ON Φ-VARIATION FOR 1-D SCALAR CONSERVATION LAWS

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Abstract. Let $\Phi : [0,\infty) \to [0,\infty)$ be a convex function satisfying $\Phi(0) = 0$, $\Phi(x) > 0$ for $x > 0$, and $\lim_{x\downarrow 0} \frac{\Phi(x)}{x} = 0$. Consider the unique entropy admissible (i.e., Kružkov) solution $u(t,x)$ of the scalar, 1-d Cauchy problem

$$\partial_t u(t,x) + \partial_x[f(u(t,x))] = 0, \qquad u(0) = \bar{u}. \tag{0.1}$$

For compactly supported data $\bar{u}$ with bounded Φ-variation, we realize the solution $u(t,x)$ as a limit of front-tracking approximations and show that the Φ-variation of (the right continuous version of) $u(t,x)$ is non-increasing in time. We also establish the following natural time-continuity estimate:

$$\int_{\mathbb{R}} \Phi(|u(t,x) - u(s,x)|)\, dx \leq C \cdot \Phi\text{-var}\, u(s) \cdot |t-s| \qquad \text{for } s,t \geq 0,$$

where C depends on f. Finally, according to a theorem of Goffman-Moran-Waterman, any regulated function of compact support has bounded Φ-variation for some Φ. As a corollary we thus have: if $\bar{u}$ is a regulated function, so is $u(t)$ for all $t > 0$.

Keywords. Scalar conservation laws, one dimensional, Phi-variation, time continuity, regulated solutions.

Contents

1. Introduction

The present work shows how the method of front tracking for scalar conservation laws of the form (0.1), with a locally Lipschitz continuous flux f, can be extended from the standard setting with $\bar{u} \in BV$ to the more general case where $\bar{u}$ is of bounded Φ-variation. Here $\Phi : [0,\infty) \to [0,\infty)$ is any convex function satisfying conditions (p1)-(p4) listed below in Section 2; the definition of Φ-variation is given in Definition 2.2. Our main results are:

(i) the spatial Φ-variation of the (right-continuous version of the) Kružkov solution $u(t,x)$ is non-increasing in time;
(ii) the natural t-continuity property of solutions with bounded Φ-variation (see (5.4));

Date: January 3, 2020.
H. K. Jenssen was partially supported by NSF grant DMS-1813283.

(iii) if $\bar{u}$ is regulated, then so is $u(t,\cdot)$ at each time $t>0$.

Properties (i) and (ii) were established by [1] for the particular case of $\Phi(u)=u^p$, $p\geq 1$. However, the details of the argument for (ii) in this case were not given in [1]. The latter work, as well as [9], focus on the nonlinear regularizing effect induced by the flux f. It turns out this can be quantified in terms of Φ-variation, with Φ depending on f.

Concerning (iii), a function defined on an interval $I\subset\mathbb{R}$ is regulated provided it has (finite) right and left limits at all points of I (see Section 2 for precise definitions). In a recent work [8] the authors established (iii) by a different argument based on the notion of ε-variation due to Fraňková [5], and her extension of Helly's Selection Principle to the space of regulated functions. (The ε-variation of a function v is defined as the infimum of variations of BV functions uniformly ε-close to v.) In [8] it was shown that the ε-variation of the Kružkov solution is non-increasing with time, and (iii) is a consequence of this fact.

It is known that the Kružkov solution belongs to $C^0(\mathbb{R}_0^+;L^1_{loc}(\mathbb{R}))$ whenever the initial data $\bar{u}$ belongs to L^∞ ([3], Chapter 6), while for BV data this is upgraded to Lipschitz continuity, with Lipschitz constant depending on $\operatorname{var}\bar{u}$. It is reasonable to expect more than mere continuity of the solution operator (but not Lipschitz continuity) for regulated initial data. However, the approach via ε-variation in [8] does not provide such information. In contrast, the present work shows that by basing the analysis on the notion of Φ-variation, and exploiting the characterization of regulated functions in terms of Φ-variation (due to Goffman-Moran-Waterman [6]), we obtain precise information about time-continuity of solutions with regulated data. In particular, any such solution defines a uniformly continuous map from $\mathbb{R}_0^+$ into $L^1_{loc}(\mathbb{R})$ (see Theorem 5.1).

In Section 2 we introduce the class of convex functions Φ under consideration, define Φ-variation (for background see [4,10]), and state various auxiliary results for later use. These are mostly either easy generalizations of corresponding results for functions of bounded p-variation (e.g., as presented in [1]), or well-known. A notable exception is the aforementioned result by Goffman-Moran-Waterman [6] which is used later to treat the case with regulated initial data. Section 3 contains detailed proofs of Helly's Selection Principle for functions with bounded Φ-variation, together with its application to sequences of functions of (t,x) that satisfy the natural time-continuity property in this setting. In Section 4 we establish a quantitative estimate on time continuity, as well as non-increase of Φ-variation, for front-tracking approximations when the data have bounded Φ-variation. With the results of Sections 2-4 in place, a standard argument yields (i) and (ii) above, see Theorem 5.1 in Section 5. Finally, we apply the result of Goffman-Moran-Waterman [6] to establish (iii). While one could extend the analysis to more general cases, for ease of exposition we formulate our main result for the case of compactly supported initial data.

2. Preliminaries on Φ-variation

We fix a function $\Phi:[0,\infty)\to[0,\infty)$ with the following properties:

(p1) $\Phi(0)=0$
(p2) Φ is convex
(p3) $\Phi(x)>0$ for $x>0$
(p4) $\lim_{x\downarrow 0}\frac{\Phi(x)}{x}=0$.

Lemma 2.1. *It follows from the properties (p1)-(p4) that*

(a) *for any $x\geq 0$ and any $t\in[0,1]$, $\Phi(tx)\leq t\Phi(x)$,*
(b) *$\Phi(x)+\Phi(y)\leq\Phi(x+y)$ for any x, $y\geq 0$,*

(c) *and more generally*

$$\Phi(x_1) + \cdots + \Phi(x_n) \leq \Phi(x_1 + \cdots + x_n) \quad \textit{for any } x_1, \ldots, x_n \geq 0,$$

(d) Φ *is strictly increasing and everywhere continuous.*

Proof. For x and t as in (a), (p2) and (p1) gives

$$\Phi(tx) = \Phi(tx + (1-t)0) \leq t\Phi(x) + (1-t)\Phi(0) = t\Phi(x),$$

establishing (a). Next, by (p1), (b) is obvious if $x = y = 0$; if not, (a) gives

$$\begin{aligned}\Phi(x) + \Phi(y) &= \Phi(\tfrac{x}{x+y}(x+y)) + \Phi(\tfrac{y}{x+y}(x+y)) \\ &\leq \tfrac{x}{x+y}\Phi(x+y) + \tfrac{y}{x+y}\Phi(x+y) = \Phi(x+y),\end{aligned}$$

establishing (b). The argument for (c) follows by induction. Next, if $0 \leq x < y$, then (a) and (p3) give

$$\Phi(x) = \Phi(\tfrac{x}{y}y) \leq \tfrac{x}{y}\Phi(y) < \Phi(y).$$

Finally, (p4) implies continuity at 0, while a standard result (convex functions are continuous on open intervals) yields continuity of Φ on $(0, \infty)$. □

Next, for any interval $I \subset \mathbb{R}$ let $\Pi(I)$ denote the set of finite partitions of I: $\pi \in \Pi(I)$ if and only if $\pi = \{x_0, \ldots, x_k\}$, for some $k \in \mathbb{N}$, with $x_0, \ldots, x_k \in I$ and $x_0 < \cdots < x_k$. We write $\Pi(a, b]$ for $\Pi((a, b])$, and similarly for other types of intervals.

