

Model Theory of Proalgebraic Groups

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Abstract

We lay the foundations for a model theoretic study of proalgebraic groups. Our axiomatization is based on the tannakian philosophy. Through a tensor analog of skeletal categories we are able to consider neutral tannakian categories with a fibre functor as many-sorted first order structures. The class of diagonalizable proalgebraic groups is analyzed in detail. We show that the theory of a diagonalizable proalgebraic group G is determined by the theory of the base field and the theory of the character group of G . Some initial steps towards a comprehensive study of types are also made.

Introduction

Our initial inspiration for this paper goes back to the model theoretic treatment of profinite groups developed by G. Cherlin, L. van den Dries, A. Macintyre and Z. Chatzidakis in the eighties. (See [CvdDM81], [CvdDM82], [Cha84], [Cha98], [Cha87].) To a profinite group G they associate an ω -sorted structure consisting of the cosets gN of the open normal subgroups N of G ; the coset gN is of sort n if $[G : N] \leq n$. In an appropriate language these structures can be axiomatized by a theory T and there is an anti-equivalence of categories between the category of profinite groups with epimorphisms as morphisms and the category of models of T with embeddings as morphisms. A certain extension T_{IP} of T is particularly well-behaved. The theory T_{IP} axiomatizes profinite groups G having the Iwasawa (or embedding) property: Any diagram

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ B & \longrightarrow & A \end{array}$$

where $B \rightarrow A$ is an epimorphism of finite groups and $G \rightarrow A$ is an epimorphism can be completed to a commutative diagram via an epimorphism $G \rightarrow B$, if B is a quotient of G . The theory of a profinite group having the Iwasawa property is ω -categorical and ω -stable. Moreover, the saturated models of T_{IP} are exactly the free profinite groups.

Some parts of the theory of free profinite groups have recently been generalized to proalgebraic groups ([Wib]). This begs the questions, which aspects, if any, of the model theory of profinite groups have a proalgebraic counterpart? To begin with, it is a priori rather unclear how to treat proalgebraic

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groups as first-order structures. One may envision that the role played by the cardinalities $|G/N|$ of the finite quotients of a profinite group G could be replaced by the degrees of defining equations of the algebraic quotients G/N of a proalgebraic group G . However, a key fact used in the axiomatization of profinite groups is that if N_1 and N_2 are open normal subgroups of a profinite group G , then $|G/(N_1 \cap N_2)|$ is bounded by $|G/N_1| \cdot |G/N_2|$. The degree does not exhibit such a behavior.

The main achievement of this paper is the introduction of a many-sorted language that allows us to axiomatize proalgebraic groups. The key idea is based on the tannakian philosophy. Instead of axiomatizing proalgebraic groups directly, we axiomatize their categories of representations, i.e., we axiomatize neutral tannakian categories together with a fibre functor. To implement this approach, certain technical challenges need to be overcome. For example, one cannot directly consider the class of all (finite dimensional, linear) representations of a proalgebraic group as a first order structure because this class is too big. Besides the fact that it is a proper class (i.e., not a set) the cardinality of the first-order structure associated to a proalgebraic group should be something algebraically meaningful, like the rank of the profinite group in the profinite setting. Therefore, one has to consider representations up to isomorphism. In other words, one has to consider skeletons of the category of representations of a proalgebraic group. To account for the fact that such a skeleton need not be closed under the tensor product, we introduce a tensor analog of skeletal categories; a notion that we deem of independent interest in the study of tensor categories.

We introduce a first-order theory PROALG in an appropriate many-sorted language such that the category of models of PROALG with the homomorphisms as morphisms is equivalent to the category of triples (k, C, ω) , where k is a field, C a neutral tannakian category over k that satisfies a tensor analog of being skeletal and ω is a fibre functor on C . We also show that the functor $(k, C, \omega) \rightsquigarrow (k, \underline{\text{Aut}}^{\otimes}(\omega))$ to the category of proalgebraic groups (over varying base fields) is full, essentially surjective and induces a bijection on isomorphism classes. Thus, to every proalgebraic group G , there is associated a model \mathcal{M} of PROALG that is unique up to an isomorphism. We can therefore unambiguously define the theory of G as the theory of \mathcal{M} .

Even for algebraic groups as innocuous as the multiplicative group \mathbb{G}_m it is a non-trivial matter to determine their theory. We show that the theory of the multiplicative group over a field k is determined by the theory of k and the theory of its character group, i.e., by the theory of $(\mathbb{Z}, +)$. Indeed, we establish a similar result for any diagonalizable proalgebraic group. If G is a proalgebraic group corresponding to a model \mathcal{M} of PROALG, then the character group of G is interpretable in \mathcal{M} . If G is diagonalizable with character group A there is a converse: The structure \mathcal{M} is interpretable in the structure (k, A) , with the language of fields on k and the language of abelian groups on A . In fact, we show that the theory of all diagonalizable proalgebraic groups is weakly bi-interpretable with the theory of pairs (k, A) , where k is a field and A an abelian group.

We consider this article to be the first step in a model theoretic treatment of proalgebraic groups. Many, even rather basic questions remain open. However, we do give a flavor of the expressive power of our theory PROALG by unveiling some of the algebraic information encoded in certain types.

There is some thematic overlap between our work and work of M. Kamensky ([Kam15]) in the sense that both articles connect model theory and tannakian categories. However, the approaches and the aims differ: Our theory PROALG axiomatizes neutral tannakian categories together with a fibre functor for the purpose of advancing the theory of proalgebraic groups using model theoretic techniques. M. Kamensky's theory $T_{\mathcal{C}}$ (in a language $\mathcal{L}_{\mathcal{C}}$ dependent on \mathcal{C}) axiomatizes fibre functors on a *fixed* neutral tannakian category \mathcal{C} for the purpose of reproving the main tannaka reconstruction theorem using model theoretic techniques. On the other hand, we feel that this article may be seen as a possible answer to the open question 0.1.2 in [Kam15].

One of the main motivations for the model theoretic treatment of profinite groups is that it has applications in the model theory of fields, in particular the model theory of pseudo algebraically closed fields. This is based on the fact that for a field K , the first-order structure corresponding to the

absolute Galois group of K is interpretable in the field K . For a differential field (K, δ) of characteristic zero with algebraically closed constants the absolute differential Galois group ([vdPS03], [BHHW]) is a proalgebraic group. It appears that at least some reduct of the structure corresponding to the absolute differential Galois group of (K, δ) is interpretable in the differential field (K, δ) . We therefore hope that our model theoretic treatment of proalgebraic groups will eventually lead to applications in the model theory of differential fields.

Typically model theorists treat algebraic and proalgebraic groups simply as definable respectively prodefinable groups in ACF, the theory of algebraically closed fields. Our approach allows us to treat proalgebraic groups as structures in their own right. One advantage of our approach is that we can handle non-reduced algebraic groups, such as, e.g., the group of p -th roots of unity in characteristic p , without difficulties, whereas the point-set approach dictated by ACF is oblivious to these groups.

The article is organized as follows: The first section is purely algebraic, i.e., does not involve any model theory. After recalling the basic definitions and results from the tannakian theory we introduce tensor skeletal tensor categories and the closely related notion of pointed skeletal neutral tannakian categories. We then proceed to define the category TANN. This category has as objects triples (k, C, ω) , where k is a field, C a pointed skeletal neutral tannakian category over k and ω a neutral fibre functor on C . We show that the functor $(k, C, \omega) \rightsquigarrow (k, \underline{\text{Aut}}^\otimes(\omega))$ from the category TANN to the category of proalgebraic groups is full, essentially surjective and induces a bijection on the isomorphism classes.

In the second section we present the axioms for PROALG in an appropriate many-sorted language. We show that the category of models of PROALG is equivalent to the category TANN and we briefly discuss some elementary classes of proalgebraic groups.

In the third section we study the theory of diagonalizable proalgebraic groups. We show that it is weakly bi-interpretable with the theory of pairs (k, A) , where k is a field and A an abelian group. From this we deduce rather directly a description of the completions of the theory of diagonalizable proalgebraic groups and a characterization of elementary extensions. It also follows that the theory of a diagonalizable proalgebraic group over an algebraically closed field is stable.

In the final section we present some initial results concerning types. The main result is that if a representation of a proalgebraic group is considered as an element of a model of PROALG, then its type over the base field determines the image of the representation.

1 Tannakian categories

In this section we first recall the main definitions and results from the theory of tannakian categories. Then we introduce a tensor version of skeletal categories and show that the isomorphism classes of pointed skeletal neutral tannakian categories with a fibre functor are in bijection with the isomorphism classes of proalgebraic groups.

Notation and Conventions:

All rings are assumed to be commutative and unital. Throughout the article k denotes an arbitrary field, usually our “base field”. The category of finite dimensional k -vector spaces is denoted by Vec_k . A *proalgebraic group*¹ over k is, by definition, an affine group scheme over k . An *algebraic group* over k is an affine group scheme of finite type over k . We will often think of a proalgebraic group G as the functor $R \rightsquigarrow G(R)$ from the category of k -algebras to the category of groups. Conversely, a functor from the category of k -algebras to the category of groups is a proalgebraic

¹It would admittedly be more accurate to use the term “pro-affine algebraic group” or “affine group scheme” instead of “proalgebraic group”. We hope the reader does not object to our choice of brevity over rigor in this instance.

group if and only if it is representable. A *closed subgroup* of a proalgebraic group is a closed subgroup scheme. Some helpful references for the theory of algebraic and proalgebraic groups are [Wat79], [DG70] and [Mil17].

For a vector space V over k we denote by GL_V the functor from the category of k -algebras to the category of groups that assigns to any k -algebra R , the group of R -linear automorphisms of $V \otimes_k R$. A *representation* of a proalgebraic group G is a pair (V, ϕ) , where V is a k -vector space and $\phi: G \rightarrow \mathrm{GL}_V$ is a morphism of functors (also called a natural transformation), i.e., $G(R)$ acts, functorially in R , on $V \otimes_k R$ through R -linear automorphisms. A morphism $f: (V, \phi) \rightarrow (V', \phi')$ of representations of G is a k -linear map $f: V \rightarrow V'$ such that

$$\begin{array}{ccc} V \otimes_k R & \xrightarrow{f \otimes R} & V' \otimes_k R \\ \phi(g) \downarrow & & \downarrow \phi'(g) \\ V \otimes_k R & \xrightarrow{f \otimes R} & V' \otimes_k R \end{array}$$

commutes for all k -algebras R and $g \in G(R)$. All representations are assumed to be finite dimensional unless the contrary is explicitly allowed. We denote the category of all finite dimensional representations of G by $\mathrm{Rep}(G)$.

1.1 Recollection

We start by recalling the basic definitions and results from the theory of tannakian categories. See [Del90] and [DM82] for more details.

Definition 1.1. A tensor category is a tuple (C, \otimes, Φ, Ψ) , where C is a category, $\otimes: C \times C \rightarrow C$ is a functor and Φ and Ψ are isomorphisms of functors, called the associativity and commutativity constraints respectively. More specifically, Φ has components $\Phi_{X,Y,Z}: X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$ and Ψ has components $\Psi_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ for objects X, Y, Z of C such that three commutative diagrams are satisfied. See [DM82, Section 1] for details. It is also required that there exists an identity object $(\mathbb{1}, u)$. This means that $C \rightarrow C$, $X \rightsquigarrow \mathbb{1} \otimes X$ is an equivalence of categories and $u: \mathbb{1} \rightarrow \mathbb{1} \otimes \mathbb{1}$ is an isomorphism.

We will often omit Φ and Ψ from the notation and refer to C or the pair (C, \otimes) as a tensor category. If $(\mathbb{1}, u)$ and $(\mathbb{1}', u')$ are identity objects of a tensor category (C, \otimes) , then there exists a unique isomorphism $a: \mathbb{1} \rightarrow \mathbb{1}'$ such that

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{u} & \mathbb{1} \otimes \mathbb{1} \\ a \downarrow & & \downarrow a \otimes a \\ \mathbb{1}' & \xrightarrow{u'} & \mathbb{1}' \otimes \mathbb{1}' \end{array}$$

commutes ([DM82, Prop. 1.3, b)). Moreover, there exists a unique isomorphism of functors l with components $l_X: X \rightarrow \mathbb{1} \otimes X$ such that certain diagrams commute ([DM82, Prop. 1.3, a)]. Similarly, for $r_X: X \rightarrow X \otimes \mathbb{1}$.

Definition 1.2. A tensor functor from a tensor category (C, \otimes) to a tensor category (C', \otimes') is a pair (T, c) , where $T: C \rightarrow C'$ is a functor and c is an isomorphism of functors with components $c_{X,Y}: T(X) \otimes' T(Y) \rightarrow T(X \otimes Y)$ for objects X, Y of C such that

(i)

$$\begin{array}{ccc}
T(X) \otimes' (T(Y) \otimes' T(Z)) & \xrightarrow{\text{id} \otimes' c} T(X) \otimes' (T(Y \otimes T(Z))) & \xrightarrow{c} T(X \otimes (Y \otimes Z)) \\
\downarrow \Phi' & & \downarrow T(\Phi) \\
(T(X) \otimes' T(Y)) \otimes' T(Z) & \xrightarrow{c \otimes' \text{id}} T(X \otimes Y) \otimes' T(Z) & \xrightarrow{c} T((X \otimes Y) \otimes Z)
\end{array}$$

commutes for all objects X, Y, Z of C ,

(ii)

$$\begin{array}{ccc}
T(X) \otimes' T(Y) & \xrightarrow{c} T(X \otimes Y) \\
\downarrow \Psi' & & \downarrow T(\Psi) \\
T(Y) \otimes' T(X) & \xrightarrow{c} T(Y \otimes X)
\end{array}$$

commutes for all objects X and Y of C and

(iii) if $(\mathbb{1}, u)$ is an identity object of (C, \otimes) , then $(T(\mathbb{1}), T(u))$ is an identity object of (C', \otimes') .

A tensor functor (T, c) is strict if c is the identity transformation. In particular, $T(X) \otimes' T(Y) = T(X \otimes Y)$ for all objects X and Y of C . A tensor functor (T, c) is a tensor equivalence if T is an equivalence of categories.

We will usually omit the isomorphism of functors c from the notation and refer to T as a tensor functor. In Section 2 we will define first order structures corresponding to neutral tannakian categories. Homomorphisms between such structures correspond to strict tensor functors. Therefore we are mostly interested in strict tensor functors. We note that for $c = \text{id}$, the above two diagrams reduce to $T(\Phi_{X,Y,Z}) = \Phi'_{T(X),T(Y),T(Z)}$ and $T(\Psi_{X,Y}) = \Psi'_{T(X),T(Y)}$.

Definition 1.3. Let (T_1, c_1) and (T_2, c_2) be tensor functors from (C, \otimes) to (C', \otimes') . A morphism $\alpha: T_1 \rightarrow T_2$ of functors is a morphism of tensor functors if

(i)

$$\begin{array}{ccc}
T_1(X) \otimes' T_1(Y) & \xrightarrow{c_1} T_1(X \otimes Y) \\
\alpha_X \otimes' \alpha_Y \downarrow & & \downarrow \alpha_{X \otimes Y} \\
T_2(X) \otimes' T_2(Y) & \xrightarrow{c_2} T_2(X \otimes Y)
\end{array}$$

commutes for all objects X, Y of C and

(ii)

$$\begin{array}{ccc}
& \mathbb{1}' & \\
\swarrow & & \searrow \\
T_1(\mathbb{1}) & \xrightarrow{\alpha_{\mathbb{1}}} & T_2(\mathbb{1})
\end{array}$$

commutes, where the downwards morphisms are deduced from the uniqueness of the identity object in (C', \otimes') .

Note that, assuming (i), condition (ii) is equivalent to $\alpha_{\mathbb{1}}$ being an isomorphism. In particular, if α is an isomorphism of tensor functors, condition (ii) is vacuous. (Cf. [EGNO15, Def. 2.4.8].)

Definition 1.4. A tensor category (C, \otimes) is rigid, if for every object X of C there exists an object X^\vee (called a dual of X) of C together with morphisms $\text{ev}: X \otimes X^\vee \rightarrow \mathbb{1}$ and $\delta: \mathbb{1} \rightarrow X^\vee \otimes X$ such that

$$X \xrightarrow{r} X \otimes \mathbb{1} \xrightarrow{\text{id} \otimes \delta} X \otimes (X^\vee \otimes X) \xrightarrow{\Phi} (X \otimes X^\vee) \otimes X \xrightarrow{\text{ev} \otimes \text{id}} \mathbb{1} \otimes X \xrightarrow{l^{-1}} X$$

and

$$X^\vee \xrightarrow{l} \mathbb{1} \otimes X^\vee \xrightarrow{\delta \otimes \text{id}} (X^\vee \otimes X) \otimes X^\vee \xrightarrow{\Phi^{-1}} X^\vee \otimes (X \otimes X^\vee) \xrightarrow{\text{id} \otimes \text{ev}} X^\vee \otimes \mathbb{1} \xrightarrow{r^{-1}} X^\vee$$

are the identity morphism.

In [Del90, Section 2] it is shown that the above definition of rigid is equivalent to the one used in [DM82]. In a rigid tensor category the dual X^\vee of an object X is uniquely determined up to an isomorphism. A rigid tensor category (C, \otimes) is *abelian* if C is an abelian category. In this case \otimes is automatically bi-additive ([DM82, Prop. 1.16]).

Let k be a field and (C, \otimes) a rigid abelian tensor category. An isomorphism between k and $\text{End}(\mathbb{1})$ induces the structure of a k -linear category on C such that \otimes is k -bilinear. (See the discussion after Def. 1.15 in [DM82].)

Let R be a ring. A basic example of a tensor category is the category Mod_R of all R -modules with the usual tensor product $(M_1, M_2) \rightsquigarrow M_1 \otimes_R M_2$ of R -modules. The associativity constraint Φ is given by

$$\Phi_{M_1, M_2, M_3}: M_1 \otimes_R (M_2 \otimes_R M_3) \rightarrow (M_1 \otimes_R M_2) \otimes_R M_3, \quad m_1 \otimes (m_2 \otimes m_3) \mapsto (m_1 \otimes m_2) \otimes m_3$$

and the commutativity constraint Ψ is given by $\Psi_{M_1, M_2}: M_1 \otimes_R M_2 \rightarrow M_2 \otimes_R M_1, \quad m_1 \otimes m_2 \mapsto m_2 \otimes m_1$. Any free module U of rank one with an isomorphism $u: U \rightarrow U \otimes_R U$ is an identity object. Note that any identity object (U, u) can be written in the form $U = Ru_0$ and $u: U \rightarrow U \otimes_R U, \quad u_0 \mapsto u_0 \otimes u_0$ for some basis element u_0 of U .

Let k be a field. Recall that Vec_k is the category of all finite dimensional vector spaces over k . As explained above, this is a tensor category. In fact, $(\text{Vec}_k, \otimes_k)$ is rigid and abelian. For a k -algebra R , the functor $\text{Vec}_k \rightarrow \text{Mod}_R, \quad V \rightsquigarrow V \otimes_k R$ together with the isomorphism of functors c with components $c_{V, W}: (V \otimes_k R) \otimes_R (W \otimes_k R) \simeq (V \otimes_k W) \otimes_k R$ is a tensor functor.

Definition 1.5. Let k be a field. A neutral tannakian category over k is a rigid abelian tensor category (C, \otimes) together with an isomorphism $k \simeq \text{End}(\mathbb{1})$ such that there exists an exact k -linear tensor functor $\omega: (C, \otimes) \rightarrow (\text{Vec}_k, \otimes_k)$. Any such functor is called a (neutral) fibre functor.

We note that a fibre functor is faithful by [Del90, Cor. 2.10]. Let (C, \otimes) be a neutral tannakian category over the field k and $\omega: C \rightarrow \text{Vec}_k$ a fibre functor. For a k -algebra R , we denote the composition of ω with the tensor functor $\text{Vec}_k \rightarrow \text{Mod}_R, \quad V \rightsquigarrow V \otimes_k R$ by ω_R . In particular, $\omega_R(X) = \omega(X) \otimes_k R$ for every object X of C . We define

$$\underline{\text{Aut}}^\otimes(\omega)(R) = \text{Aut}^\otimes(\omega_R)$$

as the group of tensor automorphisms (i.e., invertible morphisms of tensor functors) of the tensor functor $\omega_R: (C, \otimes) \rightarrow (\text{Mod}_R, \otimes_R)$. This assignment is functorial in R and so $\underline{\text{Aut}}^\otimes(\omega)$ is a functor from the category of k -algebras to the category of groups.

Let G be a proalgebraic group over k . Recall (from the beginning of this section) that $\text{Rep}(G)$ denotes the category of finite dimensional representations of G . With the associativity and commutativity constraint induced from Vec_k , $\text{Rep}(G)$ is naturally a neutral tannakian category over k with fibre functor $(V, \phi) \rightsquigarrow V$. We are now prepared to state the main tannaka reconstruction theorem.

Theorem 1.6 ([DM82, Theorem 2.11]). *Let k be a field, C a neutral tannakian category over k and $\omega: C \rightarrow \text{Vec}_k$ a fibre functor. Then $G = \underline{\text{Aut}}^\otimes(\omega)$ is a proalgebraic group over k and ω defines a tensor equivalence between C and $\text{Rep}(G)$.*

Remark 1.7. If we choose $C = \text{Rep}(G)$ for a some proalgebraic group G over a field k and $\omega: \text{Rep}(G) \rightarrow \text{Vec}_k$, $(V, \phi) \rightsquigarrow V$ in Theorem 1.6, then $\underline{\text{Aut}}^\otimes(\omega) \simeq G$ by [DM82, Prop. 2.8].

1.2 Tensor skeletons

We introduce a tensor analog of skeletal categories and show that every tensor category is tensor equivalent to a tensor skeletal tensor category. Besides the cardinality issue already mentioned in the introduction, there is another reason why it is important to work with skeletons: Two categories are ought to be considered the “same” if they are equivalent. However, the eyes of model theory are conditioned to recognize the stronger notion of isomorphic categories. Since two categories are equivalent if and only if they have isomorphic skeletons, the two points of view can be reconciled by considering skeletons. (Cf. the remark at the very end of this section.)

