

ALGEBRAIC STRUCTURE OF THE RANGE OF A TRIGONOMETRIC POLYNOMIAL

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Abstract

The range of a trigonometric polynomial with complex coefficients can be interpreted as the image of the unit circle under a Laurent polynomial. We show that this range is contained in a real algebraic subset of the complex plane. Although the containment may be proper, the difference between the two sets is finite, except for polynomials with a certain symmetry.

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1. Introduction

In 1976, Quine [6, Theorem 1] proved that the image of the unit circle \mathbb{T} under an algebraic polynomial p of degree n is contained in a real algebraic set, that is, a set $V = \{(x, y) \in \mathbb{R}^2 : q(x, y) = 0\}$, where q is a polynomial of degree $2n$. In general, $p(\mathbb{T})$ is a proper subset of V , but we will show that $V \setminus p(\mathbb{T})$ is finite and that $V = p(\mathbb{T})$ whenever V is connected.

Consider a trigonometric polynomial $P(t) = \sum_{k=-m}^n a_k e^{ikt}$, $t \in \mathbb{R}$, with complex coefficients a_k . It is natural to require $a_{-m} a_n \neq 0$. The range of P is precisely the image of the unit circle \mathbb{T} under the Laurent polynomial $p(z) = \sum_{k=-m}^n a_k z^k$. This motivates our investigation of $p(\mathbb{T})$ for Laurent polynomials. Our main result, Theorem 2.1, asserts that $p(\mathbb{T})$ is contained in the zero set V of a polynomial of degree $2 \max(m, n)$. This matches Quine's theorem in the case when p is an algebraic polynomial, that is, $m = 0$. The difference $V \setminus p(\mathbb{T})$ is finite when $m \neq n$, but may be infinite when $m = n$.

In Section 4, we investigate the exceptional case when $V \setminus p(\mathbb{T})$ is infinite and we relate it to the properties of the zero set of a certain harmonic rational function. The structure of zero sets of such functions is a topic of current interest with applications to gravitational lensing [1, 2].

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Finally, in Section 5, we use the algebraic nature of the polynomial images of \mathbb{T} to estimate the number of intersections of two such images, that is, the number of shared values of two trigonometric polynomials.

2. Algebraic nature of polynomial images of circles

A real algebraic subset of \mathbb{R}^2 is a set of the form $\{(x, y) \in \mathbb{R}^2 : q(x, y) = 0\}$, where $q \in \mathbb{R}[x, y]$ is a polynomial in x, y . Consider a Laurent polynomial

$$p(z) = \sum_{k=-m}^n a_k z^k \quad \text{for all } z \in \mathbb{C} \setminus \{0\}, \quad (2.1)$$

where $m \geq 0$, $n \geq 1$ and $a_{-m} a_n \neq 0$. This includes the case of algebraic polynomials ($m = 0$) because the condition $a_0 \neq 0$ can be ensured by adding a constant to p , which does not affect the algebraic nature of $p(\mathbb{T})$. Since we are interested in the image of the unit circle, which is invariant under the substitution of z^{-1} for z , it suffices to consider the case $m \leq n$.

THEOREM 2.1. *Let p be the Laurent polynomial (2.1) with $m \leq n$.*

- (a) *The image of \mathbb{T} under p is contained in the zero set V of some polynomial $h \in \mathbb{R}[x, y]$ of degree $2n$.*
- (b) *If h is expressed as a polynomial $h_{\mathbb{C}} \in \mathbb{C}[w, \bar{w}]$ via the substitution $w = x + iy$, the degree of $h_{\mathbb{C}}$ in each of the variables w and \bar{w} separately is $m + n$.*
- (c) *If $m < n$, then the set $V \setminus p(\mathbb{T})$ is finite.*
- (d) *In the case $m = n$ the set $V \setminus p(\mathbb{T})$ is finite if and only if V is bounded.*

The proof of Theorem 2.1 involves two polynomials

$$g(z) = z^m(p(z) - w) \quad \text{and} \quad g^*(z) = z^{n+m} \overline{g(1/\bar{z})} = z^n \overline{(p(1/\bar{z}) - w)} \quad (2.2)$$

which are the subject of the following lemma.