Definition 2.2. *For any function $u : I \to \mathbb{R}^N$ we define the* Φ-variation of u relative to $\pi \in \Pi(I)$ *as*

$$\Phi\text{-var}_I\, u[\pi] := \sum_{i=1}^{k} \Phi(|u(x_i) - u(x_{i-1})|),$$

and we define its Φ-variation *by*

$$\Phi\text{-var}_I\, u := \sup_{\pi \in \Pi(I)} \Phi\text{-var}_I\, u[\pi]. \tag{2.1}$$

We write $\Phi\text{-var}\, u$ *for* $\Phi\text{-var}_{\mathbb{R}}\, u$*, and set*

$$\Phi\text{-}BV(I) \equiv \Phi\text{-}BV(I; \mathbb{R}^N) := \{u : I \to \mathbb{R} \,:\, \Phi\text{-var}_I\, u < \infty\}. \tag{2.2}$$

Lemma 2.3. *Assume $-\infty \leq a < b < c < \infty$, and $u : (a, c] \to \mathbb{R}$. Then*

$$\Phi\text{-var}_{(a,b]}\, u + \Phi\text{-var}_{(b,c]}\, u \quad\leq\quad \Phi\text{-var}_{(a,b]}\, u + \Phi\text{-var}_{[b,c]}\, u \quad\leq\quad \Phi\text{-var}_{(a,c]}\, u. \tag{2.3}$$

Proof. The first inequality is trivial since $\Phi\text{-var}_I\, u \leq \Phi\text{-var}_J\, u$ whenever $I \subset J$. For the second inequality let $\varepsilon > 0$ be fixed. Choose $\pi \in \Pi(a, b]$ and $\pi' \in \Pi[b, c]$ such that

$$\Phi\text{-var}_{(a,b]}\, u \leq \Phi\text{-var}_{(a,b]}\, u[\pi] + \varepsilon,$$

and

$$\Phi\text{-var}_{[b,c]}\, u \leq \Phi\text{-var}_{[b,c]}\, u[\pi'] + \varepsilon.$$

Then $\pi \cup \pi' \in \Pi(a, c]$, so that

$$\begin{aligned}\Phi\text{-var}_{(a,b]}\, u + \Phi\text{-var}_{[b,c]}\, u &\leq \Phi\text{-var}_{(a,b]}\, u[\pi] + \Phi\text{-var}_{[b,c]}\, u[\pi'] + 2\varepsilon \\ &\leq \Phi\text{-var}_{(a,c]}\, u[\pi \cup \pi'] + 2\varepsilon \leq \Phi\text{-var}_{(a,c]}\, u + 2\varepsilon.\end{aligned}$$

As ε is arbitrary, the second inequality in (2.3) follows. □

Lemma 2.4. *Assume $u \in \Phi\text{-BV}(\mathbb{R})$ and let U denote its corresponding Φ-variation function defined by*

$$U(x) := \Phi\text{-var}_{(-\infty,x]}\, u = \sup\Big\{ \sum_{i=1}^{k} \Phi(|u(x_i) - u(x_{i-1})|) \,:\, k \in \mathbb{N}, \quad x_0 < \cdots < x_k \le x \Big\}.$$

Then,

$$U(x) + \text{var}_{(x,y]}\, u \le U(y) \quad \text{and} \quad U(x) + \Phi(|u(y) - u(x)|) \le U(y) \quad \text{whenever } x < y. \tag{2.4}$$

Proof. The first inequality in (2.4) is the special case of the inequality between the extreme terms in (2.3) with $a = -\infty$, $b = x$ and $c = y$. Since $\Phi(|u(y) - u(x)|) \le \Phi\text{-var}_{[x,y]}\, u$, the second inequality in (2.4) follows from the second inequality in (2.3) with the same values of a, b, and c. □

For later reference we record the following simple fact.

Lemma 2.5. *Let $u : [a, c] \to \mathbb{R}$ be such that $\operatorname{supp}(u) \subset (a, c]$, and assume $a < b < \inf \operatorname{supp}(u)$. Then*

$$\Phi\text{-var}_{(a,c]}\, u = \Phi\text{-var}_{[a,c]}\, u = \Phi\text{-var}_{(b,c]}\, u. \tag{2.5}$$

The following definition and proposition describe some useful properties of the Φ-variation, adapted from [1].

Definition 2.6. *Let $\pi = \{x_0, \dots, x_k\}$ be any partition of an interval $I \subset \mathbb{R}$, and let $u : I \to \mathbb{R}$. Then the* extremal points of π with respect to u *are x_0, x_k, and the points x_i, $1 \le i \le k-1$, with the property that either*

$$\max(u(x_{i-1}), u(x_{i+1})) < u(x_i),$$

or

$$u(x_i) < \min(u(x_{i-1}), u(x_{i+1})).$$

Given π, let $\pi[u]$ denote the partition consisting of the extremal points of π with respect to u, i.e., $\pi[u]$ consists of the points of local extrema of $u|_\pi$. The partition π is said to be extremal with respect to u *if $\pi[u] = \pi$. Finally, we let $Ext(I, u)$ denote the collection of partitions of I that are extremal with respect to u.*

Proposition 2.7. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$. Then:*

(1) *For any partition $\pi \in \Pi(I)$,*

$$\Phi\text{-var}_I\, u[\pi] \le \Phi\text{-var}_I\, u[\pi[u]].$$

(2) *We have*

$$\Phi\text{-var}_I\, u = \sup_{\pi \in Ext(I,u)} \Phi\text{-var}_I\, u[\pi].$$

(3) *If $u : I \to \mathbb{R}$ is monotone, then*

$$\Phi\text{-var}_I\, u = \Phi\big(\sup_I u - \inf_I u\big).$$

Proof. (1) Let $\pi = \{x_0, \dots, x_k\}$ be any partition of I and assume that $\pi[u] = \{y_1, \dots, y_m\}$ ($m \le k$) be the partition consisting of extremal points of π with respect to u. Let

$\phi : \{0, \dots, m\} \to \{0, \dots, k\}$ be the strictly increasing function defined by setting $x_{\phi(j)} = y_j$, for $j = 0, \dots, m$. With $u_i = u(x_i)$ we then have

$$\Phi\text{-var}_I\, u[\pi] = \sum_{i=1}^{k} \Phi(|u_i - u_{i-1}|) = \sum_{j=1}^{m} \sum_{i=\phi(j-1)+1}^{\phi(j)} \Phi(|u_i - u_{i-1}|).$$

