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Connectivity concerning the last two subconstituents of a Q -polynomial distance-regular graph



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ABSTRACT

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Let Γ be a Q -polynomial distance-regular graph of diameter $d \geq 3$. Fix a vertex γ of Γ and consider the subgraph induced on the union of the last two subconstituents of Γ with respect to γ . We prove that this subgraph is connected.

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1. Introduction

All the graphs considered here will be finite and undirected, with no loops nor multiple edges. We briefly review the key definitions and basic results involving distance-regular graphs. For other notations and definitions, see [5,6,14]. Let Γ be a connected graph with vertex set X . For $x, y \in X$, the distance between x and y is denoted by $\partial(x, y)$, and any path between x and y of length $\partial(x, y)$ is called *geodesic*. The *diameter* $\max_{x, y \in X} \partial(x, y)$ of Γ is denoted by d . For an integer $k \geq 0$, Γ is said to be *regular with valency* k whenever each vertex of Γ is adjacent to exactly k vertices of Γ . The graph Γ is called *distance-regular* whenever for all integers $0 \leq h, i, j \leq d$ there exists a nonnegative integer p_{ij}^h such that for all $x, y \in X$ with $\partial(x, y) = h$,

$$p_{ij}^h = |\{z \in X : \partial(z, x) = i, \partial(z, y) = j\}|.$$

For the rest of this paper we assume that Γ is distance-regular of diameter $d \geq 2$. Note that Γ is regular with valency $k = p_{11}^0$; to avoid trivialities we always assume $k \geq 3$. Let A_0, A_1, \dots, A_d denote the distance matrices of Γ (see [5, p.127]). Then A_0, A_1, \dots, A_d form a basis for a commutative semisimple \mathbb{R} -algebra M known as the *Bose-Mesner algebra* of Γ . The algebra M has a second basis E_0, E_1, \dots, E_d such that

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d),$$

$$I = E_0 + \dots + E_d,$$

$$E_0 = |X|^{-1} J,$$

where I is the identity matrix and J is the all ones matrix (see [5, Thm 2.6.1]). We refer to E_0, E_1, \dots, E_d as the *primitive idempotents* of Γ . The primitive idempotent E_0 is called *trivial*. The ordering E_0, E_1, \dots, E_d is said to be *Q -polynomial* whenever for $0 \leq i \leq d$ there exists a polynomial q_i of degree i such that $E_i = q_i(E_1)$ (where the matrix multiplication is done entry-wise). For a primitive idempotent E of Γ , we say that Γ is *Q -polynomial with respect to E* whenever there exists a Q -polynomial ordering E_0, E_1, \dots, E_d of the primitive idempotents such that $E = E_1$. The graph Γ is called *Q -polynomial* whenever it is Q -polynomial with respect to at least one primitive idempotent.

We now recall the antipodal property. Define a binary relation \sim on X such that for all $x, y \in X$, $x \sim y$ whenever $x = y$ or $\partial(x, y) = d$. The graph Γ is called *antipodal* whenever \sim is an equivalence relation. The graph Γ is said to be *primitive* whenever Γ is not bipartite nor antipodal (see [5, Thm 4.2.1]). A long-standing conjecture of Bannai and Ito [1, p. 312] states that if Γ is primitive and d is sufficiently large, then Γ is Q -polynomial. For more information about the Q -polynomial property, see [1,5] or [14, Chapter 5].

For $0 \leq i \leq d$ and $\gamma \in X$, let $\Gamma_i(\gamma)$ denote the set of vertices in Γ at distance i from γ . The subgraph induced by $\Gamma_i(\gamma)$ is called the *i -th subconstituent of Γ with respect to γ* .

γ . Combinatorial and algebraic properties of these subconstituents have been studied by several authors (see [9,10,16] for example). The graph Γ is called *strongly-regular* whenever $d = 2$. If Γ is strongly-regular and primitive, then with respect to any vertex the second subconstituent of Γ is connected. See [6, p. 126] for an algebraic proof, and [16] for a combinatorial proof. Answering a question of Brouwer [4], Cioabă and Koolen [10] generalized this result in the following way. Consider the dual eigenvalue sequence $\theta_0^*, \theta_1^*, \dots, \theta_d^*$ for the second largest eigenvalue of Γ (see equation (2.1) for a definition). By [5, Ch. 4] there exists a unique integer s ($1 \leq s \leq d$) such that $\theta_{s-1}^* > 0$ and $\theta_s^* \leq 0$. Then for any vertex γ of Γ the subgraph induced on $\cup_{i=s}^d \Gamma_i(\gamma)$ is connected [10]. In [10] the authors also prove that $s \geq d/2$ and pose the following problem.

Problem 1.1 (Cioabă-Koolen [10]). Assume that Γ is primitive and $d \geq 3$. Is it true that for any vertex γ , the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is connected?

