



Invariant Measures and Global Well Posedness for the SQG Equation

JURAJ FÖLDES & MOUHAMADOU SY

Communicated by J. BEDROSSIAN

Abstract

We construct an invariant measure μ for the Surface Quasi-Geostrophic (SQG) equation and show that almost all functions in the support of μ are initial conditions of global, unique solutions of SQG that depend continuously on the initial data. In addition, we show that the support of μ is infinite dimensional, meaning that it is not locally a subset of any compact set with finite Hausdorff dimension. Also, there are global solutions that have arbitrarily large initial condition. The measure μ is obtained via fluctuation–dissipation method, that is, as a limit of invariant measures for stochastic SQG with a carefully chosen dissipation and random forcing.

1. Introduction

The goal of the present manuscript is to construct an invariant measure μ for the Surface Quasi-Geostrophic (SQG) equation

$$\theta_t + \mathbf{u} \cdot \nabla \theta = 0, \quad (1.1)$$

and to prove μ almost sure global well posedness of (1.1). First, we establish the existence of invariant measures $(\mu_\alpha)_{\alpha>0}$ for the stochastic SQG

$$d\theta + \mathbf{u} \cdot \nabla \theta dt = -\alpha \Delta^2 \theta dt + \alpha \nabla \cdot (|\nabla \theta|^2 \nabla \theta) + \sqrt{\alpha} d\eta \quad \text{on } \mathbb{T}^2 \times (0, \infty), \quad (1.2)$$

and then we construct μ as a limit of μ_α as $\alpha \rightarrow 0^+$. In addition, we prove that all functions in the support of μ are initial conditions of global, regular solutions, and the support of μ is infinite dimensional. Before we precisely formulate our main results, let us fix the notation and provide a motivation for our study.

To avoid unnecessary technicalities associated with boundary conditions, we work on a two dimensional flat torus \mathbb{T}^2 , however most of our techniques could be applied to domains with boundary. Unless indicated otherwise, we always assume that θ has zero mean for all times, that is, that

$$\int_{\mathbb{T}^2} \theta(x, t) \, dx = 0 \quad t \geq 0.$$

We assume that $\theta : \mathbb{T}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ has sufficient regularity (as detailed below), and $\mathbf{u} = (-\partial_y, \partial_x)(-\Delta)^{-\frac{1}{2}}\theta = R^\perp\theta$ is the Riesz transform of θ , that is,

$$\mathbf{u} = F^{-1} \left(-i \frac{\xi}{|\xi|} F(\theta) \right),$$

where F and F^{-1} denote respectively Fourier and inverse Fourier transform. As usual, we work with cylindrical Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, \mathbb{P})$ and our stochastic forcing has the form

$$\eta(t, x) = \sum_{j=1}^{\infty} a_j e_j(x) W_j(t), \quad (1.3)$$

where e_j are zero mean eigenfunctions of $-\Delta$ on the torus \mathbb{T}^2 , ordered such that the corresponding eigenvalues $\lambda_j > 0$ form a non-decreasing sequence, and (a_j) is a sequence of real numbers such that

$$A_0 = \sum_{j=1}^{\infty} a_j^2 < \infty.$$

Finally, we complement (1.2) with appropriate initial condition specified below.

The SQG equation (1.1) appears as a model for the temperature of stratified atmosphere on the rapidly rotating planet or as a model of ocean dynamics on certain scales [6] (for derivation, applications to ocean and atmosphere dynamics, and more references see [55] or a more recent survey [44]). From a mathematical perspective, the SQG equation attracted a lot of attention due to many similarities with three dimensional Euler equation. Most notably, the vector $\nabla^\perp \theta$ satisfies an analogue of the Euler equation in the vorticity form. In particular, both equations contain vortex stretching term and a divergence free drift term, however one is posed in 2D whereas the other one in three dimensional and the constitutive laws are different; see the seminal work by Constantin, Majda, and Tabak [17] for more discussion.

Although the local existence and uniqueness of smooth solutions of (1.1) was already resolved in [17], despite many efforts, the global existence of solutions on a torus remains open. The blow-up scenario proposed in [13] was ruled out by precise numerical simulations in [54] and analytically in [18, 20, 21]. Another mechanism of gradient blow up based on the propagation of small instabilities in thin filaments was proposed in [60]. We remark that blow-up was constructed in [13] for infinite energy initial conditions on \mathbb{R}^2 .

In [36], Kiselev and Nazarov showed that there exists a solution with initial condition having arbitrarily small initial conditions that attains arbitrary big norms in finite time. Later, motivated by a construction for the Euler equation [68], it was shown in [33] that there are solutions of (1.1) with the $W^{2,\infty}$ norm growing exponentially (along sub-sequence) as time goes to infinity:

$$\sup_{t \leq T} \|\nabla^2 \theta(\cdot, t)\|_{L^\infty} \geq e^{\gamma T} \quad \text{for some } \gamma > 0. \quad (1.4)$$

Also, very little is known about non-equilibrium global smooth solutions for SQG. In fact the only example was given in [14], where with a rigorous, computer assisted proof the authors proved a global existence for initial conditions on one dimensional bifurcation branch close to a specific radial equilibrium.

We just briefly remark that one can also consider weak solutions of (1.1), which are known to be global [51, 57]. However, the uniqueness of weak solution was an challenging open problem [24] that was solved by establishing non-uniqueness in [11], see also [3, 59]. Also, several regularized models (for example additional dissipation, or smoother constitutive law) were introduced for which one can prove global well posedness of solutions, see [12, 16, 19, 37].

In the present manuscript we utilize fluctuation–dissipation method to construct global solutions of (1.1). The idea is to add a regularizing higher order differential terms, which guarantee global well posedness, and a stochastic forcing that keeps the energy balance in (1.2). Note that the strength of the forcing and the coefficients of the smoothing operators are carefully balanced. We prove that the stochastic SQG equation possesses an invariant measure supported on appropriate Sobolev spaces and by passing to the limit, we obtain an invariant measure μ for the deterministic SQG (1.1). Then, we investigate properties of μ .

Let us describe known results for problems close to (1.2). The well posedness of stochastic SQG with either additive or multiplicative noise and additional sub-critical smoothing (dissipation of the form $(-\Delta)^\beta$, $\beta > \frac{1}{2}$) was studied in [58]. The authors proved that the problem is pathwise globally well posed and under additional assumptions they showed that there exists a unique invariant measure, which is ergodic, and attracts all distributions at an algebraic rate. Later, large deviation principles for stochastic SQG were proved in [49]. Note that stochastic Quasi-geostrophic (which contains additional Laplacian compared to SQG) was earlier studied in [8, 34]. A regularization of (1.1) with help of the random diffusion was proved in [10] for sufficiently small smooth initial conditions.

In what follows, we first investigate the pathwise global well posedness of (1.2) and then we prove for each $\alpha > 0$ the existence of invariant measure μ_α supported on H^2 . The choice of bi-Laplacian in (1.1) rather than Laplacian, stems from the fact, that we need μ_α to be supported on H^2 rather than H^1 . Otherwise, after passing $\alpha \rightarrow 0$, we would obtain a measure supported on H^1 which is not sufficient for the proof of uniqueness of solutions of (1.1) (see below). Our first main result is stated in the following theorem, (for more precise formulation see Theorems 2.1 and 4.1 below):

Theorem 1.1. *Assuming $A_0 < \infty$ and appropriate moment bounds on the initial distribution (see Theorem 2.1 below), almost surely there exists a pathwise global*

solution to (1.2). Furthermore, (1.2) admits at least one stationary measure μ_α supported on $H^2(\mathbb{T}^2) \cap W^{1,4}(\mathbb{T}^2)$.

The proof of global pathwise, well posedness follows from a standard framework – Galerkin approximation and passage to the limit. Since we were not able to locate a suitable result in the literature, we provide sketch of the proof with appropriate references. Then, the existence of the invariant measure follows from moment bounds on solutions and Kryloff-Bogoliouboff theorem [38], see also [23]. Note that using standard coupling techniques, one can prove that μ_α is in fact a unique invariant measure. For proofs in settings close to ours, we refer to [9, 22, 27–29, 32, 43]

Before discussing convergence properties of measures constructed in Theorem 1.1, let us summarize known results. Passing $\alpha \rightarrow 0$ and consecutive analysis of limiting measure μ was done for Euler equation in [31, 41], where it was proved that μ is supported on $H^1 \cap L^\infty$. Moreover, it was proved that for any compact set S with finite Hausdorff dimension, one has $\mu(S) = 0$, that is, μ is infinite dimensional. The crucial property that allowed to prove the infinite dimensionality was the existence of infinitely many conservation laws. Also, it was shown that the support of μ contains solutions with large energy. Analogous results were obtained for KdV, Benjamin-Ono, Klein-Gordon, and Schrödinger equations in [42, 62, 64] (see also references therein). It is important to notice that in all previous examples the proof of the invariance of the limiting measure μ was based on the well posedness of the underlying deterministic equation, which is not known for SQG equation. Observe that the proof of invariance for Euler equation [43] does not require global well posedness of the deterministic equation, nevertheless the 2D Euler equation is significantly simpler than SQG (which resembles three dimensional Euler equation). The construction of global solutions for septic NLS [63] (not known to be globally well posed), utilizes a different strategy: the fluctuation–dissipation is used only for Galerkin approximations and the main obstacle is passage to the limit (based on an argument of Bourgain [7]).

On the other hand, different construction based on Gibbs measures was used to construct global solutions and invariant measures for various, possibly globally ill posed, Hamiltonian systems (see e.g [7, 53, 65] and references therein). However, the authors of [52] identified a serious obstruction that prevents a ‘traditional way’ (for example as for 2D Euler [1]) of building a Gibbs measure based on the conservation of L^2 norm for the SQG equation. Indeed, for functions in the support of such measure, the nonlinearity of SQG (one degree less regular than Euler) cannot be defined in the sense of distributions.

The main novelty of the paper is the proof that the set of measures (μ_α) from Theorem 1.1 has an accumulation point μ which is an invariant measure for (1.1). The invariance is understood with respect to the dynamics induced by the stochastic equation (1.2) and a passage $\alpha \rightarrow 0$. Furthermore, we prove that if the initial condition belongs to the support of μ , the corresponding solution of (1.1) is global. Hence, it is important to estimate the size of the support of μ . Although the SQG is similar to the three dimensional Euler equation, a notable difference is the existence of infinitely many conserved quantities, that allows us to prove that $\mu(K) = 0$ for

any compact set K with a finite Hausdorff dimension. Also, we show that μ is not supported only on small functions, meaning that the support of μ must contain functions with arbitrarily large $H^{\frac{3}{2}}$ norm. The last statement follows from the fact, that we can choose the noise in the fluctuation dissipation method and obtain μ with large moments (see Corollary 1.3 below).

Before we proceed, we denote

$$A_s := \sum_{j=1}^{\infty} \lambda_j^s a_j^2, \quad (1.5)$$

and recall that regular solutions of SQG equation admit, the following set of conservation laws:

$$\begin{aligned} E_{-\frac{1}{2}}(\theta) &= \frac{1}{2} \int_{\mathbb{T}^2} |(-\Delta)^{-\frac{1}{2}} \theta|^2 dx, \\ M(\theta) &= \frac{1}{2} \int_{\mathbb{T}^2} \theta^2 dx. \end{aligned}$$

The next theorem contains our main results, for more general assertions see Theorems 5.1, 6.1, and 6.2 below.

Theorem 1.2. *Assume $A_0 < \infty$. As $\alpha \rightarrow 0$, there is an accumulation point μ for the sequence (μ_α) satisfying the following properties:*

(1) μ is a probability measure concentrated on the Sobolev space $H^2(\mathbb{T}^2)$, that is,

$$\mu(H^2(\mathbb{T}^2)) = 1.$$

(2) For μ almost all data θ_0 , there is a unique function $\theta \in C(\mathbb{R}^+, H^1) \cap L^2_{loc}(\mathbb{R}^+, H^2)$ satisfying the equation (1.1) with $\theta(0, x) = \theta_0(x)$. Define a flow ρ for (1.1) as $\rho_t(\theta_0) = \theta(\cdot, t; \theta_0)$.

(3) The flow ρ_t is continuous on H^1 .

(4) μ is invariant under ρ_t .

(5) μ satisfies the estimates

$$\begin{aligned} \int_{L^2} \left(\|\theta\|_{H^{\frac{3}{2}}}^2 - \int_{\mathbb{T}^2} |\nabla \theta|^2 \nabla \theta \cdot \nabla (-\Delta)^{-\frac{1}{2}} \theta dx \right) \mu(d\theta) &= \frac{A_{-\frac{1}{2}}}{2}, \\ \int_{L^2} (\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4) \mu(d\theta) &< \infty. \end{aligned}$$

(6) If $a_m \neq 0$ for each m , μ is infinite-dimensional in the sense that it vanishes on finite-dimensional compact sets.

(7) The conservation laws of random variables $M(\theta)$ and $E_{-\frac{1}{2}}(\theta)$ are absolutely continuous with respect to the Lebesgue measure on \mathbb{R} .

The regularity of functions in the support of μ (being H^2 by Theorem 1.2, 1.) is a direct consequence of the support of μ_α , which follows from the regularizing term Δ^2 . Note that $L^2 H^2$ is a minimal smoothness required to prove uniqueness of solutions, that is, well posedness claimed in Theorem 1.2 part (2). Replacing

Δ^2 by Δ^β with $\beta < 2$, yields the existence of (μ_α) and μ supported on H^β , but solutions of (1.1) with initial conditions in the support of μ are too weak to establish uniqueness.

On the other hand, the choice of Δ^2 instead of Δ brings several obstacles. For example, for the proof of part (6) in Theorem 1.2 we needed to introduce additional smoothing term $\Delta_4\theta := \nabla \cdot (|\nabla\theta|^2 \nabla\theta)$ into (1.2) ($\Delta_p\theta := \nabla \cdot (|\nabla\theta|^{p-2} \Delta_4\theta)$ is called p -Laplacian), the rest of the proofs can be done without it. The reason for the addition of $\Delta_4\theta$ is that the expression $\langle \Delta^2\theta, f(\theta) \rangle$ is neither positive nor bounded from below for all θ , and for large set of functions f , a minimal requirement for the general framework, see details in Section 6. The addition of Δ_4 guarantees that (μ_α) , and consequently μ are supported also on $W^{1,4}$ with fourth order moment bounds. Then, we can bound $\langle \Delta^2\theta - \Delta_4\theta, f(\theta) \rangle$ from below for any f that has bounded derivatives up to fourth order, which suffices for our purposes.

