

Direct Model Reference Adaptive Control for Tracking Contracting Nonlinear Systems

Nak-seung Patrick Hyun^{*1}, Mark Petersen^{*2}, and Robert J. Wood³

Abstract—In this paper, a new direct model reference adaptive control (DMRAC) for nonlinear control affine systems is proposed to track a contracting nonlinear reference model. The structure of the uncertainty is assumed, including a multiplicative unknown parameter on the input. The contracting reference model enables the use of the certainty equivalence principle (CEP) for designing the simple controller, a tool which often cannot be used for nonlinear systems due to its weakness on these systems. The proposed controller is constructed to minimize the distance from the estimated plant vector field to the desired contracting vector field, which leverages the benefits of the contracting reference model. Even though the proposed controller does not perfectly match the true contracting vector field, the combined efforts of the adaptive controller and the adaptation law make the Riemannian distance asymptotically approach zero with bounded parameter error. In addition, the proposed algorithm is extended to the case of using a control contraction metric (CCM) based controller to construct the contracting reference model. A fully actuated pendulum example is studied to illustrate the effectiveness of the algorithm.

I. INTRODUCTION

In model reference adaptive control (MRAC) theory, there are two types of controllers: the direct adaptive controller and the indirect adaptive control. The direct MRAC does not necessarily need to learn the unknown parameters, but rather updates the controller gains directly. On the other hand, the indirect MRAC estimates the unknown parameters first, and then uses these estimates to design the controller. The certainty equivalence principle (CEP) – in which we first design the nominal controller assuming the parameters are known, and second, replace the true parameters in the nominal controller with the estimated parameters – is widely used for linear systems. However, [1](Chap 5) demonstrates the weakness of certainty equivalence, showing that if the system is nonlinear, asymptotic convergence between the system response and the model response is not guaranteed and the system can even escape to infinity in finite time. Therefore, without constructing a Lyapunov function for the CEP-based controller with some adaptation law, CEP-based controller designs have been rarely used in nonlinear adaptive control theory. In this paper, a certain class of nonlinear MRAC systems, where the reference model is contracting, are considered. This results in a simple CEP-based direct MRAC that ensures that the plant response will

asymptotically converge to the model response with bounded parameter error. Further historical discussion of linear and nonlinear adaptive control can be found in [2], [3].

In contrast to Lyapunov-based systems theory, contraction-based analysis for nonlinear systems, introduced in [4], focuses on the behavior between the trajectories of the system rather than the system behavior near the equilibrium. If the differential dynamics (or the first variational dynamics) admit a differential coordinate transform that satisfies some time-varying linear matrix inequality for all time in a so called contracting region, then every trajectory within the contracting region converge to each other. The analysis in [4] lays out the general concept of the contraction metric in nonlinear systems. More constructive ways to design a state feedback controller to make the closed-loop system contracting are considered in [5] by using a control contraction metric (CCM), and also in [6] by enforcing a skew-symmetric structure on the differential dynamics. In addition, a CCM for controlling mechanical systems was studied in [7]. In this paper, a direct MRAC problem that exploits the contracting vector field of the reference model to design the adaptive controller is considered.

The connection between contraction theory and adaptive control in general (although not to MRAC for nonlinear systems to our knowledge) have been made by many researchers in [6], [8], [9] and related literature. The main focus of [6], [9] lies in first designing the nominal controller that makes the system contracting, and proposes an added adaptive controller to cancel out the matched or extended matching disturbance. A triangular nonlinear system is assumed in [6] and the backstepping method is used to generate the nominal controller. A rather general class of nonlinear systems is considered in [9] which includes systems that cannot be feedback linearized. However, none of these works consider uncertainty in the multiplicative actuation vector gain (also known as high frequency gain in linear systems), which may cause the closed-loop system with the nominal controller to not satisfy the contracting condition. In contrast, the uncertainty in the multiplicative gain is considered in this paper, and the proposed adaptive controller is designed by applying the simple certainty equivalence principle to match the vector field of the plant to the desired contracting model (which can be generated using the CCM controller [5]). The challenge with the uncertainty in the multiplicative gain is resolved by not forcing the plant dynamics to be exactly contracting but rather driving the plant dynamics along the projection onto the desired contracting vector field.