Definition 1.8. *Let (C, \otimes) be a tensor category. An object V of C is tensor irreducible if it is not in the image of $\otimes: C \times C \rightarrow C$, i.e., V is not equal to $V_1 \otimes W_1$ for objects V_1 and W_1 of C . The tensor category (C, \otimes) has the unique tensor factorization property if*

- (i) $V_1 \otimes W_1 = V_2 \otimes W_2$ implies $V_1 = V_2$ and $W_1 = W_2$ for objects, V_1, V_2, W_1, W_2 of C , i.e., \otimes is injective on objects and
- (ii) every object of C is a finite tensor product of tensor irreducible objects.

The following example shows that a tensor category may not have any tensor irreducible objects. It also gives an example of a tensor category that does not satisfy condition (i) of the above definition.

Example 1.9. Let k be a field and let C be the category whose objects are the k -vector spaces k^n ($n \geq 0$). The morphisms of C are all k -linear maps between any two objects of C . For $m, n \geq 0$ choose an isomorphism $\eta_{m,n}: k^m \otimes_k k^n \rightarrow k^{mn}$.

On objects we define $\otimes: C \times C \rightarrow C$ by $k^m \otimes k^n = k^{mn}$ and on morphisms we define \otimes through the isomorphisms $\eta_{m,n}$, i.e., such that

$$\begin{array}{ccc} k^{m_1} \otimes_k k^{n_1} & \xrightarrow{f_1 \otimes_k f_2} & k^{m_2} \otimes_k k^{n_2} \\ \eta_{m_1 n_1} \downarrow & & \downarrow \eta_{m_2 n_2} \\ k^{m_1 n_1} & \xrightarrow{f_1 \otimes f_2} & k^{m_2 n_2} \end{array}$$

commutes for any $f_1: k^{m_1} \rightarrow k^{n_1}$ and $f_2: k^{m_2} \rightarrow k^{n_2}$. Similarly, we can use the isomorphisms $\eta_{m,n}$ to define associativity and commutativity constraints. In fact, C then becomes a neutral tannakian category over k . Note that no object of C is tensor irreducible. For example, $k^n = k^n \otimes k$ for every $n \geq 0$.

We note that C is a skeleton of the neutral tannakian category Vec_k and that some choice (namely the choice of the $\eta_{m,n}$) was involved to define a tensor product on the skeleton C . Using tensor skeletal tensor categories we will be able to avoid this choice. See Example 1.14 below.

Lemma 1.10. *Let (C, \otimes) be a tensor category with the unique tensor factorization property. Then every object of C is uniquely the tensor product of a finite completely parenthesized sequence of tensor irreducible objects of C .*

Proof. We only have to establish the uniqueness. Let V be an object of C . If V is tensor irreducible the claim is obvious. So we may assume that V is not tensor irreducible. Assume we have two presentations of V as tensor products of tensor irreducible objects. This yields two presentations $V = V_1 \otimes W_1 = V_2 \otimes W_2$, where either V_1 or W_1 is tensor irreducible and either V_2 or W_2 are tensor irreducible. Without loss of generality, let us assume that V_1 is tensor irreducible. Then, by condition (i) of Definition 1.8, we find that $V_1 = V_2$ and $W_1 = W_2$. So $V_2 = V_1$ is tensor irreducible and we have two presentations of $W_1 = W_2$ as tensor products of tensor irreducible objects. Applying the same reasoning to $W_1 = W_2$ and iterating this process, we reach, after finitely many steps, a situation where $W_1 = W_2$ is tensor irreducible. \square

Definition 1.11. Let (C, \otimes) be a tensor category with the unique tensor factorization property. The tensor length of an object V of C is the length of the unique completely parenthesized sequence of tensor irreducible objects whose tensor product equals V .

In particular, the tensor irreducible objects of C are those of tensor length one.

Definition 1.12. A tensor category (C, \otimes) is tensor skeletal if

- (i) any two isomorphic tensor irreducible objects are equal,
- (ii) every object of C is isomorphic to a tensor irreducible object and
- (iii) C has the unique tensor factorization property.

In a tensor skeletal tensor category every isomorphism class contains a unique tensor irreducible object and any object is uniquely the tensor product of tensor irreducible objects. In particular, the tensor irreducible objects are a skeleton of the category C .

Proposition 1.13. Let (C, \otimes) be a tensor category. Then there exists a tensor skeletal tensor category (C', \otimes') and a strict tensor equivalence $(C', \otimes') \rightarrow (C, \otimes)$.

Proof. We first choose a skeleton S for the category C , i.e., S is a full subcategory of C such that every object of C is isomorphic to a unique object in S . In the next step we close S under the tensor product in a generic fashion. In detail, we define the category C' as follows. Let P denote all completely parenthesized finite sequences of objects of S . Note that for a finite sequence V_1, \dots, V_n of objects in C there are $\frac{1}{n} \binom{2(n-1)}{n-1}$ possible ways to completely parenthesize the sequence. (This is the $(n-1)$ -st Catalan number.) For example, for $n=4$, we have the following possibilities

$$(((V_1, V_2), V_3), V_4) \quad ((V_1, V_2), (V_3, V_4)) \quad ((V_1, (V_2, V_3)), V_4), \quad ((V_1, (V_2, V_3)), V_4) \quad (V_1, ((V_2, V_3), V_4)).$$

To every $p \in P$ we associate an object $V(p)$ of C by evaluating the parenthesized sequence via \otimes . For example, for $p = (((V_1, V_2), V_3), V_4)$, we have $V(p) = ((V_1 \otimes V_2) \otimes V_3) \otimes V_4$. The class of objects of C' is defined as all pairs $(p, V(p))$, where $p \in P$. A morphism in C' from $(p, V(p))$ to $(q, V(q))$ is a morphism in C from $V(p)$ to $V(q)$.

We define a tensor product $\otimes': C' \times C' \rightarrow C'$ as follows. On objects we set

$$(p, V(p)) \otimes' (q, V(q)) = (pq, V(p) \otimes V(q)) = (pq, V(pq)),$$

where pq denotes the concatenation of two parenthesized sequences. For example, for $p = (((V_1, V_2), V_3), V_4)$ and $q = ((W_1, W_2), W_3)$ we have $pq = (((((V_1, V_2), V_3), V_4), ((W_1, W_2), W_3)))$.

For morphisms $f: (p_1, V(p_1)) \rightarrow (p_2, V(p_2))$ and $g: (q_1, V(q_1)) \rightarrow (q_2, V(q_2))$ in C' we define

$$f \otimes' g: (p_1, V(p_1)) \otimes' (q_1, V(q_1)) = (p_1 q_1, V(p_1) \otimes V(q_1)) \rightarrow (p_2 q_2, V(p_2) \otimes V(q_2)) = (p_2, V(p_2)) \otimes' (q_2, V(q_2))$$

as $f \otimes g$. We define an associativity constraint Φ' for \otimes' by defining

$$\Phi'_{(p,V(p)),(q,V(q)),(r,V(r))}: (p, V(p)) \otimes' ((q, (V(q)) \otimes' (r, V(r))) \rightarrow ((p, V(p)) \otimes' (q, (V(q)))) \otimes' (r, V(r))$$

as $\Phi_{V(p),V(q),V(r)}: V(p) \otimes (V(q) \otimes V(r)) \rightarrow (V(p) \otimes V(q)) \otimes V(r)$. Similarly, we define a commutativity constraint Ψ' for \otimes' by defining

$$\Psi'_{(p,V(p)),(q,V(q))}: (p, V(p)) \otimes' (q, V(q)) \rightarrow (q, V(q)) \otimes' (p, V(p))$$

as $\Psi_{V(p),V(q)}: V(p) \otimes V(q) \rightarrow V(q) \otimes V(p)$. The commutative diagrams for Φ and Ψ yield the corresponding commutative diagrams for Φ' and Ψ' . Moreover, if $\mathbb{1}$ together with $u: \mathbb{1} \rightarrow \mathbb{1} \otimes \mathbb{1}$ is an identity object for C , then $((\mathbb{1}), \mathbb{1})$ together with $u': ((\mathbb{1}), \mathbb{1}) \rightarrow ((\mathbb{1}), \mathbb{1}) \otimes' ((\mathbb{1}), \mathbb{1})$, defined as u is an identity object for C' . Thus (C', \otimes') is a tensor category.

Let us show that (C', \otimes') is tensor skeletal. Clearly, the tensor irreducible objects of C' are exactly those of the form $((V), V)$, where V belongs to S . Since S is a skeleton of C , it follows that every object of C' is isomorphic to a tensor irreducible object and that any two isomorphic tensor irreducible objects are equal. By construction, every object of C' is a tensor product of finitely many tensor irreducible objects. For $p_1, p_2, q_1, q_2 \in P$ with $p_1 q_1 = p_2 q_2$ we have $p_1 = p_2$ and $q_1 = q_2$. It follows that condition (i) of Definition 1.8 is satisfied. Thus (C', \otimes') is a tensor skeletal tensor category.

The functor $T: C' \rightarrow C$, $(p, V(p)) \rightsquigarrow V(p)$ is clearly fully faithful. Since S is a skeleton it is an equivalence of categories. Moreover, T is a strict tensor functor by construction. \square

Example 1.14. Let S denote the skeleton of $C = \text{Vec}_k$ consisting of all vector spaces of the form k^n , ($n \geq 0$). The proof of Proposition 1.13 yields a tensor skeletal tensor category C' whose objects are in bijection with the completely parenthesized finite sequences of k^n 's. The tensor product \otimes' for C' is induced from the tensor product on Vec_k . No additional choices as in Example 1.9 are required.

Let (C, \otimes) be a neutral tannakian category over a field k . Then the tensor product of two objects of C that are not zero objects is not a zero object (e.g., because this property holds in $\text{Rep}(G)$, for any proalgebraic group G). It follows that the full subcategory of all objects of C that are not zero objects is stable under the tensor product and therefore is naturally a tensor category.

Definition 1.15. A neutral tannakian category (C, \otimes) over a field k is pointed skeletal if it has exactly one zero object and the full subcategory of all objects that are not zero objects is tensor skeletal.

We note that in a pointed skeletal neutral tannakian category (C, \otimes) the zero object is not tensor irreducible. Moreover, an object of C , different from the zero object, is tensor irreducible in C if and only if it is tensor irreducible in the tensor skeletal tensor category of all objects that are not the zero object. Every object of C , different from the zero object, is uniquely the tensor product of a finite completely parenthesized sequence of tensor irreducible objects. As before, we call the length of this sequence the tensor length of the object. The tensor irreducible objects together with the zero object form a skeleton of C .

We need to introduce the above notion for a rather technical reason: If one attempts to work with tensor skeletal neutral tannakian categories one runs into trouble with axiom (20) below, because for the zero object, this axiom does not seem to be expressible as a first order statement. We will need a version of Proposition 1.13 for neutral tannakian categories.

Corollary 1.16. Let (C, \otimes) be a neutral tannakian category over k . Then there exists a pointed skeletal neutral tannakian category (C', \otimes') over k and a k -linear tensor equivalence $(C', \otimes) \rightarrow (C, \otimes)$.

Proof. Let (D, \otimes) denote the full subcategory of (C, \otimes) consisting of all objects that are not zero objects. Applying Proposition 1.13 to the tensor category (D, \otimes) yields a tensor category (D', \otimes') together with a strict tensor equivalence $T: (D', \otimes') \rightarrow (D, \otimes)$. Note that D' does not have a zero object because T is an equivalence of categories and D does not have a zero object.

We extend the category D' to a category C' by adding a zero object $\mathbf{0}$. So the objects of C' are the disjoint union of the objects of D' with $\mathbf{0}$. The morphisms between two objects in C' that both belong to D' are the same as the morphisms in D' . For an object V' of C' there is a unique morphism $0: \mathbf{0} \rightarrow V'$ in C' . Similarly, there is a unique morphism $0: V' \rightarrow \mathbf{0}$. Composition of morphisms in C' is defined such that composition with a zero morphism always yields a zero morphism. For example, the composition $V' \rightarrow \mathbf{0} \rightarrow W'$ is the unique $f \in \text{Hom}(V', W')$ such that $T(f): T(V') \rightarrow T(W')$ is the zero morphism (in C).

We extend the functor $T: D' \rightarrow D$ to a functor $T: C' \rightarrow C$ by choosing $T(\mathbf{0})$ to be a zero object of C and by defining T of a zero morphism to be a zero morphism. Then the functor $T: C' \rightarrow C$ defines an equivalence of categories. Since C is abelian, it follows that also C' is abelian.

Next we extend $\otimes': D' \times D' \rightarrow D'$ to a functor $\otimes': C' \times C' \rightarrow C'$ in the only meaningful way. Namely, $\mathbf{0} \otimes V' = \mathbf{0}$ and $V' \otimes \mathbf{0} = \mathbf{0}$ for every object V' of C' . Similarly, $f \otimes 0 = 0$ and $0 \otimes f = 0$ for any morphism f in C' . (Here 0 denotes an appropriate zero morphism.) The associativity and commutativity constraints on D' extend trivially to associativity and commutativity constraints on C' . So (C', \otimes') is a tensor category. Moreover, $T: (C', \otimes') \rightarrow (C, \otimes)$ is a tensor equivalence. Since (C, \otimes) is rigid, it follows that also (C', \otimes') is rigid. For an identity object $\mathbf{1}'$ of C' we have $\text{End}(\mathbf{1}') \simeq \text{End}(T(\mathbf{1}')) \simeq k$. For the induced k -linear structure on C' the functor T is k -linear. Composing T with a fibre functor $\omega: C \rightarrow \text{Vec}_k$ yields a fibre functor for C' . Thus (C', \otimes') is a neutral tannakian category over k . By construction (C', \otimes') is pointed skeletal. \square

The following lemma is needed in the next subsection to define the category TANN.

Lemma 1.17. *Let k be a field and C a neutral tannakian category over k . If C is pointed skeletal, then C is small, i.e., the class of objects of C is a set.*

Proof. Let $\omega: C \rightarrow \text{Vec}_k$ be a fibre functor and set $G = \underline{\text{Aut}}^\otimes(\omega)$. According to Theorem 1.6 we have an equivalence of categories $C \rightarrow \text{Rep}(G)$. Every representation of G is isomorphic to a representation of G on k^n for some $n \geq 0$. Thus the class of objects of a skeleton of $\text{Rep}(G)$ is a set. Since C is pointed skeletal, the class of all tensor irreducible objects of C together with the zero object is a skeleton of C . Since equivalent categories have isomorphic skeletons, it follows that the class of tensor irreducible objects of C is a set. Since every object of C , different from the zero object, is a finite tensor product of tensor irreducible objects it follows that the class of objects of C is a set. \square

1.3 TANN: The category of neutral tannakian categories

Our goal is to study proalgebraic groups from a model theoretic perspective by axiomatizing their categories of representations. The models of our theory PROALG will correspond to pointed skeletal neutral tannakian categories with a fibre functor. The models of PROALG together with the homomorphisms, i.e., the structure preserving maps, form a category that is equivalent to a certain category of neutral tannakian categories that we now describe in detail.

We define the category TANN as follows. The objects of TANN are triples (k, C, ω) , where k is a field, C is a pointed skeletal neutral tannakian category over k and $\omega: C \rightarrow \text{Vec}_k$ is a fibre functor. We note that by Lemma 1.17 pointed skeletal neutral tannakian categories are small, so there is no set theoretic obstruction to forming this category, like the obstruction one encounters when attempting to form the category of all categories.

A morphism in TANN from (k, C, ω) to (k', C', ω') is a pair (λ, T, α) , where $\lambda: k \rightarrow k'$ is a morphism of fields, $T: C \rightarrow C'$ is a k -linear strict tensor functor that preserves tensor irreducible objects and $\alpha: \omega_{k'} \rightarrow \omega' T$ is an isomorphism of tensor functors. Here $\omega_{k'}: C \rightarrow \text{Vec}_{k'}$ denotes the tensor functor obtained by composing ω with the tensor functor $\text{Vec}_k \rightarrow \text{Vec}_{k'}$, $V \rightsquigarrow V \otimes_k k'$ induced by $\lambda: k \rightarrow k'$.

The composition of two morphisms $(\lambda, T, \alpha): (k, C, \omega) \rightarrow (k', C', \omega')$ and $(\lambda', T', \alpha'): (k', C', \omega') \rightarrow (k'', C'', \omega'')$ in TANN is the pair $(\lambda' \lambda, T' T, \gamma): (k, C, \omega) \rightarrow (k'', C'', \omega'')$, where $\gamma: \omega_{k''} \rightarrow \omega'' T' T$ is given by

$$\gamma_V: \omega(V) \otimes_k k'' = (\omega(V) \otimes_k k') \otimes_{k'} k'' \xrightarrow{\alpha_V \otimes k''} \omega'(T(V)) \otimes_{k'} k'' \xrightarrow{\alpha'_{T(V)}} \omega''(T'(T(V)))$$

for every object V of C .

We also define a category PROALGEBRAIC GROUPS. The objects are pairs (k, G) , where k is a field and G a proalgebraic group over k . A morphism $(\lambda, \phi): (k, G) \rightarrow (k', G')$ in PROALGEBRAIC GROUPS is a pair (λ, ϕ) , where $\lambda: k \rightarrow k'$ is a morphism of fields and $\phi: G' \rightarrow G_{k'}$ is a morphism of proalgebraic groups over k' . Here $G_{k'}$ is the base change of G from k to k' via λ . The composition $(\lambda'', \phi''): (k, G) \rightarrow (k'', G'')$ of two morphism $(\lambda, \phi): (k, G) \rightarrow (k', G')$ and $(\lambda', \phi'): (k', G') \rightarrow (k'', G'')$ is defined by $\lambda'' = \lambda' \lambda$ and $\phi'': G'' \xrightarrow{\phi'} G'_{k''} \xrightarrow{\phi_{k''}} (G_{k'})_{k''} = G_{k''}$.

The following proposition is essential for establishing the close relationship between models of PROALG and proalgebraic groups.

Proposition 1.18. *The functor $(k, C, \omega) \rightsquigarrow (k, \underline{\text{Aut}}^\otimes(\omega))$ from the category TANN to the category PROALGEBRAIC GROUPS is full, essentially surjective and induces a bijection on the isomorphism classes.*

Proof. Let (k, C, ω) be an object of TANN. From Theorem 1.6 we know that $G = \underline{\text{Aut}}^\otimes(\omega)$ is a proalgebraic group over k and so we obtain an object (k, G) of PROALGEBRAIC GROUPS. A morphism $(\lambda, T, \alpha): (k, C, \omega) \rightarrow (k', C', \omega')$ in TANN defines a morphism $(\lambda, \phi): (k, G) \rightarrow (k', G')$ in PROALGEBRAIC GROUPS as follows: Let R' be a k' -algebra and $g' \in G'(R') = \text{Aut}^\otimes(\omega_{R'})$. So for every object V' of C' , we have an R' -linear automorphism $g'_{V'}: \omega'(V') \otimes_{k'} R' \rightarrow \omega'(V') \otimes_{k'} R'$. We define an element $\phi(g') \in G_{k'}(R') = G(R') = \text{Aut}^\otimes(\omega_{R'})$ by

$$\phi(g')_V: \omega(V) \otimes_k R' \simeq (\omega(V) \otimes_k k') \otimes_{k'} R' \xrightarrow{\alpha_V \otimes R'} \omega'(T(V)) \otimes_{k'} R' \xrightarrow{g'_{T(V)}} \omega'(T(V)) \otimes_{k'} R' \simeq \omega(V) \otimes_k R'$$

for every object V of C . Then $\phi_{R'}: G'(R') \rightarrow G_{k'}(R')$, $g' \mapsto \phi(g')$ is a morphism of groups that is functorial in R' and therefore defines a morphism $\phi: G' \rightarrow G_{k'}$ of proalgebraic groups over k' . We thus have a functor from TANN to PROALGEBRAIC GROUPS.

Let us show that this functor is full. We assume that a morphism $(\lambda, \phi): (k, G) \rightarrow (k', G')$ is given. We will define a morphism $(\lambda, T, \alpha): (k, C, \omega) \rightarrow (k', C', \omega')$ that induces (λ, ϕ) . Let us first explain the idea for the construction of T : We have tensor functors

$$C \xrightarrow{\omega} \text{Rep}(G) \rightarrow \text{Rep}(G_{k'}) \rightarrow \text{Rep}(G'), \quad (1)$$

where $\text{Rep}(G) \rightarrow \text{Rep}(G_{k'})$ is given by $V \rightsquigarrow V \otimes_k k'$ and $\text{Rep}(G_{k'}) \rightarrow \text{Rep}(G')$ is the restriction via $\phi: G' \rightarrow G_{k'}$. Composing the functor (1) with a quasi-inverse of the tensor equivalence $C' \xrightarrow{\omega'} \text{Rep}(G')$ yields a functor $T: C \rightarrow C'$. However, it is a priori not clear that a quasi-inverse can be chosen in such a way that T is a strict tensor functor that preserves tensor irreducible objects. Moreover, the construction of T is intertwined with the construction of α .

To define T and α , consider first a tensor irreducible object V of C . The representation of $G = \underline{\text{Aut}}^\otimes(\omega)$ on $\omega(V)$, induces a representation of $G_{k'}$ on $\omega(V) \otimes_k k'$ and by restriction via

$\phi: G' \rightarrow G_{k'}$ we obtain a representation of G' on $\omega(V) \otimes_k k'$. By Theorem 1.6 the category C' is equivalent (via ω') to the category of representations of G' . Thus there exists an object $T(V)$ of C' and an isomorphism $\alpha_V: \omega(V) \otimes_k k' \rightarrow \omega'(T(V))$ of representations of G' . In fact, since C' is tensor skeletal, we may choose $T(V)$ to be tensor irreducible.