If one wishes to construct invariant measures for (1.1) on smoother spaces, for example H^β for $\beta > 2$, and prove infinite dimensionality of such measure, then one has to correct the smoothing operator Δ^β by appropriate quasilinear positive definite operator such as p -Laplacian. Since the proofs are technically involved we decided not to present them here.

Concerning the support of μ , by part (6) of Theorem 1.2, it cannot be contained in any compact set of finite Hausdorff dimension. Moreover, (7) implies, that the support of μ is not merely a countable union of level sets of the conservation laws. Also, the following corollary asserts that there are arbitrarily large initial data that give rise to global solutions:

Corollary 1.3. *Given any constant K , denote $S_K = \{\theta : \|\theta\|_{W^{1,4}}^4 + \|\theta\|_{H^{\frac{3}{2}}}^2 \geq K\}$. Then there is $\theta_0 \in S_K$ such that the solution of (1.1) with $\theta(0) = \theta_0$ is global. More generally, there exists a sequence (a_j) (see the definition of the noise (1.3)) such that for the measure μ constructed in Theorem 1.2 one has $\mu(S_K) > 0$.*

Proof of Corollary 1.3. Choose the sequence (a_j) such that $A_{-\frac{1}{2}} = 4CK$ and $A_0 < \infty$, where C is a constant depending only on the size of \mathbb{T}^2 . If $\mu(S_K) = 0$, then, by Theorem 1.2, (5) one has by Hölder and Poincaré inequalities

$$\begin{aligned} 2CK = \frac{A_{-\frac{1}{2}}}{2} &\leq \int_{L^2} \left(\|\theta\|_{H^{\frac{3}{2}}}^2 + \int_{\mathbb{T}^2} |\nabla\theta|^3 |\nabla(-\Delta)^{\frac{1}{2}}\theta| dx \right) \mu(d\theta) \\ &\leq \int_{L^2} \|\theta\|_{H^{\frac{3}{2}}}^2 \|\nabla\theta\|_{W^{1,4}}^3 \|\theta\|_{L^4} \mu(d\theta) \\ &\leq C \int_{L^2} \|\theta\|_{H^{\frac{3}{2}}}^2 + \|\nabla\theta\|_{W^{1,4}}^4 \mu(d\theta) \\ &= C \int_{L^2 \setminus S_K} \|\theta\|_{H^{\frac{3}{2}}}^2 + \|\nabla\theta\|_{W^{1,4}}^4 \mu(d\theta) \leq CK, \end{aligned}$$

which is a contradiction. The first statement follows from the second one and Theorem 1.2 parts (1), (2). \square

Another consequence of Theorem 1.2 follows from the Poincaré recurrence theorem.

Corollary 1.4. *For μ -almost every $u_0 \in H^2$, there is a sequence $(t_k)_k$ increasing to ∞ such that*

$$\lim_{k \rightarrow \infty} \|\rho_{t_k} u_0 - u_0\|_2 = 0.$$

Corollary (1.4) could be the reason why the estimate (1.4) requires $\sum_{t \leq T}$, that is, the solution increases only along the sequence of times (solutions might return infinitely many times to a neighbourhood of the initial condition).

A natural question is whether the set of solutions constructed in Theorem 1.2 is a subset of equilibria to (1.1). This seems to be a very non-trivial question to which we do not have a definitive answer. However, we have the following alternative:

- (a) The support of μ is not a subset of the equilibria of (1.1), and then the flow ρ_t constructed in Theorem 1.2, part (2) yields non-trivial global solutions.
- (b) The support of μ coincides with the equilibria of (1.1), for any choice of (sufficiently regular) noise. In that case, we would have a remarkable stability property of the equilibria for both (1.1) and (1.2) with small α .

Recall that the linear stability of equilibria of (1.1) were studied in [30].

Let us remark that in the context of equations having only discrete set of equilibria, for instance the case of some power type nonlinearities, (6) and (7) of Theorem 1.2 imply that the alternative (a) above occurs. Also, we propose in the appendix a general example of a finite-dimensional system having continuous set of equilibria, but the support of the inviscid measure not being subset of equilibria.

Organization of the Paper. In Section 2, we prove probabilistic global well-posedness for the stochastic equation (1.2). Moment bounds for such solutions are given in Section 3 and based on moment bounds we construct stationary measures for any $\alpha > 0$ in Section 4. Section 5 contains principal results of the paper, and we prove there the existence of invariant measure for (1.1), and global well posedness on its support. In Section 6, we combine the probabilistic estimates and Krylov lemma to establish qualitative properties (infinite dimensionality of the support). Finally, Appendix A includes details about the invariant measures for finite dimensional Hamiltonian systems and in Appendices B and C we recall that Itô formula in infinite dimensions and a proof of a parabolic embedding.

1.1. General Notations

The following notation is used throughout the paper:

- * $C_0^\infty(\mathbb{R})$ is the space of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that are infinitely differentiable and compactly supported.
- * For any $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, we denote $L^p(\mathbb{T}^2)$ and $W^{s,p}(\mathbb{T}^2)$ the usual Lebesgue respectively Sobolev spaces. We also set $H^s(\mathbb{T}^2) = W^{s,2}(\mathbb{T}^2)$. Often for the clarity of presentation we do not indicate the domain \mathbb{T}^2 and we write L^p , $W^{k,p}$, and H^k .
- * When a fixed $T > 0$ is clear from context, for any Banach space X define the spaces $CX = C([0, T], X)$, $L^p X = L^p([0, T], X)$, and $W^{s,p} X = W^{s,p}([0, T], X)$. Sometime if needed, we indicate the variable of the space as

a subscript, for example $C_t H_x^s$. The spaces are equipped with usual parabolic norms denoted for example $\|\cdot\|_{CX}$.

- * We write $\|\cdot\|$ instead of $\|\cdot\|_{L^2(\mathbb{T}^2)}$.
- * $C_{\text{loc}}X$ denotes the space of functions that are locally continuous in time with values in X . Analogously we define $L_{\text{loc}}^p X$ and $W_{\text{loc}}^{s,p} X$.
- * We denote $\mathfrak{p}(X)$ the set of Borel probability measures on X .
- * The non-decreasing sequence $(\lambda_m)_{m \geq 1}$ contains all eigenvalues of $-\Delta$ on \mathbb{T}^2 and with corresponding normalized eigenvectors $(e_m)_{m \geq 1}$.
- * For a probability measure μ on X , we denote by $\mathbb{E}_\mu(f)$, the average of f with respect to μ :

$$\int_X f(x) \mu(dx).$$

- * The Riesz transform of θ is given denoted $R^\perp \theta = (-\partial_y, \partial_x)(-\Delta)^{-\frac{1}{2}} \theta$. Note that by Hörmander-Mikhlin theorem Riesz transform satisfies (see [61]), for any $p \in (1, \infty)$

$$\|R^\perp v\|_{L^p} \leq C \|v\|_{L^p}, \quad (1.6)$$

where C depends on p .

2. Global Solutions for the Stochastic SQG

In this section we establish the path-wise global well posedness of solutions of (1.2), and therefore prove the first part of Theorem 1.1. Although the proof follows from a framework used several times in the literature, we were unable to locate the precise reference that would cover our situation. Rather than providing all details, we show how to satisfy assumptions of [47, Theorem 1.3] and explain how the proof in [47] needs to be modified.

Theorem 2.1. *Fix any $\alpha > 0$, any $T > 0$, and any $p \geq 1$. Also, fix any \mathcal{F}_0 measurable (see filtration for our Brownian motion random variable θ_0 with $\mathbb{E}\|\theta_0\|_{L^2}^p < \infty$, and any noise η of the form (1.3) with $A_0 < \infty$. Then, there exists a unique adapted solution θ of (1.2) satisfying $\theta(0) = \theta_0$ and, almost surely,*

$$\theta \in C([0, T], L^2(\mathbb{T}^2)) \cap L^2([0, T], H^2(\mathbb{T}^2)) \cap L^4([0, T], W^{1,4}(\mathbb{T}^2)).$$

Furthermore,

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \|\theta(t)\|^{2p} + \alpha \mathbb{E} \int_0^T \|\theta\|^{2p-2} (\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4) ds \\ \leq C(T, \alpha, p, \|\theta_0\|). \end{aligned} \quad (2.1)$$

Proof. The proof closely follows the proof of [47, Theorem 1.3], see also [48]. However, since our differential operators have different scalings, we have to slightly modify the arguments. We only highlight differences. For easier comparison, we use the notation from [47].

Recall that $(e_n)_{n \geq 1}$ is an orthonormal basis of L^2 and denote $H_n = \text{span}\{e_1, \dots, e_n\}$. Let $P_n : H^{-2} \rightarrow H_n$ be the orthogonal projection defined by

$$P_n y = \sum_{i=1}^n \langle y, e_i \rangle e_i, \quad y \in H^{-2}.$$

For each, $n \geq 1$ consider the stochastic equation on H_n

$$dx^{(n)} = P_n(AX^{(n)})dt + \sqrt{\alpha}d\eta_n, \quad X_0^{(n)} = P_n\theta_0, \quad (2.2)$$

where

$$AX = -\alpha \Delta^2 X + \alpha \nabla(|\nabla X| \nabla X) - Y \nabla X, \quad Y = R^\perp X$$

and

$$\eta_n(x, t) = \sum_{j=1}^n a_j e_j(x) W_j(t).$$

The existence and uniqueness of solutions of (2.2) is classical and follows from [40, Section 1], see also [56, Theorem 3.1.1]. \square

We have the following a priori estimates for $X^{(n)}$:

Lemma 2.2. *For every $T > 0$ there exists C_T depending on A_0 , p and α , but independent of n such that for each $n \geq 1$,*

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, T]} \|X^{(n)}(t)\|_{L^2}^{2p} + \int_0^t \|X^{(n)}(s)\|_{L^2}^{2p-2} (\|X^{(n)}\|_{H^2}^2 + \|X^{(n)}\|_{W^{1,4}}^4) ds \\ & \leq C_T (\mathbb{E} \|X^{(n)}(0)\|_{L^2}^{2p} + 1). \end{aligned}$$

Proof of Lemma 2.2. The proof follows from [47, Lemma 2.2], see also proof of (3.2) below for the idea of the proof. \square

Define the spaces

$$Y_1 := L^2([0, T] \times \Omega, H^2) \quad Y_2 := L^4([0, T] \times \Omega, W^{1,4})$$

and

$$K = Y_1 \cap Y_2$$

and the dual of K

$$K^* = Y_1^* + Y_2^*$$

equipped with the usual intersection and sum norms

$$\begin{aligned} \|X\|_K &= \max\{\|X\|_{Y_1}, \|X\|_{Y_2}\}, \\ \|X\|_{K^*} &= \inf\{\|X_1\|_{Y_1^*} + \|X_2\|_{Y_2^*}, X_1 + X_2 = X\}. \end{aligned} \quad (2.3)$$

Note that

$$Y_1^* := L^2([0, T] \times \Omega, H^{-2}), \quad Y_2^* := L^{\frac{4}{3}}([0, T] \times \Omega, W^{-1, \frac{4}{3}}),$$

Since

$$\|AX^{(n)}\|_{K^*} \leq \alpha \|\Delta^2 X^{(n)}\|_{Y_1^*} + \alpha \|\nabla(|\nabla X^{(n)}| \nabla X^{(n)})\|_{Y_2^*} + \|Y^{(n)} \nabla X^{(n)}\|_{Y_1^*}$$

and

$$\begin{aligned} \|\Delta^2 X^{(n)}\|_{Y_1^*} &= \mathbb{E} \|X^{(n)}\|_{L^2 H^2}^2 \\ \|\nabla(|\nabla X^{(n)}|^2 \nabla X^{(n)})\|_{Y_2^*} &= \mathbb{E} \| |\nabla X^{(n)}|^3 \|_{L^{\frac{4}{3}} L^{\frac{4}{3}}}^{\frac{4}{3}} \\ &= \mathbb{E} \|\nabla X^{(n)}\|_{L^4 L^4}^3 \leq C \left(\mathbb{E} \|X^{(n)}\|_{L^4 W^{1,4}}^4 \right)^{\frac{3}{4}}, \\ \|Y^{(n)} \nabla X^{(n)}\|_{Y_1^*}^2 &\leq C \mathbb{E} \|Y^{(n)} |X^{(n)}|\|_{L^2 L^2}^2 \leq C \mathbb{E} \|Y^{(n)}\|_{L^4 L^4}^2 \|X^{(n)}\|_{L^4 L^4}^2 \\ &\leq C \mathbb{E} \|X^{(n)}\|_{L^4 W^{1,4}}^4, \end{aligned}$$

from Lemma 2.2 it follows that

$$\|AX^{(n)}\|_{K^*} \leq C, \quad (2.4)$$

with C independent of n , because $\mathbb{E} M(X_0) \leq \mathbb{E} \|\theta_0\|^2 < \infty$.