Contents. The basic contraction theory and control con-

1. All author with the John A. Paulson School of Engineering and Applied Science, Harvard University, Cambridge, MA, USA. * These authors contributed equally to this work. Email: 1. nphyun@seas.harvard.edu, 2. markpetersen@g.harvard.edu, 3. rjwood@seas.harvard.edu. This work was funded by the National Science Foundation grant IIS-1724197 and partially funded by the Department of Defense (DoD) through the National Defense Science & Engineering Graduate (NDSEG) Fellowship Program.

traction metric (CCM) based controller are summarized in (§II), and a motivating example is introduced in (§III). The direct adaptive controller for tracking the contracting system is proposed and analyzed in (§IV), and the application to CCM-based controllers with unknown actuation gain is considered in (§V), with a fully actuated toy example to illustrate the effectiveness of the proposed controller.

II. CONTRACTING NONLINEAR MODEL

Consider the following reference model for an adaptive control problem,

$$\dot{x}(t) = f(x(t)) + g(x(t))r(t) \quad (1)$$

where $f(x), g(x) \in \mathbb{R}^n$ are smooth vector fields evaluated at $x \in \mathbb{R}^n$, and $r(t) \in \mathbb{R}$ is a smooth function. One can consider (1) a closed-loop system controlled by some state feedback controller. Let $f_i(x)$ and $g_i(x)$ denotes the i -th component of each vector field f, g .

Definition 1 (Lohmiller and Slotine [4]). *A region of state space is called a contracting region with respect to a uniformly positive definite metric, $M(x, t) \in \mathbb{R}^{n \times n}$, if*

$$A(x, t)^T M + M A(x, t) + \dot{M} \leq -2\lambda M \quad (2)$$

holds in that region for all $t > 0$ where $\gamma > 0$ and

$$\begin{aligned} A(x, t) &:= \frac{\partial f(x)}{\partial x} + \frac{\partial g(x)}{\partial x} r(t) \\ \dot{M} &:= \frac{\partial M(x, t)}{\partial t} + \sum_{i=1}^n \frac{\partial M(x, t)}{\partial x_i} (f_i(x) + g_i(x)r(t)) \end{aligned}$$

Also, the dynamics (1) is denoted as contracting nonlinear system with respect to $M(x, t)$.

The fundamental idea of the contraction analysis for nonlinear systems in [4] (Theorem 2) is that if any two trajectories stay within the contracting region with respect to $M(x, t)$ for all time, the two trajectories will converge exponentially towards each other. Furthermore, if the metric $M(x, t)$ satisfying (2) is uniformly bounded, there exist $\alpha_u \geq \alpha_l > 0$ such that $\alpha_l I \leq M(x, t) \leq \alpha_u I$, then an incremental Lyapunov function can be constructed. The following subsections summarize the necessary components of Riemannian geometry to construct an incremental Lyapunov function. The preliminary overview follows the descriptions in [5] and [9].

A. Riemannian distance based contraction analysis

Suppose that the state space $\mathcal{X}$ of (1) is in $\mathbb{R}^n$. Then the tangent space, $\mathcal{T}_x \mathcal{X}$, is also identified with $\mathbb{R}^n$ and the Riemannian metric $M(x, t)$ defines an inner product $\langle \nu_1, \nu_2 \rangle_{M(x, t)} := \nu_1^T M(x, t) \nu_2$ on the tangent space $\mathcal{T}_x \mathcal{X}$ where $\nu_1, \nu_2 \in \mathcal{T}_x \mathcal{X}$. Let $c(s, t) : [0, 1] \times \mathbb{R}^+ \rightarrow \mathcal{X}$ be a smooth curve connecting two points, $x, y \in \mathcal{X}$ at time $t \in \mathbb{R}^+$, then the Riemannian length and the energy

functional of the curve c are defined as

$$L(c(s, t), t) := \int_0^1 \|c_s(s, t)\|_{M(x, t)} ds \quad (3)$$

$$E(c(s, t), t) := \int_0^1 \|c_s(s, t)\|_{M(x, t)}^2 ds \quad (4)$$

respectively, where $c_s(s, t) := \partial c(s, t) / \partial s$ and $\|\nu\|_{M(x, t)} := \sqrt{\langle \nu, \nu \rangle_{M(x, t)}}$. Suppose that $\mathcal{G}(x, y)$ is a set of regular curves connecting x and y in $\mathcal{X}$, then, by Hopf-Rinow theorem with some mild assumptions, there exists a geodesic curve $\gamma : [0, 1] \rightarrow \mathcal{X}$ such that $\gamma(0) = x$ and $\gamma(1) = y$ where $L(\gamma(\cdot, t), t) \leq L(c(\cdot, t), t)$ for any $c(\cdot, t) \in \mathcal{G}(x, y)$. Also it follows that $E(\gamma(\cdot, t), t) = L(\gamma(\cdot, t), t)^2$. By considering the first variation of energy functional with respect to time gives

$$\dot{E}(\gamma(\cdot, t), t) = 2 \left(\frac{\partial E(\gamma(\cdot, t), t)}{\partial t} + \langle \gamma_s(s, t), \dot{\gamma}(s, t) \rangle_M \Big|_{s=0}^{s=1} \right). \quad (5)$$

B. Control Contraction Metric based control

As mentioned in the beginning of this section, the contracting system (1) can be obtained by a state feedback controller. Particularly, designing a feedback controller that tracks a desired trajectory is often valuable.

