LEFSCHETZ THEORY FOR EXTERIOR ALGEBRAS AND FERMIONIC DIAGONAL COINVARIANTS

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Abstract. Let W be an irreducible complex reflection group acting on its reflection representation V. We consider the doubly graded action of W on the exterior algebra $\wedge(V \oplus V^*)$ as well as its quotient $DR_W := \wedge(V \oplus V^*)/\langle \wedge(V \oplus V^*)^W_+ \rangle$ by the ideal generated by its homogeneous W-invariants with vanishing constant term. We describe the bigraded isomorphism type of DR_W; when $W = \mathfrak{S}_n$ is the symmetric group, the answer is a difference of Kronecker products of hook-shaped $\mathfrak{S}_n$-modules. We relate the Hilbert series of DR_W to the (type A) Catalan and Narayana numbers and describe a standard monomial basis of DR_W using a variant of Motzkin paths. Our methods are type-uniform and involve a Lefschetz-like theory which applies to the exterior algebra $\wedge(V \oplus V^*)$.

1. Introduction

Let $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ be a polynomial ring in $2n$ variables equipped with the diagonal action of the symmetric group $\mathfrak{S}_n$:

$$w.x_i := x_{w(i)} \quad \text{and} \quad w.y_i := y_{w(i)} \tag{1.1}$$

for all $w \in \mathfrak{S}_n$ and $1 \leq i \leq n$. The quotient of $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ by the ideal generated by the homogeneous $\mathfrak{S}_n$-invariants of positive degree is the *diagonal coinvariant ring*; its bigraded $\mathfrak{S}_n$-structure was calculated by Haiman [11] using algebraic geometry.

In the last couple years, algebraic combinatorialists have studied variations of the diagonal coinvariants involving sets of commuting and anti-commuting variables [3, 6, 9, 16, 18, 21, 22]. In this paper we completely describe the bigraded $\mathfrak{S}_n$-structure of the diagonal coinvariants involving two sets of anti-commuting variables (but no commuting variables). Our methods apply equally well (and uniformly) to any irreducible complex reflection group W[1] as to the symmetric group $\mathfrak{S}_n$.

Let W be an irreducible complex reflection group of rank n acting on its reflection representation $V \cong \mathbb{C}^n$. The action of W on V induces an action of W on

- the dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$,
- the direct sum $V \oplus V^*$ of V with its dual space, and finally
- the exterior algebra $\wedge(V \oplus V^*)$ over the $2n$-dimensional vector space $V \oplus V^*$.

By placing V in bidegree $(1, 0)$ and V^* in bidegree $(0, 1)$, this last space $\wedge(V \oplus V^*)$ attains the structure of a doubly graded W-module.

If we let $\Theta_n = (\theta_1, \dots, \theta_n)$ be a basis for V and $\Xi_n = (\xi_1, \dots, \xi_n)$ be a basis for V^*, we have a natural identification

$$\wedge(V \oplus V^*) = \wedge\{\Theta_n, \Xi_n\} \tag{1.2}$$

of $\wedge(V \oplus V^*)$ with the exterior algebra $\wedge\{\Theta_n, \Xi_n\}$ generated by the symbols θ_i and ξ_i over $\mathbb{C}$. Following the terminology of physics, we refer to the θ_i and ξ_i as *fermionic* variables. In physics, such variables are used to model fermions, with relations $\theta_i^2 = \xi_i^2 = 0$ corresponding to the Pauli Exclusion Principle: no two fermions can occupy the same state at the same time[2]. The model

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[1] And, in fact, to a wider class of groups G; see Remark 4.5.

[2] A commuting variable x_i is called *bosonic*; the power x_i^2 corresponds to two indistinguishable bosons in State i.

$\wedge\{\Theta_n, \Xi_n\}$ for $\wedge(V \oplus V^*)$ will be helpful in our arguments. The following quotient ring is our object of study.

Definition 1.1. *The* fermionic diagonal coinvariant ring *is the quotient*

$$DR_W := \wedge(V \oplus V^*)/\langle \wedge(V \oplus V^*)_+^W \rangle \tag{1.3}$$

of $\wedge(V \oplus V^*)$ by the (two-sided) ideal generated by the subspace $\wedge(V \oplus V^*)_+^W \subseteq \wedge(V \oplus V^*)$ of W-invariant elements with vanishing constant term.

The ideal $\langle \wedge(V \oplus V^*)_+^W \rangle$ is W-stable and bihomogeneous, so the quotient ring DR_W has the structure of a bigraded W-module. We will see (Proposition 4.1) that this ideal is principal, generated by a 'Casimir element' $\delta_W \in V \otimes V^*$. Our results are as follows.

- We describe the bigraded W-isomorphism type of DR_W in terms of the isomorphism types of the exterior powers $\wedge^i V$ and $\wedge^j V^*$ (Theorem 4.2).
- We show that $\dim DR_W = \binom{2n+1}{n}$ whenever W has rank n and relate the dimensions of its graded pieces to Catalan and Narayana numbers (Corollaries 4.3 and 4.4).
- We describe an explicit monomial basis of DR_W using a variant of Motzkin paths and describe the bigraded Hilbert series of DR_W in terms of the combinatorics of these paths (Theorem 5.2).
- When $W = \mathfrak{S}_n$, in Section 6 we give variants of the above results as they apply to the n-dimensional permutation representation of $\mathfrak{S}_n$ (as opposed to its $(n-1)$-dimensional reflection representation).

The key tool in our analysis is the realization (Theorem 3.2) of the Casimir generator δ_W of the ideal defining DR_W as a kind of 'W-invariant Lefschetz element' in the ring $\wedge(V \oplus V^*)$. The ring $\wedge(V \oplus V^*)$, similar to the cohomology ring of a compact smooth complex manifold, satisfies 'bigraded' versions of Poincaré Duality and the Hard Lefschetz Theorem. This is somewhat unusual on two counts.

- Any homogeneous linear form in an exterior algebra squares to zero, and hence is not well-suited to be a (strong) Lefschetz element.
- Lefschetz elements arising in coinvariant theory are rarely W-invariant. For example, if W is a Weyl group with associated complete flag manifold G/B we may present the cohomology of G/B as

$$H^\bullet(G/B; \mathbb{C}) = \mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]_+^W \rangle \tag{1.4}$$

  where $\mathfrak{h}$ is the Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G. An element $\ell \in \mathbb{C}[\mathfrak{h}]_1 = \mathfrak{h}^*$ is a Lefschetz element if and only if it is not fixed by any element of W [14]. So the Lefschetz property is in some sense opposite to W-invariance in this case.

For examples of coinvariant-like quotients of *superspace* $\mathbb{C}[V] \otimes \wedge V^*$ satisfying other nontraditional bigraded versions of Poincaré Duality and (conjecturally) Hard Lefschetz, see [18].

The remainder of the paper is organized as follows. In **Section 2** we give background material on complex reflection groups, Gröbner theory associated to exterior algebras, and the representation theory of $\mathfrak{S}_n$. In **Section 3** we prove that $\wedge(V \oplus V^*)$ satisfies bigraded versions of the Hard Lefschetz Property and Poincaré Duality. This builds on work of Hara and Watanabe [12] showing that the incidence matrix between complementary ranks of the Boolean poset $B(n)$ is invertible. In **Section 4** we apply these Lefschetz results to determine the bigraded W-structure of DR_W. In **Section 5** we describe the standard monomial basis of DR_W using lattice paths. In **Section 6** we specialize to $W = \mathfrak{S}_n$ and translate our results to the setting of the permutation representation of $\mathfrak{S}_n$. We close in **Section 7** with some open problems.

2. Background

2.1. **Complex reflection groups.** Let $V = \mathbb{C}^n$ be an n-dimensional complex vector space. An element $t \in \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ is a *reflection* if its fixed space $V^t := \{v \in V \, : \, t(v) = v\}$ satisfies $\dim V^t = n-1$.

A finite subgroup $W \subseteq GL(V)$ is a *complex reflection group* if it is generated by reflections. The W-module V is called the *reflection representation* of W. The dimension $\dim V = n$ of V is called the *rank* of W.

If W_1 and W_2 are reflection groups with reflection representations V_1 and V_2, the direct product $W_1 \times W_2$ is naturally a reflection group with reflection representation $V_1 \oplus V_2$. A reflection group W acting on V is *irreducible* if it is impossible to express W as a direct product $W_1 \times W_2$ of reflection groups acting on $V = V_1 \oplus V_2$ unless $V_1 = 0$ or $V_2 = 0$.

2.2. **Exterior Gröbner theory.** Let $\Theta_n = (\theta_1, \dots, \theta_n)$ be a list on n anticommuting variables and let $\wedge\{\Theta_n\}$ be the exterior algebra generated by these variables over $\mathbb{C}$. For any subset $S = \{i_1 < i_2 < \cdots < i_k\} \subseteq \{1, 2, \dots, n\}$, we let

$$\theta_S := \theta_{i_1}\theta_{i_2}\cdots\theta_{i_k} \tag{2.1}$$

where the multiplication is in increasing order of subscripts. We refer to the θ_S as *monomials*; the set $\{\theta_S \, : \, S \subseteq \{1, 2, \dots, n\}\}$ is the monomial basis of $\wedge\{\Theta_n\}$. Given two monomials θ_S and θ_T, we write $\theta_S \mid \theta_T$ to mean $S \subseteq T$.

