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Newton polygon stratification of the Torelli locus in PEL-type Shimura varieties

Wanlin Li [1] and Elena Mantovan [2] and Rachel Pries [3] and Yunqing Tang [4]

[1] Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA, wanlin@math.wisc.edu and [2] Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA, mantovan@caltech.edu and [3] Department of Mathematics, Colorado State University, Fort Collins, CO 80523, USA, pries@math.colostate.edu and [4] Department of Mathematics, Princeton University, Princeton, NJ 08540, USA, yunqingt@math.princeton.edu

Correspondence to be sent to: pries@math.colostate.edu

Abstract: We study the intersection of the Torelli locus with the Newton polygon stratification of the modulo p reduction of certain PEL-type Shimura varieties. We develop a clutching method to show that the intersection of the open Torelli locus with some Newton polygon strata is non-empty. This allows us to give a positive answer, under some compatibility conditions, to a question of Oort about smooth curves in characteristic p whose Newton polygons are an amalgamate sum. As an application, we produce infinitely many new examples of Newton polygons that occur for smooth curves that are cyclic covers of the projective line. Most of these arise in inductive systems which demonstrate unlikely intersections of the open Torelli locus with the Newton polygon stratification in Siegel modular varieties. In addition, for the twenty special PEL-type Shimura varieties found in Moonen's work, we prove that all Newton polygon strata intersect the open Torelli locus (if $p >> 0$ in the supersingular cases).

Keywords: curve, cyclic cover, Jacobian, abelian variety, moduli space, Shimura variety, PEL-type, reduction, Frobenius, supersingular, Newton polygon, p-rank, Dieudonné module, p-divisible group.
MSC10 classifications: primary 11G18, 11G20, 11M38, 14G10, 14G35; secondary 11G10, 14H10, 14H30, 14H40, 14K10

1 Introduction

1.1 Overview

Consider the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g in characteristic $p > 0$. It contains the open Torelli locus $\mathcal{T}_g^\circ$, which is the image of the moduli space $\mathcal{M}_g$ of smooth genus g curves under the Torelli morphism. The generic point of $\mathcal{T}_g^\circ$ is contained in the ordinary locus of $\mathcal{A}_g$, meaning that the only

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slopes of the Newton polygon of a generic curve of genus g are 0 and 1. This was first proven by Miller for p odd [26, Proposition 1] using a computation of the Hasse–Witt matrix for hyperelliptic curves and by Koblitz for all p [16, Theorem 5, page 145] by a deformation theory argument. Faber and Van der Geer [11, Theorem 2.3] generalized the argument of Koblitz to prove, in particular, that $\mathcal{T}_g^\circ$ intersects the non-ordinary locus of $\mathcal{A}_g$. A short description of this paper is that we generalize these results for $\mathcal{A}_g$ to many Shimura varieties of PEL-type.

More generally, $\mathcal{A}_g$ can be stratified by Newton polygon. For a symmetric Newton polygon ν of height $2g$, in most cases it is not known whether the stratum $\mathcal{A}_g[\nu]$ intersects $\mathcal{T}_g^\circ$ or, equivalently, whether there exists a smooth curve of genus g in characteristic p whose Jacobian has Newton polygon ν. This question is answered only when ν is close to ordinary, meaning that the codimension of $\mathcal{A}_g[\nu]$ in $\mathcal{A}_g$ is small.

In this paper, we develop a framework to study Newton polygons of Jacobians of μ_m-covers of the projective line $\mathbb{P}^1$ for an integer $m \geq 2$. This generalizes work of Bouw [5] who studied the p-ranks of Jacobians of μ_m-covers. As an application, we find numerous infinite sequences of Newton polygons for Jacobians of smooth curves which were not previously known to occur. Most of these arise in an *unlikely intersection* of the open Torelli locus $\mathcal{T}_g^\circ$ with the Newton polygon strata of $\mathcal{A}_g$ in the sense that the codimension of the Newton stratum in $\mathcal{A}_g$ is strictly greater than the dimension of $\mathcal{M}_g$, Definition 8.2.

In essence, our strategy is to replace the system of moduli spaces $\mathcal{A}_g$ by inductive systems of PEL-type Shimura varieties. Each Hurwitz space of μ_m-covers of $\mathbb{P}^1$ determines a unitary Shimura variety Sh associated with the group algebra of μ_m, as constructed by Deligne–Mostow [8]. The Torelli morphism maps the Hurwitz space to the Shimura variety, but the codimension of the image grows quadratically with g.

This allows us to tackle the question of which Newton polygons occur for μ_m-covers from two sides. First, the structure of the Shimura variety places restrictions on the Newton polygon. By work of Kottwitz [17, 19], Wedhorn [36], Viehmann–Wedhorn [34], and Hamacher [14], the Newton polygon stratification of the modulo p reduction of Sh is well understood in terms of its signature type and the congruence class of p modulo m. There is a combinatorial description of the most generic Newton polygon u on Sh, which is called the μ-ordinary Newton polygon Definition 2.5, Section 2.6.1.

Second, using the boundary of the Hurwitz space, we can produce μ_m-covers of singular curves with prescribed Newton polygons. Under an *admissible* condition Definition 3.3, these singular curves can be deformed to smooth curves which are μ_m-covers of $\mathbb{P}^1$. The main problem is to show that this can be done *without changing the Newton polygon.* This problem disappears if the Newton polygon of the singular curve is the μ-ordinary Newton polygon on Sh. In Theorem 4.5, we show that this happens exactly when a *balanced* condition Definition 4.2 is satisfied. In Theorem 6.11, we prove a more powerful result that we can deform to a μ_m-cover of smooth curves without changing the Newton polygon under a *controlled* condition Definition 6.3.

By combining these two perspectives, we prove that the intersection of the Newton polygon stratum $Sh[\nu]$ with the open Torelli locus is non-trivial when ν is close to μ-ordinary, for infinitely many Shimura varieties Sh of PEL-type, see Sections 4.3 and 6.4. To do this, we find systems of Hurwitz spaces of μ_m-covers of $\mathbb{P}^1$ for which the admissible, balanced, and controlled conditions, together with an expected codimension condition on the Newton polygon strata, can be verified inductively.

The base cases we use involve cyclic covers of $\mathbb{P}^1$ branched at 3 points or the 20 special families found by Moonen [28]. As an application of our method, we also prove that all Newton polygon strata on Moonen's 20 special families intersect the open Torelli locus (if $p >> 0$ in the supersingular cases) Corollary 7.2.

1.2 Comparison with other work

In 2005, Oort proposed the following conjecture.

Conjecture 1.1. ([30, Conjecture 8.5.7]) For $i = 1, 2$, let $g_i \in \mathbb{Z}_{\geq 1}$ and let ν_i be a symmetric Newton polygon appearing on $\mathcal{T}_{g_i}^\circ$. Write $g = g_1 + g_2$. Let ν be the amalgamate sum of ν_1 and ν_2 as defined in Section 2.2. Then ν appears on $\mathcal{T}_g^\circ$. □

Theorems 4.5 and 6.11 show that Oort's conjecture has an affirmative answer in many cases. Our results provide the first extensive numerical support for this conjecture. They also provide theoretical support by verifying that many unlikely intersections of the Torelli locus and the Newton polygon strata occur. However, these results are not sufficient for us to judge whether Oort's conjecture is true in general.

The results in Section 4 can be viewed as a generalization of Bouw's work [5] about the intersection of $\mathcal{T}_g^\circ$ with the stratum of maximal p-rank in a PEL-type Shimura variety. For most families of μ_m-covers and most congruence classes of p modulo m, the maximal p-rank does not determine the Newton polygon.

We use clutching morphisms to study the boundary of Hurwitz spaces. This technique was also used to study the intersection of $\mathcal{T}_g^\circ$ with the p-rank stratification of $\mathcal{A}_g$ in [11, Theorem 2.3]; also [2], [13], [3].

The results in Section 6 generalize Pries' work [31, Theorem 6.4], which states that if a Newton polygon ν occurs on $\mathcal{M}_g$ with the expected codimension, then the Newton polygon $\nu \oplus (0,1)^n$ occurs on $\mathcal{M}_{g+n}$ with the expected codimension for $n \in \mathbb{Z}_{\geq 1}$. However, the expected codimension condition is difficult to verify for most Newton polygons ν.

1.3 Outline of paper and sample result

In Section 2, we review key background about Hurwitz spaces, PEL-type Shimura varieties, and Newton polygon stratifications. In Section 3, we analyze the image of a clutching morphism κ on a pair of μ_m-covers of $\mathbb{P}^1$.

In Section 4, we study whether the open Torelli locus $\mathcal{T}_g^\circ$ intersects the μ-ordinary Newton polygon stratum $Sh[u]$, see Definition 2.5, inside the Shimura variety Sh. The first main result Theorem 4.5 provides a method to leverage information about this question from lower to higher genus. Under a balanced condition on the signatures Definition 4.2, we verify that the intersection of $\mathcal{T}_g^\circ$ and $Sh[u]$ is non-trivial, for a varying family of Shimura varieties (Proposition 4.4, which we prove in Section 5).

The most powerful results in the paper are in Section 6, where we study the intersection of the open Torelli locus $\mathcal{T}_g^\circ$ with the non μ-ordinary Newton polygon strata inside the Shimura variety Sh. Theorem 6.11 also provides a method to leverage information from lower to higher genus. Under an additional controlled condition on the signatures Definition 6.3, we determine the codimension of the Newton polygon strata for a varying family of Shimura varieties (Proposition 6.8).

In Sections 4.3 and 6.4, we find situations where Theorems 4.5 and 6.11 can be implemented recursively, infinitely many times, which yields smooth curves with arbitrarily large genera and prescribed Newton polygons which were not previously known to occur. We do this by constructing suitable *infinite clutching systems* of PEL-type Shimura varieties which satisfy the admissible, balanced, (and controlled) conditions at every level. For example, we prove:

Theorem 1.2 (Special case of Corollary 4.9)**.** Let $\gamma = (m, N, a)$ be a monodromy datum as in Definition 2.1. Let p be a prime such that $p \nmid m$. Let u be the μ-ordinary Newton polygon associated to γ as in Definition 2.5. Suppose there exists a μ_m-cover of $\mathbb{P}^1$ defined over $\overline{\mathbb{F}}_p$ with monodromy datum γ and Newton polygon u.[1.1] Then, for any $n \in \mathbb{Z}_{\geq 1}$, there exists a smooth curve over $\overline{\mathbb{F}}_p$ with Newton polygon $\nu_n = u^n \oplus (0,1)^{(m-1)(n-1)}$.[1.2] □

For a symmetric Newton polygon ν of height $2g$, the open Torelli locus has an *unlikely intersection* with the Newton polygon stratum $\mathcal{A}_g[\nu]$ in $\mathcal{A}_g$ if there exists a smooth curve of genus g with Newton polygon ν and if $\dim(\mathcal{M}_g) < \operatorname{codim}(\mathcal{A}_g[\nu], \mathcal{A}_g)$, Definition 8.2. In Section 8, we study the asymptotic of $\operatorname{codim}(\mathcal{A}_g[\nu], \mathcal{A}_g)$ for the Newton polygons ν appearing in Sections 4.3 and 6.4. We verify that most of our inductive systems produce unlikely intersections once g is sufficiently large, for most congruence classes of p modulo m.

1.4 **Applications**

In Corollary 7.2 in Section 7, we prove that all the Newton polygons for the Shimura varieties associated to the 20 special families in [28, Table 1] occur for smooth curves in the family.

In Section 9, we construct explicit infinite sequences of Newton polygons that occur at odd primes for smooth curves which demonstrate unlikely intersections. For example, by Theorem 1.2 applied to $\gamma = (m, 3, (1, 1, m-2))$, we prove:

Application 1.3. (Proposition 9.2) Let $m \in \mathbb{Z}_{>1}$ be odd and $h = (m-1)/2$. Let p be a prime, $p \nmid 2m$, such that the order f of p in $(\mathbb{Z}/m\mathbb{Z})^*$ is even and $p^{f/2} \equiv -1 \bmod m$. For $n \in \mathbb{Z}_{\geq 1}$, there exists a μ_m-cover $C \to \mathbb{P}^1$ defined over $\overline{\mathbb{F}}_p$ where C is a smooth curve of genus $g = h(3n-2)$ with Newton polygon $\nu'_n = (1/2, 1/2)^{hn} \oplus (0,1)^{2h(n-1)}$. If $n \geq 34/h$, then $\operatorname{Jac}(C)$ lies in the unlikely intersection $\mathcal{T}_g^\circ \cap \mathcal{A}_g[\nu]$. □

The slopes of the Newton polygon ν'_n in Application 1.3 are $1/2$ with multiplicity $2hn$ and 0 and 1 each with multiplicity $2h(n-1)$. For the reader familiar with the Dieudonné-Manin classification, this means that the p-divisible group of $\operatorname{Jac}(C)$ is isogenous to $G_{1,1}^{hn} \oplus G_{0,1}^{2h(n-1)} \oplus G_{1,0}^{2h(n-1)}$.[1.3]

[1.1] See Proposition 4.6 for cases when this condition is satisfied.
[1.2] The slopes of ν_n are the slopes of u (with multiplicity scaled by n) and 0 and 1 each with multiplicity $(m-1)(n-1)$.
[1.3] In this description, the multiplicity of the slope 1/2 is twice the multiplicity of $G_{1,1}$ in the p-divisible group.

In Corollary 9.4, we apply Application 1.3 when $m = 3$ to verify, for $p \equiv 2 \bmod 3$ and $g \in \mathbb{Z}_{\geq 1}$, there exists a smooth curve of genus g defined over $\overline{\mathbb{F}}_p$ whose Newton polygon only has slopes $\{0, 1/2, 1\}$ and the multiplicity of slope $1/2$ is at least $2\lfloor g/3 \rfloor$. To our knowledge, this is the first time for any odd prime p that a sequence of smooth curves has been produced for every $g \in \mathbb{Z}_{\geq 1}$ such that the multiplicity of the slope $1/2$ in the Newton polygon grows linearly in g.

2 Notations and Preliminaries

More details on this section can be found in [22, Sections 2,3] and [28, Sections 2,3].

2.1 The group algebra of m-th roots of unity

Let $m, d \in \mathbb{Z}_{\geq 1}$. Let $\mu_m := \mu_m(\mathbb{C})$ denote the group of m-th roots of unity in $\mathbb{C}$. Let K_d be the d-th cyclotomic field over $\mathbb{Q}$. Let $\mathbb{Q}[\mu_m]$ denote the group algebra of μ_m over $\mathbb{Q}$. Then $\mathbb{Q}[\mu_m] = \prod_{d|m} K_d$. We endow $\mathbb{Q}[\mu_m]$ with the involution $*$ induced by the inverse map on μ_m, i.e., $\zeta^* := \zeta^{-1}$ for all $\zeta \in \mu_m$.

Set $\mathcal{T} := \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\mu_m], \mathbb{C})$. If W is a $\mathbb{Q}[\mu_m] \otimes_{\mathbb{Q}} \mathbb{C}$-module, we write $W = \oplus_{\tau \in \mathcal{T}} W_\tau$, where W_τ denotes the subspace of W on which $a \otimes 1 \in \mathbb{Q}[\mu_m] \otimes_{\mathbb{Q}} \mathbb{C}$ acts as $\tau(a)$. We fix an identification $\mathcal{T} = \mathbb{Z}/m\mathbb{Z}$ by defining, for all $n \in \mathbb{Z}/m\mathbb{Z}$,

$$\tau_n(\zeta) := \zeta^n, \text{ for all } \zeta \in \mu_m.$$

Let $m \geq 1$. For $p \nmid m$, we identify $\mathcal{T} = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\mu_m], \mathbb{Q}_p^{\mathrm{un}})$, where $\mathbb{Q}_p^{\mathrm{un}}$ is the maximal unramified extension of $\mathbb{Q}_p$ in an algebraic closure. There is a natural action of the Frobenius σ on $\mathcal{T}$, defined by $\tau \mapsto \tau^\sigma := \sigma \circ \tau$. Let $\mathfrak{O}$ be the set of all σ-orbits $\mathfrak{o}$ in $\mathcal{T}$.

2.2 Newton polygons

Let X be an abelian scheme defined over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Then there is a finite field $\mathbb{F}_0/\mathbb{F}_p$, an abelian scheme $X_0/\mathbb{F}_0$, and $\ell \in \mathbb{Z}_{\geq 1}$, such that $X \simeq X_0 \times_{\mathbb{F}_0} \mathbb{F}$ and the action of σ^ℓ on $H^1_{\mathrm{cris}}(X_0/W(\mathbb{F}_0))$ is linear; here $W(\mathbb{F}_0)$ denotes the Witt vector ring of $\mathbb{F}_0$. The *Newton polygon* $\nu(X)$ of X is the multi-set of rational numbers λ such that $\ell\lambda$ are the valuations at p of the eigenvalues of σ^ℓ acting on $H^1_{\mathrm{cris}}(X_0/W(\mathbb{F}_0))$; the Newton polygon does not depend on the choice of $(\mathbb{F}_0, X_0, \ell)$.

The *p-rank* of X is the multiplicity of the slope 0 in $\nu(X)$; it equals $\dim_{\mathbb{F}_p}(\mathrm{Hom}(\mu_p, X))$.

If ν_1 and ν_2 are two Newton polygons, the *amalgamate sum* $\nu_1 \oplus \nu_2$ is the disjoint union of the multi-sets ν_1 and ν_2. We denote by ν^d the amalgamate sum of d copies of ν.

The Newton polygon $\nu(X)$ is typically drawn as a lower convex polygon, with slopes λ occurring with multiplicity m_λ, where m_λ denotes the multiplicity of λ in the multi-set. The Newton polygon of a g-dimensional abelian variety is symmetric, with endpoints $(0,0)$ and $(2g, g)$, integral break points, and slopes in $\mathbb{Q} \cap [0,1]$. For convex polygons, we write $\nu_1 \geq \nu_2$ if ν_1, ν_2 share the same endpoints and ν_1 lies below ν_2.

We denote by *ord* the Newton polygon $(0,1)$ and by ss the Newton polygon $(1/2,1/2)$. For $s,t \in \mathbb{Z}_{\geq 1}$, with $s \leq t/2$ and $\gcd(s,t)=1$, we write $(s/t,(t-s)/t)$ for the Newton polygon with slopes s/t and $(t-s)/t$, each with multiplicity t.

Suppose Y is a semi-abelian scheme defined over $\mathbb{F}$. Then Y is an extension of an abelian scheme X by a torus T; its Newton polygon is defined to be $\nu(Y) := \nu(X) \oplus \mathrm{ord}^\epsilon$, where $\epsilon = \dim(T)$.

2.3 Cyclic covers of the projective line

Definition 2.1. Fix integers $m \geq 2$, $N \geq 3$ and an N-tuple of integers $a = (a(1), \dots, a(N))$. Then a is an *inertia type* for m and $\gamma = (m, N, a)$ is a *monodromy datum* if

1. $a(i) \not\equiv 0 \bmod m$, for each $i = 1, \dots, N$,
2. $\gcd(m, a(1), \dots, a(N)) = 1$,
3. $\sum_i a(i) \equiv 0 \mod m$.

□

For later applications, we sometimes consider a *generalized monodromy datum*, in which we allow $a(i) \equiv 0 \bmod m$. In the case that $a(i) = 0$, we set $\gcd(a(i), m) = m$.

Two monodromy data (m, N, a) and (m', N', a') are *equivalent* if $m = m'$, $N = N'$, and the images of a, a' in $(\mathbb{Z}/m\mathbb{Z})^N$ are in the same orbit under $(\mathbb{Z}/m\mathbb{Z})^* \times \mathrm{Sym}_N$.

For fixed m, we work over an irreducible scheme over $\mathbb{Z}[1/m, \zeta_m]$. Let $U \subset (\mathbb{A}^1)^N$ be the complement of the weak diagonal. For each $t = (t(1), \dots, t(N)) \in U$, the equation

$$y^m = \prod_{i=1}^{N} (x - t(i))^{a(i)} \tag{2.1}$$

defines a μ_m-cover of the projective line. Let C be the smooth projective (relative) curve over U whose fiber at each point t is the normalization of the curve defined by (2.1). Consider the μ_m-cover $\phi : C \to \mathbb{P}^1_U$ defined by the function x and the μ_m-action $\iota : \mu_m \to \mathrm{Aut}(C)$ given by $\iota(\zeta) \cdot (x, y) = (x, \zeta \cdot y)$ for all $\zeta \in \mu_m$.