Now consider a general nonlinear control affine system,

$$\dot{x} = f(x) + g(x)u, \quad (6)$$

which is not necessarily contracting. Let $(x_d(t), u_d(t))$ be the desired feasible trajectory with an open loop input. The control contraction metric (CCM) provides a tractable way to construct a smooth feedback controller, as shown in [5], which guarantees that the closed loop system is contracting towards the desired trajectory, $x_d(t)$.

A simple CCM controller, as in [5](Proposition 2), can be formulated as

$$u_{ccm} := u_d(t) - \frac{1}{2} \int_0^1 K(\gamma(s, t)) \gamma_s(s, t) ds \quad (7)$$

$$K(\gamma(s, t)) := \rho(\gamma(s, t), t) g(\gamma(s, t))^T M(\gamma(s, t), t), \quad (8)$$

where $\gamma(s, t)$ is a geodesic curve with respect to the Riemannian metric $M(x, t)$ that connects $\gamma(1, t) = x(t)$ and $\gamma(0, t) = x_d(t)$. The metric satisfies the conditions,

$$\dot{M} + \frac{\partial f^T}{\partial x} M + M \frac{\partial f}{\partial x} - \rho M g g^T M < -2\lambda M \quad (9)$$

$$\sum_{i=1}^n \frac{\partial M}{\partial x_i} g_i + \frac{\partial g^T}{\partial x} M + M \frac{\partial g}{\partial x} = 0_{n \times n}, \quad (10)$$

for some function $\rho(x, t)$ and $\lambda > 0$ over all x and t where

$$\dot{M} := \frac{\partial M(x, t)}{\partial t} + \sum_{i=1}^n \frac{\partial M(x, t)}{\partial x_i} f_i(x).$$

Observe that the CCM controller in (7) requires knowledge of ρ, M , and the geodesics γ for each x and t . In general, M and ρ are obtained offline by converting (9-10) into a dual convexified problem, and the geodesics are then solved online as explained in [10].

Remark 1 As mentioned in [5], if (9) holds for some $M(x, t)$ and $\rho_0(x, t)$ pair, then any $\rho \geq \rho_0(x, t)$ satisfies (9) for the same $M(x, t)$. On the other hand, if $\rho < \rho_0(x, t)$, then ρ and $M(x, t)$ are not guaranteed to satisfy (9). This can prevent convergence when there is a multiplicative uncertainty on the actuation vector, g . This will be revisited and solved in § V.

III. A MOTIVATING EXAMPLE

Consider a simple model reference adaptive control (MRAC) problem with scalar linear plant dynamics,

$$\dot{x}_p = a_p x_p + b_p u, \quad (11)$$

where a_p and b_p are both unknown and only the sign of b_p is known, and the desired reference model to follow is some scalar nonlinear system given by

$$\dot{x}_m = f(x_m) + g(x_m)r, \quad (12)$$

which is contracting (admitting some uniformly bounded positive definite metric $M(x, t)$ and satisfying (2)). Now, consider a nominal controller using the *certainty equivalence principle*, which indicates that if a_p and b_p are known,

$$u := -\alpha^* x_p + \beta^* (f(x_p) + g(x_p)r) \quad (13)$$

$$\alpha^* := \frac{a_p}{b_p}, \beta^* := \frac{1}{b_p}. \quad (14)$$

then the nominal closed loop system is given by

$$\dot{x}_p = f(x_p) + g(x_p)r.$$

Observe that x_m and x_p are two particular solutions of contracting system defined in (12). If the domain of the contracting region is large enough to contain both x_m and x_p for all t (which is the objective of solving (9-10) for all x), then the Riemannian distance, $d(x_m, x_p)$, will exponentially decrease to zero. Therefore, there exist two constant gains, α^* and β^* , in (13), which guarantee the asymptotic convergence between the plant trajectory to the desired model trajectory. This gives an opportunity to design a CEP-based direct adaptive controller which adaptively updates two gains in (13). A further generalization is considered in the following sections.

IV. DIRECT MRAC WITH CONTRACTING NONLINEAR MODEL

In this section, a direct model reference adaptive controller for tracking the contracting nonlinear systems is considered for the n -dimensional nonlinear plant model.