A total order $<$ on the set $\{\theta_S \, : \, S \subseteq \{1, 2, \dots, n\}\}$ is a *term order* if

- we have $1 = \theta_\varnothing \leq \theta_S$ for all S and
- for all subsets S, T, U with $U \cap S = U \cap T = \varnothing$, $\theta_S < \theta_T$ implies $\theta_{S \cup U} < \theta_{T \cup U}$.

Given a term order $<$, for any nonzero element $f \in \wedge\{\Theta_n\}$, let $\textsc{lm}(f)$ be the largest monomial θ_S under the total order $<$ such that θ_S appears with nonzero coefficient in f. If $I \subseteq \wedge\{\Theta_n\}$ is a two-sided ideal, let

$$\textsc{lm}(I) := \{\textsc{lm}(f) \, : \, f \in I - \{0\}\}$$

stand for the set of leading monomials of nonzero elements in I. The collection of *standard monomials* (or *normal forms*) for I is

$$N(I) := \{\text{monomials } \theta_S \, : \, S \subseteq \{1, 2, \dots, n\} \text{ and } \theta_S \notin \textsc{lm}(I)\}. \tag{2.2}$$

The set $N(I)$ of monomials descends to a $\mathbb{C}$-basis of the quotient $\wedge\{\Theta_n\}/I$; this is the *standard monomial basis* with respect to $<$ (see for example [5]).

2.3. **Representation Theory.** If $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a bigraded vector space with each piece $V_{i,j}$ finite-dimensional, the *bigraded Hilbert series* is $\mathrm{Hilb}(V; q, t) := \sum_{i,j \geq 0} \dim V_{i,j} \cdot q^i t^j$. This is a formal power series in q and t.

The irreducible representations of the symmetric group $\mathfrak{S}_n$ are in one-to-one correspondence with partitions $\lambda \vdash n$. Given $\lambda \vdash n$, let S^λ be the corresponding $\mathfrak{S}_n$-irreducible. For example, the trivial representation is $S^{(n)}$ and the sign representation is $S^{(1^n)}$.

Let Λ denote the ring of symmetric functions and let $\{s_\lambda \, : \, \lambda \text{ a partition}\}$ denote its Schur basis. The *Hall inner product* on Λ declares the Schur basis to be orthonormal:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \tag{2.3}$$

for any partitions λ and μ.

Any finite-dimensional $\mathfrak{S}_n$-module U may be expressed uniquely as a direct sum $U \cong \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$ for some multiplicities c_λ. The *Frobenius image* of U is the symmetric function

$$\mathrm{Frob}(U) := \sum_{\lambda \vdash n} c_\lambda s_\lambda, \tag{2.4}$$

where s_λ is the Schur function. The *Kronecker product* of two Schur functions s_λ and s_μ for $\lambda, \mu \vdash n$ is defined by

$$s_\lambda * s_\mu = \mathrm{Frob}(S^\lambda \otimes S^\mu) \tag{2.5}$$

where $\mathfrak{S}_n$ acts diagonally on $S^\lambda \otimes S^\mu$.

If $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a bigraded $\mathfrak{S}_n$-module with each piece $V_{i,j}$ finite-dimensional, the *bigraded Frobenius image* is

$$\mathrm{grFrob}(V; q, t) = \sum_{i,j \geq 0} \mathrm{Frob}(V_{i,j}) \cdot q^i t^j. \tag{2.6}$$

This is a formal power series in q and t with coefficients in the ring of symmetric functions.

3. Lefschetz theory for exterior algebras

Let $A = \bigoplus_{i=0}^n A_i$ be a commutative graded $\mathbb{C}$-algebra. The algebra A satisfies *Poincaré Duality* (PD) if $A_n \cong \mathbb{C}$ and if the multiplication map $A_i \otimes A_{n-i} \to A_n \cong \mathbb{C}$ is a perfect pairing for $0 \leq i \leq n$. In particular, this implies that the Hilbert series of A is palindromic, i.e. $\dim A_i = \dim A_{n-i}$.

If $A = \bigoplus_{i=0}^n A_i$ satisifes PD, an element $\ell \in A_1$ is called a *(strong) Lefschetz element* if for every $0 \leq i \leq n/2$, the linear map

$$\ell^{n-2i} \cdot (-) : A_i \to A_{n-i} \tag{3.1}$$

given by multiplication by ℓ^{n-2i} is bijective. If A has a Lefschetz element, it is said to satisfy the *Hard Lefschetz Property* (HL).

Algebras A which satisfy PD and HL arise naturally in geometry as the cohomology rings (with adjusted grading) of smooth complex projective varieties. HL for of the cohomology ring of the n-fold product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ of 1-dimensional complex projective space with itself was studied combinatorially by Hara and Watanabe [12].

Recall that the *Boolean poset* $B(n)$ is the partial order on all subsets $S \subseteq \{1, \ldots, n\}$ given by $S \leq T$ if and only if $S \subseteq T$. The poset $B(n)$ is graded, with the i^{th} rank given by the family $B(n)_i$ of i-element subsets of $\{1, \ldots, n\}$. The following classical result states that the incidence matrix between complementary ranks of $B(n)$ is invertible.

Theorem 3.1. *Given $r \leq s \leq n$, define a $\binom{n}{s} \times \binom{n}{r}$ matrix $M_n(r,s)$ with rows indexed by $B(n)_s$ and columns indexed by $B(n)_r$ with entires*

$$M_n(r,s)_{T,S} = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

For any $0 \leq i \leq n$, the square matrix $M_n(i, n-i)$ is invertible.

For example, if $n = 4$ and $i = 1$, Theorem 3.1 asserts that the $0,1$-matrix $M_4(1,3)$ given by

$$\begin{array}{c} \\ \{1,2,3\} \\ \{1,2,4\} \\ \{1,3,4\} \\ \{2,3,4\} \end{array} \begin{array}{c} \begin{array}{cccc} \{1\} & \{2\} & \{3\} & \{4\} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

is invertible. The origins of Theorem 3.1 are difficult to trace. This result follows easily from the fact that the 'up operator' $U : \mathbb{C}B(n)_i \to \mathbb{C}B(n)_{i+1}$ defined by

$$U(S) := \sum_{\substack{S \subset T \\ |T - S| = 1}} T$$

is injective whenever $i < n/2$; this is the main lemma in one standard proof that the coefficient sequence of the *q-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n-1)(q^{n-1}-1)\cdots(q-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)\times(q^{n-k}-1)(q^{n-k-1}-1)\cdots(q-1)}$$

is unimodal. Stanley [20, Prop. 2.9, $P = B(n)$] gave a proof of Theorem 3.1 in the context of differential posets which describes the (nonzero) eigenvalues of $M_n(i, n-i)$. Hara-Watanabe [12] gave a formula for the (nonzero) determinant of $M_n(i, n-i)$; this latter proof of Theorem 3.1 has the following geometric interpretation.

The cohomology ring of the n-fold product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ may be presented as

$$H^\bullet(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1; \mathbb{C}) = \mathbb{C}[x_1, \dots, x_n]/\langle x_1^2, \dots, x_n^2\rangle \tag{3.3}$$

where x_i represents the Chern class $c_1(\mathcal{L}_i)$ and $\mathcal{L}_i$ is the dual of the tautological line bundle over the i^{th} factor of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. Hara and Watanabe used Theorem 3.1 to give a combinatorial proof of the fact that $x_1 + \cdots + x_n$ is a Lefschetz element for the ring $H^\bullet(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1; \mathbb{C})$ [12].

We want to study PD and HL in the context of the exterior algebra $\wedge\{\Theta_n, \Xi_n\}$. This algebra satisfies a natural bigraded version of Poincaré Duality: the top bidegree $\wedge\{\Theta_n, \Xi_n\}_{n,n}$ is 1-dimensional and the multiplication map

$$\wedge\{\Theta_n, \Xi_n\}_{i,j} \otimes \wedge\{\Theta_n, \Xi_n\}_{n-i,n-j} \to \wedge\{\Theta_n, \Xi_n\}_{n,n} \cong \mathbb{C} \tag{3.4}$$

is a perfect pairing for any $0 \leq i, j \leq n$.

The notion of a Lefschetz element in $\wedge\{\Theta_n, \Xi_n\}$ is a bit more subtle because any linear form ℓ in the variables $\theta_1, \dots, \theta_n, \xi_1, \dots, \xi_n$ satisfies $\ell^2 = 0$. To get around this, we introduce the element

$$\delta_n := \theta_1\xi_1 + \theta_2\xi_2 + \cdots + \theta_n\xi_n \in \wedge\{\Theta_n, \Xi_n\}_{1,1} \tag{3.5}$$

The following result states that δ_n is a bigraded version of a Lefschetz element for the ring $\wedge\{\Theta_n, \Xi_n\}$.

