



BranchHull: Convex bilinear inversion from the entrywise product of signals with known signs



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ABSTRACT

We consider the bilinear inverse problem of recovering two vectors, \mathbf{x} and \mathbf{w} , in \mathbb{R}^L from their entrywise product. For the case where the vectors have known signs and belong to known subspaces, we introduce the convex program BranchHull, which is posed in the natural parameter space that does not require an approximate solution or initialization in order to be stated or solved. Under the structural assumptions that \mathbf{x} and \mathbf{w} are members of known K and N dimensional random subspaces, we present a recovery guarantee for the noiseless case and a noisy case. In the noiseless case, we prove that the BranchHull recovers the vectors up to the inherent scaling ambiguity with high probability when $L \gg 2(K + N)$. The analysis provides a precise upper bound on the coefficient for the sample complexity. In a noisy case, we show that with high probability the BranchHull is robust to small dense noise when $L = \Omega(K + N)$.

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1. Introduction

This paper considers a bilinear inverse problem (BIP): recover vectors \mathbf{x} and \mathbf{w} from the observation $\mathbf{y} = \mathcal{A}(\mathbf{x}, \mathbf{w})$, where \mathcal{A} is a bilinear operator. BIPs have been extensively studied in signal processing and data science literature, and comprise of fundamental problems such as blind deconvolution/demodulation [3,23,15,1], phase retrieval [9], dictionary learning [25], matrix factorization [14,16], and self-calibration [18]. Optimization problems involving bilinear terms and constraints also arise in other contexts, such as blending problems in chemical engineering [6].

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A significant challenge of BIPs is the ambiguity of solutions. For example, if $(\mathbf{x}^\natural, \mathbf{w}^\natural)$ is a solution to a BIP, then so is $(c\mathbf{x}^\natural, c^{-1}\mathbf{w}^\natural)$ for any nonzero $c \in \mathbb{R}$. Other ambiguities may also arise, including the shift ambiguity in blind deconvolution, the permutation ambiguity in dictionary learning, and the ambiguity up to multiplication by an invertible matrix in matrix factorization. These ambiguities are challenging because they cause the set of solutions to be nonconvex.

We will consider the fundamental bilinear inverse problem of recovering two L dimensional vectors \mathbf{w} and \mathbf{x} from the observations $\mathbf{y} = \mathbf{w} \circ \mathbf{x}$, where \circ denotes the entry-wise product of vectors. This is immediately recognized as the calibration problem, where one is only able to measure a signal \mathbf{x} modulo unknown multiplicative gains \mathbf{w} . A self-calibration algorithm aims to figure out the gains \mathbf{w} and the signal \mathbf{x} jointly from \mathbf{y} . The circular convolution also becomes pointwise multiplication in the Fourier domain, allowing us to reduce the important blind deconvolution problem in signal processing and wireless communications to a complex case of the above bilinear form.

In addition to the challenges of general BIPs, the BIP above is difficult because the solutions are nonunique without further structural assumptions. For example $(\mathbf{w}^\natural, \mathbf{x}^\natural)$ and $(1, \mathbf{w}^\natural \circ \mathbf{x}^\natural)$ are both consistent with the entrywise products $\mathbf{y} = \mathbf{w}^\natural \circ \mathbf{x}^\natural$. While multiple structural assumptions are reasonable, we will consider the case where \mathbf{w}^\natural and \mathbf{x}^\natural belong to known subspaces \mathbf{B} and \mathbf{C} , as in [3]. In addition, we also require \mathbf{w}^\natural and \mathbf{x}^\natural to be real and of known signs. The method can be extended to complex vectors in the case of known complex phases.

The known sign information in the real case is justified in imaging applications, where we want to recover image pixels (always non-negative) from occlusions caused by unknown multiplicative masks [7]. A stylized application of this setup also arises in the wireless communications. A source encodes a message as a series of positive magnitude shifts on tones at frequencies f_1, f_2, \dots, f_L . These real valued and positive $\mathbf{x} = [x(f_1), x(f_2), \dots, x(f_L)]^\top$ are transmitted over a linear-time invariant channel, where $x(f_\ell)$ are weighted by the frequency response of the channel $w(f_\ell)$ (in general complex valued), and in the ideal noiseless case, the receiver ends up observing $y(f_\ell) = x(f_\ell) \cdot w(f_\ell)$. The real part of the complex-valued measurements $\text{Re}\{y(f_\ell)\} = x(f_\ell) \cdot \text{Re}\{w(f_\ell)\}$ are simply the pointwise product of two unknown real numbers with known signs. In addition, in this application, the vectors \mathbf{x} , and \mathbf{w} naturally live in low-dimensional subspaces; for details, see [3] and [2].

The assumptions of sign and subspace measurements are strongly motivated by the sweep-distortion removal problem in dielectric imaging [1]. In this problem, a dielectric is imaged, and the pointwise product of an electromagnetic pulse and the reflectivity pattern is observed. The signal's nonnegativity follows from nonnegativity of the material's reflectivity, and the pulse belongs to a subspace defined by dominant wavelet coefficients of the image.

We consider the following bilinear inverse problem in the presence of multiplicative noise given by the vector $\mathbf{1} + \xi$:

$$\begin{aligned} \text{Let: } \mathbf{w}^\natural &\in \mathbf{B} \subset \mathbb{R}^L, \mathbf{x}^\natural \in \mathbf{C} \subset \mathbb{R}^L, \xi \in \mathbb{R}^L, \mathbf{s} = \text{sign}(\mathbf{w}^\natural) \\ \mathbf{y} &= \mathbf{w}^\natural \circ \mathbf{x}^\natural \circ (\mathbf{1} + \xi), \end{aligned} \tag{1}$$

Given: $\mathbf{y}, \mathbf{s}, \mathbf{B}, \mathbf{C}$

Find: $\mathbf{w}^\natural, \mathbf{x}^\natural$ up to the scaling ambiguity

One standard way to solve the BIP above¹ is to convexify it by lifting. More specifically, the bilinear inverse problem can be recast as a linear matrix recovery problem with the structural constraint that the recovered matrix is rank one. With $w_\ell^\natural = \mathbf{b}_\ell^\top \mathbf{h}^\natural$ and $x_\ell^\natural = \mathbf{c}_\ell^\top \mathbf{m}^\natural$ for $\ell = 1, \dots, L$, the underlying linear operator is given by $y_\ell = w_\ell^\natural x_\ell^\natural = \mathbf{b}_\ell^\top \mathbf{h}^\natural \mathbf{m}^{\natural\top} \mathbf{c}_\ell = \langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \mathbf{h}^\natural \mathbf{m}^{\natural\top} \rangle = \mathcal{A}_\ell(\mathbf{h}^\natural \mathbf{m}^{\natural\top})$, and the formal recovery framework

¹ As stated, this approach ignores the sign information.

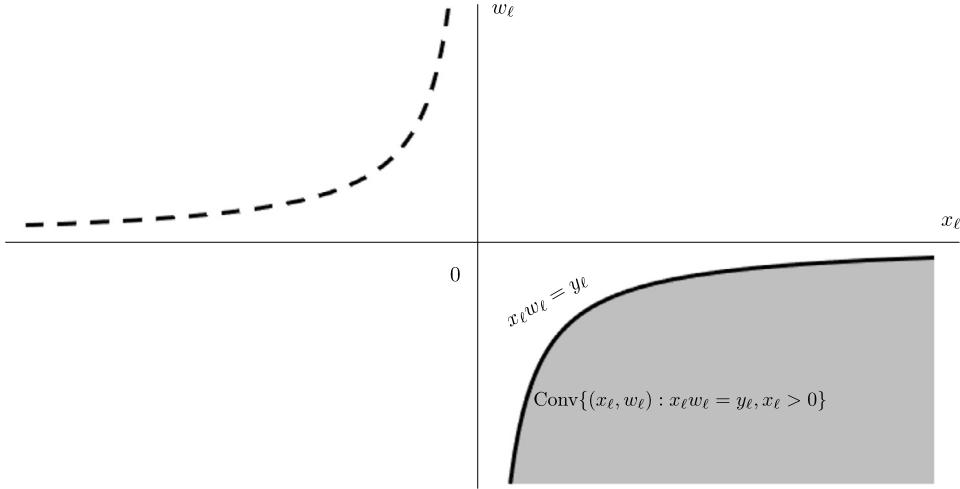


Fig. 1. Given the bilinear measurement $x_\ell w_\ell = y_\ell$, the point (x_ℓ, w_ℓ) is on a two-branch hyperbola, as depicted by the dashed and solid lines. Further information on the sign of w_ℓ identifies which branch of the hyperbola the point is on (the solid line). The convex formulation in this paper replaces the relevant branch of the hyperbola with its convex hull (the shaded region).

is to find the \mathbf{X} of minimal rank that is consistent with $\mathcal{A}(\mathbf{X}) = \mathbf{y}$. By relaxing the rank objective to the nuclear norm of \mathbf{X} , this optimization problem becomes a semidefinite program. The results in [3], which apply to the complex case, show that when \mathbf{b}_ℓ and \mathbf{c}_ℓ are Fourier and Gaussian vectors, respectively, this semidefinite program succeeds in recovering the rank-1 matrix $\mathbf{h}^\natural \mathbf{m}^{\natural \top}$ with high probability, whenever $K + N \lesssim L/\log^3 L$. Unfortunately, directly optimizing a lifted problem is prohibitively computationally expensive, as the lifted semidefinite program is posed on a space of dimensionality $K \times N$, which is much larger than the $K + N$ dimensionality of the natural parameter space.

To address the intractability of lifted methods, a recent theme of research has been to solve quadratic and bilinear recovery problems in the natural parameter space using alternating minimization and gradient descent algorithms [20,24]. These algorithms include the Wirtinger Flow (WF) and its variants for phase retrieval [5,8,28]. A Wirtinger gradient descent method was recently introduced for blind deconvolution in [17]. In the case that \mathbf{b}_ℓ are deterministic complex matrices that satisfy an incoherence property and that \mathbf{c}_ℓ are Gaussian vectors, this nonconvex method succeeds at recovering \mathbf{h}^\natural and \mathbf{m}^\natural up to the scale ambiguity with high probability when $K + N \lesssim L/\log^2 L$. While WF based methods enjoy rigorous recovery guarantees under optimal or nearly optimal sample complexity with suitable measurement models, the proofs of these results are long and technical. Also, because of the nonconvexity of the problem, the convergence of a gradient descent algorithm to the global minimum usually relies on an appropriate initialization [26,8,17].

The approach we will present in this paper will combine strengths of both of these approaches. Specifically, we introduce a convex formulation in the natural parameter space for the bilinear inverse problem of recovering two real vectors from their entrywise product, provided that the vectors live in known subspaces and have known signs. This convex formulation is called BranchHull and does not involve an initialization or approximate solution in order to be posed or solved. BranchHull is based on the following idea: The bilinear measurements $x_\ell w_\ell = y_\ell$ establish that (x_ℓ, w_ℓ) is on one of two branches of a hyperbola in \mathbb{R}^2 . Information on $\text{sign}(w_\ell)$ identifies the appropriate branch. The convex formulation is then formed by relaxing this nonconvex branch of a hyperbola to its convex hull, as shown in Fig. 1. We consider the case where the two vectors live in random subspaces of \mathbb{R}^L with dimensions K and N . Under this assumption on \mathbf{x}^\natural and \mathbf{w}^\natural , with noise $\boldsymbol{\xi}$ that does not change the sign of the measurements, we establish that the Euclidean recovery error is bounded by the ℓ_∞ norm of the noise. This result holds with high probability for $K + N \lesssim L$. In the noiseless case, we provide an explicit lower bound on the recovery probability that is nonzero when $L > 2(K + N) - 3$.