For each $j = 1, \dots, m$ we have that $i \mapsto u_i$ is monotone for $\phi(j-1) < i \le \phi(j)$, such that, by part (c) of Lemma 2.1 and monotonicity, we get

$$\begin{aligned}\sum_{i=\phi(j-1)+1}^{\phi(j)} \Phi(|u_i - u_{i-1}|) &\le \Phi\Big(\sum_{i=\phi(j-1)+1}^{\phi(j)} |u_i - u_{i-1}|\Big) \\ &= \Phi(|u_{\phi(j)} - u_{\phi(j-1)}|) \\ &\equiv \Phi(|u(y_j) - u(y_{j-1})|).\end{aligned}$$

Thus,

$$\Phi\text{-var}_I\, u[\pi] \le \sum_{j=1}^{m} \Phi(|u(y_j) - u(y_{j-1})|) = \Phi\text{-var}_I\, u[\pi[u]].$$

(2) Immediate from the definition of Φ-var u in (2.1) and part (1).
(3) If $u : I \to \mathbb{R}$ is monotone then any partition $\pi \in \text{Ext}(I, u)$ consists of only two points, viz. $\min \pi$ and $\max \pi$. According to part (2), together with continuity and monotonicity of Φ, we therefore have

$$\Phi\text{-var}_I\, u = \sup_{x,y \in I} \Phi(|u(y) - u(x)|) = \Phi\Big(\sup_{x,y \in I} |u(y) - u(x)|\Big) \equiv \Phi\big(\sup_I u - \inf_I u\big).$$

□

We next introduce the class of regulated functions on an interval.

Definition 2.8. *Let I be any interval in $\mathbb{R}$ (i.e., I may be open, closed, half open, finite, or infinite). A function $u : I \to \mathbb{R}$ is* regulated *on I provided its right and left limits exist (as finite numbers) at all points in the interior of I, it has a finite right limit at the left endpoint, and a finite left limit at the right endpoint (whether or not these endpoints are finite or belong to I). The class of regulated functions on I is denoted $\mathcal{R}(I)$.*

The following results are standard (see [4, 5]):

Lemma 2.9. *If $u \in \mathcal{R}(I)$, then u has at most a countable set of discontinuities in I.*

Lemma 2.10. *If $u \in \Phi$-BV(I) for some function Φ satisfying* (p1)-(p4)*, then $u \in \mathcal{R}(I)$.*

The work [6] established the converse result (see also [4]).

Theorem 2.11 (Goffman-Moran-Waterman [6])**.** *Assume I be a compact interval and $u \in \mathcal{R}(I)$. Then there exists a function $\Phi : [0, \infty) \to [0, \infty)$ satisfying* (p1), (p2), (p3), *and* (p4) *above, with Φ-var$_I\, u < \infty$.*

The following result shows how passing to the right-continuous version of a function with bounded Φ-variation does not increase its Φ-variation, and also how this version may be obtained via so-called Steklov averages (see also [8]).

Lemma 2.12. *Assume $u \in \Phi$-BV$(\mathbb{R})$ and let u_r denote the function*

$$u_r(x) := u(x+) \equiv \lim_{y \downarrow x} u(y).$$

According to Lemmas 2.10 and 2.9, u_r is well-defined and agrees with u except on a countable set. For $\varepsilon > 0$, define

$$u^\varepsilon(x) := \frac{1}{\varepsilon} \int_x^{x+\varepsilon} u(\xi)\, d\xi. \tag{2.6}$$

Then the following holds:

(a) $\Phi\text{-var}\, u_r \le \Phi\text{-var}\, u$,
(b) $u_r(x) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(x)$ *at all points* $x \in \mathbb{R}$, *and*
(c) u_r *is right-continuous at all points.*

Proof. For (a) consider any selection $x_0 < x_1 < \cdots < x_k$. As Φ is continuous we have

$$\sum_{i=1}^k \Phi(|u_r(x_i) - u_r(x_{i-1})|) = \lim_{\delta \downarrow 0} \sum_{i=1}^k \Phi(|u(x_i + \delta) - u(x_{i-1} + \delta)|) \le \Phi\text{-var}\, u,$$

and the result follows. For (b), fix $x \in \mathbb{R}$ and $\delta > 0$. As u is regulated there is an $h > 0$ with the property that

$$|u(\xi) - u_r(x)| < \delta \qquad \text{whenever } \xi \in (x, x+h). \tag{2.7}$$

Thus,

$$|u^\varepsilon(x) - u_r(x)| \le \frac{1}{\varepsilon} \int_x^{x+\varepsilon} |u(\xi) - u_r(x)|\, d\xi \le \delta \qquad \text{whenever } 0 < \varepsilon < h.$$

Finally, for (c), fix any $x \in \mathbb{R}$ and $\delta > 0$. Then there is an $h > 0$ such that (2.7) holds. We will show that $|u_r(x+y) - u_r(x)| \le 2\delta$ whenever $0 < y < h$. Indeed, for any such y there is a $k > 0$ with the property that

$$|u_r(x+y) - u(\eta)| < \delta \qquad \text{whenever } \eta \in (x+y, x+y+k).$$

Set $\mu := \min\{h, y+k\}$ such that $y < \mu \le h$. We then have $(x+y, x+\mu) \subset (x, x+h) \cap (x+y, x+y+k)$. Thus, for any $\eta \in (x+y, x+\mu)$ we have (using (2.7) with η for ξ) that

$$|u_r(x+y) - u_r(x)| \le |u_r(x+y) - u(\eta)| + |u(\eta) - u_r(x)| < 2\delta.$$

This shows that u_r is right continuous at x. □

We end this section with the following approximation result:

Proposition 2.13. *Assume $\bar{u} \in \Phi$-BV$(\mathbb{R})$ is right-continuous and of compact support. Then there is a sequence $(\bar{u}_n)$ of right-continuous step functions of compact support with $\bar{u}_n \to \bar{u}$ uniformly. In addition, the $\bar{u}_n$ may be chosen to agree with $\bar{u}$ at the left endpoints of their intervals of constancy.*

Proof. We first recall the fact that a regulated function on a compact interval can be realized as a uniform limit of step-functions (see [5]; in fact, this is a characterization of regulated functions of compact support). It is immediate to verify that if the regulated function in question is right-continuous, then the step functions may be chosen as right-continuous. If now $\bar{u} \in \Phi$-BV$(\mathbb{R})$, with $\operatorname{supp} \bar{u} \subset [a, b]$, we have $\bar{u}$ is regulated according to Lemma 2.10, so that there is a sequence of right continuous step functions (v_k) that converge uniformly to

