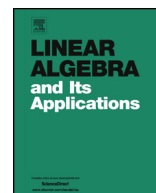Contents lists available at ScienceDirect

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On a question of Haemers regarding vectors in the nullspace of Seidel matrices

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A R T I C L E   I N F O

A B S T R A C T

In 2011, Haemers asked the following question:

If S is the Seidel matrix of a graph of order n and S is singular, does there exist an eigenvector of S corresponding to 0 which has only ± 1 elements?

In this paper, we construct infinite families of graphs which give a negative answer to this question. One of our constructions implies that for every natural number N, there exists a graph whose Seidel matrix S is singular such that for any integer vector in the nullspace of S, the absolute value of any entry in this vector is more than N. We also derive some characteristics of vectors in the nullspace of Seidel matrices, which lead to some necessary conditions for the singularity of Seidel matrices. Finally, we obtain some properties of the graphs which affirm the above question.

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1. Introduction

Throughout this paper all graphs are simple, undirected, and finite. Let G be a graph. The adjacency matrix of graph G with vertex set $V = \{v_1, \ldots, v_n\}$ is denoted by $A = [a_{ij}]_{1 \leq i,j \leq n}$, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Denote by J_n the all-ones $n \times n$ matrix and by I_n the identity matrix of order n. The *Seidel matrix* of G, denoted by S, is defined by $S = J_n - I_n - 2A$. There are close connections between Seidel matrices, equiangular lines and two-graphs [3,5,11].

It is known that $rank\, S \geq n - 1$, see [2, Lemma 3.3] or [6, p.6]. Assume $rank\, S = n - 1$. Because all elements of the Seidel matrix are integers, there exists an eigenvector corresponding to 0 with rational entries. Therefore, there exists an integral vector in the nullspace of S such that the greatest common divisor of its entries is 1, and since $null\, S = 1$, this vector is unique up to multiplying by -1. We choose that vector whose first non-zero entry is positive, and denote it by $\phi(G)$ or $\phi(S)$ or simply ϕ. The following problem on the nullspace of singular Seidel matrices was proposed by Haemers in 2011 (see [4, Problem 3.36]).

Problem 1. If S is the Seidel matrix of a graph of order n and $rank\, S = n-1$, does there exist an eigenvector of S corresponding to 0 which has only ± 1 elements?

This problem is equivalent to finding out whether $\phi(S)$ only has ± 1 entries or not.

In this paper, we construct infinite families of graphs which give a negative answer to Problem 1. Moreover, we show that for every natural number N, there exists a graph whose Seidel matrix is singular, and the absolute value of every entry of ϕ is more than N. Furthermore, we investigate some properties of ϕ, and using these properties, we obtain some necessary conditions for singularity of the Seidel matrix. Finally, we study the graphs having singular Seidel matrices with $\phi \in \{\pm 1\}^n$, and obtain some properties of such graphs.

2. Preliminaries

In this section, we introduce some notations, concepts, and results which are needed for the next sections.

By $a \overset{m}{\equiv} b$, we mean a and b are congruent modulo m. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. By *order* and *size* of G, we mean $|V(G)|$ and $|E(G)|$, respectively. The *open neighborhood* of a vertex v is denoted by $N_G(v)$ or $N(v)$, and is the set of all vertices adjacent to v. The *closed neighborhood* of v, which is $N(v) \cup \{v\}$, is denoted by $N_G[v]$ or $N[v]$. We denote the *degree* of a vertex v in a graph G by $d_G(v)$ or simply $d(v)$. A vertex v is called *even* if $d(v)$ is even, and *odd* if $d(v)$ is odd. A graph is called *even* if all vertices are even. The complement of G is denoted by $\overline{G}$. We denote the *path* and the *cycle* on n vertices by P_n and C_n, respectively.

By *switching* the graph G with respect to a vertex v, we mean deleting all the edges from v to its neighbors in G and making v adjacent to its non-neighbors in G. If G' is the graph obtained by switching with respect to v, then $N_{G'}(v) = V(G) \backslash N_G[v]$. Two graphs are called *switching-equivalent* if one can be transformed to the other one by a sequence of switchings. This is an equivalence relation on the set of all graphs, and therefore, it partitions the graphs into equivalence classes called *switching classes*. The switching class of a graph G is denoted by $[G]$.

