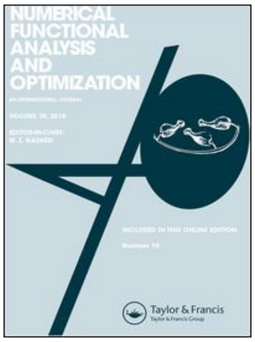

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A One Dimensional Elliptic Distributed Optimal Control Problem with Pointwise Derivative Constraints

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ABSTRACT

We consider a one dimensional elliptic distributed optimal control problem with pointwise constraints on the derivative of the state. By exploiting the variational inequality satisfied by the derivative of the optimal state, we obtain higher regularity for the optimal state under appropriate assumptions on the data. We also solve the optimal control problem as a fourth order variational inequality by a C^1 finite element method, and present the error analysis together with numerical results.

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1. Introduction

Let I be the interval $(-1, 1)$ and the function $J : L_2(I) \times L_2(I) \to \mathbb{R}$ be defined by

$$J(y,u) = \frac{1}{2}\left(\|y - y_d\|^2_{L_2(I)} + \beta\|u\|^2_{L_2(I)}\right), \tag{1.1}$$

where $y_d \in L_2(I)$ and β is a positive constant.

The optimal control problem is to

$$\text{find} \quad (\bar{y}, \bar{u}) = \mathop{\arg\min}_{(y,u)\in\mathbb{K}} J(y,u), \tag{1.2}$$

where $(y,u) \in H^1_0(I) \times L_2(I)$ belongs to $\mathbb{K}$ if and only if

$$\int_I y'z'\,dx = \int_I (u+f)z\;dx \qquad \forall z \in H^1_0(I), \tag{1.3}$$

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$$y' \le \psi \qquad \text{a.e. on } I. \tag{1.4}$$

We assume that

$$f \in H^1(I), \;\; \psi \in H^2(I) \tag{1.5}$$

and

$$\int_I \psi \; dx > 0. \tag{1.6}$$

Remark 1.1. The optimal control problem defined by Equations (1.1)–(1.4) is a one dimensional analog of the optimal control problems considered in [1–5]. It was solved by a C^1 finite element method in [6] under the assumptions that

$$f \in H^{\frac{1}{2}-\epsilon}(I) \quad \text{and} \quad \psi \in H^{\frac{3}{2}-\epsilon}(I). \tag{1.7}$$

Since the constraint Eq. (1.3) implies $y \in H^2(I)$ by elliptic regularity, we can reformulate the optimization problem Eqs. (1.1)–(1.4) as follows:

$$\text{Find} \quad \bar{y} = \underset{y \in K}{\operatorname{argmin}} \frac{1}{2}\left(\|y - y_d\|^2_{L_2(I)} + \beta\|y'' + f\|^2_{L_2(I)}\right), \tag{1.8}$$

where

$$K = \left\{y \in H^2(I) \cap H^1_0(I) : \; y' \le \psi \text{ on } I\right\}. \tag{1.9}$$

According to the standard theory [7, 8], the minimization problem defined by Eqs. (1.8)–(1.9) has a unique solution characterized by the fourth order variational inequality

$$\beta\int_I (\bar{y}'' + f)\left(y'' - \bar{y}''\right) dx + \int_I (\bar{y} - y_d)(y - \bar{y}) dx \ge 0 \qquad \forall y \in K,$$

which can also be written as

$$a(\bar{y}, y - \bar{y}) \ge \int_I y_d(y - \bar{y}) dx - \beta\int_I f\left(y'' - \bar{y}''\right) dx \qquad \forall y \in K, \tag{1.10}$$

where

$$a(y, z) = \beta\int_I y'' z'' dx + \int_I yz \; dx. \tag{1.11}$$

Remark 1.2. The reformulation of state constraint optimal control problems as fourth order variational inequalities was discussed in [9], and a nonconforming finite element based on this idea was introduced in [10]. Other finite element methods can be found in [11–17].

Remark 1.3. Note that Eq. (1.4) implies

$$\int_I \psi \; dx \geq \int_I y' \; dx = 0 \qquad \forall y \in K$$

and hence $\int_I \psi \; dx \geq 0$ is a necessary condition for K to be nonempty. It is also a sufficient condition because the function y defined by

$$y(x) = \int_{-1}^{x} \left(\psi(t) - \bar{\psi}\right) dt$$

belongs to K, where $\bar{\psi}$ is the mean of ψ over I. Furthermore,

$$0 = \int_I \psi \; dx = \int_I \left(\psi - y'\right) dx$$

together with Eq. (1.4) implies $\psi = y'$ identically on I and hence $K = \{\psi\}$ is a singleton. Therefore we impose the condition Eq. (1.6) to ensure that the optimization problem defined by Eqs. (1.8)–(1.9) is nontrivial.

Our goal is to show that $\bar{y} \in H^3(I)$ under the assumptions in Eq. (1.5) and consequently Eq. (1.8)/(1.10) can be solved by a C^1 finite element method with $O(h)$ convergence in the energy norm. Note that previously $\bar{y} \in H^{\frac{5}{2}-\epsilon}$ was the best regularity result in the literature for Dirichlet elliptic distributed optimal control problems on smooth/convex domains with pointwise constraints on the gradient of the state.