1.1. Problem formulation

We consider the bilinear inverse problem of recovering two vectors from their entrywise product. That is, let $\mathbf{w}^\natural, \mathbf{x}^\natural, \boldsymbol{\xi} \in \mathbb{R}^L$, and let $\mathbf{y} = \mathbf{w}^\natural \circ \mathbf{x}^\natural \circ (\mathbf{1} + \boldsymbol{\xi})$, where $\boldsymbol{\xi}$ corresponds to noise. From \mathbf{y} , we attempt to find \mathbf{w}^\natural and \mathbf{x}^\natural up to the scaling ambiguity $(c\mathbf{w}^\natural, \frac{1}{c}\mathbf{x}^\natural)$. To make the problem well posed, we consider the case where \mathbf{w}^\natural and \mathbf{x}^\natural belong to known subspaces \mathbf{B} and \mathbf{C} of \mathbb{R}^L . We further consider the case where the signs of the entries of \mathbf{w}^\natural , and hence those of \mathbf{x}^\natural , are known. Let $\mathbf{s} = \text{sign}(\mathbf{w}^\natural)$. This bilinear inversion problem is stated in (1).

Ideally, we could resolve the scaling ambiguity and find $(\mathbf{w}^\natural, \mathbf{x}^\natural)$ such that $\|\mathbf{w}^\natural\|_2 = \|\mathbf{x}^\natural\|_2$ by solving the following program:

$$\begin{aligned} \underset{\mathbf{w} \in \mathbf{B}, \mathbf{x} \in \mathbf{C}}{\text{minimize}} \quad & \|\mathbf{w}\|_2^2 + \|\mathbf{x}\|_2^2 \text{ subject to } w_\ell x_\ell = y_\ell \\ & s_\ell w_\ell \geq 0, \quad \ell = 1, \dots, L. \end{aligned}$$

This program is nonconvex, but it admits the following convex relaxation:

$$\begin{aligned} \underset{\mathbf{w} \in \mathbf{B}, \mathbf{x} \in \mathbf{C}}{\text{minimize}} \quad & \|\mathbf{w}\|_2^2 + \|\mathbf{x}\|_2^2 \text{ subject to } \text{sign}(y_\ell) w_\ell x_\ell \geq |y_\ell| \\ & s_\ell w_\ell \geq 0, \quad \ell = 1, \dots, L. \end{aligned}$$

Note that for fixed ℓ , the feasible set $\{(w_\ell, x_\ell) \mid \text{sign}(y_\ell) w_\ell x_\ell \geq |y_\ell|, s_\ell w_\ell \geq 0\}$ is the convex hull of $\{(w_\ell, x_\ell) \mid w_\ell x_\ell = y_\ell, s_\ell w_\ell \geq 0\}$.

We consider this problem when written in the natural parameter space. Despite the abuse of notation, let $\mathbf{B} \in \mathbb{R}^{L \times K}$ be a matrix that spans the K dimensional subspace \mathbf{B} . Similarly, let $\mathbf{C} \in \mathbb{R}^{L \times N}$ be a matrix that spans the N dimensional subspace \mathbf{C} . Let $(\mathbf{h}^\natural, \mathbf{m}^\natural) \in \mathbb{R}^K \times \mathbb{R}^N$. Let $\mathbf{w}^\natural = \mathbf{B}\mathbf{h}^\natural$ and $\mathbf{x}^\natural = \mathbf{C}\mathbf{m}^\natural$. We can write $w_\ell = \mathbf{b}_\ell^\top \mathbf{h}$, $x_\ell = \mathbf{c}_\ell^\top \mathbf{m}$, and $y_\ell = \langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \mathbf{h}^\natural \mathbf{m}^{\natural\top} \rangle$, where \mathbf{b}_ℓ^\top is the ℓ th row of \mathbf{B} and \mathbf{c}_ℓ^\top is the ℓ th row of \mathbf{C} . The recovery task is now to find $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ by the convex program called BranchHull

$$\begin{aligned} \underset{\mathbf{h} \in \mathbb{R}^K, \mathbf{m} \in \mathbb{R}^N}{\text{minimize}} \quad & \|\mathbf{h}\|_2^2 + \|\mathbf{m}\|_2^2 \text{ subject to } \text{sign}(y_\ell) \mathbf{b}_\ell^\top \mathbf{h} \cdot \mathbf{c}_\ell^\top \mathbf{m} \geq |y_\ell| \\ & \mathbf{s}_\ell \cdot \mathbf{b}_\ell^\top \mathbf{h} \geq 0, \quad \ell = 1, \dots, L. \end{aligned} \tag{BH}$$

This program is convex because for any fixed ℓ , the points consistent with both the first and second constraints is a convex set. This program has $K + N$ variables, L linear inequality constraints, and L nonlinear inequality constraints. Because the scaling $(c\mathbf{w}^\natural, \frac{1}{c}\mathbf{x}^\natural)$ is consistent with the constraints for positive c , the program will return a solution where $\|\mathbf{h}\|_2 = \|\mathbf{m}\|_2$. Thus, if recovery is successful in the noiseless case, the optimal solution is $\left(\mathbf{h}^\natural \sqrt{\frac{\|\mathbf{m}^\natural\|_2}{\|\mathbf{h}^\natural\|_2}}, \mathbf{m}^\natural \sqrt{\frac{\|\mathbf{h}^\natural\|_2}{\|\mathbf{m}^\natural\|_2}}\right)$. We implement the same convex program (BH) in the noisy case if the noise $\boldsymbol{\xi}$ does not alter the sign of the measurement \mathbf{y} . This occurs when $\xi_\ell \geq -1$ for all $\ell \in [L]$. For the case where noise alters sign or outlier case, see the discussion section for a modified program that is conjectured to tolerate sign change and significant outliers.

1.2. Main results

In this paper, we consider the bilinear recovery problem (1), where the subspaces given by \mathbf{B} and \mathbf{C} are random. Specifically, we show that if \mathbf{B} and \mathbf{C} have i.i.d. Gaussian entries, then exact recovery of $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ is possible in the noiseless case with nonzero probability when there are at least 2 times as many measurements as degrees of freedom.

Theorem 1 (Noiseless case). Fix $(\mathbf{h}^\natural, \mathbf{m}^\natural) \in \mathbb{R}^K \times \mathbb{R}^N$ such that $\mathbf{h}^\natural \neq 0$ and $\mathbf{m}^\natural \neq 0$. Let $\mathbf{B} \in \mathbb{R}^{L \times K}$, $\mathbf{C} \in \mathbb{R}^{L \times N}$ have i.i.d. $\mathcal{N}(0, 1)$ entries and $\boldsymbol{\xi} = 0$. Then $\left(\mathbf{h}^\natural \sqrt{\frac{\|\mathbf{m}^\natural\|_2}{\|\mathbf{h}^\natural\|_2}}, \mathbf{m}^\natural \sqrt{\frac{\|\mathbf{h}^\natural\|_2}{\|\mathbf{m}^\natural\|_2}}\right)$ is the unique solution to (BH) with probability at least

$$1 - \exp\left(-\frac{[L - (2N + 2K - 3)]^2}{2(L - 1)}\right),$$

provided that $L > 2N + 2K - 3$.

This theorem provides an explicit lower bound on the recovery probability by the convex program (BH). If $L > 2N + 2K - 3$, there is a nonzero probability of successful recovery. By taking $L \geq \tilde{C}(N + K)$, the probability of failure becomes at most $e^{-\tilde{c}L}$, for universal constants \tilde{C} and \tilde{c} . The scaling of L in terms of $N + K$ is information theoretically optimal up to a constant factor. The proof of Theorem 1 follows from estimating the probability of covering a sphere by random hemispheres chosen from a nonuniform distribution.

Now we will state a result that the convex program (BH) is robust to small dense noise. Let

$$\epsilon = \|\boldsymbol{\xi}\|_\infty \tag{2}$$

represent the noise level. In particular, we present a recovery theorem for $\epsilon \leq 1$. Under this assumption on the noise, we show that if the matrices \mathbf{B} and \mathbf{C} have i.i.d. Gaussian entries and there are $O(K + N)$ measurements, then the minimizer of (BH) is close to $\left(\mathbf{h}^\natural \sqrt{\frac{\|\mathbf{m}^\natural\|_2}{\|\mathbf{h}^\natural\|_2}}, \mathbf{m}^\natural \sqrt{\frac{\|\mathbf{h}^\natural\|_2}{\|\mathbf{m}^\natural\|_2}}\right)$ with high probability.

Theorem 2 (Noisy case). Fix $(\mathbf{h}^\natural, \mathbf{m}^\natural) \in \mathbb{R}^K \times \mathbb{R}^N$ such that $\mathbf{h}^\natural \neq 0$ and $\mathbf{m}^\natural \neq 0$. Let $\mathbf{B} \in \mathbb{R}^{L \times K}$, $\mathbf{C} \in \mathbb{R}^{L \times N}$ have i.i.d. $\mathcal{N}(0, 1)$ entries. Let ϵ be as defined in (2). Let $\mathbf{y} \in \mathbb{R}^L$ contain measurements that satisfy (1) with $\epsilon \in [0, 1]$. If $L \geq C(K + N)$ then the unique minimizer $(\mathbf{h}^*, \mathbf{m}^*)$ of the BranchHull program (BH) satisfies

$$\left(\left\|\mathbf{h}^* - \mathbf{h}^\natural \sqrt{\frac{\|\mathbf{m}^\natural\|_2}{\|\mathbf{h}^\natural\|_2}}\right\|_2^2 + \left\|\mathbf{m}^* - \mathbf{m}^\natural \sqrt{\frac{\|\mathbf{h}^\natural\|_2}{\|\mathbf{m}^\natural\|_2}}\right\|_2^2\right)^{\frac{1}{2}} \leq 4\sqrt{\epsilon} \sqrt{\|\mathbf{h}^\natural\|_2 \|\mathbf{m}^\natural\|_2}$$

with probability at least $1 - e^{-cL}$. Here, C and c are absolute constants.

In Theorem 2, the ℓ_2 recovery error depends on the noise level as the square root of ϵ . We suspect that this square root dependence in the power of ϵ is an artifact of the proof technique and numerical simulations presented in Fig. 4 suggests the recovery errors, for small noise, behaves linearly in ϵ .

1.3. Discussion

The BranchHull formulation is a novel convex relaxation for the bilinear recovery from the entrywise product of vectors with known signs, and it enjoys a recovery guarantee when those vectors belong to random real subspaces of appropriate dimensions. The formulation is nothing more than finding which point of a convex set is closest to the origin. Geometrically, exact recovery is possible by ℓ_2 -norm minimization because the feasible set of (\mathbf{h}, \mathbf{m}) has a ‘pointy’ ridge that corresponds to the fundamental scaling ambiguity, as illustrated in Fig. 2.

A related formulation to BranchHull was recently introduced for the phase retrieval problem. This formulation, called PhaseMax, is a linear program and was independently discovered by [4] and [10]. PhaseMax

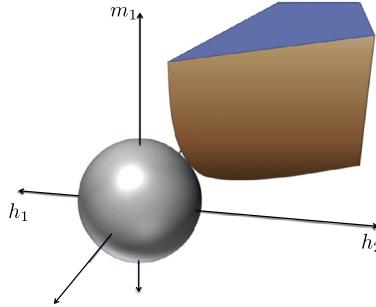


Fig. 2. An illustration of the geometry of BranchHull in the case where $\mathbf{h} \in \mathbb{R}^2$ and $\mathbf{m} \in \mathbb{R}^1$. The feasible set of BranchHull has a shape similar to the solid in the top right. The ridge of this set corresponds to the fundamental scaling ambiguity of the bilinear recovery problem. The solution to BranchHull is given by the smallest scaling of the unit ball that intersects the feasible set. The minimizer is exactly on this ridge because the ridge is ‘pointy.’

enjoys a rigorous recovery guarantee under a random data model. Existing recovery proofs are based on statistical learning theory [4], geometric probability [10], and elementary probabilistic concentration arguments [12]. As with Wirtinger Flow, successful recovery of PhaseMax with optimal sample complexity has been proven when an appropriate initialization is known. Unlike Wirtinger Flow, the initialization is used in PhaseMax’s objective function, as opposed to its algorithmic implementation. In both PhaseMax and Wirtinger Flow, an approximate solution or initialization is needed to state or solve the optimization problem. We note that BranchHull does not require an anchor vector or initialization to be stated or solved. As a trade off, BranchHull instead assumes the sign information of the signal is known.