Theorem 3.2. *Suppose $i + j \leq n$ and let $r = n - i - j$. The linear map*

$$\varphi : \wedge\{\Theta_n, \Xi_n\}_{i,j} \xrightarrow{\delta_n^r \cdot} \wedge\{\Theta_n, \Xi_n\}_{n-j,n-i} \tag{3.6}$$

given by multiplication by δ_n^r is bijective.

Proof. The idea is to introduce strategically chosen bases of the domain and target of φ and show that the matrix representing φ with respect to these bases is invertible using Theorem 3.1.

Given two subsets $A, B \subseteq \{1, \dots, n\}$, write

$$A - B = \{a_1 < a_2 < \cdots < a_r\}$$
$$B - A = \{b_1 < b_2 < \cdots < b_s\}$$
$$A \cap B = \{c_1 < c_2 < \cdots < c_t\}$$

and set

$$\mathbf{v}(A, B) := \xi_{c_1}\theta_{c_1}\xi_{c_2}\theta_{c_2}\cdots\xi_{c_t}\theta_{c_t} \cdot \theta_{a_1}\theta_{a_2}\cdots\theta_{a_r} \cdot \xi_{b_1}\xi_{b_2}\cdots\xi_{b_s}. \tag{3.7}$$

The family $\{\mathbf{v}(A, B) : A, B \subseteq \{1, \dots, n\}\}$ is a basis of $\wedge\{\Theta_n, \Xi_n\}$. For any sets A and B, a direct computation shows

$$\delta_n \cdot \mathbf{v}(A, B) = \sum_{c \notin A \cup B} \mathbf{v}(A \cup c, B \cup c). \tag{3.8}$$

The somewhat unusual variable order in the product $\mathbf{v}(A, B)$ was chosen strategically so that Equation (3.8) does not contain any signs.

Now suppose $|A| = i$ and $|B| = j$ for $i + j \leq n$ and set $r = n - i - j$. Iterating Equation (3.8) yields

$$\delta_n^r \cdot \mathbf{v}(A, B) = \sum_{\substack{|C|=n-j,\, |D|=n-i \\ A \subseteq C,\ B \subseteq D \\ |C \cap D| - |A \cap B| = r}} r! \cdot \mathbf{v}(C, D). \tag{3.9}$$

We need to show that the (square) matrix of dimensions $\binom{n}{i} \cdot \binom{n}{j} \times \binom{n}{n-j} \cdot \binom{n}{n-i}$ defined by the system (3.9) is invertible. For any $\mathbf{v}(C, D)$ appearing in on the right-hand side of (3.9) we have $A - B = C - D =: I$ and $B - A = D - C =: J$. The matrix representing φ therefore breaks up as a direct sum of smaller matrices indexed by the two sets I and J, so we only need to show that every (I, J)-submatrix is invertible.

For fixed I and J, the submatrix in the previous paragraph is determined by the system

$$\delta_n^r \cdot \mathbf{v}(A, B) = \sum_{\substack{T \subseteq S \\ |T| = r}} r! \cdot \mathbf{v}(A \cup T, B \cup T) \tag{3.10}$$

where $S := \{1, \ldots, n\} - (A \cup B)$. The system (3.10) represents a linear map

$$\begin{aligned} &\operatorname{span}\{\mathbf{v}(A, B) \,:\, |A| = i,\, |B| = j,\, A - B = I,\, B - A = J\} \to \\ &\qquad \operatorname{span}\{\mathbf{v}(C, D) \,:\, |C| = n - j,\, |D| = n - i,\, C - D = I,\, D - C = J\} \end{aligned} \tag{3.11}$$

where all sets A, B, C, D are subsets of $\{1, \ldots, n\}$. In particular, the system (3.10) represents a square matrix of size $\binom{n-|I|-|J|}{k}$ where $k := i - |I| = j - |J|$ is the size of the intersection $|A \cap B|$ for any element $\mathbf{v}(A, B)$ appearing in the LHS. If we let $S' := \{1, \ldots, n\} - (I \cup J)$, the invertibility of the system (3.10) is equivalent to the invertibility of the system

$$\delta_n^r \cdot \mathbf{v}(R, R) = \sum_{\substack{T \subseteq S' \\ |T| = r}} r! \cdot \mathbf{v}(R \cup T, R \cup T) \tag{3.12}$$

representing a linear map

$$\operatorname{span}\{\mathbf{v}(R, R) \,:\, R \subseteq S' \text{ and } |R| = k\} \to \operatorname{span}\{\mathbf{v}(R', R') \,:\, R' \subseteq S' \text{ and } |R'| = k + r\}. \tag{3.13}$$

Observe that

$$k + (k + r) = (i - |I|) + (j - |J|) + n - i - j = n - |I| - |J| = |S'| \tag{3.14}$$

so that the system (3.12) is invertible by Theorem 3.1. □

4. Casimir elements and fermionic diagonal coinvariants

Let W be an irreducible reflection group with reflection representation $V = \mathbb{C}^n$. Let $\theta_1, \ldots, \theta_n$ be a basis of V. Given the choice of $\theta_1, \ldots, \theta_n$, we let $\xi_1, \ldots, \xi_n$ be the *dual basis* of V^* characterized by

$$\xi_i(\theta_j) = \delta_{i,j}. \tag{4.1}$$

We rename the element $\delta_n = \theta_1 \xi_1 + \cdots + \theta_n \xi_n$ of $\wedge(V \oplus V^*)$ studied in the previous section as δ_W:

$$\delta_W := \delta_n = \theta_1 \xi_1 + \cdots + \theta_n \xi_n \in V \otimes V^* \subseteq \wedge(V \oplus V^*). \tag{4.2}$$

We refer to δ_W as the *Casimir element* of W.

The full general linear group $\mathrm{GL}(V)$ acts on $\wedge(V \oplus V^*)$ and it is not difficult to check using elementary matrices and the dual basis property that δ_W is invariant under this action. The $\mathrm{GL}(V)$-invariance of δ_W can be seen more conceptually by noting that δ_W corresponds to id_V under the isomorphism $V \otimes V^* \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ which sends $v \otimes f$ to the linear map $u \mapsto f(u) \cdot v$. This isomorphism $V \otimes V^* \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V, V)$ is $\mathrm{GL}(V)$-equivariant under the diagonal action on $V \otimes V^*$

and the conjugation action on $\mathrm{Hom}_{\mathbb{C}}(V,V)$, and the conjugation action of $\mathrm{GL}(V)$ on $\mathrm{Hom}_{\mathbb{C}}(V,V)$ fixes id_V. The $\mathrm{GL}(V)$-equivariance of δ_W implies that δ_W is independent of the choice of basis $\theta_1,\ldots,\theta_n$. Since $W \subseteq \mathrm{GL}(V)$, the element δ_W lies in the W-invariant subring $\wedge(V\oplus V^*)^W$ of $\wedge(V\oplus V^*)$. In fact,

Proposition 4.1. *The Casimir element δ_W generates the W-invariant subring $\wedge(V\oplus V^*)^W$ of $\wedge(V\oplus V^*)$.*

Proof. Let G be a group and let U, U' be irreducible finite-dimensional complex representations of G. The tensor product $U\otimes U'$ is a G-module by the rule $g.(u\otimes u') := (g.u)\otimes(g.u')$ for $g\in G, u\in U$, and $u'\in U'$. (This is the *Kronecker product* of the modules U and U'.) Since we are working over $\mathbb{C}$, Schur's Lemma implies

$$\dim(U\otimes U')^G = \dim \mathrm{Hom}_G(U',U^*) \tag{4.3}$$

$$= \begin{cases} 1 & U'\cong_G U^* \\ 0 & \text{otherwise.}\end{cases} \tag{4.4}$$

Since W is an irreducible complex reflection group, a result of Steinberg (see [13, Thm. A, §24-3, p. 250]) implies that the exterior powers $\wedge^0V, \wedge^1V,\ldots,\wedge^nV$ are pairwise nonisomorphic irreducible representations of W. The same is true of their duals $(\wedge^0V)^*,(\wedge^1V)^*,\ldots,(\wedge^nV)^*$. Since the (i,j)-bidegree of $\wedge(V\oplus V^*)$ is given by $\wedge(V\oplus V^*)_{i,j} = \wedge^iV\otimes\wedge^j(V^*)$, the argument of the last paragraph gives

$$\dim \wedge(V\oplus V^*)^W_{i,j} = \dim(\wedge^iV\otimes\wedge^j(V^*))^W = \dim(\wedge^iV\otimes(\wedge^jV)^*)^W = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases} \tag{4.5}$$

for any $0\le i,j\le n$ where the second equality used the W-module isomorphism $\wedge^j(V^*)\cong(\wedge^jV)^*$. On the other hand, we have $\delta_W\in\wedge(V\oplus V^*)^W_{1,1}$. A quick computation shows that δ_W^n is a nonzero scalar multiple of $\theta_1\xi_1\theta_2\xi_2\cdots\theta_n\xi_n$ so that each of the powers $\delta_W^0,\delta_W^1,\ldots,\delta_W^n$ is nonzero. The proposition follows. □

We are ready to describe the bigraded W-module structure of DR_W. We state our answer in terms of the *Grothendieck ring* of W. Recall that this is the $\mathbb{Z}$-algebra generated by the symbols $[U]$ where U is a finite-dimensional W-module and subject to a relation $[U]=[U']+[U'']$ for any short exact sequence

$$0\to U'\to U\to U''\to 0. \tag{4.6}$$

In particular, if $U\cong U'$ we have $[U]=[U']$. Multiplication in the Grothendieck ring corresponds to Kronecker product, i.e. $[U]\cdot[U'] := [U\otimes U']$.