The rest of the paper is organized as follows. The H^3 regularity of $\bar{y}$ is obtained in Section 2 through a variational inequality for $\bar{y}'$ that can be interpreted as a Neumann obstacle problem for the Laplace operator. The C^1 finite element method for Eq. (1.8)/(1.10) is analyzed in Section 3, followed by numerical results in Section 4. We end with some remarks on the extension to higher dimensions in Section 5.

2. A variational inequality for $\bar{y}'$

Observe that the set $\left\{y' : \; y \in K\right\}$ is the subset $\mathscr{K}$ of $H^1(I)$ given by

$$\mathscr{K} = \left\{v \in H^1(I) : \; \int_I v \; dx = 0 \quad \text{and} \quad v \leq \psi \text{ on } I\right\}, \tag{2.1}$$

and the variational inequality Eq. (1.10) is equivalent to

$$\int_I \left(\Phi - f'\right)(q - p) dx + \int_I p'\left(q' - p'\right) dx \\ + \left[f(1)\left(q(1) - p(1)\right) - f(-1)\left(q(-1) - p(-1)\right)\right] \geq 0 \qquad \forall q \in \mathscr{K}, \tag{2.2}$$

where $p = \bar{y}', \;\; q = y',$ and $\Phi \in H^1(I)$ is determined by

$$\beta\Phi' = y_d - \bar{y} \tag{2.3}$$

and

$$\int_I \Phi \, dx = 0. \tag{2.4}$$

Moreover Eq. (2.2) is the variational inequality that characterizes the solution of the following minimization problem:

$$\text{Find } p = \underset{q\in\mathscr{K}}{\text{argmin}} \left[\frac{1}{2}\int_I (q')^2 dx + \int_I (\Phi - f')q \, dx + f(1)q(1) - f(-1)q(-1)\right]. \tag{2.5}$$

2.1. A Neumann obstacle problem

The minimization problem Eq. (2.5), which is a Neumann obstacle problem, can be written more conveniently as

$$p = \underset{q\in\mathscr{K}}{\text{argmin}} \left[\frac{1}{2} b(q,q) + (\phi, q) + \tau q(1) - \sigma q(-1)\right], \tag{2.6}$$

where $\sigma = f(-1)$, $\tau = f(1)$,

$$b(q,r) = \int_I q'r' \, dx, \quad (\phi, q) = \int_I \phi \, q \, dx \quad \text{and} \quad \phi = \Phi - f'. \tag{2.7}$$

Note that we have a compatibility condition

$$\int_I \phi \, dx + \tau - \sigma = 0 \tag{2.8}$$

that follows from Eqs. (1.5), (2.4) and the Fundamental Theorem of Calculus for absolutely continuous functions.

Since $b(\cdot,\cdot)$ is coercive on $H^1(I)/\mathbb{R}$, the obstacle problem defined by Eqs. (2.1) and (2.6) has a unique solution p characterized by the variational inequality

$$b(p, q-p) + (\phi, q-p) + \tau\big(q(1) - p(1)\big) - \sigma\big(q(-1) - p(-1)\big) \geq 0 \quad \forall q \in \mathscr{K}. \tag{2.9}$$

Theorem 2.1. *The solution $p = \bar{y}' \in \mathscr{K}$ of (2.6)/(2.9) belongs to $H^2(I)$.*

Proof. We begin by observing that

$$b(p, q-p) + (\phi, q-p) + \tau\big(q(1) - p(1)\big) - \sigma\big(q(-1) - p(-1)\big) \geq 0 \quad \forall q \in \tilde{K}, \tag{2.10}$$

where

$$\tilde{K} = \left\{ q \in H^1(I) : \ q \leq \psi \ \text{ in } I \ \text{ and } \int_I q \ dx \geq 0 \right\}. \tag{2.11}$$

Indeed, $q \in \tilde{K}$ implies $q - G(q) \in \mathscr{K}$, where

$$G(q) = \frac{1}{2}\int_I q \ dx \tag{2.12}$$

is the mean of q over I, and hence, in view of (2.8), the definition of $b(\cdot,\cdot)$ in Eqs. (2.7) and (2.9),

$$\begin{aligned} &b(p, q-p) + (\phi, q-p) + \tau\big(q(1) - p(1)\big) - \sigma\big(\big(q(-1) - p(-1)\big) \\ &\quad = b\big(p, q - G(q) - p\big) + \big(\phi, q - G(q) - p\big) \\ &\qquad + \tau\Big(q(1) - G(q) - p(1)\Big) - \sigma\Big(q(-1) - G(q) - p(-1)\Big) \\ &\quad \geq 0 \end{aligned}$$

for all $q \in \tilde{K}$.

Let $\mathfrak{K} \subset H^1(I)$ be defined by

$$\mathfrak{K} = \big\{ q \in H^1(I) : \ q \leq \ \psi \ \text{ in } I \big\}, \tag{2.13}$$

and $G : H^1(I) \to [0, \infty)$ be defined by Eq. (2.12). Then $\tilde{K} = \{q \in \mathfrak{K} : G(q) \geq 0\}$ and the function $\psi \in \mathfrak{K}$ satisfies

$$G(\psi) > 0 \tag{2.14}$$

by Eq. (1.6).