The idea of convex relaxations in the natural parameter space for bilinear problems is not new. For example, in nonlinear programming (NLP) or mixed integer nonlinear programming (MINLP) problems with bilinear constraints and specified variable bounds, a McCormick relaxation [19] replaces bilinear terms with four linear inequality constraints that define a convex quadrilateral that contains the hyperbola of feasible points within the variable bounds. Tighter convex relaxations are possible [6], such as by using the hyperbola itself as an inequality constraint [21]. These relaxations have been studied as part of branch and bound approaches to NLPs and MINLPs. Under certain conditions and branching rules [13] these approaches can find a global minimizer; however, the branching results in many convex programs that need to be solved, and it may result in exponential time complexity. In contrast, the present paper considers only the single convex program, BranchHull, achieved by the natural convex relaxation of bilinear constraints with only sign information. This work establishes conditions — in particular, subspace conditions — under which exact recovery by an efficient convex program can be rigorously established.

This work motivates several interesting and important extensions. Most immediately, BranchHull can be extended when the phases of complex vectors are known. Because of applications in signal processing and communications, it is also important to extend the theory for BranchHull to include deterministic subspaces, such as the span of partial Fourier matrices. This paper shows that BranchHull is robust to noise that does not change the sign of any measurement. Tolerance to a general noise model, including outliers, should be established for BranchHull or a variation with slack variables, such as in [11]. Noise tolerance in the case with sign change is particularly important because even one measurement with an incorrect sign can substantially alter the shape of the feasible set. For this general noise and outlier case, we propose the Robust BranchHull program

$$\underset{\mathbf{h} \in \mathbb{R}^k, \mathbf{m} \in \mathbb{R}^N, \mathbf{e} \in \mathbb{R}^L}{\text{minimize}} \quad \|\mathbf{h}\|_2^2 + \|\mathbf{m}\|_2^2 + \lambda \|\mathbf{e}\|_1 \text{ subject to } \text{sign}(y_\ell)(\mathbf{c}_\ell^\top \mathbf{m} + e_\ell) \mathbf{b}_\ell^\top \mathbf{h} \geq |y_\ell|, \quad (\text{RBH})$$

$$s_\ell \cdot \mathbf{b}_\ell^\top \mathbf{h} \geq 0, \quad \ell = 1, \dots, L,$$

which shifts the feasibility set to include the target signal while penalizing against shift. In the outlier case, the ℓ_1 penalty promotes sparsity of slack variable \mathbf{e} , which is desired. We leave detailed empirical and

theoretical analysis of (RBH) to future work. It would also be interesting to develop convex relaxations in the natural parameter space that do not use sign information. Further, extensions to more general bilinear recovery problems are of significant interest. All of these directions are left for future publications.

1.4. Organization of the paper

The remainder of the paper is organized as follows. In Section 1.5, we present notations used throughout the paper. In Section 2.1, we present the proof of Theorem 1. In Section 2.2, we present the proof of Theorem 2. In Section 3 we observe the performance of BranchHull on synthetic data.

1.5. Notation

Vectors and matrices are written with boldface, while scalars and entries of vectors are written in plain font. For example, $c_{\ell 1}$ is the first entry of \mathbf{c}_{ℓ} . We write $\mathbf{1}$ as the vector of all ones with dimensionality appropriate for the context. Let $[L] = \{1, 2, \dots, L\}$. Let \mathbf{e}_i be the i th standard basis element. We write $K + N \lesssim L$ to mean that there exists a constant C such that $K + N \leq CL$. Given a vector in $\mathbf{x} \in \mathbb{R}^N$, let $\tilde{\mathbf{x}} \in \mathbb{R}^{N-1}$ be the subvector formed by all but the first coefficient of \mathbf{x} . Let \mathbb{S}^{n-1} be the unit sphere in \mathbb{R}^n . For matrices \mathbf{A}, \mathbf{B} , let $\langle \mathbf{A}, \mathbf{B} \rangle = \text{trace}(\mathbf{B}^T \mathbf{A})$ be the Hilbert-Schmidt inner product of \mathbf{A} with \mathbf{B} . For a set S , let $\text{Conv}(S)$ be its convex hull. Let $\text{Re}\{z\}$ be the real part of a complex z .

2. Technical proofs

In this section we provide proofs of Theorems 1 and 2. These proofs use a sphere covering type argument which is based on the idea that m random directions sampled from a symmetric distribution will cover the unit sphere \mathbb{S}^{n-1} with high probability when $m = \Omega(n)$. Another paper that uses this technique is the PhaseMax paper by [10].

2.1. Proof of Theorem 1

We will first show that BranchHull program (BH) is a convex program.

Lemma 1. *If $\mathbf{y} \in \mathbb{R}^L$ such that $y \neq 0$, $\mathbf{s} \in \{\pm 1\}^L$, $\mathbf{B} \in \mathbb{R}^{L \times K}$ and $\mathbf{C} \in \mathbb{R}^{L \times N}$ then the BranchHull program (BH) is a convex program.*

Proof. As the objective function is convex, we consider the constraints of (BH). For a fixed ℓ , let $S_{\ell} = \{(\mathbf{h}, \mathbf{m}) \in \mathbb{R}^K \times \mathbb{R}^N : \text{sign}(y_{\ell}) \mathbf{b}_{\ell}^T \mathbf{h} \cdot \mathbf{c}_{\ell}^T \mathbf{m} \geq |y_{\ell}|, s_{\ell} \cdot \mathbf{b}_{\ell}^T \mathbf{h} \geq 0\}$, $S_{\ell,1} = \{(x, w) \in \mathbb{R}^2 : \text{sign}(y_{\ell}) x w \geq |y_{\ell}|, s_{\ell} w \geq 0\}$ and $S_{\ell,2} = \{(\mathbf{h}, \mathbf{m}) \in \mathbb{R}^K \times \mathbb{R}^N : (\mathbf{b}_{\ell}^T \mathbf{h}, \mathbf{c}_{\ell}^T \mathbf{m}) \in S_{\ell,1}\}$. To show S_{ℓ} is a convex set, it is sufficient to show that the sets $S_{\ell,1}$ and $S_{\ell,2}$ are convex.

We first show that the set $S_{\ell,1}$ is convex. Let $P = \{w \in \mathbb{R} : s_{\ell} w \geq 0\}$ and consider the function $f : P \rightarrow \mathbb{R}$ such that $f(w) = \frac{w}{s_{\ell}}$. Note that if $\text{sign}(y_{\ell}) s_{\ell} \geq 0$ then f is a convex function and $S_{\ell,1}$ is the epigraph of f . Similarly, if $\text{sign}(y_{\ell}) s_{\ell} \leq 0$ then f is a concave function and $S_{\ell,1}$ is the subgraph of f . In both cases, $S_{\ell,1}$ is a convex set because the epigraph of a convex function and the subgraph of a concave function are convex.

Lastly, $S_{\ell,2}$ is convex because the inverse image of a convex set of a linear map is convex. So, S_{ℓ} is also a convex set. Since the intersection of any number of convex sets is convex, we have that the constraint of (BH) is a convex set. Thus, BranchHull program (BH) is a convex program. \square

To prove Theorem 1, we will show that $(\mathbf{h}^{\natural}, \mathbf{m}^{\natural})$ is the unique minimizer of an optimization with a larger feasible set defined by linear constraints.

Lemma 2. If $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ is the unique solution to

$$\begin{aligned} \underset{\mathbf{h} \in \mathbb{R}^K, \mathbf{m} \in \mathbb{R}^N}{\text{minimize}} \quad & \|\mathbf{h}\|_2^2 + \|\mathbf{m}\|_2^2 \text{ subject to } y_\ell \langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \mathbf{h} \mathbf{m}^{\natural\top} + \mathbf{h}^\natural \mathbf{m}^\top \rangle \geq 2y_\ell^2, \\ & \ell = 1, \dots, L, \end{aligned} \quad (3)$$

then $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ is the unique solution to (BH).

Proof of Lemma 2. It suffices to show that the feasible set of (3) contains the feasible set of (BH). We may rewrite (BH) as

$$\begin{aligned} \underset{\mathbf{h} \in \mathbb{R}^K, \mathbf{m} \in \mathbb{R}^N}{\text{minimize}} \quad & \|\mathbf{h}\|_2^2 + \|\mathbf{m}\|_2^2 \text{ subject to } y_\ell \mathbf{b}_\ell^\top \mathbf{h} \cdot \mathbf{c}_\ell^\top \mathbf{m} \geq y_\ell^2 \\ & s_\ell \cdot \mathbf{b}_\ell^\top \mathbf{h} \geq 0, \quad \ell = 1, \dots, L. \end{aligned}$$

We now use the fact that a convex set with a smooth boundary is contained in a halfspace defined by the tangent hyperplane at any point on the boundary of the set. Consider the point $(w_\ell^\natural, x_\ell^\natural) \in \mathbb{R}^2$, and observe that

$$\left\{ (w_\ell, x_\ell) \in \mathbb{R}^2 \middle| \begin{array}{l} y_\ell w_\ell x_\ell \geq y_\ell^2 \\ \text{sign}(w_\ell) = s_\ell \end{array} \right\} \subseteq \left\{ (w_\ell, x_\ell) \in \mathbb{R}^2 \middle| \begin{pmatrix} y_\ell x_\ell^\natural \\ y_\ell w_\ell^\natural \end{pmatrix} \cdot \begin{pmatrix} w_\ell - w_\ell^\natural \\ x_\ell - x_\ell^\natural \end{pmatrix} \geq 0 \right\}. \quad (4)$$

Plugging in $w_\ell = \mathbf{b}_\ell^\top \mathbf{h}$ and $x_\ell = \mathbf{c}_\ell^\top \mathbf{m}$, we have that any feasible (\mathbf{h}, \mathbf{m}) satisfies

$$y_\ell \mathbf{c}_\ell^\top \mathbf{m}^\natural \mathbf{b}_\ell^\top \mathbf{h} + y_\ell \mathbf{b}_\ell^\top \mathbf{h}^\natural \mathbf{c}_\ell^\top \mathbf{m} \geq 2y_\ell^2, \quad \ell = 1, \dots, L,$$

which implies $y_\ell \langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \mathbf{h} \mathbf{m}^{\natural\top} + \mathbf{h}^\natural \mathbf{m}^\top \rangle \geq 2y_\ell^2$ for all ℓ . \square

We now show that $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ is the unique solution to the optimization problem (3) if the unit sphere in \mathbb{R}^{N+K-2} is covered by L hemispheres given in terms of \mathbf{b}_ℓ and \mathbf{c}_ℓ . Write $\mathbf{b}_\ell = (b_{\ell 1}, \tilde{\mathbf{b}}_\ell)$, where $\tilde{\mathbf{b}}_\ell$ contains all but the first element of \mathbf{b}_ℓ . Similarly, write $\mathbf{c}_\ell = (c_{\ell 1}, \tilde{\mathbf{c}}_\ell)$.

Lemma 3. Let $\mathbf{h}^\natural = \mathbf{e}_1$ and $\mathbf{m}^\natural = \mathbf{e}_1$. The unique solution to (3) is $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ if for all $(\widetilde{\delta \mathbf{m}}, \widetilde{\delta \mathbf{h}}) \in \mathbb{R}^{N-1} \times \mathbb{R}^{K-1}$ there exists an $\ell \in [L]$ such that $b_{\ell 1} \neq 0$, $c_{\ell 1} \neq 0$, and

$$\left\langle \frac{\tilde{\mathbf{c}}_\ell}{c_{\ell 1}}, \widetilde{\delta \mathbf{m}} \right\rangle + \left\langle \frac{\tilde{\mathbf{b}}_\ell}{b_{\ell 1}}, \widetilde{\delta \mathbf{h}} \right\rangle \leq 0. \quad (5)$$

Proof of Lemma 3. Because the feasible set of (3) is closed and convex, and because a closed convex set has a unique point closest to the origin, (3) has a unique minimizer.

