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ONLINE FIRST

International Journal of Research in Undergraduate Mathematics Education

ISSN 2198-9745

Int. J. Res. Undergrad. Math. Ed.
DOI 10.1007/s40753-020-00123-8

Springer

International Journal of Research in Undergraduate Mathematics Education
https://doi.org/10.1007/s40753-020-00123-8

Investigating Combinatorial Provers' Reasoning about Multiplication

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Accepted: 4 September 2020/

Abstract
Combinatorial proof is an important topic both for combinatorics education and proof education researchers, but relatively little has been studied about the teaching and learning of combinatorial proof. In this paper, we focus on one specific phenomenon that emerged during interviews with mathematicians and students who were experienced provers as they discussed and engaged in combinatorial proof. In particular, participants used a wide variety of cognitive models to interpret multiplication by a constant when reasoning about binomial identities, some of which seemed to be more (or less) effective in helping produce a combinatorial proof. We present these cognitive models and describe episodes that illustrate implications of these cognitive models for our participants' work on proving binomial identities. Our findings both inform research on combinatorial proof and highlight the importance of understanding subtleties of the familiar operation of multiplication.

Keywords Combinatorial proof · Multiplication · Counting problems

Introduction

Combinatorics is an increasingly important branch of mathematics with applications in computer science, engineering, statistics, as well as other areas of mathematics. In addition to its applicability, combinatorics has pedagogical value for mathematics instructors due to its accessibility and ability to provide opportunities for students to use creativity, search for patterns, and generalize (e.g., Lockwood and Gibson 2016;

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Lockwood and Reed 2018; Tillema 2013). One class of combinatorics problems, combinatorial proof of binomial identities,[1] comes up in a variety of contexts, including discrete mathematics, statistics, and number theory. These problems can be tricky even for accomplished counters (e.g., Lockwood et al. In press), and yet this topic has received relatively little attention from the mathematics education research community.

A binomial identity is an equation involving one or more binomial coefficients, such as the following:

$$\binom{n}{k}k = n\binom{n-1}{k-1} \tag{1}$$

In this paper, we take *combinatorial proof* to mean any proof that establishes the veracity of a binomial identity by arguing that each side enumerates the same (finite) set.[2] The validity of these arguments is rooted in the fact that a set can have only one cardinality.

For example, to prove the binomial identity (1), one could argue that each side counts the number of committees of size k with a chairperson that can be formed from a group of n people. In this case, the right-hand side counts this set because there are n possible people who could be the chairperson, and then for every choice of one person to be the chairperson, there are $\binom{n-1}{k-1}$ ways of selecting the remaining $k – 1$ people for the committee. As a lead-in to the rest of this paper, we offer the following questions to the reader to provoke thinking about combinatorial proof (we present our research questions at the end of the section). First, w*hy does* $\binom{n}{k}k$ *also count the number of committees of size* k *with a chairperson that can be formed from* n *people?* Second*, how are you thinking of the multiplication of the binomial coefficient by* k*?*

In this paper, we report on results from a study in which we interviewed five upper-division mathematics undergraduate students and eight mathematicians to investigate the ways that experienced provers think about and engage with combinatorial proof. We particularly focus on findings related to the way that combinatorial provers conceived of and used multiplication. Students' reasoning about multiplication is a topic that has been studied extensively in the K-12[3] mathematics education literature (e.g., Greer 1992, 1994; Mulligan and Mitchelmore 1997; Steffe 1994; Tillema 2013), and, while some studies occur at the undergraduate level (e.g., Lockwood and Purdy 2019a, b), it has not received as much attention at the postsecondary level, perhaps because educators and researchers may assume that undergraduate students understand the familiar operation of multiplication. However, as we will discuss, our data show that undergraduate students' and mathematicians' conceptions of multiplication are interesting and varied, and the cognitive models of multiplication they use can have implications for their combinatorial proving activity.

[1] Combinatorial proof is a proof technique that can be applied to other types of theorems as well, but we focus on binomial identities in this paper.

[2] We acknowledge that authors such as Lockwood et al. (in press) and Rosen (2012) have articulated two types of combinatorial proof – one type is that described above, and the second type involves arguing that each side of a binomial identity counts a different set and creates a bijection between the two sets. We do not focus on bijective proofs in this paper.

[3] K-12 refers to school-age students in grades kindergarten through grade 12, which, in the United States, is education prior to university.

We attempt to address the following research questions in this paper (we will elaborate particular terminology in these questions in the following sections):

(a) What cognitive models for multiplication do undergraduate students and mathematicians use when engaging in combinatorial proof of identities involving multiplication by a scalar?
(b) What are implications of these cognitive models for students' engagement with combinatorial proof?

Here we clarify that we are considering all cognitive models for multiplication that the participants used; we were not only interested in those models that the participants produced spontaneously. We also specify multiplication by a scalar to mean multiplication by a single positive integer constant k, such as $\binom{n}{k}k$. We narrow our results to multiplication by a scalar, as opposed to other expressions involving multiplication that may occur in a binomial identity, such as $\binom{n}{k}\binom{k}{m}$, to focus our arguments and due to space limitations.[4] We consider additional kinds of multiplication as an avenue for future research.

Relevant Literature and Theoretical Perspectives

To situate our findings within the broader literature, we first look at previous work on combinatorial proof in the section "Previous Work on Combinatorial Proof." Then in "Cognitive Models for Multiplication" we elaborate our notion of cognitive models, which serves as our theoretical framework for how we think of multiplication in this paper. Finally, in "Literature on Multiplicative Reasoning" and "Multiplication within Counting at the Undergraduate Level" we describe work that has been conducted on students' reasoning about multiplication at the K-12 level and within combinatorics to help us frame our results.

Previous Work on Combinatorial Proof

We identified only two prior studies that focused on undergraduate students' thinking about and engagement in combinatorial proof. First, Engelke and CadwalladerOlsker (2010, 2011) conducted a study in which they looked at students' solutions to exam questions asking for a combinatorial proof of two binomial identities. They identified several difficulties that students faced with combinatorial proofs. Linked to these difficulties, Engelke and CadwalladerOlsker (2011) observed that the students may have engaged in *pseudo-semantic proof production*, which is based on the distinction between *semantic* and *syntactic* proof production. This distinction, articulated by Weber and Alcock (2004), describes qualitatively different approaches students can take to proof, depending on whether they use internally meaningful instantiations of the mathematical objects they are working with, or whether they are primarily manipulating mathematical symbols, facts, or definitions according to rules of logic. While it is

[4] We also note that "scalar multiplication" has a particular meaning in the domain like linear algebra; we clarify that here we simply mean multiplication by a single positive integer constant.

difficult to know for certain by looking only at student exam solutions, Engelke and CadwalladerOlsker (2010, 2011) work provides evidence that combinatorial proof can be difficult for students and that students may try to imitate enumerative arguments they previously encountered if they get stuck.

The only other study we found in the literature addressing combinatorial proof was one that we conducted more recently. We carried out a 15-session teaching experiment (Steffe and Thompson 2000) that covered a variety of topics in combinatorics with two vector-calculus students (Lockwood et al. In press). The last three sessions of the teaching experiment were centered around combinatorial proof of binomial identities, and we could study the students' reasoning on combinatorial proof based on their trajectory along the prior 12 teaching experiment sessions. In this study, we found that the students seemed to benefit from two particular instantiations: (a) focusing on a particular context (e.g., counting passwords or committees), and (b) considering specific values of n or other variables appearing in the identity to be proven. We also found that a potentially useful way to prepare students for combinatorial proof is to give them opportunities to generalize while solving counting problems and ask them to solve counting problems in two different ways.

This prior work on combinatorial proof is valuable and may help give instructors more pedagogical ideas when covering combinatorial proof. However, questions regarding student thinking about combinatorial proof still remain unanswered. In particular, while we previously looked at other aspects of student thinking about combinatorial proof (Lockwood et al. In press), we did not focus on how they thought about the mathematical operations involved in the binomial identities. While it may be easy to assume that undergraduate mathematics students understand what operations such as multiplication do, we argue that interpreting these operations combinatorially in the context of a binomial identity can introduce subtleties that are important yet not always appreciated in university-level classrooms. We now discuss previous findings from relevant literature related to multiplication.

