v^\delta(t,x)$ for a fixed x. According to (5.19) we have

$$|v^\delta(t,x) - v^\delta(s,x)| \leq \frac{1}{\delta}\int_x^{x+\delta} |\tilde{u}(t,\xi) - \tilde{u}(s,\xi)|\,d\xi \leq \frac{1}{\delta}\|\tilde{u}(t) - \tilde{u}(s)\|_1. \tag{5.20}$$

By (5.4) the map $t \mapsto \tilde{u}(t) \in L^1_{loc}(\mathbb{R})$ is continuous, and it follows from (5.20) that $v^\delta(t,x)$ is continuous with respect to t, uniformly with respect to x. Next, consider continuity of $v^\delta(t,x)$ with respect to x for a fixed t: for any $h \in (0,\delta)$ we have

$$|v^\delta(t,x+h) - v^\delta(t,x)| \leq \frac{1}{\delta}\left\{\int_x^{x+h} + \int_{x+\delta}^{x+\delta+h}\right\} |\tilde{u}(t,\xi)|\,d\xi \leq \frac{2}{\delta}\|\bar{u}\|_\infty |h|, \tag{5.21}$$

where we have used that the Kružkov solution is bounded in L^∞ by its L^∞-norm at time zero. The same estimate holds for $h \in (-\delta, 0)$. Thus, for a fixed $\delta > 0$, the function v^δ is uniformly continuous, separately in time and space, on a neighborhood of each point (t,x). It follows that v^δ is jointly continuous in (t,x). For completeness we detail the argument for this. Fix any $t \geq 0$, $x \in \mathbb{R}$, $\delta > 0$, and $\epsilon > 0$. Since the Kružkov solution $\tilde{u}$ has compact support at all times and satisfies the continuity property (5.4), there is a $\mu > 0$ such that

$$\|\tilde{u}(t) - \tilde{u}(s)\|_{L^1(\mathbb{R})} < \frac{\epsilon\delta}{2} \qquad \text{for any } s \text{ with } |t-s| < \mu,$$

and therefore, by (5.20),

$$\left|v^\delta(t,y) - v^\delta(s,y)\right| < \frac{\epsilon}{2} \qquad \text{for any } s \text{ with } |t-s| < \mu \text{ and any } y \in \mathbb{R}. \tag{5.22}$$

Hence, for any s, y such that $|t-s| + |x-y| \leq \min(\mu, \frac{\epsilon\delta}{4\|\bar{u}\|_\infty})$, (5.21) and (5.22) give

$$\begin{aligned}\left|v^\delta(t,x) - v^\delta(s,y)\right| &\leq \left|v^\delta(t,x) - v^\delta(t,y)\right| + \left|v^\delta(t,y) - v^\delta(s,y)\right| \\ &< \frac{2}{\delta}\|\bar{u}\|_\infty \cdot \frac{\epsilon\delta}{4\|\bar{u}\|_\infty} + \frac{\epsilon}{2} = \epsilon\,.\end{aligned}$$

This establishes joint continuity of $v^\delta(t,x)$. According to (5.18) the function $(t,x) \mapsto u(t,x)$ in (5.16) is therefore the pointwise limit of continuous functions and hence jointly (Borel) measurable.

Finally, thanks to the joint measurability of $u(t,x)$ we can now apply Fubini's theorem together with (5.17) and (5.15) and conclude that, for any $T > 0$,

$$\begin{aligned}\int_{[0,T]\times\mathbb{R}} |u(t,x) - \tilde{u}(t,x)|\,dt\otimes dx &= \int_{[0,T]}\int_{\mathbb{R}} |u(t,x) - \tilde{u}(t,x)|\,dxdt \\ &= \int_{[0,T]}\int_{\mathbb{R}} |v(t,x) - \tilde{u}(t,x)|\,dxdt = \int_{[0,T]} 0\,dt = 0,\end{aligned}$$

showing that $u(t,x) = \tilde{u}(t,x)$ for almost all $(t,x) \in \mathbb{R}_0^+ \times \mathbb{R}$. Therefore, viewed as an element in $L^1(\mathbb{R}_0^+ \times \mathbb{R}; dt \otimes dx)$, the function u coincides with the unique Kružkov solution of (5.1)–(5.2). This concludes the proof of Theorem 2.8.

Acknowledgment. The first author is indebted to Michael Benfield for pointing out the reference [5].

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