In [10], this was shown to be true if $d \in \{3, 4\}$. In this note, we show that it is true for all $d \geq 3$, provided that Γ is Q -polynomial. We now state our main result.

Theorem 1.1. *Let Γ be a Q -polynomial distance-regular graph of diameter $d \geq 3$. Then for any vertex γ of Γ the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is connected.*

The main tool for our proof is Terwilliger's balanced set condition (see [21,22] or Theorem 2.1 in the next section). This condition has been used by Lewis [20] to prove that the girth is at most 6 for any Q -polynomial distance-regular graph of valency at least 3.

2. Proof of the main result

For a primitive idempotent E of Γ , there exist real numbers $\theta_0^*, \theta_1^*, \dots, \theta_d^*$ (called the dual eigenvalues of Γ with respect to E) such that

$$E = |X|^{-1} \sum_{h=0}^d \theta_h^* A_h. \quad (2.1)$$

We equip the vector space \mathbb{R}^X with an inner product such that $\langle u, v \rangle = u^t v$ for all $u, v \in \mathbb{R}^X$. For $x \in X$, let \hat{x} denote the vector in \mathbb{R}^X with x -coordinate 1 and all other coordinates 0. Equation (2.1) implies that

$$\langle E\hat{x}, E\hat{y} \rangle = |X|^{-1} \theta_i^*, \quad (2.2)$$

where $i = \partial(x, y)$. The main tool for our proof is the following theorem.

Theorem 2.1 (Terwilliger [21,22]). *Let Γ be a distance-regular graph with diameter $d \geq 3$, and let E denote a nontrivial primitive idempotent of Γ with dual eigenvalues*

$\theta_0^*, \theta_1^*, \dots, \theta_d^*$. Then Γ is Q -polynomial with respect to E if and only if $\theta_0^* \notin \{\theta_1^*, \dots, \theta_d^*\}$ and

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} - \sum_{w \in \Gamma_j(x) \cap \Gamma_i(y)} E\hat{w} = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} (E\hat{x} - E\hat{y}) \quad (2.3)$$

for all integers h, i, j with $1 \leq h \leq d$ and $0 \leq i, j \leq d$ and all vertices x, y with $\partial(x, y) = h$. Furthermore, if the conditions above hold, then $\theta_0^*, \theta_1^*, \dots, \theta_d^*$ are mutually distinct.

The equation (2.3) is usually called the balanced set condition. We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let E be a primitive idempotent of Γ with respect to which Γ is Q -polynomial. We will use a proof by contradiction, and assume that there exists $\gamma \in X$ such that the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is disconnected. Let C be the vertex set of a connected component of the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$. Let the set Δ consist of the vertices in X that lie on a geodesic from γ to C . The set Δ is properly contained in X since $C \neq \Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$. We partition $\Delta = \cup_{j=0}^d \Delta_j$ where $\Delta_j = \Delta \cap \Gamma_j(\gamma)$ for $0 \leq j \leq d$. Note that for $0 \leq j \leq d-1$, each vertex in Δ_j has at least one neighbor in Δ_{j+1} .

A vertex in Δ will be called a *border* whenever it is adjacent to a vertex in $X \setminus \Delta$. Since $\Delta \neq X$ and Γ is connected, Δ contains at least one border vertex. Let t denote the maximal integer j ($0 \leq j \leq d$) such that Δ_j contains a border vertex. By the construction $1 \leq t \leq d-2$.

Pick a border vertex $z \in \Delta_t$. There exists $x \in \Delta_{t+2}$ such that $\partial(x, z) = 2$. Let $y \in X \setminus \Delta$ be a neighbor of z . Define $\xi = \partial(\gamma, y)$. By the triangle inequality $\xi \in \{t-1, t, t+1\}$. Note that $\xi \neq t-1$; otherwise y is on a geodesic from γ to C passing through z , forcing $y \in \Delta$ for a contradiction. Therefore $\xi = t$ or $\xi = t+1$.

We next show that $\partial(x, y) = 3$. Because $\partial(x, z) = 2$ and $\partial(z, y) = 1$, the triangle inequality implies that $\partial(x, y) \leq 3$. By the maximality of t and since $x \in \Delta_{t+2}$, we see that x is not a border and not adjacent to a border. Therefore Δ contains all the vertices of Γ that are at distance at most 2 from x . The vertex y is not in Δ , so $\partial(x, y) \geq 3$. We have shown that $\partial(x, y) = 3$.