For G and $A \subseteq V(G)$, switching with respect to the vertices in A in any order gives a unique graph G', which could also be obtained by applying switchings with respect to the vertices in $B = V(G) \backslash A$ in any order. In either case, G' is obtained from G by removing all edges between A and B, and adding all non-edges between A and B as edges. Therefore, applying some switchings to a graph G is equivalent to partitioning its vertex set into two parts and then complementing the edges between them. When doing so, the parity of the degree of a vertex changes if and only if the number of neighbors this vertex loses does not have the same parity as the number of neighbors it gains, which only happens if the part not including this vertex has an odd number of vertices. In a graph of odd order, there would be one part with odd and one with even cardinality; hence the parity of degrees of all the vertices in the part with even size changes, and the parity of degrees of the remaining vertices remains unchanged.

Remark 2.1. For any graph of odd order, switching with respect to the odd vertices transforms it into an even graph. Seidel [10, Theorem 3.5] proved that if G is a graph of odd order, then G contains a unique even graph in its switching class (also see [7, Theorem 3.17]).

Remark 2.2. Let G be a graph, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of S with $\alpha_1, \ldots, \alpha_n$ as their corresponding eigenvectors. If S' is the Seidel matrix of G after applying a switching to vertex v_k, then $S' = R_k S R_k$, where R_k is a diagonal matrix obtained by negating the kth main diagonal entry of the identity matrix. Since $R_k = R_k^{-1}$, we have $S' R_k = R_k S$, and therefore, $S' R_k \alpha_i = R_k S \alpha_i = \lambda_i R_k \alpha_i$. Hence, $\lambda_1, \ldots, \lambda_n$ and $\alpha'_1, \ldots, \alpha'_n$ are the eigenvalues and the corresponding eigenvectors of S', where $\alpha'_i = R_k \alpha_i$ is obtained by negating the kth entry of α_i. Therefore, applying switchings does not change the spectrum of the Seidel matrix, and only negates the entries of eigenvectors corresponding to the switched vertices. Thus, all switching-equivalent graphs have the same Seidel spectrum.

3. Some properties of $\phi(S)$ and necessary conditions for singularity of S

In this section, we study the properties of $\phi(S)$ and obtain some necessary conditions for the singularity of the Seidel matrix.

Let S be a singular Seidel matrix. Because $S\phi = 0$, we deduce that for every $1 \leq i \leq n$, the following holds:

$$\sum_{j:v_j \in N(v_i)} \phi_j - \sum_{\ell:v_\ell \notin N[v_i]} \phi_\ell = 0. \tag{1}$$

We use equation (1) to obtain some properties of ϕ.

Lemma 3.1. *Every entry of ϕ is odd.*

Proof. By equation (1), for $i = 1, \ldots, n$, the multiset $\{\phi_j \mid j \neq i\}$ can be partitioned into two multisets with equal sum. Hence, for $i = 1, \ldots, n$, $\sum_{j \neq i} \phi_j$ is even. Therefore, all the entries $\phi_1, \ldots, \phi_n$ have the same parity. Since the greatest common divisor of these entries is 1, each of them must be odd. □

Theorem 3.2. *Let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$, and singular Seidel matrix S. For $1 \leq i, j \leq n$ the following hold:*
(i) $\phi_i - \phi_j \overset{4}{\equiv} 2(d(v_i) - d(v_j))$,
(ii) If G is an even graph, then $\phi_i - \phi_j \overset{8}{\equiv} 2(d(v_i) - d(v_j))$.

Proof. (i) Let $M = \{a \mid v_a \notin N[v_i] \cup N[v_j]\}$, $N_{i \setminus j} = \{b \mid v_b \in N(v_i) \setminus N[v_j]\}$, $N_{j \setminus i} = \{c \mid v_c \in N(v_j) \setminus N[v_i]\}$, and $N_{ij} = \{d \mid v_d \in N(v_i) \cap N(v_j)\}$. Using equation (1) for vertices v_i and v_j, we get that

$$\pm \phi_j + \sum_{b \in N_{i \setminus j}} \phi_b + \sum_{d \in N_{ij}} \phi_d - \sum_{c \in N_{j \setminus i}} \phi_c - \sum_{a \in M} \phi_a = 0,$$

and

$$\pm \phi_i + \sum_{c \in N_{j \setminus i}} \phi_c + \sum_{d \in N_{ij}} \phi_d - \sum_{b \in N_{i \setminus j}} \phi_b - \sum_{a \in M} \phi_a = 0.$$

If v_i and v_j are adjacent, then both ϕ_i and ϕ_j have negative signs above. If v_i and v_j are not adjacent, then ϕ_i and ϕ_j have positive signs in the previous two equations. Either way, by subtracting these equations we get that

$$\pm(\phi_i - \phi_j) = 2\left(\sum_{b \in N_{i \setminus j}} \phi_b - \sum_{c \in N_{j \setminus i}} \phi_c\right). \tag{2}$$