This defines T on tensor irreducible objects. To define T on an object V of C , different from the zero object and of tensor length $n \geq 2$, we may assume that T has already been defined on objects of tensor length less than n . We know from Lemma 1.10 that V is uniquely of the form $V = V_1 \otimes V_2$, where V_1 and V_2 have tensor length less than n . We can thus define $T(V)$ as $T(V) = T(V_1) \otimes' T(V_2)$. Finally, we define T of the zero object of C to be the (unique) zero object of C' . This completes the definition of T on objects. Note that we have $T(V \otimes W) = T(V) \otimes' T(W)$ for all objects V, W of C .

We extend the definition of α in a similar manner: We have already defined α_V for tensor irreducible objects V . Let V be an object of C of tensor length n and assume α_V has been defined on objects of tensor length less than n . As V is of the form $V = V_1 \otimes V_2$ with V_1 and V_2 of tensor length less than n , we can define $\alpha_V = \alpha_{V_1 \otimes V_2}$ as the unique map making

$$\begin{array}{ccc} \omega(V_1 \otimes V_2) \otimes_k k' & \xrightarrow{\alpha_{V_1 \otimes V_2}} & \omega'(T(V_1 \otimes V_2)) \\ \simeq \downarrow & & \downarrow \simeq \\ (\omega(V_1) \otimes_k k') \otimes_{k'} (\omega(V_2) \otimes_k k') & \xrightarrow{\alpha_{V_1} \otimes \alpha_{V_2}} & \omega'(T(V_1)) \otimes_{k'} (\omega'(T(V_2))) \end{array} \quad (2)$$

commutative. For the zero object V , α_V is defined as the zero map. Then, by construction, the above diagram commutes for any pair of objects V_1 and V_2 of C .

We next define T on morphisms. Let $f: V \rightarrow W$ be a morphism in C . We then have a morphism $\omega(f): \omega(V) \rightarrow \omega(W)$ of representations of G , that induces a morphism $\omega(f) \otimes k': \omega(V) \otimes_k k' \rightarrow \omega(W) \otimes_k k'$ of representations of $G_{k'}$. This is also a morphism of representations of G' . In fact, we have morphisms of representations of G'

$$\begin{array}{ccc} \omega(V) \otimes_k k' & \xrightarrow{\omega(f) \otimes k'} & \omega(W) \otimes_k k' \\ \alpha_V \downarrow & & \downarrow \alpha_W \\ \omega'(T(V)) & & \omega'(T(W)) \end{array}$$

where the vertical maps are isomorphisms. Since ω' induces an equivalence of categories, there exists a unique morphism $T(f): T(V) \rightarrow T(W)$ in C' such that

$$\begin{array}{ccc} \omega(V) \otimes_k k' & \xrightarrow{\omega(f) \otimes k'} & \omega(W) \otimes_k k' \\ \alpha_V \downarrow & & \downarrow \alpha_W \\ \omega'(T(V)) & \xrightarrow{\omega'(T(f))} & \omega'(T(W)) \end{array} \quad (3)$$

commutes. This completes the definition of (λ, T, α) . Let us check that T is indeed a strict tensor functor. To see that T is compatible with the associativity constraint, let U, V, W be objects of C and $\Phi_{U,V,W}: U \otimes (V \otimes W) \rightarrow (U \otimes V) \otimes W$ the corresponding associativity isomorphism. We have the following commutative diagram:

$$\begin{array}{ccc}
\omega(U \otimes (V \otimes W)) \otimes_k k' & \xrightarrow{\omega(\Phi_{U,V,W}) \otimes k'} & \omega((U \otimes V) \otimes W) \otimes_k k' \\
\downarrow \simeq & & \downarrow \simeq \\
(\omega(U) \otimes_k k') \otimes_{k'} ((\omega(V) \otimes_k k') \otimes_{k'} (\omega(W) \otimes_k k')) & \longrightarrow & ((\omega(U) \otimes_k k') \otimes_{k'} (\omega(V) \otimes_k k')) \otimes_{k'} (\omega(W) \otimes_k k') \\
\downarrow \alpha_U \otimes (\alpha_V \otimes \alpha_W) & & \downarrow (\alpha_U \otimes \alpha_V) \otimes \alpha_W \\
\omega'(T(U)) \otimes_{k'} (\omega'(T(V)) \otimes_{k'} \omega'(T(W))) & \longrightarrow & (\omega'(T(U)) \otimes_{k'} \omega'(T(V))) \otimes_{k'} \omega'(T(W)) \\
\downarrow \simeq & & \downarrow \simeq \\
\omega'(T(U) \otimes (T(V) \otimes T(W))) & \xrightarrow{\omega'(\Phi'_{T(U),T(V),T(W)})} & \omega'((T(U) \otimes T(V)) \otimes T(W)) \\
\downarrow = & & \downarrow = \\
\omega'(T(U \otimes (V \otimes W))) & \xrightarrow{\omega'(T(\Phi_{U,V,W}))} & \omega'(T((U \otimes V) \otimes W))
\end{array}$$

Thanks to the commutativity of (2), we know that the map from the upper left to the lower left corner is $\alpha_{U \otimes (V \otimes W)}$. Similarly, the map from the upper right to the lower right corner is $\alpha_{(U \otimes V) \otimes W}$. Since, by definition, $T(\Phi_{U,V,W})$ is the unique morphism such that $\omega'(T(\Phi_{U,V,W}))$ makes the outer rectangle of the above diagram commute, we conclude that $T(\Phi_{U,V,W}) = \Phi'_{T(U),T(V),T(W)}$ as desired. In a similar fashion one shows that $T(\Psi_{U,V}) = \Psi'_{T(U),T(V)}$. Since T preserves identity objects we conclude that T is a strict tensor functor. Moreover, the commutativity of (3) shows that T is k -linear and by construction T preserves tensor irreducible objects.

The commutativity of (3) also shows that $\alpha: \omega_{k'} \rightarrow \omega' T$ is an isomorphism of functors and the commutativity of (2) shows that α is an isomorphism of tensor functors. Thus (λ, T, α) is indeed a morphism in TANN.

As the α_V 's are morphisms of representations of G' and G' is acting on $\omega(V) \otimes_k k'$ through the restriction via $\phi: G' \rightarrow G_{k'}$, it is then clear that the morphism (λ, T, α) induces the morphism (λ, ϕ) we started with. Thus the functor $(k, C, \omega) \rightsquigarrow (k, \underline{\text{Aut}}^{\otimes}(\omega))$ is full.

We next show that it is essentially surjective. Let G be a proalgebraic group over a field k . Applying Corollary 1.16 to the neutral tannakian category $\text{Rep}(G)$ yields a pointed skeletal neutral tannakian category (C, \otimes) and a k -linear tensor equivalence $F: C \rightarrow \text{Rep}(G)$. We define a fibre functor $\omega: C \rightarrow \text{Vec}_k$ by composing F with the forgetful functor $\omega_G: \text{Rep}(G) \rightarrow \text{Vec}_k$. Then (k, C, ω) is an object of TANN. Moreover, since F is a tensor equivalence, the natural morphism of functors $\underline{\text{Aut}}^{\otimes}(\omega_G) \rightarrow \underline{\text{Aut}}^{\otimes}(\omega)$ is an isomorphism. Since $\underline{\text{Aut}}^{\otimes}(\omega_G)$ is isomorphic to G (Remark 1.7) we see that $(k, C, \omega) \rightsquigarrow (k, \underline{\text{Aut}}^{\otimes}(\omega))$ is essentially surjective.

Finally, we establish the bijection on isomorphism classes. Since we already proved the essential surjectivity, it suffices to show the following: For objects (k, C, ω) and (k', C', ω') of TANN, if $(k, \underline{\text{Aut}}^{\otimes}(\omega))$ and $(k', \underline{\text{Aut}}^{\otimes}(\omega'))$ are isomorphic, then (k, C, ω) and (k', C', ω') are isomorphic. We abbreviate $G = \underline{\text{Aut}}^{\otimes}(\omega)$ and $G' = \underline{\text{Aut}}^{\otimes}(\omega')$. Let $(\lambda, \phi): (k, G) \rightarrow (k', G')$ be an isomorphism in PROALGEBRAIC GROUPS. In the above proof that $(k, C, \omega) \rightsquigarrow (k, \underline{\text{Aut}}^{\otimes}(\omega))$ is full, we have already seen how to construct a morphism $(\lambda, T, \alpha): (k, C, \omega) \rightarrow (k', C', \omega')$ from (λ, ϕ) . We claim that (λ, T, α) is an isomorphism.

We first show that T is surjective on objects. Let V' be a tensor irreducible object of C' . Then $\omega'(V')$ is a representation of G' . By assumption $\lambda: k \rightarrow k'$ is an isomorphism of fields and $\phi: G' \rightarrow G_{k'}$ is an isomorphism of proalgebraic groups. In the sequel we will use $\lambda^{-1}: k' \rightarrow k$ to base change from k' to k . For example, $\omega'(V') \otimes_{k'} k$ is a representation of G'_k . But G'_k is isomorphic

to G via $G_{k'} \xrightarrow{\phi_k} (G_{k'})_k \simeq G$ and so we can consider $\omega'(V') \otimes_{k'} k$ to be a representation of G . Since $\omega: C \rightarrow \text{Rep}(G)$ is an equivalence of categories, there exists an object V of C such that $\omega(V)$ is isomorphic to $\omega'(V') \otimes_{k'} k$ as a representation of G . Moreover, since C is tensor skeletal, we can choose V to be tensor irreducible. It follows that $\omega(V) \otimes_k k'$ is isomorphic to $\omega'(V')$ as a representation of G' . As $T(V)$ is, by definition, the unique tensor irreducible object of C' such that $\omega'(T(V))$ is isomorphic to $\omega(V) \otimes_k k'$ as a representation of G' , it follows that $T(V) = V'$. So T is surjective on tensor irreducible objects. An arbitrary object V' of C' , different from the zero object, is a finite tensor product of tensor irreducible objects. We can choose an inverse image under T for all these tensor irreducible objects, form their tensor product in C and then apply the strict tensor functor T to see that V' is in the image of T .

We next show that T is injective on objects. First let V_1 and V_2 be tensor irreducible objects of C such that $T(V_1) = T(V_2)$. Then $\omega'(T(V_1)) = \omega'(T(V_2))$ as representation of G' and $\omega'(T(V_1)) \otimes_{k'} k = \omega'(T(V_2)) \otimes_{k'} k$ as representation of G . But $\omega'(T(V_1)) \otimes_{k'} k \simeq \omega(V_1)$ as representation of G and similarly for V_2 . So $\omega(V_1)$ and $\omega(V_2)$ are isomorphic representations of G . But then V_1 and V_2 must be isomorphic objects of C . Since V_1 and V_2 are tensor irreducible, it follows that $V_1 = V_2$. Thus T is injective on tensor irreducible objects. From the uniqueness in Lemma 1.10 it then follows that T is injective on objects.

Using diagram (3), we see that T is fully faithful. Since T is bijective on objects, T is an isomorphism of categories, i.e., there exists a functor $T^{-1}: C' \rightarrow C$ such that $TT^{-1} = \text{id}_{C'}$ and $T^{-1}T = \text{id}_C$. Since T is a strict tensor functor, also T^{-1} is a strict tensor functor. Similarly, as T preserves tensor irreducible objects, also T^{-1} preserves tensor irreducible objects. Finally, $T^{-1}: C' \rightarrow C$ is k' -linear, where the k' -linear structure on C is defined via $\lambda^{-1}: k' \rightarrow k$.

For every object V' of C' we have a k' -linear isomorphism $\alpha_{T^{-1}(V')}: \omega(T^{-1}(V')) \otimes_k k' \rightarrow \omega'(V')$. The base change of this map via $\lambda^{-1}: k' \rightarrow k$ is a k -linear isomorphism $\alpha_{T^{-1}(V')} \otimes k: \omega(T^{-1}(V')) \rightarrow \omega'(V') \otimes_{k'} k$. We define $(\alpha^{-1})_{V'} = (\alpha_{T^{-1}(V')} \otimes k)^{-1}$. Then $\alpha^{-1}: \omega'_k \rightarrow \omega T^{-1}$ is an isomorphism of functors. Since α is an isomorphism of tensor functors, also α^{-1} is an isomorphism of tensor functors. Finally, $(\lambda^{-1}, T^{-1}, \alpha^{-1})$ is an inverse to (λ, T, α) in TANN. \square

The functor of Proposition 1.18 is not faithful. This is reflected in the proof of the fullness of the functor by the fact that the α_V 's, for V tensor irreducible, can be chosen arbitrarily.

We note that for Proposition 1.18 to be valid it is important to consider pointed skeletal neutral tannakian categories. For example, the neutral tannakian categories in Example 1.9 and 1.14 both correspond to the trivial proalgebraic group. However, these two categories are not isomorphic.

2 PROALG: Neutral tannakian categories as first order structures

In this section we define a many-sorted first order theory PROALG such that the isomorphism classes of models of PROALG are in bijection with the isomorphism classes of proalgebraic groups. The idea is to axiomatize pointed skeletal neutral tannakian categories with a fibre functor.

2.1 The language

We define a many-sorted first order theory PROALG as follows:

Sorts:

We have three different types of sorts: The *field* sort, the *objects* sorts and the *morphisms* sorts. The objects sorts and the morphisms sorts split further into the *base* objects/morphisms sorts and

the *total* objects/morphisms sorts. We will use the following notation:

With k we denote the universe of the field sort. For every pair $p = (m, n)$ of integers $m, n \geq 1$ we have two objects sorts: The base objects sort with universe B_p and the total objects sort with universe X_p . For every pair p, q , where, as above $p = (m, n)$ and $q = (m', n')$ with $m, n, m', n' \geq 1$, we have two morphisms sorts: The base morphisms sort with universe $B_{p,q}$ and the total morphisms sort with universe $X_{p,q}$.

The idea is that B_p , where $p = (m, n)$, represents k -vector spaces of tensor length m and dimension n , considered as objects of a category, i.e., every element of B_p corresponds to such a vector space. On the other hand, X_p contains the actual vector spaces.

Similarly, for morphisms: $B_{p,q}$ represents morphisms from vector spaces in B_p to vector spaces in B_q ; every element of $B_{p,q}$ corresponds to a morphism, the actual linear maps are encoded in $X_{p,q}$.

Constant symbols:

- We have two constant symbols 0 and 1 for the field-sort.

Relation symbols:

- For every p we have a unary relation symbol 0_p on X_p .
- For every p we have a ternary relation symbol A_p on X_p . (The “A” is for addition.)
- We have a constant symbol 1 in $B_{(1,1)}$.
- For every $p = (m, n)$ we have an n -ary relation symbol LI_p on X_p . (“LI” is for linear independence.)

Function symbols:

- We have two binary function symbols $+$ and \cdot for the field-sort.
- For every p we have a function symbol π_p with interpretation $\pi_p: X_p \rightarrow B_p$.
- For every pair p, q we have a function symbol $\pi_{p,q}$ with interpretation $\pi_{p,q}: X_{p,q} \rightarrow B_{p,q}$.
- For every p we have a function symbol SM_p with interpretation $SM_p: k \times X_p \rightarrow X_p$. (“SM” is for scalar multiplication.)
- For all p, q we have function symbols $S_{p,q}^B$ and $T_{p,q}^B$ with interpretations $S_{p,q}^B: B_{p,q} \rightarrow B_p$ and $T_{p,q}^B: B_{p,q} \rightarrow B_q$. (“S” is for source and “T” is for target of a morphism.)
- For all p, q we have function symbols $S_{p,q}^X$ and $T_{p,q}^X$ with interpretations $S_{p,q}^X: X_{p,q} \rightarrow X_p$ and $T_{p,q}^X: X_{p,q} \rightarrow X_q$.
- For $p = (m, n)$ and $q = (m', n')$ with $m, n, m', n' \geq 1$ we set $pq = (m + m', nn')$. We have function symbols $\otimes_{p,q}$ with interpretations $\otimes_{p,q}: X_p \times X_q \rightarrow X_{pq}$.

We denote this many-sorted language with \mathcal{L} .

2.2 The axioms

Rather than stating the axioms explicitly in the above language, we state their mathematical content. It is however clear that all the axioms below can be expressed as a collection of \mathcal{L} -sentences.

- (1) $(k, +, \cdot, 0, 1)$ is a field.
- (2) For every p , the map $\pi_p: X_p \rightarrow B_p$ is surjective. To simplify the notation we set $X_p(b) = \pi_p^{-1}(b)$ for $b \in B_p$.
- (3) Existence of zero: For every $V = X_p(b)$, (where $b \in B_p$), the set $V \cap 0_p$ has a unique element 0_V .
- (4) Vector space addition: For $v_1, v_2, v_3 \in X_p$, if $A_p(v_1, v_2, v_3)$ holds, then $\pi_p(v_1) = \pi_p(v_2) = \pi_p(v_3)$. Moreover, for $b = \pi_p(v_1) = \pi_p(v_2) = \pi_p(v_3) \in B_p$, and $V = X_p(b)$, the set $\{(v_1, v_2, v_3) \in V^3 \mid A_p(v_1, v_2, v_3)\}$ is the graph of a map $+_V: V \times V \rightarrow V$, that defines on V the structure of an abelian group with identity element 0_V .
- (5) Scalar multiplication: If $\lambda \in k$ and $v \in X_p(b)$, for some $b \in B_p$, then also $SM_p(\lambda, a) \in X_p(b)$. Moreover, for every $V = X_p(b)$, the restriction of SM_p to $\cdot_V: k \times V \rightarrow V$ defines a scalar multiplication on V such that V is a vector space over k with addition $+_V$.
- (6) Dimension: Every $X_p(b)$ ($b \in B_p$) is an n -dimensional k -vector space, where $p = (m, n)$.
- (7) The maps $\pi_{p,q}: X_{p,q} \rightarrow B_{p,q}$ are surjective. For $f \in B_{p,q}$ we set $X_{p,q}(f) = \pi_{p,q}^{-1}(f)$.
- (8) The diagram

$$\begin{array}{ccccc}
 X_p & \xleftarrow{S_{p,q}^X} & X_{p,q} & \xrightarrow{T_{p,q}^X} & X_q \\
 \pi_p \downarrow & & \downarrow \pi_{p,q} & & \downarrow \pi_q \\
 B_p & \xleftarrow{S_{p,q}^B} & B_{p,q} & \xrightarrow{T_{p,q}^B} & B_q
 \end{array}$$

commutes.

- (9) Morphisms: For every $f \in B_{p,q}$ the map $S_{p,q}^X: X_{p,q}(f) \rightarrow X_p(S_{p,q}^B(f))$ is bijective. The image of $(S_{p,q}^X, T_{p,q}^X): X_{p,q}(f) \rightarrow X_p(S_{p,q}^B(f)) \times X_q(T_{p,q}^B(f))$, is the graph of a k -linear map $\tilde{f}: X_p(S_{p,q}^B(f)) \rightarrow X_q(T_{p,q}^B(f))$.

Moreover, if $f, g \in B_{p,q}$ with $S_{p,q}^B(f) = S_{p,q}^B(g)$ and $T_{p,q}^B(f) = T_{p,q}^B(g)$ such that $\tilde{f} = \tilde{g}$, then $f = g$.

- (10) Existence of the identity: For all $b \in B_p$ there exists an $f \in B_{p,p}$ such that $S_{p,p}^X(a) = T_{p,p}^X(a)$ for all $a \in X_{p,p}(f)$.
- (11) Composition of morphisms: For $f \in B_{p,q}$ and $g \in B_{q,r}$ with $T_{p,q}^B(f) = S_{q,r}^B(g)$ there exists $h \in B_{p,r}$ such that $\tilde{h} = \tilde{g} \circ \tilde{f}$.
- (12) Linearity: For $f, g \in B_{p,q}$ with $S_{p,q}^B(f) = S_{p,q}^B(g)$ and $T_{p,q}^B(f) = T_{p,q}^B(g)$, there exists $h \in B_{p,q}$ such that $\tilde{f} + \tilde{g} = \tilde{h}$.

Moreover, for every $f \in B_{p,q}$ and $\lambda \in k$, there exists g in $B_{p,q}$ such that $\lambda \tilde{f} = \tilde{g}$. (In particular, for $\lambda = 0$, we see that the zero morphism is of the form \tilde{g} .)