Theorem 4.2. *Let W be an irreducible complex reflection group acting on its reflection representation $V=\mathbb{C}^n$ and let $0\le i,j\le n$. If $i+j>n$ we have $(DR_W)_{i,j}=0$. If $i+j\le n$, inside the Grothendieck group of W we have*

$$[(DR_W)_{i,j}] = [\wedge^iV]\cdot[\wedge^jV^*] - [\wedge^{i-1}V]\cdot[\wedge^{j-1}V^*] \tag{4.7}$$

where we interpret $\wedge^{-1}V=\wedge^{-1}V^*=0$.

Proof. Thanks to Proposition 4.1 we can model DR_W as

$$DR_W = \wedge(V\oplus V^*)/\langle\delta_W\rangle = \wedge\{\Theta_n,\Xi_n\}/\langle\delta_n\rangle. \tag{4.8}$$

If $i=0$ or $j=0$, the claim follows since δ_W lies in bidegree $(1,1)$, so assume $i,j>0$.

If $i+j\le n$, let $r=n-i-j+1$. Theorem 3.2 implies that the multiplication map

$$\delta_W^r\cdot(-) : \wedge(V\oplus V^*)_{i-1,j-1}\to\wedge(V\oplus V^*)_{n-j+1,n-i+1} \tag{4.9}$$

is a linear isomorphism. Whenever a composition $f \circ g$ of maps is a bijection, the map g is injective and the map f is surjective. Therefore, the map

$$\delta_W \cdot (-) : \wedge(V \oplus V^*)_{i-1,j-1} \to \wedge(V \oplus V^*)_{i,j} \tag{4.10}$$

is an injection which we know to be W-equivariant. The claimed decomposition of $[(DR_W)_{i,j}]$ follows from (4.8).

Now suppose $i + j > n$. By analogous reasoning

$$\delta_W \cdot (-) : \wedge(V \oplus V^*)_{i-1,j-1} \to \wedge(V \oplus V^*)_{i,j} \tag{4.11}$$

is a W-equivariant surjection so that $(DR_W)_{i,j} = 0$. □

Corollary 4.3. *If W has rank n, the vector space dimension of DR_W is $\dim DR_W = \binom{2n+1}{n}$.*

Proof. Since $\dim \wedge^k V = \dim \wedge^k V^* = \binom{n}{k}$ where V is the reflection representation of W, Theorem 4.2 yields

$$\dim DR_W = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} + \sum_{j=1}^{n} \binom{n}{j-1}\binom{n}{n-j} \tag{4.12}$$

$$= \binom{2n}{n} + \binom{2n}{n-1} \tag{4.13}$$

$$= \binom{2n+1}{n} \tag{4.14}$$

by the Pascal recursion. □

Recall that the *Catalan* and *Narayana* numbers are given by

$$\mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n} \quad \text{and} \quad \mathrm{Nar}(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}. \tag{4.15}$$

We have $\sum_{k=1}^{n} \mathrm{Nar}(n,k) = \mathrm{Cat}(n)$. These numbers have many combinatorial interpretations; for example, $\mathrm{Cat}(n)$ counts the number of Dyck paths from $(0,0)$ to $(2n,0)$ and $\mathrm{Nar}(n,k)$ counts the number of such paths with $k-1$ peaks. The Catalan and Narayana numbers show up as the dimensions of the 'boundary' pieces of DR_W.

Corollary 4.4. *If W has rank n, we have*

$$\dim(DR_W)_{k,n-k} = \mathrm{Nar}(n+1,k+1) \tag{4.16}$$

for $0 \leq k \leq n$ so that

$$\sum_{k=0}^{n} \dim(DR_W)_{k,n-k} = \mathrm{Cat}(n+1). \tag{4.17}$$

Proof. Thanks to Theorem 4.2 one need only verify the identity

$$\mathrm{Nar}(n+1,k+1) = \binom{n}{k}\binom{n}{n-k} - \binom{n}{k-1}\binom{n}{n-k-1}, \tag{4.18}$$

which is straightforward. □

For any reflection group W, there are Catalan and Narayana numbers attached to W (see for example [2]); the numbers appearing in Corollary 4.4 are their type A instances, and depend only on the rank of W.

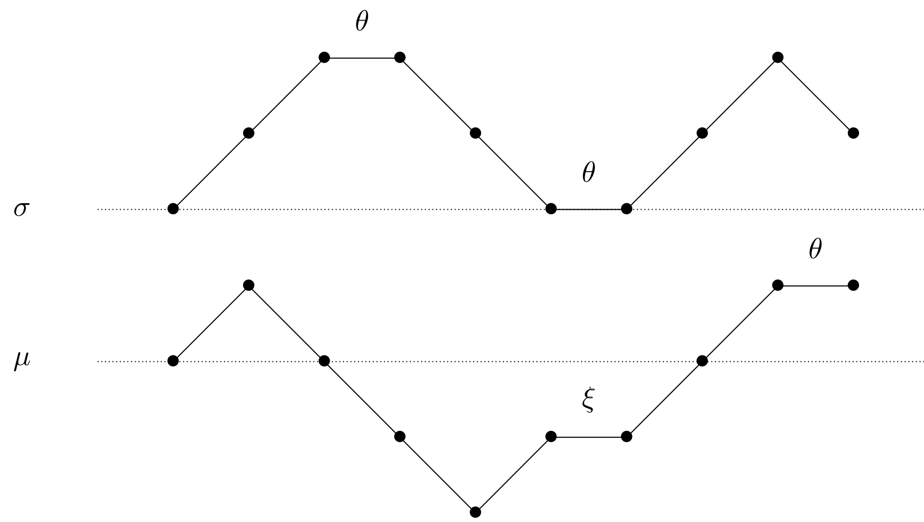

FIGURE 1. Two paths in $\Pi(9)$.

Remark 4.5. *The results in this section and the next apply equally well to any (possibly infinite) group G and any finite-dimensional G-module V for which the exterior powers $\wedge^0 V, \wedge^1 V, \ldots, \wedge^n V$ are pairwise nonisomorphic irreducibles and $n = \dim V$. The proofs go through* mutatis mutandis.

The full general linear group $\mathrm{GL}(V)$ acting on its defining representation V is one such example, and in this case Theorem 4.2 asserts that

$$\mathrm{ch}(\wedge^i V \otimes \wedge^j V^*) - \mathrm{ch}(\wedge^{i-1} V \otimes \wedge^{j-1} V^*) = e_i(x_1, \ldots, x_n) \cdot e_j(x_1^{-1}, \ldots, x_n^{-1}) - e_{i-1}(x_1, \ldots, x_n) \cdot e_{j-1}(x_1^{-1}, \ldots, x_n^{-1}) \tag{4.19}$$

is the Weyl character of a genuine (and not merely virtual) $\mathrm{GL}(V)$-module. Here e_i is the degree i elementary symmetric polynomial and the Weyl character $\mathrm{ch}(W)$ of any $\mathrm{GL}(V)$-module W is the function of $x_1, \ldots, x_n$ characterized by

$$\mathrm{ch}(W) = \mathrm{trace}_W(\mathrm{diag}(x_1, \ldots, x_n)) \tag{4.20}$$

for $x_1, \ldots, x_n \in \mathbb{C}^\times$.

5. Motzkin paths and standard bases

In this section we describe the standard monomial basis of DR_W (with respect to a term order $\prec$ which we will define) in terms of a certain family of lattice paths. A *Motzkin path* is a lattice path in $\mathbb{Z}^2$ consisting of up-steps $(1,1)$, down-steps $(1,-1)$, and horizontal steps $(1,0)$ which starts at the origin, ends on the x-axis, and never sinks below the x-axis. We consider a variant of Motzkin paths which have decorated horizontal steps and need not end on the x-axis.

Let $\Pi(n)$ be the family of n-step lattice paths $\sigma = (s_1, \ldots, s_n)$ in $\mathbb{Z}^2$ which start at the origin and consist of up-steps $(1,1)$, down-steps $(1,-1)$, and horizontal steps $(1,0)$ in which each horizontal step is decorated with a θ or a ξ. We let $\Pi(n)_{\geq 0} \subseteq \Pi(n)$ be the family of paths which never sink below the x-axis. Two paths in $\Pi(9)$ are shown in Figure 1; the top path lies in $\Pi(9)_{\geq 0}$ but the bottom path does not.