LEMMA 2.2. *The resultant $h_{\mathbb{C}} = \text{res}(g, g^*)$ of the polynomials (2.2) is a polynomial in $\mathbb{C}[w, \bar{w}]$ of degree $2n$. Moreover, $h_{\mathbb{C}}$ has degree $m + n$ in each of the variables w and \bar{w} separately. Finally, $h(x, y) := h_{\mathbb{C}}(x + iy, x - iy)$ is a polynomial of degree $2n$ in $\mathbb{R}[x, y]$.*

PROOF. Both g and g^* are polynomials of degree $m + n$ in z , except for the case $m = 0$ and $w = a_0$, which we ignore in this proof because considering a generic w is enough. By definition, the resultant of g and g^* is the determinant of the following matrix of size $2(m + n)$.

$$R = \begin{pmatrix} a_{-m} & \cdots & \cdots & a_0 - w & \cdots & a_n & 0 & 0 \\ 0 & \ddots & & & \ddots & & \ddots & 0 \\ 0 & 0 & a_{-m} & \cdots & \cdots & a_0 - w & \cdots & a_n \\ \bar{a}_n & \cdots & \bar{a}_0 - \bar{w} & \cdots & \cdots & \bar{a}_{-m} & 0 & 0 \\ 0 & \ddots & & \ddots & & & \ddots & 0 \\ 0 & 0 & \bar{a}_n & \cdots & \bar{a}_0 - \bar{w} & \cdots & \cdots & \bar{a}_{-m} \end{pmatrix}.$$

All appearances of w or \bar{w} in R are in the columns numbered $m + 1$ to $m + 2n$, which are the middle $2n$ columns of the matrix R . Therefore $h_{\mathbb{C}}$ is a polynomial of degree at most $2n$.

First, let us prove that $h_{\mathbb{C}}$ has degree $n + m$ in each variable separately. It obviously cannot be greater than $n + m$, since each of w and \bar{w} appears $n + m$ times in the matrix. The position of $a_0 - w$ in the top half of the matrix shows that the Leibniz formula for $\det R$ contains the term $\pm \bar{a}_n^m \bar{a}_{-m}^n (a_0 - w)^{n+m}$ and no other terms with the monomial w^{n+m} . Therefore the coefficient of w^{n+m} in h is $\pm \bar{a}_n^m \bar{a}_{-m}^n \neq 0$. Similarly, the coefficient of \bar{w}^{2n} in h is $\pm a_{-m}^n a_n^m \neq 0$. This proves that $h_{\mathbb{C}}$ has degree $n + m$ in w and \bar{w} separately.

When $m = n$, the preceding paragraph shows that $h_{\mathbb{C}}$ has degree $2n$ in w and \bar{w} separately, which implies that $\deg h = 2n$.

We proceed to prove that $\deg h_{\mathbb{C}} = 2n$ in the case $m < n$. Let R_1 be the matrix obtained from R by replacing all constant entries in the columns $m + 1, \dots, m + 2n$ by 0. Since the cofactor of any of the entries we replaced is a polynomial of degree less than $2n$, the difference $\det R - \det R_1$ has degree less than $2n$. Thus, it suffices to show that $\det R_1$ has degree $2n$. When deriving a formula for $\det R_1$ we may assume that $w \neq a_0$. Let us focus on the columns of R_1 numbered $m + 1, \dots, 2m$. The only nonzero entries in these columns are:

- $a_0 - w$ at $(j - m, j)$ for $m + 1 \leq j \leq 2m$; and
- $\bar{a}_0 - \bar{w}$ at $(j + m, j)$ for $n + 1 \leq j \leq 2m$.

We can use column operations to eliminate all nonzero entries in the upper-left $m \times m$ submatrix of R_1 . Since this submatrix is upper-triangular, the process only involves adding some multiples of the j th column with $m + 1 \leq j \leq 2m$ to columns numbered k , where $j - m \leq k \leq m$. Such a column operation also affects the bottom half of the matrix, where we add a multiple of the entry $(j + m, j)$ to the entry $(j + m, k)$. Since $(j + m) - k \leq j + m - (j - m) = 2m < n + m$, the affected entries of the bottom half are strictly above the diagonal $\{(n + m + j, j) : 1 \leq j \leq m\}$, which is filled with the value a_n . In conclusion, these column operations do not substantially affect the upper-triangular submatrix formed by the entries (i, j) with $n + m + 1 \leq i \leq n + 2m$, $1 \leq j \leq m$, in the sense that the submatrix remains upper-triangular and its diagonal entries remain equal to a_n .