Cognitive Models for Multiplication

Considerable work has been done to identify and elaborate ways that K-12 students think about and use multiplication to solve problems (we discuss this further in "Literature on Multiplicative Reasoning"). Less research has been conducted on how undergraduate students use and think about multiplication when counting (we present some in "Multiplication within Counting at the Undergraduate Level"). While such work is valuable and insightful, we could not identify any studies that looked at the different models of multiplication (e.g., as an array, as equivalent groups, etc.) that undergraduate students might use to solve problems, including combinatorial proof problems that involve interpreting expressions that have multiplication. In addition, it is not clear which of these models for multiplication are the most productive for students engaging in combinatorial proof, or what other implications these models may have on their combinatorial activity.

To help us investigate these questions, in our study we will use the term *cognitive models* to mean someone's personal representation of what a given instance of the operation of multiplication entails. This construct we are defining is intended to be similar to Mulligan and Mitchelmore's (1997) semantic structures, which we discuss in

"Literature on Multiplicative Reasoning"; however, our use of cognitive models is intended to go beyond a classification of pre-existing problem types—we also attempt to capture students' and mathematicians' mental representations of what multiplication is doing in a binomial identity. The construct of cognitive models is the theoretical lens we use to analyze and present the results of our study.

To frame our work on cognitive models of multiplication within existing literature, we now discuss literature on multiplicative reasoning. We begin with work that has been conducted at the K-12 level in "Literature on Multiplicative Reasoning" and "Multiplication within Counting at the Undergraduate Level" we connect to the few studies that have explored multiplication at the university level. Overall, we aim to support our claim that there are open questions in the literature about what kinds of cognitive models for multiplication undergraduate students bring to combinatorial problems (and combinatorial proof in particular).

Literature on Multiplicative Reasoning

To our knowledge, other than the studies we mention in "Multiplication within Counting at the Undergraduate Level" that look at the multiplication principle in combinatorics, there has been no research done on the mental models of multiplication that undergraduate students use when they engage in combinatorial proof activity. As noted, one reason for this could be that it is easy to assume that students understand the familiar operation of multiplication. Understandably, there has been much more work conducted in this area in mathematics education at the K-12 level. The Common Core State Standards identifies several situations (e.g., equal groups, arrays/area, and comparison) involving multiplication that K-12 students should be exposed to (National Governors Association Center for Best Practices and Council of Chief State School Officers 2010), and numerous researchers spanning several decades have studied how young children think about multiplication of positive whole numbers (e.g., Greer 1992; Mulligan and Mitchelmore 1997; Tillema 2013). This is a natural research inquiry, since primary school is typically where a student learns how to multiply. While we do not provide a comprehensive review of literature on multiplication among young students, we highlight some studies that have established that there are different ways in which students think about and approach problems involving multiplication.

Multiplication is often introduced to children around the second grade and is typically presented as an efficient calculation of repeated addition (Clark and Kamii 1996). However, the interpretation of multiplication as repeated addition is frequently inadequate in helping students navigate some multiplicative situations (Sowder et al. 1998), such as those involving a Cartesian-product context (Mulligan and Mitchelmore 1997). Specifically in combinatorics, Batanero et al. (1997) also found that 14 and 15-year-old students may find it difficult to distinguish combinatorial situations requiring addition from those requiring multiplication. Notably, Kavousian (2008) reported similar difficulties among undergraduate students.

One of the ways that researchers have tried to understand children's thinking about multiplication is by studying the *intuitive models* they employ to solve problems. In the K-12 literature, some researchers have used intuitive multiplication models to mean an internalization of multiplication as corresponding to a particular problem situation (e.g., Fischbein et al. 1985). Some researchers such as Tillema (2013) have similarly looked multiplicative concepts students use and construct when solving different types of multiplication problems, such as those involving linear or power meanings of multiplication. Other researchers, however, have found it preferable to study and define children's intuitive models of multiplication in terms of the calculation strategies that they use (e.g., Anghileri 1989; Mulligan and Mitchelmore 1997). Anghileri (1989) was one of the earliest researchers to use intuitive multiplication models to study children, and their results suggested that children use three models for whole-number multiplication: unitary counting, repeated addition, and multiplicative calculation.

Mulligan and Mitchelmore (1997) interpreted intuitive models as calculation strategies that young children use to solve multiplication problems, and they extended earlier results with a longitudinal study aimed at understanding such intuitive models and how these intuitive models develop over time. They followed Nesher (1988) in noting that "multiplication situations can be classified according to the nature of the quantities and the relation between them" (p. 310), and they referred to this classification of a multiplication problem as its *semantic structure*. They used 5 of the 10 multiplicative semantic structures identified by Greer (1992): equivalent groups, rate, comparison, array, and Cartesian product. The other 5 semantic structures were excluded because they involved measurement (a context that the 2nd- and 3rd-grade students being studied would not have been familiar with) or were more applicable to multiplication by rational numbers (instead of integers). Examples from Mulligan and Mitchelmore's (1997) study of these semantic structures are given in Table 1. After they gave the children problems from each of these semantic structures, Mulligan and Mitchelmore (1997) identified three intuitive models that the children used (p. 316): direct counting, repeated addition, and multiplicative operation.

Our notion of cognitive models is similar to Mulligan and Mitchelmore's (1997) semantic structures; however, as we noted, our use of cognitive models is intended to go beyond a classification of pre-existing problem types—we also attempt to capture students' and mathematicians' mental representations of what multiplication is doing in a binomial identity. Further, we do not consider *cognitive models* to be the same as the intuitive models construct used by Mulligan and Mitchelmore (1997) and others, since intuitive models referred to children's calculation strategies for multiplication. Thus, while our work is similar in some ways to what Mulligan and Mitchelmore studied, we

Table 1 Mulligan and Mitchelmore's (1997) multiplicative semantic structures (p. 314)

Semantic structure	Example problem
Equivalent groups	Peter had 2 drinks at lunch time every day for 3 days. How many drinks did he have altogether?
Rate	If you need 5 cents to buy 1 sticker, how much money do you need to buy 2 stickers?
Comparison	John has 3 books, and Sue has 4 times as many. How many books does Sue have?
Array	There are 4 lines of children with 3 childred in each line. How many children are there altogether?
Cartesian product	You can buy chicken chips or plain chips in small, medium, or large packets. How many different choices can you make?

are attempting to understand what multiplicative representations (and not just calculational strategies or problem types) our participants think about and use.

Multiplication within Counting at the Undergraduate Level

Multiplication is a familiar operation to undergraduate students, yet in our experience teaching combinatorics, we have found that students do not always know when to multiply while solving counting problems. This has also been found in some studies of undergraduate students; for example, researchers have found that undergraduate students may confuse situations requiring multiplication versus addition (e.g., Kavousian 2008; Sowder et al. 1998).

In combinatorics, multiplication arises as such a fundamental aspect of counting that there is a guiding principle describing when to multiply when solving counting problems – this is called the Multiplication Principle (MP). Tucker (2002) offers our preferred statement of the MP: "Suppose a procedure can be broken down into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \cdots \times r_m$ different composite outcomes" (p. 170). Despite how fundamental the MP is for counting, Lockwood et al. (2017) found that textbook statements of the MP vary significantly, and they argue, "[T]he MP is much more nuanced than instructors and students perhaps give it credit for" (p. 31).

Building off this textbook analysis, Lockwood and Purdy (2019a, b) conducted a teaching experiment with two undergraduate students, in which the students articulated their own statement of the MP, following *guided reinvention* heuristic (Freudenthal 1991). Lockwood and Purdy (2019a) demonstrated that even students who can successfully solve counting problems involving multiplication may find it challenging to characterize precisely when to multiply in counting. They additionally identified subtleties regarding the MP that textbooks do not always explicitly address and yet are critical to multiplication in counting, particularly related to handling issues of order (Lockwood and Purdy 2019b).

While Lockwood and Purdy identified central key issues related to students' understanding of the MP, they did not explore the fact that there might be other interpretations or mental models that counters may have for multiplication. Students' cognitive models of mathematical operations, including multiplication, are highly relevant in combinatorial proof, since interpreting de-contextualized expressions as having an underlying counting process lies at the heart of these types of problems.

Methods

We conducted video-recorded, semi-structured, task-based interviews (Hunting 1997) with five undergraduate students and eight mathematicians. These interviews were part of a larger study aimed at understanding mathematicians' and upper-division undergraduate students' reasoning and beliefs about combinatorial proof. We describe the data collection for both the students and the mathematicians, and then we discuss our data analysis for writing this paper.