The *depth* $d(\sigma)$ of a path $\sigma \in \Pi(n)$ is the minimum y-value attained by σ. If σ and μ are as in Figure 1 then $d(\sigma) = 0$ and $d(\mu) = -2$. We have

$$\Pi(n)_{\geq 0} = \{\sigma \in \Pi(n) \,:\, d(\sigma) = 0\} \tag{5.1}$$

and $d(\sigma) < 0$ for any $\sigma \in \Pi(n) - \Pi(n)_{\geq 0}$.

Let $\sigma = (s_1, \ldots, s_n) \in \Pi(n)$. The *weight* of the i^{th} step s_i of σ is

$$\mathrm{wt}(\sigma) := \begin{cases} 1 & \text{if } s_i = (1,1) \text{ is an up-step} \\ \theta_i & \text{if } s_i = (1,0) \text{ is decorated with } \theta \\ \xi_i & \text{if } s_i = (1,0) \text{ is decorated with } \xi \\ \theta_i \xi_i & \text{if } s_i = (1,-1) \text{ is a down-step} \end{cases} \tag{5.2}$$

and the weight of σ is the product

$$\mathrm{wt}(\sigma) := \mathrm{wt}(s_1) \cdots \mathrm{wt}(s_n) \tag{5.3}$$

of the steps of σ in the order in which they appear. For the paths σ and μ in Figure 1 we have

$$\mathrm{wt}(\sigma) = \theta_3 \cdot \theta_4\xi_4 \cdot \theta_5\xi_5 \cdot \theta_6 \cdot \theta_9\xi_9 \quad \text{and} \quad \mathrm{wt}(\mu) = \theta_2\xi_2 \cdot \theta_3\xi_3 \cdot \theta_4\xi_4 \cdot \xi_6 \cdot \theta_9. \tag{5.4}$$

A moment's thought shows that $\sigma \mapsto \mathrm{wt}(\sigma)$ gives a bijection from $\Pi(n)$ to the set of monomials in $\wedge\{\Theta_n, \Xi_n\}$, where monomials with differing signs are considered equivalent. We will identify paths σ with their monomials $\mathrm{wt}(\sigma)$.

The *(total) degree* of a path σ is

$$\deg(\sigma) := n - (\text{the terminal } y\text{-coordinate of } \sigma). \tag{5.5}$$

This is simply the total number of exterior generators θ_i and ξ_i appearing in the monomial σ. We define the *θ-degree* $\deg_\theta(\sigma)$ and *ξ-degree* $\deg_\xi(\sigma)$ analogously. Combinatorially,

$$\deg_\theta(\sigma) = (\text{number of down-steps}) + (\text{number of } \theta\text{-horizontal steps}) \tag{5.6}$$

and

$$\deg_\xi(\sigma) = (\text{number of down-steps}) + (\text{number of } \xi\text{-horizontal steps}). \tag{5.7}$$

If σ and μ are as in Figure 1 then

$$\begin{cases} \deg(\sigma) = 8 \\ \deg_\theta(\sigma) = 5 \\ \deg_\xi(\sigma) = 3 \end{cases} \quad \text{and} \quad \begin{cases} \deg(\mu) = 8 \\ \deg_\theta(\mu) = 4 \\ \deg_\xi(\mu) = 4 \end{cases} \tag{5.8}$$

We introduce the total order $\prec$ on paths $\sigma \in \Pi(n)$, or on monomials in $\wedge\{\Theta_n, \Xi_n\}$ given by

$$\sigma \prec \sigma' \Leftrightarrow \begin{cases} \deg(\sigma) < \deg(\sigma') & \text{or} \\ \deg(\sigma) = \deg(\sigma') \text{ and } d(\sigma) > d(\sigma') & \text{or} \\ \deg(\sigma) = \deg(\sigma') \text{ and } d(\sigma) = d(\sigma') \text{ and } \sigma <_{\mathrm{lex}} \sigma' \end{cases} \tag{5.9}$$

where in the last branch $<_{\mathrm{lex}}$ means the lexicographical order on the paths $\sigma = (s_1, \ldots, s_n)$ and $\sigma' = (s'_1, \ldots, s'_n)$ induced by declaring the step order

$$(1,1) < (1,0) \text{ with } \theta\text{-decoration} < (1,0) \text{ with } \xi\text{-decoration} < (1,-1). \tag{5.10}$$

The collection of paths/monomials with a given bidegree (i,j) form a subinterval of $\prec$ for all $0 \leq i, j \leq n$. In our running example of Figure 1, we have $\deg(\sigma) = \deg(\mu)$ but $d(\sigma) > d(\mu)$ so that $\sigma \prec \mu$.

Lemma 5.1. *The total order $\prec$ is a term order for $\wedge\{\Theta_n, \Xi_n\}$.*

Proof. The first branch of the definition of $\prec$ guarantees that the monomial 1 with path consisting of a sequence of n up-steps is the minimum monomial under $\prec$. Checking that $\prec$ respects multiplication amounts to the observation that total degree, depth, and lexicographical order are all respected by multiplication. $\square$

It turns out that the set $\{\mathrm{wt}(\sigma) : \sigma \in \Pi(n)_{\geq 0}\}$ descends to a $\mathbb{C}$-basis of DR_W. In fact, we prove something stronger.

Theorem 5.2. *Let W be an irreducible complex reflection group of rank n. The set $\{\mathrm{wt}(\sigma) : \sigma \in \Pi(n)_{\geq 0}\}$ is the standard monomial basis of DR_W with respect to $\prec$. The bigraded Hilbert series of DR_W is given by*

$$\mathrm{Hilb}(DR_W; q, t) = \sum_{\sigma \in \Pi(n)_{\geq 0}} q^{\deg_\theta(\sigma)} t^{\deg_\xi(\sigma)}. \tag{5.11}$$

Proof. Let $I_n = \langle \delta_n \rangle \subseteq \wedge\{\Theta_n, \Xi_n\}$ be the defining ideal of DR_W (here we apply Proposition 4.1). Identifying paths with monomials, we want to show $N(I_n) = \Pi(n)_{\geq 0}$ with respect to $\prec$. We proceed by induction on n, with the base case $n = 1$ being immediate.

Suppose $n > 1$ and $\sigma = (s_1, \ldots, s_n) \in \Pi(n) - \Pi(n)_{\geq 0}$. In particular, we have $d(\sigma) < 0$. The following lemma will show inductively that $\sigma \notin N(I_n)$.

Lemma 5.3. *The monomial σ lies in $\textsc{lm}(I_n)$ or else $\sigma = 0$ in the quotient DR_W.*

Proof. (of Lemma 5.3) Let $\sigma_0 \in \wedge\{\Theta_{n-1}, \Xi_{n-1}\}$ be the monomial σ with its last step s_n removed. The proof breaks into cases depending on the step s_n.

Case 1: *The last step s_n is a horizontal step (of either decoration θ or ξ).*

We assume the decoration of s_n is θ; the other case is similar. In this case, we have $\sigma_0 \in \Pi(n-1) - \Pi(n-1)_{\geq 0}$ and $\sigma = \sigma_0 \theta_n$. We may inductively assume that $\sigma_0 \in \textsc{lm}(I_{n-1})$ so that $\sigma_0 = \textsc{lm}(f \cdot \delta_{n-1})$ for some polynomial $f \in \wedge\{\Theta_{n-1}, \Xi_{n-1}\}$. Since

$$f \cdot \delta_n \cdot \theta_n = f \cdot \delta_{n-1}\theta_n + f \cdot \theta_n \xi_n \cdot \theta_n = f \cdot \delta_{n-1} \cdot \theta_n, \tag{5.12}$$

we conclude that $f \cdot \delta_{n-1} \cdot \theta_n \in I_n$. We have

$$\textsc{lm}(f \cdot \delta_{n-1} \cdot \theta_n) = \textsc{lm}(f \cdot \delta_{n-1}) \cdot \theta_n = \sigma_0 \cdot \theta_n = \sigma, \tag{5.13}$$

completing the proof of Case 1.

Case 2: *The last step s_n is a down-step $(1, -1)$.*

If $\sigma_0 \in \Pi(n-1) - \Pi(n-1)_{\geq 0}$ sinks below the x-axis, the proof is similar to that of Case 1. One right-multiplies $f \cdot \delta_n$ by $\theta_n \xi_n$ instead of θ_n; we leave the details to the reader.

In this case we could have $\sigma_0 \in \Pi(n-1)_{\geq 0}$, but this would imply that σ_0 ends on the x-axis, so that $\deg(\sigma) = \deg(\sigma_0) + 2 = (n-1) + 2 = n + 1$. Theorem 4.2 then forces $\sigma = 0$ in the quotient DR_W, completing the proof of Case 2.