Similar column operations on the right-hand side of the matrix eliminate all nonzero entries in the bottom right $m \times m$ submatrix of R_1 . Let R_2 be the resulting matrix: that is,

$$R_2 = \begin{pmatrix} 0 & \cdots & a_0 - w & \cdots & \cdots & 0 & 0 & 0 \\ 0 & \ddots & & \ddots & & \ddots & & 0 \\ 0 & 0 & 0 & \cdots & a_0 - w & \cdots & \cdots & a_n \\ \bar{a}_n & \cdots & \cdots & \bar{a}_0 - \bar{w} & \cdots & 0 & 0 & 0 \\ 0 & \ddots & & & \ddots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \bar{a}_0 - \bar{w} & \cdots & 0 \end{pmatrix}.$$

We claim that $\det R_2 = \pm |a_n|^{2m} |a_0 - w|^{2n}$. Indeed, the first m columns of R_2 contain only an upper-triangular submatrix with $\overline{a_n}$ on the diagonal; the last m columns contain only a lower-triangular matrix with a_n on the diagonal. After these are accounted for, we are left with a $2n \times 2n$ submatrix in which every row has exactly one nonzero element, either $a_0 - w$ or its conjugate. This completes the proof of $\deg h_{\mathbb{C}} = 2n$.

Define $h(x, y) = h_{\mathbb{C}}(x + iy, x - iy)$ for real x, y . We claim that h is real valued, and thus has real coefficients. Recall (for example, [4, page 11]) that the resultant can be expressed in terms of the roots of the polynomials g, g^* . Let z_1, \dots, z_{n+m} be the roots of g listed with multiplicity. To simplify notation, we separate the cases $m > 0$ and $m = 0$.

Case 1: $m > 0$. We have $\prod_{k=1}^{m+n} z_k = (-1)^{n+m} a_{-m}/a_n$; in particular, $z_k \neq 0$ for all k . It follows from (2.2) that g^* has roots $1/\overline{z_k}$ for $k = 1, \dots, n+m$. The leading terms of g and g^* are a_n and $\overline{a_{-m}}$, respectively. Thus,

$$\begin{aligned} \text{res}(g, g^*) &= (\overline{a_{-m}} a_n)^{m+n} \prod_{i,j=1}^{n+m} (z_i - 1/\overline{z_j}) = (\overline{a_{-m}} a_n)^{n+m} \prod_{i,j=1}^{n+m} \frac{z_i \overline{z_j} - 1}{\overline{z_j}} \\ &= (\overline{a_{-m}} a_n)^{n+m} \left(\prod_{j=1}^{m+n} \overline{z_j} \right)^{-(n+m)} \prod_{i,j=1}^{n+m} (z_i \overline{z_j} - 1) \\ &= (-1)^{n+m} (\overline{a_{-m}} a_n)^{m+n} (\overline{a_n/a_{-m}})^{n+m} \prod_{i,j=1}^{n+m} (z_i \overline{z_j} - 1) \\ &= (-1)^{n+m} |a_n|^{2(m+n)} \prod_{i,j=1}^{n+m} (z_i \overline{z_j} - 1). \end{aligned} \quad (2.3)$$

The latter product is evidently real.

Case 2: $m = 0$. We have $\prod_{k=1}^{m+n} z_k = (-1)^n (a_0 - w)/a_n$; in particular, $z_k \neq 0$ for all k provided that $w \neq a_0$. The rest of the proof goes as in case $m > 0$, with a_{-m} replaced by $a_0 - w$ throughout. Since a_{-m} cancels out at the end of (2.3), the conclusion that h is real valued still holds. \square

The following description of the local structure of the zero set of a complex-valued harmonic function is due to Sheil-Small (unpublished) and appears in [10].