Consider a feasible point $(\mathbf{h}^\natural + \delta \mathbf{h}, \mathbf{m}^\natural + \delta \mathbf{m})$. To prove that $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ is a minimizer of (3), it suffices to show

$$\langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \mathbf{h}^\natural \mathbf{m}^{\natural\top} \rangle \langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \delta \mathbf{h} \mathbf{m}^{\natural\top} + \mathbf{h}^\natural \delta \mathbf{m}^\top \rangle \geq 0 \quad \forall \ell \Rightarrow \langle \mathbf{m}^\natural, \delta \mathbf{m} \rangle + \langle \mathbf{h}^\natural, \delta \mathbf{h} \rangle \geq 0.$$

Plugging in $\mathbf{h}^\natural = \mathbf{e}_1$ and $\mathbf{m}^\natural = \mathbf{e}_1$, it suffices to show

$$b_{\ell 1} c_{\ell 1} [b_{\ell 1} c_{\ell 1} (\delta \mathbf{m}_1 + \delta \mathbf{h}_1) + b_{\ell 1} \tilde{\mathbf{c}}_\ell^\top \widetilde{\delta \mathbf{m}} + c_{\ell 1} \tilde{\mathbf{b}}_\ell^\top \widetilde{\delta \mathbf{h}}] \geq 0 \quad \forall \ell \Rightarrow \delta \mathbf{m}_1 + \delta \mathbf{h}_1 \geq 0$$

Dividing by $b_{\ell 1}^2 c_{\ell 1}^2$, it suffices to show

$$\begin{aligned} \delta m_1 + \delta h_1 + \left\langle \frac{\tilde{\mathbf{c}}_\ell}{c_{\ell 1}}, \widetilde{\delta \mathbf{m}} \right\rangle + \left\langle \frac{\tilde{\mathbf{b}}_\ell}{b_{\ell 1}}, \widetilde{\delta \mathbf{h}} \right\rangle &\geq 0 \quad \forall \ell \text{ s.t. } b_{\ell 1} \neq 0 \text{ and } c_{\ell 1} \neq 0 \\ \Rightarrow \delta m_1 + \delta h_1 &\geq 0. \end{aligned}$$

To prove this, it suffices to prove

$$\forall (\widetilde{\delta \mathbf{h}}, \widetilde{\delta \mathbf{m}}) \in \mathbb{R}^{N-1} \times \mathbb{R}^{K-1}, \exists \ell \text{ s.t. } b_{\ell 1} \neq 0, c_{\ell 1} \neq 0, \text{ and } \left\langle \frac{\tilde{\mathbf{c}}_\ell}{c_{\ell 1}}, \widetilde{\delta \mathbf{m}} \right\rangle + \left\langle \frac{\tilde{\mathbf{b}}_\ell}{b_{\ell 1}}, \widetilde{\delta \mathbf{h}} \right\rangle \leq 0. \quad \square$$

For a given vector \mathbf{a} , we will call $\{\boldsymbol{\delta} \in \mathbb{S}^{n-1} : \langle \mathbf{a}, \boldsymbol{\delta} \rangle \geq 0\}$ the hemisphere centered at \mathbf{a} . We now provide a lower bound to the probability of covering the unit sphere by hemispheres centered at m random directions under a nonuniform probability distribution that is symmetric to negation. This lemma is an immediate generalization of Lemma 2 in [10], with a nearly identical proof.

Lemma 4. *Choose m independent random vectors $\{\mathbf{a}_i\}_{i=1}^m$ in \mathbb{S}^{n-1} from a (possibly nonuniform) distribution that is symmetric with respect to negation, and is such that all subsets of size n are linearly independent with probability 1. Then, the hemispheres centered at $\{\mathbf{a}_i\}_{i=1}^m$ cover the whole sphere with probability*

$$1 - \frac{1}{2^{m-1}} \sum_{k=0}^{n-1} \binom{m-1}{k}.$$

This value is the probability of flipping at least n heads among $m-1$ tosses.

Proof of Lemma 4. Classical arguments in sphere covering [30] show² the following: If m hyperplanes containing the origin are such that the normal vectors to any subset of n hyperplanes are linearly independent, then the complement of the union of these hyperplanes is partitioned into

$$r(n, m) = 2 \sum_{k=0}^{n-1} \binom{m-1}{k}$$

connected regions. In each of these regions, every point lies on the same side of each hyperplane. Alternatively put, each region corresponds to a unique assignment of a side of each hyperplane. For a fixed set of m hyperplanes, if the half space on either side of each hyperplane is selected by independent tosses of a fair coin, then with probability given in the lemma statement, there will be no nontrivial intersection of all these half spaces.

By the assumption that the distribution of \mathbf{a}_i is symmetric with respect to negation, we have that for any $\mathbf{z} \in \mathbb{S}^{n-1}$, the conditional distribution of \mathbf{a}_i given $\mathbf{a}_i \in \{\pm \mathbf{z}\}$ is uniform over the two elements $\pm \mathbf{z}$. By independence, for any fixed $\{\mathbf{z}_i\}_{i=1}^m \in (\mathbb{S}^{n-1})^m$, the distribution of $\{\mathbf{a}_i\}$ conditioned on the event $\{\mathbf{a}_i = \pm \mathbf{z}_i\}$ is uniform over the 2^m possibilities. Thus, conditioned on this event, the probability that the sphere is covered is that of the lemma statement. Integrating over all possible $\{\mathbf{z}_i\}$, the lemma follows. \square

Our last technical lemma provides an explicit lower bound to the probability of the sphere covering given in Lemma 3.

² This article credits [22] for the proof argument.

Lemma 5. *With probability at least $1 - \exp\left(-\frac{[(L-1)-2(N+K-2)]^2}{2(L-1)}\right)$, we have that*

$$\forall (\tilde{\mathbf{h}}, \tilde{\mathbf{m}}) \in \mathbb{R}^{N-1} \times \mathbb{R}^{K-1}, \exists \ell \text{ such that } \left\langle \frac{\tilde{\mathbf{c}}_\ell}{c_{\ell 1}}, \tilde{\mathbf{m}} \right\rangle + \left\langle \frac{\tilde{\mathbf{b}}_\ell}{b_{\ell 1}}, \tilde{\mathbf{h}} \right\rangle \leq 0. \quad (6)$$

Proof of Lemma 5. To show (6), we must show that the L hemispheres (of the unit sphere in $\mathbb{R}^{N-1} \times \mathbb{R}^{K-1}$) centered at $(-\frac{\tilde{\mathbf{c}}_\ell}{c_{\ell 1}}, -\frac{\tilde{\mathbf{b}}_\ell}{b_{\ell 1}})$ cover the entire sphere. As the distribution of $(\frac{\tilde{\mathbf{c}}_\ell}{c_{\ell 1}}, \frac{\tilde{\mathbf{b}}_\ell}{b_{\ell 1}})$ is invariant to negation, and as any n samples from this distribution are linearly independent with probability 1, Lemma 4 gives that the probability that (6) holds is at least the probability of flipping at least $N + K - 2$ heads among $L - 1$ tosses of a fair coin.

We now bound the probability of getting at least n heads among m fair coin tosses. Let X be the number of heads in m tosses. By Hoeffding's inequality for Bernoulli random variables [29], for any $t \geq 0$,

$$\mathbb{P}\left(X - \frac{m}{2} > -mt\right) \geq 1 - e^{-2mt^2}.$$

By selecting $t = \frac{1}{2} - \frac{n}{m}$, we get that when $n \leq m/2$,

$$\mathbb{P}(\text{at least } n \text{ heads among } m \text{ tosses}) \geq 1 - e^{-2m(\frac{1}{2} - \frac{n}{m})^2} = 1 - e^{-\frac{(m-2n)^2}{2m}}.$$

The lemma follows by plugging in $m = L - 1$ and $n = N + K - 2$ into the above probability estimate. \square

Now, we may prove the theorem.

Proof of Theorem 1. Without loss of generality, let $\|\mathbf{h}^\natural\|_2 = \|\mathbf{m}^\natural\|_2$. This is possible because for any \mathbf{b}_ℓ and \mathbf{c}_ℓ , we have that $\left(\mathbf{h}^\natural \sqrt{\frac{\|\mathbf{m}^\natural\|_2}{\|\mathbf{h}^\natural\|_2}}, \mathbf{m}^\natural \sqrt{\frac{\|\mathbf{h}^\natural\|_2}{\|\mathbf{m}^\natural\|_2}}\right)$ and $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ give equal values of $y_\ell = \langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \mathbf{h}^\natural \mathbf{m}^{\natural\top} \rangle$.

Further, without loss of generality, let $\|\mathbf{h}^\natural\|_2 = \|\mathbf{m}^\natural\|_2 = 1$. This is possible because the scaling

$$\hat{\mathbf{h}} = \frac{\mathbf{h}^\natural}{\|\mathbf{h}^\natural\|_2}, \quad \hat{\mathbf{m}} = \frac{\mathbf{m}^\natural}{\|\mathbf{m}^\natural\|_2}, \quad \hat{\mathbf{h}}^\natural = \frac{\mathbf{h}^\natural}{\|\mathbf{h}^\natural\|_2}, \quad \hat{\mathbf{m}}^\natural = \frac{\mathbf{m}^\natural}{\|\mathbf{m}^\natural\|_2},$$

turns (BH) into

$$\begin{aligned} & \underset{\mathbf{h} \in \mathbb{R}^K, \mathbf{m} \in \mathbb{R}^N}{\text{minimize}} \quad \|\mathbf{h}^\natural\|_2^2 \|\hat{\mathbf{h}}\|_2^2 + \|\mathbf{m}^\natural\|_2^2 \|\hat{\mathbf{m}}\|_2^2 \\ & \text{subject to } \langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \hat{\mathbf{h}}^\natural \hat{\mathbf{m}}^{\natural\top} \rangle \langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \hat{\mathbf{h}} \hat{\mathbf{m}}^\top \rangle \geq \langle \mathbf{b}_\ell \mathbf{c}_\ell^\top, \hat{\mathbf{h}}^\natural \hat{\mathbf{m}}^{\natural\top} \rangle^2 \\ & \quad s_\ell \cdot \mathbf{b}_\ell^\top \hat{\mathbf{h}} \geq 0, \quad \ell = 1, \dots, L. \end{aligned} \quad (7)$$

Further, without loss of generality we may take $\mathbf{h}^\natural = \mathbf{e}_1$ and $\mathbf{m}^\natural = \mathbf{e}_1$. To see this is possible, let $\mathbf{R}_{\mathbf{h}^\natural}$ and $\mathbf{R}_{\mathbf{m}^\natural}$ be rotation matrices that map \mathbf{h}^\natural and \mathbf{m}^\natural to \mathbf{e}_1 , respectively. Letting $\bar{\mathbf{h}} = \mathbf{R}_{\mathbf{h}^\natural} \mathbf{h}^\natural$, $\bar{\mathbf{m}} = \mathbf{R}_{\mathbf{m}^\natural} \mathbf{m}^\natural$, and $\bar{s}_\ell = \text{sign}(\mathbf{b}_\ell^\top \mathbf{R}_{\mathbf{h}^\natural}^\top \mathbf{e}_1)$, problem (BH) can be written

$$\underset{\bar{\mathbf{h}} \in \mathbb{R}^K, \bar{\mathbf{m}} \in \mathbb{R}^N}{\text{minimize}} \quad \|\mathbf{R}_{\mathbf{h}^\natural} \bar{\mathbf{h}}\|_2^2 + \|\mathbf{R}_{\mathbf{m}^\natural} \bar{\mathbf{m}}\|_2^2 \quad (8)$$

$$\begin{aligned} & \text{subject to } \langle \mathbf{R}_{\mathbf{h}^\natural} \mathbf{b}_\ell \mathbf{c}_\ell^\top \mathbf{R}_{\mathbf{m}^\natural}^\top, \mathbf{e}_1 \mathbf{e}_1^\top \rangle \langle \mathbf{R}_{\mathbf{h}^\natural} \mathbf{b}_\ell \mathbf{c}_\ell^\top \mathbf{R}_{\mathbf{m}^\natural}^\top, \bar{\mathbf{h}} \bar{\mathbf{m}}^\top \rangle \geq \langle \mathbf{R}_{\mathbf{h}^\natural} \mathbf{b}_\ell \mathbf{c}_\ell^\top \mathbf{R}_{\mathbf{m}^\natural}^\top, \mathbf{e}_1 \mathbf{e}_1^\top \rangle^2 \\ & \quad \bar{s}_\ell \cdot \mathbf{b}_\ell^\top \mathbf{R}_{\mathbf{h}^\natural}^\top \bar{\mathbf{h}} \geq 0, \quad \ell = 1, \dots, L. \end{aligned}$$

As ℓ_2 norms are invariant to rotation and as $\mathbf{R}_{\mathbf{h}^\natural} \mathbf{b}_\ell$ and $\mathbf{R}_{\mathbf{m}^\natural} \mathbf{c}_\ell$ have independent $\mathcal{N}(0, 1)$ entries, we may take $(\mathbf{h}^\natural, \mathbf{m}^\natural) = (\mathbf{e}_1, \mathbf{e}_1)$.