For a closed point $t \in U$, the cover $\phi_t : C_t \to \mathbb{P}^1$ is a μ_m-cover, branched at N points $t(1), \dots, t(N)$ in $\mathbb{P}^1$, and with local monodromy $a(i)$ at $t(i)$. By the hypotheses on the monodromy datum, C_t is a geometrically irreducible curve of genus g, where

$$g = g(m, N, a) = 1 + \frac{1}{2}\Big((N-2)m - \sum_{i=1}^{N} \gcd(a(i), m)\Big). \tag{2.2}$$

Take $W = H^0(C_t, \Omega^1)$ and, under the identification $\mathcal{T} = \mathbb{Z}/m\mathbb{Z}$, let $\mathfrak{f}(\tau_n) = \dim(W_{\tau_n})$. The *signature type* of ϕ is defined as $\mathfrak{f} = (\mathfrak{f}(\tau_1), \dots, \mathfrak{f}(\tau_{m-1}))$. By [28, Lemma 2.7, §3.2],

$$\mathfrak{f}(\tau_n) = \begin{cases} -1 + \sum_{i=1}^{N} \langle \frac{-na(i)}{m} \rangle & \text{if } n \not\equiv 0 \bmod m \\ 0 & \text{if } n \equiv 0 \bmod m. \end{cases} \tag{2.3}$$

where, for any $x \in \mathbb{R}$, $\langle x \rangle$ denotes the fractional part of x. The signature type of ϕ does not depend on t; it determines and is uniquely determined by the inertia type, up to the action of Sym_N. The action of $(\mathbb{Z}/m\mathbb{Z})^*$ permutes the values $\mathfrak{f}(\tau_n)$.

2.4 Hurwitz spaces

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g in characteristic p. Its Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ is the moduli space of stable curves of genus g. For a Newton polygon ν, let $\mathcal{M}_g[\nu]$ be the subspace whose points represent objects with Newton polygon ν. We use analogous notation for other moduli spaces.

We refer to [10] and [1, Section 2.2] for a more complete description of Hurwitz spaces for cyclic covers of $\mathbb{P}^1$.[2.1] Consider the moduli functor $\overline{\mathcal{M}}_{\mu_m}$ (resp. $\widetilde{\mathcal{M}}_{\mu_m}$) on the category of schemes over $\mathbb{Z}[1/m, \zeta_m]$; its points represent admissible stable μ_m-covers $(C/U, \iota)$ of a genus 0 curve (resp. together with an ordering of the smooth branch points and the choice of one ramified point above each of these). We use a superscript $\circ$ to denote the subspace of points for which C is smooth. By [10, Theorem 3.2], see also [1, Lemma 2.2], $\overline{\mathcal{M}}_{\mu_m}$ (resp. $\widetilde{\mathcal{M}}_{\mu_m}$) is a smooth proper Deligne–Mumford stack and $\overline{\mathcal{M}}^{\circ}_{\mu_m}$ (resp. $\widetilde{\mathcal{M}}^{\circ}_{\mu_m}$) is open and dense within it.

For each irreducible component of $\widetilde{\mathcal{M}}_{\mu_m}$, the monodromy datum $\gamma = (m, N, a)$ of the μ_m-cover $(C/U, \iota)$ is constant. Conversely, the substack $\widetilde{\mathcal{M}}^{\gamma}_{\mu_m}$ of points representing μ_m-covers with monodromy datum γ is irreducible, [12, Corollary 7.5], [37, Corollary 4.2.3].

On $\overline{\mathcal{M}}_{\mu_m}$, there is no ordering of the ramification points; so only the unordered multi-set $\overline{a} = \{a(1), \dots, a(N)\}$ is well-defined. The components of $\overline{\mathcal{M}}_{\mu_m}$ are indexed by $\overline{\gamma} = (m, N, \overline{a})$. By [1, Lemma 2.4], the forgetful morphism $\widetilde{\mathcal{M}}^{\gamma}_{\mu_m} \to \overline{\mathcal{M}}^{\overline{\gamma}}_{\mu_m}$ is étale and Galois.

Definition 2.2. [2.2] If $\gamma = (m, N, a)$ is a monodromy datum, let $\widetilde{Z}(\gamma) = \widetilde{\mathcal{M}}^{\gamma}_{\mu_m}$ and let $\overline{Z}(\gamma)$ be the reduced image of $\widetilde{\mathcal{M}}^{\gamma}_{\mu_m}$ in $\overline{\mathcal{M}}_g$. We denote the subspace representing objects where C/U is smooth (resp. of compact type[2.3], resp. stable) by

$$Z^{\circ}(\gamma) \subset Z^{c}(\gamma) \subset \overline{Z}(\gamma) \text{ and } \widetilde{Z}^{\circ}(\gamma) \subset \widetilde{Z}^{c}(\gamma) \subset \widetilde{Z}(\gamma).$$

□

[2.1] The results we use from [1, Section 2.2] are true both when m is prime and when m is composite.
[2.2] This definition is slightly different from the one in our previous papers [23], [22].
[2.3] A stable curve has *compact type* if its dual graph is a tree. The Jacobian of a stable curve C is a semi-abelian variety; also C has compact type if and only if $\mathrm{Jac}(C)$ is an abelian variety.

By definition, $\overline{Z}(\gamma)$ is a reduced irreducible proper substack of $\overline{\mathcal{M}}_g$. It depends uniquely on the equivalence class of γ. The forgetful morphism $\widetilde{Z}(\gamma) \to \overline{Z}(\gamma)$ is finite and hence it preserves the dimension of any substack.

Remark 2.3. Let $\gamma' = (m, N', a')$ be a generalized monodromy datum. Assume that $a'(N') \equiv 0 \bmod m$ and $a'(i) \not\equiv 0 \bmod m$ for $1 \leq i < N'$. Consider the monodromy datum $\gamma = (m, N' - 1, a)$, where $a(i) = a'(i)$ for $1 \leq i \leq N' - 1$. Then $\overline{Z}(\gamma') = \overline{Z}(\gamma)$ and $\widetilde{Z}(\gamma') = \widetilde{Z}(\gamma)_1$, where the subscript 1 indicates that the data includes one marked point on the curve. The fibers of the morphism $\widetilde{Z}^{\circ}(\gamma') \to Z^{\circ}(\gamma')$ are of pure dimension 1. □

2.5 Shimura varieties associated to monodromy data

Consider $V := \mathbb{Q}^{2g}$ endowed with the standard symplectic form $\Psi : V \times V \to \mathbb{Q}$ and $G := \mathrm{GSp}(V, \Psi)$, the group of symplectic similitudes. Let $(G_{\mathbb{Q}}, \mathfrak{h})$ be the Siegel Shimura datum.

Fix $x \in Z^c(\gamma)(\mathbb{C})$ and let $(\mathcal{J}_x, \theta)$ denote the Jacobian of the curve represented by x together with its principal polarization θ. Choose a symplectic similitude

$$\alpha : (H_1(\mathcal{J}_x, \mathbb{Z}), \psi_\theta) \to (V, \Psi)$$

where ψ_θ denotes the Riemannian form on $H_1(\mathcal{J}_x, \mathbb{Q})$ corresponding to θ. Via α, the $\mathbb{Q}[\mu_m]$-action on $\mathcal{J}_x$ induces a $\mathbb{Q}[\mu_m]$-module structure on V, and the Hodge decomposition of $H_1(\mathcal{J}_x, \mathbb{C})$ induces a $\mathbb{Q}[\mu_m] \otimes_{\mathbb{Q}} \mathbb{C}$-linear decomposition $V_{\mathbb{C}} = V^+ \oplus V^-$.

We recall the PEL-type Shimura stack $\mathrm{Sh}(\mu_m, \mathfrak{f})$ given in [8]. The Shimura datum of $\mathrm{Sh}(\mu_m, \mathfrak{f})$ given by $(H, \mathfrak{h}_{\mathfrak{f}})$ is defined as

$$H := GL_{\mathbb{Q}[\mu_m]}(V) \cap \mathrm{GSp}(V, \Psi),$$

and $\mathfrak{h}_{\mathfrak{f}}$ the H-orbit in $\{h \in \mathfrak{h} \mid h \text{ factors through } H\}$ determined by the isomorphism class of the $\mathbb{Q}[\mu_m] \otimes_{\mathbb{Q}} \mathbb{C}$-module V^+, i.e., by the integers $\mathfrak{f}(\tau) := \dim_{\mathbb{C}}(V^+_\tau)$, for all $\tau \in \mathcal{T}$. Under the identification $\mathcal{T} = \mathbb{Z}/m\mathbb{Z}$, the formula for $\mathfrak{f}(\tau_n)$ is that given in (2.3).

For a Shimura variety $\mathrm{Sh} := \mathrm{Sh}(H, \mathfrak{h}_{\mathfrak{f}})$ of PEL type, we use Sh^* to denote the Baily-Borel (i.e., minimal) compactification and $\overline{\mathrm{Sh}}$ to denote a toroidal compactification (see [20]).

The Torelli morphism $T : \mathcal{M}_g^c \to \mathcal{A}_g$ takes a curve of compact type to its Jacobian.

Definition 2.4. We say that $Z^c(\gamma)$ is *special* if $T(Z^c(\gamma))$ is open and closed in the PEL-type Shimura stack $\mathrm{Sh}(\mu_m, \mathfrak{f})$ given in [8] (see [22, Section 3.3] for details). □

If $N = 3$, then $T(Z^c(\gamma))$ is a point of $\mathcal{A}_g$ representing an abelian variety with complex multiplication and is thus special, [23, Lemma 3.1]. By [28, Theorem 3.6], if $N \geq 4$, then $Z^c(\gamma)$ is special if and only if γ is equivalent to one of twenty examples in [28, Table 1].

2.6 The Kottwitz set and the μ-ordinary Newton polygon

Let $p \nmid m$ be a rational prime. Then the Shimura datum $(H, \mathfrak{h}_{\mathfrak{f}})$ is unramified at p. We write $H_{\mathbb{Q}_p}$ for the fiber of H at p, and $\mu_{\mathfrak{h}}$ for the conjugacy class of p-adic cocharacters μ_h associated with $h \in \mathfrak{h}_{\mathfrak{f}}$.

Following [17]-[19], we denote by $B(H_{\mathbb{Q}_p}, \mu_{\mathfrak{h}})$ the partially ordered set of μ-admissible $H_{\mathbb{Q}_p}$-isocrystal structures on $V_{\mathbb{Q}_p}$. By [34, Theorem 1.6] (see also [36]), $B(H_{\mathbb{Q}_p}, \mu_{\mathfrak{h}})$ can be canonically identified with the set of Newton polygons appearing on $\mathrm{Sh}(H, \mathfrak{h})$.[2.4] We sometimes write $\mathrm{Sh} := \mathrm{Sh}(H, \mathfrak{h})$ and $B = B(\mathrm{Sh}) := B(H_{\mathbb{Q}_p}, \mu_{\mathfrak{h}})$.

Definition 2.5. The *μ-ordinary* Newton polygon $u := u_{\mu-ord}$ is the unique maximal element (lowest Newton polygon) of $B(H_{\mathbb{Q}_p}, \mu_{\mathfrak{h}})$. □

An explicit formula for u is given below; (see also [22, Section 4.1-4.2]).

2.6.1 *Formula for slopes and multiplicities*

Let $\mathfrak{f}$ be a signature type. Fix a σ-orbit $\mathfrak{o}$ in $\mathcal{T}$ as defined in Section 2.1. We recall the formulas from [28, Section 1.2.5] for the slopes and multiplicities of the $\mathfrak{o}$-component $u(\mathfrak{o})$ of the μ-ordinary Newton polygon in terms of $\mathfrak{f}$, following the notation in [9, Section 2.8], [22, Section 4.2].

With some abuse of notation, we replace $\mathcal{T}$ by $\mathcal{T} - \{\tau_0\}$ and the set of σ-orbits $\mathfrak{O}$ by $\mathfrak{O} - \{\{\tau_0\}\}$ throughout the paper. Let $g(\tau) := \dim_{\mathbb{C}}(V_\tau)$. As the integer $g(\tau)$ depends only on the order of τ in the additive group $\mathbb{Z}/m\mathbb{Z}$, and thus only on the orbit $\mathfrak{o}$ of τ, we sometimes write $g(\mathfrak{o}) = g(\tau)$, for any/all $\tau \in \mathfrak{o}$.

Remark 2.6. For all $\tau \in \mathcal{T}$, $\dim_{\mathbb{C}}(V^+_{\tau^*}) = \dim_{\mathbb{C}}(V^-_\tau)$, and thus $\mathfrak{f}(\tau) + \mathfrak{f}(\tau^*) = g(\tau)$. □

Let $s = s(\mathfrak{o})$ be the number of distinct values of $\{\mathfrak{f}(\tau) \mid \tau \in \mathfrak{o}\}$ in the range $[1, g(\mathfrak{o}) - 1]$. We write these distinct values as

$$g(\mathfrak{o}) > E(1) > E(2) > \cdots > E(s) > 0.$$

Let $E(0) := g(\mathfrak{o})$ and $E(s+1) := 0$. Then $u(\mathfrak{o})$ has exactly $s+1$ distinct slopes, denoted by $0 \leq \lambda(0) < \lambda(1) < \cdots < \lambda(s) \leq 1$. For $0 \leq t \leq s$, the $(t+1)$-st slope is

$$\lambda(t) := \frac{1}{|\mathfrak{o}|} \#\{\tau \mid \mathfrak{f}_i(\tau) \geq E(t)\}. \tag{2.4}$$

The slope $\lambda(t)$ occurs in $u(\mathfrak{o})$ with multiplicity

$$\rho(\lambda(t)) := |\mathfrak{o}|(E(t) - E(t+1)). \tag{2.5}$$

[2.4] More precisely, following [22, §4.3], a Newton polygon appearing on $\mathrm{Sh}(H, \mathfrak{h})$ is a set $\{\nu(\mathfrak{o})\}_{\mathfrak{o} \in \mathfrak{O}}$, where each $\nu(\mathfrak{o})$ is a multi-set of slopes. On the other hand, following [34, §8.2], a Newton polygon attached to an element in $B(H_{\mathbb{Q}_p}, \mu_{\mathfrak{h}})$ is its image under the Newton map $\nu : B(H_{\mathbb{Q}_p}, \mu_{\mathfrak{h}}) \to (X_*(T) \otimes \mathbb{Q})^{\Gamma}_{\mathrm{dom}}$. Since we work with PEL-type Shimura varieties of types A and C, these two notions of Newton polygon coincide. Indeed, up to center (note that the center does not affect the Newton polygon), $H_{\mathbb{Q}_p} = \prod_{\mathfrak{o} \in \mathfrak{O}} H_{\mathfrak{o}}$, where $H_{\mathfrak{o}}$ is the restriction of scalars of a unitary group, or GL_n, or GSp_n. For such groups, one can check directly that these two notions of Newton polygons are equivalent. Moreover, $B(H_{\mathbb{Q}_p}, \mu_{\mathfrak{h}})$ can be identified with its image under ν due to [34, (8.6)] and the fact that an element in $B(H_{\mathbb{Q}_p})$ (notation as in *loc. cit.*) is determined by its image under the Newton map ν and the Kottwitz map κ.

2.7 Geometry of the Newton polygon strata on Sh

For $b \in B$, let $\mathrm{Sh}[b] := \mathrm{Sh}(H, \mathfrak{h})[b]$ denote the Newton polygon stratum for b in Sh. In other words, $\mathrm{Sh}[b]$ is the locally closed substack of Sh parametrizing abelian schemes with Newton polygon b. By Hamacher [14, Theorem 1.1, Corollary 3.12], based on the work of Chai [6], Mantovan [25], and Viehmann [33], and Kottwitz [18, Section 8],[2.5] on each irreducible component S of Sh, the substack $S[b]$ is non-empty and equidimensional and

$$\mathrm{codim}(S[b], \mathrm{Sh}) = \mathrm{length}(b), \tag{2.6}$$

where $\mathrm{length}(b) = \max\{n \mid \text{ there exists a chain } b = \nu_0 < \nu_1 < \cdots < \nu_n = u,\ \nu_i \in B\}$.

The Newton stratification extends to the toroidal and minimal compactifications $\overline{\mathrm{Sh}}, \mathrm{Sh}^*$. In [21, §3.3], the authors studied the Newton stratification on compactifications of PEL-type Shimura varieties at good primes. They proved in this case that all the Newton strata are so called *well-positioned* subschemes [21, Proposition 3.3.9]. In particular, by [21, Definition 2.2.1, Theorem 2.3.2], the set of Newton polygons on (each irreducible component of) $\overline{\mathrm{Sh}}$ is the same as that on Sh and, for any $b \in B$,

$$\mathrm{codim}(\overline{\mathrm{Sh}}[b], \overline{\mathrm{Sh}}) = \mathrm{codim}(\mathrm{Sh}[b], \mathrm{Sh}). \tag{2.7}$$

By the next remark, there exists a μ_m-cover of smooth curves having monodromy datum γ and μ-ordinary Newton polygon u if there exists such a cover of stable curves.

Lemma 2.7. The following are equivalent: $Z^\circ(\gamma)[u]$ is non-empty; $Z^\circ(\gamma)[u]$ is open and dense in $Z^c(\gamma)$; and $Z^c(\gamma)[u]$ is non-empty. □

Proof. This is clear because the Newton polygon is lower semi-continuous, $Z^c(\gamma)$ is irreducible, and $Z^\circ(\gamma)$ is open and dense in $Z^c(\gamma)$. ■

Remark 2.8. The Ekedahl–Oort type is also determined for many of the smooth curves in this paper. The reason is that the μ-ordinary Newton polygon stratum in these PEL-type Shimura varieties coincides with the unique open Ekedahl–Oort stratum. Hence one may compute the Ekedahl–Oort type of these smooth curves using [27, Section 1.2.3]. □

[2.5] Hamacher proved that $Sh[b]$ is non-empty and equidimensional of expected dimension. Since Hecke translations preserve the Newton polygon strata and act transitively on the irreducible components of Sh, we deduce the same result for $S[b]$. See [18, Section 8] for a more detailed discussion.

3 Clutching morphisms

3.1 Background

We study clutching morphisms, generalized to the context of curves that are μ_m-covers of $\mathbb{P}^1$. The clutching morphisms are the closed immersions [15, 3.9], for $1 \le i \le g$, as described below:

$$\kappa_{i,g-i} : \overline{\mathcal{M}}_{i;1} \times \overline{\mathcal{M}}_{g-i;1} \to \overline{\mathcal{M}}_g \text{ and } \lambda : \overline{\mathcal{M}}_{g-1;2} \to \overline{\mathcal{M}}_g.$$

Informally speaking, the morphism $\kappa_{i,g-i}$ takes a curve of genus i with a marked point and a curve of genus $g-i$ with a marked point and produces a singular curve by identifying the marked points in an ordinary double point. By definition, the image of $\kappa_{i,g-i}$ is the component Δ_i of the boundary of $\mathcal{M}_g$, whose generic point represents a stable curve that has two components, of genus i and $g-i$, intersecting in one point, which is an ordinary double point.

In our context, we sometimes need to clutch together two curves at more than one point. By construction, the dual graph of the resulting singular curve contains a cycle. Recall that a stable curve has *compact type* if its dual graph is a tree. By definition, Δ_0 is the component of the boundary of $\mathcal{M}_g$ whose points represent stable curves that do not have compact type. Informally speaking, the morphism λ takes a curve of genus $g-1$ with two marked points and produces a singular curve by identifying the marked points in an ordinary double point. The image of λ is the component Δ_0.

In this paper, we describe a clutching morphism, denoted κ, which shares attributes of both $\kappa_{i,g-i}$ and λ. The input for κ is a pair of cyclic covers of $\mathbb{P}^1$ and the output is a singular curve which is a cover of a tree of two projective lines. To provide greater flexibility, we include cases when the covers have different degrees and when the two covers are clutched together at several points. As a result, a curve in the image of κ is contained in Δ_i for some $1 \le i \le g$ and also may be contained in Δ_0.

Notation 3.1. Let $\gamma = (m, N, a)$ be a monodromy datum. For an integer $d \ge 1$, consider the induced datum $\gamma^{\dagger_d} = (dm, N, da)$, which we sometimes denote $\gamma^\dagger$. □

If $d > 1$, then $\gamma^{\dagger_d}$ is not a monodromy datum because it does not satisfy the gcd condition; this does not cause any difficulties. Suppose $\phi : C \to \mathbb{P}^1$ is a μ_m-cover with monodromy datum γ. Consider the induced curve $\mathrm{Ind}_m^{dm} C$, which consists of d copies of C, indexed by the cosets of $\mu_m \subset \mu_{dm}$. From the induced action of μ_{dm} on $\mathrm{Ind}_m^{dm} C$, there is a μ_{dm}-cover $\mathrm{Ind}_m^{dm}(\phi) : \mathrm{Ind}_m^{dm} C \to \mathbb{P}^1$; we say that it has induced datum $\gamma^{\dagger_d}$. The signature type of $\mathrm{Ind}_m^{dm}(\phi)$ is $\mathfrak{f}^{\dagger_d} = \mathfrak{f} \circ \pi_d$, where $\pi_d : \mathbb{Z}/dm\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ denotes the natural projection. By (2.4) and (2.5), the μ-ordinary polygon of $\mathrm{Sh}(\mu_{dm}, \mathfrak{f}^{\dagger_d})$ is $u^{\dagger_d} = u^d$.