A. Problem statement

Consider the following nonlinear plant dynamics,

$$\dot{x}_p = f(x_p) + \sum_{i=1}^N f_i(x_p) a_i + g_p(x_p) b_p u \quad (15)$$

where $f(x)$, $\{f_i(x)\}_{i=1}^N$, $g_p(x) \in \mathbb{R}^n$ for $x \in \mathbb{R}^n$ are known smooth vector fields, $\{a_i\}_{i=1}^N$ and b_p are unknown constant parameters with $N \geq 0$, u is the control input, and

$\|g_p(x)\|^2 > c$ for all x and some $c > 0$. The objective is to design a direct adaptive controller that follows the contracting nonlinear system given as

$$\dot{x}_m = f(x_m) + g(x_m)r, \quad (16)$$

where $x_m r$, and $\dot{r}$ are assumed to be bounded.

1) Certainty Equivalence Assumptions: The following condition holds for the uncertainties in (15). Let $F(x) := [f_1(x), \dots, f_N(x)] \in \mathbb{R}^{n \times N}$, and $a_p = [a_1; \dots; a_N] \in \mathbb{R}^N$ be the column vector.

Assumption 1 (matched uncertainty with respect to the contracting model). *The vector fields $\{f_i(x)\}_{i=1}^N$, $g_p(x)$ and the parameter uncertainty $\{a_i\}_{i=1}^N$ satisfy*

$$g(x)r - F(x)a_p \in \text{Span}\{g_p(x)\}. \quad (17)$$

The above assumption guarantees that if a_p and b_p were known, then the following nominal controller

$$u^* = k^* g_p(x_p)^\dagger (g(x_p)r - F(x_p)\theta^*) \quad (18)$$

$$k^* = 1/b_p, \quad \theta^* = a_p, \quad (19)$$

where $g(x_p)^\dagger = (g_p(x_p)^T g_p(x_p))^{-1} g_p(x_p)^T$ is the pseudo inverse of $g(x_p)$, makes the closed loop system become

$$\dot{x}_p = f(x_p) + g(x_p)r,$$

which fulfills our objective as in the scalar case in § III.

Remark 2 Note that if the actuation vector field of the model, $g(x_p)$, aligns with $g_p(x_p)$, then $F(x_p)a$ is also in $\text{Span}\{g_p(x_p)\}$, which is the case in [6], [9]. Therefore, Assumption 1 includes not only the simple matched uncertainty $F(x_p)a \in \text{Span}\{g_p(x_p)\}$ but allows a mismatch with $g_p(x_p)$ as long as (17) holds.

2) Contracting model assumptions: In this paper, the Riemannian metric is assumed to be state independent for the simplicity of the analysis. An existence of such state independent Riemannian metric is shown in [7] that any fully actuated mechanical system admits a constant Riemannian metric satisfying the CCM condition (9).

Assumption 2 (State independent Riemannian metric). *The reference model in (16) is contracting with respect to the state independent metric $M(t)$ which satisfies the contraction condition in (2).*

A further generalization with a state dependent Riemannian metric is left as future work and a possible extension is discussed in § VI.

B. Proposed controller

Given the nominal controller (18), the proposed adaptive feedback controller is given as

$$u(x_p, \hat{\theta}, \hat{k}) = \hat{k} g_p(x_p)^\dagger (g(x_p)r - F(x_p)\hat{\theta}) \quad (20)$$

$$\dot{\hat{\theta}} = -\Gamma_\theta F(x_p)^T \text{Proj}(g_p(x_p), M(t)e) \quad (21)$$

$$\dot{\hat{k}} = \Gamma_\phi \hat{k}^2 g_p(x_p)^T e u \quad (22)$$

where $Proj(\nu_1, \nu_2)$ is a projection of ν_2 on to $Span\{\nu_1\}$ defined as

$$Proj(\nu_1, \nu_2) = \frac{\nu_1 \nu_1^T}{\|\nu_1\|^2} \nu_2,$$

$\Gamma_\theta \in \mathbb{R}^{N \times N}$ is a diagonal matrix with all positive constant adaptive gains for θ , and $\Gamma_\phi > 0$ represents adaptive gain for k .