From Lemma 3.1 we know that all the entries of ϕ are odd. Therefore, we obtain that $\phi_i - \phi_j \overset{4}{\equiv} \pm(\phi_i - \phi_j)$ and $\sum_{b \in N_{i \setminus j}} \phi_b - \sum_{c \in N_{j \setminus i}} \phi_c \overset{2}{\equiv} |N_{i \setminus j}| - |N_{j \setminus i}|$. So

$$\phi_i - \phi_j \overset{4}{\equiv} \pm(\phi_i - \phi_j) = 2\left(\sum_{b \in N_{i \setminus j}} \phi_b - \sum_{c \in N_{j \setminus i}} \phi_c\right) \overset{4}{\equiv} 2(|N_{i \setminus j}| - |N_{j \setminus i}|).$$

Since $d(v_i) - d(v_j) = |N_{i \setminus j}| - |N_{j \setminus i}|$, we obtain that

$$\phi_i - \phi_j \overset{4}{\equiv} 2(d(v_i) - d(v_j)).$$

(ii) Since G is even, by part (i), $\phi_1 \overset{4}{\equiv} \cdots \overset{4}{\equiv} \phi_n \overset{4}{\equiv} r$ for some $r \in \{0, 1, 2, 3\}$. Lemma 3.1 implies that $r \in \{1, 3\}$. Consequently,

$$\sum_{b \in N_{i \setminus j}} \phi_b - \sum_{c \in N_{j \setminus i}} \phi_c \overset{4}{\equiv} (|N_{i \setminus j}| - |N_{j \setminus i}|)r = (d(v_i) - d(v_j))r \overset{4}{\equiv} d(v_i) - d(v_j),$$

where the last one holds because $d(v_i) - d(v_j)$ is even and $r \in \{1, 3\}$. So by equation (2), we have that

$$\phi_i - \phi_j \overset{8}{\equiv} \pm(\phi_i - \phi_j) = 2 \left(\sum_{b \in N_{i \setminus j}} \phi_b - \sum_{c \in N_{j \setminus i}} \phi_c \right) \overset{8}{\equiv} 2(d(v_i) - d(v_j)),$$

where the first one is true because $\phi_i - \phi_j \overset{4}{\equiv} 0$. The proof is complete. □

Corollary 3.3. *If G is a graph of order n with singular Seidel matrix, then $n \overset{4}{\equiv} 1$.*

Proof. From Lemma 3.1 and equation (1), we conclude that for $i = 1, \ldots, n$, $d_G(v_i) - (n - 1 - d_G(v_i)) \overset{2}{\equiv} 0$, hence $n \overset{2}{\equiv} 1$. By Seidel's result mentioned in Remark 2.1, G is switching equivalent to an even graph G'. From Remark 2.2, we know that switching does not change the Seidel spectrum of a graph, so $S(G')$ is also singular. Let $\phi' = \phi(G')$. By Theorem 3.2, part (i), we conclude $\phi'_1 \overset{4}{\equiv} \cdots \overset{4}{\equiv} \phi'_n \overset{4}{\equiv} r$, where $r \in \{1, 3\}$. Using equation (1), we deduce that for any $i = 1, \ldots, n$,

$$d_{G'}(v_i)r \overset{4}{\equiv} \sum_{j \in N_{G'}(v_i)} \phi'_j \overset{4}{\equiv} \sum_{\ell \notin N_{G'}[v_i]} \phi'_\ell \overset{4}{\equiv} (n - 1 - d_{G'}(v_i))r.$$

Since $\gcd(r, 4) = 1$ and $d_{G'}(v_i)$ is even, we get that $n \overset{4}{\equiv} 1 + 2d_{G'}(v_i) \overset{4}{\equiv} 1$. □

Note that Corollary 3.3 can also be derived from [5, Corollary 3.6].