3.2 Numerical data and hypothesis (A)

Notation 3.2. Fix integers $m_1, m_2 \ge 2$, $N_1, N_2 \ge 3$. Let $m_3 = \mathrm{lcm}(m_1, m_2)$. For $i = 1, 2$: let $d_i = m_3/m_i$; let $a_i = (a_i(1), \ldots, a_i(N_i))$ be such that $\gamma_i = (m_i, N_i, a_i)$ is a (generalized) monodromy datum; and let $g_i =$

$g(m_i, N_i, a_i)$ as in (2.2). □

Definition 3.3. A pair of monodromy data $\gamma_1 = (m_1, N_1, a_1)$, $\gamma_2 = (m_2, N_2, a_2)$ as in Notation 3.2 is *admissible* if it satisfies

$$\text{hypothesis(A)}: \ d_1 a_1(N_1) + d_2 a_2(1) \equiv 0 \bmod m_3.$$

□

Notation 3.4. Assume hypothesis (A) for the pair γ_1, γ_2. Set $r_1 = \gcd(m_1, a_1(N_1))$, and $r_2 = \gcd(m_2, a_2(1))$. Let $r_0 = \gcd(r_1, r_2)$ and let

$$\epsilon = d_1 d_2 r_0 - d_1 - d_2 + 1 \text{ and } g_3 = d_1 g_1 + d_2 g_2 + \epsilon. \tag{3.1}$$

□

Note that $d_1 r_1 = d_2 r_2 = d_1 d_2 r_0$ since $\gcd(d_1, d_2) = 1$.

Definition 3.5. If γ_1, γ_2 is an admissible pair of (generalized) monodromy data, we define $\gamma_3 = (m_3, N_3, a_3)$ by $m_3 := \operatorname{lcm}(m_1, m_2)$, $N_3 := N_1 + N_2 - 2$ and the N_3-tuple a_3 as

$$a_3(i) := \begin{cases} d_1 a_1(i) \text{ for } 1 \leq i \leq N_1 - 1, \\ d_2 a_2(i - N_1 + 2) \text{ for } N_1 \leq i \leq N_1 + N_2 - 2. \end{cases}$$

□

Lemma 3.6. The triple γ_3 from Definition 3.5 is a (generalized) monodromy datum. If $\phi_3 : C \to \mathbb{P}^1$ is a cover with monodromy datum γ_3, then the genus of C is g_3 as in (3.1). □

Proof. Most of the properties are immediate from Definition 2.1 and (2.2). The main point to check is that $\gcd(m_3, a_3(1), \ldots, a_3(N_3)) = 1$. To see this, note that $1 = \gcd(m_1, a_1(1), \ldots, a_1(N_1)) = \gcd(m_1, a_1(1), \ldots, a_1(N_1 - 1))$ because $\sum_i a_1(i) \equiv 0 \bmod m_1$. So $\gcd(m_3, d_1 a_1(1), \ldots, d_1 a_1(N_1 - 1)) = d_1$. Similarly, $\gcd(m_3, a_2(2), \ldots, a_2(N_2)) = d_2$. Also $\gcd(d_1, d_2) = 1$ since $m_3 = \operatorname{lcm}(m_1, m_2)$. Hence 1 is a $\mathbb{Z}$-linear combination of d_1 and d_2, and thus a $\mathbb{Z}$-linear combination of $m_3, d_1 a_1(1), \ldots, d_1 a_1(N_1 - 1), d_2 a_2(2), \ldots, d_2 a_2(N_2)$. ■

The signature type for ϕ_3 is given in Definition 3.13, see Lemma 3.14.

Remark 3.7. A pair γ_1, γ_2 of non-admissible monodromy data can be modified slightly to produce a pair γ_1', γ_2' of admissible generalized monodromy data by marking an extra unramified fiber. Specifically, let

1. $\gamma_1' = (m_1, N_1 + 1, a_1')$ with $a_1'(i) = a_1(i)$ for $1 \leq i \leq N_1$, and $a_1'(N_1 + 1) = 0$;
2. $\gamma_2' = (m_2, N_2 + 1, a_2')$ with $a_2'(1) = 0$ and $a_2'(i) = a_2(i - 1)$ for $2 \leq i \leq N_2 + 1$.

This does not change the geometry, because $Z(\gamma_i') = Z(\gamma_i)$ for $i = 1, 2$ by Remark 2.3. □

3.3 Clutching morphisms for cyclic covers

Notation 3.8. Let $\gamma_1 = (m_1, N_1, a_1)$, $\gamma_2 = (m_2, N_2, a_2)$ be an admissible pair of monodromy data as in Notation 3.2. Let $\gamma_3 = (m_3, N_3, a_3)$ be the monodromy datum from Definition 3.5. For $i = 1, 2, 3$, let $\widetilde{Z}_i = \widetilde{Z}(\gamma_i)$ be as in Definition 2.2. □

Recall that the points of $\widetilde{Z}_3^\circ$ represent μ_{m_3}-covers $C \to \mathbb{P}^1$ with monodromy datum γ_3, where C is smooth. The next result is well-understood but we could not find it stated in this level of generality in the literature.

Proposition 3.9. If hypothesis (A) (the admissible condition) is satisfied, there is a clutching morphism $\kappa : \widetilde{Z}_1 \times \widetilde{Z}_2 \to \widetilde{Z}_3$, and the image of κ is in the complement of $\widetilde{Z}_3^\circ$ in $\widetilde{Z}_3$. □

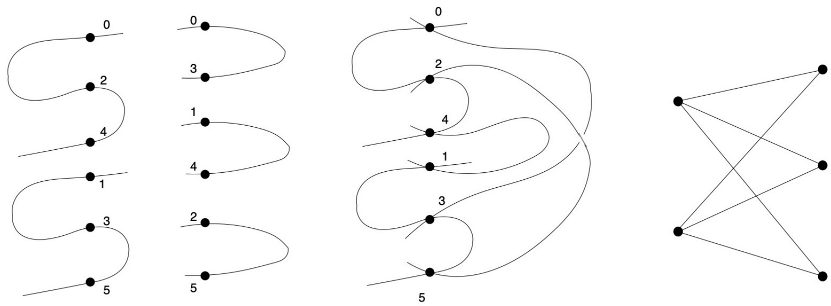
Proof. For $i = 1, 2$, let $\phi_i : C_i \to \mathbb{P}^1$ be the μ_{m_i}-cover with N_i ordered and labeled μ_{m_i}-orbits of points which is represented by a point of $\widetilde{Z}_i$. The fact that these μ_{m_i}-orbits are labeled comes from Section 2.4 since the moduli data includes the choice of one ramified point in each ramified fiber. There is a natural inclusion $\mu_{m_i} \subset \mu_{m_3}$. Let $C_i^\dagger = \mathrm{Ind}_{m_i}^{m_3}(C_i)$ be the induced curve and let $\phi_i^\dagger : C_i^\dagger \to \mathbb{P}^1$ be the induced cover from Notation 3.1. It has inertia type $a_i^\dagger = d_i a_i = (d_i a_i(1), \ldots, d_i a_i(N_i))$.

We define the morphism κ on the pair (ϕ_1, ϕ_2). Let F_1 (resp. F_2) be the set of points of $C_1^\dagger$ above $t_1(N_1)$ (resp. $C_2^\dagger$ above $t_2(1)$). Then $\#F_1 = d_1 r_1 = d_1 \gcd(m_1, a_1(N_1))$ and $\#F_2 = d_2 r_2 = d_2 \gcd(m_2, a_2(1))$. By hypothesis (A), $\#F_1 = \#F_2$. The inertia group at each point in F_1 (resp. F_2) is the unique subgroup R of order $m_3/d_1 r_1$ in μ_{m_3}. In addition, the points of F_1 (resp. F_2) are labeled by the cosets of R in μ_{m_3}. Let C_3 be the curve whose components are the d_1 components of $C_1^\dagger$ and the d_2 components of $C_2^\dagger$, formed by identifying each point in F_1 with the point in F_2 labeled by the same coset, in an ordinary double point.

Then C_3 is a μ_{m_3}-cover of a tree P of two projective lines. It has N_3 labeled μ_{m_3}-orbits with inertia type a_3 and is thus represented by a point of $\widetilde{Z}_3$. The admissible condition in Definition 3.3 is exactly the (local) admissible condition on the covers $\phi_1^\dagger$ and $\phi_2^\dagger$ at the ordinary double points formed by identifying each point of F_1 with a point in F_2. By [10, 2.2], the μ_{m_3}-cover $C_3 \to P$ is in the boundary of $\widetilde{Z}_3^\circ$ if and only if hypothesis (A) is satisfied. ■

The curve C_3 constructed in the proof of Proposition 3.9 is a μ_{m_3}-cover of type γ_3 and thus has arithmetic genus g_3 by Lemma 3.6.

Example 3.10. Let $m_1 = 3$ and $m_2 = 2$. For $i = 1, 2$, let $C_i \to \mathbb{P}^1$ be a μ_{m_i}-cover. Assume $a_1(N_1) = a_2(1) = 0$. The images below show: the induced curves $C_1^\dagger$ and $C_2^\dagger$, each with one labeled fiber with $m_3 = 6$ points; the curve C_3 constructed in the proof of Proposition 3.9; and the dual graph of C_3 with $\epsilon = 2$.



induced curves clutched curve dual graph

□

Proposition 3.11. The curve C_3 constructed in the proof of Proposition 3.9 has Newton polygon

$$\nu(C_3) \quad = \quad \nu(C_1)^{d_1} \oplus \nu(C_2)^{d_2} \oplus \mathrm{ord}^{\epsilon}. \tag{3.2}$$

It has compact type if and only if $\epsilon = 0$. □

The term ord^{ϵ} can be viewed as the *defect* of $\nu(C_3)$. It measures the number of extra slopes of 0 and 1 that arise when C_3 does not have compact type. By Notation 3.4, $\epsilon = 0$ if and only if $r_0 = 1$ and either $d_1 = 1$ or $d_2 = 1$.

Proof. By [24, Chapter 10, Proposition 1.15(b)], the toric rank of C_3 is the Euler characteristic of its dual graph. By construction, the dual graph of C_3 is a bipartite graph, with d_1 (resp. d_2) vertices in bijection with the components of $C_1^{\dagger}$ (resp. $C_2^{\dagger}$). In C_3, each of the components coming from $C_1^{\dagger}$ intersects each of the components coming from $C_2^{\dagger}$ in r_0 points. After removing $d_1 d_2(r_0 - 1)$ edges from the dual graph, there is a unique edge between each pair of vertices on opposite sides. After removing another $(d_1 - 1)(d_2 - 1)$ edges from the dual graph, it is a tree. Thus the Euler characteristic of the dual graph of C_3 is $d_1 d_2(r_0 - 1) + (d_1 - 1)(d_2 - 1)$, which equals ϵ. In particular, C_3 has compact type if and only if $\epsilon = 0$.

By [4, Section 9.2, Example 8], for some torus T of rank ϵ, there is a short exact sequence

$$0 \to T \to \mathrm{Jac}(C_3) \to \mathrm{Jac}(C_1)^{d_1} \oplus \mathrm{Jac}(C_2)^{d_2} \to 0.$$

Since $\dim_{\mathbb{F}_p}(\mu_p, T) = \epsilon$, the Newton polygon of $\mathrm{Jac}(C_3)$ is the amalgamate sum (Section 2.2) of the Newton polygon of $\mathrm{Jac}(C_1)^{d_1} \oplus \mathrm{Jac}(C_2)^{d_2}$ and ord^{ϵ}, which yields (3.2). ■

3.4 The signature

We find the signature $\mathfrak{f}_3$ for a cover with monodromy datum γ_3.

Definition 3.12. Let $d, R \in \mathbb{Z}_{\geq 1}$ with $dR | m$. For $n \in \mathbb{Z}/m\mathbb{Z}$, we define $\delta_{d,dR}(n) := 1$ if $dRn \equiv 0 \bmod m$ and $dn \not\equiv 0 \bmod m$, and $\delta_{d,dR}(n) := 0$ otherwise. □

Equivalently, if $\tau \in \mathcal{T}$, then $\delta_{d,dR}(\tau) = 1$ if the order of τ in $\mathbb{Z}/m\mathbb{Z}$ divides dR but not d, and $\delta_{d,dR}(\tau) = 0$ otherwise. Since $\delta_{d,dR}(\tau)$ only depends on the orbit $\mathfrak{o}$ of τ, we also write $\delta_{d,dR}(\mathfrak{o}) := \delta_{d,dR}(\tau)$, for any/all $\tau \in \mathfrak{o}$.

Definition 3.13. Let $\delta := \delta_{d,dr_0} + \delta_{d_1,d_1d_2} - \delta_{1,d_2}$, for $d = d_1d_2$. For $n \in \mathbb{Z}/m_3\mathbb{Z} - \{0\}$, let

$$\mathfrak{f}_3(\tau_n) = \mathfrak{f}_1^\dagger(\tau_n) + \mathfrak{f}_2^\dagger(\tau_n) + \delta(n). \tag{3.3}$$

□

By definition, $\delta(n) = 1$ if $d_1d_2r_0n \equiv 0 \bmod m_3$ and $d_1n \not\equiv 0 \bmod m_3$ and $d_2n \not\equiv 0 \bmod m_3$, and $\delta(n) = 0$ otherwise.

Lemma 3.14. If $\phi_3 : C \to \mathbb{P}^1$ is a cover with (generalized) monodromy datum γ_3, as defined in Definition 3.5, then the signature type of C_3 is $\mathfrak{f}_3$. □

Proof. We use (2.3) to compute $\mathfrak{f}_3$.[3.1] If $n \equiv 0 \bmod m_1$, then $\mathfrak{f}_1^\dagger(\tau_n) = 0$ and $\mathfrak{f}_3(\tau_n) = \mathfrak{f}_2^\dagger(\tau_n)$. If $n \equiv 0 \bmod m_2$, then $\mathfrak{f}_2^\dagger(\tau_n) = 0$ and $\mathfrak{f}_3(\tau_n) = \mathfrak{f}_1^\dagger(\tau_n)$. For $n \in \mathbb{Z}/m_3\mathbb{Z}$, with $n \not\equiv 0 \bmod m_1$ and $n \not\equiv 0 \bmod m_2$, then

$$(\mathfrak{f}_1^\dagger(\tau_n) + 1) + (\mathfrak{f}_2^\dagger(\tau_n) + 1) - (\mathfrak{f}_3(\tau_n) + 1) = \langle \frac{-nd_1a_1(N_1)}{m_3} \rangle + \langle \frac{-nd_2a_2(1)}{m_3} \rangle.$$

The right hand side is 0 or 1; it is 0 if and only if $nd_1a_1(N_1) \equiv nd_2a_2(1) \equiv 0 \bmod m_3$. ■

3.5 Compatibility with Shimura variety setting

Notation 3.15. Fix an admissible pair $\gamma_1 = (m_1, N_1, a_1)$, $\gamma_2 = (m_2, N_2, a_2)$ of monodromy data as in Notation 3.2. Consider the monodromy datum γ_3 as in Definition 3.5. In particular, let $m_3 = \mathrm{lcm}(m_1, m_2)$ and let $\mathfrak{f}_3$ be as in Definition 3.13.

For each $i = 1, 2, 3$, let $Z_i := Z(m_i, N_i, a_i)$, and similarly Z_i°, $\widetilde{Z}_i^c$, etc as in Definition 2.2. Let $\mathrm{Sh}_i := \mathrm{Sh}_i(\mu_{m_i}, \mathfrak{f}_i)$ denote the Shimura substack of $\mathcal{A}_g$ as in Section 2.5. Let $\mathcal{X}_i$ be the universal abelian scheme over Sh_i, $B_i := B(\mathrm{Sh}_i)$ the set of Newton polygons of Sh_i, and u_i the μ-ordinary Newton polygon in B_i from Definition 2.5. □

Via the Torelli map T, the clutching morphism $\kappa : \widetilde{Z}_1^c \times \widetilde{Z}_2^c \to \widetilde{Z}_3$ is compatible with a morphism into the minimal compactification of the Shimura variety

$$\iota \quad : \quad \mathrm{Sh}_1 \times \mathrm{Sh}_2 \to \mathrm{Sh}_3^*$$

[3.1] Alternatively, one may deduce the formula for $\mathfrak{f}_3$ geometrically since the extra term δ records the $\mathbb{Z}/m_3\mathbb{Z}$-action on the dual graph of the curve C_3 constructed in Proposition 3.9.

where $\iota(\mathcal{X}_1, \mathcal{X}_2) := \mathcal{X}_1^{d_1} \oplus \mathcal{X}_2^{d_2}$.[3.2] If $\epsilon = 0$ then $\mathrm{Im}(\iota)$ lies in Sh_3 and the reader may focus on this case; if $\epsilon \neq 0$, then $\mathrm{Im}(\iota)$ is contained in the boundary $\mathrm{Sh}_3^* - \mathrm{Sh}_3$.

By Proposition 3.11, $\iota(\mathrm{Sh}_1[\nu_1], \mathrm{Sh}_2[\nu_2]) \subseteq \mathrm{Sh}_3^*(\nu_1^{d_1} \oplus \nu_2^{d_2} \oplus \mathrm{ord}^\epsilon)$, which yields:

Lemma 3.16. If $\nu_i \in B_i$, then $\nu_1^{d_1} \oplus \nu_2^{d_2} \oplus \mathrm{ord}^\epsilon \in B_3$. In particular, $u_3 \geq u_1^{d_1} \oplus u_2^{d_2} \oplus \mathrm{ord}^\epsilon$. □

In Proposition 4.4, we give a necessary and sufficient condition on the pair of signature types $(\mathfrak{f}_1, \mathfrak{f}_2)$ for the equality $u_3 = u_1^{d_1} \oplus u_2^{d_2} \oplus \mathrm{ord}^\epsilon$ to hold.

4 The Torelli locus and the μ-ordinary locus of Shimura varieties

In this section, we prove theorems about the intersection of the open Torelli locus with the μ-ordinary Newton polygon stratum in a PEL-type Shimura variety. The main result, Theorem 4.5, provides a method to leverage information from smaller dimension to larger dimension. This provides an inductive method to prove that the open Torelli locus intersects the μ-ordinary stratum for certain types of families.

In Section 4.3, we use the main theorem to establish the existence of smooth curves of arbitrarily large genus with prescribed Newton polygon, see Corollary 4.7 to 4.10. For the base cases of the inductive method, we can use any instances when the μ-ordinary Newton polygon is known to occur (see Proposition 4.6).

Remark 4.1. The method in this section does not give results for every monodromy datum γ. For example, it is not known whether the μ-ordinary Newton polygon occurs on $Z^\circ(\gamma)$ for all $p \equiv 3, 5 \bmod 7$ when $\gamma = (7, 4, (1, 1, 2, 3))$. In this case, $\mathfrak{f} = (2, 1, 1, 1, 1, 0)$, $\dim(Z^c(\gamma)) = 1$, and $\dim(S(\gamma)) = 2$. The three Newton polygons on $S(\gamma)$ are $(1/6, 5/6)$, $(1/3, 2/3)^2$, and ss^6, which all have p-rank 0. None of the degenerations for this family satisfy hypothesis (B) as defined below. □

4.1 Hypothesis (B)

We fix an admissible pair $\gamma_1 = (m_1, N_1, a_1)$, $\gamma_2 = (m_2, N_2, a_2)$ of (generalized) monodromy data as in Notation 3.2. We fix a prime p such that $p \nmid m_3 = \mathrm{lcm}(m_1, m_2)$ and work over $\overline{\mathbb{F}}_p$. Recall Notation 3.4 and 3.15. So $d_i = m_3/m_i$ and $\mathfrak{f}_i^\dagger := \mathfrak{f}_i \circ \pi_i$ for $i = 1, 2$.

Definition 4.2. The pair of monodromy data γ_1, γ_2 is *balanced* if, for each orbit $\mathfrak{o} \in \mathcal{T} = \mathrm{Hom}_\mathbb{Q}(\mathbb{Q}[\mu_{m_3}], \mathbb{C})$ and all $\omega, \tau \in \mathfrak{o}$, the values of the induced signature types satisfy:

$$\text{hypothesis(B)} : \text{if } \mathfrak{f}_1^\dagger(\omega) > \mathfrak{f}_1^\dagger(\tau) \text{ then } \mathfrak{f}_2^\dagger(\omega) \geq \mathfrak{f}_2^\dagger(\tau);\ \text{if } \mathfrak{f}_2^\dagger(\omega) > \mathfrak{f}_2^\dagger(\tau) \text{ then } \mathfrak{f}_1^\dagger(\omega) \geq \mathfrak{f}_1^\dagger(\tau).$$

□

[3.2] An abelian variety of dimension less than g_3 with μ_m-action can be viewed as a point on the boundary of Sh_3^* as it comes from the pure part of some semi-abelian variety of dimension g_3 with μ_m-action, which is a point on $\overline{\mathrm{Sh}_3}$. The image of the Torelli map in the minimal compactification is determined by the Torelli map on the irreducible components of the curve and forgets the dual graph structure.

Remark 4.3. 1. If $m_1 = m_2$, then hypothesis (B) is symmetric for the pair γ_1, γ_2.