Let E be the event that (6) holds. By Lemma 5,

$$\mathbb{P}(E) \geq 1 - \exp\left(-\frac{[(L-1) - 2(N+K-2)]^2}{2(L-1)}\right).$$

By Lemma 3, on E , $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ is the unique solution to (3). By Lemma 2, on E , $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ is the unique solution to (BH). \square

2.2. Proof of Theorem 2

We will now prove that BranchHull is robust to small dense noise which does not alter the sign of the measurements. The sign of the measurements remains unchanged when $\xi_\ell \geq -1$ for all $\ell \in [L]$. We first only consider measurements y_ℓ with noise ξ_ℓ that satisfy

$$\xi_\ell \in [-1, 0] \tag{9}$$

for all $\ell \in [L]$. If all the measurements satisfy condition (9), we say the noise is “one-sided”. Note that the noise is one-sided if the convex hull of the branch of the hyperbola corresponding to the noisy measurement contain the hyperbola corresponding to the noiseless measurement, for all measurements. We first establish a recovery result for measurements with one-sided noise and show that for measurements that contain $\xi_\ell > 0$, the problem can be transformed to a related scaled problem whose corresponding measurements contain one-sided noise.

For the remainder of the paper, let $\hat{y}_\ell = \mathbf{b}_\ell^\top \mathbf{h}^\natural \mathbf{c}_\ell^\top \mathbf{m}^\natural$. Lemma 6 shows that if the measurements contain one-sided noise, the recovery error using BranchHull program (BH) is bounded by $\|\xi\|_\infty$.

Lemma 6. *Let $\mathbf{h}^\natural = \mathbf{e}_1$ and $\mathbf{m}^\natural = \mathbf{e}_1$. Let $\mathbf{B} \in \mathbb{R}^{L \times K}$, $\mathbf{C} \in \mathbb{R}^{L \times N}$ and y_ℓ satisfy (1) such that the noise is one-sided as in (9). Let $\epsilon = \|\xi\|_\infty$. The minimizer $(\mathbf{h}^*, \mathbf{m}^*)$ of the BranchHull program (BH) is unique and satisfies*

$$\|\mathbf{h}^* - \mathbf{h}^\natural\|_2^2 + \|\mathbf{m}^* - \mathbf{m}^\natural\|_2^2 \leq 4(1 - \sqrt{1 - \epsilon})$$

if for all $(\widetilde{\delta\mathbf{h}}, \widetilde{\delta\mathbf{m}}) \in \mathbb{R}^{K-1} \times \mathbb{R}^{N-1}$, there exists $\ell, k \in [L]$ such that

$$\text{sign}(b_{\ell 1}) \tilde{\mathbf{b}}_\ell^\top \widetilde{\delta\mathbf{h}} \leq 0 \text{ and } \text{sign}(c_{\ell 1}) \tilde{\mathbf{c}}_\ell^\top \widetilde{\delta\mathbf{m}} \leq 0, \tag{10}$$

$$\text{sign}(b_{k 1}) \tilde{\mathbf{b}}_k^\top \widetilde{\delta\mathbf{h}} \geq 0 \text{ and } \text{sign}(c_{k 1}) \tilde{\mathbf{c}}_k^\top \widetilde{\delta\mathbf{m}} \leq 0. \tag{11}$$

Proof. First note that the minimizer of BranchHull program (BH) is unique because the feasible set is closed and convex and a closed convex set has a unique point closest to the origin. We now prove the remainder of Lemma 6 by showing that any feasible perturbation from the candidate minimizer increases the objective value of the BranchHull program (BH).

Assume the minimizer of (BH) is $(\mathbf{h}^\natural + \delta\mathbf{h}, \mathbf{m}^\natural + \delta\mathbf{m})$. Note that $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ is feasible in (BH) because the noise is one-sided. Comparing the objective values at $(\mathbf{h}^\natural + \delta\mathbf{h}, \mathbf{m}^\natural + \delta\mathbf{m})$ and $(\mathbf{h}^\natural, \mathbf{m}^\natural)$, we get

$$\|\delta\mathbf{h}\|_2^2 + \|\delta\mathbf{m}\|_2^2 \leq -2 \left(\mathbf{h}^{\natural\top} \delta\mathbf{h} + \mathbf{m}^{\natural\top} \delta\mathbf{m} \right) = -2(\delta h_1 + \delta m_1). \tag{12}$$

We now use the second feasibility condition $s_\ell \mathbf{b}_\ell^\top \mathbf{h} \geq 0$ to show $\delta h_1 \geq -1$. Since $(\mathbf{h}^\natural + \delta \mathbf{h}, \mathbf{m}^\natural + \delta \mathbf{m})$ is feasible, the following holds for all $\ell \in [L]$.

$$\begin{aligned} s_\ell \mathbf{b}_\ell^\top (\mathbf{h}^\natural + \delta \mathbf{h}) &\geq 0 \\ \Rightarrow \text{sign}(b_{\ell 1}) (b_{\ell 1} + b_{\ell 1} \delta h_1 + \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h}) &\geq 0 \end{aligned} \quad (13)$$

$$\begin{aligned} \Rightarrow |b_{\ell 1}| \delta h_1 &\geq -|b_{\ell 1}| - \text{sign}(b_{\ell 1}) \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} \\ \Rightarrow \delta h_1 &\geq -1 - \frac{\text{sign}(b_{\ell 1}) \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h}}{|b_{\ell 1}|} \\ \Rightarrow \delta h_1 &\geq -1, \end{aligned} \quad (14)$$

where the first implication holds because $s_\ell = \text{sign}(\mathbf{b}_\ell^\top \mathbf{h}^\natural)$ and $\mathbf{h}^\natural = \mathbf{e}_1$ and the last implication holds because, by assumption (10), there exists a $\ell \in [L]$ such that $\text{sign}(b_{\ell 1}) \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} \leq 0$.

We now use the first feasibility condition on $(\mathbf{h}^\natural + \delta \mathbf{h}, \mathbf{m}^\natural + \delta \mathbf{m})$ to show that $\delta h_1 + \delta m_1$ is bounded from below. From the first feasibility condition, for all $\ell \in [L]$ we have

$$\begin{aligned} \text{sign}(y_\ell) \mathbf{b}_\ell^\top (\mathbf{h}^\natural + \delta \mathbf{h}) \mathbf{c}_\ell^\top (\mathbf{m}^\natural + \delta \mathbf{m}) &\geq |y_\ell| \\ \Rightarrow \text{sign}(y_\ell) (b_{\ell 1} + \mathbf{b}_\ell^\top \delta \mathbf{h}) (c_{\ell 1} + \mathbf{c}_\ell^\top \delta \mathbf{m}) &\geq |y_\ell| \\ \Rightarrow \text{sign}(y_\ell) (b_{\ell 1} c_{\ell 1} + b_{\ell 1} \mathbf{c}_\ell^\top \delta \mathbf{m} + \mathbf{b}_\ell^\top \delta \mathbf{h} c_{\ell 1} + \mathbf{b}_\ell^\top \delta \mathbf{h} \mathbf{c}_\ell^\top \delta \mathbf{m}) &\geq |y_\ell| \\ \Rightarrow \text{sign}(y_\ell) b_{\ell 1} c_{\ell 1} (\delta h_1 + \delta m_1) + \text{sign}(y_\ell) (\tilde{\mathbf{b}}_\ell^\top \delta \mathbf{m} + \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} c_{\ell 1} + \mathbf{b}_\ell^\top \delta \mathbf{h} \mathbf{c}_\ell^\top \delta \mathbf{m}) &\geq |y_\ell| - \text{sign}(y_\ell) b_{\ell 1} c_{\ell 1} \\ \Rightarrow |\hat{y}_\ell| (\delta h_1 + \delta m_1) + \underbrace{\text{sign}(y_\ell) (\tilde{\mathbf{b}}_\ell^\top \delta \mathbf{m} + \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} c_{\ell 1} + \mathbf{b}_\ell^\top \delta \mathbf{h} \mathbf{c}_\ell^\top \delta \mathbf{m})}_{I} &\geq |\hat{y}_\ell| \xi_\ell, \end{aligned} \quad (15)$$

where the first implication holds because $\mathbf{h}^\natural = \mathbf{e}_1$ and $\mathbf{m}^\natural = \mathbf{e}_1$ and the last implication holds because $\text{sign}(y_\ell) = \text{sign}(\hat{y}_\ell)$ and $|y_\ell| = \text{sign}(y_\ell) b_{\ell 1} c_{\ell 1} (1 + \xi_\ell)$.

We now show that term I is less than $|\hat{y}_\ell| \delta h_1 \delta m_1$ for some $\ell \in [L]$. Consider

$$\begin{aligned} I &= \text{sign}(y_\ell) (\tilde{\mathbf{b}}_\ell^\top \delta \mathbf{m} + \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} c_{\ell 1} + \mathbf{b}_\ell^\top \delta \mathbf{h} \mathbf{c}_\ell^\top \delta \mathbf{m}) \\ &= \text{sign}(y_\ell) (\tilde{\mathbf{b}}_\ell^\top \delta \mathbf{m} + \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} c_{\ell 1} + b_{\ell 1} c_{\ell 1} \delta h_1 \delta m_1 + b_{\ell 1} \delta h_1 \tilde{\mathbf{c}}_\ell^\top \delta \mathbf{m} + \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} c_{\ell 1} \delta m_1 + \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} \tilde{\mathbf{c}}_\ell^\top \delta \mathbf{m}) \\ &= |\hat{y}_\ell| \delta h_1 \delta m_1 + \text{sign}(\hat{y}_\ell) (1 + \delta m_1) \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} c_{\ell 1} + \text{sign}(\hat{y}_\ell) \tilde{\mathbf{c}}_\ell^\top \delta \mathbf{m} (b_{\ell 1} + b_{\ell 1} \delta h_1 + \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h}) \\ &= |\hat{y}_\ell| \delta h_1 \delta m_1 + \underbrace{(1 + \delta m_1) |c_{\ell 1}| \text{sign}(b_{\ell 1}) \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} + \text{sign}(c_{\ell 1}) \tilde{\mathbf{c}}_\ell^\top \delta \mathbf{m} |b_{\ell 1} + \mathbf{b}_\ell^\top \delta \mathbf{h}|}_{II}, \end{aligned}$$

where the third equality holds because $\text{sign}(y_\ell) = \text{sign}(\hat{y}_\ell) = \text{sign}(b_{\ell 1} c_{\ell 1})$ and the fourth equality holds thanks to (13). Note that because of assumptions (10) and (11), there exists a $\ell \in [L]$ such that $II \leq 0$. This is because if $(1 + \delta m_1) \geq 0$, then we have $II \leq 0$ for ℓ that satisfy (10). Similarly, if $(1 + \delta m_1) < 0$, then we have $II \leq 0$ for ℓ that satisfy (11). Thus, there exists an $\ell \in [L]$ such that

$$I = \text{sign}(y_\ell) (\tilde{\mathbf{b}}_\ell^\top \delta \mathbf{m} + \tilde{\mathbf{b}}_\ell^\top \delta \mathbf{h} c_{\ell 1} + \mathbf{b}_\ell^\top \delta \mathbf{h} \mathbf{c}_\ell^\top \delta \mathbf{m}) \leq |\hat{y}_\ell| \delta h_1 \delta m_1. \quad (16)$$