Data Collection

Student Data Collection

We recruited students from upper-division mathematics courses at a large university in the western United States. In our university system, "upper-division" implies courses typically taken by students in the last two years of their degree; they are beyond introductory courses and are geared toward students who will major or minor in mathematics. Seven total students expressed interest in participating, and we conducted an hour-long individual task-based selection interview with these students. In these selection interviews, we asked each student to solve straightforward combinatorics problems and to prove basic theorems, such as the fact that the sum of two even integers is an even integer. Selection interview tasks can be found in Appendix 2. We sought students who could navigate a mathematical proof and had some familiarity with counting problems so they could construct and interpret combinatorial proofs. Five students (pseudonyms Sidney, Riley, Adrien, Peyton, and Ash[5]) satisfied these criteria. Table 2 shows the classes each of the students had taken. Overall, the students had each taken at least one proof-based mathematics class and had each made some progress toward fulfilling the required courses for a mathematics major.

Next, these five students each participated in four hour-long individual interviews, which occurred 4–14 days apart as the students' schedules permitted. The interviews were couched within a larger study aimed at understanding experienced provers' understandings related to combinatorial proof, and so we wanted to give them opportunities to reason about combinatorial proof even if they were struggling. In line with the aims of the overall study, if the students were stuck as they worked on combinatorial tasks, the interviewer occasionally provided them hints to help facilitate productive combinatorial proving activity. During these interviews, the students were asked to solve combinatorics problems, give counting arguments for the veracity of binomial identities, and answer reflection questions about their approach to and reasoning about combinatorial proof. Student interview tasks are provided in Appendix 3.

Mathematician Data Collection

We recruited mathematicians from three different universities in the western United States. These mathematicians were a convenience sample recruited via email for our study. We included both mathematicians who did and did not conduct research in combinatorics. Table 3 shows the research background and experience of the mathematicians.

Each mathematician participated in a single, 90-min individual interview. We had the same overall goals as the student interviews, namely to understand experienced provers' understandings related to combinatorial proof, so the interviewer at times

[5] The majors of these students were Mathematics, Computer Science, Mathematics, Physics and Mathematics, and International Economics and Mathematics, respectively.

Table 2 Classes taken by student participants

	Sidney	Riley	Adrien	Peyton	Ash
Calculus I	√	√	√		√
Calculus II	√	√	√		√
Infinite series & sequences	√		√	√	√
Vector Calculus I	√	√	√	√	√
Vector Calculus II	√		√	√	√
Applied differential equations	√		√	√	√
Mathematics for management, life, and social sciences					√
Linear Algebra I	√	√	√	√	√
Linear Algebra II	√		√	√	√
Advanced Calculus	√		√		√
Introduction to modern Algebra	√		√		√
Metric spaces and topology		√*	√*		
Discrete mathematics	√	√		√*	√
Applied ordinary differential equations	√		√		
Applied partial differential equations	√				
Fundamental concepts of topology	√*	√*		√*	
Numerical linear Algebra		√			
Introduction to numerical analysis			√		
Computational number theory		√			
Mathematical modeling			√		
Actuarial mathematics			√		
Complex variables					√
Non-Euclidean geometry					√

*Indicates that the student was enrolled in this course at the time the interviews were conducted

provided hints or suggestions if the participants were at an impasse. During these interviews, we asked the mathematicians to give combinatorial proofs of various binomial identities and answer reflection questions about their approach to and reasoning about combinatorial proof. Details of the interview tasks we gave the mathematicians are provided in Appendix 1.

Data Analysis

To analyze these data, we followed the six phases of thematic analysis as outlined by Braun and Clarke (2006): familiarizing oneself with the data, generating initial codes, searching for themes, reviewing themes, defining and naming themes, and producing the report (p. 87). First, all of the videos were transcribed, and the first author then re-watched all of the interview videos, making note of key episodes related to our research questions. The first author flagged every episode in the data where a participant (student or mathematician) interpreted an expression involving multiplication, and then she exhaustively coded each of these episodes according to which cognitive model the participant used. These episodes included both participants' work on combinatorial

Table 3 Mathematician participants' research and teaching experience information

Name[a]	Research experience	Regularly taught combinatorics
Ridley	Algebraic combinatorics & bijective combinatorics (13 years)	Yes
Dominique	Competitive coloring algorithms and parameters defined on graphs (20 years)	Yes
Jaiden	Computability, computable analysis, and algorithmic information theory (3 years)	Yes
Skyler	Dynamical systems and number theory (15 years)	No
Emery	Modular forms and partition functions (17 years)	Yes
Lake	Partial differential equations and related functional analysis (60 years)	No
Justice	Representation theory of finite groups (6 years)	Yes
Robin	Geometry, algebra, and mathematics education (40 years)	No

[a] These are pseudonyms

proof tasks involving multiplication and participants' answers to reflection questions concerning their reasoning about instances of multiplication. Our initial list of codes was based off Mulligan and Mitchelmore's (1997) semantic structures (Table 1), and we added new cognitive models to our list as they arose in the data.

To decide which code to apply for a given episode in the interviews, she examined the participants' utterances about how they were conceiving of an instance of multiplication and any additional representations they gave of their thought process if they wrote anything down. For instance, if a participant alluded to "copies" or "duplicates," this could indicate they were using an equivalent groups cognitive model of multiplication; likewise, a mention of "Cartesian products" might correspond to the Cartesian product cognitive model. After the first author identified all of the cognitive models within the data, both authors discussed key episodes and findings that emerged from the initial analysis and reviewed parts of the interviews that warranted additional analysis. We discussed the codes that were being used to ensure that they faithfully represented the data, and any episodes in which it was difficult to determine the participant's cognitive model for multiplication were discussed thoroughly until agreement was reached.

Results

In this section, we discuss the results of our investigation into how the students and mathematicians conceived of multiplication when engaged in combinatorial proof. Ultimately, the main points we want to emphasize are 1) the mathematicians and students used a variety of cognitive models of multiplication, and 2) those different cognitive models were not all equally effective in helping the participants correctly construct combinatorial proofs. In fact, while six cognitive models arose in the interviews, only two of the cognitive models were used productively by participants in any of the interviews. For the purposes of this paper, when we characterize a student or mathematician's use of a multiplicative cognitive model as *productive*, we mean that their use of that cognitive model resulted in a logically and mathematically correct combinatorial argument for the identity. By this, we do not intend to imply that a participant's work was not valuable or enriching if their work did not result in a correct proof; we simply use the term in this section to distinguish between cognitive models that did (and did not) ultimately result in correct proofs.

In the section "Combinatorial Provers Used a Variety of Cognitive Models for Multiplication," we present the cognitive models for multiplication that the mathematicians and students used in their interviews, which addresses our first research question. Some of the student and mathematician participants used a cognitive model that aligned with one of the five semantic structures given by Mulligan and Mitchelmore (1997), but some of them used cognitive models that were not discussed in the literature. In "Cognitive Models that were Productively Used for Combinatorial Proof," we highlight participant work that demonstrates the two cognitive models that were used on combinatorial proofs that were correctly proven. In particular, in "Element Selection as a Productive Cognitive Model" we show a student's use of the element selection model, which was by far the most productively used cognitive model, and in "A Combinatorial Proof Utilizing Repeated Addition and Equivalent Groups" we present a mathematician's use of the equivalent groups model, which was the only instance of a productive model other than element selection. Then, to better understand why other models were perhaps not productively used, in "Instances of Cognitive Models Not Being Used Productively" we show two episodes in which the participants were not successful in leveraging cognitive models for the purpose of combinatorial proof. In "Weight as a Cognitive Model that was not Productive" we show how a student used a cognitive model of weighted sums but could not find a way to apply that model as part of a correct combinatorial proof. Finally, in "Conflating Models of Multiplication and Exponentiation" we discuss an interpretation of multiplication that arose in the data, but actually represents a model of *exponentiation* rather than multiplication.

Combinatorial Provers Used a Variety of Cognitive Models for Multiplication

In this section, we present the cognitive models for multiplication by a constant k in the context of binomial identities that emerged during the interviews: *equivalent groups*, *Cartesian product*, *scaling factor*, *inverse of a probability*, *weight*, and *element selection*. See Table 4 for further details; we use the term k-committee to refer to a committee of size k formed from a group of n (distinct) people. As stated previously, we were interested in documenting all cognitive models of multiplication that the participants used, regardless if these models originated with the participant or interviewer. All of the cognitive models given in Table 4 were produced spontaneously by the participants, with the exception of element selection for some participants. Some participants did use this cognitive model spontaneously, while others eventually used it when it was suggested by the interviewer. During subsequent discussion of results, we will specify whether the participant was using the model spontaneously or after receiving a hint from the interviewer.