Case 3: *The last step s_n is an up-step $(1, 1)$.*

This is the most involved case. We have $\sigma_0 = \sigma$ and $\sigma_0 \in \Pi(n-1) - \Pi(n-1)_{\geq 0}$. By induction, we may assume that there is $f \in \wedge\{\Theta_{n-1}, \Xi_{n-1}\}$ with $\sigma_0 = \textsc{lm}(f \cdot \delta_{n-1})$. Now consider

$$f \cdot \delta_n = f \cdot \delta_{n-1} + f \cdot \theta_n \xi_n \in I_n. \tag{5.14}$$

By discarding redundant terms if necessary, we may assume that f is bi-homogeneous. The monomial $\sigma = \sigma_0$ is the $\prec$-largest monomial appearing in $f \cdot \delta_{n-1}$. Since σ does not involve θ_n or ξ_n, it does not appear in $f \cdot \theta_n \xi_n$. We will have $\sigma = \textsc{lm}(f \cdot \delta_n)$ unless some monomial μ appearing in $f \cdot \theta_n \xi_n$ satisfies $\mu \succ \sigma$.

Let μ be the $\prec$-largest element of $f \cdot \theta_n \xi_n$ and assume $\sigma \prec \mu$. Let $\mu_0 \in \Pi(n-1)$ be the path obtained from μ by removing its last step (which is necessarily a down-step since μ appears in $f \cdot \theta_n \xi_n$). Since $\sigma \prec \mu$, the bihomogeneity of f forces $d(\mu) \leq d(\sigma) < 0$.

Subcase 3.1: *We have $\mu_0 \in \Pi(n-1)_{\geq 0}$, or equivalently $d(\mu_0) \geq 0$.*

Since $d(\mu) = d(\mu_0) + 1 < 0$, this can only happen if $d(\mu_0) = 0$ and μ_0 ends at the lattice points $(n-1, 0)$. This implies that $\deg(\mu) = \deg(\mu_0) + 2 = (n-1) + 2 = n + 1$ and Theorem 4.2 forces $\mu \in I_n$. We may therefore discard the term involving μ from (5.14) and still have an element of I_n involving σ.

Subcase 3.2: *We have $\mu_0 \in \Pi(n-1) - \Pi(n-1)_{\geq 0}$, or equivalently $d(\mu_0) < 0$.*

In this case, we induct on n to obtain some polynomial $g \in \wedge\{\Theta_{n-1}, \Xi_{n-1}\}$ whose leading monomial is $\mu_0 = \textsc{lm}(g \cdot \delta_{n-1})$. We calculate

$$\textsc{lm}(g \cdot \delta_n \cdot \theta_n \xi_n) = \textsc{lm}(g \cdot \delta_{n-1} \cdot \theta_n \xi_n) = \textsc{lm}(g \cdot \delta_{n-1}) \cdot \theta_n \xi_n = \mu_0 \cdot \theta_n \xi_n = \mu \tag{5.15}$$

where the second equality used the fact that $g \cdot \delta_{n-1}$ does not involve θ_n or ξ_n. Since σ does not involve θ_n or ξ_n, it does not appear in $g \cdot \delta_n \cdot \theta_n \xi_n$. We may therefore replace (5.14) by

$$f \cdot \delta_{n-1} + (f - g \cdot \delta_{n-1}) \cdot \theta_n \xi_n \in I_n \tag{5.16}$$

to obtain another element of I_n which involves σ only in its first term, still satisfies $\sigma = \textsc{lm}(f \cdot \delta_{n-1})$, but now only involves monomials $\prec \mu$.

Iterating the arguments of Subcases 3.1 and 3.2, we see that $\sigma \in \textsc{lm}(I_n)$, proving both Case 3 and the lemma. $\square$

We complete the proof of Theorem 5.2 using Lemma 5.3. Lemma 5.3 implies $N(I_n) \subseteq \Pi(n)_{\geq 0}$, and to force equality it suffices to verify

$$\dim DR_W = |\Pi(n)_{\geq 0}|. \tag{5.17}$$

In fact, we verify the following equality of polynomials in q and t:

$$\mathrm{Hilb}(DR_W; q, t) = \sum_{\sigma \in \Pi(n)_{\geq 0}} q^{\deg_\theta(\sigma)} t^{\deg_\xi(\sigma)} =: P_n(q, t). \tag{5.18}$$

If we let $\Pi(n)_{=0} \subseteq \Pi(n)_{\geq 0}$ be the subset of paths that end on the x-axis and let

$$P'_n(q, t) := \sum_{\sigma \in \Pi(n)_{=0}} q^{\deg_\theta(\sigma)} t^{\deg_\xi(\sigma)}, \tag{5.19}$$

considering the addition of one more step to a path yields

$$P_{n+1}(q, t) = (1 + q + t + qt) \cdot P_n(q, t) - (qt) \cdot P'_n(q, t). \tag{5.20}$$

On the other hand (adopting the notation $DR_{W(n)}$ for DR_W whenever W has rank n) Theorem 4.2 yields

$$\dim(DR_{W(n+1)})_{i,j} = \begin{cases} \binom{n+1}{i} \cdot \binom{n+1}{j} - \binom{n+1}{i-1} \cdot \binom{n+1}{j-1} & \text{if } i, j > 0 \text{ and } i + j \leq n + 1 \\ \binom{n+1}{i} \cdot \binom{n+1}{j} & \text{if } i = 0 \text{ or } j = 0 \\ 0 & \text{if } i + j > n + 1 \end{cases} \tag{5.21}$$

It can be shown using the Pascal identity and Equation (5.21) that

$$\mathrm{Hilb}(DR_{W(n+1)}; q, t) = (1+q+t+qt) \cdot \mathrm{Hilb}(DR_{W(n)}; q, t) - (qt) \cdot \sum_{i+j=n+1} \dim(DR_{W(n)})_{i,j} \cdot q^i t^j, \tag{5.22}$$

which matches the combinatorial recursion in Equation (5.20). $\square$

6. The permutation representation of $\mathfrak{S}_n$

In the coinvariant theory of the symmetric group $\mathfrak{S}_n$, it is more common to consider its n-dimensional permutation representation U as opposed to its $(n-1)$-dimensional reflection representation V. In this section we describe how to translate our results into this setting.

The following decompositions of U and U^* into $\mathfrak{S}_n$-irreducibles are well-known:

$$U = V \oplus U^{\mathfrak{S}_n} \quad \text{and} \quad U^* = V^* \oplus (U^*)^{\mathfrak{S}_n}. \tag{6.1}$$

It follows that

$$\wedge(U \oplus U^*) \cong \wedge[(V \oplus U^{\mathfrak{S}_n}) \oplus (V^* \oplus (U^*)^{\mathfrak{S}_n})] \tag{6.2}$$

$$\cong \wedge[(V \oplus V^*) \oplus (U^{\mathfrak{S}_n} \oplus (U^*)^{\mathfrak{S}_n})] \tag{6.3}$$

$$\cong [\wedge(V \oplus V^*)] \otimes [\wedge(U^{\mathfrak{S}_n} \oplus (U^*)^{\mathfrak{S}_n})]. \tag{6.4}$$

Modding out by ideals generated by $\mathfrak{S}_n$-invariants with vanishing constant term, we see that

$$\wedge(U \otimes U^*)/\langle \wedge(U \otimes U^*)_+^{\mathfrak{S}_n} \rangle \cong \wedge(V \otimes V^*)/\langle \wedge(V \otimes V^*)_+^{\mathfrak{S}_n} \rangle. \tag{6.5}$$

Let $\mathfrak{S}_n$ act on $\wedge\{\Theta_n, \Xi_n\}$ diagonally, viz. $w.\theta_i := \theta_{w(i)}$ and $w.\xi_i := \xi_{w(i)}$. Expressing the left-hand side of (6.5) in terms of coordinates, we have the following translation of Theorem 4.2, Corollary 4.3, and Corollary 4.3.

Theorem 6.1. *Let DR_n be the bigraded $\mathfrak{S}_n$-module*

$$DR_n := \wedge\{\Theta_n, \Xi_n\}/\langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n} \rangle. \tag{6.6}$$

We have $(DR_n)_{i,j} = 0$ whenever $i + j \geq n$. If $i + j < n$, we have

$$\mathrm{Frob}(DR_n)_{i,j} = s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})} \tag{6.7}$$

where $*$ denotes Kronecker product. Here we interpret $s_{(n+1,-1)} = 0$. We have

$$\dim DR_n = \binom{2n-1}{n} \tag{6.8}$$

and, for $1 \leq k \leq n$, we have

$$\dim(DR_n)_{k-1,n-k} = \mathrm{Nar}(n,k) \tag{6.9}$$

so that $\sum_{k=1}^n \dim(DR_n)_{k-1,n-k} = \mathrm{Cat}(n)$.