Lemma 1. Suppose that $\hat{\theta}(t)$ and $\hat{k}(t)$ are the current estimate of a_p and $1/b_p$, and let $\hat{x}_p(\hat{\theta}(t), \hat{k}(t), u(t))$ be the estimate of the vector field of the plant at time t ,

$$\hat{x}_p(\hat{\theta}, \hat{k}, u(t)) = f(x_p(t)) + F(x_p(t))\hat{\theta} + g_p(x_p) \frac{1}{\hat{k}} u(t),$$

and $\dot{x}_d(t) := f(x_p(t)) + g(x_p(t))r(t)$ be the vector field for the desired closed loop system, then the above controller $u(t)$ in (20) minimizes $\|\psi(u(t))\|^2$ where $\psi(u(t)) = \dot{x}_d(t) - \hat{x}_p(\hat{\theta}(t), \hat{k}(t), u(t))$,

$$u(t) = \arg \min_{u(t) \in \mathbb{R}} \|\psi(u(t))\|^2. \quad (23)$$

Proof. By direct calculation, $\psi(u) = g(x_p)r - F(x_p)\hat{\theta} - g_p(x_p)u/\hat{k}$, where all the t 's are dropped, $\psi(u)$ has an affine relation to $u(t)$. Therefore, the minimum of (23) is computed by the left pseudo inverse of $g_p(x_p)/\hat{k}$, which leads to the same form as in (20). $\square$

The result of Lemma 1 implies that the proposed adaptive controller in (20) makes the vector field of the plant with the estimated parameters closest to the desired contracting vector field.

C. Analysis of the proposed controller

As in prior work, let the error be defined as $e = x_m - x_p$, and the parameter error be defined as $\tilde{\theta} = \hat{\theta} - \theta^*$ and $\tilde{k} = \hat{k} - k^*$. The main theorem is stated as follows.

Theorem 1. Suppose that the nonlinear system in (15) satisfies Assumption 1 and the reference model in (16) is contracting with respect to the uniformly bounded positive definite metric $M(t)$ satisfying (2), then by using the proposed controller and the adaptive rule in (20-22) with initial sign constraint $\text{sgn}(\hat{k}(0)) = \text{sgn}(b_p)$, $\|e(t)\|^2$ asymptotically converges to 0 and $\hat{\theta}$ and $\hat{k}$ remain bounded.

Proof. First, let $\hat{\phi} := \hat{k}^{-1}$ and $\phi^* := k^{*-1}$. Define the new parameter error as $\tilde{\phi}(t) = \hat{\phi} - \phi^*$, and now by computing the error dynamics,

$$\begin{aligned} \dot{e} &= f(x_m) + g(x_m)r - f(x_p) - F(x_p)\theta^* - g_p(x_p)\phi^*u \\ &= f(x_m) + g(x_m)r - f(x_p) - F(x_p)\theta^* + g_p(x_p)\tilde{\phi}u \\ &\quad - g_p(x_p)\hat{\phi}u \end{aligned}$$

Now substitute u from (20) into the last equation only for the $g_p(x_p)\hat{\phi}u$ term. Then

$$\begin{aligned} \dot{e} &= f(x_m) + g(x_m)r - f(x_p) - F(x_p)\theta^* + g_p(x_p)\tilde{\phi}u \\ &\quad - g_p(x_p)g_p(x_p)^\dagger (g(x_p)r - F(x_p)\hat{\theta}) \end{aligned}$$

since $\hat{\phi}\hat{k} = 1$. By adding and subtracting $g(x_p)r$ as well as $g_p(x_p)g_p(x_p)^\dagger F(x_p)\theta^*$,

$$\begin{aligned} \dot{e} &= f(x_m) - f(x_p) + (g(x_m) - g(x_p))r + g_p(x_p)\tilde{\phi}u \\ &\quad + \left(I_{n \times n} - \frac{g_p(x_p)g_p(x_p)^T}{\|g_p(x_p)\|^2} \right) (g(x_p)r - F(x_p)\theta^*) \\ &\quad + \frac{g_p(x_p)g_p(x_p)^T}{\|g_p(x_p)\|^2} F(x_p)(\hat{\theta} - \theta^*) \end{aligned}$$

where $I_{n \times n}$ is an identity matrix. Observe that for $\mu \in \mathbb{R}^n$

$$Op(\mu) := \left(I_{n \times n} - \frac{g_p(x_p)g_p(x_p)^T}{\|g_p(x_p)\|^2} \right) \mu$$

defines a projection of μ to the orthogonal complement of $g_p(x_p)$, namely $Span(g_p(x_p))^\perp$. By using Assumption 1, this term can be removed from the error dynamics yielding

$$\begin{aligned} \dot{e} &= f(x_m) - f(x_p) + (g(x_m) - g(x_p))r + g_p(x_p)\tilde{\phi}u \\ &\quad + \frac{g_p(x_p)g_p(x_p)^T}{\|g_p(x_p)\|^2} F(x_p)\tilde{\theta} \end{aligned} \quad (24)$$