Table 4 Cognitive models for multiplication by a scalar used by students and mathematicians

Cognitive models	Brief description (applied to $\binom{n}{k} \times k$)	Example from the data
Equivalent groups	k copies of each k-committee	*Emery (considering $\binom{n}{k} \times k$)*: I guess what I'm really doing is I'm starting with n objects. I am taking one out, and then choosing $k - 1$ of what's left. *Int.*: Uh huh. *Emery*: And then I will get exactly—I will get repeats exactly k—each choice will be repeated exactly k times, and so that's why I'm getting n choose k times k.
Cartesian product	Coordinate pairs with $\binom{n}{k}$ and k ways to fill the positions	*Emery (considering $\binom{n}{k} \times k$)*: I could specify, okay so I guess I can think of it as counting pairs of the smallest element and the rest? Wait I'm choosing k. I can think of it as counting pairs where the first element in the pair is the smallest element and the second is a set of k-elements. *Int.*: And you think of picking that special element first, and then making the group of k? Or, what order is that being done in? *Emery*: What I was thinking a Cartesian product. It wouldn't necessarily be in order.
Scaling factor	Each k-committee is scaled by a factor of k	*Adrien (considering $15 \times \binom{14}{3}$)*: Well, because in that case it's... When I was reading it like this, the way I was reading it was like as was taking the combination. Its you're taking, you have 14, you choose three. I was thinking of scaling that number somehow.
Inverse of a probability	The multiplicative inverse of the solution to, "If there is a 1-in-k chance that a committee will form at all, what is the probability that a certain committee will form?"	*Riley (considering $15 \times \binom{14}{3}$)*: There is a 1 over 15 chance that a congressional committee will be formed. Given that probability and the fact that there are 14 candidates for the council and three positions, what is the multiplicative inverse of the probability that a given council will be selected? Because you have to overcome the probability that it won't happen at all.
Weight	$\binom{n}{k}$ counts the number of bit strings of length n with k 1 s, and these bit strings are assigned a weight k.	*Riley (considering $\binom{5}{k} \times k$)*: So the thing that's occurring to me is some kind of idea of like a weighted bit string. *Int.*: What do you mean by weighted bit string? *Riley*: Oh, something weird where like you say, "Okay, I have a bunch of five-bit integers. Something like, pick me ones with an associate like k 1 s, and then multiply the result of picking that by the number of 1 s," which is kind of weird. I'm trying to think of a more or like a less abstract example, because that's almost just kind of like a definitional like, "Well, you can choose ones."
Element selection	Interpreting k as $\binom{k}{1}$, that is, selecting one from k people after forming a k-committee	*Adrien (considering $\binom{n}{i} \times i$)*: So, you have a group of n people, and you're trying to select one of them in two stages specifically.

Table 4 (continued)

Cognitive models	Brief description (applied to $\binom{n}{k} \times k$)	Example from the data
		So, you have your first stage, where you just select some group of people—it doesn't matter how large it is—and then out of those candidates you then select the final one.

The Cartesian product cognitive model was exclusive to Emery (one of the mathematicians), and the scaling factor cognitive model was exclusive to Adrien (a student). The cognitive models inverse of a probability and weight were both exclusive to Riley (a student). Finally, at least some of the mathematicians (Emery, Skyler, Jaiden, and Robin) and at least one of the students (Riley) used the equivalent groups cognitive model, and every participant at some point used the element selection cognitive model.

We now offer a couple more comments about these cognitive models, and we elaborate some examples of participants' work that demonstrate these models in the following sections. Overall, there was a notable amount of variety of cognitive models for multiplication that the mathematicians and students leveraged while engaged in combinatorial proof. As can be seen from Table 4, the participants' multiplicative cognitive models were varied and differed from those used by Mulligan and Mitchelmore (1997). This aligns with previous findings (e.g., Lockwood et al. 2017; Lockwood and Purdy 2019a, b) showing that while multiplication is a familiar operation for undergraduates, its representations can vary when used in counting, and it can involve subtleties not always internalized by students and instructors. Additionally, while some of the multiplication models that students and mathematicians used overlapped with those identified as important to K-12 education and research (e.g., National Governors Association Center for Best Practices and Council of Chief State School Officers 2010; Mulligan and Mitchelmore 1997), several others did not. Perhaps this is not surprising since postsecondary students and mathematicians often use multiplication in more complicated and wide-ranging situations than K-12 students, but we nevertheless did not expect the variety that emerged in the data among the undergraduates and mathematicians.

Cognitive Models that were Productively Used for Combinatorial Proof

Only two of the previously-described cognitive models led to correct combinatorial proofs for the participants. We provide examples of these two cognitive models to demonstrate work that successfully implemented these ideas.

Element Selection as a Productive Cognitive Model

The element selection cognitive model of multiplication lends itself nicely to combinatorial proof of binomial identities, as it relates to the Multiplication Principle, a fundamental concept in combinatorics. To clarify, when we say the *element selection cognitive model of multiplication*, we mean that a counter interprets an instance of multiplication by a scalar, such as "$\times k$," as a stage in the Multiplication Principle with k

options. While the cognitive model was very useful for our participants and often applies to solutions of combinatorial-proof problems, surprisingly it often was not our participants' go-to cognitive model. In fact, for each of our student participants, at some point during the interviews they got stuck on a binomial identity involving multiplication, and a particular instantiation helped them to see that they could use the element selection cognitive model. Specifically, we reminded the students that they could represent k as $\binom{k}{1}$ in binomial identities involving multiplication by k. To illustrate how this instantiation seemed productive for students engaging in combinatorial proof, we turn to the work of one of the students, Adrien.

Adrien was working on the Reverse Counting Problem, which asks, "Write down a counting problem whose answer is $15 \times \binom{14}{3}$." After they entertained some ideas that were not productive, we suggested that they recall that $15 = \binom{15}{1}$. The moment we drew their attention to this fact, they immediately articulated that $15 \times \binom{14}{3}$ could count the number of ways to elect a club president and then a 3-person committee from a set of 15 club members. We were impressed that they could articulate a correct counting problem corresponding to $15 \times \binom{14}{3}$ so quickly, and we asked them if writing 15 as $\binom{15}{1}$ was helpful. They responded as follows.

> *Adrien*: So when I was reading it like this, the way I was reading it was as taking the combination—you have 14 you choose 3. **I was thinking of, like, scaling that number somehow.** So, I was still thinking of it in terms of that number. **But then, when you write it like this, it's like, oh, so you started with 15, took 1 specifically, so now it seems like a 2-step process**—like, two separate parts of the problem, rather than one part of the problem and, like, oh, how do I scale that?

We note that in the quote, Adrien said that they initially were using a scaling factor cognitive model of multiplication, which was not productive for them. However, once they thought of the multiplication by a scalar as element selection, prompted by representing 15 as $\binom{15}{1}$, Adrien could quickly and easily interpret the factors in 15 $\times \binom{14}{3}$ as representing two stages in the Multiplication Principle, and hence corresponding to a particular counting process. This is a critical step in constructing a combinatorial proof as argued by Lockwood et al. (in press).

Progressing through subsequent problems in our interviews, Adrien proved multiple binomial identities with the aid of the $k = \binom{k}{1}$ instantiation. These include one of the more challenging identities we gave students in the interviews, $\sum_{i=1}^{n} \binom{n}{i} i = n2^{n-1}$. When we first gave Adrien this identity, they began by writing down the identity with the substitution $n = 5$. Once Adrien did this, they then re-wrote the summation replacing i with $\binom{i}{1}$ and were able to recognize the terms in the summation as choosing i from a set of n distinct things, and then selecting one of those i chosen things (Figure 1.) Adrien then gave a nice combinatorial proof of the identity in the context of selecting a finalist from n people after two selection rounds:

> *Int.*: What would you say both sides are counting?
> *Adrien*: So, you have a group of n people, and you're trying to select one of them in two stages specifically. So, you have your first stage, where you just select

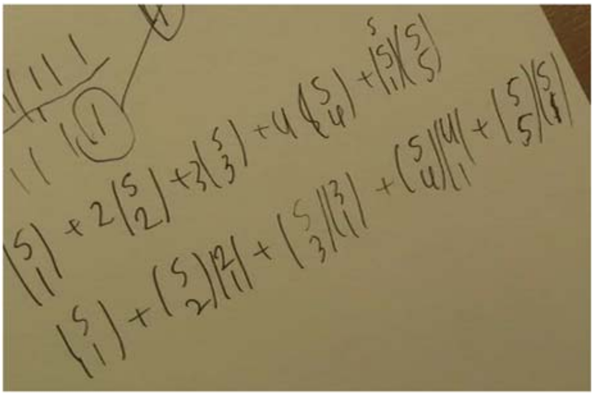

Fig. 1 Adrien's work proving $n2^{n-1} = \sum_{i=1}^{n}\binom{n}{i}i$

> some group of people—it doesn't matter how large it is—and then out of those candidates you then select the final one. And this [left-hand side] sort of does that in the opposite direction. It's, like, it counts, okay, who was the final one? And then who made it to the second round?