The continuity of the map (assumption (H1) in [47]) $s \mapsto \langle A(X_1 + sX_2), X \rangle$, $i = 1, 2$ is easy to verify for any $X_1, X_2, X \in H^2$. Also, the local monotonicity assumption ((H2) in [47])

$$\langle A(X_1) - A(X_2), X_1 - X_2 \rangle \leq (K + \kappa(X_2)) \|X_1 - X_2\|_{L^2}^2 \quad (2.5)$$

is valid for our operator A . Indeed, note that p -Laplacian is monotone operator (see for example [67, Proposition 30.10]), thus (2.5) holds true for $AX = \nabla(|\nabla X|^2 \nabla X)$ with $f = \rho = \kappa \equiv 0$. Since Y_1 is divergence free, (1.6), $H^2 \hookrightarrow L^\infty$, and Hölder and the Young inequalities yield

$$\begin{aligned} \langle A(X_1) - A(X_2), X_1 - X_2 \rangle &\leq -\alpha \|X_1 - X_2\|_{H^2}^2 + \langle Y_2 \nabla X_2 - Y_1 \nabla X_1, X_1 - X_2 \rangle \\ &= -\alpha \|X_1 - X_2\|_{H^2}^2 + \langle Y_2 \nabla X_2 - Y_1 \nabla X_2, X_1 - X_2 \rangle \\ &= -\alpha \|X_1 - X_2\|_{H^2}^2 + \langle (Y_2 - Y_1) \nabla X_2, X_1 - X_2 \rangle \\ &\leq -\alpha \|X_1 - X_2\|_{H^2}^2 + \|Y_2 - Y_1\|_{L^2} \|\nabla X_2\|_{L^2} \|X_1 - X_2\|_{L^\infty} \\ &\leq C_\alpha \|X_2 - X_1\|_{L^2}^2 \|\nabla X_2\|_{L^2}^2, \end{aligned}$$

where we used that

$$Y_1 \nabla X_1 = Y_1 \nabla (X_1 - X_2) + Y_1 \nabla X_2$$

and that, by an integration by part and the property $\nabla \cdot Y_2 = 0$,

$$-\langle Y_1 \nabla (X_1 - X_2), X_1 - X_2 \rangle = \langle \nabla \cdot Y_1, \frac{1}{2} (X_1 - X_2)^2 \rangle = 0.$$

Thus, assumption (H2) in [47] holds with $K = 0$ and $\rho(X) = \|X\|_{H^1}^2$. Next, the assumption (H3) in [47],

$$2\langle A(X), X \rangle + \delta \|X\|_{H^2}^\gamma \leq K \|X\|_{L^2}^2 + f(t),$$

holds in our case, due to cancellation in the non-linear term, for $\delta < 2\alpha$, $f \equiv 0$, and $\gamma = 2$.

Finally, as stated in [47, Remark 3.2, (4)] the growth assumption in [47, (H4)] is only needed to prove that $\mathbb{E}\|A(X^{(n)})\|_{K^*}^2$ is uniformly bounded in n , which was already established in (2.4).

The rest of the roof follows line by line the same as in [47, proof of Theorem 1.1]. \square

3. Probabilistic Estimates for the Stochastic Flow

In this section, we derive moment bounds on solutions of (1.2), that were constructed in Section 2. Our choice of norms is dictated by the conserved quantities of (1.1), and it is essential to keep track of dependencies of constants on α . The proofs are based on energy estimates and Itô lemma recalled in “Appendix B”.

For A_0 defined in (1.5), observe that, for any $p > 0$

$$\int_{\mathbb{T}^2} \left(\sum_{j=1}^{\infty} a_j^2 (e_j(x))^2 \right)^{\frac{p}{2}} dx \leq \frac{1}{(2\pi)^p} \int_{\mathbb{T}^2} \left(\sum_{j=1}^{\infty} a_j^2 \right)^{\frac{p}{2}} dx = (2\pi)^{2-p} A_0^{\frac{p}{2}} < \infty,$$

and recall the notation

$$M(\theta) = \frac{1}{2} \int_{\mathbb{T}^2} \theta^2 dx.$$

Theorem 3.1. *Assume $A_0 < \infty$. Then, the solution θ constructed in Theorem 2.1 satisfies the following properties:*

(1) *If $\mathbb{E}\|\theta(0)\|_{H^{-1/2}}^2 < \infty$, then for any $t \geq 0$*

$$\begin{aligned} \mathbb{E}\|\theta(t)\|_{H^{-1/2}}^2 + 2\alpha\mathbb{E} \int_0^t \|\theta(s)\|_{H^{\frac{3}{2}}}^2 ds + 2\alpha\mathbb{E} \int_0^t \int_{\mathbb{T}^2} |\nabla\theta(s)|^2 \nabla\theta \cdot \nabla(-\Delta^{-\frac{1}{2}})\theta dx ds \\ = \mathbb{E}\|\theta_0\|_{H^{-1/2}}^2 + \alpha A_{-1}t. \end{aligned} \quad (3.1)$$

(2) *If $\mathbb{E}M^q(\theta(0)) < \infty$ for some $q \geq 1$, then for any $t \geq 0$*

$$\begin{aligned} \mathbb{E}M^q(\theta(t)) + 2\alpha q\mathbb{E} \int_0^t M^{q-1}(\theta)(\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4) ds = \mathbb{E}M^q(\theta(0)) \\ + \alpha q\mathbb{E} \int_0^t A_0 M^{q-1}(\theta) ds + 2(q-1)M^{q-2}(\theta) \sum_{j=1}^{\infty} a_j^2 \left(\int_{\mathbb{T}^2} e_j \theta dx \right)^2 ds. \end{aligned} \quad (3.2)$$

In particular, when $q = 1$, we have for any $t \geq 0$

$$\mathbb{E} \|\theta(t)\|^2 + 2\alpha \int_0^t \mathbb{E} (\|\theta(s)\|_{H^2}^2 + \|\theta(s)\|_{W^{1,4}}^4) ds = \mathbb{E} \|\theta_0\|^2 + \alpha A_0 t. \quad (3.3)$$

Proof. We proceed in the following steps:

Proof of (3.1).

We use that $H^{-\frac{1}{2}}$ is conserved for the SQG equation (1.1) (see (3.4) below). Clearly, the function $\theta \mapsto \|\theta\|_{H^{-1/2}}^2 = \|\Delta^{-1/4}\theta\|^2$ satisfies assumptions (B.1) and (B.2) of Theorem B.1 with $s = -\frac{1}{2}$. To satisfy (B.3), notice that

$$\begin{aligned} & \sum_{j=1}^{\infty} a_j^2 \int_0^t \mathbb{E} \left(\int_{\mathbb{T}^2} (-\Delta)^{-\frac{1}{4}} \theta (-\Delta)^{-\frac{1}{4}} e_j dx \right)^2 ds \\ &= \sum_{j=1}^{\infty} a_j^2 \int_0^t \mathbb{E} \left(\int_{\mathbb{T}^2} \theta (-\Delta)^{-\frac{1}{2}} e_j dx \right)^2 ds \\ &= \sum_{j=1}^{\infty} \frac{a_j^2}{\lambda_j} \int_0^t \mathbb{E} \left(\int_{\mathbb{T}^2} \theta e_j dx \right)^2 ds \\ &\leq C \sum_{j=1}^{\infty} \frac{a_j^2}{\lambda_j} \int_0^t \mathbb{E} \int_{\mathbb{T}^2} \theta^2 dx ds \\ &= C A_{-1} \mathbb{E} \int_0^t \|\theta\|^2 ds < \infty, \end{aligned}$$

where the last inequality follows from (2.1). Thus, heorem B.1 yields

$$\begin{aligned} \mathbb{E} \|\theta(t)\|_{H^{-1/2}}^2 &= \mathbb{E} \|\theta(0)\|_{H^{-1/2}}^2 \\ &+ 2\mathbb{E} \int_0^t \int_{\mathbb{T}^2} (-\Delta)^{-\frac{1}{4}} \theta (-\Delta)^{-\frac{1}{4}} (-\alpha \Delta^2 \theta + \alpha \nabla \cdot (|\nabla \theta|^2 \nabla \theta) - \mathbf{u} \nabla \theta) \\ &\quad + \alpha \sum_{j=1}^{\infty} a_j^2 ((-\Delta)^{-\frac{1}{4}} e_j)^2 dx ds. \end{aligned}$$

Using that $\mathbf{u} = \nabla^\perp (-\Delta)^{\frac{1}{2}} \theta$ is divergence free, we obtain

$$\begin{aligned} \int_{\mathbb{T}^2} (-\Delta)^{-\frac{1}{4}} \theta (-\Delta)^{-\frac{1}{4}} (\mathbf{u} \nabla \theta) dx &= \int_{\mathbb{T}^2} (-\Delta)^{-\frac{1}{2}} \theta \mathbf{u} \nabla \theta dx \\ &= - \int_{\mathbb{T}^2} (\nabla (-\Delta)^{-\frac{1}{2}} \theta \cdot \nabla^\perp (-\Delta)^{-\frac{1}{2}} \theta) \theta dx = 0. \end{aligned} \quad (3.4)$$

Hence, using integration by parts and Fourier representation of fractional Laplacian we obtain

$$\begin{aligned} \mathbb{E}\|\theta(t)\|_{H^{-1/2}}^2 &= \mathbb{E}\|\theta(0)\|_{H^{-1/2}}^2 \\ &+ 2\alpha\mathbb{E}\int_0^t\int_{\mathbb{T}^2}(-\Delta)^{-\frac{1}{4}}\theta(-\Delta)^{-\frac{1}{4}}(-\Delta^2\theta + \nabla \cdot (|\nabla\theta|^2\nabla\theta))\,dx\,ds + \alpha\sum_{j=1}^{\infty}\frac{a_j^2}{\lambda_j}t \\ &= \mathbb{E}\|\theta(0)\|_{H^{-1/2}}^2 - 2\alpha\mathbb{E}\int_0^t\|\theta\|_{\frac{3}{2}}^2\,ds - 2\alpha\mathbb{E}\int_0^t\int_{\mathbb{T}^2}(-\Delta)^{-\frac{1}{2}}\nabla\theta|\nabla\theta|^2\nabla\theta\,dx\,ds + \alpha A_{-1}t, \end{aligned}$$

and (3.1) follows.

Estimate for $M^q(\theta)$. Next, we turn our attention to the moment bounds for M^q with $q \geq 1$. Clearly, the function $\theta \mapsto M^q(\theta) = \frac{1}{2}\|\theta\|^{2q}$ satisfies assumptions (B.1) and (B.2) of Theorem (B.1) with $s = 0$. In order to obtain (B.3), we need to estimate the quadratic variation of the martingale term. Since (e_j) are bounded,

$$\begin{aligned} \sum_{j=1}^{\infty} a_j^2 \mathbb{E} \int_0^t \left(\|\theta\|^{q-1} \int_{\mathbb{T}^2} \theta e_j(x) dx \right)^2 ds &\leq C A_0 \mathbb{E} \int_0^t \|\theta\|^{2q-2} \int_{\mathbb{T}^2} |\theta|^2 dx ds \\ &= C A_0 \mathbb{E} \int_0^t \|\theta\|^{2q} ds < \infty, \end{aligned}$$

where the last inequality follows from (2.1). Hence, by Theorem B.1 we obtain, for any $q \geq 1$,

$$\begin{aligned} \mathbb{E}M^q(\theta(t)) &= \mathbb{E}M^q(\theta(0)) \\ &+ q\mathbb{E}\int_0^t M^{q-1}(\theta) \int_{\mathbb{T}^2} 2\theta(-\alpha\Delta^2\theta + \alpha\nabla \cdot (|\nabla\theta|^2\nabla\theta) - \mathbf{u}\nabla\theta) \\ &+ \alpha\sum_{j=1}^{\infty} a_j^2 e_j^2 dx ds + 2\alpha(q-1)M^{q-2}(\theta) \sum_{j=1}^{\infty} a_j^2 \left(\int_{\mathbb{T}^2} e_j \theta dx \right)^2 ds. \end{aligned}$$

An integration by parts and the property $\nabla \cdot \mathbf{u} = 0$ imply that $\int_{\mathbb{T}^2} \theta \mathbf{u} \nabla \theta dx = 0$, and (3.2) follows after integration by parts. \square

4. Stationary Measures for the Stochastic SQG

In this section we construct invariant measures for the stochastic SQG equation (1.2) and establish its moment bounds, which finishes the second part of Theorem 1.1. As above, it is necessary to keep track of the parameter α , since below we pass α to zero. Also note that the moment estimates are equalities, which will be important in the proof of non-degeneracy of the limiting measure. The proof of existence of invariant measures is based on the Kryloff-Bogoliouboff theorem, and the moment bounds follow from bounds on solutions established in Section 3.

4.1. Construction and Basic Estimates

Theorem 4.1. Assume $A_0 < \infty$. For any $\alpha \in (0, 1)$, the equation (1.2) admits at least one stationary measure μ_α supported on L^2 and satisfying the following properties:

$$\int_{L^2} \left(\|\theta\|_{H^{\frac{3}{2}}}^2 + \int_{\mathbb{T}^2} |\nabla \theta|^2 \nabla \theta \cdot \nabla (-\Delta)^{-\frac{1}{2}} \theta dx \right) \mu_\alpha(d\theta) = \frac{A_{\frac{-1}{2}}}{2}, \quad (4.1)$$

$$\int_{L^2} (\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4) \mu_\alpha(d\theta) = \frac{A_0}{2}. \quad (4.2)$$

More generally, for any $q \geq 1$,

$$\begin{aligned} & \int_{L^2} M^{q-1}(\theta) (\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4) \mu_\alpha(d\theta) \\ &= \int_{L^2} \frac{A_0}{2} M^{q-1}(\theta) + (q-1) M^{q-2}(\theta) \sum_{j=1}^{\infty} a_j^2 \left(\int_{\mathbb{T}^2} e_j \theta dx \right)^2 \mu_\alpha(d\theta). \end{aligned} \quad (4.3)$$

In particular, $\mu_\alpha(H^2) = 1$, any $\alpha > 0$, and any $q \geq 1$ there is C independent of α such that

$$\int_{L^2} M^{q-1}(\theta) (\|\theta\|_{H^2}^2 + \|\theta(s)\|_{W^{1,4}}^4) \mu_\alpha(d\theta) \leq C. \quad (4.4)$$

Proof. Tightness and existence of stationary measures. Let θ_α be the solution of (1.2) with $\theta_\alpha(0) = 0$ almost surely, that is $\theta_\alpha(0)$ is distributed as the Dirac measure concentrated at 0. Then, by (3.3), one has

$$2\alpha \mathbb{E} \int_0^t \|\theta\|_{H^2}^2 ds \leq \alpha A_0 t,$$

and consequently,

$$\frac{1}{t} \mathbb{E} \int_0^t \|\theta\|_{H^2}^2 ds \leq C.$$

For each $t > 0$ define the Borel probability measure on $L^2(\mathbb{T}^2)$ as

$$\mu_\alpha^t(A) = \frac{1}{t} \int_0^t \mathbb{P}(\theta_\alpha(s) \in A) ds,$$

where A is any Borel set in $H^2(\mathbb{T}^2)$. Then,

$$\int_{L^2(\mathbb{T}^2)} \|\theta\|_{H^2}^2 \mu_\alpha^t(d\theta) = \frac{1}{t} \mathbb{E} \int_0^t \|\theta_\alpha(s)\|_{H^2}^2 ds \leq C. \quad (4.5)$$

In particular, if B_R is a ball in H^2 of radius R centered at 0 and $B_R^c = H^2 \setminus B_R$, then by Chebyshev inequality and (4.5) one has

$$\begin{aligned}\mu_\alpha^t(B_R^c) &= \frac{1}{t} \int_0^t \mathbb{P}(\|\theta_\alpha(s)\|_{H^2} \geq R) \, ds \\ &\leq \frac{1}{t} \int_0^t \frac{\mathbb{E}\|\theta(s)\|_{H^2}^2}{R^2} \, ds \leq \frac{C}{R^2}.\end{aligned}$$

Since B_R is compact in $H^{2-\delta}$, $\delta > 0$, then for each $\alpha > 0$, the set of measures $(\mu_\alpha^t)_{t>0}$ is tight, and therefore by Pokhorov theorem it is compact. For any sequence (t_n) with $t_n \rightarrow \infty$, one has that $\mu_\alpha^{t_n}$ has a weakly convergent subsequence converging to μ_α^* . The Bogoliubov-Krylov argument (see for example [23]) implies that μ_α^* is stationary for (1.2).