THEOREM 2.3 [10, Theorem 3]. *Let $\Omega \subset \mathbb{C}$ be a domain and let $f: \Omega \rightarrow \mathbb{C}$ be a harmonic function. Suppose that the points $\{z_k\}_{k=1}^{\infty}$ are distinct zeros of f which converge to a point $z^* \in \Omega$. Then z^* is an interior point of a simple analytic arc γ which is contained in $f^{-1}(0)$ and contains infinitely many of the points z_k .*

The fact that $z_k \in \gamma$ for infinitely many k is not stated in [10, Theorem 3] but is a consequence of the proof.

PROOF OF THEOREM 2.1. (a)–(b) Suppose that $w \in p(\mathbb{T})$. Then the rational functions $p(z) - w$ and $p(1/\overline{z}) - w$ have a common zero, namely, any preimage of w that lies on \mathbb{T} . Consequently, the polynomials (2.2) have a common zero, which implies that their

resultant $h_{\mathbb{C}} = \text{res}(g, g^*)$ vanishes at w . Claims (a) and (b) follow from Lemma 2.2. For future reference, note that the zero set of h can be written as

$$V = h^{-1}(0) = p(E) \quad \text{where } E = \{z \in \mathbb{C} \setminus \{0\} : p(z) = p(1/\bar{z})\}. \quad (2.4)$$

(c) In view of (2.4), to prove that $V \setminus p(\mathbb{T})$ is finite it suffices to show that $E \setminus \mathbb{T}$ is finite. Let $q(z) = p(z) - p(1/\bar{z})$, which is a harmonic Laurent polynomial. Since $m < n$, it follows that $q(z) = p(z) + O(|z|^m) = a_n z^n + O(|z|^{n-1})$ as $|z| \rightarrow \infty$. Thus E is a bounded set. By symmetry, E is also bounded away from zero.

Suppose that $E \setminus \mathbb{T}$ is infinite. Then it contains a convergent sequence of distinct points $z_k \rightarrow z^* \neq 0$. By Theorem 2.3, there exists a simple analytic arc Γ such that $g|_{\Gamma} = 0$ and z^* is an interior point of Γ . In the case $z^* \in \mathbb{T}$, the arc Γ is not a subarc of \mathbb{T} because it contains infinitely many of the points z_k which are not on \mathbb{T} . By virtue of its analyticity, γ has finite intersection with \mathbb{T} . By shrinking γ we can achieve that $\gamma \cap \mathbb{T} = \{z^*\}$ if $z^* \in \mathbb{T}$, and $\gamma \cap \mathbb{T} = \emptyset$ otherwise.

Since the endpoints of γ lie in $E \setminus \mathbb{T}$, the process described above can be iterated to extend γ further in both directions. This continuation process can be repeated indefinitely. Since E is bounded, we conclude that E contains a simple closed analytic curve Γ , as in the proof of [10, Theorem 4].

If Γ does not surround zero, then the maximum principle yields $q \equiv 0$ in the domain enclosed by Γ , which is impossible since q is nonconstant. If Γ surrounds zero, then the complement of $\Gamma \cup \mathbb{T}$ has a connected component G such that $0 \notin G$. The maximum principle yields $q \equiv 0$ in G , which is a contradiction. The proof of (c) is complete.

(d) The proof of (c) used the assumption that $m < n$ only to establish that the set E in (2.4) is bounded. Thus the conclusion still holds if $m = n$ and E is a bounded set. Recalling that $V = p(E)$ and $|p(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$, we find that E is bounded whenever V is bounded.

Finally, if V is an unbounded set, then $V \setminus p(\mathbb{T})$ must be infinite because $p(\mathbb{T})$ is bounded. \square

Since a real algebraic set has finitely many connected components [9, Theorem 3], it follows from Theorem 2.1 that when $V \setminus p(\mathbb{T})$ is finite, the set $p(\mathbb{T})$ coincides with one of the connected components of V and the other components of V are singletons. The number of singleton components of V can be arbitrarily large, even when p is an algebraic polynomial.

REMARK 2.4. For every integer N , there exists a polynomial p such that the set $V \setminus p(\mathbb{T})$ described in Theorem 2.1 contains at least N points.