$\bar{u}$. We then define the sequence $(\bar{u}_n)$ as follows (see also [8]). For each n, let $k(n)$ be such that

$$\|v_{k(n)} - \bar{u}\| \leq \tfrac{1}{2n}\,. \tag{2.8}$$

Let $\{x_{k,i}\}_{i=1}^{N_k}$ denote the jump set of v_k, such that

$$v_k(x) \equiv v_k(x_{k,i}) \qquad \text{for } x \in [x_{k,i}, x_{k,i+1}),\ i = 1, \ldots, N_k - 1, \tag{2.9}$$

and $v_k(x)$ vanishes for $x < x_{k,1}$ and for $x \geq x_{k,N_k}$. We then define

$$\bar{u}_n(x) := \bar{u}(x_{k(n),i}) \qquad \text{for } x \in [x_{k(n),i}, x_{k(n),i+1}),\ i = 1, \ldots, N_{k(n)} - 1,$$

and $\bar{u}_n(x) := 0$ everywhere else. As a consequence of (2.8) and (2.9), given any $x \in [a,b]$, $x \in [x_{k(n),i}, x_{k(n),i+1})$, we have

$$\begin{aligned}
|\bar{u}_n(x) - \bar{u}(x)| &= |\bar{u}(x_{k(n),i}) - \bar{u}(x)| \\
&\leq |\bar{u}(x_{k(n),i}) - v_{k(n)}(x_{k(n),i})| + |v_{k(n)}(x_{k(n),i}) - v_{k(n)}(x)| + |v_{k(n)}(x) - \bar{u}(x)| \\
&\leq \tfrac{1}{2n} + 0 + \tfrac{1}{2n} = \tfrac{1}{n}.
\end{aligned}$$

As $\bar{u}_n(x) = \bar{u}(x) = 0$ for all $x \notin [a,b]$, this shows that $\bar{u}_n$ converges uniformly to $\bar{u}$. □

3. Helly's selection principle for Φ-BV(ℝ)

Theorem 3.1 (Helly)**.** *Let (u_n) be a sequence of functions $u_n : \mathbb{R} \to \mathbb{R}^N$ which are uniformly bounded in magnitude and Φ-variation, i.e. there are finite numbers M and V such that*

- (H1) $|u_n(x)| \leq M$ *for all x and all n, and*
- (H2) $\Phi\text{-var}\, u_n \leq V$ *for all n.*

Then there is a subsequence of (u_n), denoted (u_m), and a function $u : \mathbb{R} \to \mathbb{R}^N$ such that

- (C1) (u_m) *converges pointwise to u at every point of* $\mathbb{R}$*,*
- (C2) $|u(x)| \leq M$ *for all x, and*
- (C3) $\Phi\text{-var}\, u \leq V$*.*

Proof. The proof follows that of the standard case of BV [2, 11]. For each n define the variation function

$$U_n(x) := \Phi\text{-var}_{(-\infty,x]}\, u_n. \tag{3.1}$$

For later use we note that $(2.4)_2$ and monotonicity of U_n give

$$\Phi(|u_n(x) - u_n(y)|) \leq U_n(q) - U_n(p) \qquad \text{whenever } p \leq y \leq x \leq q.$$

Since Φ is strictly increasing, it is invertible, and its inverse function Φ^{-1} is also increasing. It follows that

$$|u_n(x) - u_n(y)| \leq \Phi^{-1}(U_n(q) - U_n(p)) \qquad \text{whenever } p \leq y \leq x \leq q. \tag{3.2}$$

Now, each U_n is non-decreasing and satisfies, according to (H2),

$$0 \leq U_n(x) \leq V \qquad \text{for all } x \in \mathbb{R}. \tag{3.3}$$

Applying a diagonal argument to the sequence (U_n) we extract a subsequence U_{n_k} which converges at all rational points, and we define

$$U(x) := \lim_k U_{n_k}(x) \qquad \text{for each } x \in \mathbb{Q}. \tag{3.4}$$

As each U_{n_k} is non-negative, non-decreasing, and pointwise bounded by V, it follows that $U : \mathbb{Q} \to \mathbb{R}$ has the same properties. This implies that the jump set of U, i.e.,

$$J := \big\{x \in \mathbb{R} : \lim_{\mathbb{Q}\ni z\downarrow x} U(z) > \lim_{\mathbb{Q}\ni y\uparrow x} U(y)\big\},$$

is countable.

Next apply a diagonal argument to the sequence (u_{n_k}) to extract a subsequence, for convenience denoted (u_m), which converges at each point in the countable set $\mathbb{Q} \cup J$. Let the limiting function be denoted $u : \mathbb{Q} \cup J \to \mathbb{R}$, i.e.,

$$u(x) := \lim_m u_m(x) \qquad \text{for each } x \in \mathbb{Q} \cup J. \tag{3.5}$$

The claim now is that the sequence $(u_m(x))$ converges for every $x \in \mathbb{R}$. To verify the claim, we fix any $x \notin \mathbb{Q} \cup J$ and show that the sequence $(u_m(x))$ is Cauchy. Choose any $\varepsilon > 0$. As $x \notin J$ we have that

$$\lim_{\mathbb{Q}\ni z\downarrow x} U(z) = \lim_{\mathbb{Q}\ni y\uparrow x} U(y),$$

such that there exist $y, z \in \mathbb{Q}$ with

$$y < x < z \qquad \text{and} \qquad 0 \le U(z) - U(y) < \delta, \tag{3.6}$$

where

$$\delta := \tfrac{1}{3}\Phi(\varepsilon) > 0. \tag{3.7}$$

Let (U_m) denote the sequence of variation functions corresponding to the subsequence we have denoted (u_m). From here on, any U_k and u_k are understood to be from the subsequences denoted (U_m) and (u_m), respectively. According to (3.5) and (3.4) there is an index $N \in \mathbb{N}$ (depending on y and z, and thus on x) such that whenever $k \ge N$, then

$$|u_k(y) - u(y)| < \varepsilon, \tag{3.8}$$

and

$$|U_k(y) - U(y)| < \delta, \quad |U_k(z) - U(z)| < \delta. \tag{3.9}$$

Hence, for k, $l \ge N$, we have

$$\begin{aligned}
|u_l(x) - u_k(x)| &\le |u_l(x) - u(y)| + |u(y) - u_k(x)| \\
&\le |u_l(x) - u_l(y)| + |u_l(y) - u(y)| + |u(y) - u_k(y)| + |u_k(y) - u_k(x)| \\
&\le |u_l(x) - u_l(y)| + |u_k(x) - u_k(y)| + 2\varepsilon \qquad \text{(by (3.8))} \\
&\le \Phi^{-1}(U_l(z) - U_l(y)) + \Phi^{-1}(U_k(z) - U_k(y)) + 2\varepsilon \qquad \text{(by (3.2))} \\
&\le \Phi^{-1}(|U_l(z) - U(z)| + |U(z) - U(y)| + |U(y) - U_l(y)|) \\
&\quad + \Phi^{-1}(|U_k(z) - U(z)| + |U(z) - U(y)| + |U(y) - U_k(y)|) + 2\varepsilon \\
&\le 2\Phi^{-1}(3\delta) + 2\varepsilon = 4\varepsilon \qquad \text{(by (3.6), (3.9), and (3.7)).}
\end{aligned}$$

This shows that $(u_m(x))$ is Cauchy for every $x \in \mathbb{R}$, verifying the claim and establishing (C1). (C2) is an immediate consequence of (C1) and (H1). Finally, to verify (C3), fix any partition $\{x_0, \dots, x_k\}$, and use continuity of Φ and (C1) to get

$$\sum_{i=1}^{k} \Phi(|u(x_i) - u(x_{i-1})|) = \lim_m \sum_{i=1}^{k} \Phi(|u_m(x_i) - u_m(x_{i-1})|) \le V.$$

□

3.1. **Application to sequences** $(u_n(t,x))$**.** We first include an observation that will be used in the proof of the theorem below.