It follows from the Slater condition Eq. (2.14) and the theory of Lagrange multipliers [18, Chapter 1, Theorem 1.6] that there exists a non-negative number λ such that

$$\begin{aligned} &b(p, q-p) + (\phi, q-p) + \tau\big(\big(q(1) - p(1)\big) - \sigma\big(q(-1) - p(-1)\big) \\ &\quad -\lambda\int_I (q-p)dx \geq 0 \end{aligned} \tag{2.15}$$

for all $q \in \mathfrak{K}$.

Finally, we observe that Eq. (2.15) can be written as

$$\tilde{b}(p, q-p) + (F, q-p) + \tau\big(q(1) - p(1)\big) - \sigma\big(q(-1) - p(-1)\big) \geq 0 \quad \forall q \in \mathfrak{K}, \tag{2.16}$$

where

$$\tilde{b}(q, r) = \int_I q' r' dx + \int_I qr \ dx \tag{2.17}$$

and

$$F = \phi - \lambda - p. \tag{2.18}$$

The variational inequality defined by Eqs. (2.13), (2.16) and (2.17) characterizes the solution of a coercive Neumann obstacle problem on $H^1(I)$. Since $F \in L_2(I)$ and $\psi \in H^2(I)$, we can apply the result in [19, Chapter 5, Theorem 3.4] to conclude that $p \in H^2(I)$.

We can deduce the regularity of $(\bar{y}, \bar{u})$ from the relations $p = \bar{y}'$ and $\bar{u} = -(\bar{y}'' + f)$. □

Corollary 2.2. *Under the assumption (1.5) on the data, the solution $(\bar{y}, \bar{u})$ of the optimal control problem Eqs. (1.1)–(1.6) belongs to $H^3(I) \times H^1(I)$.*

Remark 2.3. The result in [19], which is for dimensions ≥ 2, requires a compatibility condition between $\partial\psi/\partial n$ and the Neumann boundary condition so that the boundary trace of the normal derivative of the solution of the obstacle problem belongs to the correct Sobolev space. This is not needed in one dimension since the boundary values of the normal derivative are just numbers.

2.2. The Karush-Kuhn-Tucker conditions

It follows from Eq. (2.7), Theorem 2.1 and integration by parts that

$$b(p,q) + (\phi, q) + \tau q(1) - \sigma q(-1) - \lambda \int_I q \; dx + \int_I q \; d\nu = 0 \qquad \forall q \in H^1(I), \tag{2.19}$$

where the regular Borel measure ν is given by

$$d\nu = \big(p'' - \phi + \lambda\big) dx + \big[p'(-1) + \sigma\big] d\delta_{-1} - \big[p'(1) + \tau\big] d\delta_1, \tag{2.20}$$

and δ_{-1} (resp., δ_1) is the Dirac point measure at -1 (resp., 1).

Let $\mathscr{A}$ be the active set of the derivative constraint Eq. (1.4), i.e.,

$$\mathscr{A} = \{x \in [-1,1] : \; \bar{y}'(x) = \psi(x)\} = \{x \in [-1,1] : \; p(x) = \psi(x)\}. \tag{2.21}$$

By a standard argument, p satisfies Eq. (2.15) if and only if

$$\nu \text{ is nonnegative and supported on } \mathscr{A}. \tag{2.22}$$

We can translate Eqs. (2.19)–(2.22) into necessary conditions for the solution $\bar{y}' = p \in \mathscr{K}$ of Eq. (2.2)/(2.5), which is summarized in the following theorem.

Theorem 2.4. *There exists a nonnegative number λ such that*

$$\begin{aligned} \int_I p' q' dx + \int_I (\Phi - f') q \; dx + f(1) q(1) - f(-1) q(-1) + \int_I q \; d\nu \\ = \lambda \int_I q \; dx \qquad\qquad \forall q \in H^1(I), \end{aligned} \tag{2.23}$$

$$\int_{[-1,1]} (p-\psi)d\nu = 0, \tag{2.24}$$

$$d\nu = \rho \ dx + \gamma d\delta_{-1} + \zeta d\delta_1, \tag{2.25}$$

where

$$\rho = p'' + f' - \Phi + \lambda \in L_2(I) \text{ is nonnegative a.e.,} \tag{2.26}$$

$$\gamma = p'(-1) + f(-1) \text{ and } \zeta = -\big[p'(1) + f(1)\big] \text{ are nonnegative numbers,} \tag{2.27}$$

and $\Phi \in H^1(I)$ *satisfies Eqs. (2.3)–(2.4).*

Remark 2.5. It can be checked that the necessary conditions Eqs. (2.23)–(2.27) are also sufficient conditions for Eq. (2.2)/(2.5). Indeed, they imply, for any $q \in \mathscr{K}$,

$$\begin{aligned}
&\int_I p'(q'-p')dx + \int_I (\Phi - f')q \ dx + f(1)\big(q(1)-p(1)\big) - f(-1)\big(q(-1)-p(-1)\big)\\
&\quad = \lambda\int_I (q-p)dx - \int_I (q-p)d\nu\\
&\quad = -\int_I (q-\psi)d\nu \geq 0,
\end{aligned}$$

which then also implies $\bar{y}(x) = \int_{-1}^{x} p(t)dt$ is the solution of Eq. (1.8). We will refer to these conditions as the Karush-Kuhn-Tucker (KKT) conditions for Eq. (2.2)/(2.5).