PROOF. Let a_1, \dots, a_N be distinct complex numbers with $0 < |a_k| < 1$ for $k = 1, \dots, N$. Using Lagrange interpolation, we get a polynomial q of degree $2N - 1$ such that $q(a_k) = q(1/\bar{a}_k) = k$ for $k = 1, \dots, N$. Let r be a polynomial of degree $2N$ with zeros at the points a_k and $1/\bar{a}_k$, $k = 1, \dots, N$. Since $\inf_{\mathbb{T}} |r| > 0$, for sufficiently large constant M the polynomial $p = q + Mr$ satisfies $q(a_k) = q(1/\bar{a}_k) = k$ for $k = 1, \dots, N$ as well as $|p(z)| > N$ for $z \in \mathbb{T}$. It follows that the algebraic set V , as described by (2.4), contains the points $1, \dots, N$, none of which lie on the curve $p(\mathbb{T})$. \square

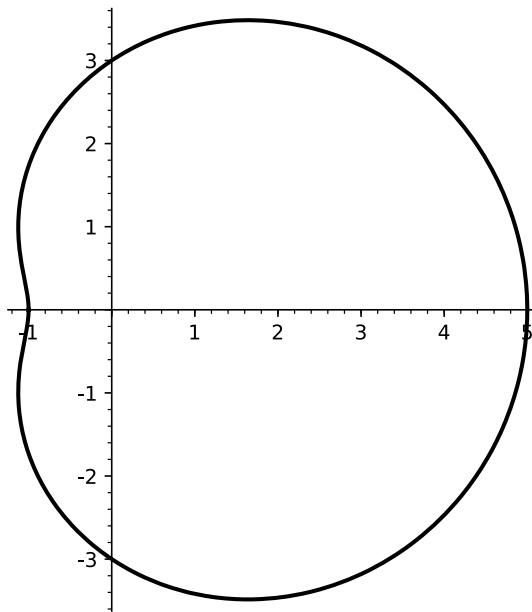


FIGURE 1. Nonalgebraic image of the circle.

3. Examples

First, we observe that $p(\mathbb{T})$ need not be a real algebraic set, even for a quadratic polynomial p .

EXAMPLE 3.1. Let $p(z) = z^2 + 3z + 1$. Then $p(\mathbb{T})$ is not a real algebraic set.

PROOF. Direct computation of the polynomial h in Theorem 2.1 yields

$$h(x, y) = \det \begin{pmatrix} 1 - w & 3 & 1 & 0 \\ 0 & 1 - w & 3 & 1 \\ 1 & 3 & 1 - \bar{w} & 0 \\ 0 & 1 & 3 & 1 - \bar{w} \end{pmatrix} \\ = x^4 + 2x^2y^2 + y^4 - 4x^3 - 4xy^2 - 5x^2 - 9y^2,$$

where $w = x + iy$. By Theorem 2.1, the set $h^{-1}(0)$ contains $p(\mathbb{T})$. Since $p \neq 0$ on \mathbb{T} , we have $0 \in h^{-1}(0) \setminus p(\mathbb{T})$. If $p(\mathbb{T})$ was an algebraic set, then V would be reducible. However, h is an irreducible polynomial. Indeed, the fact that the zero set of h is bounded implies that any nontrivial factorisation $h = fg$ would have $\deg f = \deg g = 2$. This means that V is the union of two conic sections, which it evidently is not, as $p(\mathbb{T})$ is not an ellipse (see Figure 1). \square

According to Theorem 2.1, the set $p(\mathbb{T})$ can be completed to a real algebraic set by adding finitely many points, provided that p is either an algebraic polynomial or a

Laurent polynomial with $m < n$. The following example shows that the case $m = n$ is indeed exceptional.

EXAMPLE 3.2. Let $p(z) = z + z^{-1}$. Then $p(\mathbb{T})$ is the line segment $[-2, 2]$. The smallest real algebraic set containing $p(\mathbb{T})$ is the real line \mathbb{R} .

The claimed properties of Example 3.2 are straightforward to verify. In addition, the polynomial h from Theorem 2.1 can be computed as $h(x, y) = -4y^2$, which shows that h is not necessarily irreducible.