Lemma 3.4. *Let G be an even graph of order n and size m. If the Seidel matrix of G is singular, then*

$$m \overset{4}{\equiv} \frac{n - 1}{4}.$$

Proof. By Corollary 3.3, we have $n \overset{4}{\equiv} 1$. Since G is an even graph, by Theorem 3.2, Part (i) and Lemma 3.1, we conclude that $\phi_1 \overset{4}{\equiv} \cdots \overset{4}{\equiv} \phi_n \overset{4}{\equiv} r$, for some $r \in \{1, 3\}$. We define ϕ' as follows:

$$\phi' := \begin{cases} \phi & r = 1 \\ -\phi & r = 3. \end{cases}$$

So we have $\phi'_1 \overset{4}{\equiv} \cdots \overset{4}{\equiv} \phi'_n \overset{4}{\equiv} 1$. Also, since $\phi' \in \{-\phi, \phi\}$, equation (1) holds for ϕ'. If we add these equations for $i = 1, \ldots, n$, we find that,

$$\sum_{i=1}^{n} \left(\sum_{j: v_j \in N(v_i)} \phi'_j - \sum_{\ell: v_\ell \notin N[v_i]} \phi'_\ell \right) = 0.$$

Since for $i = 1, \ldots, n$, ϕ'_i appears $d(v_i)$ times with positive sign and $n - 1 - d(v_i)$ times with negative sign, we conclude that,

$$\sum_{i=1}^{n} d(v_i)\phi'_i - \sum_{i=1}^{n} (n - 1 - d(v_i))\phi'_i = 0,$$

thus

$$\sum_{i=1}^{n} d(v_i)\phi'_i = \frac{n-1}{2} \sum_{i=1}^{n} \phi'_i. \tag{3}$$

Because $\phi'_i \overset{4}{\equiv} 1$ and $d(v_i)$ is even for $i = 1, \ldots, n$, we deduce that $d(v_i)\phi'_i \overset{8}{\equiv} d(v_i)$. Hence, by equation (3),

$$\sum_{i=1}^{n} d(v_i) \overset{8}{\equiv} \frac{n-1}{2} \sum_{i=1}^{n} \phi'_i.$$

Therefore,

$$m = \frac{1}{2} \sum_{i=1}^{n} d(v_i) \overset{4}{\equiv} \frac{n-1}{4} \sum_{i=1}^{n} \phi'_i \overset{4}{\equiv} \frac{n-1}{4} n \overset{4}{\equiv} \frac{n-1}{4},$$

where the last one is true since $n \overset{4}{\equiv} 1$. The proof is complete.   □

Theorem 3.5. *Let G be a graph of order n and size m. If the Seidel matrix of G is singular, then the following holds:*

$$m + n_{odd} \overset{4}{\equiv} \frac{n-1}{4}, \tag{4}$$

where n_{odd} is the number of odd vertices of G.

Proof. Let $O(G)$ be the set of odd vertices of G. Suppose that G' is the graph obtained from G by switching with respect to the vertices in $O(G)$. Let m' be the size of G'. By

Remark 2.1, G' is an even graph, and by Remark 2.2, the Seidel matrix of G' is singular. Lemma 3.4 implies that $n \stackrel{4}{\equiv} 1$ and $m' \stackrel{4}{\equiv} \frac{n-1}{4}$. Let n_{even} be the number of even vertices of G and let $e_G(v)$ be the number of even vertices adjacent to v in G. Since G' is obtained from G by switching the edges and non-edges of G between $O(G)$ and $V(G)\backslash O(G)$, we have that

$$m' - m = \sum_{v : v \in O(G)} (n_{even} - e_G(v)) - \sum_{v : v \in O(G)} e_G(v) = \sum_{v : v \in O(G)} (n_{even} - 2e_G(v)).$$

Thus,

$$m' = m + n_{odd}n_{even} - 2\sum_{v : v \in O(G)} e_G(v) = m + n_{odd}(n - n_{odd}) - 2m_{oe},$$

where m_{oe} is the number of edges of G between $O(G)$ and $V(G)\backslash O(G)$. The Handshaking lemma implies that n_{odd} is even. Also, by adding up the degrees of the vertices in $O(G)$ or in its complement, we deduce that m_{oe} is also even. Therefore,

$$m' = m + n_{odd}n - n_{odd}^2 - 2m_{oe} \stackrel{4}{\equiv} m + n_{odd}n \stackrel{4}{\equiv} m + n_{odd},$$

and the proof is complete. □

Now, we have an immediate corollary.

Corollary 3.6. *Let G be a graph of order n and size m. If the Seidel matrix of G is singular, then $m \stackrel{2}{\equiv} \frac{n-1}{4}$.*

It is worth mentioning that Corollary 3.6 is weaker than Theorem 3.5. For example, for each $k \geq 0$, non-singularity of the Seidel matrix of P_{16k+1} can be deduced from Theorem 3.5 but cannot be derived from Corollary 3.6. Note that Corollary 3.6 can also be concluded from [5, Theorem 3.5].