Finally, we observe that Theorem 2.4 implies

$$\beta\int_I (\bar{y}'' + f)z''dx + \int_I (\bar{y} - y_d)z \ dx + \int_{[-1,1]} z' d\mu = 0 \quad \forall z \in H^2(I) \cap H_0^1(I), \tag{2.28}$$

where

$$\mu = \beta\nu \text{ is a nonnegative Borel measure,} \tag{2.29}$$

and

$$\int_{[-1,1]} (\bar{y}' - \psi)d\mu = 0. \tag{2.30}$$

3. The discrete problem

Let $V_h \subset H^2(I) \cap H_0^1(I)$ be the cubic Hermite finite element space (cf. [20, 21]) associated with a triangulation/partition $\mathcal{T}_h$ of I with mesh size h. The discrete problem is to find

$$\bar{y}_h = \underset{y_h \in K_h}{\operatorname{argmin}} \frac{1}{2}\left(\|y_h - y_d\|^2_{L_2(I)} + \beta\|y_h'' + f\|^2_{L_2(I)}\right), \tag{3.1}$$

where

$$K_h = \{y_h \in V_h : \ P_h y_h' \le P_h \psi\}, \tag{3.2}$$

and P_h is the interpolation operator associated with the P_1 finite element space associated with $\mathcal{T}_h$, i.e., the constraint Eq. (1.3) is only enforced at the grid points.

The nodal interpolation operator from $C^1([-1,1])$ onto V_h is denoted by Π_h.

We will use the following standard estimates for P_h and Π_h (cf. [20, 21]) in the error analysis:

$$\|\zeta - P_h\zeta\|_{L_2(I)} \le Ch|\zeta|_{H^1(I)} \qquad \forall\, \zeta \in H^1(I), \tag{3.3}$$

$$\|\zeta - P_h\zeta\|_{L_2(I)} \le Ch^2|\zeta|_{H^2(I)} \qquad \forall\, \zeta \in H^2(I), \tag{3.4}$$

$$|\zeta - \Pi_h\zeta|_{H^1(I)} + h|\zeta - \Pi_h\zeta|_{H^2(I)} \le Ch^2|\zeta|_{H^3(I)} \qquad \forall\, \zeta \in H^3(I). \tag{3.5}$$

Here and below we use C to denote a generic positive constant that is independent of the mesh size h.

The unique solution $\bar{y}_h \in K_h$ of the minimization problem defined by Eqs. (3.1) and (3.2) is characterized by the discrete variational inequality

$$\beta\int_I (\bar{y}_h'' + f)(y_h'' - \bar{y}_h'')dx + \int_I (\bar{y}_h - y_d)(y_h - \bar{y}_h)dx \ge 0 \qquad \forall\, y_h \in K_h,$$

which can also be written as

$$a(\bar{y}_h, y_h - \bar{y}_h) \ge \int_I y_d(y_h - \bar{y}_h)dx - \beta\int_I f(y_h'' - \bar{y}_h'')dx \qquad \forall\, y_h \in K_h, \tag{3.6}$$

where the bilinear form $a(\cdot,\cdot)$ is defined in Eq. (1.11).

The error analysis of the finite element method is based on the approach in [22] for state-constrained optimal control problems that was extended to one dimensional problems with constraints on the derivative of the state in [6].

We will use the energy norm $\|\cdot\|_a$ defined by

$$\|v\|_a^2 = a(v,v) = \|v\|^2_{L_2(I)} + \beta|v|^2_{H^2(I)}. \tag{3.7}$$

Note that

$$\|v\|_a \approx \|v\|_{H^2(I)} \qquad \forall\, v \in H^2(I) \cap H^1_0(I) \tag{3.8}$$

by a Poincaré-Friedrichs inequality [23].

3.1. An abstract error estimate

In view of Eqs. (3.6), (3.7) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\bar{y}-\bar{y}_h\|_a^2 &= a(\bar{y}-\bar{y}_h, \bar{y}-y_h) + a(\bar{y}-\bar{y}_h, y_h-\bar{y}_h) \\ &\le \frac{1}{2}\|\bar{y}-\bar{y}_h\|_a^2 + \frac{1}{2}\|\bar{y}-y_h\|_a^2 + a(\bar{y}, y_h-\bar{y}_h) \\ &\quad - \int_I y_d(y_h-\bar{y}_h)dx + \beta\int_I f\left(y_h''-\bar{y}_h''\right)dx \qquad \forall\, y_h \in K_h. \end{aligned} \tag{3.9}$$