We see here again that using the element selection cognitive model was productive in Adrien's combinatorial proof activity.

These excerpts illustrate that 1) the element selection cognitive model of multiplication was productive for students solving combinatorial-proof problems, and 2) reminding students that $k = \binom{k}{1}$ is a useful instantiation that can help students to see that they can use this model. Although we only discussed Adrien's work in detail, again, this instantiation was helpful for all five of our student participants, and all eight of our mathematician participants commonly used element selection productively in their work as well.

A Combinatorial Proof Utilizing Repeated Addition and Equivalent Groups

The only other cognitive model (other than element selection) that the participants used successfully was equivalent groups. Only Emery, one of the mathematicians, successfully proved $\binom{\mathrm{n}}{\mathrm{k}} \times \mathrm{k} = \mathrm{n} \times \binom{\mathrm{n-1}}{\mathrm{k-1}}$ using equivalent groups. We describe parts of their work here, although we do not have space to include all the details.

When we first gave Emery the prompt to provide a combinatorial proof of the identity, they quickly saw that $\mathrm{n}\binom{\mathrm{n-1}}{\mathrm{k-1}}$ could be thought of as counting all the way to select one of n objects, and then choosing $k - 1$ of the remaining $n - 1$ objects, ultimately resulting in a subset of size k. Here Emery utilized the element selection cognitive model of multiplication, because to them this instance of multiplication by n represented a choice of one object out of n. Notice also that this counting process does not create distinct outcomes, that is, distinct subsets of size k from n distinct objects. If the objects are numbered 1 to n, consider for example the outcome $\{1, 2, \ldots, k\}$. This subset can be generated by first selecting the item '1' and then choosing $\{2, \ldots, k\}$, or by first selecting the item '2' and then choosing {1, …, *k*}. For each outcome, there are indeed *k* ways that outcome is generated by the process Emery articulated. We did not point this out in the moment during our interview with Emery, as we wanted to see how they would resolve this on their own. As Emery tried to continue the problem, they were unsure why $\binom{n}{k} \times k$ would count the same thing. When they reached this point and were stuck, the interviewer asked:

Int.: What might multiplying by *k* be doing?
Emery: Well, I know what it's doing in terms of the factorials.
Int.: Uh huh, right.
Emery: But in terms of the counting it's just doing it *k* times, right?
Int.: Mmhm. So you're thinking of this as like *k* copies of *n* choose *k*?
Emery: Yeah.

From this exchange, we infer that Emery conceived of the multiplication by *k* as generating *k* copies of whatever $\binom{n}{k}$ is counting, rather than thinking of the multiplication as picking one of the *k* objects chosen in the $\binom{n}{k}$ step. Because we felt that making a combinatorial proof involving a multiset (rather than a set where all the outcomes are distinct) would be challenging, we decided to try to direct Emery back to the element selection cognitive model. To do this, we asked if they could conceive of the multiplication by *k* as picking one of the *k* objects (chosen from *n*) to be special in some way. We thought that then Emery might see that on the right-hand side of the identity, they could think of first designating one of the *n* items to be special and then apply the Multiplication Principle to form the rest of the *k*-group around the special element. (Hence, both sides count the number of groups of size *k* with a specially designated element.) However, while Emery said they could conceive of the multiplication by *k* in that manner, they continued to struggle with the problem. Even after suggesting that they try using committees (a more concrete context than sets and subsets) to solve the problem, Emery still did not make progress on counting *k*-subsets (or *k*-committees) with a specially designated element.

Finally, we encouraged Emery to revisit their conception of $n \times \binom{n-1}{k-1}$ as counting the number of ways to make a *k*-subset from *n* things by picking a first object and then $k-1$ objects to round out the rest of the subset. As mentioned previously, the set of outcomes generated by this process (Lockwood 2013) contains duplicates. We asked Emery if they could write down anything that would represent this idea, and then asked why this process (picking one, then $k-1$ objects) would generate *k* copies of every *k*-subset. After a little more thought, they realized that, for instance, the subset {1, 2, …, *k*} would be generated exactly *k* times by this process—once for each of the 1, 2, …, *k* objects chosen first (see Fig. 2). Highlighting the fact that counting a set with duplicates may have been easier for Emery from the start, they said the following:

Emery: I didn't count how many ways I was double-counting. That's the problem. If I had done that I probably would have been done. If I had actually figured out exactly how much I'm double-counting, then I'd be done. Because if I knew I was doing that *k* times I'd have the *k*.

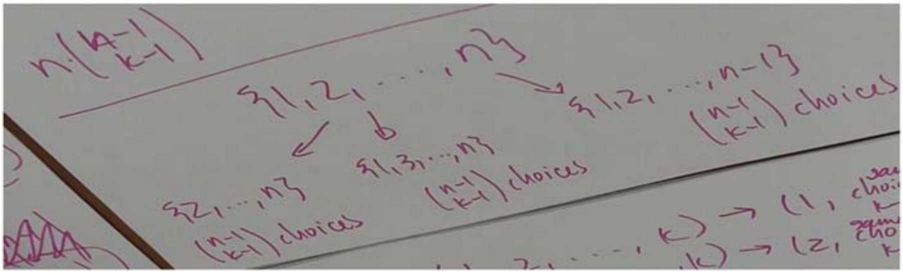

Fig. 2 Emery's work proving $\binom{n}{k}k = n\binom{n-1}{k-1}$ using repeated addition and equivalent groups

Emery summarized their final combinatorial proof in the following exchange:

Int.: So maybe, would you mind just summarizing for me what then is your combinatorial argument for why the identity holds?
Emery: Okay. So my combinatorial argument for why the identity holds is—now I have to think all the way back—to interpret the right-hand side as a choice of k objects from n objects where the way I'm getting that choice is by taking $k-1$ from $n-1$ and plugging one of n that wasn't there.
Int.: Uh huh.
Emery: Well, I know what it's doing in terms of the factorials.
Int.: Uh huh, right.
Emery: I guess what I'm really doing is I'm starting with n objects. I am taking one out, and then choosing $k-1$ of what's left.
Int.: Uh huh.
Emery: And then I will get exactly—I will get repeats exactly k—each choice will be repeated exactly k times, and so that's why I'm getting n choose k times k.

To summarize their work, while Emery initially used the element selection cognitive model when interpreting the multiplication by n on the right-hand side of the equation, they additionally used the equivalent groups cognitive model to successfully complete the combinatorial proof. For the left-hand side, rather than conceiving of the multiplication by k as selecting one out of k objects (that is, using element selection) they could interpret the multiplication by k on the left-hand side of the equation as making k copies each of all possible subsets of size k, and then they argued why the process they articulated on the right-hand side (choosing 1 out of n objects and then $k-1$ out of the remaining $n-1$ objects) generates the same (multi)set of outcomes.

While this combinatorial proof is correct, we hypothesize that constructing an argument that enumerates a multiset with duplicate objects may be challenging for students. It is not trivial to see that $n \times \binom{n-1}{k-1}$ counts a collection of size-k subsets each with k copies, and generally combination problems that allow for repetition can be more difficult for students than those that do not allow repetition. Indeed, in our interviews with the upper-division mathematics students, none of their combinatorial proof attempts using an equivalent groups cognitive model of multiplication were successful, and ultimately the student participants (and every mathematician participant except Emery) only used the element selection cognitive model of multiplication to successfully prove binomial identities combinatorially.

To summarize, only two of the multiplicative cognitive models we encountered in the data actually led to successful combinatorial proof attempts: equivalent groups and element selection. However, our findings also suggest that these two cognitive models may not be equally useful. Only one participant—one of the mathematicians—was able to use the equivalent groups cognitive model successfully, while all eight of the mathematicians and all five of the undergraduate student participants eventually used the element selection cognitive model productively.