Corollary 3.6 implies that if G is a tree with singular Seidel matrix, then $n \stackrel{8}{\equiv} 1$. By a computer search, we noted that there is no tree of order 9, whose Seidel matrix is singular. Among 48629 non-isomorphic trees of order 17 (see [8]), there are 15 trees with singular Seidel matrix. Also, we checked that Problem 1 has an affirmative answer for only 2 of these trees. These two trees can be seen in Fig. 1. In every labeling of these trees that v_1 is an even vertex, $\phi_i = (-1)^{d(v_i)}$, for $i = 1, \ldots, 17$. Our program, along with the other 13 trees with singular Seidel matrices, are available in GitHub [1].

4. Graph families with unbounded entries of ϕ

Let G be a graph of order n with a singular Seidel matrix. By Corollary 3.3, we know that $n = 4k+1$, for some $k \geq 0$. By a computer search, we verified that for every graph of
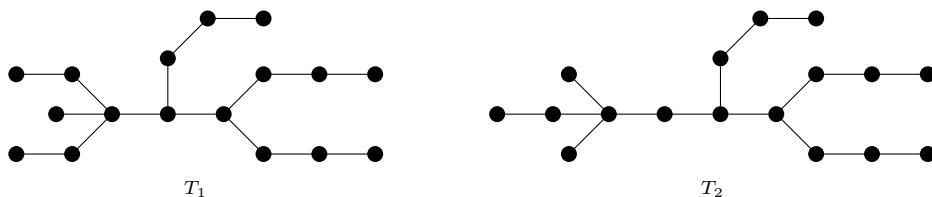
Fig. 1. The only two trees of order 17 with singular Seidel matrices, where $\phi \in \{\pm 1\}^{17}$.
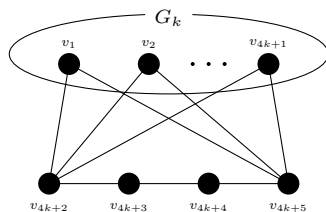

Fig. 2. Structure of Graph G_{k+1}.

order 5, Problem 1 has an affirmative answer. However, the following theorem shows that for $k \geq 2$, there is a graph of order $4k+1$, which gives a negative answer to Problem 1. The presented family of graphs also shows the unboundedness of entries of ϕ. Furthermore, in Theorem 4.3, we show that there is no upper bound for the minimum absolute value of the entries of ϕ either. Thus, for every positive integer N, there exists a graph whose Seidel matrix is singular, and the absolute value of every entry of ϕ is at least N.

Theorem 4.1. *For every $k \geq 2$, there exists a graph of order $4k + 1$ with singular Seidel matrix, which gives a negative answer to Problem 1.*

Proof. For $k \geq 1$, we construct a graph G_k of order $4k+1$ with singular Seidel matrix for which the maximum entry of ϕ is 5^{k-1}. Hence, G_k gives a negative answer to Problem 1 for $k \geq 2$. The construction is recursive. Let $G_1 = C_5$, so $S(G_1) = 2J_5 - I_5 - A(C_5)$ and $\phi(G_1) = j_5$, the all-ones vector of dimension 5. The structure of G_{k+1} in terms of G_k is represented in Fig. 2. Also, the structures of $S(G_{k+1})$ and $\phi(G_{k+1})$ in terms of $S(G_k)$ and $\phi(G_k)$ are as follows:

$$
\underbrace{\left[
\begin{array}{cccc|cccc}
 & & & & -1 & +1 & +1 & -1 \\
 & & & & -1 & +1 & +1 & -1 \\
 & \multicolumn{2}{c}{S(G_k)} & & \vdots & \vdots & \vdots & \vdots \\
 & & & & -1 & +1 & +1 & -1 \\
\hline
-1 & -1 & \cdots & -1 & 0 & -1 & +1 & +1 \\
+1 & +1 & \cdots & +1 & -1 & 0 & -1 & +1 \\
+1 & +1 & \cdots & +1 & +1 & -1 & 0 & -1 \\
-1 & -1 & \cdots & -1 & +1 & +1 & -1 & 0
\end{array}
\right]}_{S(G_{k+1})}
\underbrace{\left[
\begin{array}{c}
\phi(G_k) \\
\hline
c_k \\
c_k \\
c_k \\
c_k
\end{array}
\right]}_{\phi(G_{k+1})}
=
\left[
\begin{array}{c}
0 \\
0 \\
\vdots \\
0
\end{array}
\right],
$$

where c_k is the sum of the entries of $\phi(G_k)$. Clearly, the greatest common divisor of entries of $[\,\phi(G_k)\,|\,c_k\ c_k\ c_k\ c_k\,]^T$ is 1 and its first entry is positive; thus, it is consistent with the definition of ϕ. It can be seen that $c_{k+1} = 5c_k$, and since $c_1 = 5$, we have $c_{k+1} = 5^{k+1}$. Furthermore, it is clear that the maximum entry of $\phi(G_{k+1})$ is c_k, and the proof is complete. $\square$