Now consider the Lyapunov function candidate

$$V(x_p, t) = \frac{1}{2}(E(\gamma(\cdot, t)) + \Gamma_\phi^{-1}\tilde{\phi}^2 + \tilde{\theta}^T \Gamma_\theta^{-1} \tilde{\theta}), \quad (25)$$

where $\gamma_s(s, t) = x_m - x_p = e$ for all $s \in [0, 1]$ and so $E(\gamma(\cdot, t)) := e^T M(t)e$ by computing (4). The derivative of V with respect to time gives

$$\begin{aligned} \dot{V} &= \frac{1}{2}e^T M(t)(f(x_m) - f(x_p) + (g(x_m) - g(x_p))r) \\ &\quad + \frac{1}{2}(f(x_m) - f(x_p) + (g(x_m) - g(x_p))r)^T M(t)e \\ &\quad + \frac{1}{2}e^T \dot{M}(t)e + (e^T M(t)g(x_p)u + \Gamma_\phi^{-1}\dot{\tilde{\phi}}\tilde{\phi}) \\ &\quad + (e^T M(t)g_p(x_p)g_p(x_p)^\dagger + \Gamma_\theta\dot{\tilde{\theta}})\tilde{\theta}. \end{aligned}$$

Observe that $\dot{\tilde{\phi}} = -1/\hat{k}^2 \dot{\hat{k}}$. Now by following the adaptation rule in (21-22), the derivative of V becomes,

$$\begin{aligned} \dot{V} &= \frac{1}{2}e^T M(t)(f(x_m) - f(x_p) + (g(x_m) - g(x_p))r) \\ &\quad + \frac{1}{2}(f(x_m) - f(x_p) + (g(x_m) - g(x_p))r)^T M(t)e \\ &\quad + \frac{1}{2}e^T \dot{M}(t)e \end{aligned}$$

By invoking the fundamental theorem of calculus,

$$\begin{aligned} f(x_m) - f(x_p) &+ (g(x_m) - g(x_p))r \\ &= \int_0^1 A(\gamma(s, t), t) \gamma_s(s, t) ds \\ &= \left(\int_0^1 A(\gamma(s, t), t) ds \right) e. \end{aligned}$$

By substituting it to $\dot{V}$ where t is dropped to have a compact form,

$$\begin{aligned} \dot{V} &= \frac{1}{2}e^T \left(\int_0^1 \dot{M} + MA(\gamma(s)) + A(\gamma(s))^T M ds \right) e \\ &\leq -\lambda M e^2 \leq 0. \end{aligned}$$

Since $V(x_p, t)$ is lower bounded and $\dot{V} \leq 0$, $\lim_{t \rightarrow \infty} V(x_p(t), t)$ exists. This also implies that e is bounded and parameter estimation errors $\hat{\theta}$ and $\hat{\phi}$ are also bounded. Now, observe that $\dot{M}$ is also bounded by the assumption that M is uniformly bounded and M and $\dot{M}$ are continuously differentiable. Also, note that x_p is bounded as e is bounded and $x_m, r, \dot{r}$ are bounded by the model assumption. This implies that $\dot{e}$ in (24) is bounded as f, g, F and g_p are continuous differentiable. Therefore, $|\dot{V}|$ is bounded for all t , and so $\dot{V}$ is uniformly continuous. By invoking Barbalat's lemma [11] (Lemma 4.3), $\lim_{t \rightarrow \infty} \dot{V}(t) = 0$, which also implies that $e(t)$ asymptotically converges to 0 as t goes to infinity using the condition that $M(t)$ is lower bounded and $\lambda > 0$. $\square$

V. APPLICATION TO CCM BASED CONTROLLER WITH UNCERTAIN HIGH FREQUENCY GAIN

In § IV, a new adaptive controller that projects the plant vector field to a desired contracting vector field is proposed with the adaptation rule for structured uncertainty in the actuation vector field. One way to construct a contracting reference would be using the CCM controller proposed in [5] and recapped in § II.

Let the desired contracting nonlinear reference model be generated by the CCM controller, $u_{ccm}(x_m, t)$ in (7) with respect to the uniformly positive definite metric $M(t)$,

$$\dot{x}_m = f(x_m) + g(x_m)u_{ccm}(x_m, t). \quad (26)$$

In particular, consider the model starts from $x_m(0) = x_d(0)$, then $x_m(t) = x_d(t)$ for all time $t > 0$, as $u_{ccm}(x_d, t) = u_d(t)$.