- (13) Tensor is compatible with projections: For $a \in X_p$ and $b \in X_q$ we write $a \otimes b$ or $a \otimes_{p,q} b$ for $\otimes_{p,q}(a, b)$.
 If $a_1, a_2 \in X_p$ with $\pi_p(a_1) = \pi_p(a_2)$ and $b_1, b_2 \in X_q$ with $\pi_q(b_1) = \pi_q(b_2)$, then $\pi_{pq}(a_1 \otimes b_1) = \pi_{pq}(a_2 \otimes b_2)$.
- (14) Bilinearity of tensor product: If $a_1, a_2 \in X_p$ with $\pi_p(a_1) = \pi_p(a_2)$ and $b \in X_q$, then $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$. Moreover, for $\lambda \in k$ we have $\lambda a_1 \otimes b = \lambda(a_1 \otimes b)$. Similarly for left- and right-hand side interchanged.
- (15) Tensor product: For $b \in B_p$ and $c \in B_q$, let $b \otimes c = \pi_{pq}(v \otimes w) \in B_{pq}$, where $v \in X_p(b)$ and $w \in X_q(c)$. (Note that, by axiom [13](#), $b \otimes c$ does not depend on the choice of v and w .)
 The map $X_p(b) \otimes_k X_q(c) \rightarrow X_{pq}(b \otimes c)$ induced by the bilinear map $\otimes_{p,q}: X_p(b) \times X_q(c) \rightarrow X_{pq}(b \otimes c)$ is an isomorphism. We set $X_p(b) \otimes X_q(c) = X_{pq}(b \otimes c)$.
- (16) Functoriality of tensor product: For $b_1 \in B_{p_1}$, $b_2 \in B_{p_2}$, $c_1 \in B_{q_1}$, $c_2 \in B_{q_2}$, $f \in B_{p_1, q_1}$ with $S_{p_1, q_1}^B(f) = b_1$ and $T_{p_1, q_1}^B(f) = c_1$, and $g \in B_{p_2, q_2}$ with $S_{p_2, q_2}^B(g) = b_2$ and $T_{p_2, q_2}^B(g) = c_2$, there exists $h \in B_{p_1 p_2, q_1 q_2}$ such that
- $$\tilde{h} = \tilde{f} \otimes \tilde{g}: X_{p_1 p_2}(b_1 \otimes b_2) = X_{p_1}(b_1) \otimes_k X_{p_2}(b_2) \rightarrow X_{q_1}(c_1) \otimes_k X_{q_2}(c_2) = X_{q_1 q_2}(c_1 \otimes c_2).$$
- (17) Associativity of the tensor product: For $b \in B_p$, $c \in B_q$ and $d \in B_r$, there exists $f \in B_{pqr, pqr}$ such that $\tilde{f}: X_p(b) \otimes (X_q(c) \otimes X_r(d)) \rightarrow (X_p(b) \otimes X_q(c)) \otimes X_r(d)$ equals the map defined by $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$.
- (18) Commutativity of the tensor product: For $b \in B_p$ and $c \in B_q$, there exists $f \in B_{pq, pq}$ such that $\tilde{f}: X_p(b) \otimes X_q(c) \rightarrow X_q(c) \otimes X_p(b)$ equals the map defined by $v \otimes w \mapsto w \otimes v$.
- (19) Uniqueness of tensor factorization: If $p_1 q_1 = p_2 q_2$, $b_1 \in B_{p_1}$, $c_1 \in B_{q_1}$, $b_2 \in B_{p_2}$ and $c_2 \in B_{q_2}$ are such that $b_1 \otimes c_1 = b_2 \otimes c_2$, then $p_1 = p_2$, $q_1 = q_2$, $b_1 = b_2$ and $c_1 = c_2$.
- (20) Existence of tensor factorization: For every $b \in B_{(m, n)}$, there exist elements b_1, \dots, b_m with $b_i \in B_{(1, n_i)}$ for some n_i with $n_1 \dots n_m = n$ and a complete parenthesization of the sequence b_1, \dots, b_m such that the corresponding tensor product of the sequence is equal to b .
- (21) Tensor skeletal: For $b \in B_{(m, n)}$, there exists a $c \in B_{(1, n)}$ and $f \in B_{((m, n), (1, n))}$ such that $\tilde{f}: X_{(m, n)}(b) \rightarrow X_{(1, n)}(c)$ is bijective.
 Moreover, if $b, c \in B_{(1, n)}$ are such that there exists an f in $B_{(1, n), (1, n)}$ with $\tilde{f}: X_{(1, n)}(b) \rightarrow X_{(1, n)}(c)$ bijective, then $b = c$.
- (22) Existence of the identity object: Recall that 1 is a constant in $B_{(1, 1)}$. For every non-zero element u_0 of $\mathbb{1} = X_{(1, 1)}(1)$ and $b \in B_{(m, n)}$, there exists f in $B_{((m, n), (m+1, n))}$ such that $\tilde{f}: X_{(m, n)}(b) \rightarrow \mathbb{1} \otimes X_{(m, n)}(b)$ is the map $v \mapsto u_0 \otimes v$.
- (23) Existence of duals: For every $b \in B_{(m, n)}$ there exists b^\vee in $B_{(1, n)}$, $f \in B_{(m+1, n^2), (1, 1)}$ and $g \in B_{(1, 1), (m+1, n^2)}$ with $\tilde{f}: V \otimes V^\vee \rightarrow \mathbb{1}$ and $\tilde{g}: \mathbb{1} \rightarrow V^\vee \otimes V$, where $V = X_{(m, n)}(b)$ and $V^\vee = X_{(1, n)}(b^\vee)$, such that the maps

$$V \rightarrow V \otimes \mathbb{1} \xrightarrow{V \otimes \tilde{g}} V \otimes (V^\vee \otimes V) \rightarrow (V \otimes V^\vee) \otimes V \xrightarrow{\tilde{g} \otimes V} \mathbb{1} \otimes V \rightarrow V$$

$$V^\vee \rightarrow \mathbb{1} \otimes V^\vee \xrightarrow{\tilde{g} \otimes V^\vee} (V^\vee \otimes V) \otimes V^\vee \rightarrow V^\vee \otimes (V \otimes V^\vee) \xrightarrow{V^\vee \otimes \tilde{f}} V^\vee \otimes \mathbb{1} \rightarrow V^\vee$$

are the identity maps.

- (24) Existence of direct sums (biproducts): For $b \in B_{(m,n)}$ and $c \in B_{(m',n')}$, there exists $d \in B_{(1,n+n'), (m,n)}$, $P_b \in B_{((1,n+n'), (m,n))}$, $P_c \in B_{(1,n+n'), (m',n')}$, $I_b \in B_{(m,n), (1,n+n')}$ and $I_c \in B_{(m',n'), (1,n+n')}$ such that $\widetilde{I}_b \circ \widetilde{P}_b + \widetilde{I}_c \circ \widetilde{P}_c = \text{id}_{X_{(1,n+n')}(d)}$, $\widetilde{P}_b \circ \widetilde{I}_b = \text{id}_{X_{(m,n)}(b)}$, $\widetilde{P}_c \circ \widetilde{I}_c = \text{id}_{X_{(m',n')}}(c)$, $\widetilde{P}_c \circ \widetilde{I}_b = 0$ and $\widetilde{P}_b \circ \widetilde{I}_c = 0$.
- (25) Existence of kernels: For every f in $B_{p,q}$ and f' in $B_{r,q}$ with $\widetilde{f}: V \rightarrow W$ injective and $\widetilde{f}': U \rightarrow W$ such that $\widetilde{f}'(U) \subseteq \widetilde{f}(V)$, there exists $f'' \in B_{r,p}$ with $\widetilde{f}'': U \rightarrow V$ such that $\widetilde{f} \circ \widetilde{f}'' = \widetilde{f}'$.
Moreover, for every $f \in B_{p,q}$ with $\widetilde{f}: V \rightarrow W$ and $\dim(\ker(\widetilde{f})) = \ell \geq 1$, there exists $f' \in B_{(1,\ell),p}$ with $\widetilde{f}': U \rightarrow V$ such that \widetilde{f}' is injective and $\widetilde{f} \circ \widetilde{f}' = 0$.
- (26) Existence of cokernels: For $f \in B_{p,q}$ and $f' \in B_{p,r}$ with $\widetilde{f}: V \rightarrow W$ surjective and $\widetilde{f}': V \rightarrow U$ such that $\ker(\widetilde{f}) \subseteq \ker(\widetilde{f}')$, there exists $f'' \in B_{q,r}$ with $\widetilde{f}'': W \rightarrow U$ such that $\widetilde{f}'' \circ \widetilde{f} = \widetilde{f}'$.
Moreover, for every $f \in B_{p,q}$ with $\widetilde{f}: V \rightarrow W$ and $\dim(\text{Im}(\widetilde{f})) = \ell < n$, where $q = (m,n)$, there exists $\widetilde{f}' \in B_{(q,(1,n-\ell))}$, with $\widetilde{f}': W \rightarrow U$, such that \widetilde{f}' is surjective and $\widetilde{f}' \circ \widetilde{f} = 0$.
- (27) Linear independence: For $v_1, \dots, v_n \in X_p$, we have $LI_p(v_1, \dots, v_n)$, where $p = (m,n)$, if and only if $\pi_p(v_i) = \pi_p(v_j)$ for $1 \leq i, j \leq n$ and v_1, \dots, v_n are linearly independent (in $X_p(b)$, where $b = \pi_p(v_i)$).

Remark 2.1. The relations LI_p are definable from the other symbols of the language. So the relation symbols LI_p could in principle be omitted from the language. It is however convenient to work with the LI_p 's, because they imply that a homomorphism of models of PROALG has certain desirable properties. See Theorem 2.2 and its proof for details.

2.3 Equivalence of PROALG and TANN

Let \mathcal{M} and \mathcal{M}' be models of PROALG. Recall that a homomorphism $h: \mathcal{M} \rightarrow \mathcal{M}'$ is a sequence of maps, one for each sort s , that maps the \mathcal{M} -universe of the s -sort to the \mathcal{M}' -universe of the s -sort, such that all constants, relations, and functions are preserved.

Theorem 2.2. *The category of models of PROALG with the homomorphisms as morphisms is equivalent to the category TANN.*

Proof. Let $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ be an object of PROALG. We will associate an object $(k(\mathcal{M}), C(\mathcal{M}), \omega(\mathcal{M}))$ of TANN to \mathcal{M} . We set $k(\mathcal{M}) = k$ (including the field structure) and we define a category $D = D(\mathcal{M})$ as follows: The set of objects of D is the disjoint union of all B_p 's. For $b \in B_p$ and $c \in B_q$, the set of morphisms from b to c is defined as

$$\text{Hom}(b, c) = \{f \in B_{p,q} \mid S_{p,q}^B(f) = b, T_{p,q}^B(f) = c\}.$$

To define the composition $g \circ f$ for $f \in \text{Hom}(b, c)$ and $g \in \text{Hom}(c, d)$ we use axioms 9 and 11. We define $g \circ f$ as the unique element of $B_{p,r}$ such that $\widetilde{f \circ g} = \widetilde{g} \circ \widetilde{f}$. As the composition of k -linear maps is associative, it follows that our composition is also associative. For $b \in B_p$ we define the identity $\text{id}_b \in \text{Hom}(b, b)$ to be the unique element of $\text{Hom}(b, b)$ with $\widetilde{\text{id}_b} = \text{id}_{X_p(b)}$ (axiom 10). It is then clear that D is a category.

We define a tensor product $\otimes: D \times D \rightarrow D$ as follows: On objects, say $b \in B_p$ and $c \in B_q$, we define $b \otimes c \in B_{pq}$ as in axiom 15. For morphisms $f: b_1 \rightarrow c_1$ and $g: b_2 \rightarrow c_2$ we define $f \otimes g: b_1 \otimes b_2 \rightarrow c_1 \otimes c_2$ as the unique element of $\text{Hom}(b_1 \otimes b_2, c_1 \otimes c_2)$ with $\widetilde{f \otimes g} = \widetilde{f} \otimes \widetilde{g}$ (axiom 16). It is then clear that \otimes is a functor.

We define an associativity constraint Φ with components $\Phi_{b,c,d}: b \otimes (c \otimes d) \rightarrow (b \otimes c) \otimes d$ such that $\widetilde{\Phi}_{b,c,d}$ corresponds to the usual associativity constraint in Vec_k (axiom [17](#)). Similarly, we define a commutativity constraint Ψ using axiom [18](#). The required diagrams for Φ and Ψ commute because the corresponding diagrams commute in Vec_k .

Let u_0 be any non-zero element of $\mathbf{1}$ (see axiom [22](#)) and let u be the unique element of $\text{Hom}(\mathbf{1}, \mathbf{1} \otimes \mathbf{1})$ such that $\tilde{u}: \mathbf{1} \rightarrow \mathbf{1} \otimes \mathbf{1}$, $u_0 \mapsto u_0 \otimes u_0$. We claim that $(\mathbf{1}, u)$ is an identity object for (D, \otimes) . Note that by axiom [25](#) every $f \in B_{p,q}$ such that \tilde{f} is bijective is an isomorphism. So it follows from axiom [22](#) that $b \rightsquigarrow \mathbf{1} \otimes b$ is an equivalence of categories. Thus D is a tensor category. Axioms [19](#), [20](#) and [21](#) imply that D is tensor skeletal. The tensor irreducible objects are those belonging to some $B_{(1,n)}$.

Note that the category D does not have a zero object, because all the $X_p(b)$'s are vector spaces of dimension greater or equal to $n \geq 1$. We now add a zero object $\mathbf{0}$ to D to form a category $C = C(\mathcal{M})$. This is done in a similar fashion as in the proof of Corollary [1.16](#). So the objects of C are the disjoint union of the objects of D with $\mathbf{0}$. The morphisms between two objects in C that both belong to D are the same as the morphisms in D . For an object b of D there is a unique morphism $0: \mathbf{0} \rightarrow b$ in C . Similarly, there is a unique morphism $0: b \rightarrow \mathbf{0}$. Composition of morphisms in C is defined in the obvious way. For example, the composition $b \rightarrow \mathbf{0} \rightarrow c$ is the unique $f \in \text{Hom}(b, c)$ such that \tilde{f} is the zero map (cf. axiom [12](#)).

For consistence reasons we extend some of our notation to include zero: We define $X(\mathbf{0})$ to be the zero vector space (over k) and we set $\tilde{0}$ to be the zero map.

As in the proof of Corollary [1.16](#) we extend $\otimes: D \times D \rightarrow D$ to a functor $\otimes: C \times C \rightarrow C$ in the only meaningful way. The associativity and commutativity constraints on D extend trivially to associativity and commutativity constraints on C . So (C, \otimes) is a tensor category. It follows from axiom [23](#) that (C, \otimes) is rigid.

Let us next show that C is an abelian category. Clearly C has a zero object, namely $\mathbf{0}$. By axiom [24](#) the category C has biproducts. It follows from axioms [25](#) and [26](#) that C has kernels and cokernels. Let $f: b \rightarrow c$ be a monomorphism in C . We claim that $f: X_p(b) \rightarrow X_q(c)$ is injective. For a kernel $g: a \rightarrow b$ of f we have $fg = 0 = f0$ and therefore $g = 0$. Since the image of \tilde{g} is the kernel of f we see that \tilde{f} is injective. It then follows from axiom [26](#) that f is the kernel of its cokernel. So f is normal. Similarly, we see that also every epimorphism in C is normal. Thus C is an abelian category.

For $f \in \text{Hom}(b, c)$ and $\lambda \in k$ we define λf as the unique element of $\text{Hom}(b, c)$ such that $\tilde{\lambda f} = \lambda \tilde{f}$ (axiom [12](#)). Thus C becomes a k -linear category.

We now define the fibre functor $\omega = \omega(\mathcal{M})$ from C to Vec_k : For $b \in B_p$ we set $\omega(b) = X_p(b)$. For a morphism $f: b \rightarrow c$ in C we set $\omega(f) = \tilde{f}$. By construction f is an exact k -linear functor. The isomorphism $\gamma: \omega(-) \otimes_k \omega(-) \rightarrow \omega(- \otimes -)$ of functors, with components

$$\gamma_{b,c}: \omega(b) \otimes_k \omega(c) = X_p(b) \otimes_k X_p(c) \rightarrow X_{pq}(b \otimes c) = \omega(b \otimes c), \quad v \otimes w \mapsto v \otimes_{p,q} w$$

turns ω into a tensor functor (cf. axiom [15](#)). Thus C is a neutral tannakian category over k . Since (D, \otimes) is tensor skeletal, we see that, as desired, C is a pointed skeletal neutral tannakian category over k . Thus $(k(\mathcal{M}), C(\mathcal{M}), \omega(\mathcal{M}))$ is an object of TANN.

Let $h: \mathcal{M} \rightarrow \mathcal{M}'$ be a homomorphism of models of PROALG. Where $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$, $\mathcal{M}' = (k', B'_p, X'_p, B'_{p,q}, X'_{p,q})$ and $h = (h_{\text{field}}, h_p^B, h_p^X, h_{p,q}^B, h_{p,q}^X)$

We claim that h induces a morphism $(\lambda(h), T(h), \alpha(h)): (k, C, \omega) \rightarrow (k', C', \omega')$ in TANN. We set $\lambda(h) = h_{\text{field}}$ and consider k' as a field extension of k via $\lambda(h)$.

We define the functor $T = T(h): C \rightarrow C'$ through the action of h on the base sorts: We set $T(b) = h_p^B(b)$ for $b \in B_p$ and $T(f) = h_{p,q}^B(f)$ for $f \in B_{p,q}$. We also set $T(\mathbf{0}) = \mathbf{0}'$ and $T(0) = 0$. The

commutativity of

$$\begin{array}{ccccc}
B_p & \xleftarrow{S_{p,q}^B} & B_{p,q} & \xrightarrow{T_{p,q}^B} & B_q \\
h_p \downarrow & & \downarrow h_{p,q} & & \downarrow h_q \\
B'_p & \xleftarrow{S_{p,q}^B} & B'_{p,q} & \xrightarrow{T_{p,q}^B} & B'_q
\end{array}$$

shows that T maps $\text{Hom}(b, c)$ into $\text{Hom}(T(b), T(c))$. The commutativity of

$$\begin{array}{ccc}
X_p & \xrightarrow{h_p^X} & X'_p \\
\pi_p \downarrow & & \downarrow \pi_p \\
B_p & \xrightarrow{h_p^B} & B'_p
\end{array}$$

shows that h_p^X maps $X_p(b)$ into $X'_p(T(b))$. Similarly, the commutativity of

$$\begin{array}{ccc}
X_{p,q} & \xrightarrow{h_{p,q}^X} & X'_{p,q} \\
\pi_{p,q} \downarrow & & \downarrow \pi_{p,q} \\
B_{p,q} & \xrightarrow{h_{p,q}^B} & B'_{p,q}
\end{array}$$

shows that $h_{p,q}^X$ maps $X_{p,q}(f)$ into $X_{p,q}(T(f))$. For a morphism $f: b \rightarrow c$ in C the left, right, upper and other squares in

$$\begin{array}{ccc}
X_{p,q}(f) & \xrightarrow{h_{p,q}^X} & X'_{p,q}(T(f)) \\
\downarrow S_{p,q}^X & & \downarrow S_{p,q}^X \\
X_p(b) & \xrightarrow{h_p^X} & X'_p(T(b)) \\
\downarrow \tilde{f} & & \downarrow \widetilde{T(f)} \\
X_q(c) & \xrightarrow{h_q^X} & X'_q(T(c))
\end{array}
\begin{array}{c}
\curvearrowleft T_{p,q}^X \quad \quad \quad \curvearrowright T_{p,q}^X
\end{array}$$

commute. Thus also the lower square

$$\begin{array}{ccc}
X_p(b) & \xrightarrow{h_p^X} & X'_p(T(b)) \\
\downarrow \tilde{f} & & \downarrow \widetilde{T(f)} \\
X_q(c) & \xrightarrow{h_q^X} & X'_q(T(c))
\end{array} \tag{4}$$

commutes. Since h preserves A_p and SM_p it follows that the maps $h_p^X: X_p(b) \rightarrow X'_p(T(b))$ are k -linear. Furthermore, since h preserves LI_p and $X_p(b)$ and $X'_p(T(b))$ have the same dimension, we see that the induced map $\alpha_b: X_p(b) \otimes_k k' \rightarrow X'_p(T(b))$ is an isomorphism of k' -vector spaces. Diagram (4) extends to

$$\begin{array}{ccc}
X_p(b) \otimes_k k' & \xrightarrow{\alpha_b} & X'_p(T(b)) \\
\downarrow \tilde{f} \otimes k' & & \downarrow \widetilde{T(f)} \\
X_q(c) \otimes_k k' & \xrightarrow{\alpha_c} & X'_q(T(c))
\end{array} \tag{5}$$

a diagram of k' -linear maps.

This implies that $T(\text{id}_b) = \text{id}_{T(b)}$. Moreover, for morphisms $f: b \rightarrow c$ and $g: c \rightarrow d$ in C , the commutativity of the diagram

$$\begin{array}{ccccc}
& & \widetilde{gf} & & \\
& \nearrow & & \searrow & \\
X_p(b) & \xrightarrow{\tilde{f}} & X_q(c) & \xrightarrow{\tilde{g}} & X_r(d) \\
\downarrow h_p^X & & \downarrow h_q^X & & \downarrow h_r^X \\
X'_p(b) & \xrightarrow{\widetilde{T(f)}} & X'_q(c) & \xrightarrow{\widetilde{T(g)}} & X'_r(d) \\
& \nwarrow & & \nearrow & \\
& & \widetilde{T(g)T(f)} & &
\end{array}$$

shows that $T(gf) = T(g)T(f)$. Thus T is a functor. We claim that T is a strict tensor functor.

Since the diagram

$$\begin{array}{ccc}
X_p(b) \times X_q(c) & \xrightarrow{\otimes_{p,q}} & X_{pq}(b \otimes c) \\
\downarrow h_p^X \times h_q^X & & \downarrow h_{pq}^X \\
X'_p(T(b)) \times X'_q(T(c)) & \xrightarrow{\otimes_{p,q}} & X'_{pq}(T(b) \otimes T(c))
\end{array} \tag{6}$$

commutes, we see that $T(b \otimes c) = T(b) \otimes T(c)$. Note that the above diagram can also be expressed as $h_p^X \otimes h_q^X = h_{pq}^X$. The diagram

$$\begin{array}{ccc}
X(b \otimes (c \otimes d)) & \xrightarrow{\widetilde{\Phi_{b,c,d}}} & X((b \otimes c) \otimes d) \\
\parallel & & \parallel \\
X(b) \otimes (X(c) \otimes X(d)) & \xrightarrow{\quad} & (X(b) \otimes X(c)) \otimes X(d) \\
\downarrow h^X \otimes (h^X \otimes h^X) & & \downarrow (h^X \otimes h^X) \otimes h^X \\
X'(T(b)) \otimes (X'(T(c)) \otimes X'(T(d))) & \xrightarrow{\quad} & (X'(T(b)) \otimes X'(T(c))) \otimes X'(T(d)) \\
\parallel & & \parallel \\
X'(T(b \otimes (c \otimes d))) & \xrightarrow{\widetilde{\Phi'_{T(b),T(c),T(d)}}} & X'(T((b \otimes c) \otimes d))
\end{array}$$

commutes, where for simplicity we have omitted the p, q, r indices. By (6) the map from the upper left to the lower right corner is h^X . Similarly, the map from the upper right corner to the lower right corner is h^X . It thus follows that $T(\Phi_{b,c,d}) = \Phi'_{T(b),T(c),T(d)}$. In a similar fashion one shows that $T(\Psi_{b,c}) = \Psi'_{T(b),T(c)}$. Since $h_{(1,1)}^B(1) = 1'$ we have $T(1) = 1'$ and so T is a strict tensor functor. From diagram (2) it follows that T is k -linear. Clearly T preserves tensor irreducible objects. In summary, as desired, T is a k -linear strict tensor functor that preserves tensor irreducible objects.