4. Zero set of harmonic Laurent polynomials

The relation (2.4) highlights the importance of the zero set of the harmonic Laurent polynomial $P(z) = p(z) - p(1/\bar{z})$, where p is a Laurent polynomial. It is not a trivial task to determine whether a given harmonic Laurent polynomial has unbounded zero set: for example, Khavinson and Neumann [2] remarked on the varied nature of zero sets for rational harmonic functions in general. In this section, we develop a necessary condition, in terms of the coefficients of p , for the function P to have an unbounded zero set.

Suppose that p is a Laurent polynomial (2.1) such that the associated function $P(z) = p(z) - p(1/\bar{z})$ has unbounded zero set. Consider the algebraic part of P , namely,

$$q(z) = \sum_{k=1}^n a_k z^k - \sum_{k=1}^m a_{-k} \bar{z}^k. \quad (4.1)$$

Then q is a harmonic polynomial such that $\liminf_{z \rightarrow \infty} |q(z)|$ is finite. In other words, q is not a proper map of the complex plane.

One necessary condition is immediate: if $m < n$, then $|q(z)| = a_n |z|^n + o(|z|^n)$ as $z \rightarrow \infty$. Thus P can only have an unbounded zero set if $m = n$.

We now look for further conditions on a harmonic polynomial which will ensure that it is a proper map of \mathbb{R}^2 to \mathbb{R}^2 . More generally, given a polynomial map $F = (F_1, \dots, F_n): \mathbb{R}^n \rightarrow \mathbb{R}^n$, let us decompose each component F_k into homogeneous polynomials and let $\mathcal{H}(F_k)$ be the homogeneous term of highest degree in F_k . Write $\mathcal{H}(F)$ for $(\mathcal{H}(F_1), \dots, \mathcal{H}(F_n))$ so that $\mathcal{H}(F)$ is also a polynomial map of \mathbb{R}^n . The following result is from [7, Lemma 10.1.9].

LEMMA 4.1 (L. Andrew Campbell). *If $\mathcal{H}(F)$ does not vanish in $\mathbb{R}^n \setminus \{0\}$, then the map $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a proper map, that is, $|F(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$.*

Lemma 4.1 can be restated in a form adapted to harmonic polynomials in \mathbb{C} .

LEMMA 4.2. *Consider a harmonic polynomial $q(z) = \sum_{k=0}^n (a_k z^k + b_k \bar{z}^k)$ of degree $n \geq 1$ as a map from \mathbb{C} to \mathbb{C} .*

(a) *If $|a_n| \neq |b_n|$, then q is proper.*

(b) If $|a_n| = |b_n|$, let $\eta \in \mathbb{T}$ be such that $\eta a_n = \overline{\eta b_n}$. If $\eta a_k = \overline{\eta b_k}$ for $k = 1, \dots, n$, then q is not proper. Otherwise, let K be the largest value of k such that $\eta a_k \neq \overline{\eta b_k}$. If there is no $z \neq 0$ such that

$$\operatorname{Re}(\eta a_n z^n) = 0 = \operatorname{Im}((\eta a_K - \overline{\eta b_K}) z^K),$$

then q is proper.

PROOF. Part (a) follows from the reverse triangle inequality: that is,

$$|q(z)| \geq ||a_n| - |b_n|| |z|^n + o(|z|^n) \quad \text{as } n \rightarrow \infty.$$

To prove part (b), observe that

$$\operatorname{Im}(\eta q(z)) = \sum_{k=0}^n \operatorname{Im}((\eta a_k - \overline{\eta b_k}) z^k). \quad (4.2)$$

If $\eta a_k = \overline{\eta b_k}$ for $k = 1, \dots, n$, then $\operatorname{Im}(\eta q)$ is constant, which means that, up to a constant term, ηq is a real-valued harmonic function. By Harnack's inequality, a nonconstant harmonic function $h: \mathbb{C} \rightarrow \mathbb{R}$ must be unbounded from above and from below, and therefore $q^{-1}(0)$ is an unbounded set. Since q is constant on an unbounded set, it is not a proper map.