Combining (15) and (16), we get there exists a ℓ such that

$$\delta h_1 + \delta m_1 + \delta h_1 \delta m_1 \geq \frac{|\hat{y}_\ell| \xi_\ell}{|\hat{y}_\ell|} \geq \xi_\ell \geq -\epsilon. \quad (17)$$

The last inequality holds because $\epsilon = \|\xi\|_\infty$. Lastly, $\delta h_1 + \delta m_1 \geq -2(1 - \sqrt{1 - \epsilon})$ because for all $\epsilon \in [0, 1]$,

$$\begin{aligned} & \{(\delta h_1, \delta m_1) \in \mathbb{R}^2 \mid \delta h_1 + \delta m_1 \geq -2(1 - \sqrt{1 - \epsilon})\} \\ & \supset \{(\delta h_1, \delta m_1) \in \mathbb{R}^2 \mid \delta h_1 + \delta m_1 + \delta h_1 \delta m_1 \geq -\epsilon, \delta h_1 \geq -1\}. \end{aligned} \quad (18)$$

Thus, combining (12) with $\delta h_1 + \delta m_1 \geq -2(1 - \sqrt{1 - \epsilon})$, we get the desired result $\|\delta \mathbf{h}\|_2^2 + \|\delta \mathbf{m}\|_2^2 \leq 4(1 - \sqrt{1 - \epsilon})$. \square

The next lemma shows that if $\mathbf{B} \in \mathbb{R}^{L \times K}$ and $\mathbf{C} \in \mathbb{R}^{L \times N}$ contain i.i.d. $\mathcal{N}(0, 1)$ entries, then (10) and (11) holds with high probability if $L = \Omega(K + N)$.

Lemma 7. *Let $\mathbf{B} \in \mathbb{R}^{L \times K}$ and $\mathbf{C} \in \mathbb{R}^{L \times N}$ contain i.i.d. $\mathcal{N}(0, 1)$ entries. If $L \geq C(K + N)$ then*

$$\min_{\mathbf{x} \in \mathbb{S}^{K-1}, \mathbf{y} \in \mathbb{S}^{N-1}} \sum_{\ell=1}^L \mathbb{1}_{\mathbf{b}_\ell^\top \mathbf{x} \leq 0} \mathbb{1}_{\mathbf{c}_\ell^\top \mathbf{y} \leq 0} \geq 0.2L$$

with probability at least $1 - e^{-cL}$. Here, C and c are absolute constants.

Proof. Let $f(\mathbf{x}, \mathbf{y}) = \sum_{\ell=1}^L \mathbb{1}_{\mathbf{b}_\ell^\top \mathbf{x} \leq 0} \mathbb{1}_{\mathbf{c}_\ell^\top \mathbf{y} \leq 0}$. We will consider a continuous relaxation of $f(\mathbf{x}, \mathbf{y})$. Let

$$w(z) = \begin{cases} 1 & z < -0.1 \\ -\frac{z}{0.1} & -0.1 \leq z \leq 0 \\ 0 & z > 0 \end{cases}$$

and $g(\mathbf{x}, \mathbf{y}) = \sum_{\ell=1}^L w(\mathbf{b}_\ell^\top \mathbf{x}) w(\mathbf{c}_\ell^\top \mathbf{y})$. Note that $f(\mathbf{x}, \mathbf{y}) \geq g(\mathbf{x}, \mathbf{y})$ for all (\mathbf{x}, \mathbf{y}) . So, it is sufficient to show that with probability at least $1 - e^{-cL}$,

$$\min_{\mathbf{x} \in \mathbb{S}^{K-1}, \mathbf{y} \in \mathbb{S}^{N-1}} \sum_{\ell=1}^L w(\mathbf{b}_\ell^\top \mathbf{x}) w(\mathbf{c}_\ell^\top \mathbf{y}) \geq 0.2L, \quad (19)$$

if $L \geq C(K + N)$.

Let $\beta_\ell(\mathbf{x}, \mathbf{y}) = w(\mathbf{b}_\ell^\top \mathbf{x}) w(\mathbf{c}_\ell^\top \mathbf{y})$. We first compute $\mathbb{E}[\beta_\ell(\mathbf{x}, \mathbf{y})]$ for a fixed $\mathbf{x} \in \mathbb{S}^{K-1}$ and $\mathbf{y} \in \mathbb{S}^{N-1}$. Without loss of generality, let $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{y} = \mathbf{e}_1$.

$$\begin{aligned} \mathbb{E}[\beta_\ell(\mathbf{x}, \mathbf{y})] &= \mathbb{E}[w(b_{\ell 1}) w(c_{\ell 1})] \\ &= (\mathbb{E}[w(b_{\ell 1})])^2 \\ &= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-\frac{s^2}{2}} ds + \frac{1}{\sqrt{2\pi}} \int_{-1}^0 \left(-\frac{s}{0.1}\right) e^{-\frac{s^2}{2}} ds \right)^2 \\ &\geq 0.23, \end{aligned}$$

where the second inequality follows by independence of $b_{\ell 1}$ and $c_{\ell 1}$. So, for a fixed $(\mathbf{x}, \mathbf{y}) \in \mathbb{S}^{K-1} \times \mathbb{S}^{N-1}$, we have $\mathbb{E}[g(\mathbf{x}, \mathbf{y})] \geq 0.23L$.

We will now show that for a fixed $(\mathbf{x}, \mathbf{y}) \in \mathbb{S}^{K-1} \times \mathbb{S}^{N-1}$, $g(\mathbf{x}, \mathbf{y}) \geq 0.22L$ with high probability. Fix $(\mathbf{x}, \mathbf{y}) \in \mathbb{S}^{K-1} \times \mathbb{S}^{N-1}$. Since $g(\mathbf{x}, \mathbf{y})$ is bounded, $g(\mathbf{x}, \mathbf{y})$ is sub-gaussian. Let α be the

sub-gaussian norm of β_ℓ after centering. Thus, by Hoeffding-type inequality (see Proposition 5.10 in Vershynin [27]), $\mathbb{P}\{|g(\mathbf{x}, \mathbf{y}) - \mathbb{E}[g(\mathbf{x}, \mathbf{y})]| \geq t\} \leq e \cdot e^{-\frac{ct^2}{\alpha^2 L}}$, where $c > 0$ is a absolute constant. So, $\mathbb{P}\{g(\mathbf{x}, \mathbf{y}) \leq 0.23L - \delta L\} \leq e \cdot e^{-\frac{c\delta^2 L}{\alpha^2}}$. Pick $\delta = 0.01$, then for any fixed $(\mathbf{x}, \mathbf{y}) \in \mathbb{S}^{K-1} \times \mathbb{S}^{N-1}$, we have

$$\mathbb{P}\{g(\mathbf{x}, \mathbf{y}) \leq 0.22L\} \leq e \cdot e^{-cL}$$

for some $c > 0$.

We will now show that for all (\mathbf{x}, \mathbf{y}) in an ϵ -net, $g(\mathbf{x}, \mathbf{y}) \geq 0.22L$ with high probability. Let \mathcal{N}_ϵ be an ϵ -net of $\mathbb{S}^{K-1} \times \mathbb{S}^{N-1}$ such that $|\mathcal{N}_\epsilon| \leq (1 + \frac{2\sqrt{2}}{\epsilon})^{K+N}$. By lemma 5.2 in [27], such an ϵ -net exists. So

$$\mathbb{P}\left\{\min_{(\mathbf{x}, \mathbf{y}) \in \mathcal{N}_\epsilon} g(\mathbf{x}) \geq 0.22L\right\} \geq 1 - e \cdot e^{-cL + (N+K) \log(1 + \frac{2\sqrt{2}}{\epsilon})}. \quad (20)$$

If $L \geq \frac{2}{c}(1 + \log(1 + \frac{2\sqrt{2}}{\epsilon}))(K + N)$ then

$$\mathbb{P}\left\{\min_{(\mathbf{x}, \mathbf{y}) \in \mathcal{N}_\epsilon} g(\mathbf{x}, \mathbf{y}) \geq 0.22L\right\} \geq 1 - e \cdot e^{-\frac{cL}{2}} \quad (21)$$

Lastly, we will show that for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{S}^{K-1} \times \mathbb{S}^{N-1}$, $g(\mathbf{x}, \mathbf{y}) \geq 0.2L$ with high probability. We first show that $g(\mathbf{x}, \mathbf{y})$ is $30\sqrt{2}L$ -Lipschitz with high probability. This holds because if $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in \mathbb{R}^{K-1} \times \mathbb{R}^{N-1}$ then

$$|g(\mathbf{x}_1, \mathbf{y}_1) - g(\mathbf{x}_2, \mathbf{y}_2)| \leq \sum_{\ell=1}^L |w(\mathbf{b}_\ell^\top \mathbf{x}_1)w(\mathbf{c}_\ell^\top \mathbf{y}_1) - w(\mathbf{b}_\ell^\top \mathbf{x}_2)w(\mathbf{c}_\ell^\top \mathbf{y}_2)| \quad (22)$$

$$= \sum_{\ell=1}^L |(w(\mathbf{b}_\ell^\top \mathbf{x}_1) - w(\mathbf{b}_\ell^\top \mathbf{x}_2))w(\mathbf{c}_\ell^\top \mathbf{y}_1) + (w(\mathbf{c}_\ell^\top \mathbf{y}_1) - w(\mathbf{c}_\ell^\top \mathbf{y}_2))w(\mathbf{b}_\ell^\top \mathbf{x}_2)| \quad (23)$$

$$\leq \sum_{\ell=1}^L |(w(\mathbf{b}_\ell^\top \mathbf{x}_1) - w(\mathbf{b}_\ell^\top \mathbf{x}_2))w(\mathbf{c}_\ell^\top \mathbf{y}_1)| + |(w(\mathbf{c}_\ell^\top \mathbf{y}_1) - w(\mathbf{c}_\ell^\top \mathbf{y}_2))w(\mathbf{b}_\ell^\top \mathbf{x}_2)| \quad (24)$$

$$\leq \sum_{\ell=1}^L |w(\mathbf{b}_\ell^\top \mathbf{x}_1) - w(\mathbf{b}_\ell^\top \mathbf{x}_2)| + |w(\mathbf{c}_\ell^\top \mathbf{y}_1) - w(\mathbf{c}_\ell^\top \mathbf{y}_2)| \quad (25)$$

$$\leq 10 \sum_{\ell=1}^L |\mathbf{b}_\ell^\top (\mathbf{x}_1 - \mathbf{x}_2)| + 10 \sum_{\ell=1}^L |\mathbf{c}_\ell^\top (\mathbf{y}_1 - \mathbf{y}_2)| \quad (26)$$

$$\leq 10\sqrt{L} \cdot \sqrt{\sum_{\ell=1}^L (\mathbf{b}_\ell^\top (\mathbf{x}_1 - \mathbf{x}_2))^2} + 10\sqrt{L} \cdot \sqrt{\sum_{\ell=1}^L (\mathbf{c}_\ell^\top (\mathbf{y}_1 - \mathbf{y}_2))^2} \quad (27)$$

$$= 10\sqrt{L} (\|\mathbf{B}(\mathbf{x}_1 - \mathbf{x}_2)\|_2 + \|\mathbf{C}(\mathbf{y}_1 - \mathbf{y}_2)\|_2) \quad (28)$$

$$\leq 10\sqrt{L} (\|\mathbf{B}\| \|\mathbf{x}_1 - \mathbf{x}_2\|_2 + \|\mathbf{C}\| \|\mathbf{y}_1 - \mathbf{y}_2\|_2), \quad (29)$$

where the fourth line follows because $|w(z)| \leq 1$ for all $z \in \mathbb{R}$, the fifth line follows because w is 10-Lipschitz and the sixth line follows from Cauchy-Schwarz inequality. By Corollary 5.35 in [27], there exist events E_1 and E_2 each with probability at least $1 - 2e^{-\frac{L}{2}}$, on which $\|\mathbf{B}\| \leq 3\sqrt{L}$ and $\|\mathbf{C}\| \leq 3\sqrt{L}$, respectively. So, on $E_1 \cap E_2$ we have

$$\begin{aligned} |g(\mathbf{x}_1, \mathbf{y}_1) - g(\mathbf{x}_2, \mathbf{y}_2)| &\leq 30L(\|\mathbf{x}_1 - \mathbf{x}_2\|_2 + \|\mathbf{y}_1 - \mathbf{y}_2\|_2) \\ &\leq 30\sqrt{2}L\|(\mathbf{x}_1, \mathbf{y}_1) - (\mathbf{x}_2, \mathbf{y}_2)\|_2. \end{aligned}$$