Equation (6.8) was conjectured by Mike Zabrocki [23]. We also have a lattice path basis of the $\mathfrak{S}_n$-module DR_n in Theorem 6.1. For a partition $\lambda \vdash n$, work of Rosas [19] implies that

$$\langle \mathrm{grFrob}(DR_n; q, t), s_\lambda \rangle = 0 \tag{6.10}$$

unless the partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots)$ satisfies $\lambda_3 \leq 2$ (i.e. the Young diagram of λ is a union of two possibly empty hooks). While these multiplicities can be less than aesthetic in general, they are nice when λ is a hook. Recall that the *q, t-analog* of n is given by

$$[n]_{q,t} := \frac{q^n - t^n}{q - t} = q^{n-1} + q^{n-2}t + \cdots + qt^{n-2} + t^{n-1}. \tag{6.11}$$

Proposition 6.2. *The graded multiplicities of the trivial and sign representations in DR_n are given by*

$$\langle \mathrm{grFrob}(DR_n; q, t), s_{(n)} \rangle = 1 \quad \textit{and} \quad \langle \mathrm{grFrob}(DR_n; q, t), s_{(1^n)} \rangle = [n]_{q,t}. \tag{6.12}$$

If $0 < k < n - 1$ we have

$$\langle \mathrm{grFrob}(DR_n; q, t), s_{(n-k,1^k)} \rangle = [k+1]_{q,t} + (qt) \cdot [k]_{q,t}. \tag{6.13}$$

Proof. The equation $\langle \mathrm{grFrob}(DR_n; q, t), s_{(n)} \rangle = 1$ is immediate since DR_n is obtained from $\wedge\{\Theta_n, \Xi_n\}$ by modding out by $\mathfrak{S}_n$-invariants with vanishing constant term. The multiplicity of the sign representation follows from Theorem 6.1 and the fact that for any partitions $\lambda, \mu \vdash n$

$$\text{multiplicity of } s_{(1^n)} \text{ in } s_\lambda * s_\mu = \begin{cases} 1 & \text{if } \mu = \lambda' \\ 0 & \text{otherwise} \end{cases} \tag{6.14}$$

where λ' is the conjugate (transpose) partition of λ.

We turn our attention to Equation (6.13). For any statement P, let $\chi(P) = 1$ if P is true and $\chi(P) = 0$ if P is false. Rosas proves [19, Proof of Thm. 13 (4)] that the multiplicity of the Schur function $s_{(n-c,1^c)}$ in the Kronecker product $s_{(n-a,1^a)} * s_{(n-b,1^b)}$ is

$$\langle s_{(n-a,1^a)} * s_{(n-b,1^b)}, s_{(n-c,1^c)} \rangle = \chi(|b-a| \leq c) \times \chi(c \leq a+b \leq 2n-c-2) \tag{6.15}$$

whenever $0 < a, b < n$ and $0 < c < n-1$.

For any $0 \leq k \leq n-1$ and all $i + j < n$, we have

$$\begin{aligned} \langle \mathrm{Frob}(DR_n)_{i,j}, s_{(n-k,1^k)} \rangle = \\ \langle s_{(n-i,1^i)} * s_{(n-j,1^j)}, s_{(n-k,1^k)} \rangle - \langle s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})}, s_{(n-k,1^k)} \rangle \end{aligned} \tag{6.16}$$

A somewhat tedious casework using Equation (6.15) yields

$$\langle \mathrm{Frob}(DR_n)_{i,j}, s_{(n-k,1^k)} \rangle = \begin{cases} 1 & \text{if } i+j = k \\ 1 & \text{if } i+j = k+1 \text{ and } i,j > 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.17}$$

which is equivalent to Equation (6.13). $\square$

In order to state a DR_n-analog of Theorem 5.2, we need some notation. We define the *primed weight* $\mathrm{wt}'(s)$ of a step s to be

$$\begin{cases} 1 & \text{if } s = (1,1) \text{ is an up-step} \\ \theta_i & \text{if } s = (1,0) \text{ is decorated with } \theta \\ \xi'_i & \text{if } s = (1,0) \text{ is decorated with } \xi \\ \theta_i \xi'_i & \text{if } s = (1,-1) \text{ is a down-step} \end{cases} \tag{6.18}$$

where

$$\xi'_i := \xi_i + \sum_{j=2}^{n} \xi_j. \tag{6.19}$$

The *primed weight* $\mathrm{wt}'(\sigma)$ of a path $\sigma \in \Pi(n)$ with steps $\sigma = (s_1, \ldots, s_n)$ is $\mathrm{wt}'(\sigma) := \mathrm{wt}'(s_1) \cdots \mathrm{wt}'(s_n)$. Let $\Pi(n)_{>0} \subseteq \Pi(n)$ consist of those paths which only meet the x-axis at their starting point $(0,0)$ and stay strictly above the x-axis otherwise.

Theorem 6.3. *The set* $\{\mathrm{wt}'(\sigma) : \sigma \in \Pi(n)_{>0}\}$ *descends to a basis of* DR_n. *Consequently, we have*

$$\mathrm{Hilb}(DR_n; q, t) = \sum_{\sigma \in \Pi(n)_{>0}} q^{\deg_\theta(\sigma)} t^{\deg_\xi(\sigma)}. \tag{6.20}$$

Proof. Proposition 4.1 and the discussion prior to Theorem 6.1 imply that the invariant subalgebra $\wedge\{\Theta_n, \Xi_n\}^{\mathfrak{S}_n}$ is generated by the three elements

$$\theta_1 + \cdots + \theta_n, \ \xi_1 + \cdots + \xi_n, \quad \text{and} \quad \theta_1\xi_1 + \cdots + \theta_n\xi_n$$

and consequently

$$DR_n = \wedge\{\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n\} / \langle \theta_1 + \cdots + \theta_n, \xi_1 + \cdots + \xi_n, \theta_1\xi_1 + \cdots + \theta_n\xi_n \rangle. \tag{6.21}$$

We express DR_n as a successive quotient

$$DR_n = \wedge\{\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n\} / \langle \theta_1 + \cdots + \theta_n, \xi_1 + \cdots + \xi_n, \theta_1\xi_1 + \cdots + \theta_n\xi_n \rangle. \tag{6.22}$$

$$= \left(\wedge\{\theta_1, \ldots, \theta_n\} / \langle \sum_{i=1}^{n} \theta_i \rangle \otimes \wedge\{\xi_1, \ldots, \xi_n\} / \langle \sum_{i=1}^{n} \xi_i \rangle \right) / \langle \sum_{i=1}^{n} \theta_i \otimes \xi_i \rangle \tag{6.23}$$

Then as graded vector spaces, we identify $\theta_1 = -\theta_2 - \cdots - \theta_n$ and $\xi_1 = -\xi_2 - \cdots - \xi_n$ to obtain

$$DR_n \cong (\wedge\{\theta_2, \ldots, \theta_n\} \otimes \wedge\{\xi_2, \ldots, \xi_n\}) / \langle (-\theta_2 - \cdots - \theta_n) \otimes (-\xi_2 - \cdots - \xi_n) + \sum_{i=2}^{n} \theta_i \otimes \xi_i \rangle \tag{6.24}$$

$$= (\wedge\{\theta_2, \ldots, \theta_n\} \otimes \wedge\{\xi_2, \ldots, \xi_n\}) / \langle \sum_{i=2}^{n} \theta_i \otimes (\xi_i + \sum_{j=2}^{n} \xi_j) \rangle \tag{6.25}$$

The transition matrix from the set $\{\xi_2 + \sum_{j=2}^n \xi_j, \ldots, \xi_n + \sum_{j=2}^n \xi_j\} = \{\xi_2', \ldots, \xi_n'\}$ to the standard basis $\{\xi_2, \ldots, \xi_n\}$ of the degree 1 component of $\wedge\{\xi_2, \ldots, \xi_n\}$ is

$$\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}$$

which is easily checked to be invertible. Therefore, the set $\{\xi_2', \ldots, \xi_n'\}$ is also a basis of the degree 1 component of $\wedge\{\xi_2, \ldots, \xi_n\}$ and we may write

$$DR_n \cong \wedge\{\theta_2, \ldots, \theta_n, \xi_2', \ldots, \xi_n'\} / \langle \theta_2 \xi_2' + \cdots + \theta_n \xi_n' \rangle. \tag{6.26}$$

Theorem 5.2 applies to complete the proof. □

7. Open Problems

The key result underpinning our analysis of DR_W and DR_n was the Lefschetz Theorem 3.2. Our proof was combinatorial and ultimately relied on the Boolean poset $B(n)$. Given the importance of Lefschetz elements in geometry, it is natural to ask the following.

Question 7.1. *Is there a geometric proof of Theorem 3.2?*

Modern variants of HL and PD were used to great effect in the work of Adiprasito, Huh, and Katz on the Chow rings of matroids [1]. Is there a deeper meaning to HL and PD as they apply to exterior algebras? Perhaps the realization of $\wedge\{\Theta_n, \Xi_n\}$ as the exterior algebra over the holomorphic tangent space at the origin in $\mathbb{C}^n \oplus \mathbb{C}^n$ would be relevant here.

It may also be interesting to consider combining two sets of commuting and anticommuting variables to get a ring

$$\mathbb{C}[X_n, Y_n] \otimes \wedge\{\Theta_n, \Xi_n\} := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \otimes \wedge\{\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n\} \tag{7.1}$$

which may be identified with the algebra of polynomial-valued holomorphic differential forms on $\mathbb{C}^n \oplus \mathbb{C}^n$. This ring is quadruply graded, and the diagonal action of $\mathfrak{S}_n$ gives rise to a coinvariant space $\mathbb{C}[X_n, Y_n, \Theta_n, \Xi_n] / \langle \mathbb{C}[X_n, Y_n, \Theta_n, \Xi_n]_+^{\mathfrak{S}_n} \rangle$. Setting the ξ-variables to zero, Zabrocki [22] conjectured that the triply graded Frobenius image of this quotient is given by the *Delta Conjecture* of Haglund, Remmel, and Wilson [8]. Furthermore, again when the ξ-variables are set to zero, Haglund and Sergel [10] have a conjectural monomial basis of this quotient which would extend a basis of the diagonal coinvariants due to Carlsson and Oblomkov [7].