Lemma 3.2. *Assume* $u \in L^\infty([0,\infty)\times\mathbb{R};\mathbb{R}^N)$ *has the property that for each compact* $K \subset \mathbb{R}_x$*, there a constant* L_K *such that*

$$\int_K \Phi(|u(t,x)-u(s,x)|)\,dx \le L_K|t-s|. \tag{3.10}$$

Then $t \mapsto u(t,\cdot)$ *is a uniformly continuous map of* $[0,\infty)$ *into* $L^1_{loc}(\mathbb{R}_x;\mathbb{R}^N)$*, and specifically,*

$$\int_K |u(t,x)-u(s,x)|\,dx \le \Psi_K(|t-s|), \tag{3.11}$$

where

$$\Psi_K(\xi) := |K|\Phi^{-1}\Big(\frac{L_K}{|K|}\xi\Big), \qquad \xi \ge 0. \tag{3.12}$$

Proof. This is a direct consequence of Jensen's inequality applied to the convex function Ψ: for a fixed compact $K \subset \mathbb{R}$, (3.10) gives

$$\Phi\Big(\frac{1}{|K|}\int_K |u(t,x)-u(s,x)|\,dx\Big) \le \frac{1}{|K|}\int_K \Phi(|u(t,x)-u(s,x)|)\,dx \le \frac{L_K}{|K|}|t-s|.$$

As Φ is increasing, it follows that

$$\int_K |u(t,x)-u(s,x)|\,dx \le |K|\Phi^{-1}\Big(\frac{L_K}{|K|}|t-s|\Big).$$

Since Φ is one-to-one and continuous, so is Φ^{-1}. In particular, Φ^{-1} is continuous at 0, and the uniform continuity of $t \mapsto u(t,\cdot)$, as a map into $L^1_{loc}(\mathbb{R}_x;\mathbb{R}^N)$, follows. □

Theorem 3.3. *Assume* (u_n) *is a sequence of functions* $u_n : [0,\infty)\times\mathbb{R} \to \mathbb{R}^N$ *for which the following holds. There are constants* M *and* V*, and for each compact* $K \subset \mathbb{R}$ *there is a constant* L_K*, such that for all* n*:*

- (h1) $|u_n(t,x)| \le M$ *for all* t*,* x*;*
- (h2) $\Phi\text{-var}\,u_n(t) \le V$ *for all* t*; and*
- (h3) *for any* $t,s \ge 0$*,*

$$\int_K \Phi(|u_n(t,x)-u_n(s,x)|)\,dx \le L_K|t-s|. \tag{3.13}$$

Then there exists a subsequence of (u_n)*, denoted* (u_m)*, and a function* $u : [0,\infty)\times\mathbb{R} \to \mathbb{R}^N$ *such that*

- (c0) $(u_m(t))$ *converges to* $u(t)$ *in* $L^1_{loc}(\mathbb{R};\mathbb{R}^N)$ *for all times* $t \ge 0$*;*
- (c1) (u_m) *converges to* u *in* $L^1_{loc}([0,\infty)\times\mathbb{R};\mathbb{R}^N)$*;*
- (c2) $|u(t,x)| \le M$ *for all* t*,* x*;*
- (c3) $\Phi\text{-var}\,u(t) \le V$ *for all* t*;*
- (c4) *for each fixed* t*,* $x \mapsto u(t,x)$ *is right-continuous at every point* x*; and*
- (c5) *for all* $t,s \ge 0$ *and any compact* $K \subset \mathbb{R}$*, we have*

$$\int_K \Phi(|u(t,x)-u(s,x)|)\,dx \le L_K|t-s|. \tag{3.14}$$

Proof. First, for each time $t \in \mathbb{Q}_0^+ \equiv \mathbb{Q} \cap [0, \infty)$ the sequence $(u_n(t, \cdot))$ satisfies the hypotheses (H1) and (H2) of Theorem 3.1. Therefore, for each $t \in \mathbb{Q}_0^+$, there is a subsequence $(u_{n_k}(t, \cdot))$ and a function $\bar{u}(t, \cdot)$, satisfying

$$|\bar{u}(t,x)| \leq M, \qquad \Phi\text{-var}\, \bar{u}(t) \leq V, \tag{3.15}$$

and such that $u_{n_k}(t,x) \to \bar{u}(t,x)$ at all points $x \in \mathbb{R}$, as $k \to \infty$. Performing a diagonal argument we extract a subsequence (u_m) with the property that

$$u_m(t,x) \to \bar{u}(t,x) \qquad \text{for all } x \in \mathbb{R}, \text{ for all } t \in \mathbb{Q}_0^+. \tag{3.16}$$

This (u_m) is the subsequence in the statement of the theorem. The issue now is that the resulting function $\bar{u}$ is only defined for rational times. The remainder of the proof concerns extending $\bar{u}$ to irrational times, and then modifying it, in such a way that (c0)-(c5) all hold.

We note that $\bar{u}$ satisfies (c2) and (c3), with t restricted to $\mathbb{Q}_0^+$. Also, whenever $K \subset\subset \mathbb{R}_x$, then, according to (h1), dominated convergence, (h3), and continuity of Φ, we obtain

$$\begin{aligned}\int_K \Phi(|\bar{u}(t,x) - \bar{u}(s,x)|)\, dx &= \lim_m \int_K \Phi(|u_m(t,x) - u_m(s,x)|)\, dx \\ &\leq L_K|t-s| \qquad \text{for } t, s \in \mathbb{Q}_0^+.\end{aligned} \tag{3.17}$$

We want to exploit this time-continuity to define a suitable extension $\hat{u}$ of $\bar{u}$, with $\hat{u}(t,x)$ defined on all of $\mathbb{R}_0^+ \times \mathbb{R}$. Of course, we set

$$\hat{u}(t,x) := \bar{u}(t,x) \qquad \text{for all } x \in \mathbb{R}, \text{ whenever } t \in \mathbb{Q}_0^+. \tag{3.18}$$

We also want $\hat{u}$ to satisfy the bound in (c3) for all times $t \in \mathbb{R}_0^+$. We therefore need to be precise about the pointwise values of $\hat{u}(t,x)$ also at irrational times. To do so we proceed as follows. For each $t \notin \mathbb{Q}_0^+$ we fix a sequence $(t_l) \subset \mathbb{Q}^+$ with $t_l \to t$, and then apply Theorem 3.1 to the sequence $(\bar{u}(t_l, \cdot))_l$. This yields a subsequence (t_k) of (t_l) with the property that $(\bar{u}(t_k, x))$ converges for all x, and we define

$$\hat{u}(t,x) := \lim_k \bar{u}(t_k, x) \qquad \text{for all } x \in \mathbb{R}. \tag{3.19}$$