It follows from Eqs. (2.28), (2.30) and (3.2) that

$$\begin{aligned} & a(\bar{y}, y_h-\bar{y}_h) - \int_I y_d(y_h-\bar{y}_h)dx + \beta\int_I f\left(y_h''-\bar{y}_h''\right)dx \\ &= \int_{[-1,1]} \left(\bar{y}_h'-y_h'\right)d\mu \\ &= \int_{[-1,1]} \left(\bar{y}_h'-P_h\bar{y}_h'\right)d\mu + \int_{[-1,1]} \left(P_h\bar{y}_h'-P_h\psi\right)d\mu + \int_{[-1,1]} (P_h\psi-\psi)d\mu \\ &\quad + \int_{[-1,1]} \left(\psi-\bar{y}'\right)d\mu + \int_{[-1,1]} \left(\bar{y}'-y_h'\right)d\mu \\ &\le \int_{[-1,1]} \left(\bar{y}_h'-P_h\bar{y}_h'\right)d\mu + \int_{[-1,1]} (P_h\psi-\psi)d\mu + \int_{[-1,1]} \left(\bar{y}'-y_h'\right)d\mu \end{aligned} \tag{3.10}$$

for all $y_h \in K_h$.

Putting Eqs. (3.9) and (3.10) together, we arrive at the abstract error estimate

$$\begin{aligned} \|\bar{y}-\bar{y}_h\|_a^2 &\le 2\left(\int_{[-1,1]} \left(\bar{y}_h'-P_h\bar{y}_h'\right)d\mu + \int_{[-1,1]} (P_h\psi-\psi)d\mu\right) \\ &\quad + \inf_{y_h\in K_h}\left(\|\bar{y}-y_h\|_a^2 + 2\int_{[-1,1]} \left(\bar{y}'-y_h'\right)d\mu\right). \end{aligned} \tag{3.11}$$

3.2. Concrete error estimates

The three terms on the right-hand side of Eq. (3.11) can be estimated as follows.

First of all, we have

$$\begin{aligned} \int_{[-1,1]} \left(\bar{y}_h'-P_h\bar{y}_h'\right)d\mu &= \int_{[-1,1]} \left[\left(\bar{y}_h'-\bar{y}'\right)-P_h\left(\bar{y}_h'-\bar{y}'\right)\right]d\mu + \int_{[-1,1]} \left(\bar{y}'-P_h\bar{y}'\right)d\mu \\ &= \beta\left(\int_I \left[\left(\bar{y}_h'-\bar{y}'\right)-P_h\left(\bar{y}_h'-\bar{y}'\right)\right]\rho\,dx + \int_I \left(\bar{y}'-P_h\bar{y}'\right)\rho\,dx\right) \\ &\le C\left(h\|\bar{y}-\bar{y}_h\|_a + h^2|y|_{H^3(I)}\right), \end{aligned} \tag{3.12}$$

by Corollary 2.2, (2.25), (2.29), (3.3), (3.4), (3.8) and the fact that $\zeta - P_h\zeta$ vanishes at the points ± 1 for any $\zeta \in H^1(I)$. The generic positive constant C in Eq. (3.12) (and other estimates below) is allowed to depend on ρ.

Similarly, we can derive

$$\int_{[-1,1]} (P_h\psi - \psi)d\mu = \beta\int_I (P_h\psi - \psi)\rho\, dx \le Ch^2|\psi|_{H^2(I)} \tag{3.13}$$

by Eqs. (1.5) and (3.4).

Finally, we have

$$\begin{aligned}
&\inf_{y_h\in K_h}\left(\|\bar{y} - y_h\|_a^2 + 2\int_{[-1,1]} (\bar{y}' - y_h')d\mu\right) \\
&\le \|\bar{y} - \Pi_h\bar{y}\|_a^2 + 2\int_{[-1,1]} \left[\bar{y}' - (\Pi_h\bar{y})'\right]d\mu \\
&= \|\bar{y} - \Pi_h\bar{y}\|_a^2 + 2\beta\int_I \left[\bar{y}' - (\Pi_h\bar{y})'\right]\rho\ dx \le Ch^2\left[|\bar{y}|_{H^3(I)}^2 + |\bar{y}|_{H^3(I)}\right],
\end{aligned} \tag{3.14}$$

by Corollary 2.2, (2.25), (2.29), (3.5), (3.8) and the fact that $\bar{y}' - (\Pi_h\bar{y})'$ vanishes at ± 1.

It follows from Eqs. (3.11)–(3.14) and Young's inequality that

$$\|\bar{y} - \bar{y}_h\|_a \le Ch, \tag{3.15}$$

which immediately implies the following result, where $\bar{u}_h = -\left(\bar{y}_h'' + f\right)$ is the approximation for $\bar{u} = -(\bar{y} + f)$.

Theorem 3.1. *Under the assumptions on the data in Eq. (1.5), we have*

$$|\bar{y} - \bar{y}_h|_{H^1(I)} + \|\bar{u} - \bar{u}_h\|_{L_2(I)} \le Ch. \tag{3.16}$$

Remark 3.2. Numerical results in Section 4 indicate that the estimate for $\|\bar{u} - \bar{u}_h\|_{L_2(I)}$ in Theorem 3.1 is sharp.