2. If $\gamma_1 = \gamma_2$, then hypothesis (B) is automatically satisfied.
3. Hypothesis (B) depends (only) on the congruence of p modulo m_3. If $p \equiv 1 \bmod m_3$, then each orbit $\mathfrak{o}$ in $\mathcal{T}$ has size one and hypothesis (B) is vacuously satisfied.
4. Let γ_3 be as in Definition 3.5. If the pair γ_1, γ_2 satisfies hypotheses (A) and (B), then γ_i, γ_3 satisfies hypothesis (B) for $i = 1$ and for $i = 2$.

□

Proposition 4.4 below gives a geometric interpretation of hypothesis (B). From Notation 3.4, recall the formula $\epsilon = d_1 d_2 r_0 - d_1 - d_2 + 1$.

Proposition 4.4. Let γ_1, γ_2 be an admissible pair of monodromy data. Consider the monodromy datum γ_3 as in Definition 3.5. For $1 \leq i \leq 3$, let u_i be the μ-ordinary Newton polygon of the Shimura variety Sh_i as in Definitions 2.5 and 3.15. Then the equality $u_3 = u_1^{d_1} \oplus u_2^{d_2} \oplus \mathrm{ord}^{\epsilon}$ holds if and only if the pair γ_1, γ_2 is balanced. □

We postpone the proof of Proposition 4.4 to the independent Section 5.

4.2 A first main result

In this subsection, we assume that the pair γ_1, γ_2 is admissible and balanced, meaning that it satisfies hypotheses (A) and (B) as in Definitions 3.3 and 4.2. Let $\gamma_3 = (m_3, N_3, a_3)$ and $\mathfrak{f}_3$ be as in Definitions 3.5 and 3.13.

The next result provides a partial positive answer to Conjecture 1.1 when $\epsilon = 0$.

Theorem 4.5. Let γ_1, γ_2 be an admissible, balanced pair of monodromy data. If $Z_1^\circ[u_1]$ and $Z_2^\circ[u_2]$ are both non-empty, then $Z_3^\circ[u_3]$ is non-empty. □

Proof. By Lemma 3.6 and 3.14, the signature for the monodromy datum γ_3 is given in (3.3). By Proposition 4.4, hypothesis (B) implies that $u_3 = u_1^{d_1} \oplus u_2^{d_2} \oplus \mathrm{ord}^{\epsilon}$. From the hypothesis, $\widetilde{Z}_1^\circ[u_1]$ and $\widetilde{Z}_2^\circ[u_2]$ are both non-empty. By Proposition 3.9, the image of κ on $\widetilde{Z}_1^\circ[u_1] \times \widetilde{Z}_2^\circ[u_2]$ is in $\widetilde{Z}_3$. By Proposition 3.11, the Newton polygon of a curve C_3 represented by a point in the image of κ is given by $\nu(C_3) = u_1^{d_1} \oplus u_2^{d_2} \oplus \mathrm{ord}^{\epsilon}$, which is u_3. Thus $Z_3[u_3]$ is non-empty and applying Lemma 2.7 finishes the proof. ■

4.3 Infinite clutching for μ-ordinary

In this section, we find situations in which Theorem 4.5 can be implemented recursively, infinitely many times, to verify the existence of smooth curves of arbitrarily large genus with prescribed Newton polygon. The required input is a family (or a compatible pair of families) of cyclic covers of $\mathbb{P}^1$ for which the μ-ordinary Newton polygon at a prime p is known to occur (see Proposition 4.6). Section 9 contains concrete implementations of these results.

4.3.1 *Base cases*

We recall instances when the the μ-ordinary Newton polygon u is known to occur for the Jacobian of a (smooth) curve in the family Z.

Proposition 4.6. The μ-ordinary Newton polygon stratum $Z(m, N, a)[u]$ is non-empty if either:

1. $N = 3$; or
2. $N \geq 4$ and (m, N, a) is equivalent to one of the twenty examples in [28, Table 1]; or
3. u is the only Newton polygon in $B(H_{\mathbb{Q}_p}, \mu_{\mathfrak{h}})$ of maximum p-rank and either $N = 4$ or $p \geq m(N-3)$ or $p \equiv \pm 1 \bmod m$.

□

Proof. 1. When $N = 3$, then Z is 0-dimensional and thus special as in Definition 2.4.

2. This is [22, Proposition 5.1].
3. For any monodromy datum $\gamma = (m, N, a)$, define

$$\beta(\gamma) := \sum_{\tau \in \mathcal{T}} \min_{j \in \mathbb{N}} \{\mathfrak{f}(\tau^{\sigma^j})\} = \sum_{\mathfrak{o} \in \mathfrak{O}} \#\mathfrak{o} \cdot \min\{\mathfrak{f}(\tau) \mid \tau \in \mathfrak{o}\}. \tag{4.1}$$

By [5, Equation (1)], $\beta(\gamma)$ is an upper bound for the p-rank of curves in $Z^\circ(\gamma)$. By [5, Theorem 6.1, Propositions 7.7, 7.4, 7.8], if $p \geq m(N-3)$ or $N = 4$ or $p \equiv \pm 1 \bmod m$, then there exists a μ_m-cover $C \to \mathbb{P}^1$ defined over $\overline{\mathbb{F}}_p$ with monodromy datum γ, for which the p-rank of C equals $\beta(\gamma)$. The p-rank is the multiplicity of 1 as a slope of the Newton polygon. By the formulas (2.4) and (2.5) for the slopes and multiplicities of the μ-ordinary Newton polygon $u = u(\gamma)$, the p-rank of u equals $\beta(\gamma)$. ■

To determine the μ-ordinary formula u, we refer to the Shimura–Taniyama formula [32, Section 5] (see also [23, Theorem 3.2]) when $N = 3$ and to [22, Section 6] and Section 10 for the special families of [28].

If $p \equiv -1 \bmod m$, then all Newton polygons in $B = B(H_{\mathbb{Q}_p}, \mu_{\mathfrak{h}})$ have slopes in $\{0, 1/2, 1\}$. For $0 \leq f \leq g$, the unique symmetric Newton polygon of height g and p-rank f with slopes in $\{0, 1/2, 1\}$ is $ord^f \oplus ss^{g-f}$. Hence, the Newton polygons in B are uniquely determined by their p-ranks. For examples of families where the μ-ordinary Newton polygon is not ordinary, see [22, Section 7.2].

4.3.2 *Adding slopes 0 and 1*

By implementing Theorem 4.5 recursively, we obtain a method to increase the genus and the multiplicity of the slopes 0 and 1 in the Newton polygon by the same amount. Because of this, in later results we will aim to minimize the multiplicity of $\{0, 1\}$ in the Newton polygon.

Corollary 4.7. Let $\gamma = (m, N, a)$ be a monodromy datum. Assume that $Z^\circ(\gamma)[u]$ is non-empty. Then for any n in the semi-group of $(\mathbb{Z}, +)$ generated by $\{m - t\colon\ t \mid m\}$, there exists a μ_m-cover $C \to \mathbb{P}^1$ over $\overline{\mathbb{F}}_p$ where C is a smooth curve with Newton polygon $u \oplus \mathrm{ord}^n$. □

Proof. For $c \in \mathbb{Z}_{\geq 1}$ with $c \leq m-1$, let $t = \gcd(m, c)$ and consider the monodromy datum $\gamma_1 = (m/t, 3, (c/t, (m-c)/t, 0))$. Note $g_1(\tau) = 0$, and $\mathfrak{f}_1(\tau) = 0$, for all $\tau \in \mathcal{T}$.

Let $\gamma_2 = (m, N+1, a')$, where $a'(1) = 0$, and $a'(i) = a(i-1)$, for $i = 2, \ldots, N+1$. By construction, the pair γ_1, γ_2 is admissible and balanced. By Remark 2.3, $Z^\circ(\gamma_2)[u]$ is non-empty. Set $a_3 = (c, m-c, a(1), \ldots, a(N))$; then $\gamma_3 = (m, N+2, a_3)$ is the monodromy datum from Definition 3.5 for the pair γ_1, γ_2. By (3.1), $\epsilon = m - t$. By Theorem 4.5, $Z_3^\circ[u_3]$ is non-empty, where $u_3 = u \oplus \mathrm{ord}^{m-t}$. The statement follows by iterating this construction, letting c vary. ■

4.3.3 *Single Induction*

We consider inductive systems generated by a single monodromy datum γ_1. The next result follows from the observation that if the pair γ_1, γ_1 is admissible and if $Z_1^\circ[u_1]$ is non-empty, then all hypotheses of Theorem 4.5 are satisfied, and continue to be after iterations.

Corollary 4.8. Assume there exist $1 \leq i < j \leq N$ such that $a(i) + a(j) \equiv 0 \bmod m$.[4.1] Let $r = \gcd(a(i), m)$. If $Z^\circ(\gamma)[u]$ is non-empty, then there exists a smooth curve over $\overline{\mathbb{F}}_p$ with Newton polygon $u^n \oplus \mathrm{ord}^{(n-1)(r-1)}$, for any $n \in \mathbb{Z}_{\geq 1}$. □

Proof. After reordering the branch points, we can suppose that $i = 1$, $j = N$. We define a sequence of families $Z^{\times n}$ as follows: let $Z^{\times 1} = Z$; for $n \geq 2$, let $Z^{\times n}$ be the family constructed from the monodromy datum produced by applying Definition 3.5 to the monodromy data of $Z^{\times (n-1)}$ and $Z^{\times 1}$. For $n \in \mathbb{Z}_{\geq 1}$, the pair of monodromy data for $Z^{\times n}$ and $Z^{\times 1}$ satisfies hypotheses (A) and (B). Then $u_n := u^n \oplus \mathrm{ord}^{(n-1)(r-1)}$ is the μ-ordinary Newton polygon for $Z^{\times n}$. The statement follows by applying Theorem 4.5 repeatedly. ■

The first hypothesis of Corollary 4.8 appears restrictive. However, from any monodromy datum γ with $Z^\circ(\gamma)[u]$ non-empty, we can produce a new monodromy datum which satisfies this hypothesis by clutching with a μ_m-cover branched at only two points. As a result, Corollary 4.8 can be generalized to Corollary 4.9 which holds in much greater generality, at the expense of making the defect slightly larger.

Corollary 4.9. Assume that $Z^\circ(\gamma)[u]$ is non-empty and let t be a positive divisor of m. Then there exists a smooth curve over $\overline{\mathbb{F}}_p$ with Newton polygon $u^n \oplus \mathrm{ord}^{mn-n-t+1}$, for any $n \in \mathbb{Z}_{\geq 1}$. □

Proof. For $t = m$, consider the family $Z^{\times 1}$ with monodromy datum $(m, N+2, a')$, where $a'(i) = a(i)$ for $1 \leq i \leq N$, $a'(N+1) = a'(N+2) = 0$. Then, the statement follows from Corollary 4.8 applied to $Z^{\times 1}$ for $r = m$.

For $t < m$, consider the family $Z^{\times 1}$ with monodromy datum $(m, N+2, a')$, where $a'(i) = a(i)$ for $1 \leq i \leq N$, $a'(N+1) = t$ and $a'(N+2) = m - t$. The μ-ordinary polygon u' of the associated Shimura variety is $u \oplus \mathrm{ord}^{m-t}$. By Corollary 4.7, $(Z^{\times 1})^\circ[u']$ is non-empty. Then, the statement follows from Corollary 4.8 applied to $Z^{\times 1}$ for $r = t$, by observing that $(u')^n \oplus \mathrm{ord}^{(n-1)(t-1)} = u^n \oplus \mathrm{ord}^{mn-n-t+1}$. ■

[4.1] This condition implies that $\overline{Z}(\gamma)$ intersects the boundary component Δ_0 of $\mathcal{M}_g$.

4.3.4 *Double Induction*

We next consider inductive systems constructed from a pair of monodromy data satisfying hypotheses (A) and (B). For clarity, we state Corollary 4.10 under the simplifying assumption $m_1 = m_2$. Corollary 9.9 contains an example of this result; it also applies to the pair of monodromy data in the proof of Corollary 9.7.

Corollary 4.10. Let γ_1 and γ_2 be a pair of monodromy data with $m_1 = m_2$ satisfying hypotheses (A) and (B). Let $r = \gcd(m, a_1(N_1))$ and recall Notation 3.15. Assume that $Z_1^\circ[u_1]$ and $Z_2^\circ[u_2]$ are both non-empty. Then there exists a smooth curve over $\overline{\mathbb{F}}_p$ with Newton polygon $u_1^{n_1} \oplus u_2^{n_2} \oplus \mathrm{ord}^{(n_1+n_2-2)(m-1)+(r-1)}$ for any $n_1, n_2 \in \mathbb{Z}_{\geq 1}$. □

Proof. We apply Corollary 4.9 to the family Z_1 (resp. Z_2) with $t = m$. The result is a family $Z_1^{\times n_1}$ (resp. $Z_2^{\times n_2}$) of smooth curves with Newton polygon $u_1^{n_1} \oplus \mathrm{ord}^{(n_1-1)(m-1)}$ (resp. $u_2^{n_2} \oplus \mathrm{ord}^{(n_2-1)(m-1)}$). Since Z_1 and Z_2 satisfy hypotheses (A) and (B), due to the construction in the proof of Corollary 4.9, $Z_1^{\times n_1}$ and $Z_2^{\times n_2}$ also satisfy hypothesizes (A) and (B). The result then follows from Theorem 4.5. ■

5 Hypothesis (B) and the μ-ordinary Newton polygon

In this section, we prove Proposition 4.4, namely that hypothesis (B) for a pair of signatures $(\mathfrak{f}_1, \mathfrak{f}_2)$ is a necessary and sufficient condition for the associated Shimura variety $\mathrm{Sh}_1 \times \mathrm{Sh}_2$ to intersect the μ-ordinary Newton polygon stratum of Sh_3^*. Recall that σ is Frobenius and $\mathfrak{O}$ is the set of orbits of σ in $\mathcal{T}$.

Notation 5.1. For a σ-orbit $\mathfrak{o} \in \mathfrak{O}$, let $\mathfrak{p}_\mathfrak{o}$ be the prime of $\mathbb{Q}[\mu_m]$ above p associated with $\mathfrak{o}$ and let $|\mathfrak{o}|$ be the size of the orbit. For each $\tau \in \mathfrak{o}$, the order of τ in $\mathbb{Z}/m\mathbb{Z}$ is constant and denoted $e_\mathfrak{o}$; by definition, $e_\mathfrak{o} \mid m$. Let $\mathbb{Q}[\mu_m]_{\mathfrak{p}_\mathfrak{o}}$ denote the local field which is the completion of $K_{e_\mathfrak{o}}$ along the prime $\mathfrak{p}_\mathfrak{o}$. □

Let $\mathcal{X}$ denote the universal abelian scheme over $\mathrm{Sh} = \mathrm{Sh}(H, \mathfrak{h})$[5.1], and $\mathcal{X}[p^\infty]$ the associated p-divisible group scheme. Let $x \in \mathrm{Sh}(\overline{\mathbb{F}}_p)$ and consider the abelian variety $X := \mathcal{X}_x$. Let $\nu = \nu(X)$ be the Newton polygon of X. We omit the proof of the following.

Lemma 5.2. The $\mathbb{Q}[\mu_m]$-action of $\mathcal{X}$ induces a $\mathbb{Q}[\mu_m] \otimes_\mathbb{Q} \mathbb{Q}_p$-action on $\mathcal{X}[p^\infty]$. Thus it induces canonical decompositions

$$\mathcal{X}[p^\infty] = \bigoplus_{\mathfrak{o} \in \mathfrak{O}} \mathcal{X}[\mathfrak{p}_\mathfrak{o}^\infty] \text{ and } \nu = \bigoplus_{\mathfrak{o} \in \mathfrak{O}} \nu(\mathfrak{o}),$$

where, for each $\mathfrak{o} \in \mathfrak{O}$, the group scheme $\mathcal{X}[\mathfrak{p}_\mathfrak{o}^\infty]$ is a p-divisible $\mathbb{Q}[\mu_m]_{\mathfrak{p}_\mathfrak{o}}$-module and $\nu(\mathfrak{o}) = \nu(X[\mathfrak{p}_\mathfrak{o}^\infty])$ is its Newton polygon. □

For each $\nu \in B = B(H_{\mathbb{Q}_p}, \mu_\mathfrak{h})$, we write $\nu(\mathfrak{o})$ for its $\mathfrak{o}$-component, hence $\nu = \bigoplus_{\mathfrak{o} \in \mathfrak{O}} \nu(\mathfrak{o})$. For all $\nu, \nu' \in B$, note that $\nu \leq \nu'$ if and only if $\nu(\mathfrak{o}) \leq \nu'(\mathfrak{o})$, for all $\mathfrak{o} \in \mathfrak{O}$.

By the next lemma, to prove Proposition 4.4, it suffices to consider each σ-orbit $\mathfrak{o}$ in $\mathcal{T} = \mathbb{Z}/m_3\mathbb{Z}$ separately.

[5.1] or more generally the universal semi-abelian variety over its toroidal compactification,

Lemma 5.3. With the same notation and assumption as in Proposition 4.4: the equality $u_3 = u_1^{d_1} \oplus u_2^{d_2} \oplus \mathrm{ord}^{\epsilon}$ is equivalent to the system of equalities, for every orbit $\mathfrak{o}$ in $\mathcal{T}$,

$$u_3(\mathfrak{o}) = u_1^{d_1}(\mathfrak{o}) \oplus u_2^{d_2}(\mathfrak{o}) \oplus \mathrm{ord}^{\epsilon_{\mathfrak{o}}}, \tag{5.1}$$

where $\epsilon_{\mathfrak{o}} := |\mathfrak{o}|$ if $e_{\mathfrak{o}}$ is a divisor of $d_1 d_2 r_0$ but not a divisor of d_1 or d_2, and $\epsilon_{\mathfrak{o}} := 0$ otherwise. □

Proof. Note that $\epsilon = d_1 d_2 r_0 - d_1 - d_2 + 1$ is equal to the number of $\tau \in \mathcal{T}$ whose order is a divisor of $d_1 d_2 r_0$ but not a divisor of d_1 or d_2; recall $\gcd(d_1, d_2) = 1$. Thus $\epsilon = \sum_{\mathfrak{o} \in \mathfrak{O}} \epsilon_{\mathfrak{o}}$, and the statement follows from Lemma 5.2 and the discussion below the lemma. ■

Proof of Proposition 4.4. Fix an orbit $\mathfrak{o}$ in $\mathcal{T}$. For $i = 1, 2$, let $u_i^\dagger$ denote the μ-ordinary Newton polygon of $\mathrm{Sh}(\mu_{m_3}, \mathfrak{f}_i^\dagger)$. By definition, $u_i^\dagger = u_i^{d_i}$, and $u_i^\dagger(\mathfrak{o}) = u_i^{d_i}(\mathfrak{o})$. That is, the Newton polygons $u_i^\dagger(\mathfrak{o})$ and $u_i(\mathfrak{o})$ have the same slopes, with the multiplicity of each slope in $u_i^\dagger(\mathfrak{o})$ being d_i times its multiplicity in $u_i(\mathfrak{o})$. Recall the formulas for the slopes and multiplicities of $u(\mathfrak{o})$ from Section 2.6.1.

Reduction to a combinatorial problem. The formulas for the slopes and multiplicities rely only on the signature type $\mathfrak{f}$ viewed as a $\mathbb{N}$-valued function on $\mathcal{T}$, and do not require $\mathfrak{f}$ to be a signature associated with a Shimura variety. We regard each signature type $\mathfrak{f}$ as an $\mathbb{N}$-valued function on $\mathfrak{o}$, and denote by $u(\mathfrak{o})$ the Newton polygon defined by the data of slopes in (2.4) and multiplicities in (2.5).[5.2] We use subscripts and superscripts to identify various $\mathbb{N}$-valued functions and their Newton polygons, for example, $\mathfrak{f}_1^\dagger$ and $u_1^\dagger(\mathfrak{o})$.

Proposition 4.4 follows from the claim below taking $R = r_0$, using Lemma 5.3.

Claim *Let R be a positive integer which divides $\gcd(m_1, m_2)$. Set $\delta := \delta_{d,dR} + \delta_{d_1, d_1 d_2} - \delta_{1, d_2}$, with $d := m_3 / \gcd(m_1, m_2) = d_1 d_2$ and notations as in Definition 3.12; set $\epsilon_{\mathfrak{o}} := |\mathfrak{o}|$ if $e_{\mathfrak{o}}$ divides dR but not d_1 or d_2, and $\epsilon_{\mathfrak{o}} := 0$ otherwise. Define $\mathfrak{f}_3 := \mathfrak{f}_1^\dagger + \mathfrak{f}_2^\dagger + \delta$. Then the equality*

$$u_3(\mathfrak{o}) = u_1^\dagger(\mathfrak{o}) \oplus u_2^\dagger(\mathfrak{o}) \oplus \mathrm{ord}^{\epsilon_{\mathfrak{o}}} \tag{5.2}$$

holds if and only if the pair $\mathfrak{f}_1, \mathfrak{f}_2$ satisfies hypothesis (B).

Reduction of claim to the case $d = R = 1$ We first prove that if $\mathfrak{f}_3 = \mathfrak{f}_1^\dagger + \delta$ then

$$u_3(\mathfrak{o}) = u_1^\dagger(\mathfrak{o}) \oplus \mathrm{ord}^{\epsilon_{\mathfrak{o}}}. \tag{5.3}$$

[5.2] the integer $g(\mathfrak{o})$ will be specified in each part of the proof.