According to Theorem 3.1, (3.18), and (3.15) we have that

$$\Phi\text{-var}\, \hat{u}(t) \leq V \quad \text{ and } \quad |\hat{u}(t,x)| \leq M \quad \text{for all } t \in \mathbb{R}_0^+ \text{ and all } x \in \mathbb{R}. \tag{3.20}$$

Next, fix any two times $t, s \in \mathbb{R}_0^+$. If t and s are both irrational, let (t_k) and (t'_k) be the rational sequences used in (3.19) to define $\hat{u}(t)$ and $\hat{u}(s)$, respectively. If t is rational, we let (t_k) be the constant sequence $t_k \equiv t$; similarly for s. Thanks to (3.19), (3.18), (3.17), continuity of Φ, and dominated convergence, we have

$$\int_K \Phi(|\hat{u}(t,x) - \hat{u}(s,x)|)\, dx = \lim_k \int_K \Phi(|\bar{u}(t_k,x) - \bar{u}(t'_k,x)|)\, dx \leq L_K|t-s| \tag{3.21}$$

for any $K \subset\subset \mathbb{R}_x$ and any t, $s \geq 0$. In particular, applying Lemma 3.2 to $\hat{u}$, we have that

$$\|\hat{u}(t) - \hat{u}(s)\|_{L^1(K;\mathbb{R}^N)} \leq \Psi_K(|t-s|). \tag{3.22}$$

We finally modify $\hat{u}$ by setting

$$u(t,x) := \hat{u}(t, x+), \tag{3.23}$$

which is well-defined according to $(3.20)_1$ and Lemma 2.10. This is the function u in the statement of the theorem. We proceed to check that it is (jointly) Borel measurable in (t,x) and satisfies properties (c0)-(c5).

To verify joint Borel measurability of u we fix the point $(t,x)\in\mathbb{R}_0^+\times\mathbb{R}$ and observe that $u(t,x)$, according to (3.23) and part (b) of Lemma 2.12, is given by

$$u(t,x)=\lim_{\varepsilon\downarrow 0}\hat{u}^\varepsilon(t,x),\qquad \text{where}\quad \hat{u}^\varepsilon(t,x)=\frac{1}{\varepsilon}\int_x^{x+\varepsilon}\hat{u}(t,\xi)\,d\xi.$$

It follows from (3.22) that

$$|\hat{u}^\varepsilon(t,x)-\hat{u}^\varepsilon(s,x)|\le\frac{1}{\varepsilon}\int_x^{x+\varepsilon}|\hat{u}(t,\xi)-\hat{u}(s,\xi)|\,d\xi\le\frac{1}{\varepsilon}\cdot\Psi_{x,\varepsilon}(|t-s|),$$

where $\Psi_{x,\varepsilon}\equiv\Psi_{[x,x+\varepsilon]}$. Also, for any $h\in(0,\varepsilon)$, assumption (h1) gives that

$$|\hat{u}^\varepsilon(t,x+h)-\hat{u}^\varepsilon(t,x)|\le\frac{1}{\varepsilon}\left\{\int_x^{x+h}+\int_{x+\varepsilon}^{x+\varepsilon+h}\right\}|\hat{u}(t,\xi)|\,d\xi\le\frac{2M}{\varepsilon}\cdot|h|.$$

The same estimate holds for $h\in(-\varepsilon,0)$. Thus, for a fixed $\varepsilon>0$, the function $\hat{u}^\varepsilon$ is uniformly continuous, separately in time and space, on a neighborhood of each point (t,x). It follows that $\hat{u}^\varepsilon$ is jointly continuous in (t,x). The function u is therefore a pointwise limit of jointly continuous functions, and hence Borel measurable with respect to (t,x).

Since, by (3.23) and Lemma 2.9, $u(t,x)=\hat{u}(t,x)$ for almost every $x\in\mathbb{R}$ at each fixed time $t\ge 0$, (c5) follows directly from (3.21). Next, (c2) is immediate from $(3.20)_2$, and the definition (3.23) of u. Property (c3) follows from $(3.20)_1$, part (a) of Lemma 2.12, and the definition of u. Property (c4) holds according to $(3.20)_1$, part (c) of Lemma 2.12, and the definition of u. To verify (c0) we need to argue that

$$u_m(t,\cdot)\to u(t,\cdot)\qquad\text{in } L^1_{loc}(\mathbb{R}_x)\text{ at every time } t\in\mathbb{R}_0^+. \tag{3.24}$$

For this we fix $t\ge 0$ and any compact $K\subset\mathbb{R}_x$, and use that $u(t,x)$ agrees x-a.e. with $\hat{u}(t,x)$, such that

$$\int_K|u_m(t,x)-u(t,x)|\,dx=\int_K|u_m(t,x)-\hat{u}(t,x)|\,dx. \tag{3.25}$$

If $t\in\mathbb{Q}_0^+$ then, by (3.18), $\hat{u}(t,x)=\bar{u}(t,x)$ for all x, and the right-hand side of (3.25) tends to zero as $m\to\infty$ thanks to (3.16) and dominated convergence. For irrational times t we argue as follows. Let (t_k) be the sequence of rational times converging to t which is used in the definition (3.19) of $\hat{u}(t,x)$, and estimate the right-hand side of (3.25):

$$\begin{aligned}\int_K|u_m(t,x)-\hat{u}(t,x)|\,dx&\le\int_K|u_m(t,x)-u_m(t_k,x)|\,dx+\int_K|u_m(t_k,x)-\bar{u}(t_k,x)|\,dx\\&\quad+\int_K|\bar{u}(t_k,x)-\hat{u}(t,x)|\,dx\\&\le\Psi_K(|t-t_k|)+\int_K|u_m(t_k,x)-\bar{u}(t_k,x)|\,dx\\&\quad+\int_K|\bar{u}(t_k,x)-\hat{u}(t,x)|\,dx,\end{aligned}\tag{3.26}$$

where we have used (h3) and Lemma 3.2 (applied to u_m). For $\varepsilon>0$, choose k large so that:

- by continuity of Ψ_K at 0, the first term on the right-hand side of (3.26) is less than $\varepsilon/3$,
- according to (3.19), boundedness of $\bar{u}$ and $\hat{u}$, and dominated convergence, the last term on the right-hand side of (3.26) is less than $\varepsilon/3$.

Then, for this k, apply the definition of $\bar{u}(t_k, x)$ in (3.16), together with boundedness and dominated convergence, to choose m so large that the second term on the right-hand side of (3.26) is less than $\varepsilon/3$. Together with (3.25) this yields (3.24), i.e., (c0). Finally, by boundedness of u and u_m, and dominated convergence, (c1) follows from (c0). □

4. Time continuity and Φ-variation of front tracking approximations

In this section we consider piecewise constant solutions generated by front tracking [2,7]. We fix a piecewise affine and continuous function $f : \mathbb{R} \to \mathbb{R}$, and a right-continuous step function $\bar{u}(x)$ with compact support and with range in the set of break points of f. We denote by L the Lipschitz constant of $f|_{\mathrm{range}(\bar{u})}$, and by $u(t, x)$ the version of the Kružkov solution with the property that $x \mapsto u(t, x)$ is right-continuous at each time $t \geq 0$. Each jump in $\bar{u}$ defines a so-called Riemann problem at time $t = 0$, whose Kružkov solution yields a fan of fronts (i.e., straight line segments across which $u(t, \cdot)$ suffer discontinuities), emanating from each point of discontinuity of $\bar{u}$. Within each fan, $x \mapsto u(t, x)$ is monotone. Whenever two or more fronts meet ("interact"), a new Riemann problem is defined, resolved, and its solution propagated forward in time. It is well-known that in the present setup (i.e., f piecewise affine and $\mathrm{range}(\bar{u}) \subset \{\text{breakpoints of } f\}$), only finitely many Riemann problems need to be solved in order to determine $u(t, x)$ globally. For details see [2,7].