Remark 3.3. For comparison, the error estimate

$$|\bar{y} - \bar{y}_h|_{H^1(I)} + \|\bar{u} - \bar{u}_h\|_{L_2(I)} \le C_\epsilon h^{\frac{1}{2}-\epsilon}$$

was obtained in [6] under the assumptions in Eq. (1.7).

4. A numerical experiment

We begin by constructing an example for the problem Eq. (1.8)/(1.10) with a known exact solution.

4.1. An example

Let $\beta = 1$,

$$\psi(x) = \begin{cases} 1 - \frac{9}{2}x^2 & -1 \le x \le 0 \\ 1 & 0 \le x \le 1 \end{cases}, \tag{4.1}$$

and

$$\bar{y}(x) = \int_{-1}^{x} p(t)dt, \tag{4.2}$$

where

$$p(x) = \begin{cases} 1 - \frac{81}{32}\left(x - \frac{1}{3}\right)^2 & -1 \le x \le \frac{1}{3} \\ 1 & \frac{1}{3} \le x \le 1 \end{cases}. \tag{4.3}$$

We have $\psi \in H^2(I)$,

$$\int_I \psi \; dx = \frac{1}{2}, \tag{4.4}$$

$p \in H^2(I)$,

$$p''(x) = \begin{cases} -81/16 & -1 < x < \frac{1}{3} \\ 0 & \frac{1}{3} < x < 1 \end{cases}, \tag{4.5}$$

$p'(1) = 0,\; p'(-1) = 27/4,$

$$\int_I p \; dx = 0, \quad p \le \psi \quad \text{and} \quad \mathscr{A} = \{-1\} \cup [1/3, 1]. \tag{4.6}$$

Let $f \in H^1(I)$ be defined by

$$f(x) = \begin{cases} \frac{2}{9\pi}\sin\left(\pi(3x-1)\right) & -1 < x \le \frac{1}{3} \\ -\left(x - \frac{1}{3}\right)^2 & \frac{1}{3} \le x < 1 \end{cases}. \tag{4.7}$$

We have $f(-1) = 0,\; f(1/3) = 0,\; f'_{-}(1/3) = 2/3,\; f'_{+}(1/3) = 0$ and $f(1) = -4/9$. Therefore the function

$$\Phi(x) = \begin{cases} f'(x) & -1 < x < \frac{1}{3} \\ f'(x) + \frac{2}{3} & \frac{1}{3} < x < 1 \end{cases} \tag{4.8}$$

belongs to $H^1(I)$ and

$$\int_I \Phi \; dx = \int_I f'(x) + \int_{\frac{1}{3}}^{1} \frac{2}{3} \; dx = f(1) - f(-1) + \frac{4}{9} = 0. \tag{4.9}$$

Finally, we take $\lambda = 81/16$ and $y_d = \bar{y} + \Phi'$. Then the KKT conditions Eqs. (2.23)–(2.27) are satisfied with

Table 1. Numerical results for the example in Section 4.1.

$2/h$	$\|\bar{y}-\bar{y}_h\|_{L_2(I)}$	$\|\bar{y}-\bar{y}_h\|_{L_\infty(I)}$	$\|\bar{y}-\bar{y}_h\|_{H^1(I)}$	$\|\bar{y}-\bar{y}_h\|_{H^2(I)}$
$1+2^0$	1.430334 e-01	1.625937 e-01	2.581989 e-01	8.660252 e-01
$1+2^1$	1.216070 e-01	1.385037 e-01	2.199480 e-01	7.486796 e-01
$1+2^2$	4.306657 e-02	4.679253 e-02	8.061916 e-02	4.485156 e-01
$1+2^3$	1.613494 e-02	1.850729 e-02	2.919318 e-02	2.573315 e-01
$1+2^4$	3.439341 e-03	3.849954 e-03	6.315816 e-03	1.266029 e-01
$1+2^5$	9.590453 e-04	1.087740 e-03	1.741244 e-03	6.470514 e-02
$1+2^6$	2.256478 e-04	2.542346 e-04	4.125212 e-04	3.223430 e-02
$1+2^7$	5.874304 e-05	6.639870 e-05	1.067193 e-04	1.618687 e-02
$1+2^8$	1.425640 e-05	1.608790 e-05	2.549283 e-05	8.086258 e-03
$1+2^9$	3.657433 e-06	4.124680 e-06	6.430499 e-06	4.047165 e-03

$$\begin{aligned} d\nu &= \big[p''+f'-\Phi+\lambda\big]dx+\big[p'(-1)+f(-1)\big]d\delta_{-1}-\big[p'(1)+f(1)\big]d\delta_1 \\ &= (211/48)\chi_{[1/3,1]}dx+(27/4)d\delta_{-1}+(4/9)d\delta_1, \end{aligned}$$

where $\chi_{[1/3,1]}$ is the characteristic function of the interval $[1/3,1]$.

4.2. Numerical results

We solved the problem in Section 4.1 by the finite element method in Section 3 on uniform meshes. The results are displayed in Table 1.