To obtain a morphism in TANN, we also need to specify an isomorphism $\alpha = \alpha(h): \omega_{k'} \rightarrow \omega' T$ of tensor functors. But the collection of all $\alpha_b: X_p(b) \otimes_k k' \rightarrow X'_p(T(b))$ defined above exactly yields such an isomorphism: The commutativity of (5) shows that α is a morphism of functors, whilst the commutativity of (6) implies that α is an isomorphism of tensor functors. So we indeed have a functor from PROALG to TANN.

We next show that the functor $\mathcal{M} \rightsquigarrow (k(\mathcal{M}), C(\mathcal{M}), \omega(\mathcal{M}))$ is faithful. Let $h, g: \mathcal{M} \rightarrow \mathcal{M}'$ be homomorphisms such that $(\lambda(h), T(h), \alpha(h)) = (\lambda(g), T(g), \alpha(g))$. Then $h_{\text{field}} = \lambda(f) = \lambda(g) = g_{\text{field}}$. Moreover, $h_p^B = g_p^B$ and $h_{p,q}^B = g_{p,q}^B$ for all p, q since $T(h) = T(g)$.

On $X_p(b)$, h_p^X agrees with $\alpha(h)_b$ and g_p^X agrees with $\alpha(g)_b$. Thus $h_p^X = g_p^X$. To show that also $h_{p,q}^X = g_{p,q}^X$, consider $f \in B_{(p,q)}$. The vertical maps in the diagram

$$\begin{array}{ccc} X_{p,q}(f) & \xrightarrow{\quad\quad\quad} & X'_{p,q}(T(f)) \\ S_{p,q}^X \downarrow & & \downarrow S_{p,q}^X \\ X_p(b) & \xrightarrow{h_p^X = g_p^X} & X'_p(T(b)) \end{array}$$

are bijective. Thus there exists a unique map $X_{p,q}(f) \rightarrow X'_{p,q}(T(f))$ that makes this diagram commutative. As the restrictions to $X_{p,q}(f)$ of both, $h_{p,q}^X$ and $g_{p,q}^X$ indeed make this diagram commutative, we see that $h_{p,q}^X = g_{p,q}^X$.

To show that the functor $\mathcal{M} \rightsquigarrow (k(\mathcal{M}), C(\mathcal{M}), \omega(\mathcal{M}))$ is full, consider a morphism

$$(\lambda, T, \alpha): (k(\mathcal{M}), C(\mathcal{M}), \omega(\mathcal{M})) \rightarrow (k(\mathcal{M}'), C(\mathcal{M}'), \omega(\mathcal{M}'))$$

in TANN. We have to construct a homomorphism $h: \mathcal{M} \rightarrow \mathcal{M}'$ that induces (λ, T, α) . We set $h_{\text{field}} = \lambda: k \rightarrow k'$.

Note that for $b \in B_p = B_{(m,n)}$, the k -vector space $\omega(b) = X_{(m,n)}(b)$ has dimension n . Since $\alpha_b: X_p(b) \otimes_k k' \rightarrow \omega'(T(b))$ is an isomorphism of k' -vector spaces we see that $\omega'(T(b))$ also has dimension n . Moreover, as T preserves tensor irreducible objects, we see that $T(b)$ has tensor length m . So $T(b) \in B'_p$. Thus T induces maps $h_p^B: B_p \rightarrow B'_p$. Since $B_{p,q}$ is the set of all morphism in C with source in B_p and target in B_q , it then also follows that T induces maps $h_{p,q}^B: B_{p,q} \rightarrow B'_{p,q}$, $f \mapsto T(f)$.

We define $h_p^X: X_p \rightarrow X'_p$ by $h_p^X(v) = \alpha_b(v \otimes 1)$, where $b = \pi_p(v)$. To define $h_{p,q}^X$, consider $f \in B_{p,q}$ with $f: b \rightarrow c$. We define $h_{p,q}^X: X_{p,q} \rightarrow X'_{p,q}$ to be the unique map whose restriction to any $X_{p,q}(f)$ makes

$$\begin{array}{ccc} X_{p,q}(f) & \xrightarrow{\quad\quad\quad} & X'_{p,q}(T(f)) \\ S_{p,q}^X \downarrow & & \downarrow S_{p,q}^X \\ X_p(b) & \xrightarrow{h_p^X} & X'_p(T(b)) \end{array} \tag{7}$$

commutative. We need to check that $h = (h_{\text{field}}, h_p^B, h_p^X, h_{p,q}^B, h_{p,q}^X)$ is a homomorphism. Clearly f_{field} preserves $+$, \cdot , 0 and 1 . Since the α_b are k' -linear isomorphisms, 0_p , A_p , SM_p and LI_p are preserved by h .

We note that $C = C(\mathcal{M})$ has several identity objects. However, since C is pointed skeletal there exists a unique tensor irreducible object 1 of C such that $(1, u)$ is an identity object, for some isomorphism $u: 1 \rightarrow 1 \otimes 1$; similarly for $C' = C(\mathcal{M}')$. Since T preserves identity objects and tensor irreducible objects, we see that $T(1) = 1'$, i.e., h preserves 1 .

Since T is a functor, h preserves $S_{p,q}^B$ and $T_{p,q}^B$. As $\alpha_b: X_p(b) \otimes_k k' \rightarrow X'_p(T(b))$ we see that h preserves π_p . Using diagram (7) we see that h also preserves $\pi_{p,q}$. Diagram (7) shows that h

preserves $S_{p,q}^X$. For a morphism $f: b \rightarrow c$ in C , the diagram

$$\begin{array}{ccccc}
& & h_{p,q}^X & & \\
& \swarrow & \text{---} & \searrow & \\
X_{p,q}(f) & \xrightarrow{S_{p,q}^X \otimes 1} & X_p(b) \otimes_k k' & \xrightarrow{\alpha_b} & X'_p(T(b)) \xleftarrow{S_{p,q}^X} X_{p,q}(T(f)) \\
& \searrow & \downarrow \tilde{f} \otimes k' & & \downarrow \widetilde{T(f)} \\
& & X_q(c) \otimes_k k' & \xrightarrow{\alpha_c} & X'_q(T(c))
\end{array}$$

$T_{p,q}^X \otimes 1$ (from $X_{p,q}(f)$ to $X_q(c) \otimes_k k'$) and $T_{p,q}^X$ (from $X_{p,q}(T(f))$ to $X'_q(T(c))$)

shows that f also preserves $T_{p,q}^X$. As T is strict and α an isomorphism of tensor functors, we have for equivalence classes $b \in B_p$ and $c \in B_q$ a commutative diagram:

$$\begin{array}{ccc}
(X_p(b) \otimes X_q(c)) \otimes_k k' & \xrightarrow{\alpha_{b \otimes c}} & X'_{pq}(T(b \otimes c)) \\
\downarrow \simeq & & \parallel \\
(X_p(b) \otimes_k k') \otimes_k (X_q(c) \otimes_k k') & \xrightarrow{\alpha_b \otimes \alpha_c} & X'_p(T(b)) \otimes X'_q(T(c))
\end{array}$$

Thus for $v \in X_p(b)$ and $w \in X_q(c)$ we obtain $h_{pq}^X(v \otimes w) = \alpha_{b \otimes c}(v \otimes w \otimes 1) = \alpha_b(v \otimes 1) \otimes \alpha_c(w \otimes 1) = h_p^X(v) \otimes h_q^X(w)$. Thus h also preserves $\otimes_{p,q}$. Therefore $f: \mathcal{M} \rightarrow \mathcal{M}'$ is a homomorphism of models of PROALG. It is then clear that h induces the morphism (λ, T, α) in TANN.

Finally, we show that the functor $\mathcal{M} \rightsquigarrow (k(\mathcal{M}), C(\mathcal{M}), \omega(\mathcal{M}))$ is essentially surjective. Let (k, C, ω) be an object of TANN. We will construct a model \mathcal{M} of PROALG such that $(k(\mathcal{M}), C(\mathcal{M}), \omega(\mathcal{M}))$ is isomorphic to (k, C, ω) in TANN. We define the field sort of \mathcal{M} to be k (including the field structure). For $p = (m, n)$, with $m, n \geq 1$, let B_p denote the set of all objects b of C of tensor length m and such that $\omega(b)$ has dimension n . Let $B_{p,q}$ denote the set of all morphisms in C from objects in B_p to objects in B_q . Let $S_{p,q}^B: B_{p,q} \rightarrow B_p$ denote the map that assigns the source to a morphism and similarly for the target. Let X_p denote the disjoint union of the k -vector spaces $\omega(b)$, $b \in B_p$ and let $\pi_p: X_p \rightarrow B_p$ be the map such that $\pi_p(v) = b$ for $v \in \omega(b)$. We define $X_{p,q}$ as the disjoint union of the graphs of the k -linear maps $\omega(f): \omega(b) \rightarrow \omega(c)$, where $f: b \rightarrow c$ is a morphism in C with $b \in B_p$ and $c \in B_q$. The maps $\pi_{p,q}: X_{p,q} \rightarrow B_{p,q}$ are defined by $\pi_{p,q}(a) = f$, if a belongs to the graph of $\omega(f)$. The maps $S_{p,q}^X: X_{p,q} \rightarrow X_p$ and $T_{p,q}^X: X_{p,q} \rightarrow X_q$ are defined by $S_{p,q}^X((v, w)) = v$ and $T_{p,q}^X((v, w)) = w$, where $(v, w) = (v, \omega(f)(v))$ is an element of the graph $\{(v, \omega(f)(v)) \mid v \in X_p(b)\}$ of $\omega(f)$, for a morphism $f: b \rightarrow c$ in $B_{p,q}$. We define 0_p to be the subset of X_p consisting of all zero vectors of all vector spaces in X_p . We define A_p through $A_p(v_1, v_2, v_3)$ if and only if all three elements v_1, v_2, v_3 of X_p belong to the same vector space $\omega(b)$ and $v_3 = v_1 + v_2$, where the $+$ here is vector space addition in $\omega(b)$. The map $SM_p: k \times X_p \rightarrow X_p$ is scalar multiplication. For elements v_1, \dots, v_n of X_p , where $p = (m, n)$, we define LI_p through $LI_p(v_1, \dots, v_n)$ if and only if all $\pi_p(a_i) = \pi_p(a_j)$ for all i, j and v_1, \dots, v_n are k -linearly independent (in $X_p(b)$, where $b = \pi_p(a_i)$).

There exists a unique tensor irreducible object 1_C in C such that $(1_C, u)$ is an identity object of C for some isomorphism $u: 1_C \rightarrow 1_C \otimes 1_C$. We set $1 = 1_C \in B_{(1,1)}$.

Finally, we define $\otimes_{p,q}: X_p \times X_q \rightarrow X_{pq}$ by sending $(v, w) \in \omega(b) \times \omega(c)$ to the image of $v \otimes w$ under $\omega(b) \otimes_k \omega(c) \rightarrow \omega(b \otimes c)$, where the latter map is part of the functorial isomorphism defining the tensor functor ω . (As C is pointed skeletal, $\otimes_{p,q}$ is well defined.) It is now straight forward to check that our structure \mathcal{M} satisfies all 27 axioms of PROALG. Moreover, $(k(\mathcal{M}), C(\mathcal{M}), \omega(\mathcal{M})) = (k, C, \omega)$. So the isomorphism (λ, T, α) can be chosen to be the identity. \square

For a model \mathcal{M} of PROALG we define $(k, C, \omega) = (k(\mathcal{M}), C(\mathcal{M}), \omega(\mathcal{M}))$ as in the proof of Theorem 2.2 and we call (k, C, ω) the object of TANN associated to \mathcal{M} . In the sequel, when given a model $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ of PROALG we will use this notation without further ado. For example, for $b \in B_p$ we will usually write $\omega(b)$ instead of $X_p(b)$.

We also set

$$G = G(\mathcal{M}) = \underline{\text{Aut}}^\otimes(\omega(\mathcal{M})).$$

Combining Theorem 2.2 and Proposition 1.18 we obtain:

Corollary 2.3. *The functor $\mathcal{M} \mapsto G(\mathcal{M})$ from the category of models of PROALG with homomorphism as the morphisms to the category PROALGEBRAIC GROUPS is full, essentially surjective and induces a bijection on the isomorphism classes.* \square

Example 2.4. We describe the object (k, C, ω) of TANN that corresponds to the trivial proalgebraic group. The objects of C are the zero object $\mathbf{0}$ together with all completely parenthesized finite sequences of integers greater or equal to 1. For an object b of C corresponding to a complete parenthesization of the sequence (n_1, \dots, n_m) we set $\omega(b) = k^{n_1} \otimes_k \dots \otimes_k k^{n_m}$. We also define $\omega(\mathbf{0})$ to be the zero vector space. For objects b_1, b_2 of C the set of morphisms $\text{Hom}(b_1, b_2)$ is defined as the set of k -linear maps from $\omega(b_1)$ to $\omega(b_2)$. The tensor product $\otimes: C \times C \rightarrow C$ is defined on non-zero objects as the concatenation of parenthesized sequences. We also set $b \otimes \mathbf{0} = \mathbf{0} \otimes b = \mathbf{0}$ for any object b of C . On morphisms \otimes is defined as the usual tensor product of k -linear maps.

Let G be a proalgebraic group. There does not seem to be a *canonical* way to construct from G a model \mathcal{M} of PROALG such that $G(\mathcal{M}) \simeq G$. However, according to Corollary 2.3, there always exists such a model \mathcal{M} . Moreover, if \mathcal{M}_1 and \mathcal{M}_2 are two models of PROALG such that $G(\mathcal{M}_1)$ and $G(\mathcal{M}_2)$ are isomorphic to G , then \mathcal{M}_1 and \mathcal{M}_2 are isomorphic. We can therefore safely define the theory

$$\text{Th}(G)$$

of G as $\text{Th}(\mathcal{M})$ for any model \mathcal{M} of PROALG such that $G(\mathcal{M}) \simeq G$. Two proalgebraic groups G and H (not necessarily living over the same base field) are *elementarily equivalent* if $\text{Th}(G) = \text{Th}(H)$. We may also express this as $G \equiv H$. In a similar spirit, a class \mathcal{C} of proalgebraic groups (potentially over varying base fields) is called *elementary* if the class of all models of PROALG such that the associated proalgebraic group lies in \mathcal{C} is elementary.

We conclude this section with a discussion of some elementary classes of proalgebraic groups.

Proposition 2.5. *The following classes of proalgebraic groups are elementary:*

- *The class of all diagonalizable proalgebraic groups.*
- *The class of all unipotent proalgebraic groups.*
- *The class of all linearly reductive proalgebraic groups.*

Proof. The definition of diagonalizable proalgebraic groups is recalled in the beginning of the next section. Let us simply mention here that according to Prop. 1.6, Chapter IV, §1, in [DG70] a proalgebraic group is diagonalizable if and only if every representation of G is a direct sum of one-dimensional representations and the latter condition can be axiomatized.

A unipotent proalgebraic group can be defined as a proalgebraic group G such that every representation of G has a fixed vector (cf. [Wat79, Section 8.3]). This condition can be axiomatized by saying that for every representation V there exists a morphism $\mathbf{1} \rightarrow V$.

Recall that a proalgebraic group G is linearly reductive if and only if every representation of G is a direct sum of irreducible representations. This condition can be axiomatized. \square

In the following section we will show that the class of all algebraic groups is not elementary (Corollary 3.5).

3 Diagonalizable proalgebraic groups

In this section we determine the theory of the multiplicative group \mathbb{G}_m and deduce some basic consequences for the theory PROALG from this. We show that $\text{Th}(\mathbb{G}_m)$ is determined by the theory of the base field and the theory of the abelian group $(\mathbb{Z}, +)$. In fact, we establish a similar result for any diagonalizable proalgebraic group: The theory of a diagonalizable proalgebraic group G is determined by the theory of the base field and the theory of the character group of G . Indeed, we show that the theory of all diagonalizable proalgebraic groups is weakly bi-interpretable with the two-sorted theory of pairs (k, A) , where k is a field and A an abelian group.

Let us first recall some basic facts about diagonalizable proalgebraic groups. See e.g., [Wat79, Section 2.2], [Mil17, Section 12, c] or [DG70, Chapter IV, §1, 1]. Let k be a field and let A be an abelian group (usually written additively). The proalgebraic group $D(A)_k$ over k is defined by $D(A)_k(R) = \text{Hom}(A, R^\times)$ for any k -algebra R , where $\text{Hom}(A, R^\times)$ denotes the abelian group of all morphisms of abelian groups from A to R^\times . (Here, as usual, R^\times denotes the multiplicative group of a ring R .) For example, $D(\mathbb{Z})_k \simeq \mathbb{G}_m$, or more generally, $D(\mathbb{Z}^n)_k \simeq \mathbb{G}_m^n$. A proalgebraic group is *diagonalizable* if it is isomorphic to $D(A)_k$ for some abelian group A . The functor $A \mapsto D(A)_k$ is an equivalence of categories from the category of abelian groups to the category of diagonalizable proalgebraic groups over k . The quasi-inverse is the functor that associates the character group to a diagonalizable proalgebraic group. Recall that the *character group* $\chi(G)$ of a proalgebraic group G is the abelian group of all morphisms of proalgebraic groups from G to \mathbb{G}_m . Note that $\chi(G)$ is isomorphic to the abelian group of isomorphism classes of one-dimensional representations of G under the tensor product.

As noted before, in general, for a proalgebraic group G , there does not seem to be a *canonical* way to define a model \mathcal{M} of PROALG such that $G(\mathcal{M}) \simeq G$. (Recall that by Corollary 2.3 such an \mathcal{M} always exists and is unique up to an isomorphism.) However, if $G = D(A)_k$ is diagonalizable, there is a canonical choice which we will now describe. This will be helpful later on (Theorem 3.2) for showing that \mathcal{M} is interpretable in the structure (k, A) .

Given a field k and an abelian group A we will now define a model

$$\mathcal{M}(k, A) = (k(k, A), B_p(k, A), X_p(k, A), B_{p,q}(k, A), X_{p,q}(k, A))$$

of PROALG such that $G(\mathcal{M}(k, A)) \simeq D(A)_k$. We set $k(k, A) = k$ (including the field structure). The isomorphism classes of representations of $D(A)_k$ of dimension n are in bijection with multisets of elements of A of cardinality n (see e.g., [DG70, Prop. 1.6, Chapter IV, §1]). In more detail, for $a \in A$ let k_a denote the one-dimensional representation of $D(A)_k$ given by the morphism $\chi_a : D(A)_k \rightarrow \mathbb{G}_m$ with $\chi_a(g) = g(a)$ for all $g \in D(A)_k(R) = \text{Hom}(A, R^\times)$ and all k -algebras R . Then the map

$$\{a_1, \dots, a_n\} \mapsto W_{\{a_1, \dots, a_n\}} = k_{a_1} \oplus \dots \oplus k_{a_n}$$

induces a bijection between the set of multisets of cardinality n of elements of A and the set of isomorphism classes of n -dimensional representations of $D(A)_k$. So, for $n \geq 1$, we can define $B_{(1,n)}(k, A)$ as the set of multisets of elements from A of cardinality n . In general, for $p = (m, n)$ we define $B_p(k, A)$ as the set of all completely parenthesized sequences of m -multisets A_1, \dots, A_m formed from elements from A such that $|A_1| \dots |A_m| = n$. The representation corresponding to such a parenthesized sequence of multisets would be $W_{A_1} \otimes \dots \otimes W_{A_m}$ with the corresponding parenthesization of the tensor product, where the W_{A_i} 's are understood to be tensor irreducible. Explicitly, for an element b of $B_p(k, A)$ determining a parenthesization of A_1, \dots, A_m , we let \underline{v}_b denote the multiset $A_1 \times \dots \times A_m$, where an element (a_1, \dots, a_m) of \underline{v}_b is considered as a parenthesized sequence with the same parenthesization pattern as the sequence $A_1 \times \dots \times A_m$. We let V_b denote the n -dimensional k -vector space with basis \underline{v}_b . In other words, V_b is the k -vector space of functions from \underline{v}_b to k .

We define $X_p(k, A)$ to be the (disjoint) union of the V_b 's and we let $\pi_p: X_p(k, A) \rightarrow B_p(k, A)$ denote the map that maps an element in V_b to b . We use the vector space structure on the V_b 's to define 0_p , A_p , LI_p , and SM_p . We also define the interpretation of the constant symbol 1 to correspond to the neutral element of A , considered as an element of $B_{(1,1)}(k, A)$.

We next want to define $\otimes_{p,\hat{p}}$, where $p = (m, n)$ and $\hat{p} = (\hat{m}, \hat{n})$. Note that two elements $b \in B_p(k, A)$ and $\hat{b} \in B_{\hat{p}}(k, A)$ can be concatenated to an element $b \otimes \hat{b} \in B_{p\hat{p}}(k, A)$. Similarly, two elements $v \in \underline{v}_b$ and $\hat{v} \in \underline{v}_{\hat{b}}$ can be concatenated to an element $v\hat{v} \in \underline{v}_{b\hat{b}}$. This defines bilinear maps $V_b \times V_{\hat{b}} \rightarrow V_{b\hat{b}}$ that combine to a map $\otimes_{p,\hat{p}}: X_p(k, A) \times X_{\hat{p}}(k, A) \rightarrow X_{p\hat{p}}(k, A)$.

We next want to define the morphism sorts. Note that for $a_1, a_2 \in A$, there is a non-zero morphism (of $D(A)_k$ -representations) from k_{a_1} to k_{a_2} if and only if $a_1 = a_2$. Moreover, for $a \in A$, every linear map $k_a^{n_1} \rightarrow k_a^{n_2}$ is a morphism. This yields a description of the morphisms from $W_{A_1} = \oplus_{a_1 \in A_1} k_{a_1}$ to $W_{A_2} = \oplus_{a_2 \in A_2} k_{a_2}$ for finite multisets A_1 and A_2 consisting of elements of A : For $a \in A$ and $i = 1, 2$ let

$$W_{i,a} = \bigoplus_{\substack{a_1 \in A_1 \\ a_1 = a}} k_{a_1}$$

and let $\text{Hom}_k(W_{1,a}, W_{2,a})$ denote the set of k -linear maps from $W_{1,a}$ to $W_{2,a}$. Then the set of morphisms of $D(A)_k$ -representations from W_{A_1} to W_{A_2} can be identified with $\prod_{a \in A} \text{Hom}_k(W_{1,a}, W_{2,a})$.