Problem 7.2. *Find a basis of the quotient $\mathbb{C}[X_n, Y_n, \Theta_n, \Xi_n] / \langle \mathbb{C}[X_n, Y_n, \Theta_n, \Xi_n]_+^{\mathfrak{S}_n} \rangle$ which generalizes the basis of $\mathbb{C}[X_n, Y_n] / \langle \mathbb{C}[X_n, Y_n]_+^{\mathfrak{S}_n} \rangle$ due to Carlsson-Oblomkov* [7] *and the conjectural basis of $\mathbb{C}[X_n, Y_n, \Theta_n] / \langle \mathbb{C}[X_n, Y_n, \Theta_n]_+^{\mathfrak{S}_n} \rangle$ due to Haglund-Sergel* [10].

A solution to Problem 7.2 might be obtained by interpolating between the parking function 'schedules' present in [7, 10] and our Motzkin-like paths $\Pi(n)_{>0}$.

Let $X_{k \times n} = (x_{i,j})_{1 \leq i \leq k, 1 \leq j \leq n}$ be a $k \times n$ matrix of commuting variables and let $\mathbb{C}[X_{k \times n}]$ be the polynomial ring in these variables. The ring $\mathbb{C}[X_{k \times n}]$ carries a $\mathfrak{S}_n$-module structure inherited from column permutation and the quotient $\mathbb{C}[X_{k\times n}]/\langle \mathbb{C}[X_{k\times n}]_+^{\mathfrak{S}_n}\rangle$ is a $(\mathbb{Z}_{\geq 0})^k$-graded $\mathfrak{S}_n$-module. When $k=2$, we recover the classical diagonal coinvariants. F. Bergeron has many fascinating conjectures about this object obtained by letting the parameter k grow [3].

We can carry out the construction of the previous paragraph with a matrix $\Theta_{k\times n} = (\theta_{i,j})_{1\leq i\leq k, 1\leq j\leq n}$ of anticommuting variables. We still have an action of $\mathfrak{S}_n$ on columns and can still consider the quotient

$$R(k\times n) := \wedge\{\Theta_{k\times n}\}/\langle \wedge\{\Theta_{k\times n}\}_+^{\mathfrak{S}_n}\rangle. \tag{7.2}$$

In the case $k=2$ we recover DR_n. For stability results involving such quotients, and corresponding quotients using both commuting and anticommuting variables, see [17].

Question 7.3. *Find the multigraded $\mathfrak{S}_n$-isomorphism type of $R(k\times n)$.*

The *Diagonal Supersymmetry Conjecture* of F. Bergeron [4, Conj. 1] predicts that a solution to Question 7.3 for all values of n and k would determine the multigraded $\mathfrak{S}_n$-isomorphism type of

$$\mathbb{C}[X_{m\times n}]\otimes \wedge\{\Theta_{k\times n}\}/\langle (\mathbb{C}[X_{m\times n}]\otimes \wedge\{\Theta_{k\times n}\})_+^{\mathfrak{S}_n}\rangle \tag{7.3}$$

for all values of n, m, and k. In particular, this includes the classical diagonal coinvariant ring $(k=0, m=2)$ as well as the not-yet-understood case of $k=0, m>2$. In light of [4, Conj. 1] it is interesting that the $m=0, k=2$ case of DR_n was so much easier to analyze than the case $m=2, k=0$. Furthermore, [4, Conj. 1] suggests that Question 7.3 will become very difficult as k grows.

It is unclear how to use Lefschetz Theory to solve Question 7.3 for $k>2$. For any set $S\subseteq \{1,2,\dots,k\}$ of rows, we have a $\mathfrak{S}_n$-invariant

$$\delta_S := \prod_{i\in S}\theta_{i,1} + \prod_{i\in S}\theta_{i,2} + \cdots + \prod_{i\in S}\theta_{i,n} \in \wedge\{\Theta_{k\times n}\} \tag{7.4}$$

where the products are taken in increasing order of $i\in S$. Orellana and Zabrocki proved [15] that the elements δ_S generate the invariant subring $\wedge\{\Theta_{k\times n}\}^{\mathfrak{S}_n}$ where S ranges over all nonempty subsets of $\{1,2,\dots,k\}$. In fact, the results of [15] give an explicit generating set of the $\mathfrak{S}_n$-invariant subring of the tensor product $\mathbb{C}[X_{m\times n}]\otimes\wedge\{\Theta_{k\times n}\}$ for any n, m, and k. When $|S|$ is even, the element δ_S has the potential to be Lefschetz, but $\delta_S^2=0$ when $|S|$ is odd. For $|S|=1$, the row sum δ_S may be easy to handle, but the situation becomes more complicated as an odd-sized set S grows. Furthermore, one would have to understand how the various images of multiplication by the δ_S between bidegrees intersect as S varies.

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References

[1] K. Adiprasito, J. Huh, and E. Katz. Hodge theory for combinatorial geometries. *Ann. of Math.*, **3** (2018), 3079–3098.

[2] D. Armstrong, V. Reiner, and B. Rhoades. Parking spaces. *Adv. Math.*, **269** (2015), 647–706.

[3] F. Bergeron. $(GL_k\times S_n)$-Modules of Multivariate Diagonal Harmonics. Preprint, 2020. `arXiv:2003.07402`.

[4] F. Bergeron. The bosonic-fermionic diagonal coinvariant modules conjecture. Preprint, 2020. `arXiv:2005.00924`.

[5] M. Bremner. Free associative algebras, noncommutative Gröbner bases, and universal associative envelopes for nonassociative structures. `arXiv:1303.0920`.

[6] S. Billey, B. Rhoades, and V. Tewari. Boolean product polynomials, Schur positivity, and Chern plethysm. To appear, *Int. Math. Res. Notices (IMRN)*, 2020. `arXiv:1902.11165`.

[7] E. Carlsson and A. Oblomkov. Affine Schubert calculus and double coinvariants. Preprint, 2018. arXiv:1801.09033.

[8] J. Haglund, J. Remmel, and A. T. Wilson. The Delta Conjecture. *Trans. Amer. Math. Soc.*, **370**, (2018), 4029–4057.

[9] J. Haglund, B. Rhoades, and M. Shimozono. Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture. *Adv. Math.*, **329**, (2018), 851–915.

[10] J. Haglund and E. Sergel. Schedules and the Delta Conjecture. Preprint, 2019. arXiv:1908.04732

[11] M. Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. *Invent. Math.*, **149 (2)**, (2002), 371–407.

[12] M. Hara and J. Watanabe. The determinants of certain matrices arising from the Boolean lattice. *Discrete Math.*, **308 (23)**, (2008), 5815–5822.

[13] R. Kane. *Reflection Groups and Invariant Theory.* CMS Books in Mathematics. Springer-Verlag, New York. 2001.

[14] T. Maeno, Y. Numata, and A. Wachi. Strong Lefschetz elements of the coinvariant rings of finite Coxeter groups. *Algebr. Represent. Th.*, **14 (4)**, (2007), 625–638.

[15] R. Orellana and M. Zabrocki. A combinatorial model for the decomposition of multivariate polynomial rings as an S_n-module. To appear, *Electron. J. Combin.*, 2020. arXiv:1906.01125.

[16] B. Pawlowski and B. Rhoades. A flag variety for the Delta Conjecture. *Trans. Amer. Math. Soc.*, **372**, (2019), 8195–8248.

[17] B. Pawlowski, E. Ramos, and B. Rhoades. Spanning subspace configurations and representation stability. Preprint, 2019. arXiv:1907.07268.

[18] B. Rhoades and A. T. Wilson. Vandermondes in superspace. *Trans. Amer. Math. Soc.*, **373**, (2020), 4483–4516.

[19] M. Rosas. The Kronecker product of Schur functions indexed by two-row shapes or hook shapes. *J. Algebraic Combin.*, **14 (2)**, (2001), 153–173.

[20] R. Stanley. Variations on differential posets. In *Invariant Theory and Tableaux* (D. Stanton, ed.), The IMA Volumes in Mathematics and Its Applications, vol. 19, Springer-Verlag, New York, 1990, 145–165.

[21] J. Swanson and N. Wallach. Harmonic differential forms for pseudo-reflection groups I. Semi-invariants. Preprint, 2020. arXiv:2001.06076

[22] M. Zabrocki. A module for the Delta conjecture. Preprint, 2019. arXiv:1902.08966

[23] M. Zabrocki. Coinvariants and harmonics. Blog for Open Problems in Algebraic Combinatorics 2020. https://realopacblog.wordpress.com/2020/01/26/coinvariants-and-harmonics/.

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