We observe $O(h)$ convergence in the H^2 norm which agrees with Theorem 3.1. On the other hand, the convergence in the H^1 norm is $O(h^2)$, better than the $O(h)$ convergence predicted by Theorem 3.1. The convergence in L_2 and L_∞ is also $O(h^2)$.

5. Concluding remarks

We have shown that higher regularity for the solutions of one dimensional Dirichlet elliptic distributed optimal control problems with pointwise constraints on the derivative of the state can be obtained through a variational inequality satisfied by the derivative of the optimal state. A similar result for one dimensional optimal control problems with mixed boundary conditions was obtained earlier in [6]. A natural question is: Can these results be extended to higher dimensions?

For analogs of Eqs. (1.1)–(1.4) on a smooth/convex domain $\Omega \in \mathbb{R}^d$ $(d=2,\,3)$, where $f\in H^1(\Omega)$ and $\boldsymbol{\Psi}\in[H^2(\Omega)]^d$, one can also derive a variational inequality for the gradient of the optimal state. Observe that the space $\boldsymbol{G}$ of the gradients of the states is characterized by (cf. [24, Chapter I, Section 2.3])

$$\begin{aligned} \boldsymbol{G} &= \{\nabla y:\ y\in H^2(\Omega)\cap H_0^1(\Omega)\} \\ &= \Big\{\boldsymbol{q}\in[H^1(\Omega)]^d:\ \text{curl }\boldsymbol{q}=0 \text{ on } \Omega \text{ and } \boldsymbol{n}\times\boldsymbol{q}=0 \text{ on } \partial\Omega\Big\}, \end{aligned}$$

where $\boldsymbol{n}$ is the unit outward normal along $\partial\Omega$.

Let $\boldsymbol{K}$ be the subset of $\boldsymbol{G}$ defined by

$$\boldsymbol{K} = \{\boldsymbol{q} \in \boldsymbol{G} : \ \boldsymbol{q} \le \boldsymbol{\Psi} \text{ a.e. in } \Omega\}.$$

We assume that $\boldsymbol{K}$ is nonempty, which is the case for example if $\boldsymbol{\Psi} \ge 0$.

The analog of Eq. (2.2) is given by

$$\int_\Omega (\boldsymbol{\Phi} - \nabla f)\cdot(\boldsymbol{q}-\boldsymbol{p})dx + \int_\Omega \operatorname{div}\boldsymbol{p}\ \operatorname{div}(\boldsymbol{q}-\boldsymbol{p})\ dx + \int_{\partial\Omega} f(\boldsymbol{q}-\boldsymbol{p})\cdot\boldsymbol{n}\ dS \ge 0 \tag{5.1}$$

for all $\boldsymbol{q} \in \boldsymbol{K}$, where $\boldsymbol{p} = \nabla\bar{y} \in \boldsymbol{K}$, and $\boldsymbol{\Phi} \in \boldsymbol{G}$ is defined by $\beta \operatorname{div} \boldsymbol{\Phi} = y_d - \bar{y}$, which is an analog of Eq. (2.3). The variational inequality Eq. (5.1) is uniquely solvable because (cf. [24, Chapter I, Sections 3.2 and 3.4])

$$\int_\Omega (\operatorname{div}\ \boldsymbol{q})^2 dx \ge C_\Omega |\boldsymbol{q}|^2_{H^1(\Omega)} \qquad \forall\, \boldsymbol{q} \in \boldsymbol{G}.$$

We can also write Eq. (5.1) as

$$\begin{aligned}&\int_\Omega (\boldsymbol{\Phi} - \nabla f)\cdot(\boldsymbol{q}-\boldsymbol{p})dx + \int_\Omega \left[\operatorname{div}\ \boldsymbol{p}\ \operatorname{div}\ (\boldsymbol{q}-\boldsymbol{p}) + \operatorname{curl}\ \boldsymbol{p}\cdot\operatorname{curl}\ (\boldsymbol{q}-\boldsymbol{p})\right]\ dx \\ &\quad + \int_{\partial\Omega} f(\boldsymbol{q}-\boldsymbol{p})\cdot\boldsymbol{n}\ dS \ge 0 \qquad \forall\, \boldsymbol{q} \in \boldsymbol{K},\end{aligned} \tag{5.2}$$

which can be interpreted as an obstacle problem for the vector Laplacian operator with natural boundary conditions.

In order to obtain higher regularity for the optimal state $\bar{y}$, one will need regularity results for Eq. (5.1)/(5.2), which unfortunately are not available. Therefore the problem of extending the results in this paper to higher dimensions remains open.

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References

[1] Casas, E., Bonnans, J. F. (1988). Contrôle de systèmes elliptiques semilinéares comportant des contraintes sur l'état. In Brezzis, H., Lions, J. L., eds., *Nonlinear Partial Differential Equations and Their Applications*, vol. 8. New York: Longman, pp. 69–86.

[2] Casas, E., Fernandez, L. A. (1993). Optimal control of semilinear elliptic equations with pointwise constraints on the gradient of the state. *Appl. Math. Optim*. 27(1): 35–56. DOI: 10.1007/BF01182597.