Instances of Cognitive Models Not Being Used Productively

Four other cognitive models emerged during the interviews with students and mathematicians, and none of them were used productively in helping the participants correctly prove combinatorial identities. In this section, we briefly describe an episode in which a certain cognitive model arose but was not ultimately productive for the successful completion of a combinatorial proof. Our goal in this section is not to criticize the participants, but rather to illustrate that cognitive models are important, underscoring the idea that some cognitive models may be more productive than others in thinking about proving combinatorial identities.

Weight as a Cognitive Model that was not Productive

Another cognitive model of multiplication that occurred in our data was multiplication as a weight. Here, we define the weight cognitive model as when a counter conceives of the multiplication as assigning a weight to objects being counted. To illustrate the weight cognitive model in our data, we show Riley's (a student participant) work on proving $\sum_{i=1}^{n}\binom{n}{i}\times i=n\times 2^{n-1}$ combinatorially.

Initially, Riley struggled with this problem, and so we encouraged them to consider the case $n = 5$. This intervention of encouraging students to consider a particular case of n was found to be helpful in Lockwood et al. (in press), and so we felt it could help Riley to make some progress on the problem. One benefit of considering a specific case when proving a binomial identity involving a summation is that it allowed Riley to write out all the terms of the summation. Riley did this and then expressed that the sum could be counting weighted bit strings. This bit strings context was one that Riley used to prove several binomial identities throughout the interviews, but the idea of weights was unique to this problem. (Figure 3).

> *Riley*: So the thing that's occurring to me is some kind of idea of like a weighted bit string.
> *Int.*: What do you mean by weighted bit string?
> *Riley*: Oh, something weird where like you say, "Okay, I have a bunch of five-bit integers. Something like, pick me ones with an associate like k 1s, and then multiply the result of picking that by the number of 1s," which is kind of weird.

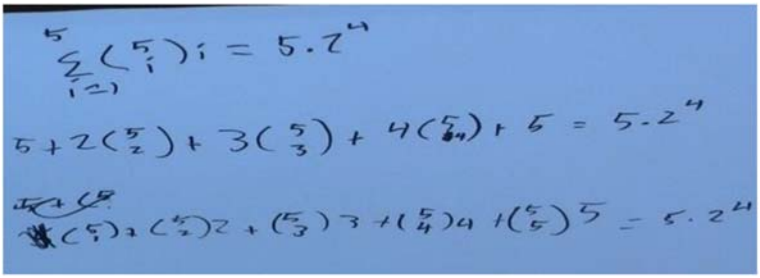

Fig. 3 Riley's work considering $\sum_{i=1}^{n}\binom{n}{i}\times i = n\times 2^{n-1}$ in the case where $n = 5$

> I'm trying to think of a more or like a less abstract example, because that's almost just kind of like a definitional like, "Well, you can choose ones."

After spending some more time thinking about this context, Riley did not come up with a way that the right-hand side $n \times 2^{n-1}$ could count the same weighted bit strings. This is important, because it suggests that the issue was not that the weighted sum cognitive model of the sum of $\binom{n}{i} \times i$ is necessarily incorrect, but rather that Riley could then not connect it to the other side of the binomial identity. A bit later, Riley articulated thinking about the scalar terms as weights, this time in terms of money. Again, they could think of the left-hand side as a weighted sum, but they struggled to make sense of the right-hand side in terms of these weights.

> *Riley:* I'm thinking now in terms of like the weight of these different sets, and this five to me is a matter of the weight of this set. Oh, okay. So now I'm feeling like I'm starting to maybe get somewhere, because you can say, "Okay, well what if … what if we gave the maximum number of dollars this \$5 out, in some different distribution?" So, like what if, essentially instead of … Yeah, what if instead of giving all these people this whole combination of dollars, we instead eliminate one person from the group, and do every other combination of them, such that somebody gets \$2 and everyone else gets one. So this is, one, \$2 … Or I guess, yeah, two, \$2 since we've eliminated someone from the group. And then two, or wait, no, sorry. That's right, \$2 for \$1. No, no. Oh, sorry, yeah, three. Three, \$1.
> *Int.:* So what does the five represent in terms of like the \$1 and the \$2?
> *Riley:* The weight of the set, in other words the number of dollars being distributed. So over here we're thinking of just like a bit string, but like over here we're saying, "Okay, well I have four people and you're allowed to give one of them more than \$1," although now that I'm thinking more about it, there's two to the four that I've been playing around with is pretty dependent on the idea of a bit string. So I might've done something wrong in translating that idea.

Here, Riley was still considering the case where $n = 5$. Note that this context of distributing money could lend itself to a correct combinatorial proof.[6] However, we

[6] In particular, both sides of the identity count the number of ways to distribute money to a group of n people where one person gets \$2 and everyone else can receive either \$1 or \$0. We omit the details due to space constraints.

interpreted Riley's utterances to mean that they conceived of the $\times i$ in the identity as a weight, where the value of the weight was equal to the number of dollars distributed. Ultimately, Riley was not able to make sense of the right-hand side $n \times 2^{n-1}$ using weights, and so their attempts to prove the identity combinatorially using the weight cognitive model of multiplication were not successful.

This episode illustrates a couple of ways that Riley tried to prove $\sum_{i=1}^{n}\binom{n}{i} \times i = n \times 2^{n-1}$ using the weight cognitive model for multiplication. By showing these excerpts, our intention is not to criticize Riley's thinking and work. It is understandable that they tried to use this cognitive model, since weighted sums are often useful when solving problems in upper-division mathematics (such as probability), but in this instance the weight cognitive model of multiplication was not productive for Riley. To us, this highlights the fact that counters at the university level have seen and used multiplication in a wide variety of situations, and so when they are faced with a combinatorial task, they bring with them a variety of multiplicative cognitive models that may affect their combinatorial reasoning. It also reinforces the idea that the particular multiplicative model that one uses matters, and some cognitive models may not be as effective in supporting reasoning about combinatorial proof.

Conflating Models of Multiplication and Exponentiation

Finally, we turn our attention to an interpretation of multiplication that occurred a few times in interviews both with students and with mathematicians. We did not classify these interpretations as multiplication cognitive models, because these instead exemplify models of exponentiation.

We first consider Riley's work on the Reverse Counting problem, interpreting $15 \times \binom{14}{3}$ as the solution to a counting problem to be relevant to our research questions about combinatorial proof. When we first gave this problem to Riley, they said, "there are, let's say 15 congressional committees, each should be sized of three. Um, and there are 14 candidates. For each county council and a membership in one council doesn't preclude membership in another council.[7]" See Figs. 4 and 5. After we asked Riley to talk more about the outcomes of their counting problem, we had the following exchange.

> *Riley:* So, um, yeah, so these are committees, um, where like, you know, J uh, Jean, Tim and Bob are, uh, and the fact that they're on two different committees in the exact same combination isn't significant to us in this question….
> *Int.*: And what do the one and two represent? They represent different committees, different committees they could be one?
> *Riley*: Yeah, so this would be dot, dot, dot 15.
> *Int.*: Okay.
> *Riley*: Um, and maybe these congresspeople are super popular, so they're on every single one, but, okay.
> *Int.*: So what exactly…do the outcomes look like for this counting problem? Like, could you give a couple of example outcomes.

[7] During the interviews, Riley used the terms "committee" and "council" interchangeably and meant them to be synonymous.

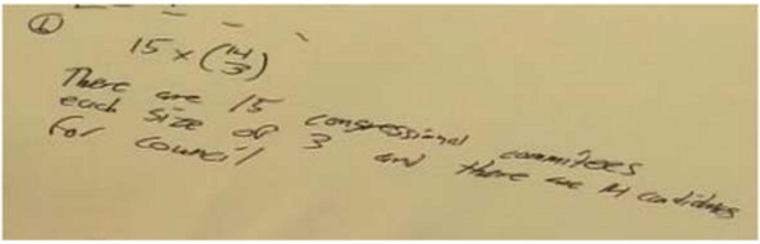

Fig. 4 Riley's solution to the Reverse Counting Problem: "There are 15 congressional committees each size of 3 and there are 14 candidates for council"

Riley: Um, in terms of like just individual councils or like–?
Int.: Yeah, or if you—like is this an example outcome or do you also have to like tack on people from other committees?
Riley: I see what you're saying. I think, uh, yeah, so this entire thing is one set.
Int.: Okay. That whole thing is an outcome?
Riley: Yeah.