For $b \in B_p(k, A)$ and $v \in \underline{v}_b$ determining a parenthesization of $(a_1, \dots, a_m) \in A^m$ we set $|v| = a_1 + \dots + a_m$. Furthermore, for $a \in A$ let $V_{b,a}$ denote the subspace of V_b generated by all $v \in \underline{v}_b$ such that $|v| = a$.

For $b \in B_p(k, A)$ and $\hat{b} \in B_{\hat{p}}(k, A)$ let $H_{b,\hat{b}}$ denote $\prod_{a \in A} \text{Hom}_k(V_{b,a}, V_{\hat{b},a})$ considered as a subset of $\text{Hom}_k(V_b, V_{\hat{b}})$. We set

$$B_{p,\hat{p}}(k, A) = \left\{ (b, \hat{b}, H_{b,\hat{b}}) \mid b \in B_p(k, A), \hat{b} \in B_{\hat{p}}(k, A) \right\}$$

and

$$X_{p,\hat{p}}(k, A) = \left\{ (b, \hat{b}, h, v, h(v)) \mid b \in B_p(k, A), \hat{b} \in B_{\hat{p}}(k, A), h \in H_{b,\hat{b}}, v \in V_b \right\}.$$

We define $\pi_{p,\hat{p}}: X_{p,\hat{p}}(k, A) \rightarrow B_{p,\hat{p}}(k, A)$ to be the projection onto the first three factors while $S_{p,\hat{p}}^B: B_{p,\hat{p}}(k, A) \rightarrow B_p(k, A)$ and $T_{p,\hat{p}}^B: B_{p,\hat{p}}(k, A) \rightarrow B_{\hat{p}}(k, A)$ are defined as the projections onto the first and second factor respectively. Similarly, $S_{p,\hat{p}}^X: X_{p,\hat{p}}(k, A) \rightarrow X_p(k, A)$ and $T_{p,\hat{p}}^X: X_{p,\hat{p}}(k, A) \rightarrow X_{\hat{p}}(k, A)$ are defined as the projections onto the third and fourth factor respectively.

This completes our definition of the \mathcal{L} -structure $\mathcal{M}(k, A)$. It is clear from the construction (and the proof of Corollary 2.3) that $\mathcal{M}(k, A)$ is a model of PROALG and that $G(\mathcal{M}(k, A)) \simeq D(A)_k$.

Since the addition in the character group can be described through the tensor product it is not surprising that the character group of $G(\mathcal{M})$ is interpretable in \mathcal{M} for any model \mathcal{M} of PROALG:

Lemma 3.1. *Let \mathcal{M} be a model of PROALG. Then the character group of $G(\mathcal{M})$ is definably interpreted in \mathcal{M} .*

Proof. Let $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$. Then $B_{(1,1)}$ can be identified with the isomorphism classes of one-dimensional representations of $G = G(\mathcal{M})$, i.e., with $\chi(G)$. The graph of addition in $\chi(G) = B_{(1,1)}$ consists of all $(b_1, b_2, b_3) \in B_{(1,1)}^3$ such that there exist $v_1 \in \omega(b_1)$, $v_2 \in \omega(b_2)$ and $v_3 \in \omega(b_3)$ and an isomorphism between $\pi_{(2,1)}(v_1 \otimes v_2)$ and b_3 . This set is \emptyset -definable. The identity element of $B_{(1,1)}$ is given by the constant symbol 1. \square

To proceed, let us recall the notion of *interpretation* in the many-sorted context (cf. [Hod93, Chapter 5] for the one-sorted case). Let \mathcal{L} and \mathcal{L}' be many-sorted languages with sorts S and S' respectively. An *interpretation* Ξ of \mathcal{L}' in \mathcal{L} is comprised of the following data:

- For every sort $s' \in S'$ an \mathcal{L} -formula $\partial_{\Xi, s'}(x_1^{s_1}, \dots, x_n^{s_n})$ (the domain formula for the sort s') in the free variables $\bar{x}_{s'} = (x_1^{s_1}, \dots, x_n^{s_n})$, where n and $s_1, \dots, s_n \in S$ depend on s' . (Here the notation x^s means that the variable x belongs to the sort s .)
- For every $s' \in S'$ an \mathcal{L} -formula $=_{\Xi, s'}(\bar{x}_{s'}, \bar{y}_{s'})$ (the equivalence formula for the sort s').
- For every function symbol $f' \in \mathcal{L}'$ that maps sorts s'_1, \dots, s'_r into sort s'_{r+1} an \mathcal{L} -formula $f'_{\Xi}(\bar{x}_{s'_1}, \dots, \bar{x}_{s'_{r+1}})$.
- For every relation symbol R' of \mathcal{L}' between sorts s'_1, \dots, s'_r an \mathcal{L} -formula $R'_{\Xi}(\bar{x}_{s'_1}, \dots, \bar{x}_{s'_r})$.
- For every constant symbol c' in \mathcal{L}' with sort s' an \mathcal{L} -formula $c'_{\Xi}(\bar{x}_{s'})$.

The *admissibility conditions* of Ξ are the \mathcal{L} -sentences expressing that for every \mathcal{L} -structure $M = (M_s)_{s \in S}$ the following holds:

- For all $s' \in S'$ the formula $=_{\Xi, s'}$ defines an equivalence relation on $\partial_{\Xi, s'}(M)$. We will denote this equivalence relation simply by \sim (even though it depends on Ξ, s' and M).
- For every function symbol f' of \mathcal{L}' that maps sorts s'_1, \dots, s'_r into sort s'_{r+1} we have
 - $M \models f'_{\Xi}(\bar{a}_{s'_1}, \dots, \bar{a}_{s'_{r+1}})$ if and only if $M \models f'_{\Xi}(\bar{b}_{s'_1}, \dots, \bar{b}_{s'_{r+1}})$ for all tuples $\bar{a}_{s'_1}, \dots, \bar{a}_{s'_{r+1}}, \bar{b}_{s'_1}, \dots, \bar{b}_{s'_{r+1}}$ from M with $\bar{a}_{s'_1} \sim \bar{b}_{s'_1}, \dots, \bar{a}_{s'_{r+1}} \sim \bar{b}_{s'_{r+1}}$ and
 - the induced subset of $\partial_{\Xi, s'_1}(M)/\sim \times \dots \times \partial_{\Xi, s'_{r+1}}(M)/\sim$ is the graph of a function $f'(M): \partial_{\Xi, s'_1}(M)/\sim \times \dots \times \partial_{\Xi, s'_{r+1}}(M)/\sim \rightarrow \partial_{\Xi, s'_{r+1}}(M)/\sim$.
- For every relation symbol R' of \mathcal{L}' between sorts s'_1, \dots, s'_r we have $M \models R'_{\Xi}(\bar{a}_{s'_1}, \dots, \bar{a}_{s'_r})$ if and only if $M \models R'_{\Xi}(\bar{b}_{s'_1}, \dots, \bar{b}_{s'_r})$ for all tuples $\bar{a}_{s'_1}, \dots, \bar{a}_{s'_r}, \bar{b}_{s'_1}, \dots, \bar{b}_{s'_r}$ from M with $\bar{a}_{s'_1} \sim \bar{b}_{s'_1}, \dots, \bar{a}_{s'_r} \sim \bar{b}_{s'_r}$. We thus have an induced relation $R'(M)$ on $\partial_{\Xi, s'_1}(M)/\sim \times \dots \times \partial_{\Xi, s'_r}(M)/\sim$.
- For every constant symbol c' in \mathcal{L}' with sort s' the realizations of $c'_{\Xi}(\bar{x}_{s'})$ in M are an equivalence class $c'(M)$ in $\partial_{\Xi, s'}(M)$.

Note that if M is an \mathcal{L} -structure that satisfies the admissibility conditions of Ξ , then $\Xi(M) = (\partial_{\Xi, s'}(M)/\sim)_{s' \in S'}$ is an \mathcal{L}' -structure.

Now let T be an \mathcal{L} -theory and let T' be an \mathcal{L}' -theory. An interpretation Ξ of \mathcal{L}' in \mathcal{L} is a *left total interpretation* of T' in T if every model of T satisfies the admissibility conditions of Ξ and if for every model M of T the \mathcal{L}' -structure $\Xi(M)$ is a model of T' . Thus, from every model M of T we get a model of $\Xi(M)$ of T' .

Finally, following [Vis06, Section 3.3.], the theories T and T' are *weakly bi-interpretable* if there exists a left total interpretation Ξ of T' in T and a left total interpretation Ω of T in T' such that for every model M of T the \mathcal{L} -structures $\Omega(\Xi(M))$ and M are isomorphic and for every model M' of T' the \mathcal{L}' -structures $\Xi(\Omega(M'))$ and M' are isomorphic. (Bi-interpretability is the slightly stronger notion where the above isomorphisms are required to be uniformly definable.)

By the *theory of diagonalizable proalgebraic groups* we mean the set of all \mathcal{L} -sentences true in all models \mathcal{M} of PROALG such that $G(\mathcal{M})$ is diagonalizable (cf. Proposition 2.5).

Theorem 3.2. *The theory of diagonalizable proalgebraic groups is weakly bi-interpretable with the two-sorted theory of pairs (k, A) , where k is a field and A an abelian group.*

Proof. Let \mathcal{L}' denote the two-sorted language with the language of rings on the first sort (the field sort) and the language of abelian groups on the second sort (the group sort). Let T' denote the \mathcal{L}' -theory of all pairs (k, A) where k is a field and A an abelian group. Moreover, let T denote the \mathcal{L} -theory of diagonalizable proalgebraic groups.

The interpretation Ξ of \mathcal{L}' in \mathcal{L} is relatively easy to describe (cf. Lemma 3.1): The domain formula for the field sort of \mathcal{L}' is trivial (i.e., equal to $x_1 = x_1$, where x_1 belongs to the field sort of \mathcal{L}) so that it returns the field sort of \mathcal{L} . Similarly, the domain formula for the group sort of \mathcal{L}' is trivial so that it returns the sort $B_{(1,1)}$ of \mathcal{L} . The two equivalence formulas are also trivial, so that the corresponding equivalence relation simply expresses equality of elements. The interpretation of the ring language on the field sort of \mathcal{L}' is the ring language on the field sort of \mathcal{L} . The \mathcal{L}' -symbol for the identity element of the group is to be interpreted as the \mathcal{L} -symbol 1 (corresponding to the trivial representation). Finally, the addition symbol $+$ on the group sort, yields the formula $+\Xi$ that defines in every model \mathcal{M} of T the set of all $(b_1, b_2, b_3) \in B_{(1,1)}^3$ such that there exist $v_1 \in \omega(b_1), v_2 \in \omega(b_2)$ and $v_3 \in \omega(b_3)$ and an isomorphism between $\pi_{(2,1)}(v_1 \otimes v_2)$ and b_3 . Clearly Ξ is a left total interpretation of T' in T .

We will next construct an interpretation Ω of \mathcal{L} in \mathcal{L}' . The idea for the construction is rather simple but a little tedious to implement. The formulas for Ω boil down to interpreting the \mathcal{L} -structure $\mathcal{M}(k, A)$ defined above in the \mathcal{L}' -structure (k, A) . We begin with the domain formulas $\partial_{\Omega, s}$, where s is a sort from \mathcal{L} . The domain formula for the field sort of \mathcal{L} simply returns the field sort of \mathcal{L}' .

Definition of ∂_{Ω, B_p} : For $p = (m, n)$, we consider, for every \mathcal{L}' -structure (k, A) , the set $P_p(k, A)$ of completely parenthesized sequences

$$(a_1^1, \dots, a_{n_1}^1), \dots, (a_1^m, \dots, a_{n_m}^m)$$

of sequences in A with $n_1 \dots n_m = n$. To describe $P_p(k, A)$ inside (k, A) we can encode the pattern associated with such a parenthesization of a sequence of sequences in a sequence of zero's and one's belonging to k . While there are different ways to do this, for the sake of concreteness, let us fix the following decoding. A sequence of zero's and one's always has to be read by blocks of two elements according to the following convention:

- A block 10 is to be read as an opening parenthesis “(”.
- A block 01 is to be read as a closing parenthesis “)”.
- A block 00 is to be interpreted as a place holder for an element of A .

We also use parenthesis to separate place holders that correspond to different sequences in A . For example, the element $((a_1, a_2, a_3)(a_4))(a_5, a_6)$ of $P_{(3,6)}(k, A)$ yields the pattern

$$(((\bullet \bullet \bullet)(\bullet))(\bullet \bullet))$$

that is encoded in the sequence

$$10 \ 10 \ 10 \ 00 \ 00 \ 00 \ 01 \ 10 \ 00 \ 01 \ 01 \ 10 \ 00 \ 00 \ 01 \ 01.$$

Note that different patterns may yield binary sequences of different lengths. Let $r = r(p)$ denote the maximal length of all theses binary sequences and let $D_p(k, A) \subseteq \{0, 1\}^r \subseteq k^r$ denote the set of all binary sequences that are derived from elements of $P_p(k, A)$. Here a binary sequence of length less than r is extended to a sequence of length r by adding 11-blocks. Let $s = s(p)$ denote the maximum number of 00-blocks that occur in any element of $D_p(k, A)$ and let $F_p(k, A)$ denote the

subset of $k^r \times A^s$ consisting of all elements of the form $(d, a_1, \dots, a_t, 0, \dots, 0)$, where $d \in D_p(k, A)$, t is the number of 00-blocks occurring in d and $a_1, \dots, a_t \in A$. By construction, the set $F_p(k, A)$ is in bijection with $P_p(k, A)$.

We let ∂_{Ω, B_p} denote the \mathcal{L}' -formula that defines in every \mathcal{L}' -structure (k, A) the subset $F_p(k, A)$ of $k^r \times A^s$. An element f of $F_p(k, A)$ corresponding to a complete parenthesization of a sequence

$$(a_1^1, \dots, a_{n_1}^1), \dots, (a_1^m, \dots, a_{n_m}^m)$$

of sequences in A with $n_1 \dots n_m = n$ yields a totally ordered multiset \underline{v}_f of cardinality n consisting of all sequences of length m in A that are of the form $(a_{i_1}^1, \dots, a_{i_m}^m)$ with $1 \leq i_1 \leq n_1, \dots, 1 \leq i_m \leq n_m$. Alternatively, we can define \underline{v}_f as the multiset product $\{a_1^1, \dots, a_{n_1}^1\} \times \dots \times \{a_1^m, \dots, a_{n_m}^m\}$. The order on \underline{v}_f is obtained by stipulating that $a_1^j < a_2^j < \dots < a_{n_j}^j$ for $j = 1, \dots, m$ and then using the lexicographic order. Let V_f denote the k -vector space with basis \underline{v}_f . We think of an element f of $F_p(k, A)$ as determining the pair (V_f, \underline{v}_f) , where \underline{v}_f is an ordered basis of V_f .

Definition of ∂_{Ω, X_p} : We let ∂_{Ω, X_p} denote the formula that defines the set $F_p(k, A) \times k^n$ in every \mathcal{L}' -structure (k, A) . We think of an element $(f, \xi) \in F_p(k, A) \times k^n$ as determining the element $\underline{v}_f \xi = \xi_1 v_{f,1} + \dots + \xi_n v_{f,n}$ of V_f , where $\xi = (\xi_1, \dots, \xi_n)$ and $\underline{v}_f = \{v_{f,1}, \dots, v_{f,n}\}$ with $v_{f,1} < v_{f,2} < \dots < v_{f,n}$.

Definition of $\partial_{\Omega, B_{p,\hat{p}}}$: Let $p = (m, n)$ and $\hat{p} = (\hat{m}, \hat{n})$. For every \mathcal{L}' -structure (k, A) and $(a_1, \dots, a_m) \in A^m$ we set $|a| = a_1 + \dots + a_m$ for a fixed but arbitrary parenthesization of this sum². For $f \in F_p(k, A)$ we let $\Sigma(f) = \{|v| \mid v \in \underline{v}_f\}$ denote the multiset of all sums of all tuples in \underline{v}_f . Furthermore, for $a \in A$ we let $m_f(a)$ denote the multiplicity of a in $\Sigma(f)$. Of course $m_f(a) = 0$ for all but finitely many $a \in A$. For $(f, \hat{f}) \in F_p(k, A) \times F_{\hat{p}}(k, A)$ we set $r(f, \hat{f}) = \sum_{a \in A} m_f(a) m_{\hat{f}}(a)$. Let

$$r = \max \left\{ r(f, \hat{f}) \mid (f, \hat{f}) \in F_p(k, A) \times F_{\hat{p}}(k, A) \right\}.$$

We define $H_{p,\hat{p}}(k, A)$ to be the subset of $F_p(k, A) \times F_{\hat{p}}(k, A) \times k^r$ consisting of all elements of the form (f, \hat{f}, λ) , where $f \in F_p(k, A)$, $\hat{f} \in F_{\hat{p}}(k, A)$, $\lambda = (\lambda_1, \dots, \lambda_r) \in k^r$ and $\lambda_i = 0$ for $i > r(f, \hat{f})$. Let $\partial_{\Omega, B_{p,\hat{p}}}$ denote the \mathcal{L}' -formula that defines in every \mathcal{L}' -structure (k, A) the set $H_{p,\hat{p}}(k, A)$.

We think of an element (f, \hat{f}, λ) of $H_{p,\hat{p}}(k, A)$ as defining a morphism $\psi_{(f,\hat{f},\lambda)} : (V_f, \underline{v}_f) \rightarrow (V_{\hat{f}}, \underline{v}_{\hat{f}})$ as follows: To simplify the formulas we set $m(a) = m_f(a)$ and $\hat{m}(a) = m_{\hat{f}}(a)$. For each of the finitely many $a \in A$ such that $m(a)\hat{m}(a) \geq 1$ let

$$I_a = \{i_{1,a}, \dots, i_{m(a),a}\} = \{i \in \{1, \dots, n\} \mid |v_{f,i}| = a\}$$

and

$$J_a = \{j_{1,a}, \dots, j_{\hat{m}(a),a}\} = \{j \in \{1, \dots, \hat{n}\} \mid |v_{\hat{f},j}| = a\}$$

with $v_{f,i_{1,a}} < \dots < v_{f,i_{m(a),a}}$ and $v_{\hat{f},j_{1,a}} < \dots < v_{\hat{f},j_{\hat{m}(a),a}}$. We order the sets of the form I_a by comparing the $v_{f,i_{1,a}}$. Say $I(a_1) < \dots < I(a_s)$. For every $\ell = 1, \dots, s$ let $\underline{v}_{f,I_{a_\ell}}$ denote the (ordered) sequence of elements of \underline{v}_f whose index belongs to I_{a_ℓ} . We define $\underline{v}_{\hat{f},J_{a_\ell}}$ similarly. We define a k -linear map ψ_ℓ from the subspace of V_f generated by $\underline{v}_{f,I_{a_\ell}}$ to the subspace of $V_{\hat{f}}$ generated by $\underline{v}_{\hat{f},J_{a_\ell}}$ by setting $\psi_\ell(\underline{v}_{f,I_{a_\ell}}) = \underline{v}_{\hat{f},J_{a_\ell}} M_{a_\ell}$, where M_{a_ℓ} is the $m(a_\ell) \times \hat{m}(a_\ell)$ -matrix obtained by putting, row by row, the entries of $\lambda = (\lambda_1, \dots, \lambda_r)$ that start at index $m(a_1)\hat{m}(a_1) + \dots + m(a_{\ell-1})\hat{m}(a_{\ell-1}) + 1$ and end at index $m(a_1)\hat{m}(a_1) + \dots + m(a_\ell)\hat{m}(a_\ell)$ into a matrix. Finally, we define the linear map $\psi_{(f,\hat{f},\lambda)} : V_f \rightarrow V_{\hat{f}}$ by

$$\psi_{(f,\hat{f},\lambda)}(v_{f,i}) = \begin{cases} \psi_\ell(v_{f,i}) & \text{if } i \in I_{a_\ell} \\ 0 & \text{otherwise.} \end{cases}$$

²The parenthesization will ultimately not matter since only the case when A is an abelian group is relevant for us.

Definition of $\partial_{\Omega, X_{p, \widehat{p}}}$: We define³ $\partial_{\Omega, X_{p, \widehat{p}}}$ to be the \mathcal{L}' -formula that defines in every \mathcal{L}' -structure (k, A) the set of all $(f, \widehat{f}, \lambda, \xi, \widehat{\xi})$ such that $(f, \widehat{f}, \lambda) \in H_{p, \widehat{p}}(k, A)$, $(f, \xi) \in \partial_{\Omega, X_p}(k, A)$, $(\widehat{f}, \widehat{\xi}) \in \partial_{\Omega, X_{\widehat{p}}}(k, A)$ and $\psi_{(f, \widehat{f}, \lambda)}(\underline{v}_f \xi) = \underline{v}_{\widehat{f}} \widehat{\xi}$. This concludes the definition of the domain formulas $\partial_{\Omega, s}$ for all sorts s of \mathcal{L} .

We will next define the equivalence formulas $=_{\Omega, s}$. We define $=_{\Omega, k}$ to be the \mathcal{L}' -formula $x_1 = x_2$, where x_1 and x_2 are variables from the field sort. (So the equivalence relation on the field sort is trivial.)