Remark 4.2. In the graphs G_k constructed in the proof of Theorem 4.1, while the minimum absolute value of the entries of ϕ is 1, the maximum absolute value of its entries tends to infinity as k tends to infinity.

Theorem 4.3. *There is no constant upper bound for the minimum absolute value of the entries of ϕ.*

Proof. By induction on $k \geq 0$, we construct a graph H_k of order $8k + 5$ with singular Seidel matrix such that the minimum absolute value of the entries of $\phi(H_k)$ is 3^k, and the sum of its entries is 5×7^k. Thus, for any natural number N, there is k such that the minimum absolute value of entries of the vector ϕ of H_k is greater than N.

Let $H_0 = C_5$. As before, $\phi(H_0) = j_5$, where j_5 is the all-ones vector of order 5, so the minimum absolute value of the entries of $\phi(H_0)$ is 1, and the sum of its entries is 5. The structure of graph H_{k+1} in terms of H_k is shown in Fig. 3.

Also, the structures of $S(H_{k+1})$ and $\phi(H_{k+1})$ in terms of $S(H_k)$ and $\phi(H_k)$ are as follows:

$$
\underbrace{\left[
\begin{array}{cccc|cccccccc}
 & & & & +1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 \\
 & \multicolumn{3}{c}{S(H_k)} & +1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 \\
 & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & +1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 \\
\hline
+1 & +1 & \cdots & +1 & 0 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \\
+1 & +1 & \cdots & +1 & +1 & 0 & +1 & +1 & +1 & +1 & -1 & -1 \\
+1 & +1 & \cdots & +1 & +1 & +1 & 0 & +1 & +1 & +1 & +1 & -1 \\
-1 & -1 & \cdots & -1 & +1 & +1 & +1 & 0 & +1 & +1 & +1 & +1 \\
-1 & -1 & \cdots & -1 & +1 & +1 & +1 & +1 & 0 & +1 & +1 & +1 \\
+1 & +1 & \cdots & +1 & -1 & +1 & +1 & +1 & +1 & 0 & +1 & +1 \\
+1 & +1 & \cdots & +1 & -1 & -1 & +1 & +1 & +1 & +1 & 0 & +1 \\
+1 & +1 & \cdots & +1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & 0 \\
\end{array}
\right]}_{S(H_{k+1})}
\underbrace{\left[
\begin{array}{c}
3\phi(H_k) \\
\hline
5c_k \\
-c_k \\
-3c_k \\
c_k \\
c_k \\
-3c_k \\
-c_k \\
5c_k \\
\end{array}
\right]}_{\phi(H_{k+1})}
=
\left[
\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
\end{array}
\right],
$$

where c_k is the sum of the entries of $\phi(H_k)$. Note that $c_k = 5 \times 7^k$, so the greatest common divisor of entries of $[\,3\phi(H_k)\,|\,5c_k\ -c_k\ -3c_k\ c_k\ c_k\ -3c_k\ -c_k\ 5c_k\,]^T$ is $gcd(3, c_k) = 1$ and its first entry is positive. Hence, it is consistent with the definition of ϕ. The structure of $\phi(H_{k+1})$ indicates that $c_{k+1} = 7c_k$, thus $c_{k+1} = 5 \times 7^{k+1}$. Furthermore, the minimum absolute value of entries of $\phi(H_{k+1})$ is equal to the minimum absolute value of $3\phi(H_k)$, which is 3^{k+1}, and it tends to infinity as k tends to infinity. $\square$
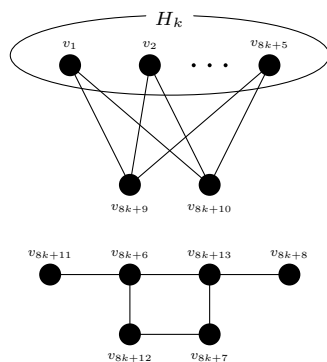
Fig. 3. Structure of Graph H_{k+1}.