We interpret Riley's utterances and work on the paper (in Figs. 4 and 5) to mean that they thought of carrying out a counting process where a committee of size three is chosen from a set of fourteen people 15 times (without a person's position on one committee precluding their position on another committee). Indeed, Riley articulated correctly in the quotes we provided that this process would create outcomes that are ordered 15-tuples where each element is a committee of people. Interpreting multiplication as repeating a process several times came up not only in interviews with students, but with mathematicians as well, and it is an easy error to make. We note, however, that this process and the resulting outcomes are not a model of multiplication, but of exponentiation, and the solution to Riley's counting problem would be $\binom{14}{3}^{15}$.

Another participant who tried to represent multiplication with an exponentiation model was Skyler, a mathematician. In particular, when we gave Skyler the combinatorial identity $\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$, they said the following and represented their thought process in the case where $n = 7$ and $k = 3$ (Fig. 6).

Skyler: I have like a matrix of boxes. So, the *n* choose *k* times *k*, there's now *n* columns and *k* rows. And so, each row is now *n* boxes I want to put *k* things into. And now, it's how many ways can I fill up that entire grid with making sure

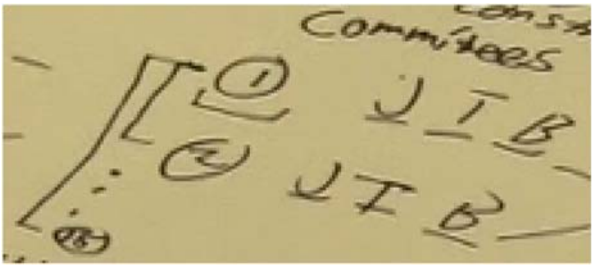

Fig. 5 Riley represents their outcomes for the Reverse Counting Problem

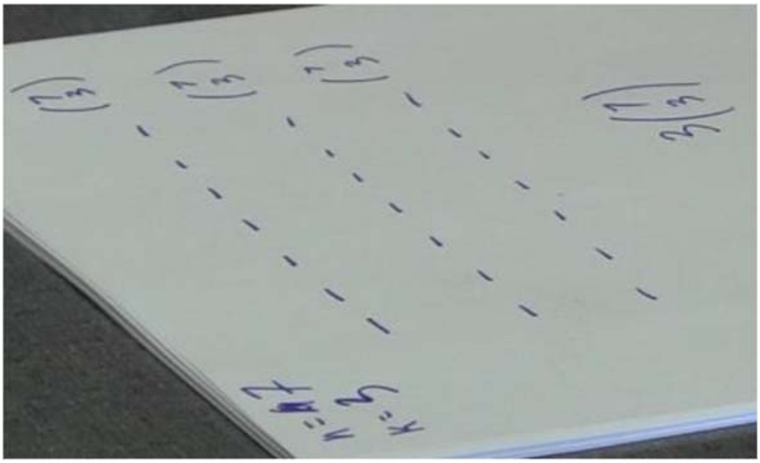

Fig. 6 Skyler represents their thought process for proving $\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$ in the case where $n = 7$ and $k = 3$

> there's k things in each row. So, the more I think about this, it makes me think like Sudoku kind of matrix.

This interpretation of multiplication is interesting and may be related to Mulligan and Mitchelmore's (1997) *array* semantic structure. However, a key difference here is that in order for multiplication to be faithfully represented as an array, each column (or row) must be identical. The interpretation that Skyler articulated where each row may have k things placed in different locations (like in a Sudoku puzzle) would not be a model for multiplication but for exponentiation. Indeed, the number of ways to fill an $n \times k$ array with k things in each row would be $\binom{n}{k}^k$.

Again, the purpose of discussing these episodes is not just to point out that Riley and Skyler's work was wrong (and both overall were very successful at tackling the combinatorial-proof tasks throughout the interviews). Rather, it is noteworthy to see why their errors occurred, and in particular we want to point out how easy it can be for counters—even experts who conduct mathematical research—to conflate situations that involve multiplicative or exponential structures. This corroborates findings from previous research (such as Batanero et al. 1997; Kavousian 2008; and Sowder et al. 1998) which show that counters can struggle to correctly use multiplication when doing combinatorial tasks. The subtle differences between combinatorial situations involving multiplication and exponentiation is a topic that should be discussed in combinatorics classrooms, since, as we have shown, even experts can conflate the two.

Discussion and Conclusions

Here we provide a summary of the results of our study, framed by our research questions. We also discuss implications of this work for teaching practice and future research. Our first research question was, *What cognitive models do undergraduate* *students and mathematicians use when engaging in combinatorial proof of identities involving multiplication by a scalar?* We found our participants used two (equivalent groups and Cartesian product) of Mulligan and Mitchelmore's (1997) five semantic structures as cognitive models when interpreting binomial identities involving multiplication by a scalar. In addition, scaling factor, weight, inverse of a probability, and element selection also emerged from the data as cognitive models for multiplication that our participants used. Equivalent groups and, mainly, element selection were the only multiplicative cognitive models that were used in successful combinatorial proofs by participants. This makes sense, since element selection involves a person interpreting multiplication by a scalar as a stage in the Multiplication Principle, a fundamental counting concept that is taught in nearly all university-level courses that cover counting. Only one of the mathematicians successfully articulated a combinatorial proof using the equivalent groups cognitive model for multiplication, though a couple of the mathematicians and students also made unsuccessful attempts to do this. The multiplicative cognitive models used by our participants were more varied than we had expected, supporting the finding from previous studies that multiplication is a nuanced operation and can be challenging to apply in combinatorics (e.g., Batanero et al. 1997; Kavousian 2008; Lockwood and Purdy 2019a, b, and Sowder et al. 1998). This suggests that upper-division students and mathematicians have a more varied and nuanced understanding of multiplication in combinatorial contexts than previously thought, and the variety in our data suggest that this could be an interesting avenue for further research.

Our second research question was, *What are the implications of these cognitive models for students' engagement with combinatorial proof?* Because not a single student and only one mathematician successfully used the equivalent groups cognitive model in their combinatorial argument, this suggests that some students may find this approach to be challenging when trying to come up with a combinatorial proof. The element selection cognitive model for multiplication was used in all the students' successful combinatorial proof attempts and nearly all of those of the mathematicians, which is understandable given the connection to the Multiplication Principle. However, we also saw that not all of the students spontaneously thought to use this multiplicative cognitive model. Indeed, many of them only saw that they could successfully use element selection as a cognitive model for multiplication after we pointed out they could reconceive of multiplication by k as multiplication by $\binom{k}{1}$. We hypothesize that this reformulation is helpful for students because it enables them to more easily recognize the multiplication by a scalar as a stage in a two-step counting process, and indeed we have some data (such as Adrien's work on the Reverse Counting Problem) that seems to support this hypothesis.

The fact that the upper-division mathematics students in this study (who had all taken discrete mathematics) frequently did not think to apply the Multiplication Principle was curious, and it corroborates previous work that suggests undergraduate students—even upper-division mathematics students—do not always recognize situations where multiplication is used while counting (e.g., by Lockwood and Purdy 2019a, b). Upper-division mathematics does frequently use multiplication in a variety of contexts. For example, students that are writing proofs in advanced calculus often have to "scale" epsilon to find delta and prove something is continuous, a limit, etc., or it might arise with weighted sums and probability. However, none of these cognitive models for multiplication involved in these contexts (scaling factor, inverse of probability, and weight) were helpful when our participants were attempting to engage in combinatorial proof. This suggests that combinatorics instructors, rather than assuming that undergraduate students know when to multiply and what multiplication does when counting, should have discussions with their students about multiplication and highlight that it can be used to count the number of ways to complete a two-stage process using the Multiplication Principle.

In conclusion, multiplication is a familiar operation for undergraduate students, and yet we have shown that the particular ways in which they reason about it may have implications for their success at combinatorial proof. Combinatorial proof and similar types of problems are a context where the subtleties of multiplication emerge, and we see that it is not a trivial topic, even for upper-division mathematics students. While much work has been done examining the ways that K-12 students reason about multiplication (e.g., Greer 1992; Mulligan and Mitchelmore 1997; Tillema 2013), our study indicates that examining undergraduate students' conceptions of multiplication in combinatorial contexts may be fruitful, as these conceptions are varied and have implications for their combinatorial proof activity.

Future avenues of research include continuing to explore the cognitive models of multiplication students and mathematicians may use in combinatorial contexts, and a natural extension of our research would be to investigate cognitive models of other mathematical operations as well (or other instances of multiplication besides multiplication of a scalar). Furthermore, our results also suggest that when researchers consider multiplication in combinatorial settings in the future, they should attend to different cognitive models that participants might be using as they multiply. It is perhaps best not to assume that participants are adopting any particular cognitive model, but rather explicit attention should be paid to how people might be reasoning about multiplication as they engage with combinatorial tasks. Finally, given that distinguishing between multiplication and exponentiation arose for multiple participants in our study, future work could seek to better understand the conflation of multiplication and exponentiation, potentially investigating ways to help students differentiate between the two operations in combinatorial settings.