Now consider the same plant model as in (15) with a_p and b_p unknown and assume the certainty equivalence matching that

$$g(x)u_{ccm}(x, t) - F(x)a_p \in \text{Span}\{g_p(x)\}. \quad (27)$$

holds for all x and $t > 0$. Once again this assumption, (27), becomes equivalent to the matched uncertainty in [9]. The following adaptive controller is proposed,

$$u = \hat{k}g_p(x_p)^\dagger(g(x_p)u_{ccm}(x_p, t) - F(x_p)\hat{\theta}). \quad (28)$$

Theorem 2. Suppose that there exist u_{ccm} of the form (7) satisfying (9-10) with respect to state independent uniformly bounded positive definite metric, $M(t)$, and the above assumption in (27) holds. Then by using the adaptive controller in (28) with the same adaptation rule (21-22) and the initial sign constraint on $\text{sgn}(\hat{k}(0)) = \text{sgn}(b_p)$, $\|e(t)\|^2$ asymptotically converges to 0 and $\hat{\theta}$ and $\hat{k}$ remain bounded.

Proof. Let $\hat{\phi} := \hat{k}^{-1}$ and $\phi^* := k^{*-1}$ and define the parameter error as $\tilde{\phi}(t) = \hat{\phi} - \phi^*$. By computing the error dynamics,

$$\begin{aligned} \dot{e} &= f(x_m) + g(x_m)u_{ccm}(x_m, t) - f(x_p) \\ &\quad - F(x_p)\theta^* + g_p(x_p)\tilde{\phi}u - g_p(x_p)\hat{\phi}u \end{aligned}$$

Now substitute u in (28) into the last equation only for the $g_p(x_p)\hat{\phi}u$ term. Then

$$\begin{aligned} \dot{e} &= f(x_m) + g(x_m)u_{ccm}(x_m, t) - f(x_p) - F(x_p)\theta^* \\ &\quad - g_p(x_p)g_p(x_p)^\dagger(g(x_p)u_{ccm}(x_p, t) - F(x_p)\hat{\theta}) \\ &\quad + g_p(x_p)\tilde{\phi}u \end{aligned}$$

since $\hat{\phi}\hat{k} = 1$. By following the same procedure as in Theorem 1,

$$\begin{aligned} \dot{V} &= 1/2(e^T M(t)(f(x_m) + g(x_m)u_{ccm}(x_m, t) \\ &\quad - (f(x_p) + g(x_p)u_{ccm}(x_p, t))) \\ &\quad + (f(x_m) + g(x_m)u_{ccm}(x_m, t) \\ &\quad - (f(x_p) + g(x_p)u_{ccm}(x_p, t)))^T M(t)e) + \frac{1}{2}e^T \dot{M}e \end{aligned}$$

holds. Now by invoking the fundamental theorem of calculus along the geodesic, γ , as before,

$$\begin{aligned} f(x_m) - f(x_p) &= \left(\int_0^1 \frac{f(\gamma(s, t))}{\partial x} ds \right) e \\ g(x_m)u_{ccm}(x_m) - g(x_p)u_{ccm}(x_p) \\ &= \left(\int_0^1 \frac{\partial g(\gamma)}{\partial x} u_{ccm}(\gamma) - \frac{1}{2}\rho(\gamma)g(\gamma)g(\gamma)^T M(t) ds \right) e \end{aligned}$$

where the t is all dropped to have a compact form. Now by using the condition for the CCM in (9) and the Killing vector for M condition in (10) gives

$$\begin{aligned} \dot{V} &:= e^T \left(\int_0^1 \dot{M} + \frac{\partial f^T}{\partial x} M + M \frac{\partial f}{\partial x} - \rho M g g^T M ds \right) e \\ &\leq -\lambda e^T M e \leq 0 \end{aligned}$$

Finally, since the u_{ccm} is a smooth feedback controller [5], the same argument as in the proof of Theorem 1 holds. Hence, $\|e(t)\|^2$ asymptotically converges to 0 while the parameter estimates are bounded. $\square$

Remark 3 Similar adaptive controllers using CCM were previously considered in [9] for the case when the unknowns are restricted to the uncertainty in the drift dynamics, $F(x_p)a_p$, meaning b_p is known. In this case, the adaptive controller has a separable form

$$u_{adapt} = u_{ccm} + u_{reject}$$

where u_{ccm} is guaranteed to make the whole system contracting and u_{reject} cancels out the unknown disturbance. Based on the exact b_p , the ρ in the CCM controller is computed by satisfying the contracting condition (9). However, if the actual b_p is enough smaller than the nominal b_p , then (9) would be violated and u_{ccm} would no longer guarantee the contraction property. Suppose that $M(t)$, $\rho(b_p^{nom})$ is a pair satisfying (9),

$$\dot{M} + \frac{\partial f^T}{\partial x} M + M \frac{\partial f}{\partial x} + 2\lambda M < \rho M g g^T M (b_p^{nom})^2$$