Also, by using (3.2) with $\theta(0) = 0$, we obtain for any $q \geq 1$ that

$$\begin{aligned}\int_{L^2} M^{q-1}(\theta)(\|\theta\|_{H^2}^2 + \|\theta(s)\|_{W^{1,4}}^4) \mu_\alpha^t(d\theta) &\leq C \int_{L^2} M^{q-1}(\theta) \mu_\alpha^t(d\theta) \\ &= C \int_{\|\theta\|_{H^2}^2 \geq R} M^{q-1}(\theta) \frac{\|\theta\|_{H^2}^2}{\|\theta\|_{H^2}^2} \mu_\alpha^t(d\theta) + C \int_{\|\theta\|_{H^2}^2 < R} M^{q-1}(\theta) \mu_\alpha^t(d\theta) \\ &\leq \frac{C}{R} \int_{\|\theta\|_{H^2}^2 \geq R} M^{q-1}(\theta) \|\theta\|_{H^2}^2 \mu_\alpha^t(d\theta) + C \int_{\|\theta\|_{H^2}^2 < R} M^{q-1}(\theta) \mu_\alpha^t(d\theta).\end{aligned}$$

Choosing $R = 2C$ and using $M(\theta) \leq C\|\theta\|_{H^2}$, we have

$$\begin{aligned}\int_{L^2} M^{q-1}(\theta)(\|\theta\|_{H^2}^2 + \|\theta(s)\|_{W^{1,4}}^4) \mu_\alpha^t(d\theta) \\ \leq C \int_{\|\theta\|_{H^2}^2 < 2C} M^{q-1}(\theta) \mu_\alpha^t(d\theta) \leq C,\end{aligned}$$

where C is independent of t . Then, by the Portmanteau theorem, the same inequality holds true with μ_α^t replaced by μ_α and (4.4) follows.

Estimates for the stationary measures. Denote by μ_α any invariant measure constructed by the above procedure. Let θ_0^α be a random variable with the law μ_α and let θ^α be the solution of (1.2) with initial condition θ_0^α . Then, by (4.4) one has $\mathbb{E}\|\theta_0^\alpha\|^{q-1} < \infty$ and $\mathbb{E}\|\theta_0^\alpha\|_{H^{-\frac{1}{2}}}^2 < \infty$ for any $q \geq 1$.

Also, the invariance of μ_α implies $\mathbb{E}\|\theta^\alpha(t)\|_X^2 = \mathbb{E}\|\theta_0^\alpha\|_X^2$ for X being L^2 , $W^{1,4}$, or H^2 , and consequently, by (3.3),

$$\begin{aligned}t \int_{L^2} \|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4 \mu_\alpha(d\theta) &= \mathbb{E} \int_0^t \|\theta^\alpha(s)\|_{H^2}^2 + \|\theta^\alpha(s)\|_{W^{1,4}}^4 \, ds \\ &= t \frac{A_0}{2},\end{aligned}\tag{4.6}$$

and (4.2) follows.

Similarly, in (3.1) and (3.2), using $\mathbb{E}\|\theta^\alpha(t)\|_{H^{-\frac{1}{2}}}^2 = \mathbb{E}\|\theta_0^\alpha\|_{H^{-\frac{1}{2}}}^2$ and also $\mathbb{E}M^q(\theta(t)) = \mathbb{E}M^q(\theta(0))$ (by the invariance of μ_α) we obtain respectively (4.1) and (4.3). Recall that (4.4) was already proved. \square

5. Inviscid Limit

This section contains proofs of essential assertions of the manuscript detailed in Theorem 1.2 parts 1–5. In particular, we prove convergence of measures μ_α constructed in Theorem 4.1 to an invariant measure μ for the deterministic SQG equation. Furthermore we show that almost all points in the support of μ are initial conditions for regular global solutions.

Theorem 5.1. *If $A_0 < \infty$, there exists a measure μ supported on H^2 with the following properties:*

- (1) *For almost every $\theta_0 \in \text{supp}(\mu)$, there exists a unique, global (existing for all positive times) solution θ of (1.1) with $\theta \in C_t H_x^1 \cap L_t^2 H^2$. Furthermore, for any $t \geq 0$, the map $\theta_0 \mapsto \theta(t, \theta_0)$ is continuous on H^1 .*
- (2) *The measure μ is invariant for (1.1), meaning that for every Borel set A in H^2 , one has $\mu\{\theta_0 : \theta(t, \theta_0) \in A\} = \mu(A)$.*
- (3) *For any $q \geq 1$ we have the moment bounds*

$$\int_{L^2} \left(\|\theta\|_{H^{\frac{3}{2}}}^2 - \int_{\mathbb{T}^2} |\nabla \theta|^2 \nabla \theta \cdot \nabla (-\Delta)^{-\frac{1}{2}} \theta dx \right) \mu(d\theta) = \frac{A_{\frac{-1}{2}}}{2}, \quad (5.1)$$

$$\int_{L^2} M^{q-1}(\theta) (\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4) \mu(d\theta) \leq C. \quad (5.2)$$

Before providing details of the proof, let us first sketch the general strategy.

Remark 5.2. Compared to the known results we face different challenges since it is not known whether the equation (1.1) is globally well posed. This poses several challenges. After verifying the tightness of measures (μ_α) and passing $\mu_\alpha \rightarrow \mu$ as $\alpha \rightarrow 0$, we obtain moment bounds for μ , however we cannot immediately conclude that almost all functions in the support are initial conditions of global solutions. This problem is not solved even if we prove that μ is invariant. For example, there can be a set M_t of measure zero that contain functions that cease to exist at time t . Since $(M_t)_{t>0}$ form an uncountable family, we cannot conclude that the union $\cup M_t$ has zero measure. For that reason we use the “lifted” measures ν_α supported on solutions of (1.2) rather than on initial conditions. To pass $\alpha \rightarrow 0$ and conclude that the limiting measure ν is supported on solutions of (1.1), we have to obtain compactness (tightness) of (ν_α) in spaces of time dependent functions. This follows from improved temporal bounds for the solutions of (1.2). Also, these bounds imply that the restriction of the measure ν at the initial time is μ . Using the Skorokhod theorem we find stationary random variables θ_α distributed as ν_α that converge almost surely to θ , which solves (1.1). In addition, $\theta(0)$ is distributed as μ . To prove the uniqueness of $t \mapsto \theta(t)$ we crucially use that the operator in the fluctuation–dissipation method is bi-Laplacian instead of Laplacian, and therefore θ is supported on $L^2 H^2$, a regularity space sufficient to guarantee uniqueness and continuous dependence on initial conditions.

Proof of Theorem 5.1. The proof is divided into several parts. Proof of 1. follows from Proposition 5.11, part 2 follows from Lemma 5.10 and the proof of 3 follows from Proposition 5.12. \square

If θ_α is a solution of (1.2) with $\theta_\alpha(0)$ distributed as μ_α (see Theorem 4.1), then due to the invariance, $\theta_\alpha(t)$ is distributed as μ_α for any $t \geq 0$. We can either view $t \mapsto \theta_\alpha(t)$ as a random process with range in a space of x dependent functions or alternatively, we can view θ_α as a random variable on a space of (x, t) dependent functions in $L^2_{loc}(\mathbb{R}_+, H^2)$ (see (4.5)).

Denote ν_α the distribution of θ_α and by the invariance of μ_α one has $\mathbb{P}(\theta_\alpha(t_0) \in A) = \mu_\alpha(A)$ for any $t_0 \geq 0$, and any Borel set A in $H^{-\delta}$, $\delta \in [0, 1)$. Observe that μ_α is supported on H^2 and we can trivially (by zero) extend it to the larger space $H^{-\delta}$. Hence,

$$\int \chi_{A \times \{t_0\}} d\nu_\alpha = \nu_\alpha(A \times \{t_0\}) = \mathbb{P}(\theta_\alpha(t_0) \in A) = \mu_\alpha(A) = \int_A \chi_A d\mu_\alpha,$$

where χ_Z denotes the characteristic function that a set Z . The linearity of integrals and the dominated convergence theorem also implies

$$\int_{C_t H_x^{-\delta}} g(\theta(t_0)) d\nu_\alpha(\theta) = \int_{H_x^{-\delta}} g(\theta) d\mu_\alpha(\theta) \quad (5.3)$$

for any bounded continuous function $g : H_x^{-\delta} \rightarrow \mathbb{R}$.

Fix $T > 0$ and define $I = (0, T) \subset \mathbb{R}$.

Remark 5.3. In this section we implicitly assume that all spaces are defined on the time interval I . For example, $L^2 H^2 = L^2(I, H^2(\mathbb{T}^2))$ or $H_t^1 L_x^4 = H^1(I, L^4(\mathbb{T}^2))$. We use the notation, say L^4 (single space), to denote $L^4(\mathbb{T}^2)$, that is, we do not specify regularity in time.

Also, θ_α denotes the solution of (1.2), that is, a function depending on x and t , whereas θ denotes the integration variable, that is, a function depending on x only.

By (4.4), and the invariance of μ_α , for any $q \geq 0$,

$$\begin{aligned} C &\geq \int_0^T \int_{L_x^2} \|\theta\|^q \|\theta\|_{H^2}^2 d\mu_\alpha(\theta) ds = \int_0^T \mathbb{E} \|\theta_\alpha(s)\|^q \|\theta_\alpha(s)\|_{H^2}^2 ds = \mathbb{E} \|\theta_\alpha\|^{\frac{q}{2}} \|\theta_\alpha\|_{H_x^2}^2 \|L_t^2\|^2 \\ &= \int_{L_{t,x}^2} \|\theta\|^{\frac{q}{2}} \|\theta\|_{H_x^2}^2 d\nu_\alpha(\theta), \end{aligned} \quad (5.4)$$

where here and below, C is allowed to implicitly depend on T . To gain the temporal compactness in time, we prove the regularity of θ_α in time.

Lemma 5.4. Set $\mathcal{X} = H^1 L^2 + H^1 H^{-2} + W^{1, \frac{4}{3}} W^{-1, \frac{4}{3}} + W^{\kappa, 4} L^2$ with $\kappa \in (1/4, 1/2)$ equipped with the standard sum norm (cf. (2.3)). Then

$$C \geq \mathbb{E} \|\theta_\alpha\|_{\mathcal{X}}^{\frac{4}{3}} = \int_{L_{t,x}^2} \|\theta\|_{\mathcal{X}}^{\frac{4}{3}} d\nu_\alpha(\theta),$$

Proof. Observe that

$$\begin{aligned} \theta_\alpha(t) = & \underbrace{\theta_\alpha(0) - \int_0^t \mathbf{u} \cdot \nabla \theta_\alpha ds}_I - \underbrace{\alpha \int_0^t \Delta^2 \theta_\alpha ds}_{II} \\ & + \underbrace{\alpha \int_0^t \nabla \cdot (|\nabla \theta_\alpha|^2 \nabla \theta_\alpha) ds}_{III} + \underbrace{\sqrt{\alpha} \int_0^t d\eta}_{IV}. \end{aligned} \quad (5.5)$$

First, by interpolation, (1.6), and embeddings we have (spatial norm)

$$\begin{aligned} \|\mathbf{u} \cdot \nabla \theta\|_{L_x^2} &\leq C \|\mathbf{u}\|_{L^4} \|\nabla \theta\|_{L^4} \\ &\leq C \|\theta\|_{H^{\frac{1}{2}}} \|\theta\|_{H^{\frac{3}{2}}} \leq C \|\theta\| \|\theta\|_{H^2}, \end{aligned}$$

and by (4.2) and (4.4) with $q = 3$,

$$\mathbb{E} \|I\|_{H_t^1 L_x^2}^2 \leq 2 \left(\mathbb{E} \|\theta_\alpha(0)\|_{L_x^2}^2 + \mathbb{E} \|\mathbf{u} \cdot \nabla \theta_\alpha\|_{L_{t,x}^2}^2 \right) \leq C.$$

Second, using (4.2), we have

$$\mathbb{E} \|II\|_{H_t^1 H_x^{-2}}^2 \leq C \mathbb{E} \int_0^T \|\Delta^2 \theta\|_{H^{-2}}^2 dt = \mathbb{E} \int_0^T \|\theta\|_{H^2}^2 \leq CT. \quad (5.6)$$

Moreover,

$$\|\nabla \cdot (|\nabla \theta|^2 \nabla \theta)\|_{W^{-1, \frac{4}{3}}} \leq C \|\nabla \theta\|_{L^{\frac{4}{3}}}^3 = C \|\nabla \theta\|_{L^4}^3 = C \|\theta\|_{W^{1,4}}^3,$$

and therefore, by (4.4),

$$\begin{aligned} \mathbb{E} \|III\|_{W_t^{1, \frac{4}{3}} W_x^{-1, \frac{4}{3}}}^{\frac{4}{3}} &\leq \mathbb{E} \|\nabla \cdot (|\nabla \theta_\alpha|^2 \nabla \theta_\alpha)\|_{L^{\frac{4}{3}} W^{-1, \frac{4}{3}}}^{\frac{4}{3}} \\ &\leq C \mathbb{E} \int_0^T \|\theta_\alpha\|_{W^{1,4}}^4 \leq C. \end{aligned} \quad (5.7)$$