Note that $\Gamma_1(x) \cap \Gamma_2(y) \subset \Gamma_{t+1}(\gamma)$ and $\Gamma_2(x) \cap \Gamma_1(y) \subset \Gamma_t(\gamma)$. We apply the balanced set condition (2.3) to x and y using $h = 3, i = 1, j = 2$ and then take the inner product of each side with $E\hat{y}$; this gives

$$p_{12}^3 (\theta_{t+1}^* - \theta_t^*) = p_{12}^3 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_3^*} (\theta_{t+2}^* - \theta_\xi^*). \quad (2.4)$$

There exists $y' \in \Gamma_{t-1}(\gamma) \cap \Gamma_1(z)$. We have $\partial(x, y') = 3$ and $\Gamma_1(x) \cap \Gamma_2(y') \subset \Gamma_{t+1}(\gamma)$ and $\Gamma_2(x) \cap \Gamma_1(y') \subset \Gamma_t(\gamma)$. We apply the balanced set condition (2.3) to x and y' using $h = 3, i = 1, j = 2$ and then take the inner product of each side with $E\hat{y}$; this gives

$$p_{12}^3(\theta_{t+1}^* - \theta_t^*) = p_{12}^3 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_3^*} (\theta_{t+2}^* - \theta_{t-1}^*). \quad (2.5)$$

Comparing (2.4) and (2.5) we obtain $\theta_\xi^* = \theta_{t-1}^*$. We have $\xi = t-1$ since $\theta_0^*, \theta_1^*, \dots, \theta_d^*$ are mutually distinct. We mentioned earlier that $\xi \neq t-1$, for a contradiction. We conclude that the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is connected. \square

To see how Theorem 1.1 is best possible, assume that Γ is the Odd graph O_{d+1} with $d \geq 3$. Recall that the vertices of Γ are the d -subsets of a set Ω of size $2d+1$. Two vertices α and β are adjacent whenever $\alpha \cap \beta = \emptyset$. The diameter of Γ is d and its intersection numbers are known (see [2] or [5, Prop 9.1.7]). For $0 \leq h \leq d$, we have $p_{1h}^h = 0$ if $h < d$ and $p_{1h}^h = \lceil \frac{d+1}{2} \rceil$ if $h = d$. So with respect to any vertex of Γ , the h -th subconstituent has no edges if $h < d$ and is regular with valency $\lceil \frac{d+1}{2} \rceil$ if $h = d$.

Lemma 2.2. *Assume that Γ is the Odd graph O_{d+1} with $d \geq 3$. For any $\gamma \in X$, the number of connected components in the d -th subconstituent of Γ with respect to γ is equal to $\binom{2m}{m}/2$, where $m = d/2$ if d is even and $m = (d+1)/2$ if d is odd. Moreover, this d -th subconstituent is not connected.*

Proof. From the intersection numbers of Γ we obtain $|\Gamma_d(\gamma)| = \binom{d}{m} \binom{d+1}{m}$. Using the results of Biggs [2], each connected component of $\Gamma_d(\gamma)$ is isomorphic to the bipartite double (see [5, Section 1.11]) of O_{r+1} , where $r = d/2$ if d is even and $r = (d-1)/2$ if d is odd. This bipartite double has $2\binom{2r+1}{r}$ vertices. The result follows after some routine algebra. Note that the lemma also follows by observing that $\Gamma_d(\gamma)$ consists of the vertices at distance m from γ in the Johnson graph $J(2d+1, d)$. \square

Note also that for O_{d+1} the subgraph induced on $\Gamma_1(\gamma) \cup \Gamma_2(\gamma)$ is disconnected. Next assume that Γ is the folded $(2d+1)$ -cube. It has diameter d and for $1 \leq h \leq d-1$, the h -subconstituent of Γ with respect to any vertex has no edges (see [5, p. 264]), and consequently not connected. Gardiner, Godsil, Hensel and Royle [16] proved that the diameter of the second subconstituent of a primitive strongly-regular graph is at most three. It would be interesting to extend this result to distance-regular graphs with diameter $d \geq 3$. For example, if Γ is a distance-regular with $d = 3$, then what is the diameter of $\Gamma_3(\gamma)$ when $\Gamma_3(\gamma)$ is connected? Another related problem from [10] is to classify the distance-regular graphs Γ of diameter 3 such that $\Gamma_3(\gamma)$ is disconnected for some vertex γ . See [19] for related results.

The vertex-connectivity of a primitive distance-regular graph is equal to its valency, as proved by Brouwer and Mesner [8] for diameter $d = 2$, and by Brouwer and Koolen [7] for $d \geq 3$. Brouwer and Haemers [6, p. 127] observed that for certain strongly-regular graphs constructed by Haemers [17, p. 76] the vertex-connectivity of their second subconstituent is strictly less than the valency. It would be interesting to determine lower bounds for the vertex-connectivity and edge-connectivity of the subconstituents for a distance-regular

graph with $d \geq 3$. See [3,11–13,15,18] for related connectivity results concerning distance-regular graphs and association schemes.

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