5. Properties of graphs with $\phi(S) \in \{\pm 1\}^n$

In this section, we study the graphs G which give an affirmative answer to the Problem 1. This means that $\phi(S) \in \{\pm 1\}^n$. By Corollary 3.3, we know that if G is a graph of order n with singular Seidel matrix, then $n = 4k + 1$, for some non-negative integer k. Hence, we only consider graphs of order $n = 4k + 1$. Our first result gives a characterization of the switching class of such graphs.

Theorem 5.1. *Let G be a graph of order $n = 4k+1$. The Seidel matrix S of G is singular and $\phi(S) \in \{\pm 1\}^n$ if and only if G belongs to the switching class of a $2k$-regular graph.*

Proof. Suppose that G is a graph with Seidel matrix S such that $\phi(S) \in \{\pm 1\}^n$. We define $L := \{\, v_i \,|\, (\phi(S))_i = -1 \,\}$. By switching G with respect to L, we obtain a graph G', with S' and A' as its Seidel matrix and adjacency matrix, respectively. By Remark 2.2, S' is singular and $\phi(S') = j_n$, where j_n is the all-ones vector of dimension n. Therefore,

$$0 = S' j_n = (J_n - I_n - 2A') j_n = (4k + 1) j_n - j_n - 2A' j_n.$$

Thus, $A' j_n = 2k j_n$, and so G' is a $2k$-regular graph.

To prove the other direction, suppose that H is a $2k$-regular graph in the switching class of G. We have that

$$S(H) j_n = (J_n - I_n - 2A(H)) j_n = (4k + 1) j_n - j_n - 2(2k) j_n = 0.$$

So j_n is an eigenvector of $S(H)$ corresponding to 0, and therefore $\phi(H) = j_n$. The graph G is obtained from H by some switching. Since switching with respect to any subset of vertices only negates the corresponding entries of its Seidel eigenvectors, it follows that $\phi(S) \in \{\pm 1\}^n$, which finishes the proof. $\square$

Theorem 5.1 provides a way for constructing graphs with singular Seidel matrices, which are rare in general. It is shown in [9] that the proportion of graphs of order n with singular Seidel matrices tends to zero as n tends to infinity.

The next results present some structural properties of graphs with $\phi(S) \in \{\pm 1\}^n$.

Lemma 5.2. *Let G be a graph of order $n = 4k + 1$ with singular Seidel matrix S, and $\phi(S) \in \{\pm 1\}^n$. If G has a leaf, then G has exactly $2k$ or $2k + 2$ odd vertices.*

Proof. By Theorem 5.1, there exists a $2k$-regular graph H such that G is obtained from H by some switching. Let v be a leaf of G, and u be its neighbor in G. There are two cases to consider.

Case 1: The vertices u and v are adjacent in H. In this case, H is transformed into G by switching with respect to $A = N_H(v) \backslash \{u\}$ or $B = V \backslash A$. Because $|A| = 2k - 1 \overset{2}{\equiv} 1$ and $|B| = 2k + 2 \overset{2}{\equiv} 0$, complementing the edges between A and B changes the parity of the degree of vertices in B. Therefore, in G, the vertices in B are odd and the vertices in A are even. So G has $2k + 2$ odd vertices.

Case 2: The vertices u and v are not adjacent in H. In this case, H is transformed into G by switching with respect to $A = N_H(v) \cup \{u\}$ or $B = V \backslash A$. In this case, $|A| = 2k + 1 \overset{2}{\equiv} 1$ and $|B| = 2k \overset{2}{\equiv} 0$. Thus, G has $|B| = 2k$ odd vertices.   □

Corollary 5.3. *If G is a tree of order $n = 16k + r$ $(0 \leq r \leq 15)$ with a singular Seidel matrix, then G has exactly $8k + s$ odd vertices, where (r, s) is either $(1, 0)$ or $(9, 6)$.*

Proof. Let n_{odd} be the number of odd vertices of G. By Theorem 3.5 and Corollary 3.3, we conclude that $n_{odd} \overset{4}{\equiv} \frac{n-1}{4}$. By Lemma 5.2, n_{odd} is either $\frac{n-1}{2}$ or $\frac{n+3}{2}$. Thus, $n_{odd} = \frac{n-1}{2}$, if $\frac{n-1}{2} \overset{4}{\equiv} \frac{n-1}{4}$, which means $n \overset{16}{\equiv} 1$, and $n_{odd} = \frac{n+3}{2}$, if $\frac{n+3}{2} \overset{4}{\equiv} \frac{n-1}{4}$, which means $n \overset{16}{\equiv} 9$.   □