Note that the function $\delta(\tau)$ is constant on $\mathfrak{o}$, with value $\delta(\mathfrak{o})$ equal to 1 if $e_{\mathfrak{o}}$ divides dR but not d_1 or d_2, and to 0 otherwise. In particular, $\epsilon_{\mathfrak{o}} = \sum_{\tau\in\mathfrak{o}} \delta(\tau)$, $\delta(\mathfrak{o}) = \delta(\mathfrak{o}^*)$, and by Remark 2.6, $g_3(\mathfrak{o}) = g_1^\dagger(\mathfrak{o}) + 2\delta(\mathfrak{o})$.

If $\delta(\mathfrak{o}) = 0$, then $g_3(\mathfrak{o}) = g_1^\dagger(\mathfrak{o})$, and $\mathfrak{f}_3(\tau) = \mathfrak{f}_1^\dagger(\tau)$ for all $\tau \in \mathfrak{o}$. Hence, $u_3(\mathfrak{o}) = u_1^\dagger(\mathfrak{o})$ which agrees with equality (5.3) for $\epsilon_{\mathfrak{o}} = 0$.

If $\delta(\mathfrak{o}) = 1$, then $g_3(\mathfrak{o}) = g_1^\dagger(\mathfrak{o}) + 2$, and $\mathfrak{f}_3(\tau) = \mathfrak{f}_1^\dagger(\tau) + 1$ for all $\tau \in \mathfrak{o}$. In particular, $g_3(\mathfrak{o}) > \mathfrak{f}_3(\tau) \geq 1$, for all $\tau \in \mathfrak{o}$. By (2.4) and (2.5), both 0 and 1 occur as slopes of $u_3(\mathfrak{o})$, with multiplicity respectively $\rho_3(0) = \rho_1(0) + |\mathfrak{o}|$ and $\rho_3(1) = \rho_1(1) + |\mathfrak{o}|$. Each of the slopes λ of $u_1^\dagger(\mathfrak{o})$, with $\lambda \neq 0, 1$, also occurs for $u_3(\mathfrak{o})$, with multiplicity $\rho_3(\lambda) = \rho_1(\lambda)$. Thus $u_3(\mathfrak{o}) = u_1^\dagger(\mathfrak{o}) \oplus \mathrm{ord}^{|\mathfrak{o}|}$, which agrees with equality (5.3) for $\epsilon_{\mathfrak{o}} = |\mathfrak{o}|$.

Equality (5.3) is equivalent to the claim for $\mathfrak{f}_2^\dagger(\tau) = 0$, for all $\tau \in \mathfrak{o}$. Indeed, hypothesis (B) for the pair $(\mathfrak{f}_1^\dagger, 0)$ holds trivially, and equality (5.2) for $\mathfrak{f}_3 = \mathfrak{f}_1^\dagger + \delta$ specializes to (5.3).

By (5.3), we deduce that equality (5.2) holds if and only if it holds in the special case of $d = R = 1$, that is if $u_3(\mathfrak{o}) = u_1^\dagger(\mathfrak{o}) + u_2^\dagger(\mathfrak{o})$ when $\mathfrak{f}_3 = \mathfrak{f}_1^\dagger + \mathfrak{f}_2^\dagger$. By definition, hypothesis (B) holds for the pair $(\mathfrak{f}_1^\dagger, \mathfrak{f}_2^\dagger)$ if and only if it holds for $(\mathfrak{f}_1^\dagger + \delta, \mathfrak{f}_2^\dagger)$. Hence, without loss of generality, we may assume $d = R = 1$ and $\mathfrak{f}_3 = \mathfrak{f}_1 + \mathfrak{f}_2$. In this case, $g_3(\mathfrak{o}) = g_1(\mathfrak{o}) + g_2(\mathfrak{o})$ by Remark 2.6 and the claim reduces to the following:

Specialized claim: *Assume $\mathfrak{f}_3 = \mathfrak{f}_1 + \mathfrak{f}_2$. Then, the equality*

$$u_3(\mathfrak{o}) = u_1(\mathfrak{o}) \oplus u_2(\mathfrak{o}). \tag{5.4}$$

holds if and only if the pair $\mathfrak{f}_1, \mathfrak{f}_2$ satisfies hypothesis (B).

Converse direction: assume hypothesis (B) We shall prove that the equality (5.4) holds, by induction on the integer $s_3 + 1$, the number of distinct slopes of $u_3(\mathfrak{o})$. More precisely, we shall proceed as follows. First, we shall establish the base case of induction, for $s_3 = 0$. Next, we shall prove the equality of multiplicities

$$\rho_3(\lambda) = \rho_1(\lambda) + \rho_2(\lambda) \tag{5.5}$$

for $\lambda = \lambda_3(0)$ the first (smallest) slope of $u_3(\mathfrak{o})$. Then, we shall assume $s_3 \geq 1$ and show that the inductive hypothesis and equality (5.5) imply equality (5.4), which will complete the argument. In the induction process, $g_i(\mathfrak{o})$ remains unchanged.

Base case: Assume $s_3 = 0$. Then, for all $\tau \in \mathfrak{o}$, either $\mathfrak{f}_3(\tau) = g_3(\mathfrak{o})$ or $\mathfrak{f}_3(\tau) = 0$. The equalities $\mathfrak{f}_3(\tau) = \mathfrak{f}_1(\tau) + \mathfrak{f}_2(\tau)$ and $g_3(\mathfrak{o}) = g_1(\mathfrak{o}) + g_2(\mathfrak{o})$ imply that $\mathfrak{f}_3(\tau) = g_3(\mathfrak{o})$ (resp. $\mathfrak{f}_3(\tau) = 0$) if and only if $\mathfrak{f}_i(\tau) = g_i(\mathfrak{o})$ (resp. $\mathfrak{f}_i(\tau) = 0$) for both $i = 1, 2$. We deduce that both hypothesis (B) and (5.4) hold in this situation.

Equality (5.5) For $i = 1, 2, 3$, let $E_i(\max)$ denote the maximal value of $\mathfrak{f}_i$ on $\mathfrak{o}$. By definition, $E_i(\max)$ is equal to either $E_i(0)$ or $E_i(1)$. In the first case, $\lambda_i(0) > 0$; in the second case, $\lambda_i(0) = 0$. We claim that hypothesis (B) implies $E_3(\max) = E_1(\max) + E_2(\max)$. First, note that hypothesis (B) implies that, for $\omega, \tau \in \mathfrak{o}$:

$$\mathfrak{f}_3(\omega) = \mathfrak{f}_3(\tau) \text{ if and only if } \mathfrak{f}_i(\omega) = \mathfrak{f}_i(\tau) \text{ for both } i = 1, 2. \tag{5.6}$$

For $i = 1, 2, 3$, set $S_i = \{\tau \in \mathfrak{o} \mid \mathfrak{f}_i(\tau) = E_i(\max)\}$. Then, property (5.6) implies that $S_3 = S_1 \cap S_2$, which in turn implies $E_3(\max) = E_1(\max) + E_2(\max)$.

We claim that hypothesis (B) implies that $S_3 = S_i$ for some $i \in \{1, 2\}$. Without loss of generality, assume that S_2 properly contains S_3, and let $\tau_0 \in S_2 - S_3$. Then $\tau_0 \notin S_1$. For any $\omega \in S_1$: $\mathfrak{f}_1(\omega) = E_1(\max) > \mathfrak{f}_1(\tau_0)$. Thus, by hypothesis (B), $\mathfrak{f}_2(\omega) \geq \mathfrak{f}_2(\tau_0) = E_2(\max)$. We deduce that $\mathfrak{f}_2(\omega) = E_2(\max)$, hence $S_1 \subseteq S_2$.

By the formulas for slopes (2.4), the equality $S_1 = S_3$ implies that $\lambda_1(0) = \lambda_3(0)$, and the inclusion $S_1 \subseteq S_2$ implies $\lambda_1(0) \leq \lambda_2(0)$ (and the equality holds if and only if $S_2 = S_3$).

For $i = 1, 2, 3$, let $E_i(\text{next})$ denote the maximal value of $\mathfrak{f}_i$ on $\mathfrak{o} - S_3$. For $i = 1, 3$, $E_i(\text{next}) < E_i(\max)$; for $i = 2$, $E_2(\text{next}) \leq E_2(\max)$ and the equality holds if and only if S_2 properly contains S_3.

As before by property (5.6), we deduce that hypothesis (B) implies $E_3(\text{next}) = E_1(\text{next}) + E_2(\text{next})$. By the formulas for multiplicities (2.5), the two identities, $E_3(\max) = E_1(\max) + E_2(\max)$ and $E_3(\text{next}) = E_1(\text{next}) + E_2(\text{next})$, imply the desired equality (5.5).

Assume $s_3 \geq 1$ Then the polygon $u_3(\mathfrak{o})$ has at least two distinct slopes. Our plan is to introduce auxiliary functions $\widetilde{\mathfrak{f}}_1(\tau), \widetilde{\mathfrak{f}}_2(\tau)$ such that polygon $\widetilde{u}_3(\mathfrak{o})$ for the function $\widetilde{\mathfrak{f}}_3(\tau) := \widetilde{\mathfrak{f}}_1(\tau) + \widetilde{\mathfrak{f}}_2(\tau)$ has s_3 distinct slopes.

For $i = 1, 2$, define $\widetilde{\mathfrak{f}}_i(\tau) := \mathfrak{f}_i(\tau)$ for all $\tau \notin S_3$ and $\widetilde{\mathfrak{f}}_i(\tau) := E_i(\text{next})$ for $\tau \in S_3$. Note that $\widetilde{\mathfrak{f}}_2 = \mathfrak{f}_2$ unless $S_2 = S_3$. By definition, for $i = 1, 2, 3$, $\widetilde{E}_i(\max) = E_i(\text{next})$.

For $i = 1, 3$, and for $i = 2$ if $S_2 = S_3$, the polygon $\widetilde{u}_i(\mathfrak{o})$ shares the same slopes as $u_i(\mathfrak{o})$ except $\lambda_i(0)$ which no longer occurs. For each $t = 2, \ldots, s_i$, the slope $\lambda_i(t)$ occurs in $\widetilde{u}_i(\mathfrak{o})$ with multiplicity equal to $\rho_i(\lambda_i(t))$; while the slope $\lambda_i(1)$ occurs in $\widetilde{u}_i(\mathfrak{o})$ with multiplicity $\rho_i(\lambda_i(0)) + \rho_i(\lambda_i(1))$. For $i = 2$, if $S_2 \neq S_3$, then $\widetilde{u}_2(\mathfrak{o}) = u_2(\mathfrak{o})$.

Note that $\widetilde{u}_3$ has exactly s_3 slopes. Hence, by the inductive hypothesis, we deduce that $\widetilde{u}_3(\mathfrak{o}) = \widetilde{u}_1(\mathfrak{o}) \oplus \widetilde{u}_2(\mathfrak{o})$. This identity, together with (5.5) and the above computation of $\widetilde{\rho}_i(\lambda_i(1))$, implies that $u_3(\mathfrak{o}) = u_1(\mathfrak{o}) \oplus u_2(\mathfrak{o})$.

Forward direction: assume (5.4) We shall prove that the pair $(\mathfrak{f}_1, \mathfrak{f}_2)$ satisfies hypothesis (B), arguing by contradiction. Supposing hypothesis (B) does not hold, we shall define auxiliary functions $\bar{\mathfrak{f}}_1, \bar{\mathfrak{f}}_2$ obtained by precomposing $\mathfrak{f}_1, \mathfrak{f}_2$ with a permutation of $\mathfrak{o}$, such that the Newton polygon $\overline{u}_3(\mathfrak{o})$ associated with the function $\bar{\mathfrak{f}}_3(\tau) := \bar{\mathfrak{f}}_1(\tau) + \bar{\mathfrak{f}}_2(\tau)$ is strictly above $u_3(\mathfrak{o})$, i.e., $\overline{u}_3(\mathfrak{o}) < u_3(\mathfrak{o})$. By the formulas for slopes and multiplicities (2.4) and (2.5), we see that precomposing a function $\mathfrak{f}$ with a permutation of $\mathfrak{o}$ does not change the associated polygon $u(\mathfrak{o})$. Hence, for $i = 1, 2$ we deduce $\overline{u}_i(\mathfrak{o}) = u_i(\mathfrak{o})$. On the other hand, by repeating the permutation process, we eventually end up with a pair $(\bar{\mathfrak{f}}_1, \bar{\mathfrak{f}}_2)$ which satisfies hypothesis (B) and by the above argument, hypothesis (B) implies that $\overline{u}_1(\mathfrak{o}) \oplus \overline{u}_2(\mathfrak{o}) = \overline{u}_3(\mathfrak{o})$. Hence $\overline{u}_3(\mathfrak{o}) = u_1(\mathfrak{o}) \oplus u_2(\mathfrak{o})$, which equals $u_3(\mathfrak{o})$ by (5.2) and contradicts the conclusion that $\overline{u}_3(\mathfrak{o}) < u_3(\mathfrak{o})$.

Contradict hypothesis (B) Thus, there exist $\omega_0, \eta_0 \in \mathfrak{o}$ such that

$$\mathfrak{f}_2(\omega_0) > \mathfrak{f}_2(\eta_0) \text{ and } \mathfrak{f}_1(\omega_0) < \mathfrak{f}_1(\eta_0).$$

Let γ denote the permutation of $\mathfrak{o}$ which switches ω_0 and η_0, and define $\bar{\mathfrak{f}}_1(\tau) := \mathfrak{f}_1(\gamma(\tau))$ and $\bar{\mathfrak{f}}_2(\tau) := \mathfrak{f}_2(\tau)$. Set $\bar{\mathfrak{f}}_3(\tau) := \bar{\mathfrak{f}}_1(\tau) + \bar{\mathfrak{f}}_2(\tau)$. Then, $\bar{\mathfrak{f}}_3(\tau) = \mathfrak{f}_3(\tau)$ except for $\tau = \omega_0, \eta_0$. Note that $\bar{\mathfrak{f}}_3(\omega_0) > \mathfrak{f}_3(\omega_0)$, $\bar{\mathfrak{f}}_3(\eta_0) < \mathfrak{f}_3(\eta_0)$, and $\bar{\mathfrak{f}}_3(\omega_0) > \mathfrak{f}_3(\eta_0)$ (also, $\bar{\mathfrak{f}}_3(\eta_0) < \mathfrak{f}_3(\omega_0)$).

We claim that $\overline{u}_3(\mathfrak{o}) < u_3(\mathfrak{o})$, meaning that $\overline{u}_3(\mathfrak{o})$ and $u_3(\mathfrak{o})$ share the same endpoints (this follows from the equality $\overline{g}_3(\mathfrak{o}) = g_3(\mathfrak{o})$), and that $\overline{u}_3(\mathfrak{o})$ lies strictly above $u_3(\mathfrak{o})$.

We first show that possibly after sharing the first several slopes, $\overline{u}_3(\mathfrak{o})$ admits a slope which is strictly larger than the corresponding one in $u_3(\mathfrak{o})$. Let us consider the value $A = \bar{\mathfrak{f}}_3(\omega_0)$. Note that $A > \bar{\mathfrak{f}}_3(\omega_0) \geq 0$. If $A = \mathfrak{f}_3(\tau)$ for some $\tau \in \mathfrak{o}$, say $A = E_3(t)$ for some $t \in \{0, \dots, s_3\}$. Then by the formulas for slopes and multiplicities (2.4) and (2.5), we deduce that the first t slopes, and their multiplicities, of $\overline{u}_3(\mathfrak{o})$ and $u_3(\mathfrak{o})$ agree, but the $(t+1)$-st slope of $\overline{u}_3(\mathfrak{o})$ is strictly larger that the $(t+1)$-st slope of $u_3(\mathfrak{o})$. If $\mathfrak{f}_3(\tau) \neq A$ for all $\tau \in \mathfrak{o}$, let $t \in \{0, \dots, s_3\}$ be such that $E_3(t) > A > E_3(t+1)$. Thus the first t slopes of $\overline{u}_3(\mathfrak{o})$ and $u_3(\mathfrak{o})$ agree, and so do the multiplicities of the first $t-1$ slopes, but the multiplicity of the t-th slope of $\overline{u}_3$ is strictly smaller than that of $u_3(\mathfrak{o})$.

By similar arguments for the subsequent slopes and multiplicities, $\overline{u}_3(\mathfrak{o})$ never drops strictly below $u_3(\mathfrak{o})$, but it might (and often does) agree with $u_3(\mathfrak{o})$ for large slopes. ■

6 The Torelli locus and the non μ-ordinary locus of Shimura varieties

In this section, we study the intersection of the open Torelli locus with Newton polygon strata which are not μ-ordinary in PEL-type Shimura varieties. The main result, Theorem 6.11, provides a method to leverage information from smaller dimension to larger dimension. This theorem is significantly more difficult than Theorem 4.5; we add an extra condition to maintain control over the codimensions of the Newton polygon strata. This is the first systematic result on this topic that we are aware of.

For applications, we find situations where we can apply Theorem 6.11 infinitely many times; from this, we produce systems of infinitely many PEL-type Shimura varieties for which we can verify that the open Torelli locus intersects non μ-ordinary Newton polygon strata. See Corollary 6.14 to 6.16 and Section 9 for details.

Notation 6.1. Let $\gamma_1 = (m_1, N_1, a_1)$, $\gamma_2 = (m_2, N_2, a_2)$ be an ordered pair of (generalized) monodromy data which satisfies hypothesis (A). Assume that $m_1 | m_2$. Set $d := m_2/m_1$ and $r := \gcd(m_1, a_1(N_1))$. Then, (3.1) specializes to $\epsilon = d(r-1)$ and $g_3 = dg_1 + g_2 + \epsilon$. In particular, $\epsilon = 0$ if and only if $r = 1$. □

Remark 6.10 explains why we restrict to the case $m_1 | m_2$.

6.1 Hypothesis (C)

To study the Newton polygons beyond the μ-ordinary case, we introduce an extra hypothesis.

Definition 6.2. Given a Newton polygon ν, the *first slope* $\lambda_{1st}(\nu)$ is the smallest slope of ν and the *last slope* $\lambda_{last}(\nu)$ is the largest slope of ν. If ν is symmetric with q distinct slopes, the *middle slope* $\lambda_{mid}(\nu)$ is the $\lfloor \frac{q+1}{2} \rfloor$-st slope of ν. $\square$

Definition 6.3. An ordered pair of monodromy data γ_1, γ_2 is *controlled* if the slopes of the μ-ordinary Newton polygons u_1 and u_2 satisfy:

hypothesis (C): for each orbit $\mathfrak{o} \in \mathfrak{O}$, every slope of $u_1(\mathfrak{o})$ is in the range $[0,1] \setminus (\lambda_{1st}(u_2(\mathfrak{o})), \lambda_{last}(u_2(\mathfrak{o})))$.

By convention, the condition in the previous line holds for $\mathfrak{o}$ if $u_i(\mathfrak{o})$ is empty for either $i = 1, 2$. If a pair of monodromy data is controlled, then we write $u_1 \ll_{(\mathrm{C})} u_2$. $\square$

Remark 6.4.

1. Hypothesis (C) holds for $\mathfrak{o}$ if either $u_1(\mathfrak{o})$ has slopes in $\{0, 1\}$ or $u_2(\mathfrak{o})$ is supersingular. In particular, if u is ordinary or supersingular, then $u \ll_{(\mathrm{C})} u$.
2. If $\mathfrak{o} = \mathfrak{o}^*$, then hypothesis (C) holds for $\mathfrak{o}$ if and only if $\lambda_{mid}(u_1(\mathfrak{o})) \leq \lambda_{1st}(u_2(\mathfrak{o}))$.
3. If $u_1 \ll_{(\mathrm{C})} u_2$ then $u_1^d \ll_{(\mathrm{C})} u_2$ for all $d \in \mathbb{Z}_{\geq 1}$.

$\square$

Remark 6.5. Let γ_1, γ_2 be a pair of monodromy data as in Notation 6.1. By Section 2.6.1,

$$\lambda_{1st}(u_2(\mathfrak{o})) = \frac{1}{|\mathfrak{o}|}\#\{\tau \in \mathfrak{o} \mid \mathfrak{f}_2(\tau) = g_2(\mathfrak{o})\} \text{ and } \lambda_{last}(u_2(\mathfrak{o})) = \frac{1}{|\mathfrak{o}|}\#\{\tau \in \mathfrak{o} \mid \mathfrak{f}_2(\tau) > 0\}.$$

Hypothesis (C) holds for $\mathfrak{o}$ if and only if there exists an integer $E(\mathfrak{o}) \in [0, g_1(\mathfrak{o})]$ such that

$$\#\{\tau \in \mathfrak{o} \mid \mathfrak{f}_1^\dagger(\tau) > E(\mathfrak{o})\} \leq \#\{\tau \in \mathfrak{o} \mid \mathfrak{f}_2(\tau) = g_2(\mathfrak{o})\}, \text{ and}$$

$$\#\{\tau \in \mathfrak{o} \mid \mathfrak{f}_1^\dagger(\tau) \geq E(\mathfrak{o})\} \geq \#\{\tau \in \mathfrak{o} \mid \mathfrak{f}_2(\tau) > 0\}.$$

$\square$

The next statement follows from Definition 6.3 and Remark 6.5.