[3] Deckelnick, K., Günther, A., Hinze, M. (2009). Finite element approximation of elliptic control problems with constraints on the gradient. *Numer. Math*. 111(3): 335–350. DOI: 10.1007/s00211-008-0185-3.

[4] Ortner, C., Wollner, W. (2011). A priori error estimates for optimal control problems with pointwise constraints on the gradient of the state. *Numer. Math*. 118(3): 587–600. DOI: 10.1007/s00211-011-0360-9.

[5] Wollner, W. (2012). Optimal control of elliptic equations with pointwise constraints on the gradient of the state in nonsmooth polygonal domains. *SIAM J. Control Optim*. 50(4):2117–2129. DOI: 10.1137/110836419.

[6] Brenner, S. C., Sung, L.-Y., Wollner, W. (2020). Finite element methods for one dimensional elliptic distributed optimal control problems with pointwise constraints on the derivative of the state. *Optim Eng*. published online 18 February 2020. doi.10.1007/s11081-020-09491-1).

[7] Ekeland, I., Témam, R. (1999). Convex Analysis and Variational Problems. Classics in Applied Mathematics. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).

[8] Kinderlehrer, D., Stampacchia, G. (2000). *An Introduction to Variational Inequalities and Their Applications*. Philadelphia, PA: Society for Industrial and Applied Mathematics.

[9] Pierre, M., Sokołowski, J. (1996). Differentiability of projection and applications. In *Control of Partial Differential Equations and Applications (Laredo, 1994), Volume 174 of Lecture Notes in Pure and Applied Mathematics*. New York: Dekker, pp. 231–240.

[10] Liu, W., Gong, W., Yan, N. (2009). A new finite element approximation of a state-constrained optimal control problem. *J. Comput. Math*. 27:97–114.

[11] Brenner, S. C., Davis, C. B., Sung, L.-Y. (2014). A partition of unity method for a class of fourth order elliptic variational inequalities. *Comp. Methods Appl. Mech. Eng*. 276:612–626. DOI: 10.1016/j.cma.2014.04.004.

[12] Brenner, S. C., Gedicke, J., Sung, L.-Y. (2018). C^0 interior penalty methods for an elliptic distributed optimal control problem on nonconvex polygonal domains with pointwise state constraints. *SIAM J. Numer. Anal*. 56(3):1758–1785. DOI: 10.1137/17M1140649.

[13] Brenner, S. C., Gudi, T., Porwal, K., Sung, L.-Y. (2018). A Morley finite element method for an elliptic distributed optimal control problem with pointwise state and control constraints. *ESAIM: COCV*. 24(3):1181–1206. DOI: 10.1051/cocv/2017031.

[14] Brenner, S. C., Oh, M., Pollock, S., Porwal, K., Schedensack, M., Sharma, N. (2016). A C^0 interior penalty method for elliptic distributed optimal control problems in three dimensions with pointwise state constraints. In Brenner, S. C., eds., *Topics in Numerical Partial Differential Equations and Scientific Computing, Volume 160 of the IMA Volumes in Mathematics and Its Applications*. Cham: Springer, pp. 1–22.

[15] Brenner, S. C., Sung, L.-Y., Zhang, Y. (2013). A quadratic C^0 interior penalty method for an elliptic optimal control problem with state constraints. In: Karakashian, O., Feng, X., and Xing, Y., eds., *Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations, Volume 157 of the IMA Volumes in Mathematics and Its Applications*. Cham: Springer, pp. 97–132. (2012 John H. Barrett Memorial Lectures).

[16] Brenner, S. C., Sung, L.-Y., Zhang, Y. (2019). C^0 interior penalty methods for an elliptic state-constrained optimal control problem with Neumann boundary condition. *J. Comput. Appl. Math*. 350:212–232. DOI: 10.1016/j.cam.2018.10.015.

[17] Gong, W., Yan, N. (2011). A mixed finite element scheme for optimal control problems with pointwise state constraints. *J. Sci. Comput*. 46(2):182–203. DOI: 10.1007/s10915-010-9392-z.

[18] Ito, K., Kunisch, K. (2008). *Lagrange Multiplier Approach to Variational Problems and Applications*. Philadelphia, PA: Society for Industrial and Applied Mathematics.

[19] Rodrigues, J.-F. (1987). *Obstacle Problems in Mathematical Physics*. Amsterdam: North-Holland Publishing Co.

[20] Brenner, S. C., Scott, L. R. (2008). *The Mathematical Theory of Finite Element Methods*, 3rd ed. New York: Springer-Verlag.

[21] Ciarlet, P. G. (1978). *The Finite Element Method for Elliptic Problems*. Amsterdam: North-Holland.

[22] Brenner, S. C., Sung, L.-Y. (2017). A new convergence analysis of finite element methods for elliptic distributed optimal control problems with pointwise state constraints. *SIAM J. Control Optim*. 55(4):2289–2304. DOI: 10.1137/16M1088090.

[23] Nečas, J. (2012). *Direct Methods in the Theory of Elliptic Equations*. Heidelberg: Springer.

[24] Girault, V., Raviart, P.-A. (1986). *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*. Berlin: Springer-Verlag.