Finally, since for any m and $0 \leq s \leq t$, $W_m(t) - W_m(s) \sim \mathcal{N}(0, t - s)$, we have $\mathbb{E}|W_m(t) - W_m(s)|^2 = t - s$ and $\mathbb{E}|W_m(t) - W_m(s)|^4 = 3(t - s)^2$. By the independence of W_j and W_k for $j \neq k$, one has

$$\begin{aligned}
\mathbb{E}\|IV(t) - IV(s)\|^4 &= \mathbb{E}\left(\sum_{j=1}^{\infty} a_j^2 |W_j(t) - W_j(s)|^2\right)^2 \\
&= \mathbb{E}\sum_{j,k=1}^{\infty} a_j^2 a_k^2 |W_j(t) - W_j(s)|^2 |W_k(t) - W_k(s)|^2 \\
&= 3|t-s|^2 \sum_{j,k=1}^{\infty} a_j^2 a_k^2 (1 + 2\delta_{jk}) \leq 9|t-s|^2 \sum_{j,k=1}^{\infty} a_j^2 a_k^2 \leq 9|t-s|^2 A_0^2,
\end{aligned}$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. Consequently, if $\kappa < \frac{1}{2}$,

$$\begin{aligned}
\mathbb{E}\|IV\|_{W^{\kappa,4}L^2}^4 &= \mathbb{E}\int_0^T \|IV\|_{L_x^2}^4 dt + \mathbb{E}\int_0^T \int_0^T \frac{\|IV(t) - IV(s)\|^4}{|t-s|^{1+4\kappa}} dt ds \\
&\leq C \int_0^T t^2 dt + C \int_0^T \int_0^T |t-s|^{1-4\kappa} dt ds \leq C(T).
\end{aligned}$$

Overall,

$$\mathbb{E}\|\theta_\alpha\|_{\mathcal{X}}^{\frac{4}{3}} \leq \mathbb{E}\|I\|_{H^1L^2}^2 + \alpha\|II\|_{H^1H^{-2}}^2 + \alpha\|III\|_{W^{1,\frac{4}{3}}W^{-1,\frac{4}{3}}}^{\frac{4}{3}} + \mathbb{E}\|IV\|_{W^{\kappa,4}L^2}^4 \leq C,$$

where C is independent of $\alpha \in (0, 1)$ and the result follows. \square

Proposition 5.5. *For any $\delta > 0$ denote $\mathcal{Y}_\delta = L^2H^{2-\delta} \cap CH^{-\delta}$. Let \mathcal{X} be as in Lemma 5.4 for some $\kappa \in (\frac{1}{4}, \frac{1}{2})$. Then, for any $q \geq 0$ there is a constant C independent of α such that*

$$\int \|\theta\|_{\mathcal{X}}^{\frac{4}{3}} dv_\alpha = \mathbb{E}\|\theta_\alpha\|_{\mathcal{X}}^{\frac{4}{3}} \leq C, \quad (5.8)$$

$$\int \|\theta\|_{\mathcal{X}}^{\frac{q}{2}} \|\theta\|_{H_x^2}^2 \|L_t^2 dv_\alpha = \mathbb{E}\|\theta_\alpha\|_{\mathcal{X}}^{\frac{q}{2}} \|\theta_\alpha\|_{H_x^2}^2 \|L_t^2 \leq C. \quad (5.9)$$

Moreover, for any $\delta > \frac{1}{3}$ the set of measures $(v_\alpha)_\alpha$ is tight in \mathcal{Y}_δ . Consequently, there is a sequence $(v_k) := (v_{\alpha_k})$ with $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$, and a measure ν supported on \mathcal{Y}_δ such that v_k converges weakly to ν as $k \rightarrow \infty$.

Proof. The estimates (5.8) and (5.9) follow from Lemma 5.4 and (5.4) respectively.

We claim that \mathcal{Y}_δ is compactly embedded in $\mathcal{X} \cap L^2H^2$ for any $\delta > \frac{1}{3}$. Indeed, by [45, Theorem 5.1 and 5.2], for any $\delta > 0$, $L^2H^2 \cap \mathcal{X}$ is compactly embedded in $L^2H^{2-\delta}$. Also, for any $\delta > 0$, [46, Theorem 3.1], [39, Lemma 11.2.4], and standard Sobolev embedding imply that $L^2H^2 \cap H^1H^{-2}$ and $L^2H^2 \cap W^{\kappa,4}L^2$ are compactly embedded in $CH^{-\delta}$. Finally, by Appendix C, for any $\delta > \frac{1}{3}$, $L^2H^2 \cap W^{1,\frac{4}{3}}W^{-1,\frac{4}{3}}$ is compactly embedded in $CH^{-\delta}$, and the claim follows.

Let B_R be the ball in $\mathcal{X} \cap L^2H^2$ of radius R centered at the origin. By the just proved compactness, B_R is compact in \mathcal{Y}_δ . Furthermore, by Chebyshev inequality

$$v_\alpha(B_R^c) = \mathbb{P}(\|\theta_\alpha\|_{\mathcal{X} \cap L^2H^2} \geq R) \leq \frac{\mathbb{E}\|\theta_\alpha\|_{\mathcal{X} \cap L^2H^2}^{\frac{4}{3}}}{R^{\frac{4}{3}}} \leq \frac{C}{R^{\frac{4}{3}}}$$

and therefore the set of measures $(\nu_\alpha)_\alpha$ is tight in \mathcal{Y}_δ . The existence of appropriate sequence follows from Prokhorov theorem. \square

Lemma 5.6. *Let $\nu_k \rightarrow \nu$ in be as in Proposition 5.5. Then, there is a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$, on which is defined a sequence of random variables $(\tilde{\theta}_k)$ and a random variable $\tilde{\theta}$ having the following properties:*

- (1) *The law of $\tilde{\theta}$ is ν and for every k , the law of $\tilde{\theta}_k$ is ν_k .*
- (2) *For any $\delta > \frac{1}{3}$, the sequence $\tilde{\theta}_k$ converges to $\tilde{\theta}$ almost surely, that is, for $\tilde{\mathbb{P}}$ almost every $\omega \in \tilde{\Omega}$ one has $\|\tilde{\theta}_k(\omega) - \tilde{\theta}(\omega)\|_{\mathcal{Y}_\delta} \rightarrow 0$ as $k \rightarrow \infty$.*
- (3) *For each k , $\tilde{\theta}_k$ satisfies (5.5).*

Furthermore, by passing to a sub-sequence if necessary, $\tilde{\theta}_k$ converges weakly to $\tilde{\theta}$ in $L^{\frac{4}{3}}(\tilde{\Omega}, \mathcal{X}) \cap L^2(\tilde{\Omega}, L^2 H^2)$ and, for any $q \geq 0$,

$$\int \|\theta\|_{\mathcal{X}}^{\frac{4}{3}} d\nu(\theta) = \mathbb{E} \|\tilde{\theta}\|_{\mathcal{X}}^{\frac{4}{3}} \leq \liminf_{k \rightarrow \infty} \mathbb{E} \|\tilde{\theta}_k\|_{\mathcal{X}}^{\frac{4}{3}} \leq C_T, \quad (5.10)$$

$$\begin{aligned} \int \|\theta\|^{\frac{q}{2}} \|\theta\|_{H_x^2}^2 \|\theta\|_{L_t^2}^2 d\nu(\theta) &= \mathbb{E} \|\tilde{\theta}\|^{\frac{q}{2}} \|\tilde{\theta}\|_{H_x^2}^2 \|\tilde{\theta}\|_{L_t^2}^2 \\ &\leq \liminf_{k \rightarrow \infty} \mathbb{E} \|\tilde{\theta}_k\|^{\frac{q}{2}} \|\tilde{\theta}_k\|_{H_x^2}^2 \|\tilde{\theta}_k\|_{L_t^2}^2 \leq C_T. \end{aligned} \quad (5.11)$$

Proof. Since \mathcal{Y}_δ with $\delta \in (\frac{1}{3}, \frac{1}{2})$ is a separable metric space, Skorokhod theorem (see [26, Theorem 11.7.2]) implies (1) and (2). Moreover, (3) follows analogously as in [4, Section 4.3.4].

By Proposition 5.5, $(\tilde{\theta}_k)$ is uniformly bounded in the space $\mathcal{Z} = L^{\frac{4}{3}}(\tilde{\Omega}, \mathcal{X}) \cap L^2(\tilde{\Omega}, L^2 H^2)$, and therefore, up to a subsequence, $(\tilde{\theta}_k)$ weakly converges in \mathcal{Z} to some $\hat{\theta}$. In particular, $(\tilde{\theta}_k)$ converges weakly to $\hat{\theta}$ in $L^{\frac{4}{3}}(\tilde{\Omega}, \mathcal{Y}_\delta)$. Since $(\tilde{\theta}_k)$ converges almost surely (up to subsequence) in $L^{\frac{4}{3}}(\tilde{\Omega}, \mathcal{Y}_\delta)$ to $\tilde{\theta}$, then by [25, Proposition 9.1c] (trivially modified to Banach spaces), weak and almost sure limits are equal, and therefore $\hat{\theta} = \tilde{\theta}$. For another approach see [66, Proposition 16.6].

Finally, (5.10) and (5.11) follows from the weak lower semi-continuity of norms and (5.8), (5.9) respectively. \square

Next, we prove that $\tilde{\theta}$ satisfies (1.1) almost surely. Before proceeding, we prove the following auxiliary result:

Lemma 5.7. *Fix $\delta \in (\frac{1}{3}, \frac{2}{3})$ and recall $\mathcal{Y}_\delta = C_t H^{-\delta} \cap L_t^2 H^{2-\delta}$. For any sufficiently smooth $\theta \in \mathcal{Y}_\delta$ one has*

$$\|R^\perp \theta \cdot \nabla \theta\|_{L_t^2 H^{-1}} \leq C \|\theta\|_{\mathcal{Y}_\delta}^2. \quad (5.12)$$

Also, the map $B : \mathcal{Y}_\delta \rightarrow L_t^2 H_x^{-1}$ defined as $B(\theta) = R^\perp \theta \cdot \nabla \theta$ is continuous.

Proof. It suffices to prove the assertion for smooth functions and then use a standard argument to pass to the limit.

For any smooth θ_1, θ_2 , with help of (1.6), Agmon's inequality, and interpolation, one has

$$\begin{aligned}
\|B(\theta_1) - B(\theta_2)\|_{H_x^{-1}} &\leq \|R^\perp(\theta_1 - \theta_2) \cdot \nabla \theta_1\|_{H_x^{-1}} + \|R^\perp \theta_2 \cdot \nabla(\theta_1 - \theta_2)\|_{H_x^{-1}} \\
&\leq \| |R^\perp(\theta_1 - \theta_2)| \theta_1 \| + \| |R^\perp \theta_2| (\theta_1 - \theta_2) \| \\
&\leq C(\|R^\perp(\theta_1 - \theta_2)\| \|\theta_1\|_{L_x^\infty} + \|R^\perp \theta_2\|_{L_x^\infty} \|(\theta_1 - \theta_2)\|) \\
&\leq C\|\theta_1 - \theta_2\| (\|\theta_1\|_{H_x^{-\delta}}^{\frac{1}{2}-\frac{\delta}{4}} \|\theta_1\|_{H_x^{2-\delta}}^{\frac{1}{2}+\frac{\delta}{4}} + \|R^\perp \theta_2\|_{H_x^{-\delta}}^{\frac{1}{2}-\frac{\delta}{4}} \|R^\perp \theta_2\|_{H_x^{2-\delta}}^{\frac{1}{2}+\frac{\delta}{4}}) \\
&\leq C\|\theta_1 - \theta_2\|_{H_x^{-\delta}}^{1-\frac{\delta}{2}} \|\theta_1 - \theta_2\|_{H_x^{2-\delta}}^{\frac{\delta}{2}} (\|\theta_1\|_{H_x^{-\delta}}^{\frac{1}{2}-\frac{\delta}{4}} \|\theta_1\|_{H_x^{2-\delta}}^{\frac{1}{2}+\frac{\delta}{4}} \\
&\quad + \|\theta_2\|_{H_x^{-\delta}}^{\frac{1}{2}-\frac{\delta}{4}} \|\theta_2\|_{H_x^{2-\delta}}^{\frac{1}{2}+\frac{\delta}{4}}).
\end{aligned}$$

To prove (5.12) we set $\theta_2 \equiv 0, \theta_1 = \theta$. After integration in time and an application of Jensen's inequality we obtain, for $\delta \in (0, \frac{2}{3})$,

$$\begin{aligned}
\int_0^T \|B(\theta)\|_{H_x^{-1}}^2 ds &\leq C \int_0^T \|\theta\|_{H_x^{-\delta}}^{3-\frac{3\delta}{2}} \|\theta\|_{H_x^{2-\delta}}^{1+\frac{3\delta}{2}} ds \\
&\leq C_T \|\theta\|_{L_t^\infty H_x^{-\delta}}^{3-\frac{3\delta}{2}} \|\theta\|_{L_t^2 H_x^{2-\delta}}^{1+\frac{3\delta}{2}} \leq C_T \|\theta\|_{\mathcal{Y}_\delta}^4.
\end{aligned}$$

To prove the continuity of B we observe that, for $\delta \in (0, \frac{2}{3})$,

$$\begin{aligned}
&\int_0^T \|\theta_1 - \theta_2\|_{H_x^{-\delta}}^{2-\delta} \|\theta_1 - \theta_2\|_{H_x^{2-\delta}}^\delta \|\theta_1\|_{H_x^{-\delta}}^{1-\frac{\delta}{2}} \|\theta_1\|_{H_x^{2-\delta}}^{1+\frac{\delta}{2}} ds \\
&\leq \|\theta_1 - \theta_2\|_{L_t^\infty H_x^{-\delta}}^{2-\delta} \|\theta_1\|_{L_t^\infty H_x^{-\delta}}^{1-\frac{\delta}{2}} \int_0^T \|\theta_1 - \theta_2\|_{H_x^{2-\delta}}^\delta \|\theta_1\|_{H_x^{2-\delta}}^{1+\frac{\delta}{2}} ds \\
&\leq C_T \|\theta_1 - \theta_2\|_{L_t^\infty H_x^{-\delta}}^{2-\frac{\delta}{2}} \|\theta_1\|_{L_t^\infty H_x^{-\delta}}^{1-\frac{\delta}{2}} \|\theta_1 - \theta_2\|_{L_t^2 H_x^{2-\delta}}^\delta \|\theta_1\|_{L_t^2 H_x^{2-\delta}}^{1+\frac{\delta}{2}} \\
&\leq C_T \|\theta_1 - \theta_2\|_{\mathcal{Y}_\delta}^2 \|\theta_1\|_{\mathcal{Y}_\delta}^2.
\end{aligned}$$

Thus,

$$\|B(\theta_1) - B(\theta_2)\|_{L_t^2 H_x^{-1}} \leq C_T \|\theta_1 - \theta_2\|_{\mathcal{Y}_\delta} (\|\theta_1\|_{\mathcal{Y}_\delta} + \|\theta_1\|_{\mathcal{Y}_\delta}),$$

as desired. \square

Lemma 5.8. For $\tilde{\theta}$ defined in Lemma 5.6 one has almost surely $\tilde{\theta} \in C_t H_x^1 \cap L_t^2 H_x^2 \cap H_t^1 L_x^2$ locally in time. Furthermore, $\tilde{\theta}$ almost surely satisfies

$$\tilde{\theta}(t) = \tilde{\theta}(0) - \int_0^t \mathbf{u} \cdot \nabla \tilde{\theta} ds \quad t \geq 0,$$

that is, $\tilde{\theta}$ is a strong solution of (1.1) on $[0, T]$.