Finally, suppose that K , as defined in (b), exists. It follows from (4.2) that

$$\mathcal{H}(\operatorname{Im}(\eta q(z))) = \operatorname{Im}((\eta a_K - \overline{\eta b_K}) z^K).$$

Since also

$$\mathcal{H}(\operatorname{Re}(\eta p(z))) = \operatorname{Re}((\eta a_n + \overline{\eta b_n}) z^n) = 2 \operatorname{Re}(\eta a_n z^n),$$

the last statement in (b) follows by applying Lemma 4.1 to $(\operatorname{Re}(\eta q), \operatorname{Im}(\eta q))$ considered as a map of \mathbb{R}^2 to \mathbb{R}^2 . \square

We are now ready to apply Lemma 4.2 to the special case $P(z) = p(z) - p(1/\bar{z})$, where p is a Laurent polynomial. Recall that, in view of Theorem 2.1 and the relation (2.4), the following result describes when the image $p(\mathbb{T})$ has infinite complement in the real algebraic set V containing it.

THEOREM 4.3. Suppose $p(z) = \sum_{k=-n}^n a_k z^k$ is a Laurent polynomial with $a_n a_{-n} \neq 0$. Let $P(z) = p(z) - p(1/\bar{z})$. If the zero set of P is unbounded, then one of the following holds.

- (a) $p(\mathbb{T})$ is contained in a line.
- (b) There exists $\eta \in \mathbb{T}$ such that $\eta a_n + \overline{\eta a_{-n}} = 0$. Furthermore, there is an integer $k \in \{1, \dots, n-1\}$ such that the harmonic polynomial $\operatorname{Im}((\eta a_k + \overline{\eta a_{-k}}) z^k)$ is nonconstant and shares a nonzero root with the harmonic polynomial $\operatorname{Re}(\eta a_n z^n)$.

As a partial converse, if (a) holds, then the zero set of P is unbounded.

Although part (b) of Theorem 4.3 is convoluted, it is not difficult to check, in practice, because η is uniquely determined (up to irrelevant sign) and the zero sets of both harmonic polynomials involved are simply unions of equally spaced lines through the origin.

PROOF. We apply Lemma 4.2 to the polynomial q in (4.1), which means letting $b_k = -a_{-k}$ for $k = 1, \dots, n$. Since q is not proper, part (b) of the lemma provides two possible scenarios, which are considered below.

One possibility is that there exists a unimodular constant η such that $\eta a_k = -\overline{\eta a_{-k}}$ for $k = 1, \dots, n$. Therefore, for $z \in \mathbb{T}$,

$$\operatorname{Re}(\eta p(z)) = \operatorname{Re}(a_0) + \sum_{k=1}^n \operatorname{Re}(\eta a_k z^k + \overline{\eta a_{-k}} z^{-k}) = \operatorname{Re}(a_0),$$

which means that $p(\mathbb{T})$ is contained in a line. The converse is true as well. If $p(\mathbb{T})$ is contained in a line, then there exists a unimodular constant η such that $\operatorname{Re}(\eta p)$ is constant on \mathbb{T} . Considering the Fourier coefficients of $\operatorname{Re}(\eta p)$, we find $\eta a_k + \overline{\eta a_{-k}} = 0$ for all $1 \leq k \leq n$.

The other possibility described in Lemma 4.2(b) transforms into part (b) of Theorem 4.3 with the substitution $b_k = -a_{-k}$. \square

5. Intersection of polynomial images of the circle

As an application of Theorem 2.1, we establish an upper bound for the number of intersections between two images of the unit circle \mathbb{T} under Laurent polynomials. It is necessary to exclude some pairs of polynomials from consideration because, for example, the images of \mathbb{T} under any two of the Laurent polynomials

$$p_\alpha(z) = z + z^{-1} + \alpha \quad \text{for } -2 < \alpha < 2,$$

have infinite intersection. This is detected by the computation of the polynomial h in Theorem 2.1, according to which $h(x, y) = -4y^2$ regardless of α .