Take $\epsilon = \frac{0.01}{30\sqrt{2}}$. For any $(\mathbf{x}_1, \mathbf{y}_1) \in \mathbb{S}^{K-1} \times \mathbb{S}^{N-1}$, pick $(\mathbf{x}_2, \mathbf{y}_2) \in \mathcal{N}_\epsilon$ such that $\|(\mathbf{x}_1, \mathbf{y}_1) - (\mathbf{x}_2, \mathbf{y}_2)\|_2 \leq \epsilon$. On the event $E_1 \cap E_2$ and the event given by (20), we have that

$$g(\mathbf{x}_1, \mathbf{y}_1) \geq \min_{(\mathbf{x}_2, \mathbf{y}_2) \in \mathcal{N}_\epsilon} g(\mathbf{x}_2, \mathbf{y}_2) - 30\sqrt{2}L\|(\mathbf{x}_1, \mathbf{y}_1) - (\mathbf{x}_2, \mathbf{y}_2)\|_2 \quad (30)$$

$$\geq 0.2L. \quad (31)$$

This occurs with the probability at least $1 - e^{-cL}$, provided $L \geq \frac{2}{C}(1 + \log(1 + \frac{2\sqrt{2}}{\epsilon}))(K + N)$. \square

We now extend Lemma 6 to the case with arbitrary but non-zero \mathbf{h}^\natural and \mathbf{m}^\natural .

Lemma 8. *Fix $\mathbf{h}^\natural \in \mathbb{R}^K$ and $\mathbf{m}^\natural \in \mathbb{R}^N$ such that $\mathbf{h}^\natural \neq 0$ and $\mathbf{m}^\natural \neq 0$. Let $\mathbf{B} \in \mathbb{R}^{L \times K}$, $\mathbf{C} \in \mathbb{R}^{L \times N}$ contain i.i.d $\mathcal{N}(0, 1)$ entries and y_ℓ satisfy (1) such that the noise is one-sided as in (9). Let $\epsilon = \|\boldsymbol{\xi}\|_\infty$. The minimizer $(\mathbf{h}^*, \mathbf{m}^*)$ of the BranchHull program (BH) is unique and if $L \geq C(K + N)$ then the minimizer satisfies*

$$\left\| \mathbf{h}^* - \mathbf{h}^\natural \sqrt{\frac{\|\mathbf{m}^\natural\|_2}{\|\mathbf{h}^\natural\|_2}} \right\|_2^2 + \left\| \mathbf{m}^* - \mathbf{m}^\natural \sqrt{\frac{\|\mathbf{h}^\natural\|_2}{\|\mathbf{m}^\natural\|_2}} \right\|_2^2 \leq 4(1 - \sqrt{1 - \epsilon})\|\mathbf{h}^\natural\|_2\|\mathbf{m}^\natural\|_2$$

with probability at least $1 - e^{-cL}$. Here, C and c are absolute constants.

Proof. Without loss of generality let $\|\mathbf{h}^\natural\|_2 = \|\mathbf{m}^\natural\|_2$, which is possible because for any b_ℓ, c_ℓ and ξ_ℓ , we have that $\left(\mathbf{h}^\natural \sqrt{\frac{\|\mathbf{m}^\natural\|_2}{\|\mathbf{h}^\natural\|_2}}, \mathbf{m}^\natural \sqrt{\frac{\|\mathbf{h}^\natural\|_2}{\|\mathbf{m}^\natural\|_2}}\right)$ and $(\mathbf{h}^\natural, \mathbf{m}^\natural)$ give equal values of $y_\ell = b_\ell^\top \mathbf{h}^\natural c_\ell^\top \mathbf{m}^\natural (1 + \xi_\ell)$. Further, without loss of generality, we may take $\|\mathbf{h}^\natural\|_2 = \|\mathbf{m}^\natural\|_2 = 1$. This is possible because of a similar line of argument as (7).

Further, without loss of generality we may take $\mathbf{h}^\natural = \mathbf{e}_1$ and $\mathbf{m}^\natural = \mathbf{e}_1$. To see this is possible, let $R_{\hat{\mathbf{h}}^\natural}$ and $R_{\hat{\mathbf{m}}^\natural}$ be rotation matrices that map $\hat{\mathbf{h}}^\natural$ and $\hat{\mathbf{m}}^\natural$ to \mathbf{e}_1 , respectively. Letting $\bar{\mathbf{h}} = R_{\hat{\mathbf{h}}^\natural} \hat{\mathbf{h}}^\natural$, $\bar{\mathbf{m}} = R_{\hat{\mathbf{m}}^\natural} \hat{\mathbf{m}}^\natural$, and $\bar{s}_\ell = \text{sign}(b_\ell^\top R_{\hat{\mathbf{h}}^\natural}^\top \mathbf{e}_1)$, BranchHull can be written as

$$\begin{aligned} & \min_{\bar{\mathbf{h}} \in \mathbb{R}^K, \bar{\mathbf{m}} \in \mathbb{R}^N} \|\mathbf{h}^\natural\|_2^2 \|R_{\hat{\mathbf{h}}^\natural}^\top \bar{\mathbf{h}}\|_2^2 + \|\mathbf{m}^\natural\|_2^2 \|R_{\hat{\mathbf{m}}^\natural}^\top \bar{\mathbf{m}}\|_2^2 \\ & \text{s.t. } \text{sign}\left((R_{\hat{\mathbf{h}}^\natural} \mathbf{b}_\ell)^\top \mathbf{e}_1 (R_{\hat{\mathbf{m}}^\natural} \mathbf{c}_\ell)^\top \mathbf{e}_1\right) (R_{\hat{\mathbf{h}}^\natural} \mathbf{b}_\ell)^\top \bar{\mathbf{h}} (R_{\hat{\mathbf{m}}^\natural} \mathbf{c}_\ell)^\top \bar{\mathbf{m}} \geq \left|(R_{\hat{\mathbf{h}}^\natural} \mathbf{b}_\ell)^\top \mathbf{e}_1 (R_{\hat{\mathbf{m}}^\natural} \mathbf{c}_\ell)^\top \mathbf{e}_1 (1 + \xi_\ell)\right| \\ & \quad \bar{s}_\ell \cdot \mathbf{b}_\ell^\top R_{\hat{\mathbf{h}}^\natural}^\top \bar{\mathbf{h}} \geq 0, \quad \ell \in [L]. \end{aligned} \quad (32)$$

As ℓ_2 norms are invariant to rotation and $R_{\hat{\mathbf{h}}^\natural} \mathbf{b}_\ell$ and $R_{\hat{\mathbf{m}}^\natural} \mathbf{c}_\ell$ have independent $\mathcal{N}(0, 1)$ entries, we may take $(\mathbf{h}^\natural, \mathbf{m}^\natural) = (\mathbf{e}_1, \mathbf{e}_1)$.

By Lemma 6, the minimizer $(\bar{\mathbf{h}}^*, \bar{\mathbf{m}}^*)$ of (32) is unique and satisfies

$$\left\| \bar{\mathbf{h}}^* - \mathbf{e}_1 \right\|_2^2 + \left\| \bar{\mathbf{m}}^* - \mathbf{e}_1 \right\|_2^2 \leq 4(1 - \sqrt{1 - \epsilon}) \quad (33)$$

if for all $(\widetilde{\delta\mathbf{h}}, \widetilde{\delta\mathbf{m}}) \in \mathbb{R}^{K-1} \times \mathbb{R}^{N-1}$, there exists $\ell, k \in [L]$ such that

$$\text{sign}((R_{\hat{\mathbf{h}}^\natural} \mathbf{b}_\ell)^\top \mathbf{e}_1) (\widetilde{\delta\mathbf{h}})^\top \widetilde{\delta\mathbf{h}} \leq 0 \text{ and } \text{sign}((R_{\hat{\mathbf{m}}^\natural} \mathbf{c}_\ell)^\top \mathbf{e}_1) (\widetilde{\delta\mathbf{m}})^\top \widetilde{\delta\mathbf{m}} \leq 0 \quad (34)$$

$$\text{sign}((R_{\hat{\mathbf{h}}^\natural} \mathbf{b}_k)^\top \mathbf{e}_1) (\widetilde{\delta\mathbf{h}})^\top \widetilde{\delta\mathbf{h}} \geq 0 \text{ and } \text{sign}((R_{\hat{\mathbf{m}}^\natural} \mathbf{c}_k)^\top \mathbf{e}_1) (\widetilde{\delta\mathbf{m}})^\top \widetilde{\delta\mathbf{m}} \leq 0. \quad (35)$$

By Lemma 7, there exists $\ell, k \in [L]$ that satisfy (34) and (35), respectively, with probability at least $1 - e^{-cL}$ if $L \geq C(K + N)$. \square

We now present a proof of Theorem 2. In Theorem 2, the noise $\xi_\ell \in [-1, 1]$ which is in contrast to $\xi_\ell \in [-1, 0]$ in Lemma 8. The key idea is measurements with noise that satisfy $\xi_\ell \geq -1$ can be converted to measurements with noise in the interval $[-1, 1]$. In order to see this, let

$$\bar{s} = \max_{\ell \in [L]} \frac{y_\ell}{\hat{y}_\ell} = 1 + \max_{\ell \in [L]} \xi_\ell, \quad (36)$$

$$s = \max\{\bar{s}, 1\} \leq 1 + \|\boldsymbol{\xi}\|_\infty, \quad (37)$$

$$\eta_\ell = \frac{1}{s}(1 - s + \xi_\ell). \quad (38)$$

We then consider the measurements $y_\ell = s\hat{y}_\ell(1 + \eta_\ell)$ for $\ell \in [L]$. Because $s\hat{y}_\ell(1 + \eta_\ell) = \hat{y}_\ell(1 + \xi_\ell)$, the noisy measurements are the same, however the noise may be different.