Lemma 6.6. The following are equivalent: $u \ll_{(\mathrm{C})} u$; for each $\mathfrak{o} \in \mathfrak{O}$, the Newton polygon $u(\mathfrak{o})$ has at most two distinct slopes; and, for each $\mathfrak{o} \in \mathfrak{O}$, there exists an integer $E(\mathfrak{o}) \in [0, g(\mathfrak{o})]$ such that $\mathfrak{f}(\tau) \in \{0, E(\mathfrak{o}), g(\mathfrak{o})\}$ for all $\tau \in \mathfrak{o}$. $\square$

Unlike hypothesis (B), hypothesis (C) does not behave well under induction in general. Lemma 6.7 identifies two instances when it does. We omit the proof.

Lemma 6.7.

1. If $u_1 \ll_{(\mathrm{C})} u_2$ then $u_1^n \oplus \mathrm{ord}^l \ll_{(\mathrm{C})} u_2$, for any $n, l \in \mathbb{Z}_{\geq 1}$.

2. If $u_1 \ll_{(\mathrm{C})} u_2$ and $u_2 \ll_{(\mathrm{C})} u_2$, then $u_1^n \oplus u_2^m \oplus \mathrm{ord}^l \ll_{(\mathrm{C})} u_2$, for any $n, m, l \in \mathbb{Z}_{\geq 1}$.

□

6.2 The significance of hypothesis (C)

Hypothesis (C) is sufficient to prove the geometric condition on the Newton polygon stratification in Proposition 6.8 below. We use hypothesis (C) to prove the surjectivity of the map in (6.2).

Proposition 6.8. Let $\gamma_1 = (m_1, N_1, a_1), \gamma_2 = (m_2, N_2, a_2)$ be an ordered pair of monodromy data as in Notation 6.1. Assume the pair satisfies hypotheses (A), (B), and (C). Consider the monodromy datum γ_3 as in Definition 3.5. Then, for any Newton polygon $\nu_2 \in B_2$,

$$\mathrm{codim}(\mathrm{Sh}_2[\nu_2], \mathrm{Sh}_2) = \mathrm{codim}(\overline{\mathrm{Sh}}_3[u_1^d \oplus \nu_2 \oplus \mathrm{ord}^\epsilon], \overline{\mathrm{Sh}}_3). \tag{6.1}$$

□

The following lemma is a reformulation of Proposition 4.4.

Lemma 6.9. Hypothesis (B) is equivalent to the assumption that (6.1) holds for $\nu_2 = u_2$. □

In fact, the proof below shows that if hypothesis (B) does not hold, then (6.1) is false for all $\nu_2 \in B_2$.

Proof of Proposition 6.8. For any Kottwitz set B and any $\nu \in B$, let $B(\nu) = \{t \in B \mid t \geq \nu\}$.

We first prove the case when $\epsilon = 0$. Consider the map $\Sigma : B_2 \to B_3$, where $t \mapsto u_1^d \oplus t$.

We first note that Σ is an order-preserving injection.[6.1] Let $t \in B_2$. For each orbit $\mathfrak{o}$ in $\mathfrak{O}$, let $q_1 = q_1(\mathfrak{o})$ (resp. $q_1' = q_1'(\mathfrak{o})$) be the number of distinct slopes of $u_1(\mathfrak{o})$ in $[0, \lambda_{1st}(u_2(\mathfrak{o})]$ (resp. $[\lambda_{last}(u_2(\mathfrak{o})), 1]$). By hypothesis (C), for the Newton polygon $u_1^d \oplus t$, the first q_1 and the last q_1' slopes of $u_1(\mathfrak{o})^d \oplus t(\mathfrak{o})$ are the slopes of $u_1(\mathfrak{o})^d$ with the same multiplicities and the rest are the slopes of $t(\mathfrak{o})$ with the same multiplicities. So, if $t \leq t'$ in B_2, then $u_1^d \oplus t \leq u_1^d \oplus t'$ in B_3. In particular, the map Σ induces an injection on the ordered sets

$$B_2(\nu_2) \to B_3(u_1^d \oplus \nu_2), \ t \mapsto u_1^d \oplus t. \tag{6.2}$$

By (2.6), to conclude, it suffices to prove that under hypotheses (B) and (C) the map in (6.2) is also surjective. By Proposition 4.4, hypothesis (B) implies that $u_3 = u_1^d \oplus u_2$. Hence, for a Newton polygon $v \in B_3(u_1^d \oplus \nu_2)$, then $u_1^d \oplus \nu_2 \leq v \leq u_3 = u_1^d \oplus u_2$. By the paragraph after Lemma 5.2,

$$u_1(\mathfrak{o})^d \oplus \nu_2(\mathfrak{o}) \leq v(\mathfrak{o}) \leq u_1(\mathfrak{o})^d \oplus u_2(\mathfrak{o}).$$

[6.1] For convenience, we use hypothesis (C) to construct the order-preserving map (6.2); however, this part can be proved without using this hypothesis.

By hypothesis (C), the inequalities above imply that $v(\mathfrak{o})$ and $u_1(\mathfrak{o})^d$ share the first q_1 and last q_1' slopes, with the same multiplicities except for the q_1-th slope (resp. q_2-th slope) which may occur with higher multiplicity in the former if $u_1(\mathfrak{o})$ has slope $\lambda_{1st}(u_2(\mathfrak{o}))$ (resp. $\lambda_{last}(u_2(\mathfrak{o}))$). We deduce that each $v \in B_3(u_1^d \oplus \nu_2)$ is of the form $v = u_1^d \oplus t$ for some $t \in B_2(\nu_2)$; thus the map in (6.2) is surjective.

If $\epsilon \neq 0$, by (2.7), the same argument still applies, with u_1^d replaced by $u_1^d \oplus \mathrm{ord}^\epsilon$. ■

Remark 6.10. Let γ be a monodromy datum, of signature $\mathfrak{f}$. For $d > 1$, let $\gamma^\dagger$ be the induced datum, of signature $\mathfrak{f}^\dagger$, as in Notation 3.1. The map $(\cdot)^\dagger : B(\mathfrak{f}) \to B(\mathfrak{f}^\dagger)$, $\nu \mapsto \nu^d$, is injective and order-preserving, but is not surjective in general. If $(\cdot)^\dagger$ is not surjective, then, by (2.6), there exists $\nu \in B$ such that $\mathrm{codim}(\mathrm{Sh}[\nu], \mathrm{Sh}) \neq \mathrm{codim}(\mathrm{Sh}^\dagger[\nu^d], \mathrm{Sh}^\dagger)$. For example, if $\mathfrak{f} = (1)$, and $d = 2$, this happens when $\nu = (1/2, 1/2)$. □

6.3 The second main result

The next result also provides a partial positive answer to Conjecture 1.1 when $\epsilon = 0$.

Theorem 6.11. Let γ_1, γ_2 be an ordered pair of monodromy data as in Notation 6.1. Assume it satisfies hypotheses (A), (B), and (C). Let $\epsilon = d(r-1)$. Consider the monodromy datum γ_3 from Definition 3.5. Let $\nu_2 \in B_2$. If $Z_1^\circ[u_1]$ and $Z_2^\circ[\nu_2]$ are non-empty, and $Z_2^\circ[\nu_2]$ contains an irreducible component Γ_2 such that

$$\mathrm{codim}(\Gamma_2, Z_2) = \mathrm{codim}(\mathrm{Sh}_2[\nu_2], \mathrm{Sh}_2), \tag{6.3}$$

then $Z_3^\circ[u_1^d \oplus \nu_2 \oplus \mathrm{ord}^\epsilon]$ is non-empty and contains an irreducible component Γ_3 such that

$$\mathrm{codim}(\Gamma_3, Z_3) = \mathrm{codim}(\mathrm{Sh}_3[u_1^d \oplus \nu_2 \oplus \mathrm{ord}^\epsilon], \mathrm{Sh}_3).$$

□

Remark 6.12. As seen in Section 6.5, hypothesis (C) is not a necessary condition and it can occasionally be removed. Specifically, Theorem 6.11 still holds with hypothesis (C) replaced by the weaker (but harder to verify) assumption that (6.1) holds for the given non μ-ordinary Newton polygon $\nu_2 \in B_2$. □

Remark 6.13. If Z_2 is one of Moonen's special families from [28] and $Z_2^\circ[\nu_2]$ is non-empty, then every irreducible component of $Z_2^\circ[\nu_2]$ satisfies the codimension condition (6.3). □

Proof of Theorem 6.11. By Remark 2.3, without loss of generality, we may assume that the inertia types a_1 and a_2 contain no zero entries if $r < m_1$ and no zero entries other than $a_1(N_1) = a_2(1) = 0$ if $r = m_1$. Set $l := \mathrm{codim}(\mathrm{Sh}_2[\nu_2], \mathrm{Sh}_2)$. Recall the clutching morphism $\kappa : \widetilde{Z}_1 \times \widetilde{Z}_2 \to \widetilde{Z}_3$ from Proposition 3.9. Note that

$$\dim(Z_3) = N_3 - 3 = (N_1 - 3) + (N_2 - 3) + 1. \tag{6.4}$$

We distinguish three cases: $\epsilon = 0$, $\epsilon \neq 0$ and $r < m_1$, and $\epsilon \neq 0$ and $r = m_1$.

Assume $\epsilon = 0$

Then (6.4) implies that

$$\dim(Z_3) = \dim(Z_1) + \dim(Z_2) + 1 = \dim(\widetilde{Z}_1) + \dim(\widetilde{Z}_2) + 1. \tag{6.5}$$

By Notation 3.4, $g_3 = dg_1 + g_2$ and the formula for $\mathfrak{f}_3$ is in (3.3). The clutching morphism κ is compatible with the morphism $\iota : \mathrm{Sh}_1 \times \mathrm{Sh}_2 \to \mathrm{Sh}_3$ given by $\iota(\mathcal{X}_1, \mathcal{X}_2) = \mathcal{X}_1^d \oplus \mathcal{X}_2$. Since the map $\widetilde{Z}_i \to \overline{Z}_i$ is finite, $\dim(\widetilde{Z}_i[\nu_i]) = \dim(\overline{Z}_i[\nu_i])$ for any $\nu_i \in B_i$.

By Proposition 6.8, $l = \mathrm{codim}(\mathrm{Sh}_3[u_1^d \oplus \nu_2], \mathrm{Sh}_3)$. Let $\widetilde{\Gamma}_2$ denote the Zariski closure of the preimage of Γ_2 in $\widetilde{Z}_2[\nu_2]$. Apriori, $\widetilde{\Gamma}_2$ may not be irreducible, in which case we replace it by one of its irreducible components. Then $\dim(\widetilde{\Gamma}_2) = \dim(\Gamma_2)$.

Let $W := \kappa(\widetilde{Z}_1[u_1], \widetilde{Z}_2[\nu_2])$. Since $W \subseteq \widetilde{Z}_3[u_1^d \oplus \nu_2]$, then $\widetilde{Z}_3[u_1^d \oplus \nu_2]$ is non-empty. By (6.5), $\kappa(\widetilde{Z}_1, \widetilde{Z}_2)$ has codimension 1 in $\widetilde{Z}_3$. In addition, W is an open and closed substack of the intersection of $\kappa(\widetilde{Z}_1, \widetilde{Z}_2)$ and $\widetilde{Z}_3[u_1^d \oplus \nu_2]$. By [35, page 614], every irreducible component of W has codimension at most 1 in the irreducible component of $\widetilde{Z}_3[u_1^d \oplus \nu_2]$ which contains it. Note that $\kappa(\widetilde{Z}_1[u_1], \widetilde{\Gamma}_2)$ is an irreducible component of W. Let $\widetilde{\Gamma}_3$ be the irreducible component of $\widetilde{Z}_3[u_1^d \oplus \nu_2]$ which contains $\kappa(\widetilde{Z}_1[u_1], \widetilde{\Gamma}_2)$. It follows that $\mathrm{codim}(\kappa(\widetilde{Z}_1[u_1], \widetilde{\Gamma}_2), \widetilde{\Gamma}_3) \le 1$. So

$$\dim(\widetilde{\Gamma}_3) = \begin{cases} \dim(Z_1) + \dim(Z_2) - l & \text{if } \widetilde{\Gamma}_3 = \kappa(\widetilde{Z}_1[u_1], \widetilde{\Gamma}_2), \\ \dim(Z_1) + \dim(Z_2) - l + 1 & \text{otherwise.} \end{cases}$$

On the other hand, for all $b \in B_3$, by (2.6) and the de Jong–Oort purity theorem [7, Theorem 4.1], the codimension of any irreducible component of $\widetilde{Z}_3[b]$ in $\widetilde{Z}_3$ is no greater than $\mathrm{length}(b) = \mathrm{codim}(\mathrm{Sh}_3[b], \mathrm{Sh}_3)$. For $b = u_1^d \oplus \nu_2$, by (6.5), this yields

$$\dim(\widetilde{\Gamma}_3) \ge \dim(Z_3) - l = \dim(Z_1) + \dim(Z_2) + 1 - l.$$

We deduce that $\mathrm{codim}(\widetilde{\Gamma}_3, \widetilde{Z}_3) = l$ and that $\widetilde{\Gamma}_3$ strictly contains $\kappa(\widetilde{Z}_1[u_1], \widetilde{\Gamma}_2)$.

Let $\overline{\Gamma}_3$ denote the image of $\widetilde{\Gamma}_3$ via the forgetful map $\widetilde{Z}_3 \to \overline{Z}_3$. Define $\Gamma_3 = \overline{\Gamma}_3 \cap Z_3^\circ$. To finish the proof, we only need to show that Γ_3 is non-empty. Therefore, it suffices to show that $\widetilde{\Gamma}_3$ is not contained in the image of any other clutching map from Proposition 3.9. Since $r = 1$, by Proposition 3.11 the points in W represent curves of compact type, thus $\widetilde{\Gamma}_3 \cap \widetilde{Z}_3^c$ is non-empty.

To finish, we argue by contradiction; suppose $\widetilde{\Gamma}_3$ is contained in the image of any of the other clutching maps in $\widetilde{Z}_3^c$. This would imply that all points of $\kappa(\widetilde{Z}_1[u_1], \widetilde{\Gamma}_2)$ represent μ_m-covers of a curve of genus 0 comprised of at least 3 projective lines. This is only possible if all points of either $\widetilde{Z}_1[u_1]$ or $\widetilde{\Gamma}_2$ represent μ_m-covers of a curve of genus 0 comprised of at least 2 projective lines. This would imply that either $Z_1^\circ[u_1]$ or $\Gamma_2 \subset Z_2^\circ[\nu_2]$ is empty, which contradicts the hypotheses of the theorem.

Assume $\epsilon \neq 0$ and $r < m_1$

By the same argument as when $\epsilon = 0$, there exists an irreducible component $\widetilde{\Gamma}_3$ of $\widetilde{Z}_3[u_1^d \oplus \nu_2 \oplus \mathrm{ord}^\epsilon]$ of codimension l such that $\widetilde{\Gamma}_3$ strictly contains $\kappa(\widetilde{Z}_1[u_1], \widetilde{\Gamma}_2)$. To finish the proof, we only need to show that $\widetilde{\Gamma}_3$ is not contained in the boundary of $\widetilde{\mathcal{M}}_{\mu_m}$. As before, $\widetilde{\Gamma}_3$ is not contained in the image of any of the clutching maps in $\widetilde{Z}_3^c$. Suppose that $\widetilde{\Gamma}_3$ is contained in the image of any of the clutching maps not in $\widetilde{Z}_3^c$. By keeping careful track of the toric rank, one can check that this implies that the points of either $\widetilde{Z}_1[u_1]$ or $\widetilde{\Gamma}_2$ represent μ_m-covers of curves that are not of compact type. This would imply that either $Z_1^\circ[u_1]$ or $\Gamma_2 \subset Z_2^\circ[\nu_2]$ is empty, which contradicts the hypotheses of the theorem.

Assume $\epsilon \neq 0$ and $r = m_1$

By Remark 2.3, for $i = 1, 2$, the fibers of the forgetful map $f_i : \widetilde{Z}_i^\circ \to Z_i^\circ$ have pure dimension 1. Let $\widetilde{\Gamma}'_2$ be an irreducible component of the preimage via f_2 of Γ_2; it is in $\widetilde{Z}_2^\circ[\nu_2]$. Let $\widetilde{\Gamma}'_1$ be an irreducible component of the preimage via f_1 of $Z_1^\circ[u_1]$; it is in $\widetilde{Z}_1^\circ[u_1]$. Then $\dim(\widetilde{\Gamma}'_2) = \dim(\Gamma_2) + 1$. Similarly, $\dim(\widetilde{\Gamma}'_1) = \dim(Z_1[u_1]) + 1$.

Let $\widetilde{\Gamma}_3$ be the irreducible component of $\widetilde{Z}_3[u_1^d \oplus \nu_2 \oplus \mathrm{ord}^\epsilon]$ that contains the image $\kappa(\widetilde{\Gamma}'_1, \widetilde{\Gamma}'_2)$. As before, $\dim(\widetilde{\Gamma}_3) \geq \dim(\kappa(\widetilde{\Gamma}'_1, \widetilde{\Gamma}'_2)) + 1$. The rest of the proof follows in the same way as when $r < m_1$, by taking $\Gamma_3 = \overline{\Gamma}_3 \cap Z_3^\circ$, where $\overline{\Gamma}_3$ is the image of $\widetilde{\Gamma}_3$ via the forgetful map. To obtain the dimension inequality, note that

$$\dim(\kappa(\widetilde{\Gamma}'_1, \widetilde{\Gamma}'_2)) = 1 + \dim(Z_1[u_1]) + 1 + \dim(\Gamma_2) = 2 + \dim(Z_1) + \dim(Z_2) - l,$$

where $l = \mathrm{codim}(\mathrm{Sh}_2[\nu_2], \mathrm{Sh}_2)$. In this case, $\dim(Z_i) = N_i - 4$ for $i = 1, 2$. By (6.4),

$$\dim(Z_3) = (N_1 - 4) + (N_2 - 4) + 3 = \dim(Z_1) + \dim(Z_2) + 3. \tag{6.6}$$

On the other hand, by the de Jong–Oort purity theorem [7, Theorem 4.1],

$$\dim(\widetilde{\Gamma}_3) \geq \dim(Z_3) - l = \dim(Z_1) + \dim(Z_2) + 3 - l = \dim(\kappa(\widetilde{\Gamma}'_1, \widetilde{\Gamma}'_2)) + 1.$$ ■

6.4 Infinite clutching for non μ-ordinary

This section is similar to Section 4.3, in that we find situations in which Theorem 6.11 can be implemented recursively, infinitely many times, except that we now focus on non μ-ordinary Newton polygons.

Let $\gamma = (m, N, a)$ be a monodromy datum and let $\nu \in B(\gamma)$.