Definition of $=_{\Omega, B_p}$: We let $=_{\Omega, B_p}$ denote the \mathcal{L}' -formula that defines in every \mathcal{L}' -structure (k, A) the following equivalence relation on $F_p(k, A)$: For $(d, a_1, \dots, a_t, 0, \dots, 0), (d', a'_1, \dots, a'_{t'}, 0, \dots, 0) \in F_p(k, A)$ with $d, d' \in D_p(k, A)$ and $a_1, \dots, a_t, a'_1, \dots, a'_{t'} \in A$ we have

$$(d, a_1, \dots, a_t, 0, \dots, 0) \sim (d', a'_1, \dots, a'_{t'}, 0, \dots, 0)$$

if $d = d'$ (so that also $t = t'$) and $a'_1, \dots, a'_{t'}$ is a permutation of a_1, \dots, a_t , where elements corresponding to the same string of 00-blocks in $d = d'$ are permuted among themselves. Note that the map $f \mapsto b(f)$ that assigns to an $f \in F_p(k, A)$ corresponding to a complete parenthesization of a sequence

$$(a_1^1, \dots, a_{n_1}^1), \dots, (a_1^m, \dots, a_{n_m}^m)$$

of sequences in A the corresponding parenthesization of the sequence

$$\{a_1^1, \dots, a_{n_1}^1\}, \dots, \{a_1^m, \dots, a_{n_m}^m\}$$

of multisets, induces a bijection between $\partial_{\Omega, B_p}(k, A)/\sim$ and $B_p(k, A)$.

We note that V_f only depends on the equivalence class of $f \in F_p(k, A)$. Indeed, the multiset underlying \underline{v}_f only depends on the equivalence class of f . Only the ordering on the multiset \underline{v}_f depends on f . Moreover, if the equivalence class of f maps to $b \in B_p(k, A)$ under $\partial_{\Omega, B_p}(k, A)/\sim \simeq B_p(k, A)$, then, with the notation introduced in the beginning of this section, $V_f = V_b$.

Definition of $=_{\Omega, X_p}$: Let $=_{\Omega, X_p}$ denote the formula that defines on every $\partial_{\Omega, X_p}(k, A) = F_p(k, A) \times k^n$ the equivalence relation

$$(f, \xi) \sim (f', \xi') \Leftrightarrow f \sim f' \text{ and } \xi' \text{ is a permutation of } \xi \text{ such that } \underline{v}_f \xi = \underline{v}_{f'} \xi'.$$

We note that the map $(f, \xi) \rightarrow \underline{v}_f \xi$ induces a bijection $\partial_{\Omega, X_p}(k, A)/\sim \rightarrow \oplus V_f$, where the disjoint union is taken over all equivalence classes in $F_p(k, A)$. In other words, $\partial_{\Omega, X_p}(k, A)/\sim$ is in bijection with $X_p(k, A)$.

Definition of $=_{\Omega, B_{p, \widehat{p}}}$: Let $=_{\Omega, B_{p, \widehat{p}}}$ denote the \mathcal{L}' -formula that defines on every $H_{p, \widehat{p}}(k, A)$ the equivalence relation

$$(f_1, \widehat{f}_1, \lambda_1) \sim (f_2, \widehat{f}_2, \lambda_2) \Leftrightarrow f_1 \sim f_2, \widehat{f}_1 \sim \widehat{f}_2 \text{ and } \lambda_2 \text{ is a permutation of } \lambda_1 \text{ such that } \psi_{(f_1, \widehat{f}_1, \lambda_1)} = \psi_{(f_2, \widehat{f}_2, \lambda_2)}.$$

Then the map $(f, \widehat{f}, \lambda) \rightarrow (b(f), b(\widehat{f}), \psi_{(f, \widehat{f}, \lambda)})$ induces a bijection between $\partial_{\Omega, B_{p, \widehat{p}}}(k, A)/\sim$ and $B_{p, \widehat{p}}(k, A)$.

Definition of $=_{\Omega, X_{p, \widehat{p}}}$: Let $=_{\Omega, X_{p, \widehat{p}}}$ denote the \mathcal{L}' -formula that defines on every $\partial_{\Omega, X_{p, \widehat{p}}}(k, A)$ the equivalence relation

$$(f_1, \widehat{f}_1, \lambda_1, \xi_1, \widehat{\xi}_1) \sim (f_2, \widehat{f}_2, \lambda_2, \xi_2, \widehat{\xi}_2) \Leftrightarrow (f_1, \widehat{f}_1, \lambda_1) \sim (f_2, \widehat{f}_2, \lambda_2), (f_1, \xi_1) \sim (f_2, \xi_2) \text{ and } (\widehat{f}_1, \widehat{\xi}_1) \sim (\widehat{f}_2, \widehat{\xi}_2).$$

³Again the implicit use of matrix multiplication in this definition need not concern us, since ultimately we are only interested in the case when k is a field and A an abelian group.

Then the map $(f, \widehat{f}, \lambda, \xi, \widehat{\xi}) \mapsto (b(f), b(\widehat{f}), \psi_{(f, \widehat{f}, \lambda)}, v_f \xi, v_{\widehat{f}} \widehat{\xi})$ induces a bijection between $\partial_{\Omega, X_p, \widehat{p}}(k, A) / \sim$ and $X_{p, \widehat{p}}(k, A)$.

This concludes the definition of the equivalence formulas for Ω . Note that for every model (k, A) of T' we have a bijection between $(\partial_{\Omega, s}(k, A) / \sim)_{s \in S}$ (where S denotes the set of sorts for \mathcal{L}) and $\mathcal{M}(k, A)$. Using this bijection, the interpretation of the symbols of \mathcal{L} in $\mathcal{M}(k, A)$ gives rise to an interpretation of the symbols of \mathcal{L} in $(\partial_{\Omega, s}(k, A) / \sim)_{s \in S}$. It is now straight forward to check that these interpretations can be defined (uniformly in (k, A)) by appropriate \mathcal{L}' -formulas. This completes the definition of Ω . Note that $\Omega(k, A) \simeq \mathcal{M}(k, A)$ for every model (k, A) of T' .

It is clear that Ω is a left total interpretation of T in T' . Moreover, $\Xi(\Omega(k, A)) \simeq (k, A)$ for every model (k, A) of T' .

For a model $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ of T , let us consider $B_{(1,1)}$ as an abelian group (via the identification of $B_{(1,1)}$ with the character group of $G(\mathcal{M})$ as in the definition of Ξ). Then $\mathcal{M} \simeq \mathcal{M}(k, B_{(1,1)})$ because $G(\mathcal{M})$ and $G(\mathcal{M}(k, B_{(1,1)}))$ are both isomorphic to $D(B_{(1,1)})_k$. Moreover, $\Omega(\Xi(\mathcal{M})) = \Omega(k, B_{(1,1)}) \simeq \mathcal{M}(k, B_{(1,1)})$. Thus $\Omega(\Xi(\mathcal{M})) \simeq \mathcal{M}$ as desired. \square

We note that the above isomorphism $\Omega(\Xi(\mathcal{M})) \simeq \mathcal{M}$ is not canonical. For example, on the p -total objects sort we need a bijection between $\partial_{\Omega, X_p}(k, B_{(1,1)}) / \sim = (F_p(k, B_{(1,1)}) \times k^n) / \sim$ and X_p . Specifying such a bijection involves the choice of appropriate bases. This is why we have weak bi-interpretability rather than bi-interpretability in Theorem 3.2.

In the course of the proof of Theorem 3.2 we have seen the following:

Corollary 3.3. *Let \mathcal{M} be a model of PROALG such that $G(\mathcal{M})$ is diagonalizable. Then \mathcal{M} is interpretable in (k, A) , where k is the field sort of \mathcal{M} and A the character group of $G(\mathcal{M})$. \square*

Based on Theorem 3.2 we can now characterize elementary equivalence and elementary extensions for diagonalizable proalgebraic groups.

Corollary 3.4. *Let k be a field and G a diagonalizable proalgebraic group over k . Then a proalgebraic group G' over a field k' is elementarily equivalent to G if and only if k' is elementarily equivalent to k , G' is diagonalizable and $\chi(G')$ is elementarily equivalent to $\chi(G)$.*

Proof. First assume that $G' \equiv G$. Then clearly $k' \equiv k$. Moreover, we know from Proposition 2.5 that G' must be diagonalizable. Let $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ and $\mathcal{M}' = (k', B'_p, X'_p, B'_{p,q}, X'_{p,q})$ be models of PROALG such that $G(\mathcal{M}) \simeq G$ and $G(\mathcal{M}') \simeq G'$. Since interpretations preserve elementary equivalence we see that $\Xi(\mathcal{M}') \equiv \Xi(\mathcal{M})$, where Ξ is defined as in the proof of Theorem 3.2. So $(k', B'_{(1,1)}) \equiv (k, B_{(1,1)})$. Since $B'_{(1,1)}$ and $B_{(1,1)}$ are isomorphic with $\chi(G')$ and $\chi(G)$ respectively, we see that $\chi(G) \equiv \chi(G')$.

Conversely, assume now that G' is diagonalizable, $k' \equiv k$ and $\chi(G') \equiv \chi(G)$. Then also $(k', \chi(G')) \equiv (k, \chi(G))$. Therefore $\Omega(k', \chi(G')) \equiv \Omega(k, \chi(G))$, where Ω is defined as in the proof of Theorem 3.2. But $\mathcal{M}' \simeq \Omega(k', \chi(G'))$ and $\mathcal{M} \simeq \Omega(k, \chi(G))$. Thus $\mathcal{M}' \equiv \mathcal{M}$, i.e., $G' \equiv G$. \square

In particular, for a field k , a proalgebraic group G over k is elementarily equivalent to the multiplicative group \mathbb{G}_m over k if and only if G is isomorphic to $D(A)_k$ and A is elementarily equivalent to $(\mathbb{Z}, +)$. Since $\text{Th}(\mathbb{Z}, +)$ has models that are not finitely generated as abelian groups (e.g., $\mathbb{Z} \oplus \mathbb{Q}$, see [EF72]) and $D(A)_k$ is algebraic if and only if A is finitely generated we obtain:

Corollary 3.5. *The class of all algebraic groups is not elementary. \square*

Corollary 3.6. *Let \mathcal{M} be a model of PROALG such that $G(\mathcal{M})$ is diagonalizable and let A denote the character group of $G(\mathcal{M})$. If $\mathcal{M}' \succeq \mathcal{M}$ is an elementary extension of \mathcal{M} , then $G(\mathcal{M}') \simeq D(A')_{k'}$, where $k' \succeq k$ and $A' \succeq A$.*

Conversely, if $k' \succeq k$ and $A' \succeq A$ are elementary extensions, then there exists an elementary extension $\mathcal{M}' \succeq \mathcal{M}$ such that $G(\mathcal{M}') \simeq D(A')_{k'}$.

Proof. Again, let Ξ and Ω be the interpretations defined in the proof of Theorem 3.2. We observe that $G(\mathcal{M}')$ is diagonalizable by Proposition 2.5. Let A' denote the character group of $G(\mathcal{M}')$. Since interpretations preserve elementary embeddings (cf. [Hod93, Theorem 5.3.4] for the one-sorted case) we see that $\Xi(\mathcal{M}') \succeq \Xi(\mathcal{M})$, i.e., $(k', A') \succeq (k, A)$. It follows that $k' \succeq k$ and $A' \succeq A$.

Conversely, assume we start with elementary extensions $k' \succeq k$ and $A' \succeq A$. Then $(k', A') \succeq (k, A)$ and $\Omega(k', A') \succeq \Omega(k, A)$. As $\Omega(k, A) \simeq \mathcal{M}$ and $G(\Omega(k', A')) \simeq D(A')_{k'}$ the claim follows. \square

Corollary 3.7. *Let G be a diagonalizable proalgebraic group over an algebraically closed field k . Then $\text{Th}(G)$ is stable, but not necessarily superstable.*

Proof. Let \mathcal{M} be a model of PROALG such that $G(\mathcal{M}) \simeq G$ and let A denote the character group of $G(\mathcal{M})$. Since $\text{Th}(k)$ is stable and $\text{Th}(A)$ is stable, it follows that also $\text{Th}(k, A)$ is stable. As \mathcal{M} can be interpreted in (k, A) by Corollary 3.3 it follows that $\text{Th}(\mathcal{M})$ is stable.

There are abelian groups whose theory is not superstable (e.g., an infinite direct sum of copies of \mathbb{Z} , see [Hod93, Theorem A.2.13]). Since these can be interpreted in a model \mathcal{M} of PROALG with $G(\mathcal{M})$ diagonalizable, it follows that $\text{Th}(\mathcal{M})$ cannot be superstable. \square

4 Types

We postpone a more comprehensive study of types for certain models of PROALG to a future publication. Here we establish some initial algebraic results that illustrate the expressive power of PROALG:

- For a model $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ of PROALG the type of an element $b \in B_p$ over the empty set determines the minimal degree of defining equations of the image of the representation $G(\mathcal{M}) \rightarrow \text{GL}_{\omega(b)}$ associated with b .
- The type of b over k determines the image of $G(\mathcal{M}) \rightarrow \text{GL}_{\omega(b)}$.

The key to these results is the fact that the type of b over k knows which subspaces of representations of $G(\mathcal{M})$ obtained from $\omega(b)$ by forming tensor products, duals and direct sums are $G(\mathcal{M})$ -stable (i.e., subrepresentations). Moreover, the image of $G(\mathcal{M}) \rightarrow \text{GL}_{\omega(b)}$ is determined by these stable subspaces.

4.1 Stable subspaces and defining equations

The results in this subsection are of a preparatory nature and purely algebraic, i.e., do not involve any model theory. It is well known that any closed subgroup G of GL_V , for a finite dimensional vector space V , is the stabilizer of some subspace of a representation of GL_V obtained from V by forming tensor products, duals and direct sums. We will need to understand this result in full detail. In particular, we would like to know how the degree of the polynomials defining the stabilizer is related to the constructions (tensor product, duals, direct sums) applied to V .

Let G be a closed subgroup of GL_n . Then the defining ideal $\mathbb{I}(G)$ of G is a Hopf ideal of the Hopf algebra $k[\text{GL}_n] = k[Z, 1/\det(Z)] = k[Z, Z^{-1}]$, where Z is an $n \times n$ matrix of indeterminates. For every $d \geq 0$ let $k[Z, Z^{-1}]_{\leq d}$ denote the finite dimensional k -subspace of $k[Z, Z^{-1}]$ consisting of all elements of the form $P(Z, Z^{-1})$, where P is a polynomial over k in $2n^2$ variables of degree at most d .

Lemma 4.1. *The subspace $k[Z, Z^{-1}]_{\leq d}$ is a subcoalgebra of $k[Z, Z^{-1}]$.*

Proof. Let $\Delta: k[Z, Z^{-1}] \rightarrow k[Z, Z^{-1}] \otimes_k k[Z, Z^{-1}]$, $Z \mapsto Z \otimes Z$ denote the comultiplication. Here $Z \otimes Z$ is the $n \times n$ matrix whose ij -entry is $\sum_{\ell=1}^n Z_{i\ell} \otimes Z_{\ell j}$. In other words, $Z \otimes Z$ is the (matrix) product of the matrices $Z \otimes 1$ and $1 \otimes Z$, where $Z \otimes 1 = (Z_{i,j} \otimes 1)_{1 \leq i,j \leq n} \in (k[Z, Z^{-1}] \otimes_k k[Z, Z^{-1}])^{n \times n}$ and $1 \otimes Z = (1 \otimes Z_{i,j})_{1 \leq i,j \leq n} \in (k[Z, Z^{-1}] \otimes_k k[Z, Z^{-1}])^{n \times n}$. We have $\Delta(Z^{-1}) = \Delta(Z)^{-1} = ((Z \otimes 1)(1 \otimes Z))^{-1} = (1 \otimes Z)^{-1}(Z \otimes 1)^{-1} = (1 \otimes Z^{-1})(Z^{-1} \otimes 1)$. So $(\Delta(Z^{-1}))_{ij} = \sum_{\ell=1}^n (Z^{-1})_{\ell j} \otimes (Z^{-1})_{i\ell}$. Consequently, if P is a polynomial of degree at most d , then $\Delta(P(Z, Z^{-1})) = P(\Delta(Z), \Delta(Z^{-1})) \in k[Z, Z^{-1}]_{\leq d} \otimes_k k[Z, Z^{-1}]_{\leq d}$. \square

Lemma 4.2. *The ideal of $k[\mathrm{GL}_n] = k[Z, Z^{-1}]$ generated by $\mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d}$ is a Hopf ideal.*

Proof. If C is a coalgebra with a coideal V and a subcoalgebra D , then $V \cap D$ is a coideal of D , and so, in particular, a coideal of C . (To see this note that $D \rightarrow C/V$ is a morphism of coalgebras with kernel $V \cap D$ and kernels of morphisms of coalgebras are coideals.) It follows, using Lemma 4.1 that $\mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d}$ is a coideal of $k[Z, Z^{-1}]$.

Let $I = (\mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d})$ denote the ideal of $k[Z, Z^{-1}]$ generated by $\mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d}$. In any commutative Hopf algebra, the ideal generated by a coideal is a coideal. It follows that I is a coideal.

Thus it only remains to check that I is stable under the antipode $S: k[Z, Z^{-1}] \rightarrow k[Z, Z^{-1}]$, $Z \mapsto Z^{-1}$. However, since $k[Z, Z^{-1}]_{\leq d}$ and $\mathbb{I}(G)$ are stable under S this is immediate. \square

Let V be a finite dimensional k -vector space. The choice of a basis of V allows us to identify GL_V with GL_n and $k[\mathrm{GL}_V]$ with $k[Z, Z^{-1}]$. We note that $k[Z, Z^{-1}]_{\leq d}$, considered as a subspace of $k[\mathrm{GL}_V]$, does not depend on the choice of the basis and we may therefore safely denote it by $k[\mathrm{GL}_V]_{\leq d}$.

The following notation will be useful: For a polynomial $P \in \mathbb{N}[X, Y]$ in two variables and a finite dimensional k -vector space V we define $P(V, V^\vee)$ as the k -vector space obtained from V and P by replacing X by V , Y by the dual vector space V^\vee of V , addition in P by the direct sum of vector spaces and multiplication by the tensor product of vector spaces. The constant term of P has to be interpreted as the appropriate direct sum of copies of k . Note that if V is a representation of some proalgebraic group G , then $P(V, V^\vee)$ is also naturally a representation of G . (The constant term has to be interpreted as a trivial representation.) The choice of a basis $v = (v_1, \dots, v_n)$ of V determines, for every $P \in \mathbb{N}[X, Y]$, a basis of $P(V, V^\vee)$, which we will call the *v -canonical basis* of $P(V, V^\vee)$. It can be defined recursively as follows:

- The v -canonical basis of V is v .
- The v -canonical basis of V^\vee is the basis $v^\vee = (v_1^\vee, \dots, v_n^\vee)$ dual to v .
- If w_1, \dots, w_m is the v -canonical basis of W and $w'_1, \dots, w'_{m'}$ is the v -canonical basis of W' , then $w_1, \dots, w_m, w'_1, \dots, w'_{m'}$ is the v -canonical basis of $W \oplus W'$.
- If w_1, \dots, w_m is the v -canonical basis of W and $w'_1, \dots, w'_{m'}$ is the v -canonical basis of W' , then $(w_i \otimes w'_j)_{1 \leq i \leq m, 1 \leq j \leq m'}$ is the v -canonical basis of $W \otimes_k W'$.

Let V be a not necessarily finite dimensional representation of a proalgebraic group G . For a k -algebra R , a subspace W of V is *stable* under $g \in G(R)$ (or g *stabilizes* W) if $g: V \otimes_k R \rightarrow V \otimes_k R$ maps $W \otimes_k R$ into $W \otimes_k R$. If W is stable under all $g \in G(R)$ for all R , then W is *G -stable*. The subgroup of G consisting of all g that stabilize W is a closed subgroup of G called the *stabilizer* of W .

For $n \geq 1$ and $d \geq 0$ we define

$$P_d = \sum_{a+b \leq d} \binom{a+n-1}{a} \binom{b+n-1}{b} X^a Y^b \in \mathbb{N}[X, Y],$$

where the sum is taken over all pairs $(a, b) \in \mathbb{N}^2$ with $a + b \leq d$. The significance of this polynomial is explained in the following proposition.

Proposition 4.3. *Let V be a finite dimensional k -vector space and let G be a closed subgroup of GL_V . For $d \geq 0$ the following closed subgroups of GL_V are equal:*

- (i) *The subgroup of GL_V defined by the ideal of $k[\mathrm{GL}_V]$ generated by $\mathbb{I}(G) \cap k[\mathrm{GL}_V]_{\leq d}$ (cf. Lemma 4.2).*
- (ii) *The subgroup of GL_V consisting of all elements that stabilize all G -stable subspaces of $P(V, V^\vee)$ for all $P \in \mathbb{N}[X, Y]$ of degree at most d .*
- (iii) *The subgroup of GL_V consisting of all elements that stabilize all G -stable subspaces of $P_d(V, V^\vee)$.*

Moreover, there exists a G -stable subspace W of $P_d(V, V^\vee)$ such that the stabilizer of W (in GL_V), agrees with the group defined in (i), (ii) and (iii).

Proof. For $j = 1, 2, 3$ let G_j denote the group defined in point j above. Clearly $G_2 \leq G_3$. Fixing a basis $v = (v_1, \dots, v_n)$ of V , we may identify GL_V with GL_n .

To show that $G_1 \leq G_2$ let $P \in \mathbb{N}[X, Y]$ be a polynomial of degree at most d and let W be a subspace of $P(V, V^\vee)$. Let u_1, \dots, u_r denote the v -canonical basis of $P(V, V^\vee)$. Then there exist $P_{ij} \in k[Z, Z^{-1}]_{\leq d}$ such that $g(u_i) = \sum_{j=1}^r P_{ij}(g)u_j$ for all $g \in \mathrm{GL}_n(R)$ and all k -algebras R . It follows that for any basis w_1, \dots, w_r of $P(V, V^\vee)$ there exist $Q_{ij} \in k[Z, Z^{-1}]_{\leq d}$ such that $g(w_i) = \sum_{j=1}^r Q_{ij}(g)w_j$ for all $g \in \mathrm{GL}_n(R)$ and all k -algebras R . We may extend a basis w_1, \dots, w_s of W to a basis $w_1, \dots, w_s, w_{s+1}, \dots, w_r$ of $P(V, V^\vee)$. Then, using the above notation, an element $g \in \mathrm{GL}_n(R)$ stabilizes W if and only if $Q_{ij}(g) = 0$ for all i and $j > s$. Thus, an element $g \in \mathrm{GL}_n(R)$ such that $Q(g) = 0$ for all $Q \in \mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d}$, will stabilize every subspace W of $P(V, V^\vee)$ stabilized by G . Hence $G_1 \leq G_2$.