Proof. Let (α_k) be as in Proposition 5.5. Replacing α by α_k in (5.5) and setting $\tilde{\theta}_k = \tilde{\theta}_{\alpha_k}$ (see property Lemma 5.6 part (3)), we have

$$\begin{aligned}\tilde{\theta}_k(t) &= \tilde{\theta}_k(0) - \int_0^t \mathbf{u}_k \cdot \nabla \tilde{\theta}_k ds + \alpha_k \int_0^t \nabla \cdot (|\nabla \tilde{\theta}_k|^2 \nabla \tilde{\theta}_k) - \Delta^2 \tilde{\theta}_k ds + \sqrt{\alpha_k} \zeta \\ &= I + \alpha_k II + \sqrt{\alpha_k} III.\end{aligned}$$

By Proposition 5.5 (cf. (5.6) and (5.7))

$$\alpha_k II, \sqrt{\alpha_k} III \rightarrow 0, \quad (5.13)$$

where the convergence is in $L^{\frac{4}{3}}(\tilde{\Omega}, \mathcal{X})$. Then, by Chebyshev inequality, (5.13) holds in probability and by passing to a sub-sequence, we can assume that (5.13) holds almost surely in \mathcal{X} . Since $\mathcal{X} \hookrightarrow C_t H_x^{-2}$, (5.13) holds almost surely in $C_t H_x^{-2}$.

If $\delta \in (\frac{1}{3}, \frac{2}{3})$, then Lemma 5.7 yields that $\tilde{\theta} \mapsto R^\perp \tilde{\theta} \cdot \nabla \tilde{\theta}$ is continuous as map from \mathcal{Y}_δ into $L^2(I, H_x^{-1})$. Hence, as $k \rightarrow \infty$,

$$\int_0^t \mathbf{u}_k \cdot \nabla \tilde{\theta}_k ds \rightarrow \int_0^t \mathbf{u} \cdot \nabla \tilde{\theta} ds \quad \text{in } H_t^1 H_x^{-1} \quad \text{a.s.,}$$

From Lemma 5.6, part (2) follows almost surely:

$$\tilde{\theta}_k \rightarrow \tilde{\theta}, \quad \text{in } C_t H_x^{-\delta}.$$

Overall, almost surely we have, for any $t \in I$, that

$$\tilde{\theta}(t) = \tilde{\theta}(0) - \int_0^t \mathbf{u} \cdot \nabla \tilde{\theta} ds \quad \text{in } C_t H_x^{-2}.$$

To obtain the regularity of $\tilde{\theta}$, observe that (5.11) implies almost surely $\tilde{\theta} \in L^2 H^2$. Also, by interpolation, properties of Riesz transform, and Agmon's inequality,

$$\begin{aligned}\|\mathbf{u} \cdot \nabla \tilde{\theta}\|_{L_x^2} &\leq \|\mathbf{u}\|_{L_x^\infty} \|\nabla \tilde{\theta}\|_{L_x^2} \\ &\leq C \|\mathbf{u}\|_{L_x^2}^{\frac{1}{2}} \|\mathbf{u}\|_{H_x^2}^{\frac{1}{2}} \|\tilde{\theta}\|_{L_x^2}^{\frac{1}{2}} \|\tilde{\theta}\|_{H_x^2}^{\frac{1}{2}} \\ &\leq C \|\tilde{\theta}\|_{L_x^2} \|\tilde{\theta}\|_{H_x^2}.\end{aligned}$$

Consequently, by (5.11),

$$\mathbb{E} \|\mathbf{u} \cdot \nabla \tilde{\theta}\|_{L_{x,t}^2}^2 \leq C \mathbb{E} \|\tilde{\theta}\|_{H_x^2} \|\tilde{\theta}\|_{L_t^2}^2 \leq C,$$

and therefore, almost surely $\partial_t \tilde{\theta} \in L_{t,x}^2$. Then, the Lions-Magenes lemma (see [46, Theorem 3.1]) yields that $\tilde{\theta}$ belongs a.s. locally to $C_t H_x^1 \cap L_t^2 H_x^2 \cap H_t^1 L_x^2$. \square

The proved regularity is exactly a borderline case for the proof of uniqueness. As such we cannot use direct energy estimates, but we have to employ more subtle argument of Judovich, who used it for Euler equation, see [35, 50]. In particular, we need a precise estimates on the Sobolev embedding constants.

Lemma 5.9. *Solution of (1.1) with $\theta(0) = \theta_0 \in H^1$ that belongs to $C_t H^1 \cap L_t^2 H^2 \cap H_t^1 L_x^2$ is unique.*

Proof. Let $\theta_i \in C_t H^1 \cap L_t^2 H^2 \cap H_t^1 L_x^2$, $i = 1, 2$ be two solution of (1.1) with $\theta_1(0) = \theta_2(0)$. Then, $w = \theta_1 - \theta_2$ satisfies

$$w_t = -R^\perp w \cdot \nabla \theta_1 - R^\perp \theta_1 \cdot \nabla w. \quad (5.14)$$

Testing with w and using (1.6) yields

$$\frac{d}{dt} \|w\|^2 \leq 2|(w, R^\perp w \cdot \nabla \theta_1)| \leq C \|w\|_{L^{2p}} \|w\| \|\nabla \theta_1\|_{L^{\frac{2p}{p-1}}}.$$

where C is independent of p . By interpolation, we have, for any $p \in (1, 2)$,

$$\|w\|_{L^{2p}} \leq \|w\|^{2-p} \|w\|_{L^{\frac{2p}{p-1}}}^{p-1}. \quad (5.15)$$

Using Hölder inequality, $p < 2$, and Sobolev inequality with a precise constant (see for example [15, Remark 1.5]), we obtain

$$\begin{aligned} \frac{d}{dt} \|w\|^2 &\leq C \|w\|^{3-p} \|w\|_{L^{\frac{2p}{p-1}}}^{p-1} \|\nabla \theta_1\|_{L^{\frac{2p}{p-1}}} \\ &\leq C \sqrt{\frac{2p}{p-1}} \|w\|^{2\delta} (\|\theta_1\|_{H^2}^2 + \|\theta_2\|_{H^2}^2)^{\frac{p}{2}} \\ &\leq \frac{C}{\sqrt{1-\delta}} \|w\|^{2\delta} (1 + \|\theta_1\|_{H^2}^2 + \|\theta_2\|_{H^2}^2), \end{aligned}$$

where $\delta = \frac{3-p}{2} < 1$ and $p \in (1, 2)$. Then, after recalling that $w(0) = 0$, we have

$$\frac{1}{1-\delta} \|w(t)\|^{2(1-\delta)} \leq \frac{C}{\sqrt{1-\delta}} \int_0^t (1 + \|\theta_1\|_{H^2}^2 + \|\theta_2\|_{H^2}^2) ds,$$

and consequently,

$$\|w(t)\|^2 \leq C \left(\sqrt{1-\delta} \int_0^t (1 + \|\theta_1\|_{H^2}^2 + \|\theta_2\|_{H^2}^2) ds \right)^{\frac{1}{1-\delta}}.$$

Since $\theta_i \in L_t^2 H^2$, then for any $t_0 \geq 0$ one has $\sqrt{1-\delta} \int_0^{t_0} (1 + \|\theta_1\|_{H^2}^2 + \|\theta_2\|_{H^2}^2) ds < 1$ for any $\delta < 1$ sufficiently close to 1. Passing $p \rightarrow 1$ (or equivalently $\delta \rightarrow 1$), we arrive at

$$\|w(t)\| = 0 \quad \text{for any } t \leq t_0.$$

Since t_0 was arbitrary, $\|w(t)\| = 0$ for any $t \geq 0$, as desired. \square

Lemma 5.10. *The law of $\tilde{\theta}(t)$ is independent of t and is equal to μ . Here, μ is a weak limit of (a sub-sequence) (μ_α) as $\alpha \rightarrow 0$ in the space $H^{2-\gamma}$, $\gamma > 0$, where μ_α was defined in Theorem 4.1. Furthermore, μ is concentrated on H^2 .*

Proof. From Chebyshev inequality and (4.4) it follows that

$$\mu_\alpha(B_R^c) \leq \frac{1}{R^2} \int_{L^2} \|\theta\|_{H^2}^2 d\mu_\alpha(\theta) \leq \frac{C}{R^2}, \quad (5.16)$$

where B_R is a ball of radius R in H^2 and C is independent of α . Since H^2 is compactly embedded in $H^{2-\gamma}$, $\gamma > 0$, the Prokhorov theorem implies that there exists a weakly convergent sequence (μ_{α_k}) in $H^{2-\gamma}$ to μ . To prove that μ is supported on H^2 note that, by (5.16),

$$\mu_\alpha(B_R) \geq 1 - \frac{C}{R^2} \quad (5.17)$$

and by the Portmanteau theorem, (5) holds with μ_α replaced by μ . Passing $R \rightarrow \infty$, one obtain $\mu(H^2) = 1$.

Fix $\tau \in [0, T)$ and a bounded continuous function $g : H^{-\delta} \rightarrow \mathbb{R}$ and define $G(\theta) = g(\theta(\tau))$. We claim that for any $\delta \in (\frac{1}{3}, \frac{1}{2})$, $G : \mathcal{Y}_\delta \rightarrow \mathbb{R}$ is bounded continuous. Indeed, if $\|\theta_1 - \theta_2\|_{\mathcal{Y}_\delta} < \varepsilon$, then $\|\theta_1(\tau) - \theta_2(\tau)\|_{H^{-\delta}} < \varepsilon$ and

$$|G(\theta_1) - G(\theta_2)| = |g(\theta_1(\tau)) - g(\theta_2(\tau))|,$$

and the boundedness and continuity of G follows from the boundedness and continuity of g .

By Proposition 5.5,

$$\lim_{k \rightarrow \infty} \int_{\mathcal{Y}_\delta} G(\theta) dv_k(\theta) = \int_{\mathcal{Y}_\delta} G(\theta) dv(\theta)$$

and by using (5.3) and weak converges of (v_k) and (μ_k) , one obtains

$$\begin{aligned} \mathbb{E}g(\tilde{\theta}(\tau)) &= \int_{\mathcal{Y}_\delta} g(\theta(\tau)) dv(\theta) = \lim_{k \rightarrow \infty} \int_{\mathcal{Y}_\delta} g(\theta(\tau)) dv_k(\theta) \\ &= \lim_{k \rightarrow \infty} \int_{H^{-\delta}} g(\theta) d\mu_k(\theta) = \int_{H^{-\delta}} g(\theta) d\mu(\theta). \end{aligned}$$

Thus, for any τ , the law of $\tilde{\theta}$ is μ , as desired. \square

Proposition 5.11. *For $\tilde{\theta}$ defined in Lemma 5.6 one has almost surely $\tilde{\theta} \in C(\mathbb{R}_+, H_x^1) \cap L^2(\mathbb{R}_+, H_x^2) \cap H^1(\mathbb{R}_+, L_x^2)$ and $\tilde{\theta}$ satisfies (1.1). Furthermore, $t \mapsto \tilde{\theta}(t)$ is unique, and $\tilde{\theta}$ depends continuously on initial conditions, that is,*

$$\lim_{\|\theta_1(T_0) - \theta_2(T_0)\|_{H_x^1} \rightarrow 0} \sup_{t \in [T_0, T_1]} \|\theta_1(t) - \theta_2(t)\|_{H_x^1} = 0. \quad (5.18)$$

We remark that by changing t to $-t$, we can define solutions for all times, positive or negative.

Proof. By Lemmas 5.8 and 5.9, for each integer $N > 0$ there exists almost surely a unique strong solution $\tilde{\theta}^N$ of (1.1) on interval $[0, N)$. Since (1.1) is a deterministic equation, by almost surely we mean for μ almost every initial condition $\tilde{\theta}(0)$ (see Lemma 5.10). Thus for each integer $N > 0$ there exists a set \mathcal{M}_N , with $\mu(\mathcal{M}_N) = 1$ such that for each $\tilde{\theta}_0 \in \mathcal{M}_N$, there exists a unique solution of (1.1) on $[0, N)$ with $\tilde{\theta}(0) = \tilde{\theta}_0$.

If we denote $\mathcal{M} = \cap_N \mathcal{M}_N$, then $\mu(\mathcal{M}) = 1$ and for each $\tilde{\theta}_0 \in \mathcal{M}$, and each N , there exists a unique solution of (1.1) on $[0, N)$ (see Lemma 5.9), and global existence follows.

By slightly modifying the argument of Lemma 5.9, we could prove (5.18) with H^1 replaced by L^2 . However, we need to modify the argument to obtain continuity with respect to the H^1 topology.