Corollary 6.14. (Extension of Corollary 4.7) Assume $Z^\circ(\gamma)[\nu]$ is non-empty and contains an irreducible component Γ such that $\mathrm{codim}(\Gamma, Z^c(\gamma)) = \mathrm{codim}(\mathrm{Sh}[\nu], \mathrm{Sh})$. Then for any n in the semi-group of $(\mathbb{Z}, +)$ generated by $\{m-t\colon\ t \mid m\}$, there exists a μ_m-cover $C \to \mathbb{P}^1$ over $\overline{\mathbb{F}}_p$ where C is a smooth curve with Newton polygon $\nu \oplus \mathrm{ord}^n$. $\square$

Proof. Let γ_1 be as in the proof of Corollary 4.7. Note that $u_1(\mathfrak{o})$ is empty for all $\mathfrak{o}$. So the pair γ_1, γ satisfies hypothesis (C), in addition to (A) and (B). The proof is then the same as for Corollary 4.7, replacing Theorem 4.5 with Theorem 6.11. $\blacksquare$

Corollary 6.15. (Extension of Corollaries 4.8 and 4.9) Let $\epsilon = (n-1)(r-1)$ if there exist $1 \leq i < j \leq N$ such that $a(i) + a(j) \equiv 0 \bmod m$, and $\epsilon = (n-1)(m-1)$ otherwise. Assume $Z^\circ(\gamma)[\nu]$ is non-empty and contains an irreducible component Γ such that $\mathrm{codim}(\Gamma, Z^c(\gamma)) = \mathrm{codim}(\mathrm{Sh}[\nu], \mathrm{Sh})$. Assume $u \ll_{(\mathrm{C})} u$. Then for any $n \in \mathbb{Z}_{\geq 1}$, there exists a smooth curve with Newton polygon $u^{n-1} \oplus \nu \oplus \mathrm{ord}^\epsilon$. $\square$

Proof. The result is true when $n = 1$ by hypothesis. For $n \geq 2$, we use Corollary 4.8 (resp. Corollary 4.9 with $t = m$) to construct a family $Z^{\times n-1}$ with Newton polygon $u^{n-1} \oplus \mathrm{ord}^{(n-2)(r-1)}$ (resp. $u^{n-1} \oplus \mathrm{ord}^{(n-2)(m-1)}$). The pair of monodromy data of the families $Z^{\times n-1}$ and Z satisfies hypotheses (A) and (B). Since $u \ll_{(\mathrm{C})} u$, by Lemma 6.7 (1), the pair also satisfies hypothesis (C). Hence we conclude by Theorem 6.11. $\blacksquare$

Corollary 6.16. With notation and hypotheses as in Corollary 4.10, assume furthermore that for some $\nu_2 \in B(\gamma_2)$, $Z_2^\circ[\nu_2]$ is non-empty and contains an irreducible component Γ such that $\mathrm{codim}(\Gamma, Z_2) = \mathrm{codim}(\mathrm{Sh}_2[\nu_2], \mathrm{Sh}_2)$. Also assume that $u_1 \ll_{(\mathrm{C})} u_2$ and $u_2 \ll_{(\mathrm{C})} u_2$. Then there exists a smooth curve with Newton polygon $u_1^{n_1} \oplus u_2^{n_2-1} \oplus \nu_2 \oplus \mathrm{ord}^{(n_1+n_2-2)(m-1)+(r-1)}$. $\square$

Proof. If $n_2 = 1$, we first apply Corollary 4.9 with $t = m$ to produce a family Z_3 with Newton polygon $u_1^{n_1} \oplus \mathrm{ord}^{(n_1-1)(m-1)}$. Note that Z_3 and Z_2 satisfy hypotheses (A) and (B). Since $u_1 \ll_{(\mathrm{C})} u_2$, by Lemma 6.7 (1), Z_3 and Z_2 also satisfy hypothesis (C). Applying Theorem 6.11 produces a smooth curve with Newton polygon $u_1^{n_1} \oplus \nu_2 \oplus \mathrm{ord}^{(n_1-1)(m-1)+(r-1)}$.

For $n_2 \geq 2$, we apply Corollary 4.10 to produce a family Z_4 with Newton polygon $u_1^{n_1} \oplus u_2^{n_2-1} \oplus \mathrm{ord}^{(n_1+n_2-3)(m-1)}$. Since $u_1 \ll_{(\mathrm{C})} u_2$ and $u_2 \ll_{(\mathrm{C})} u_2$, by Lemma 6.7 (2), Z_4 and Z_2 satisfy hypotheses (A), (B), and (C). Applying Remark 3.7, we produce generalized monodromy data for Z_4 and Z_2 by marking an additional unramified fiber. In this situation, the toric rank is $\epsilon' = m - 1$. Applying Theorem 6.11 to the generalized families for Z_4 and Z_2 completes the proof. $\blacksquare$

Corollary 6.16 applies to the pair of monodromy data in the proof of Corollary 9.7.

6.5 An exceptional example

We give an example of a pair of monodromy data, and non μ-ordinary Newton polygon ν_2, satisfying hypotheses (A) and (B), but not (C), for which (6.1) can be verified directly. Furthermore, as the Kottwitz set B_2 has size 2, this example also shows that hypothesis (C) is sufficient but not necessary for Proposition 6.8 to hold.

Recall that ss is the Newton polygon $(1/2, 1/2)$.

Proposition 6.17. If $p \equiv 7 \bmod 8$ is sufficiently large, then there exists a smooth curve over $\overline{\mathbb{F}}_p$ of genus 9 with Newton polygon $\mathrm{ss}^7 \oplus \mathrm{ord}^2$. □

Proof. Let $Z_2 = Z(8, 4, (4, 2, 5, 5))$. Then Z_2 is the special family $M[15]$ in [28, Table 1], and the associated Shimura variety Sh_2 has signature type $\mathfrak{f}_2 = (1, 1, 0, 0, 2, 0, 1)$. At any prime $p \equiv 7 \bmod 8$, the μ-ordinary Newton polygon is $u_2 = \mathrm{ord}^2 \oplus \mathrm{ss}^3$ and the basic Newton polygon is $\nu_2 = \mathrm{ss}^5$ [22, Section 6.2].

Let $Z_1 = Z(4, 3, (1, 1, 2))$, which has signature $(1, 0, 0)$. At any prime $p \equiv 7 \bmod 8$, the μ-ordinary Newton polygon is $u_1 = \mathrm{ss}$ [23, Section 4, $m = 4$]. Then $d = 2$ and $r = 2$. By Section 3.1, the induced signature type is $\mathfrak{f}_1^\dagger = (1, 0, 0, 0, 1, 0, 0)$.

The pair of monodromy data for Z_1 and Z_2 satisfies hypothesis (A). Let $p \equiv 7 \bmod 8$; then it also satisfies hypothesis (B). For the orbit $\mathfrak{o} = \{1, 7\}$, by [22, Example 4.5], $u_1(\mathfrak{o})$ has slopes $1/2$ and $u_2(\mathfrak{o})$ has slopes 0 and 1. Thus the pair does not satisfy hypothesis (C).

The image of $\widetilde{Z}_1 \times \widetilde{Z}_2$ under the clutching morphism lies in the family $\widetilde{Z}_3$ of curves with monodromy datum $(8, 5, (2, 2, 2, 5, 5))$. The Shimura variety Sh_3 has signature type $\mathfrak{f}_3 = (2, 2, 0, 0, 3, 1, 1)$ and its μ-ordinary Newton polygon is $u_3 = u_1^2 \oplus u_2 \oplus \mathrm{ord}^2 = \mathrm{ss}^5 \oplus \mathrm{ord}^4$ by Proposition 4.4. By [22, Section 4.3], there is only one element u_3 in $B(\mathrm{Sh}_3)$ which is strictly larger than $\mathrm{ss}^7 \oplus \mathrm{ord}^2 = u_1^2 \oplus \nu \oplus \mathrm{ord}^2$. From (2.6), we see that the codimension of $\mathrm{Sh}_3[u_1 \oplus \nu_2 \oplus \mathrm{ord}^2]$ in Sh_3 is 1. Thus, we conclude by Remark 6.12 and Theorem 6.11. ■

7 Supersingular cases in Moonen's table

In [28, Theorem 3.6], Moonen proved there are exactly 20 positive-dimensional special families arising from cyclic covers of $\mathbb{P}^1$. In [22, Section 6], we computed all of the Newton polygons ν that occur on the corresponding Shimura varieties using the Kottwitz method, see Section 10. Moreover, in [22, Theorem 1.1], we proved that the open Torelli locus intersects each non-supersingular (resp. supersingular) Newton polygon stratum (resp. as long as the family has dimension 1 and p is sufficiently large).

In this section, we extend [22, Theorem 1.1] to include the supersingular Newton polygon strata in the five remaining cases when the dimension of the family is greater than 1, using results from Section 6. Case (5) is note-worthy since it was not previously known that there exists a smooth supersingular curve of genus 6 when $p \equiv 2, 3, 4 \bmod 5$, see [23, Theorem 1.1] and [22, Theorem 1.1] for related results.

Theorem 7.1. There exists a smooth supersingular curve of genus g defined over $\overline{\mathbb{F}}_p$ for all sufficiently large primes satisfying the given congruence condition in the following families:

1. $g = 3$, when $p \equiv 2 \bmod 3$, in the family $M[6]$;
2. $g = 3$, when $p \equiv 3 \bmod 4$, in the family $M[8]$;
3. $g = 4$, when $p \equiv 2 \bmod 3$, in the family $M[10]$;
4. $g = 4$, when $p \equiv 5 \bmod 6$, in the family $M[14]$; and
5. $g = 6$, when $p \equiv 2, 3, 4 \bmod 5$, in the family $M[16]$.

□

Corollary 7.2. Let $\gamma = (m, N, a)$ denote the monodromy datum for one of Moonen's special families from [28, Table 1]. Assume $p \nmid m$. Let $\nu \in \nu(B(\mu_m, \mathfrak{f}))$ be a Newton polygon occurring on $\mathrm{Sh}(\gamma)$ as in Section 2.6. Then ν occurs as the Newton polygon of a smooth curve in the family $Z^\circ(\gamma)$, as long as p is sufficiently large when ν is supersingular. □

Proof. The proof is immediate from [22, Theorem 1.1] and Theorem 7.1. ■

Proof of Theorem 7.1 in cases (1), (2), (4), and (5). Let γ denote the monodromy datum, let Z denote the special family of curves and let Sh denote the corresponding Shimura variety and suppose that $p \not\equiv 1 \bmod m$. Then $\dim(Z) = \dim(\mathrm{Sh}) = 2$, and the basic locus $\mathrm{Sh}[\nu]$ is supersingular with codimension 1 in Sh.

Following [22, Section 5.2], a point of $\mathrm{Sh}[\nu]$ which is not in the image of Z° is the Jacobian of a singular curve of compact type. This point arises from an admissible clutching of points from two families Z_1 and Z_2. This yields an admissible degeneration of the inertia type, see [22, Definition 5.4]. A complete list of admissible degenerations of the inertia type for Moonen's families can be found in [22, Lemma 6.4]. In each of these cases, there exists an admissible degeneration such that $\dim(Z_1) = 0$ and the μ-ordinary Newton polygon u_1 for Z_1 is supersingular, and $m_1 = m_2$ (so $d = 1$).

In the degenerations from [22, Lemma 6.4], one checks using [22, Sections 6.1-6.2] that Z_2 is a special family with $\dim(Z_2) = 1$ and that Z_2 has exactly two Newton polygons, the μ-ordinary one u_2 and the basic one ν_2 which is supersingular. By [22, Theorem 1.1], for p sufficiently large, $Z_2^\circ[\nu_2]$ is non-empty. Since there are exactly two Newton polygons on Z, we conclude that these are $u = u_1 \oplus u_2$ and $\nu = u_1 \oplus \nu_2$. By Proposition 4.4, the pair of monodromy data for Z_1 and Z_2 satisfies hypothesis (B). The codimension condition in (6.1) is satisfied since the basic locus has codimension 1 in both Z and Z_2. By Remark 6.12 and Theorem 6.11, there exists a 1-dimensional family of smooth curves in Z with the basic Newton polygon ν, which is supersingular. ■

Proof of Theorem 7.1 in case (3). We use the same notation as in the first 2 paragraphs of the proof of the other cases. The only difference in case (3) is that $\dim(Z) = \dim(\mathrm{Sh}) = 3$ and the basic locus is supersingular with codimension 2 in Sh. In case (3), the only admissible degeneration comes from the pair of monodromy data $\gamma_1 = (3, 3, (1, 1, 1))$ and $\gamma_2 = (3, 5, (2, 1, 1, 1, 1))$. The latter of these is the monodromy datum for the special family $M[6]$. The basic locus $\mathrm{Sh}[\nu]$ has dimension 1. The codimension condition in (6.1) is not satisfied in this situation: $\mathrm{codim}(\mathrm{Sh}_2[\nu_2], \mathrm{Sh}_2) = 1$, while $\mathrm{codim}(\mathrm{Sh}[\nu], \mathrm{Sh}) = 2$.

For p sufficiently large, we claim that the number of irreducible components of $\mathrm{Sh}[\nu]$ exceeds the number that arise from the boundary of Z. Let W be a 1-dimensional family of supersingular singular curves in $Z \setminus Z^\circ$. The only way to construct such a family W is to clutch a genus 1 curve with μ_3-action together with a 1-dimensional family of supersingular curves in $M[6]$. In other words, W arises as the image under κ of $T_1 \times T_2$, for some component T_1 of $\mathrm{Sh}(3, 3, (1, 1, 1))$ and some component T_2 of the supersingular locus of $M[6]$. The number of choices for T_1, for the μ_3-actions, and for the labelings of the ramification points is a fixed constant that does not depend on p.

Thus it suffices to compare the number $s_{M[10]}$ of irreducible components of the supersingular locus in $M[10]$ with the number $s_{M[6]}$ of irreducible components T_2 of the supersingular locus in $M[6]$ when $p \equiv 2 \bmod 3$. The signature type for $M[10]$ is $(1,3)$. By [22, Theorem 8.1], the number $s_{M[10]}$ grows with respect to p.

The signature type for $M[6]$ is $(1,2)$. By [22, Remark 8.2], we see that $s_{M[6]}$ is the same for all odd $p \equiv 2 \bmod 3$. More precisely, note that $\dim(\mathrm{Sh}_2) = 2\dim(\mathrm{Sh}_2(\nu_2))$ when $p \equiv 2 \bmod 3$, that the center of the associated reductive group is connected, and that the supersingular locus is the basic locus. Thus by [38, Remark 1.1.5 (2)], all odd $p \equiv 2 \bmod 3$ satisfy the hypothesis of [38, Theorem 1.1.4 (1), Proposition 7.4.2], which provides an expression for $s_{M[6]}$ over $\overline{\mathbb{F}}_p$ in terms of objects independent of p.

Hence there exist irreducible components of $\mathrm{Sh}[\nu]$ which contain the Jacobian of a smooth curve, for p sufficiently large. ■

8 Unlikely intersections

In this section, we prove that the non-trivial intersection of the open Torelli locus with the Newton polygon strata found in most of the results of the paper is unexpected.

Recall that ss denotes the Newton polygon $(1/2, 1/2)$.

Definition 8.1. Let ν be a symmetric Newton polygon of height $2g$, and let $\mathcal{A}_g[\nu]$ be its Newton polygon stratum in the Siegel variety $\mathcal{A}_g$. Then ν satisfies condition (U) if $\dim(\mathcal{M}_g) < \mathrm{codim}(\mathcal{A}_g[\nu], \mathcal{A}_g)$. □

Definition 8.2. The open Torelli locus has an *unlikely intersection* with $\mathcal{A}_g[\nu]$ in $\mathcal{A}_g$ if there exists a smooth curve of genus g with Newton polygon ν, and ν satisfies condition (U). □

8.1 The codimension of Newton polygon strata in Siegel varieties

We study the codimension of the Newton strata in $\mathcal{A}_g$. By [29, Theorem 4.1], see also (2.6),

$$\mathrm{codim}(\mathcal{A}_g[\nu], \mathcal{A}_g) = \#\Omega(\nu), \tag{8.1}$$

where $\Omega(\nu) := \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \le x, y \le g,\ (x,y) \text{ strictly below } \nu\}$.

Remark 8.3. By (8.1), if ν is non-ordinary, then $\mathrm{codim}(\mathcal{A}_{ng}[\nu^n], \mathcal{A}_{ng})$ grows quadratically in n. In particular, if $\nu = \mathrm{ss}$, then $\mathrm{codim}(\mathcal{A}_n[\mathrm{ss}^n], \mathcal{A}_n) = n(n+1)/2 - \lfloor n^2/4 \rfloor > n^2/4$. □

Proposition 8.4. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of symmetric Newton polygons. Let $2g_n$ be the height of u_n. Suppose there exists $\lambda \in \mathbb{Q} \cap (0,1)$ such that the multiplicity of λ as a slope of u_n is at least n for each $n \in \mathbb{N}$. Then $\mathrm{codim}(\mathcal{A}_{g_n}[u_n], \mathcal{A}_{g_n})$ grows at least quadratically in n. □

Proof. Let $\nu = (\lambda, 1-\lambda)$ and let h be the height of ν. By hypothesis, $u_n = \nu^n \oplus \nu_n$ for some symmetric Newton polygon ν_n for each $n \in \mathbb{N}$ and $g_n \geq nh$. Since ν_n lies on or above $\mathrm{ord}^{g_n - nh}$, then $u_n = \nu^n \oplus \nu_n$ lies on or above $\nu^n \oplus \mathrm{ord}^{g_n - nh}$. Hence

$$\mathrm{codim}(\mathcal{A}_{g_n}[u_n], \mathcal{A}_{g_n}) \geq \mathrm{codim}(\mathcal{A}_{g_n}[\nu^n \oplus \mathrm{ord}^{g_n - nh}], \mathcal{A}_{g_n}).$$

By (8.1), or alternatively Proposition 6.8,

$$\mathrm{codim}(\mathcal{A}_{g_n}[\nu^n \oplus \mathrm{ord}^{g_n - nh}], \mathcal{A}_{g_n}) \geq \mathrm{codim}(\mathcal{A}_{nh}[\nu^n], \mathcal{A}_{nh}).$$

Thus $\mathrm{codim}(\mathcal{A}_{g_n}[u_n], \mathcal{A}_{g_n}) \geq \mathrm{codim}(\mathcal{A}_{nh}[\nu^n], \mathcal{A}_{nh})$, which is sufficient by Remark 8.3. ∎

8.2 Verifying condition (U)

Given a sequence $\{u_n\}_{n \in \mathbb{N}}$ of symmetric Newton polygons of increasing height, we state simple criteria to ensure that all but finitely many of them satisfy condition (U). Let $2g_n$ be the height of u_n.

Proposition 8.5. Assume that g_n grows linearly in n and that there exists $\lambda \in \mathbb{Q} \cap (0,1)$ such that the multiplicity of λ as a slope of u_n grows linearly in n, for all sufficiently large $n \in \mathbb{N}$. Then, for all sufficiently large n, the Newton polygon u_n satisfies condition (U). □

Proof. By Proposition 8.4, $\mathrm{codim}(\mathcal{A}_{g_n}[u_n], \mathcal{A}_{g_n})$ is quadratic in n while $\dim(\mathcal{M}_{g_n}) = 3g_n - 3$ is linear in n by hypothesis. Thus $\dim(\mathcal{M}_{g_n}) < \mathrm{codim}(\mathcal{A}_{g_n}[u_n], \mathcal{A}_{g_n})$ for $n \gg 0$. ∎

Proposition 8.6. If there exists $t \in \mathbb{R}_{>0}$ such that the multiplicity of $1/2$ as a slope of u_n is at least $2tg_n$, for all $n \in \mathbb{N}$, then u_n satisfies condition (U) for each $n \in \mathbb{N}$ such that $g_n \geq 12/t^2$. □

Proof. By the proof of Proposition 8.4 and Remark 8.3,

$$\mathrm{codim}(\mathcal{A}_{g_n}[u_n], \mathcal{A}_{g_n}) \geq \mathrm{codim}(\mathcal{A}_{\lceil tg_n \rceil}[\mathrm{ss}^{\lceil tg_n \rceil}], \mathcal{A}_{\lceil tg_n \rceil}) > (tg_n)^2/4.$$

So condition (U) for u_n is true when $(tg_n)^2/4 \geq (3g_n - 3)$ and thus when $g_n \geq 12/t^2$. ∎

Proposition 8.7. Let ν_1, ν_2 be two symmetric Newton polygons, respectively of height $2g \geq 2$, and $2h \geq 0$. Assume ν_1 is not ordinary. Then

1. for all sufficiently large $n \in \mathbb{N}$, the Newton polygon $\nu_1^n \oplus \nu_2$ satisfies condition (U);
2. if $1/2$ occurs as a slope of ν_1 with multiplicity $2\delta > 0$, then the Newton polygon $\nu_1^n \oplus \nu_2$ satisfies condition (U), for each $n \geq \max\{15g/\delta^2, 9\sqrt{h}/\delta\}$.[8.1]

□

[8.1] This bound is not sharp, but it is written so that the asymptotic dependency on g, δ, h is more clear.

Proof. 1. Let $\lambda \in \mathbb{Q} \cap (0,1)$ be a slope of ν_1, occurring with multiplicity $m_\lambda \geq 1$. Then, for each $n \in \mathbb{N}$, the Newton polygon $u_n = \nu_1^n \oplus \nu_2$ has height $2g_n = 2(ng+h)$ and slope λ occurring with multiplicity at least $m_\lambda n$. Taking $u_n = \nu_1^n \oplus \nu_2$, the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies the hypotheses of Proposition 8.5. Hence, part (1) holds.

2. As for Proposition 8.4, $\mathrm{codim}(\mathcal{A}_{ng}[\nu_1^n], \mathcal{A}_{ng}) \leq \mathrm{codim}(\mathcal{A}_{ng+h}[\nu_1^n \oplus \nu_2], \mathcal{A}_{ng+h})$. Therefore, condition (U) for $\nu_1^n \oplus \nu_2$ is implied by the inequality

$$\dim(\mathcal{M}_{ng+h}) < \mathrm{codim}(\mathcal{A}_{ng}[\nu_1^n], \mathcal{A}_{ng}). \tag{8.2}$$

Following the proof of Proposition 8.6, if the slope 1/2 occurs in ν_1 with multiplicity 2δ, then inequality (8.2) is true if $3(ng+h-1) \leq (n\delta)^2/4$, which holds for $n \geq N := 6g\delta^{-2}(1+(1+\delta^2(h-1)3^{-1}g^{-2})^{1/2})$. The asserted bound follows by noticing that $N < \max\{6(1+\sqrt{2})g/\delta^2, 2\sqrt{3}(1+\sqrt{2})\sqrt{h}/\delta\}$. ■

Remark 8.8. For $g \gg 0$, Proposition 8.7 implies that the non-trivial intersections of $\mathcal{T}_g^\circ$ with $\mathcal{A}_g[\nu]$ in Corollaries 4.8, 4.9, and 6.15 (resp. 4.10 and 6.16) are unlikely if the μ-ordinary Newton polygon u is not ordinary. (resp. if either u_1 or u_2 is not ordinary). □

Remark 8.9. Consider the following refinement of Definition 8.2: a non-empty substack U of $\mathcal{T}_g \cap \mathcal{A}_g[\nu]$ is an *unlikely intersection* if $\mathrm{codim}(U, \mathcal{M}_g) < \mathrm{codim}(\mathcal{A}_g[\nu], \mathcal{A}_g)$.