for nominal b_p^{nom} . If the actual $b_p \neq b_p^{nominal}$, then

$$\dot{M} + \frac{\partial f^T}{\partial x} M + M \frac{\partial f}{\partial x} + 2\lambda M < \rho M g g^T M (b_p b_p^{nom})$$

holds if $b_p \geq b_p^{nom}$, but is not guaranteed otherwise. In contrast, the proposed method in this paper solves the problem of arbitrary b_p by using the adaptive gain $\hat{k}(t)$ multiplied by u_{ccm} (based on b_p^{nom}) to minimize the distance from the desired contracting vector field as explained in Lemma 1.

A. Fully Actuated System Example

As shown in [7], for any fully actuated system, there exist a constant contraction metric M which satisfies (9-10). Consider the simple reference pendulum model

$$\ddot{q}_d + d\dot{q}_d + \sin(q_d) = u_d, \quad (29)$$

where d is a damping coefficient and $u_d(t) := 4\sin(t)$ is sinusoidal input. Then a CCM controller, u_{ccm} can be constructed to follow $(q_d(t), u_d(t))$. The following CCM parameters are chosen

$$M := \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \lambda = 1.5, \rho = 21.24 \quad (30)$$

to satisfy (9-10) for the reference model (29).

Now suppose that the actual plant has different actuation strength and gravity which is unknown to the controller designer, then the uncertain plant is given as

$$\ddot{q}_p + d\dot{q}_p + g\sin(q_p) = bu, \quad (31)$$

where b is an unknown actuation strength and g is an unknown gravity.

To compare the effectiveness of the adaptive controller given by (28) to that of the adaptive controller given in [9], both are used on the pendulum plant given in (31) alongside the CCM controller, u_{ccm} , calculated using the pendulum model (29).

The performance results are shown in Figure 1. The system parameters are chosen as $b = 0.25$, $g = 9.81$, $d = 0.1$ for Figure 1. This makes $b_p = 0.25$, strictly smaller than $b_p^{nom} = 1$ in (29). The results of using the CCM adaptive controller in [9] are shown in Figure 1 (a), which is unable to properly track the reference trajectory because the decreased b has resulted in violation of (9), removing the guarantee that the system will be contracting. The adaptive controller proposed in this paper explicitly adapts to changes in actuator strength, enabling the closed loop system to track the reference trajectory as shown in Figure 1 (b). When actuator strength is not different between the plant and model, both controllers asymptotically converge to the model system, demonstrating that the CCM adaptive controller is sensitive to the multiplicative gain b_p .

VI. CONCLUSION AND FUTURE WORK

In this paper, a new direct model reference adaptive controller for a nonlinear system with parameter uncertainties is proposed and analyzed when the reference model is contracting with respect to a state independent metric. The contracting nature of the reference model gives an intuitive nominal controller design by simply matching the contracting vector field. The proposed adaptive controller is then designed to minimize the distance of the estimated vector field

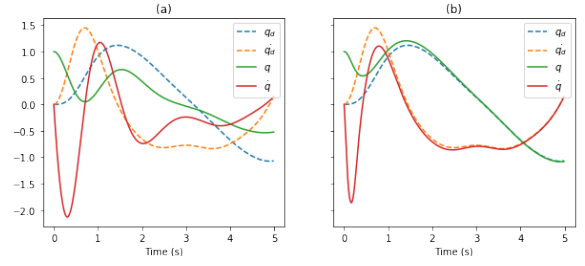


Fig. 1. Error between the state of the reference system (29) (dashed) and the state of the plant system with reduced actuator strength (31). System (a) utilizes the CCM controller alongside the the adaptive rule from [9] while system (b) utilizes the CCM controller alongside the adaptive rule given in (28).

of the plant given current parameter estimates to the desired contracting vector field. The proposed adaptive controller asymptotically minimizes the Riemannian energy functional which leads to the asymptotic convergence between the plant and model trajectories. In addition, the proposed controller can be combined with a CCM controller that uses smooth feedback control to create the contracting reference model.

As the proposed controller uses the Euclidean projection between two vector fields, the current study works well when the Riemannian metric remain state independent, making the geodesic curve a straight line. An extension to the case where the Riemannian metric is state dependent may be achieved by considering parallel transportation of two different tangent vector in $T_{x_m(t)}\mathcal{X}$ and $T_{x_p(t)}\mathcal{X}$ along the geodesic curve and then taking a projection with respect to the Riemannian metric. In addition, combining Nussbaum functions in [12] with the contracting reference model will be another possible extension to this study.

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