THEOREM 5.1. *Consider two Laurent polynomials*

$$p(z) = \sum_{k=-m}^n a_k z^k \quad \text{and} \quad \tilde{p}(z) = \sum_{k=-r}^s b_k z^k,$$

where $m, r \geq 0$, $n, s \geq 1$ and $a_{-m} a_n b_{-r} b_s \neq 0$. Then the intersection $p(\mathbb{T}) \cap \tilde{p}(\mathbb{T})$ consists of at most $4ns - 2(n-m)(s-r)$ points unless the corresponding polynomials h and \tilde{h} from Theorem 2.1 have a nontrivial common factor.

In the special case of algebraic polynomials, $m = r = 0$, the estimate in Theorem 5.1 simplifies to $2ns$. In this case, the theorem is due to Quine [6, Theorem 3], where the bound $2ns$ is shown to be sharp. A related problem of counting the self-intersections of $p(\mathbb{T})$ was addressed in [5] for algebraic polynomials and in [3] for Laurent polynomials.

PROOF. Let $h_{\mathbb{C}} \in \mathbb{C}[w, \overline{w}]$ be the polynomial associated to p by Theorem 2.1(b). Consider its homogenisation

$$H(w, \overline{w}, \zeta) = \zeta^{2n} h_{\mathbb{C}}(w/\zeta, \overline{w}/\zeta).$$

Since $h_{\mathbb{C}}$ has degree $m+n$ in the variable w , it follows that H has a zero of order at least $2n - (m+n) = n - m$ at the point $(1, 0, 0)$ of the projective space \mathbb{CP}^2 . Similarly, it has a zero of order at least $n - m$ at the point $(0, 1, 0)$.

The homogeneous polynomial \tilde{H} associated with \tilde{p} has zeros of order at least $s - r$ at the same two points. Therefore the projective curves $H = 0$ and $\tilde{H} = 0$ intersect with multiplicity at least $(n - m)(s - r)$ at each of the points $(1, 0, 0)$ and $(0, 1, 0)$ [8, Theorem 5.10, page 114].

Bezout's theorem implies that, unless H and \tilde{H} have a nontrivial common factor, the projective curves $H = 0$ and $\tilde{H} = 0$ have at most $\deg H \deg \tilde{H} = 4ns$ intersections in \mathbb{CP}^2 , counted with multiplicity. Subtracting the intersections at the two aforementioned points, we are left with at most $4ns - 2(n - m)(s - r)$ points of intersection in the affine plane. \square

References

- [1] P. M. Bleher, Y. Homma, L. L. Ji and R. K. W. Roeder, 'Counting zeros of harmonic rational functions and its application to gravitational lensing', *Int. Math. Res. Not. IMRN* **2014**(8) (2014), 2245–2264.
- [2] D. Khavinson and G. Neumann, 'On the number of zeros of certain rational harmonic functions', *Proc. Amer. Math. Soc.* **134**(4) (2006), 1077–1085.
- [3] L. V. Kovalev and S. Kalmykov, 'Self-intersections of Laurent polynomials and the density of Jordan curves', *Proc. Amer. Math. Soc.* (2019), to appear, arXiv:1902.02468.
- [4] G. Orzech and M. Orzech, *Plane Algebraic Curves: An Introduction Via Valuations*, Monographs and Textbooks in Pure and Applied Mathematics, 61 (Marcel Dekker, Inc., New York, 1981).
- [5] J. R. Quine, 'On the self-intersections of the image of the unit circle under a polynomial mapping', *Proc. Amer. Math. Soc.* **39** (1973), 135–140.
- [6] J. R. Quine, 'Some consequences of the algebraic nature of $p(e^{i\theta})$ ', *Trans. Amer. Math. Soc.* **224**(2) (1976), 437–442.
- [7] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, Progress in Mathematics, 190 (Birkhäuser, Basel, 2000).
- [8] R. J. Walker, *Algebraic Curves* (Springer, New York, 1978). Reprint of the 1950 edition.
- [9] H. Whitney, 'Elementary structure of real algebraic varieties', *Ann. of Math.* (2) **66** (1957), 545–556.
- [10] A. S. Wilmshurst, 'The valence of harmonic polynomials', *Proc. Amer. Math. Soc.* **126**(7) (1998), 2077–2081.

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