Proof of Theorem 2. As the noise of measurements $y_\ell = \hat{y}_\ell(1 + \xi_\ell)$ may not be one-sided as in (9), we consider equivalent measurements $y_\ell = s\hat{y}_\ell(1 + \eta_\ell)$, where s and η_ℓ are as defined in (37) and (38), respectively. This turns the BranchHull program (BH) into

$$\begin{aligned} \min_{\mathbf{h} \in \mathbb{R}^K, \mathbf{m} \in \mathbb{R}^N} \|\mathbf{h}\|_2^2 + \|\mathbf{m}\|_2^2 \quad & \text{s.t.} \quad \text{sign}(y_\ell) \mathbf{b}_\ell^\top \mathbf{h} \mathbf{c}_\ell^\top \mathbf{m} \geq |s\hat{y}_\ell(1 + \eta_\ell)|, \\ & s_\ell \cdot \mathbf{b}_\ell^\top \mathbf{h} \geq 0, \quad \ell \in [L]. \end{aligned} \quad (39)$$

First, we note that for all $\ell \in [L]$,

$$\eta_\ell = \frac{1}{s}(1 + \xi_\ell - s) \quad (40)$$

$$\leq \frac{1}{s}(s - s) \quad (41)$$

$$= 0, \quad (42)$$

where the first inequality holds because $1 + \xi_\ell \leq 1 + \max_{\ell \in [L]} \xi_\ell \leq s$. Second, we have $\eta_\ell \geq -1$ for all $\ell \in [L]$, which follows directly from $\xi_\ell \geq -1$ for all ℓ . Thus, the noise $\boldsymbol{\eta}$ is one-sided and by Lemma 8, the minimizer $(\mathbf{h}^*, \mathbf{m}^*)$ of (39) is unique and if $L \geq C(K + N)$, the minimizer satisfies

$$\left\| \mathbf{h}^* - \sqrt{s} \mathbf{h}^\natural \sqrt{\frac{\|\mathbf{m}^\natural\|_2}{\|\mathbf{h}^\natural\|_2}} \right\|_2^2 + \left\| \mathbf{m}^* - \sqrt{s} \mathbf{m}^\natural \sqrt{\frac{\|\mathbf{h}^\natural\|_2}{\|\mathbf{m}^\natural\|_2}} \right\|_2^2 \leq 4(1 - \sqrt{1 - \delta})s \|\mathbf{h}^\natural\|_2 \|\mathbf{m}^\natural\|_2 \quad (43)$$

with probability at least $1 - e^{-cL}$. In (43),

$$\begin{aligned} \delta & \equiv \|\boldsymbol{\eta}\|_\infty \\ & = - \min_{\ell \in [L]} \left(\frac{1}{s} - 1 + \frac{\xi_\ell}{s} \right) \\ & = -\frac{1}{s}(1 + \min_{\ell \in [L]} \xi_\ell) + 1 \\ & \leq -\frac{1 - \epsilon}{s} + 1, \end{aligned} \quad (44)$$

where the first equality holds because $\epsilon = \|\xi\|_\infty \geq -\min_{\ell \in [L]} \xi_\ell$. Let $(\mathbf{h}^\circ, \mathbf{m}^\circ) = \left(\mathbf{h}^\natural \sqrt{\frac{\|\mathbf{m}^\natural\|_2}{\|\mathbf{h}^\natural\|_2}}, \mathbf{m}^\natural \sqrt{\frac{\|\mathbf{h}^\natural\|_2}{\|\mathbf{m}^\natural\|_2}} \right)$. We now compute

$$\begin{aligned} & \left(\|\mathbf{h}^* - \mathbf{h}^\circ\|_2^2 + \|\mathbf{m}^* - \mathbf{m}^\circ\|_2^2 \right)^{\frac{1}{2}} \\ &= \left(\|\mathbf{h}^* - \sqrt{s}\mathbf{h}^\circ + \sqrt{s}\mathbf{h}^\circ - \mathbf{h}^\circ\|_2^2 + \|\mathbf{m}^* - \sqrt{s}\mathbf{m}^\circ + \sqrt{s}\mathbf{m}^\circ - \mathbf{m}^\circ\|_2^2 \right)^{\frac{1}{2}} \end{aligned} \quad (45)$$

$$\leq \left(\|\mathbf{h}^* - \sqrt{s}\mathbf{h}^\circ\|_2^2 + \|\mathbf{m}^* - \sqrt{s}\mathbf{m}^\circ\|_2^2 \right)^{\frac{1}{2}} + \left(\|\sqrt{s}\mathbf{h}^\circ - \mathbf{h}^\circ\|_2^2 + \|\sqrt{s}\mathbf{m}^\circ - \mathbf{m}^\circ\|_2^2 \right)^{\frac{1}{2}} \quad (46)$$

$$\leq \left(4(1 - \sqrt{1 - \delta})s\|\mathbf{h}^\natural\|_2\|\mathbf{m}^\natural\|_2 \right)^{\frac{1}{2}} + (\sqrt{s} - 1)(\|\mathbf{h}^\circ\|_2^2 + \|\mathbf{m}^\circ\|_2^2)^{\frac{1}{2}} \quad (47)$$

$$\leq \left(2 \left(s \left(1 - \sqrt{\frac{1 - \epsilon}{s}} \right) \right)^{\frac{1}{2}} + \sqrt{2}(\sqrt{s} - 1) \right) \sqrt{\|\mathbf{h}^\natural\|_2\|\mathbf{m}^\natural\|_2} \quad (48)$$

$$\leq \left(2 \left(1 + \epsilon - \sqrt{1 - \epsilon^2} \right)^{\frac{1}{2}} + \sqrt{2}(\sqrt{1 + \epsilon} - 1) \right) \sqrt{\|\mathbf{h}^\natural\|_2\|\mathbf{m}^\natural\|_2} \quad (49)$$

$$\leq 4\sqrt{\epsilon} \sqrt{\|\mathbf{h}^\natural\|_2\|\mathbf{m}^\natural\|_2}, \quad (50)$$

where (45) holds because of triangle inequality, (46) holds because of (43), (47) holds because of (44) and $\|\mathbf{h}^\circ\|_2 = \|\mathbf{m}^\circ\|_2 = \sqrt{\|\mathbf{h}^\natural\|_2\|\mathbf{m}^\natural\|_2}$, (48) holds because of (37) and (49) and (50) holds because for all $\epsilon \in [0, 1]$, we have $1 + \epsilon - \sqrt{1 - \epsilon^2} \leq 2\epsilon$, $\sqrt{1 + \epsilon} - 1 \leq \frac{\epsilon}{2}$ and $\epsilon \leq \sqrt{\epsilon}$. \square

3. Numerical results

In this section, we provide two numerical studies on synthetic data. The first study numerically verifies Theorem 1 and the second study shows that the BranchHull program (BH) is robust to small dense noise. For both simulations, we used an interior point solver available in Matlab to solve the corresponding BranchHull program.

For the first simulation, consider the following measurements: fix $N \in \{10, 20, \dots, 150\}$, $L \in \{10, 70, \dots, 850\}$ and let $K = N$. Let the target signal $(\mathbf{h}^\natural, \mathbf{m}^\natural) \in \mathbb{R}^K \times \mathbb{R}^N$ be such that $\mathbf{h}^\natural = e_1$ and $\mathbf{m}^\natural = e_1$. Let $\mathbf{B} \in \mathbb{R}^{L \times K}$ and $\mathbf{C} \in \mathbb{R}^{L \times N}$ such that $B_{ij} \sim \mathcal{N}(0, 1)$ and $C_{ij} \sim \mathcal{N}(0, 1)$. Lastly, let $y_\ell = \mathbf{B}\mathbf{h}^\natural \circ \mathbf{C}\mathbf{m}^\natural$ and $s = \text{sign}(\mathbf{B}\mathbf{h}^\natural)$.

Fig. 3 shows the fraction of successful recoveries from 10 independent trials for the bilinear inverse problem (1) from data as described above. Black squares correspond to no successful recovery and white squares correspond to 100% successful recovery. Let $(\mathbf{h}^*, \mathbf{m}^*)$ be the output of (BH). For each trial, we say (BH) successfully recovers the target signal if $\|(\mathbf{h}^*, \mathbf{m}^*) - (e_1, e_1)\|_2 < 10^{-5}$. The area to the left of the line corresponds to the oversampling required for successful recovery stated in Theorem 1. The figure shows the linear relationship between number of measurements L and size of target signals $K + N$ for successful recovery.

For the noisy simulation, consider the following measurements: fix $N = K = 20$ and $L \in \{10, 20, \dots, 200\}$. Let the target signal $(\mathbf{h}^\natural, \mathbf{m}^\natural) \in \mathbb{R}^K \times \mathbb{R}^N$ be such that $\mathbf{h}^\natural \sim \mathcal{N}(0, 1)$ and $\mathbf{m}^\natural \sim \mathcal{N}(0, 1)$. Let $\mathbf{B} \in \mathbb{R}^{L \times K}$ and $\mathbf{C} \in \mathbb{R}^{L \times N}$ be such that $B_{ij} \sim \mathcal{N}(0, 1)$ and $C_{ij} \sim \mathcal{N}(0, 1)$. Fix $\alpha \in \{0, 0.1, \dots, 1\}$ and let $\xi \in \mathbb{R}^L$ such that $\xi_\ell \sim \text{Uniform}([- \alpha, \alpha])$. Lastly, let $y_\ell = \mathbf{B}\mathbf{h}^\natural \circ \mathbf{C}\mathbf{m}^\natural \circ (\mathbf{1} + \xi)$ and $s = \text{sign}(\mathbf{B}\mathbf{h}^\natural)$.

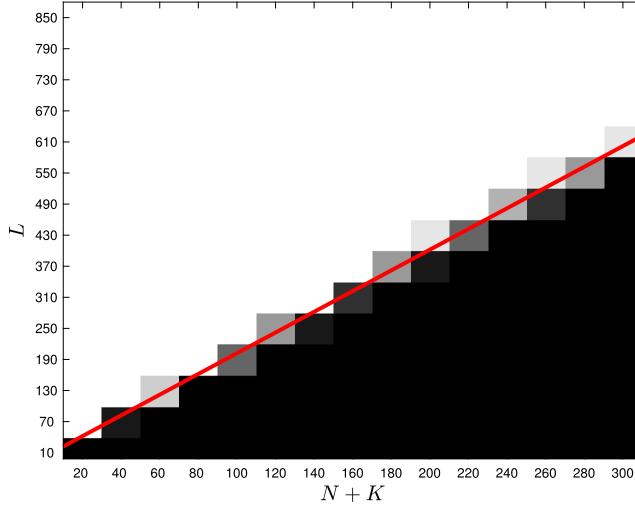


Fig. 3. The empirical recovery probability from synthetic data with total measurements L as a function of size of the target signals $K + N$. The shades of black and white represents the fraction of successful simulation. White blocks correspond to successful recovery and black blocks correspond to unsuccessful recovery. Each block corresponds to the average from 10 independent trials. The area to the left of the line satisfies $L > 2(K + N)$, which is the theoretical successful recovery bound stated in Theorem 1.

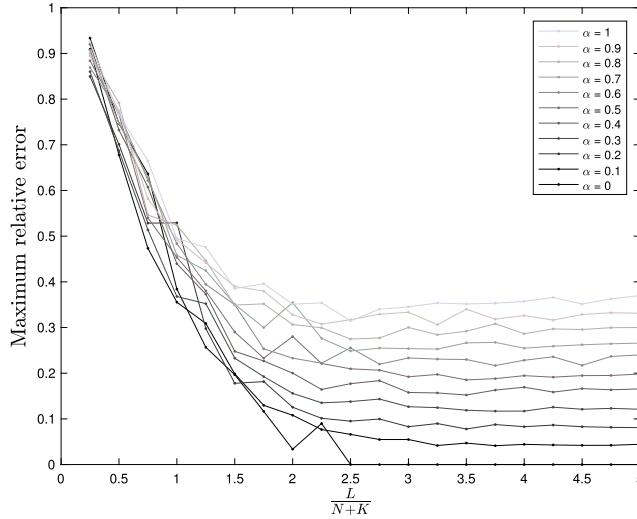


Fig. 4. The empirical recovery error from synthetic data as a function of sampling ratio $\frac{L}{K+N}$. The size of the signals, N and K , are fixed to 20. Different piecewise line corresponds to different noise level α .

Fig. 4 shows the maximum relative error from 10 independent trials for the bilinear inverse problem (1) from data as described above. Each curve corresponds to different noise level α , which controls the size of noise level ϵ defined in (2). The plot shows the effect of different levels of noise on the maximum relative error as a function of the sampling ratio $\frac{L}{K+N}$. Empirically, sampling ratio of about 2.5 is sufficient for stable estimation of the target signal. Additionally, the spacing between the piecewise lines for large sampling ratio is uniform in Fig. 4. This suggests that the relationship between recovery error and the noise level ϵ is linear, which is in contrast to Theorem 2 which shows that the recovery error depends as $\sqrt{\epsilon}$ for small noise.

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