The results in Sections 4.3 and 6.4 yield families Z of cyclic covers of $\mathbb{P}^1$ such that $Z^\circ[\nu]$ is non-empty and has the expected codimension in Z. This produces an unlikely intersection as in Remark 8.9 for $g \gg 0$, when the initial Newton polygon u is not ordinary. □

9 Applications

We apply the results in Sections 4.3 and 6.4 to construct smooth curves of arbitrarily large genus g with prescribed Newton polygon ν. By Proposition 8.7, when g is sufficiently large, the curves in this section lie in the unlikely intersection $\mathcal{T}_g^\circ \cap \mathcal{A}_g[\nu]$.

Notation 9.1. For $s, t \in \mathbb{N}$, with $s \leq t/2$ and $\gcd(s,t) = 1$, we write $(s/t, (t-s)/t)$ for the Newton polygon of height $2t$ with slopes s/t and $(t-s)/t$, each with multiplicity t. □

9.1 Newton polygons with many slopes of 1/2

We obtain examples of smooth curves of arbitrarily large genus g such that the only slopes of the Newton polygons are $0, \frac{1}{2}, 1$. We focus on examples where the multiplicity of 1/2 is large relative to g.

Corollary 9.2. Let $m \in \mathbb{Z}_{\geq 1}$ be odd and $h = (m-1)/2$. Let p be a prime, $p \nmid 2m$, such that the order f of p in $(\mathbb{Z}/m\mathbb{Z})^*$ is even and $p^{f/2} \equiv -1 \bmod m$. For $n \in \mathbb{Z}_{\geq 1}$, there exists a μ_m-cover $C \to \mathbb{P}^1$ defined over $\overline{\mathbb{F}}_p$ where C is a smooth curve of genus $g = h(3n-2)$ with Newton polygon $\nu = ss^{hn} \oplus ord^{2h(n-1)}$. If $n \geq 34/h$, then $\mathrm{Jac}(C)$ lies in the unlikely intersection $\mathcal{T}_g^\circ \cap \mathcal{A}_g[\nu]$. □

Proof. Let $C \to \mathbb{P}^1$ be a μ_m-cover with $\gamma = (m, 3, a)$ where $a = (1, 1, m-2)$. Without loss of generality, an equation for C is $y^m = x^2 - 1$. By [39, Theorem 6.1], the Newton polygon of C is ss^h. The first claim follows from applying Corollary 4.9 to $Z(m, 3, \gamma)$ with $t = m$. As in the proof of Proposition 8.6, the second claim follows from the inequalities:

$$\mathrm{codim}(\mathcal{A}_{nh}[\mathrm{ss}^{nh}], \mathcal{A}_{nh}) \geq (nh)^2/4 + (nh)/2 > \dim(\mathcal{M}_{3nh-2h}) = 9nh - 6h - 3. \qquad \blacksquare$$

Remark 9.3. The Newton polygons in Proposition 9.2 are μ-ordinary; they do not appear in the literature, but the result also follows from Proposition 4.6(3) if $p \equiv -1 \bmod m$ or if $p \geq m(N-3)$ where N is the (increasingly large) number of branch points. □

We highlight the case $m = 3$ below. To our knowledge, for any odd prime p, this is the first time that a sequence of smooth curves has been produced for every $g \in \mathbb{Z}_{\geq 1}$ such that the multiplicity of the slope $1/2$ in the Newton polygon grows linearly in g.

Corollary 9.4. Let $p \equiv 2 \bmod 3$ be an odd prime. Let $g \in \mathbb{Z}_{\geq 1}$. There exists a smooth curve C_g of genus g defined over $\overline{\mathbb{F}}_p$, whose Newton polygon ν_g only has slopes $0, \frac{1}{2}, 1$ and such that the multiplicity of the slope $1/2$ is at least $2\lfloor g/3 \rfloor$. If $g \geq 107$, the curve C_g demonstrates an unlikely intersection of the open Torelli locus with the Newton polygon stratum $\mathcal{A}_g[\nu_g]$ in $\mathcal{A}_g$. □

Proof. If $g = 3n - 2$ for some n, the result is immediate from Proposition 9.2. For $g = 3n - 2 + 2\epsilon$ with $\epsilon = 1$ (resp. $\epsilon = 2$), we apply Corollary 4.7 with $t = 1$ (resp. twice) and obtain a smooth curve with Newton polygon $\mathrm{ss}^n \oplus \mathrm{ord}^{2n-2+2\epsilon}$. ■

Working with Moonen's families gives examples of families of curves where the multiplicity of the slope $1/2$ is particularly high relative to the genus.

Corollary 9.5. Let $p \equiv 4 \bmod 5$. For $n \in \mathbb{Z}_{\geq 1}$, there exists a smooth curve of genus $g = 10n - 4$ in $Z = Z(5, 5n, (2, 2, \ldots, 2))$ over $\overline{\mathbb{F}}_p$ with μ-ordinary Newton polygon $u_n = \mathrm{ss}^{4n} \oplus \mathrm{ord}^{6n-4}$. □

For $n \geq 7$, the curves with Newton polygon u_n from Corollary 9.5 lie in the unlikely intersection $\mathcal{T}_g^\circ \cap \mathcal{A}_g[\nu]$.

Proof. When $p \equiv 4 \bmod 5$, $M[16]$ has μ-ordinary Newton polygon $u_1 = \mathrm{ord}^2 \oplus \mathrm{ss}^4$.[9.1] The claim is immediate from Corollary 4.9. ■

Corollary 9.6. Under the given congruence condition on p, and with $p \gg 0$, there exists a smooth curve in $Z = Z(m, N, a)$ over $\overline{\mathbb{F}}_p$ with Newton polygon ν and $\mathrm{codim}(Z[\nu], Z) = 1$.

[9.1] The codimension condition in (6.1) does not hold for $\nu = \mathrm{ss}^6$.

construction	(m,N,a)	genus	congruence	Newton Polygon ν
$M[9]+M[9]$	$(6,6,(1,1,4,4,4,4))$	8	2 mod 3	$\mathrm{ss}^4 \oplus \mathrm{ord}^4$
$M[9]+M[12]$	$(6,6,(1,1,1,1,4,4))$	9	2 mod 3	$\mathrm{ss}^5 \oplus \mathrm{ord}^4$
$M[12]+M[12]$	$(6,6,(1,1,1,1,1,1))$	10	2 mod 3	$\mathrm{ss}^7 \oplus \mathrm{ord}^3$
$M[18]+M[18]$	$(10,6,(3,3,6,6,6,6))$	16	9 mod 10	$\mathrm{ss}^{10} \oplus \mathrm{ord}^6$
$M[20]+M[20]$	$(12,6,(4,4,7,7,7,7))$	19	11 mod 12	$\mathrm{ss}^{12} \oplus \mathrm{ord}^7$

□

9.2 Newton polygons with slopes 1/3, 1/4, and beyond

Corollary 9.7. Let $n \in \mathbb{Z}_{\geq 1}$. The following Newton polygons occur for Jacobians of smooth curves over $\overline{\mathbb{F}}_p$ under the given congruence condition on p.

congruence	ν (μ-ordinary)	ν (non μ-ordinary) for $p \gg 0$
$2,4$ mod 7	$(1/3,2/3)^n \oplus \mathrm{ord}^{6n-6}$	NA
$3,5$ mod 7	$(1/3,2/3)^{2n} \oplus \mathrm{ord}^{6n-6}$	$(1/3,2/3)^{2n-2} \oplus \mathrm{ss}^6 \oplus \mathrm{ord}^{6n-6}$
$2,5$ mod 9	$(1/3,2/3)^{2n} \oplus \mathrm{ss}^n \oplus \mathrm{ord}^{8n-8}$	$(1/3,2/3)^{2n-2} \oplus \mathrm{ss}^{n+6} \oplus \mathrm{ord}^{8n-8}$
$4,7$ mod 9	$(1/3,2/3)^{2n} \oplus \mathrm{ord}^{9n-8}$	$(1/3,2/3)^{2n-2} \oplus \mathrm{ss}^6 \oplus \mathrm{ord}^{9n-8}$

□

We remark that none of the last three lines follows from [5, Theorem 6.1] because there are at least two Newton polygons in $B(\mu_m, \mathfrak{f})$ having the maximal p-rank.

Proof. Lines 1, 2, and 3 are obtained from applying both Corollaries 4.9 and 6.15 to the families $(7,3,(1,1,5))$, $M[17]$, and $M[19]$, respectively.

For the last line, let $m=9$ and $p \equiv 4,7 \bmod 9$. There are four orbits $\mathfrak{o}_1=(1,4,7)$, $\mathfrak{o}_2=(2,5,8)$, $\mathfrak{o}_3=(3)$, and $\mathfrak{o}_4=(6)$. The μ-ordinary Newton polygon for the family $M[19]$ is $u=(1/3,2/3)^2 \oplus \mathrm{ord}$, and $\nu=\mathrm{ss}^6 \oplus \mathrm{ord}$ also occurs for a smooth curve in the family. By [22, Section 6.2], for each $\mathfrak{o} \in \mathfrak{O}$, $u(\mathfrak{o})$ has at most 2 slopes, hence hypothesis (C) is satisfied, and we obtain the Newton polygons in line 4 from Corollary 4.9 and 6.15.[9.2] ■

Corollary 9.8. Let $n \in \mathbb{Z}_{\geq 1}$. The following Newton polygons occur for Jacobians of smooth curves over $\overline{\mathbb{F}}_p$ under the given congruence condition on p.

congruence	ν (μ-ordinary)	ν (non μ-ordinary) for $p \gg 0$
$2,3$ mod 5	$(1/4,3/4)^n \oplus \mathrm{ord}^{4n-4}$	$(1/4,3/4)^{n-1} \oplus \mathrm{ss}^4 \oplus \mathrm{ord}^{4n-4}$
$3,7$ mod 10	$(1/4,3/4)^n \oplus \mathrm{ss}^{2n} \oplus \mathrm{ord}^{9n-9}$	$(1/4,3/4)^{n-1} \oplus \mathrm{ss}^{2n+4} \oplus \mathrm{ord}^{9n-9}$

□

[9.2] Alternatively, applying Corollary 4.10 and 6.16 produces the Newton polygons $(1/3,2/3)^{n_1+2n_2} \oplus \mathrm{ord}^{8n_1+9n_2-14}$ and $(1/3,2/3)^{n_1+2n_2-2} \oplus \mathrm{ss}^6 \oplus \mathrm{ord}^{8n_1+9n_2-13}$ for $n_1, n_2 \in \mathbb{Z}_{\geq 1}$.

Proof. The proof follows from Corollary 4.9 and 6.15 applied to $M[11]$ and $M[18]$. ■

Corollary 9.9. Let $p \equiv 2, 3 \bmod 5$. For any $n_1, n_2 \in \mathbb{Z}_{\geq 1}$, there exists a smooth curve of genus $g = 6n_1 + 8n_2$ defined over $\overline{\mathbb{F}}_p$ with Newton polygon $(1/4, 3/4)^{n_2+1} \oplus \mathrm{ss}^{2n_1} \oplus \mathrm{ord}^{4(n_1+n_2-1)}$. □

Proof. We apply Corollary 4.10 to $Z_1 = Z(5, 3, (2, 2, 1))$ and $Z_2 = M[11]$. By [22, Section 6.2] and [23, Section 4], if $p \equiv 2, 3 \bmod 5$, then $u_1 = \mathrm{ss}^2$ and $u_2 = (1/4, 3/4)$. [9.3] ■

Example 9.10. Let m be prime and p have odd order modulo m. The Newton polygon ν_1 for a μ_m-cover with monodromy datum $\gamma = (m, 3, a)$ has no slopes of 1/2 by [23, Section 3.2]. Applying Corollary 4.9 to $Z = Z^c(\gamma)$ with $t = m$ shows that the Newton polygon $\nu_n = \nu_1^n \oplus \mathrm{ord}^{(m-1)(n-1)}$ occurs for a smooth curve over $\overline{\mathbb{F}}_p$, for any $n \in \mathbb{Z}_{\geq 1}$. □

Examples of γ and ν_1 can be found in [23, Theorem 5.4]. For example, when $m = 11$, $a = (1, 1, 9)$ and $p \equiv 3, 4, 5, 9 \bmod 11$, then $\nu_1 = (1/5, 4/5)$. As another example, let $m = 29$, $a = (1, 1, 27)$, and $p \equiv 7, 16, 20, 23, 24, 25 \bmod 29$, then $\nu_1 = (2/7, 5/7) \oplus (3/7, 4/7)$, yielding another infinite family that cannot be studied using [5, Theorem 6.1].

10 Appendix: Newton polygons for Moonen's families

For convenience, we provide the full list of Newton polygons on Moonen's special families from [22, Section 6]. These occur for a smooth curve in the family by Corollary 7.2. The label $M[r]$ is from [28, Table 1]. The notation † means **we further assume** $p \gg 0$.

[9.3] The pair Z_1 and Z_2 does not satisfy hypothesis (C) and the codimension condition in (6.1) does not hold inductively.

Label	m	a, $\mathfrak{f}$	Newton Polygon [congruence on p]
$M[1]$	2	(1,1,1,1), (1)	ord, ss [1 mod 2]
$M[2]$	2	(1,1,1,1,1,1), (2)	ord^2, ord $\oplus$ ss, ss^2 [1 mod 2]
$M[3]$	3	(1,1,2,2), $(1,1)$	ord^2 $[1,2 \bmod 3]$, ss^2 [1 mod 3], ss^2 [2 mod 3]†
$M[4]$	4	(1,2,2,3), $(1,0,1)$	ord^2 $[1,3 \bmod 4]$, ss^2 [1 mod 4], ss^2 [3 mod 4]†
$M[5]$	6	(2,3,3,4), $(1,0,0,0,1)$	ord^2 $[1,5 \bmod 6]$, ss^2 [1 mod 6], ss^2 [5 mod 6]†
$M[6]$	3	(1,1,1,1,2), $(2,1)$	ord^3 [1 mod 3], $\mathrm{ord}^2 \oplus$ ss [2 mod 3], ord $\oplus\, \mathrm{ss}^2$, $(1/3,2/3)$ [1 mod 3], ss^3 [2 mod 3]†
$M[7]$	4	(1,1,1,1), $(2,1,0)$	ord^3 [1 mod 4], ord $\oplus\, \mathrm{ss}^2$ [3 mod 4] $\mathrm{ord}^2 \oplus$ ss [1 mod 4], ss^3 [3 mod 4]†
$M[8]$	4	(1,1,2,2,2), $(2,0,1)$	ord^3 [1 mod 4], $\mathrm{ord}^2 \oplus$ ss [3 mod 4] ord $\oplus\, \mathrm{ss}^2$, $(1/3,2/3)$ [1 mod 4], ss^3 [3 mod 4]†
$M[9]$	6	(1,3,4,4), $(1,1,0,0,1)$	ord^3 [1 mod 6], $\mathrm{ord}^2 \oplus$ ss [5 mod 6] ord $\oplus\, \mathrm{ss}^2$ [1 mod 6], ss^3 [5 mod 6]†
$M[10]$	3	(1,1,1,1,1,1) $(3,1)$	ord^4 [1 mod 3], $\mathrm{ord}^2 \oplus \mathrm{ss}^2$ [2 mod 3] $\mathrm{ord}^2 \oplus \mathrm{ss}^2$ [1 mod 3], $(1/4,3/4)$ [2 mod 3] ord $\oplus$ $(1/3,2/3)$ [1 mod 3], ss^4 [2 mod 3]†, $(1/4,3/4)$ [1 mod 3]
$M[11]$	5	(1,3,3,3), $(1,2,0,1)$	ord^4 [1 mod 5], $(1/4,3/4)$ $[2,3 \bmod 5]$, $\mathrm{ord}^2 \oplus \mathrm{ss}^2$ [4 mod 5] $\mathrm{ord}^2 \oplus \mathrm{ss}^2$ [1 mod 5], ss^4 $[2,3,4 \bmod 5]^\dagger$
$M[12]$	6	(1,1,1,3), $(2,1,1,0,0)$	ord^4 [1 mod 6], ord $\oplus\, \mathrm{ss}^3$ [5 mod 6] $\mathrm{ord}^3 \oplus$ ss [1 mod 6], ss^4 [5 mod 6]†
$M[13]$	6	(1,1,2,2), $(2,1,0,1,0)$	ord^4 [1 mod 6], $\mathrm{ord}^2 \oplus \mathrm{ss}^2$ [5 mod 6] $\mathrm{ord}^2 \oplus \mathrm{ss}^2$ [1 mod 6], ss^4 [5 mod 6]†
$M[14]$	6	(2,2,2,3,3), $(2,0,0,1,1)$	ord^4 [1 mod 6], $\mathrm{ord}^2 \oplus \mathrm{ss}^2$ [5 mod 6] $\mathrm{ord}^2 \oplus \mathrm{ss}^2$ [1 mod 6], ss^4 [5 mod 6]† ord $\oplus$ $(1/3,2/3)$ [1 mod 6]
$M[15]$	8	(2,4,5,5), $(1,1,0,0,2,0,1)$	ord^5 [1 mod 8], $\mathrm{ord}^2 \oplus \mathrm{ss}^3$ $[3,7 \bmod 8]$, $\mathrm{ord}^3 \oplus \mathrm{ss}^2$ [5 mod 8] $\mathrm{ord}^3 \oplus \mathrm{ss}^2$ [1 mod 8], $(1/4,3/4) \oplus$ ss [3 mod 8], ord $\oplus$ $(1/4,3/4)$ [5 mod 8], ss^5 [7 mod 8]†
$M[16]$	5	(2,2,2,2,2), $(2,0,3,1)$	ord^6 [1 mod 5], $(1/4,3/4) \oplus \mathrm{ss}^2$ $[2,3 \bmod 5]$, $\mathrm{ord}^2 \oplus \mathrm{ss}^4$ [4 mod 5] $\mathrm{ord}^4 \oplus \mathrm{ss}^2$ [1 mod 5], ss^6 $[2,3,4 \bmod 5]^\dagger$ $\mathrm{ord}^3 \oplus (1/3,2/3)$ [1 mod 5]
$M[17]$	7	(2,4,4,4), $(1,2,0,2,0,1)$	ord^6 [1 mod 7], $\mathrm{ord}^3 \oplus (1/3,2/3)$ $[2,4 \bmod 7]$ $(1/3,2/3)^2$ $[3,5 \bmod 7]$, $\mathrm{ord}^2 \oplus \mathrm{ss}^4$ [6 mod 7] $\mathrm{ord}^4 \oplus \mathrm{ss}^2$ [1 mod 7], $(1/6,5/6)$ $[2,4 \bmod 7]$, ss^6 $[3,5,6 \bmod 7]^\dagger$
$M[18]$	10	(3,5,6,6), $(1,1,0,1,0,0,2,0,1)$	ord^6 [1 mod 10], $(1/4,3/4) \oplus \mathrm{ss}^2$ $[3,7 \bmod 10]$, $\mathrm{ord}^2 \oplus \mathrm{ss}^4$ [9 mod 10] $\mathrm{ord}^4 \oplus \mathrm{ss}^2$ [1 mod 10], ss^6 $[3,7,9 \bmod 10]^\dagger$
$M[19]$	9	(3,5,5,5), $(1,2,0,2,0,1,0,1)$	ord^7 [1 mod 9], $(1/3,2/3)^2 \oplus$ ss $[2,5 \bmod 9]$ ord $\oplus$ $(1/3,2/3)^2$ $[4,7 \bmod 9]$, $\mathrm{ord}^2 \oplus \mathrm{ss}^5$ [8 mod 9] $\mathrm{ord}^5 \oplus \mathrm{ss}^2$ [1 mod 9], ss^7 $[2,5,8 \bmod 9]^\dagger$, ord $\oplus\, \mathrm{ss}^6$ $[4,7 \bmod 9]$
$M[20]$	12	(4,6,7,7), $(1,1,0,1,0,0,$ $2,0,1,0,1)$	ord^7 [1 mod 12], $\mathrm{ord}^3 \oplus \mathrm{ss}^4$ [5 mod 12], $\mathrm{ord}^4 \oplus \mathrm{ss}^3$ [7 mod 12], $\mathrm{ord}^2 \oplus \mathrm{ss}^5$ [11 mod 12] $\mathrm{ord}^5 \oplus \mathrm{ss}^2$ [1 mod 12], ord $\oplus$ $(1/4,3/4) \oplus \mathrm{ss}^2$ [5 mod 12] $\mathrm{ord}^2 \oplus \mathrm{ss}^5$ [7 mod 12], ss^7 [11 mod 12]†

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