Test (5.14) by Δw , and let us first focus on the right hand side (using summation convention)

$$(\partial_{ii}^2 w, \mathbf{u}^j \partial_j \theta_1) + (\partial_{ii}^2 w, \mathbf{u}_1^j \partial_j w) = (1) + (2),$$

where $\mathbf{u} = \mathcal{R}^\perp(w)$ and $\mathbf{u}_1 = \mathcal{R}^\perp(\theta_1)$. To estimate (2), we use an integration by parts and $\nabla \cdot \mathbf{u} = 0$ to obtain

$$\begin{aligned} (\partial_{ii}^2 w, \mathbf{u}_1^j \partial_j w) &= -(\partial_i w, \partial_i \mathbf{u}_1^j \partial_j w) - (\partial_i w, \mathbf{u}_1^j \partial_{ij}^2 w) \\ &= -(\partial_i w, \partial_i \mathbf{u}_1^j \partial_j w) - \frac{1}{2}(\partial_j (\partial_i w)^2, \mathbf{u}_1^j) \\ &= -(\partial_i w, \partial_i \mathbf{u}_1^j \partial_j w), \end{aligned}$$

and consequently, for any $p \in (1, 2)$,

$$|(\partial_i w, \partial_j w \partial_i \mathbf{u}_1^j)| \leq \|w\|_{H^1} \|D\mathbf{u}_1\|_{L^{\frac{2p}{p-1}}} \|Dw\|_{L^{2p}}. \quad (5.19)$$

Using (5.15), (1.6), and precise constant of embedding as in the proof of Lemma 5.9, one has

$$\begin{aligned} |(2)| &= |(\partial_i w, \partial_j w \partial_i \mathbf{u}_1^j)| \leq C \|w\|_{H^2}^{3-p} \|Dw\|_{L^{\frac{2p}{p-1}}}^{p-1} \|D\mathbf{u}_1\|_{L^{\frac{2p}{p-1}}} \\ &\leq \frac{C}{(p-1)^{\frac{p}{2}}} \|w\|_{H^1}^{3-p} (\|\theta_1\|_{H^2} + \|\theta_2\|_{H^2})^p. \end{aligned} \quad (5.20)$$

On the other hand,

$$|(1)| \leq |(\partial_i w, \partial_i \mathbf{u}^j \partial_j \theta_1)| + |(\partial_i w, \mathbf{u}^j \partial_{ij}^2 \theta_1)| = |(3)| + |(4)|.$$

As in (5.19) and (5.20) we obtain,

$$\begin{aligned} |(3)| &= |(\partial_i w, \partial_i \mathbf{u}^j \partial_j \theta_1)| \\ &\leq \frac{C}{(p-1)^{\frac{p}{2}}} \|w\|_{H^1}^{3-p} (\|\theta_1\|_{H^2} + \|\theta_2\|_{H^2})^p. \end{aligned}$$

To estimate (4), the embedding $H^1 \hookrightarrow L^q$ (with precise constant [15, Remark 1.5]), (1.6) and (5.15) imply

$$\begin{aligned} |(4)| &= |(\partial_i w, \mathbf{u}^j \partial_{ij}^2 \theta_1)| \leq \|\theta_1\|_{H^2} \|Dw\|_{L^{2p}} \|\mathbf{u}\|_{L^{\frac{2p}{p-1}}} \leq \|\theta_1\|_{H^2} \|Dw\|_{L^{2p}} \|w\|_{L^{\frac{2p}{p-1}}} \\ &\leq C \|\theta_1\|_{H^2} \|w\|_{H^1}^{2-p} \|Dw\|_{L^{\frac{2p}{p-1}}}^{p-1} \|w\|_{L^{\frac{2p}{p-1}}} \\ &\leq \frac{C}{(p-1)^{\frac{p}{2}}} \|w\|_{H^1}^{3-p} (\|\theta_1\|_{H^2} + \|\theta_2\|_{H^2})^p, \end{aligned}$$

where C is independent of p . Combining all the estimates, and using that $p < 2$, we have

$$\frac{d}{dt} \|w(t)\|_{H^1}^2 \leq \frac{C}{(p-1)^{\frac{p}{2}}} \|w(t)\|_{H^1}^{2\delta} (1 + \|\theta_1\|_{H^2}^2 + \|\theta_2\|_{H^2}^2),$$

where $\delta = \frac{3-p}{2}$ and $p \in (1, 2)$. Thus, for any $T_0 < T_1$,

$$\begin{aligned} \sup_{t \in [T_0, T_1]} \|w(t)\|_{H^1}^2 &\leq \left(\|w(T_0)\|_{H^1}^{2(1-\delta)} \right. \\ &\quad \left. + C(1-\delta)^{\delta-\frac{1}{2}} \int_{T_0}^{T_1} (1 + \|\theta_1\|_{H^2}^2 + \|\theta_2\|_{H^2}^2) ds \right)^{\frac{1}{1-\delta}}. \end{aligned}$$

Passing to $\|w(T_0)\|_1 \rightarrow 0$ implies, for any $\delta \in (0, 1)$, that

$$\lim_{\|w(T_0)\|_1 \rightarrow 0} \sup_{t \in [T_0, T_1]} \|w(t)\|_1^2 \leq \left(C(1-\delta)^{\delta-\frac{1}{2}} \int_{T_0}^{T_1} (1 + \|\theta_1\|_2^2 + \|\theta_2\|_2^2) ds \right)^{\frac{1}{1-\delta}}.$$

Finally, letting $\delta \rightarrow 1^+$, or equivalently $p \rightarrow 1^+$, we arrive at

$$\lim_{\|w(T_0)\|_1 \rightarrow 0} \sup_{t \in [T_0, T_1]} \|w(t)\|_1 = 0,$$

as desired. \square

Proposition 5.12. *Under the assumption of Theorem 5.1, the relations (5.1) and (5.2).*

Proof. Recall that by Lemma 5.10 $\mu_k \rightarrow \mu$ as measures on $H^{2-\gamma}$, $\gamma > 0$ and μ is supported on H^2 .

The inequality (5.2) follows from (4.4) and the Portmanteau theorem, since C in (4.4) is independent of α .

To establish (5.1), fix $R \geq 1$ and let $\psi_R : \mathbb{R} \rightarrow [0, 1]$ be a C^∞ cut off function with $\psi_R(r) = 1$ for $|r| \leq R$ and $\psi_R(r) = 0$ for $|r| \geq R+1$. Denote B_R the ball in L^2 centred at 0 with radius R , and B_R^c the complement of B_R in L^2 and define

$$I(\theta) = \|\theta\|_{\frac{3}{2}}^2 - \int_{\mathbb{T}^2} |\nabla \theta|^2 \nabla \theta \cdot \nabla (-\Delta)^{-\frac{1}{2}} \theta \, dx. \quad (5.21)$$

Then, by (4.1),

$$\begin{aligned} \frac{A_{-\frac{1}{2}}}{2} - \int_{B_R^c} |I(\theta)| \mu_k(d\theta) &\leq \int_{L^2} \psi_R(I(\theta)) I(\theta) \mu_k(d\theta) \\ &\leq \frac{A_{-\frac{1}{2}}}{2} + \int_{B_R^c} |I(\theta)| \mu_k(d\theta). \end{aligned}$$

Also, Hölder's inequality, interpolation, (5.16), and (4.4) imply that

$$\begin{aligned} \int_{B_R^c} \|\theta\|_{\frac{2}{3}}^2 \mu_k(d\theta) &\leq \left(\int_{L^2} \|\theta\|_{\frac{8}{3}}^{\frac{8}{3}} \mu_k(d\theta) \right)^{\frac{3}{4}} (\mu_k(B_R^c))^{\frac{1}{4}} \\ &\leq \left(\int_{L^2} \|\theta\|_{\frac{2}{3}}^{\frac{2}{3}} \|\theta\|_2^2 \mu_k(d\theta) \right)^{\frac{3}{4}} (\mu_k(B_R^c))^{\frac{1}{4}} \leq \frac{C}{R^{\frac{1}{2}}}, \end{aligned}$$

and by Hölder the and Gagliardo-Nirenberg inequalities,

$$\begin{aligned} \left| \int_{\mathbb{T}^2} |\nabla \theta|^2 \nabla \theta \cdot \nabla (-\Delta)^{-\frac{1}{2}} \theta dx \right| &\leq \| |\nabla \theta|^2 \nabla \theta \|_{L^{\frac{4}{3}}} \| \nabla (-\Delta)^{-\frac{1}{2}} \theta \|_{L^4} \leq C \| \nabla \theta \|_{L^4}^3 \| \theta \|_{L^4} \\ &\leq C \| \nabla \theta \|_{L^4}^{\frac{10}{3}} \| \theta \|_{L^2}^{\frac{2}{3}}. \end{aligned} \quad (5.22)$$

Hence, by (4.4) and (5.16),

$$\begin{aligned} \int_{B_R^c} \left| \int_{\mathbb{T}^2} |\nabla \theta|^2 \nabla \theta \cdot \nabla (-\Delta)^{-\frac{1}{2}} \theta dx \right| \mu_k(d\theta) &\leq C \int_{B_R^c} \| \nabla \theta \|_{L^4}^{\frac{10}{3}} \| \theta \|_{L^2}^{\frac{2}{3}} \mu_k(d\theta) \\ &\leq C \left(\int_{L^2} \| \nabla \theta \|_{L^4}^4 \| \theta \|_{L^2}^{\frac{4}{3}} \mu_k(d\theta) \right)^{\frac{5}{6}} (\mu_k(B_R^c))^{\frac{1}{6}} \\ &\leq \frac{C}{R^{\frac{1}{3}}}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{A_{-\frac{1}{2}}}{2} - C \left(\frac{1}{R^{\frac{1}{2}}} + \frac{1}{R^{\frac{1}{3}}} \right) &\leq \int_{L^2} \psi_R(I(\theta)) I(\theta) \mu_k(d\theta) \\ &\leq \frac{A_{-\frac{1}{2}}}{2} + C \left(\frac{1}{R^{\frac{1}{2}}} + \frac{1}{R^{\frac{1}{3}}} \right). \end{aligned} \quad (5.23)$$

Furthermore, Hölder's inequality and the Sobolev embedding imply that

$$\begin{aligned} \left| \int_{\mathbb{T}^2} |\nabla \theta|^2 \nabla \theta \cdot \nabla (-\Delta)^{-\frac{1}{2}} \theta dx \right| &\leq \| |\nabla \theta|^2 \nabla \theta \|_{L^{\frac{7}{6}}} \| \nabla (-\Delta)^{-\frac{1}{2}} \theta \|_{L^7} \\ &\leq C \| \nabla \theta \|_{L^{\frac{7}{2}}}^3 \| \theta \|_{L^7} \leq C \| \theta \|_{H^{\frac{10}{7}}}^3 \| \theta \|_{L^7}. \end{aligned}$$

Therefore, $I : H^{\frac{10}{7}} \rightarrow \mathbb{R}$, and by passing $k \rightarrow \infty$ in (5.23) and using of weak convergence $\mu_k \rightarrow \mu$ on $H^{\frac{10}{7}}$ (see Lemma 5.10) and the boundedness of $\psi_R(I(\theta))I(\theta)$, we obtain

$$\begin{aligned} \frac{A_{-\frac{1}{2}}}{2} - C \left(\frac{1}{R^{\frac{2}{3}}} + \frac{1}{R^{\frac{1}{6}}} \right) &\leq \int_{L^2} \psi_R(I(\theta))I(\theta)\mu(d\theta) \\ &\leq \frac{A_{-\frac{1}{2}}}{2} + C \left(\frac{1}{R^{\frac{2}{3}}} + \frac{1}{R^{\frac{1}{6}}} \right). \end{aligned} \quad (5.24)$$

Finally, by (5.22), one has

$$|\psi_R(I(\theta))I(\theta)| \leq |I(\theta)| \leq C(\|\nabla\theta\|_{L^4}^{\frac{10}{3}}\|\theta\|_{L^2}^{\frac{2}{3}} + \|\theta\|_{\frac{3}{2}}^2), \quad (5.25)$$

and by (5.2), the right hand side is μ integrable. Since $\psi_R(I(\theta))I(\theta) \rightarrow I(\theta)$ everywhere as $R \rightarrow \infty$, by the dominated convergence theorem, and (5.24) implies (5.1). \square

6. Qualitative Properties

In this section we complete the proof of the main result, Theorem 1.2 by showing that parts 6 and 7 holds true.

In particular, we show that the the distributions via μ of the functionals below admit densities with respect to the Lebesgue measure on \mathbb{R} ; that is

$$\begin{aligned} E_{-\frac{1}{2}}(\theta) &= \frac{1}{2}\|\theta\|_{H^{-\frac{1}{2}}}^2, \\ M(\theta) &= \frac{1}{2}\|\theta\|^2. \end{aligned}$$

Also, using other conservation laws of the SQG equation, we show the infinite-dimensional nature of the measure μ . The proofs follow general framework developed for analogous problems, however the adaptation is not straightforward. Since our smoothing operator is not Laplacian, but bi-Laplacian, we lost several important properties. For example, unlike $\langle -\Delta\theta, f(\theta) \rangle \geq 0$, for any increasing function f , it is not clear that $\langle \Delta^2\theta, f(\theta) \rangle$ is bounded from below for any $\theta \in H^2$ and for sufficiently many functions f . This obstacle was solved by introducing the p -Laplacian to the equation, that lead to stronger moment bounds, see Theorem 5.1. In such case, after nontrivial integration by parts we can show that $\langle \Delta^2\theta, f(\theta) \rangle$ is bounded from below if f has bounded derivative up to fourth order. Also, compared to Euler equation we have to choose differently the set of functions f . Let us provide the details.

Theorem 6.1. *Assume $A_0 < \infty$. The laws of the functionals $M(\theta)$ and $E_{-\frac{1}{2}}(\theta)$ under μ are absolutely continuous with respect to the Lebesgue measure on $(0, \infty)$.*

Analogous statement can be proved for other invariants, but it is more technically involved, and we decided to skip it for the clarity of presentation.

To obtain the following result in a cleaner form, we redefine the measure μ constructed in Theorem 5.1 so that it does not have an atom at the origin.