Proposition 4.1. *With the setup described above we have*

$$\int_{\mathbb{R}} \Phi(|u(t,x) - u(s,x)|)\,dx \leq 2L \cdot \Phi\text{-var}\, u(s) \cdot |t-s| \qquad \textit{for any } s, t \geq 0. \tag{4.1}$$

Proof. Assume $t > s$ and set $\Delta := t - s$ and $h := L\Delta$. Also, let M be such that

$$\operatorname{supp} u(s, \cdot) \subset (-M, M). \tag{4.2}$$

Setting

$$I(x, h) := [x - h, x + h],$$

the maximum principle and finite speed of propagation (Section 6.2 in[3]), imply

$$\min_{y \in I(x,h)} u(s, y) \leq u(t, x) \leq \max_{y \in I(x,h)} u(s, y).$$

It follows that

$$|u(t, x) - u(s, x)| \leq \max_{y \in I(x,h)} u(s, y) - \min_{y \in I(x,h)} u(s, y),$$

so that

$$\Phi(|u(t, x) - u(s, x)|) \leq \Phi\text{-var}_{I(x,h)}\, u(s).$$

We thus have

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(|u(t,x) - u(s,x)|)\,dx &\equiv \int_{-M-h}^{M+h} \Phi(|u(t,x) - u(s,x)|)\,dx \\
&\leq \int_{-M-h}^{M+h} \Phi\text{-var}_{I(x,h)}\, u(s)\,dx \\
&= \int_{-M-h}^{-M} \Phi\text{-var}_{I(x,h)}\, u(s)\,dx + \int_{-M}^{M+h} \Phi\text{-var}_{I(x,h)}\, u(s)\,dx \\
&= \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned} \tag{4.3}$$

For $\mathcal{J}_1$ we use (4.2) and Lemma 2.5 (applied to $u|_{I(x,h)}$ and with $a = x - h$, $b = -M$, $c = x + h$, and for $-M - h < x < -M$) to obtain

$$\mathcal{J}_1 = \int_{-M-h}^{-M} \Phi\text{-var}_{(-M,x+h]}\, u(s)\, dx. \tag{4.4}$$

For $\mathcal{J}_2$ we use Lemma 2.3 and Lemma 2.5 to get

$$\begin{aligned}\Phi\text{-var}_{I(x,h)}\, u(s) &\le \Phi\text{-var}_{(-M-h,x+h]}\, u(s) - \Phi\text{-var}_{(-M-h,x-h]}\, u(s)\\ &= \Phi\text{-var}_{(-M,x+h]}\, u(s) - \Phi\text{-var}_{(-M-h,x-h]}\, u(s),\end{aligned}$$

so that

$$\mathcal{J}_2 \le \int_{-M}^{M+h} \Phi\text{-var}_{(-M,x+h]}\, u(s)\, dx - \int_{-M}^{M+h} \Phi\text{-var}_{(-M-h,x-h]}\, u(s)\, dx.$$

Combining this with the expression for $\mathcal{J}_1$ in (4.4), (4.3) gives

$$\begin{aligned}\int_{\mathbb{R}} \Phi(|u(t,x) - u(s,x)|)\, dx &\le \mathcal{J}_1 + \mathcal{J}_2\\ &\le \int_{-M-h}^{M+h} \Phi\text{-var}_{(-M,x+h]}\, u(s)\, dx - \int_{-M}^{M+h} \Phi\text{-var}_{(-M-h,x-h]}\, u(s)\, dx = \mathcal{I}_1 - \mathcal{I}_2.\end{aligned}$$

For $\mathcal{I}_2$, Lemma 2.5 gives

$$\Phi\text{-var}_{(-M-h,x-h]}\, u(s) = \begin{cases} 0 & x - h < -M\\ \Phi\text{-var}_{(-M,x-h]} & x - h > -M,\end{cases}$$

so that

$$\mathcal{I}_2 = \int_{-M+h}^{M+h} \Phi\text{-var}_{(-M,x-h]}\, u(s)\, dx.$$

Finally, using the integration variables $x' := x + h$ in $\mathcal{I}_1$ and $x' := x - h$ in $\mathcal{I}_2$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \Phi(|u(t,x) - u(s,x)|)\, dx &\le \mathcal{I}_1 - \mathcal{I}_2\\ &= \int_{-M}^{M+2h} \Phi\text{-var}_{(-M,x']}\, u(s)\, dx' - \int_{-M}^{M} \Phi\text{-var}_{(-M,x']}\, u(s)\, dx'\\ &= \int_{M}^{M+2h} \Phi\text{-var}_{(-M,x']}\, u(s)\, dx' \le 2h \cdot \Phi\text{-var}\, u(s).\end{aligned}$$

□

Remark 4.2. *Presumably, the factor 2 in (4.1) is superfluous; we have not pursued an optimal estimate.*

Proposition 4.3. *With the setup described above we have*

$$\Phi\text{-var}\, u(t,\cdot) \le \Phi\text{-var}\, \bar{u} \qquad \textit{at each time } t \ge 0. \tag{4.5}$$

Proof. Let $0 < t_1 < \cdots < t_M$ denote the times of front interactions in the solution $u(t,x)$. Due to the structure of the solution, it suffices to argue for (4.5) for times $t \in (0, t_1]$.

First consider a time $t \in (0, t_1)$. Fix any extremal partition $\pi = \{x_0, ..., x_k\}$ with respect to $u(t,\cdot)$, and let $u(t,\cdot)$ is constant on the open interval $C_0 := (-\infty, a_1)$, and the half-open intervals $C_j := [b_j, a_{j+1})$, $j = 1, \ldots, N$ (with $a_{N+1} = +\infty$). The C_j are called "constancy zones." In the remaining "fan-zones" $F_j := [a_j, b_j)$, $j = 1, \ldots, N$, the function $u(t,\cdot)$ is monotone.

Set $u_i := u(t, x_i)$, $i = 1, \dots, k$, and write "$u_i = \max$" (respectively, "$u_i = \min$") to mean that the value u_i is strictly larger (respectively, smaller) than the neighboring values $u_{i\pm1}$. As π is extremal with respect to $u(t, \cdot)$, we have either $u_i = \max$ or $u_i = \min$ for each i. The goal is to build a new partition $\pi' = \{x'_0, ..., x'_k\}$ with the following properties:

(I) each x'_i lies in one of the constancy zones C_j, and
(II) $\Phi\text{-var}\, u(t)[\pi] \le \Phi\text{-var}\, u(t)[\pi']$.