The most difficult part now is to show that $G_3 \leq G_1$. This follows from a detailed analysis of the proofs of two basic theorems on representations of algebraic groups (Theorems 4.14 and 4.27 in [Mil17]): We consider the regular representation of GL_n on $k[Z, Z^{-1}]$. This can be defined as the not necessarily finite dimensional representation of GL_n corresponding to the comodule $k[Z, Z^{-1}]$ with comodule map $\Delta: k[Z, Z^{-1}] \rightarrow k[Z, Z^{-1}] \otimes_k k[\mathrm{GL}_n]$, $Z \mapsto Z \otimes Z$. Explicitly, the action of an element $g \in \mathrm{GL}_n(R)$ on $k[Z, Z^{-1}] \otimes_k R$ is given by $g(f(Z)) = f(Zg)$ for $f \in k[Z, Z^{-1}]$. We note that $k[Z, Z^{-1}]_{\leq d}$ is a GL_n -stable subspace of $k[Z, Z^{-1}]$. For $i = 1, \dots, n$ let V_i denote the k -subspace of $k[Z, Z^{-1}]_{\leq d}$ generated by the i -th row of Z . Then V_i is GL_n -stable. In fact, V_i is isomorphic to V as a GL_n -representation, under the isomorphism that identifies Z_{ij} with v_j for $j = 1, \dots, n$. Similarly, for $j = 1, \dots, n$ let W_j denote the subspace of $k[Z, Z^{-1}]_{\leq d}$ generated by the j -th column of Z^{-1} . Then W_j is GL_n -stable and indeed is isomorphic to V^\vee as a GL_n -representation, under the isomorphism that identifies $(Z^{-1})_{ij}$ with v_i^\vee for $i = 1, \dots, n$. Let

$$Q = \sum_{d_1 + \dots + d_n + e_1 + \dots + e_n \leq d} X_1^{d_1} \dots X_n^{d_n} Y_1^{e_1} \dots Y_n^{e_n} \in \mathbb{N}[X_1, \dots, X_n, Y_1, \dots, Y_n]$$

be the full polynomial of degree d all of whose coefficients are equal to 1. We have a surjective map $Q(V_1, \dots, V_n, W_1, \dots, W_n) \rightarrow k[Z, Z^{-1}]_{\leq d}$ of GL_n -representations, where $Q(V_1, \dots, V_n, W_1, \dots, W_n)$ is defined in a fashion similar to the definition of $P(V, V^\vee)$ above. Since $V_i \simeq V$ and $W_j \simeq V^\vee$ this can be interpreted as a surjective map $\pi: P_d(V, V^\vee) \rightarrow k[Z, Z^{-1}]_{\leq d}$ of GL_n -representations, where $P_d = Q(X, \dots, X, Y, \dots, Y) = \sum_{a+b \leq d} \binom{a+n-1}{a} \binom{b+n-1}{b} X^a Y^b \in \mathbb{N}[X, Y]$.

Since $\mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d}$ is a G -stable subspace of $k[Z, Z^{-1}]_{\leq d}$, it follows that $W = \pi^{-1}(\mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d})$ is a G -stable subspace of $P_d(V, V^\vee)$. Thus, by the definition of G_3 , the subspace W is G_3 -stable. Therefore $\pi(W) = \mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d}$ is G_3 -stable. From this we deduce that

$(\mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d}) = \mathbb{I}(G_1)$ is G_3 -stable. However, only elements of G_1 stabilize $\mathbb{I}(G_1)$ (cf. [Spr09](#) Lemma 2.3.8). Therefore $G_3 \leq G_1$ as desired.

Finally, the G -stable subspace $W = \pi^{-1}(\mathbb{I}(G) \cap k[Z, Z^{-1}]_{\leq d})$ of $P_d(V, V^\vee)$ has the property required in the last statement of the proposition. \square

Let V be a finite dimensional k -vector space and let G be a closed subgroup of GL_V . For every $d \geq 0$ let

$$G_{\leq d}$$

denote the closed subgroup of GL_V characterized in Proposition [4.3](#). We then have a descending chain

$$\mathrm{GL}_V = G_{\leq 0} \supseteq G_{\leq 1} \supseteq G_{\leq 2} \supseteq \dots$$

of closed subgroups of GL_V that eventually stabilizes at G .

Definition 4.4. *The defining degree of G is the smallest d such that $G = G_{\leq d}$.*

The following lemma will be used in the next subsection. Roughly speaking, it shows that stabilizers of subspaces of $P(V, V^\vee)$ are uniformly definable.

Lemma 4.5. *Let $P \in \mathbb{N}[X, Y]$ be a polynomial of degree d , $n \geq 1$ and let s denote the dimension of the vector space $P(V, V^\vee)$, where V is an n -dimensional vector space. Furthermore let $1 \leq r \leq s$ and $1 \leq i_1 < \dots < i_r \leq s$. Then there exist polynomials $Q_1, \dots, Q_{(s-r)r} \in \mathbb{Z}[T, 1/\det((T_{i_\ell, j})_{1 \leq \ell, j \leq r}), Z, Z^{-1}]$ of degree at most d in Z and Z^{-1} , where $T = (T_{i_\ell, j})_{1 \leq i \leq s, 1 \leq j \leq r}$ and $Z = (Z_{i, j})_{1 \leq i, j \leq n}$ such that the following holds: For every field k , every k -vector space V of dimension n with basis $\underline{v} = (v_1, \dots, v_n)$, all matrices $A \in k^{s \times r}$ such that $\det(A_{i_\ell, j})_{1 \leq \ell, j \leq r} \neq 0$, all k -algebras R and all $g \in \mathrm{GL}_V(R) \simeq \mathrm{GL}_n(R)$ (via \underline{v}) the k -subspace of $P(V, V^\vee)$ generated by $\underline{u}A$, where \underline{u} is the \underline{v} -canonical basis of $P(V, V^\vee)$, is stable under g if and only if $Q_i(A, g) = 0$ for $i = 1, \dots, (s-r)r$.*

Proof. Let $S = \mathbb{Z}[T, 1/\det((T_{i_\ell, j})_{1 \leq \ell, j \leq r})]$ and consider a free S -module V_S of rank n with basis \underline{v}_S . The definition of $P(V, V^\vee)$ and the \underline{v} -canonical bases of $P(V, V^\vee)$ extends from vector spaces to finite rank free modules in a straight forward manner. So let \underline{u}_S denote the \underline{v}_S -canonical basis of $P(V_S, V_S^\vee)$. We extend the matrix $T \in S^{s \times r}$ to a matrix $\tilde{T} \in \mathrm{GL}_s(S)$ by adding the standard basis vectors e_1, \dots, e_{s-r} of length $s-r$ in the rows $\{1, \dots, s\} \setminus \{i_1, \dots, i_r\}$. The group scheme $\mathrm{GL}_{n, S} = \mathrm{Spec}(S[Z, Z^{-1}])$ over S , acts linearly on V_S and on $P(V_S, V_S^\vee)$. Moreover, there exist a matrix $B \in \mathbb{Z}[Z, Z^{-1}]^{s \times s}$ with entries of at most degree d such that $g(\underline{u}_S) = \underline{u}_S B(g)$ for any S -algebra S' and $g \in \mathrm{GL}_n(S')$. It follows that

$$g(\underline{u}_S \tilde{T}) = \underline{u}_S B(g) \tilde{T} = \underline{u}_S \tilde{T} (\tilde{T}^{-1} B(g) \tilde{T}).$$

Thus the submodule of V_S with basis $\underline{u}_S T$ is stable under g if and only if the $(s-r) \times r$ submatrix in the lower left corner of $(\tilde{T}^{-1} B(g) \tilde{T}) \in S[Z, Z^{-1}]^{s \times s}$ is zero.

We claim that the entries $Q_1, \dots, Q_{(s-r)r}$ of this matrix have the required property. Since the entries of B have degree at most d in Z and Z^{-1} , also $Q_1, \dots, Q_{(s-r)r}$ have degree at most d in Z and Z^{-1} . The choice of a field k and a matrix $A \in k^{s \times r}$ with $\det(A_{i_\ell, j})_{1 \leq \ell, j \leq r} \neq 0$ defines a morphism of rings $S \rightarrow k$, i.e., a k -valued point of $\mathrm{Spec}(S)$. The claim now follows by considering the fiber over this point. In more detail: The k -vector space $V_S \otimes_S k$ has basis $\underline{v}_S \otimes 1$ and we can define an isomorphism $V \rightarrow V_S \otimes_S k$ by $\underline{v} \mapsto \underline{v}_S \otimes 1$. Similarly, we have an isomorphism $P(V, V^\vee) \rightarrow P(V_S, V_S^\vee) \otimes_S k$, $\underline{u} \mapsto \underline{u}_S \otimes 1$. We extend A to a matrix \tilde{A} in $\mathrm{GL}_s(k)$ in a similar fashion as we did with T . Then, for a k -algebra R and $g \in \mathrm{GL}_n(R)$ we have $g(\underline{u} \tilde{A}) = \underline{u} \tilde{A} (\tilde{A}^{-1} B(g) \tilde{A})$. Thus the subspace of $P(V, V^\vee)$ generated by $\underline{u}A$ is stable under g if and only if $Q_i(A, g) = 0$ for $i = 1, \dots, (s-r)r$. \square

4.2 Types and stable subspaces

Let \mathcal{M} be a model of PROALG and let (k, C, ω) be the associated object of TANN. For an object b of C we let

$$G(b) \leq \mathrm{GL}_{\omega(b)}$$

denote the (scheme-theoretic) image of the representation $G(\mathcal{M}) \rightarrow \mathrm{GL}_{\omega(b)}$ defined by b . Note that an algebraic group is a quotient of $G(\mathcal{M})$ if and only if it is isomorphic to some $G(b)$. Moreover, $G(\mathcal{M})$ is the projective limit of the $G(b)$'s. Our main result is that $\mathrm{tp}(b/k)$ determines $G(b)$. We begin by translating the main findings of the previous subsection into a statement about models of PROALG. The point of the following corollary is that this somewhat clumsy characterization of when $G(b)_{\leq d} = G(b)_{\leq d'}$ can be expressed by an \mathcal{L} -formula.

Corollary 4.6. *Let \mathcal{M} be a model of PROALG and (k, C, ω) the associated object of TANN. Let b be an object of C and for $0 \leq d < d'$ let s and s' denote the dimensions of $P_d(\omega(b), \omega(b)^\vee)$ and $P_{d'}(\omega(b), \omega(b)^\vee)$ respectively. Then $G(b)_{\leq d} = G(b)_{\leq d'}$ if and only if the following condition is satisfied: For all bases \underline{v} of $\omega(b)$, for all r' with $1 \leq r' \leq s'$, all $1 \leq i'_1 < \dots < i'_{r'} \leq s$ and all $A' \in k^{s' \times r'}$ with $\det((A'_{i'_\ell, j})_{1 \leq \ell, j \leq r'}) \neq 0$ such that the subspace of $P_{d'}(\omega(b), \omega(b)^\vee)$ generated by $\underline{u}' A'$ is $G(\mathcal{M})$ -stable, where \underline{u}' is the \underline{v} -canonical basis of $P_{d'}(V, V^\vee)$, there exist $1 \leq r \leq s$, $1 \leq i_1 < \dots < i_r \leq s$ and $A \in k^{s \times r}$ with $\det((A_{i_\ell, j})_{1 \leq \ell, j \leq r}) \neq 0$ such that the subspace of $P_d(V, V^\vee)$ generated by $\underline{u} A$ is $G(\mathcal{M})$ -stable, where \underline{u} denotes the \underline{v} -canonical basis of $P_d(V, V^\vee)$, and the polynomials $Q'_1(A', Z), \dots, Q'_{(s'-r')r'}(A', Z) \in k[Z, Z^{-1}]_{\leq d'}$ lie in the ideal of $k[Z, Z^{-1}]$ generated by $Q_1(A, Z), \dots, Q_{(s-r)r}(A, Z) \in k[Z, Z^{-1}]_{\leq d}$. Here the polynomials $Q'_1, \dots, Q'_{(s'-r')r'}$ and $Q_1, \dots, Q_{(s-r)r}$ are defined as in Lemma 4.5.*

Proof. Note that a subspace of some $P(\omega(b), \omega(b)^\vee)$ is $G(\mathcal{M})$ -stable if and only if it is $G(b)$ -stable. Thus, according to Proposition 4.3 the closed subgroup $G(b)_{\leq d'}$ of $\mathrm{GL}_{\omega(b)}$ is the stabilizer of some $G(\mathcal{M})$ -stable subspace W' of $P_{d'}(V, V^\vee)$. If \underline{v} is a basis of $\omega(b)$ and \underline{u}' is the \underline{v} -canonical basis of $P_{d'}(V, V^\vee)$, then W' has a basis of the form $\underline{u}' A'$, for some $A' \in k^{s' \times r'}$ with linearly independent columns, i.e., $\det((A'_{i'_\ell, j})_{1 \leq \ell, j \leq r'}) \neq 0$ for an appropriate choice of $1 \leq i'_1 < \dots < i'_{r'} \leq s$. So the defining ideal $\mathbb{I}(G(b)_{\leq d'})$ of $G(b)_{\leq d'}$ is generated by $Q'_1(A', Z), \dots, Q'_{(s'-r')r'}(A', Z)$ according to Lemma 4.5. By construction, the polynomials $Q_1(A, Z), \dots, Q_{(s-r)r}(A, Z)$ lie in the ideal $\mathbb{I}(G(b)_{\leq d})$. Thus, if $Q'_1(A', Z), \dots, Q'_{(s'-r')r'}(A', Z)$ lie in the ideal generated by $Q_1(A, Z), \dots, Q_{(s-r)r}(A, Z)$, then $\mathbb{I}(G(b)_{\leq d'}) \subseteq \mathbb{I}(G(b)_{\leq d})$. So $G(b)_{\leq d} = G(b)_{\leq d'}$ as desired.

Conversely, assume that $G(b)_{\leq d} = G(b)_{\leq d'}$. Similarly to the above argument, there exists appropriate integers, r, i_1, \dots, i_r and a matrix $A \in k^{s \times r}$ such that $\det((A_{i_\ell, j})_{1 \leq \ell, j \leq r}) \neq 0$ and $\mathbb{I}(G(b)_{\leq d})$ is generated by $Q_1(A, Z), \dots, Q_{(s-r)r}(A, Z)$. Since $\mathbb{I}(G(b)_{\leq d'}) = \mathbb{I}(G(b)_{\leq d})$ it follows that $Q'_1(A', Z), \dots, Q'_{(s'-r')r'}(A', Z)$ lie in the ideal generated by $Q_1(A, Z), \dots, Q_{(s-r)r}(A, Z)$. \square

To verify that the above statement can be expressed by an \mathcal{L} -formula we need two lemmas. Roughly speaking, the following lemma shows that we can quantify over the $G(\mathcal{M})$ -stable subspaces of $P(\omega(b), \omega(b)^\vee)$.

Lemma 4.7. *Let $p = (m, n)$ and $P \in \mathbb{N}[X, Y]$. Let s be the dimension of the vector space $P(V, V^\vee)$, where V is an n -dimensional vector space. Then, for every r with $1 \leq r \leq s$, there exists an \mathcal{L} -formula $\varphi(T, y_1, \dots, y_n)$, where $T = (T_{i,j})$ is an $s \times r$ matrix of variables from the field sort and y_1, \dots, y_n are variables from the p -total objects sort such that the following holds:*

For every model $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ of PROALG, all $b \in B_p$, all bases v_1, \dots, v_n of $\omega(b)$ and all $A \in k^{s \times r}$ with linearly independent columns we have $\mathcal{M} \models \varphi(A, v_1, \dots, v_n)$ if and only if the k -subspace of $P(\omega(b), \omega(b)^\vee)$ generated by $\underline{u} A$ is $G(\mathcal{M})$ -stable, where \underline{u} is the \underline{v} -canonical basis of $P(\omega(b), \omega(b)^\vee)$.

Proof. Let $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ be a model of PROALG and let (k, C, ω) be the associated object of TANN. Moreover let $b \in B_p$ and let $P(b, b^\vee) \in B_{(1,s)}$ denote the unique tensor irreducible object of C such that $\omega(P(b, b^\vee)) \simeq P(\omega(b), \omega(b)^\vee)$ as $G(\mathcal{M})$ -representations. A subspace W of $P(\omega(b), \omega(b)^\vee)$ of dimension r is $G(\mathcal{M})$ -stable if and only if there exist $b' \in B_{(r,1)}$ and a morphism $h: b' \rightarrow P(b, b^\vee)$ in C such that $\omega(h): \omega(b') \rightarrow \omega(P(b, b^\vee)) \simeq P(\omega(b), \omega(b)^\vee)$ has image W . So, in coordinates, if \underline{v} is a basis of $\omega(b)$, and $A \in k^{s \times r}$ is such that $\underline{v}A$ is a basis of W , where \underline{v} is the \underline{v} -canonical basis of $P(\omega(b), \omega(b)^\vee)$, then W is $G(\mathcal{M})$ -stable if and only if there exists $b' \in B_{(r,1)}$ with a basis \underline{v}' of $\omega(b')$ and a morphism $h: b' \rightarrow P(b, b^\vee)$ in C such that $\omega(h): \omega(b') \rightarrow \omega(P(b, b^\vee)) \simeq P(\omega(b), \omega(b)^\vee)$ maps \underline{v}' to $\underline{v}A$. Since the \underline{v} -canonical basis of $P(\omega(b), \omega(b)^\vee)$ can be characterized in terms of \underline{v} by \mathcal{L} -formulas, the claim follows. \square

We will also need a classical result of G. Hermann (cf. [Asc04, Theorem 3.4]).

Lemma 4.8 (G. Hermann [Her26]). *For every field k , if $f, g_1, \dots, g_m \in k[X_1, \dots, X_n]$ are polynomials of degree at most d such that f lies in the ideal generated by g_1, \dots, g_m , then there exist polynomials $f_1, \dots, f_m \in k[X_1, \dots, X_n]$ of degree at most $(2d)^{2^n}$ such that $f = f_1g_1 + \dots + f_mg_m$. \square*

Combining the above results we see that the set of all $b \in B_p$ such that $G(b)_{\leq d} = G(b)_{\leq d'}$ is definable:

Proposition 4.9. *For given p and integers $0 \leq d < d'$ there exists an \mathcal{L} -formula $\varphi(x)$ in one free variable x belonging to the p -basic objects sort such that for every model $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ of PROALG and every $b \in B_p$ we have $\mathcal{M} \models \varphi(b)$ if and only if $G(b)_{\leq d} = G(b)_{\leq d'}$.*

Proof. It suffices to see that the statement from Corollary 4.6 can be expressed by an \mathcal{L} -formula. This is guaranteed by Lemmas 4.7 and 4.8. \square

Corollary 4.10. *Let $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ be a model of PROALG. If $b, b' \in B_p$ are such that $\text{tp}(b/\emptyset) = \text{tp}(b'/\emptyset)$, then the defining degree of $G(b)$ agrees with the defining degree of $G(b')$.*

Proof. Clear from Proposition 4.9. \square

Theorem 4.11. *Let $\mathcal{M} = (k, B_p, X_p, B_{p,q}, X_{p,q})$ be a model of PROALG and let $b, b' \in B_p$. If $\text{tp}(b/k) = \text{tp}(b'/k)$, then there exists an isomorphism $\omega(b) \rightarrow \omega(b')$ of k -vector spaces that induces an isomorphism between $G(b)$ and $G(b')$.*

Proof. According to Corollary 4.10 the defining degree of $G(b)$ agrees with the defining degree of $G(b')$. Let us denote it with d . Let s denote the dimension of $P_d(\omega(b), \omega(b)^\vee)$ and let x be a variable from the p -objects sort. Let $1 \leq r \leq s$ and $A \in k^{s \times r}$ with $\det((A_{i_\ell, j})_{1 \leq \ell, j \leq r}) \neq 0$ for $1 \leq i_1 < \dots < i_r \leq s$. Moreover let $Q_1, \dots, Q_{(s-r)r} \in \mathbb{Z}[T, 1/\det((T_{i_\ell, j})_{1 \leq \ell, j \leq r}), Z, Z^{-1}]$ be defined as in Lemma 4.5. Consider the formula $\varphi_A(x)$ with parameters from k such that $\varphi_A(b)$ expresses the following statement:

There exists a basis \underline{v} of $\omega(b)$ such that for all r' with $1 \leq r' \leq s$, all $1 \leq i'_1 < \dots < i'_{r'} \leq s$ and all $A' \in k^{s \times r'}$ with $\det((A'_{i'_\ell, j})_{1 \leq \ell, j \leq r'}) \neq 0$ such that the subspace of $P_d(\omega(b), \omega(b)^\vee)$ generated by $\underline{v}A'$ is $G(\mathcal{M})$ -stable, where \underline{v} is the \underline{v} -canonical basis of $P_d(\omega(b), \omega(b)^\vee)$, the polynomials $Q'_1(A', Z), \dots, Q'_{(s-r')r'}(A', Z) \in k[Z, Z^{-1}]_{\leq d}$ (with $Q'_1, \dots, Q'_{(s-r')r'}$ defined as in Lemma 4.5) lie in the ideal of $k[Z, Z^{-1}]$ generated by $Q_1(A, Z), \dots, Q_{(s-r)r}(A, Z) \in k[Z, Z^{-1}]_{\leq d}$.

If $\varphi_A(x)$ lies in $\text{tp}(b/k)$ the formula $\varphi_A(x)$ determines $G(b)$ because it shows that $\mathbb{I}(G(b)) = \mathbb{I}(G(b)_{\leq d})$ is generated by $Q_1(A, Z), \dots, Q_{(s-r)r}(A, Z) \in k[Z, Z^{-1}]_{\leq d}$. (Here $\text{GL}_{\omega(b)} \simeq \text{GL}_n$ via the basis \underline{v} deemed to exist by $\varphi_A(x)$.)

On the other hand, $\varphi_A(x)$ lies in $\text{tp}(b/k)$ for some choice of r , A and i_1, \dots, i_r . \square

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