Theorem 5.4. *Let G be a graph of order $n = 4k + 1$. If the Seidel matrix S of G is singular, and $\phi(S) \in \{\pm 1\}^n$, then $3k \leq |E(G)| \leq 8k^2 - k$, and both given bounds are tight.*

Proof. Let $\delta(G)$ denote the minimum degree of G. If $\delta(G) \geq 2$, then $|E(G)| \geq 4k+1 > 3k$. Suppose that $\delta(G) < 2$, and let v be a vertex of minimum degree. By Theorem 5.1, there exists a $2k$-regular graph H such that G is obtained from H by some switching. Let V be the vertex set of G and H. There are two possibilities for $d_G(v)$:

- $d_G(v) = 0$. Let α be the number of edges in H with endpoints in $N_H(v)$, β be the number of edges in H with endpoints in $V \backslash N_H[v]$, and γ be the number of edges in H with one endpoint in $N_H(v)$ and the other one in $V \backslash N_H[v]$. The $2k$ edges adjacent to v are the only edges of H that are not counted in neither α, β nor γ. Thus, we have
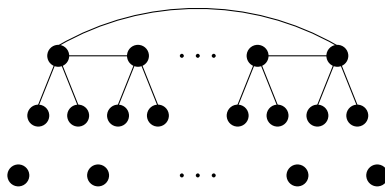
Fig. 4. Graph H_k.

$$\alpha + \beta + \gamma = \frac{2k(4k+1)}{2} - 2k = 4k^2 - k. \tag{5}$$

Every vertex u in $N_H(v)$ has at most $2k-1$ neighbors in $V \backslash N_H[v]$, so $\gamma \leq (2k)(2k-1) = 4k^2 - 2k$. Thus, we have

$$4k^2 - 2\gamma \geq 4k^2 - 2(4k^2 - 2k) = 4k - 4k^2. \tag{6}$$

There exists a set $A \subseteq V$ such that switching with respect to A transforms H into G. This switching makes v an isolated vertex. Therefore, A is equal to either $N_H(v)$ or $V \backslash N_H(v)$. By applying this switching, the edges with endpoints in $N_H(v)$ and the edges with endpoints in $V \backslash N_H[v]$ remain as they were, and the edges and non-edges between the two sets are switched. Since $|N_H(v)| = |V \backslash N_H[v]| = 2k$, by equations (5) and (6), we get that

$$|E(G)| = \alpha + \beta + 4k^2 - \gamma = (\alpha + \beta + \gamma) + (4k^2 - 2\gamma) \geq 4k^2 - k + 4k - 4k^2 = 3k.$$

- $d_G(v) = 1$. By Lemma 5.2, there are at most $2k + 2$ odd and at least $2k - 1$ even vertices in G. Since there is no vertex of degree 0, the degree of any even vertex in G is at least 2. As a result,

$$|E(G)| = \frac{1}{2} \sum_{v \in V} d_G(v) \geq \frac{1}{2}(1(2k+2) + 2(2k-1)) = 3k.$$

Since $S(\overline{G}) = -S(G)$, we have $\phi(\overline{G}) = \phi(G)$, so by the first inequality for graph $\overline{G}$, $|E(\overline{G})| \geq 3k$, and therefore, $|E(G)| \leq 8k^2 - k$.

In the sequel, we show that the given lower bound and upper bound are tight. Let k be a natural number and G_k be the disjoint union of k copies of P_4 and one isolated vertex. Note that the order and size of G_k are $4k + 1$ and $3k$, respectively. By switching with respect to the $2k$ leaves of G_k, we obtain a $2k$-regular graph of order $4k + 1$. By Theorem 5.1, the Seidel matrix of G_k is singular, and $\phi(G_k) \in \{\pm 1\}^n$. The tightness of the upper bound can be shown by considering $\overline{G_k}$. $\square$

Remark 5.5. For any natural number $k \geq 3$, we construct a graph H_k of order $4k + 1$ and size $3k$, shown in Fig. 4, containing one C_k, $2k$ leaves, and $k + 1$ isolated vertices. It

can be checked that switching the leaves transforms this graph into a $2k$-regular graph, and so by Theorem 5.1, $\phi(H_k) \in \{\pm1\}^n$.

The structure given in Remark 5.5 indicates that there are some graphs with $\phi \in \{\pm1\}^n$ and minimum possible size, which are not a forest. Also, it shows that for any $k \geq 3$, there is a graph containing C_k, which has an affirmative answer to the Problem 1.

Declaration of competing interest

There is no competing interest.

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