Assuming these for now, it follows from (I) and the structure of front-tracking solutions, that there there is a partition $\pi'' = \{x''_0, \dots, x''_k\}$ such that $u(t, x'_i) = \bar{u}(x''_i)$ for each i. Thus, $\Phi\text{-var}\, u(t)[\pi'] = \Phi\text{-var}\, \bar{u}[\pi''] \le \Phi\text{-var}\, \bar{u}$, so that property (II) and Proposition 2.7 give

$$\Phi\text{-var}\, u(t) = \sup_{\pi \in \mathrm{Ext}(\mathbb{R}, u)} \Phi\text{-var}\, u(t)[\pi] \le \Phi\text{-var}\, \bar{u}. \tag{4.6}$$

It remains to define π' and argue for (I) and (II). The $x'_i \in \pi'$ are specified as follows. First, for each i, $1 \le i \le k$, let $j(i)$ be the unique index such that $x_i \in C_{j(i)} \cup F_{j(i)}$. Then:

(1) if $x_i \in C_{j(i)}$, we set $x'_i := x_i$;
(2) if $x_i \in F_{j(i)}$ we define x'_i to be a point in either $C_{j(i)-1}$ or in $C_{j(i)}$, according to the following rules:
   (a) if $u_i = \max$ and $u(t)$ is increasing on $F_{j(i)}$, then $x'_i \in C_{j(i)}$
   (b) if $u_i = \max$ and $u(t)$ is decreasing on $F_{j(i)}$, then $x'_i \in C_{j(i)-1}$
   (c) if $u_i = \min$ and $u(t)$ is increasing on $F_{j(i)}$, then $x'_i \in C_{j(i)-1}$
   (d) if $u_i = \min$ and $u(t)$ is decreasing on $F_{j(i)}$, then $x'_i \in C_{j(i)}$.

In other words, the x'_i are obtained from the x_i by, first, leaving unchanged any x_i in a constancy zone, and, second, moving any x_i located in a fan zone to one of the constancy zones adjacent to it, according to the rules (a)-(d). It is clear that it is possible to do this in such a way that the x'_i satisfy $x'_0 < \cdots < x'_k$. (In fact, an additional argument shows that, since π is extremal for $u(t, x)$, there can be at most one x'_i in any constancy zone C_j.) Also, (I) is satisfied by construction, while (II) is a direct consequence of the following inequality:

$$|u'_{i+1} - u'_i| \ge |u_{i+1} - u_i| \qquad \text{for each } i = 0, \dots, k-1. \tag{4.7}$$

To argue for (4.7), we observe that the rules (a)-(d) above are such that: if $u_i = \max$ then $u'_i \ge u_i$, and if $u_i = \min$ then $u'_i \le u_i$. Thus, if $u_i = \max$, then $u_{i+1} = \min$ and we have $u'_i \ge u_i > u_{i+1} \ge u'_{i+1}$, and if $u_i = \min$, then $u_{i+1} = \max$ and we have $u'_{i+1} \ge u_{i+1} > u_i \ge u'_i$. In either case (4.7) holds.

Finally, assume $t = t_1$ is the first time when two or more front meet, let $\pi = \{x_0, \dots, x_k\}$ be any partition, and fix a time $t' \in (0, t_1)$. It is clear from the structure of front-tracking solutions that we can find a partition $\pi' = \{x'_0, \dots, x'_k\}$ with the property that $u(t, x_i) = u(t', x'_i)$ for $i = 0, \dots, k$. It follows that $\Phi\text{-var}\, u(t)[\pi] = \Phi\text{-var}\, u(t')[\pi'] \le \Phi\text{-var}\, u(t')$. As π is arbitrary, we obtain $\Phi\text{-var}\, u(t) \le \Phi\text{-var}\, u(t')$. Combining this with the first part of the proof, shows that (4.5) holds also in this case. □

5. Scalar conservation laws with data in Φ-BV(ℝ)

Theorem 5.1. *Assume $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, and let $\bar{u}$ be a compactly supported function with $\Phi\text{-var}\, \bar{u} < \infty$ for some $\Phi : [0, \infty) \to \mathbb{R}$ satisfying (p1)-(p4). Let L be the Lipschitz constant of $f|_{range(\bar{u})}$. Then the right-continuous version u of the Kružkov* *solution of the Cauchy problem*

$$\partial_t u(t,x) + \partial_x[f(u(t,x))] = 0 \qquad t>0,\, x\in\mathbb{R} \tag{5.1}$$

$$u(0,x) = \bar{u}(x), \tag{5.2}$$

satisfies

$$\Phi\text{-var}\, u(t,\cdot) \le \Phi\text{-var}\, \bar{u}, \tag{5.3}$$

and

$$\int_{\mathbb{R}} \Phi(|u(t,x)-u(s,x)|)\, dx \le 2L\cdot \Phi\text{-var}\, \bar{u}\cdot |t-s| \qquad \textit{for any times } s,t\ge 0. \tag{5.4}$$

Proof. With the results from the previous sections in place, the proof follows closely the argument for the standard case of BV initial data (see [2]), and we therefore provide only a brief sketch.

First, without loss of generality we may assume that $\bar{u}$ is right continuous. Applying Proposition 2.13, we let $(\bar{u}_n)$ be a sequence of right-continuous step functions with compact support, agreeing with $\bar{u}$ at the left endpoints of their intervals of constancy, and such that $\bar{u}_n \to \bar{u}$ uniformly. It follows that $\Phi\text{-var}\, \bar{u}_n \le \Phi\text{-var}\, \bar{u}$. Let f_n denote the piecewise affine and continuous function that coincides with the given flux f at the points $\frac{1}{n}\mathbb{Z}\cup \mathrm{range}(\bar{u}_n)$, and let $u_n(t,x)$ denote the corresponding right continuous version of the Kružkov solution of (5.1) with f replaced by f_n, and with initial data $\bar{u}_n$. According to Propositions 4.1 and 4.3, and the maximum principle for scalar conservation laws, the sequence $(u_n(t,x))$ satisfy assumptions (h1)-(h3) in Theorem 3.3 (with $L_K := 2L\cdot \Phi\text{-var}\, \bar{u}$). It is now standard (see [2, 7]) to verify that the limit $u(t,x)$, the existence of which is guaranteed by Theorem 3.3, is the right-continuous version of the unique Kružkov solution of (5.1)-(5.2). According to the same theorem, u satisfies (5.3) and (5.4). □

Finally, by combining this result with Theorem 2.11 we obtain the following.

Corollary 5.2. *Assume $\bar{u}$ is a regulated function with compact support. Then the Kružkov solution u of (5.1)-(5.2) is such that the function $x\mapsto u(t,x)$ is regulated at each time $t\ge 0$.*

Proof. According to Theorem 2.11 there is a function Φ with properties (p1)-(p4), and such that $\Phi\text{-var}\, \bar{u} < \infty$. Theorem 5 gives $\Phi\text{-var}\, u(t,\cdot) < \infty$ for each $t\ge 0$, and Lemma 2.10 gives that $u(t,\cdot)$ is regulated. □

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