Finally, this work suggests that from a pedagogical perspective, some cognitive models of multiplication seem to be more productive for students engaging in combinatorial proof activity than others. Encouraging students to think of multiplication by k as $\binom{k}{1}$ if they are stuck may help them more easily see the binomial identity they are working with as corresponding to an underlying counting process that uses the Multiplication Principle (Lockwood 2013). We do not necessarily advocate that instructors should present all binomial identities involving multiplication by k as $\binom{k}{1}$ in all instances. Students who engage in combinatorial proof will eventually encounter expressions that involve multiplication by a scalar, and they will have to interpret multiplication by k as element selection, even it is not presented with the binomial coefficient $\binom{k}{1}$. As a short-term instructional intervention, however, we found that rewriting multiplication by a scalar in this way was helpful in encouraging students to understand and use element selection.

Acknowledgments This paper is based upon work supported by the National Science Foundation under Grant No. 1650943.

Compliance with Ethical Standards

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix 1. Mathematician Interview Protocol

1. Prior experience with combinatorial proof

   First we asked the following questions:

- What is your research area? How long have you been conducting mathematics research?
- Do you ever use combinatorial proof in your research? How important is combinatorial proof in your field?
- Do you ever teach classes that cover combinatorial proof of binomial identities? How frequently? When did you teach combinatorial proof most recently?

2. Combinatorial proof

   Next, we gave the expert a subset of the following binomial identities (one at a time) and ask them to provide a combinatorial proof.

Table 5 Identities given to the mathematicians to provide a combinatorial argument

$\binom{n}{k} = \binom{n}{n-k}$	$2^n = \sum_{i=0}^{n} \binom{n}{i}$
$\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$	$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
$\sum_{i=1}^{n} \binom{n}{i} i = n \cdot 2^{n-1}$	$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$
$\sum_{i \geq 0} \binom{n}{2i} = \sum_{i \geq 0} \binom{n}{2i+1}$	$\sum_{i=k}^{n} \binom{i}{k} = \binom{n=1}{k=1}$
$\frac{n}{k} \binom{n-1}{k-1} = \binom{n}{k}$	$\frac{n+1-k}{k} \binom{n}{k-1} = \binom{n}{k}$

Appendix 2. Student Selection Interview Protocol

1. Solving counting problems.

   We gave the student the following counting problems, one at a time.

   Domino Problem. *A domino is a small, thin rectangular tile that has dots on one of its broad faces. That face is split into two halves, and there can be zero through six dots on each of those halves. Suppose you want to make a set of dominos (i.e.,* *include every possible domino). How many distinguishable dominos would you make for a complete set?*[8]

Committees Problem. *A university department has 30 faculty members.*

a) *How many ways could a 5-member hiring committee be formed?*
b) *How many ways could a 5-member hiring committee be formed if one of the committee members must be the chairperson?*
c) *In the university department, 17 faculty members are professors and 13 are instructors. How many ways could a 5-member hiring committee be formed if the committee must consist of 3 professors and 2 instructors? (The committee won't have a chairperson.)*

Power Set Problem. *Let S be a set containing 5 (distinct) elements. How many subsets are there of the set S? (That is, what is the cardinality of P(S), the power set of S?)*

Binary Strings Problem. *A binary string is a finite sequence containing only 1s and 0s.*

a) *How many binary strings of length 8 contain exactly 5 0's?*
b) *How many binary strings of length n contain exactly k 0's?*

2. Writing basic proofs.

Next, we asked the students to prove some or all of the following theorems.

Theorem 1. *The sum of two even integers is an even integer.*
Theorem 2. *Let* n *be a nonnegative integer. Then,*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Theorem 3.*Let n and k be nonnegative integers such that n ≥ k. Then,*

$$\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$$

3. Additional Questions

We asked the students the following questions:

[8] This problem is used with permission from (Lockwood, Swinyard, Caughman, 2015).

- What are the things that come to mind for you when you see (n choose k) (e.g. do you think only about its formula? Does it make you think of a class of or specific counting problem(s)? Do you think of Pascal's Triangle?)
- What is your major, and what year are you (freshman, sophomore, junior, senior)?
- What math classes are you currently taking? What other math classes have you taken in college?

Appendix 3. Student Interview Protocol

1 Solving more counting problems and finding bijections.

We asked students to solve counting problems involving combinations. As they solved these, we asked the students to articulate what each of these things mean regarding sets of outcomes, and we asked how they know when to use addition versus multiplication, or subtraction versus division. We asked them to solve all of the following counting problems. We also asked them to list outcomes and to create explicit bijections between outcomes.

Table 6 Combinatorial tasks for students to scaffold combinatorial proof

Task	Intended Purpose
1. Spoonbill Problem. The scientific name of the roseate spoonbill (a species of large, wading bird) is *Platalea ajaja*. How many arrangements are there of the letters in the word AJAJA? Can you list all of the outcomes?	Ensure students are familiar (or to familiarize them) with combination problems involving ordered sequences of two indistinguishable objects. Encourage students to use a set-oriented perspective (Lockwood 2014) when counting.
2. Subsets Problem. How many 3-element subsets are there of the set {1, 2, 3, 4, 5}? Can you list all of the outcomes?	Ensure students are familiar (or to familiarize them) with combination problems involving unordered selections of distinguishable objects. Encourage students to use a set-oriented perspective (Lockwood 2014) when counting.
3. Find-a-Bijection Problem. Describe a bijection between the outcomes in the Spoonbill Problem and the Subsets Problem.	Facilitate a robust, flexible understanding of combinations. Lay groundwork for students to solve bijective combinatorial-proof problems[a].
4. Even- and Odd-Sized Sets Problem. Let S = {1, 2, 3, 4, 5, 6}. (a) List all of the even-sized subsets of S. How many should there be? (b) List all of the odd-sized subsets of S. How many should there be? (c) Find a bijection between the subsets in parts (a) and (b) by considering whether the subsets contain the item 1.	Continue to facilitate a solid understanding of combinations. Provide scaffolding for students to eventually prove the identity $\sum_{i\geq 0}\binom{n}{2i} = \sum_{i\geq 0}\binom{n}{2i+1}$ using a bijective combinatorial proof.
5. Reverse Counting Problem. (a) Write down a counting problem whose answer is 2^5. (b) Write down a counting problem whose answer is $15 \times \binom{14}{3}$.	Provide scaffolding for the concept of a combinatorial proof by asking students to interpret expressions in a combinatorial context.

[a] These problems were part of the interview protocol for a larger study aimed at investigating students' understandings of combinatorial proof, including bijective proofs. We did not include information about the students' work finding bijective combinatorial proofs in this paper due to space constraints

2. Write combinatorial proof

In this section, we asked the students to justify why some or all of the following binomial identities hold by coming up with a counting problem that each side of the identity enumerates:

$$\binom{n}{k}\cdot\binom{k}{r}=\binom{n}{r}\cdot\binom{n-r}{k-r}$$

$$\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$$

$$\sum_{i=1}^{n}\binom{n}{i}\cdot i=n\cdot 2^{n-1}$$

During this section, we asked students some or all of the following questions about the identities and their reasoning about them:

- What could this be counting?
- What if you tried plugging in specific numbers for n, k, or r?
- How are you thinking of the multiplication/addition in that identity?
- Why might you multiply/add?

3. Write more combinatorial proofs

   In this section, we gave the students some or all of the following more challenging binomial identities to prove and observe their activity. Again, we encouraged the utilization of specific numbers and explicitly asking what an expression might be counting if they get stuck, following Lockwood et al. (in press).

Table 7 Additional identities given to the students to provide a combinatorial argument

$\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}=\binom{2n}{n}$	$\sum_{i\geq 0}\binom{n}{2i}=\sum_{i\geq 0}\binom{n}{2i+1}$
$\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}$	$\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}=\binom{m+n}{k}$
$\frac{n}{k}\binom{n-1}{k-1}=\binom{n}{k}$	$\frac{n+1-k}{k}\binom{n}{k-1}=\binom{n}{k}$
$\sum_{i=k}^{n}\binom{i}{k}=\binom{n+1}{k+1}$	$\sum_{i=0}^{n}\binom{n}{i}\cdot 2^i=3^n$

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