By (5.1) and $A_{-\frac{1}{2}} \neq 0$ one has $\mu(\{0\}) < 1$, and therefore $S := \mu(H^2 \setminus \{0\}) > 0$. Define the probability measure

$$\tilde{\mu}(A) = \frac{\mu(A \setminus \{0\})}{S}.$$

The SQG equation preserves the L^2 norm of solutions, and therefore it preserves the set $\text{supp}(\mu) \setminus \{0\}$. The invariance of $\tilde{\mu}$ comes readily from the invariance of μ .

Theorem 6.2. *If $A_0 < \infty$ and $a_m \neq 0$ for each m , then the measure $\tilde{\mu}$ is infinite-dimensional in the sense that if $K \subset H^1$ is a compact set of finite Hausdorff dimension, then $\mu(K) = 0$.*

Proof of Theorem 6.1. Let $F(\theta)$ be either $M(\theta)$ or $E_{-\frac{1}{2}}(\theta)$. Thanks to the Portmanteau theorem, it suffices to prove the theorem for measures μ_α with bounds that are uniform in α . Also, according to the non-negativity of F , our analysis shall be reduced to the interval $[0, \infty)$.

Step 1 : The pilot relation. Fix any $f \in C_0^\infty$ and define

$$\begin{aligned} \Phi_\delta(x) &= \frac{1}{\sqrt{2\delta}} \int_{-\infty}^{\infty} f(y) e^{-|x-y|\sqrt{2\delta}} dy \\ &= \frac{1}{\sqrt{2\delta}} \left(\int_{-\infty}^x f(y) e^{-(x-y)\sqrt{2\delta}} dy + \int_x^{\infty} f(y) e^{(x-y)\sqrt{2\delta}} dy \right). \end{aligned}$$

Then,

$$\Phi'_\delta(x) = \int_x^{\infty} f(y) e^{(x-y)\sqrt{2\delta}} dy - \int_{-\infty}^x f(y) e^{-(x-y)\sqrt{2\delta}} dy.$$

Computing the second derivative of Φ_δ , we obtain that

$$\frac{1}{2} \Phi''_\delta + f = \delta \Phi_\delta.$$

Since Φ_δ is bounded uniformly in δ (as f is compactly supported), for every x , $\delta \Phi_\delta(x) \rightarrow 0$ as $\delta \rightarrow 0$ and

$$\begin{aligned} \Phi'_\delta(x) &\rightarrow \int_x^{\infty} f(y) dy - \int_{-\infty}^x f(y) dy \quad \text{as } \delta \rightarrow 0, \\ \Phi''_\delta(x) &\rightarrow -2f(x) \quad \text{as } \delta \rightarrow 0. \end{aligned}$$

Assume that θ is a solution of (1.2) with $\theta(0)$ distributed as μ_α , and therefore $\theta(t)$ is distributed as μ_α for any $t \geq 0$. Note that by Theorem 2.1, θ is a global solution and to simplify the notation, we will not indicate explicitly the dependence of θ on α . Denote \mathbb{E}_{μ_α} to integral with respect to the measure $\mu_\alpha(d\theta)$.

Let us apply the Itô formula to $\Phi_\delta(F(\theta))$, take the expectation, and use $\nabla_\theta F(\theta; \mathbf{u} \cdot \nabla \theta) = 0$ and stationarity of $\theta(t)$ to obtain

$$\begin{aligned} \mathbb{E}_{\mu_\alpha} \left(\Phi'_\delta(F(\theta)) \left[\nabla_\theta F(\theta; \nabla \cdot (|\nabla \theta|^2 \nabla \theta - \Delta^2 \theta)) \right. \right. \\ \left. \left. + \frac{1}{2} \sum_m a_m^2 \nabla_\theta^2 F(\theta; e_m, e_m) \right] \right) \\ + \frac{1}{2} \mathbb{E}_{\mu_\alpha} \left(\Phi''_\delta(F(\theta)) \sum_m a_m^2 (\nabla_\theta F(\theta, e_m))^2 \right) = 0. \end{aligned} \quad (6.1)$$

If $F(\theta) = M(\theta)$, then

$$\nabla_\theta F(\theta; \Delta^2 \theta - \nabla \cdot (|\nabla \theta|^2 \nabla \theta)) = \|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4,$$

and if $F(\theta) = E_{-\frac{1}{2}}(\theta)$, then

$$\nabla_\theta F(\theta; \Delta^2 \theta - \nabla \cdot (|\nabla \theta|^2 \nabla \theta)) = I(\theta),$$

where I is defined by (5.21).

By (3.3), μ_α is supported on H^2 . Since $f \in C_0^\infty$, $|\Phi'_\delta|, |\Phi''_\delta|$ are bounded independently of δ , we can use the Lebesgue dominated convergence theorem to pass $\delta \rightarrow 0$ in (6.1) and obtain, for $F = M$, that

$$\begin{aligned} \mathbb{E}_{\mu_\alpha} \left(\left[\int_{F(\theta)}^\infty f(y) dy - \int_{-\infty}^{F(\theta)} f(y) dy \right] \times \right. \\ \left. \times \left[\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4 + \frac{1}{2} \sum_m a_m^2 \nabla_\theta^2 F(\theta; e_m, e_m) \right] \right) \\ - \mathbb{E}_{\mu_\alpha} \left(f(F(\theta)) \sum_m a_m^2 |\nabla_\theta F(\theta, e_m)|^2 \right) = 0, \end{aligned} \quad (6.2)$$

and for $F = E_{-\frac{1}{2}}$ we just replace $\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4$ by $I(\theta)$.

By a standard approximation argument combined with the Lebesgue dominated convergence theorem, we can extend (6.2) to $f = \chi_\Gamma$ being the characteristic function of a Borel set $\Gamma \subset \mathbb{R}$. Then, $F \geq 0$ and (4.4) imply that there is C independent of α and Γ such that

$$\begin{aligned} \mathbb{E}_{\mu_\alpha} \left(\left[\int_{F(\theta)}^\infty \chi_\Gamma(y) dy - \int_{-\infty}^{F(\theta)} \chi_\Gamma(y) dy \right] \left[\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4 + \frac{1}{2} \sum_m a_m^2 \nabla_\theta^2 F(\theta; e_m, e_m) \right] \right) \\ \leq \left(\int_0^\infty \chi_\Gamma(y) dy \right) \mathbb{E}_{\mu_\alpha} \left[\|\theta\|_{H^2}^2 + \|\theta\|_{W^{1,4}}^4 + \frac{1}{2} \sum_m a_m^2 \right] \\ \leq C \int_{-\infty}^\infty \chi_\Gamma(y) dy \leq C \ell(\Gamma), \end{aligned}$$

where $\ell(\Gamma)$ denotes the Lebesgue measure of Γ . If $F = E_{-\frac{1}{2}}$, one obtains a similar bound by using (5.25) and (4.4).

By (6.2),

$$\mathbb{E}_{\mu_\alpha} \left(\chi_\Gamma(F(\theta)) \sum_m a_m^2 |\nabla_\theta F(\theta, e_m)|^2 \right) \leq C \ell(\Gamma). \quad (6.3)$$

In the remaining part of the proof, we estimate the left hand side from below.

Step 2: Absolute continuity on $(0, \infty)$. Recalling that F is either $E_{-\frac{1}{2}}(\theta)$ or $M(\theta)$, then $\nabla_\theta F(\theta; e_m)$ is

$$\lambda_m^{-1}(\theta, e_m) \quad \text{or} \quad (\theta, e_m).$$

First, focus on $F(\theta) = E_{-\frac{1}{2}}(\theta)$. For any $\delta > 0$ denote the set $\Gamma_\delta = \{\|\theta\|_{-1}^2 \geq \delta\} \cap \{\|\theta\|^2 \leq 1/\delta\}$.

Let $\bar{a}_N = \min\{|a_m|, 1 \leq |m| \leq N\}$ and recall that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N \leq \dots$ are eigenvalues of $(-\Delta)$. Let P_N be the projection on the space spanned by the first N eigenfunctions of the Laplacian. Then, by the inverse Poincaré inequality, for any $\theta \in \Gamma_\delta$ one has

$$\begin{aligned} \sum_m a_m^2 \lambda_m^{-2}(\theta, e_m)^2 &= \sum_{|m| \leq N} a_m^2 \lambda_m^{-2}(\theta, e_m)^2 + \sum_{|m| > N} a_m^2 \lambda_m^{-2}(\theta, e_m)^2 \\ &\geq \bar{a}_N^2 \lambda_N^{-1} \|P_N \theta\|_{-1}^2 \geq \bar{a}_N^2 \lambda_N^{-1} (\|\theta\|_{-1}^2 - \lambda_N^{-1} \|(I - P_N)\theta\|^2) \\ &\geq \bar{a}_N^2 \lambda_N^{-1} \left(\delta - \frac{1}{\delta \lambda_N} \right). \end{aligned}$$

Fix any $\bar{\delta} > 0$ and any Borel set $\Gamma^* \subset (\bar{\delta}, \infty)$. For any $\delta \in (0, \bar{\delta})$, using (6.2) the left hand side of (6.3) can be estimated as

$$\begin{aligned} \mathbb{E}_{\mu_\alpha} \left(\chi_{\Gamma^*}(F(\theta)) \sum_m a_m^2 |\nabla_\theta F(\theta, e_m)|^2 \right) \\ \geq \bar{a}_N^2 \lambda_N^{-1} \left(\delta - \frac{1}{\delta \lambda_N} \right) \mu_\alpha(F^{-1}(\Gamma^*) \cap \Gamma_\delta). \end{aligned}$$

Since the sequence λ_N increases to infinity, we can find N such that $\delta - \frac{1}{\delta \lambda_N} > 0$, and therefore, by (6.3),

$$\mu_\alpha(F^{-1}(\Gamma^*) \cap \Gamma_\delta) \leq \frac{C \lambda_N}{\bar{a}_N^2 \left(\delta - \frac{1}{\delta \lambda_N} \right)} \ell(\Gamma^*).$$

Using the Portmanteau theorem, we pass to the limit $\alpha \rightarrow 0$ and obtain

$$\mu(F^{-1}(\Gamma^*) \cap \Gamma_\delta) \leq \frac{C \lambda_N}{\bar{a}_N^2 \left(\delta - \frac{1}{\delta \lambda_N} \right)} \ell(\Gamma^*).$$

Consequently, if $\ell(\Gamma^*) = 0$, then for any $\delta \in (0, \bar{\delta})$,

$$\mu(F^{-1}(\Gamma^*) \cap \Gamma_\delta) = 0.$$

Since $\bigcup_{n \geq 1} \Gamma_{1/n} = L^2 \setminus \{0\}$ and $0 \notin F^{-1}(\Gamma^*)$, one obtains

$$\mu_\alpha(F^{-1}(\Gamma^*)) = \mu_\alpha(F^{-1}(\Gamma^*) \setminus \{0\}) = 0.$$

Thus, $\mu_\alpha(F^{-1}(\Gamma^*)) = 0$ whenever $\ell(\Gamma^*) = 0$, and the claimed absolute continuity follows.

Next, we focus on $F(\theta) = M(\theta)$. By the Cauchy–Schwartz inequality,

$$\begin{aligned} \|\theta\|^2 &= (\theta, \theta) = \sum_m (\theta, e_m)^2 = \sum_{|m| \leq N} (\theta, e_m)^2 + \sum_{|m| > N} (\theta, e_m)^2 \\ &\leq \frac{\|\theta\|}{\bar{a}_N} \left(\sum_{|m| \leq N} a_m^2 (\theta, e_m)^2 \right)^{\frac{1}{2}} + \|\theta\| \left(\sum_{|m| > N} (\theta, e_m)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Also,

$$\sum_{|m| > N} (\theta, e_m)^2 \leq \frac{1}{\lambda_N^2} \|\theta\|_1^2,$$

and therefore,

$$\bar{a}_N \left(\|\theta\| - \frac{1}{\lambda_N} \|\theta\|_1 \right) \leq \left(\sum_m a_m^2 (\theta, e_m)^2 \right)^{\frac{1}{2}}.$$

For any $\epsilon > 0$ denote $I_\epsilon = \{\|\theta\| \geq \epsilon, \|\theta\|_1 \leq \frac{1}{\epsilon}\}$. Then, for any $\theta \in I_\epsilon$,

$$\bar{a}_N \left(\epsilon - \frac{1}{\lambda_N \epsilon} \right) \leq \left(\sum_m a_m^2 (\theta, e_m)^2 \right)^{\frac{1}{2}}. \quad (6.4)$$

Fix $\bar{\epsilon} > 0$ and a Borel set $\Gamma \subset [\bar{\epsilon}, \infty)$. Since θ is distributed as μ_α , and $M(\theta) \in \Gamma$ implies $\|\theta\| \geq \bar{\epsilon}$, then for any $\epsilon \in (0, \bar{\epsilon})$,

$$\begin{aligned} \mu_\alpha(M^{-1}(\Gamma)) &= \mu_\alpha(\{M(\theta) \in \Gamma\} \cap \{\theta \in I_\epsilon\}) \\ &\quad + \mu_\alpha\left(\{M(\theta) \in \Gamma\} \cap \left\{\|\theta\|_1 \geq \frac{1}{\epsilon}\right\}\right) = I + II. \end{aligned}$$

Using the Chebyshev inequality and (4.4), we obtain

$$II \leq C \frac{A_0}{2} \epsilon^2.$$

Since $\lambda_N \rightarrow \infty$ as $N \rightarrow \infty$, we can suppose that $(\epsilon - \frac{1}{\epsilon \lambda_N}) > 0$. By (6.3) and (6.4),

$$I \leq \frac{C}{\bar{a}_N(\epsilon - \frac{1}{\lambda_N \epsilon})} \ell(\Gamma).$$

Consequently, that Portmanteau theorem yields that

$$\mu(M^{-1}(\Gamma)) \leq \frac{C}{\bar{a}_N(\varepsilon - \frac{1}{\varepsilon \lambda_N})} \ell(\Gamma) + \frac{A_0}{2} \varepsilon^2,$$

and the rest of the proof follows as in the previous case. \square

Next, let us turn to the proof of Theorem 6.2.

Proof of Theorem 6.2. For any positive integer k denote

$$F_k(\theta) = \frac{1}{|\mathbb{T}^2|} \int_{\mathbb{T}^2} f_k(\theta(x)) dx,$$

where f_k is a smooth function on \mathbb{R} . Then, by $\nabla \cdot \mathbf{u} = 0$, we have for any $\theta \in H^2$ that

$$(f'_k(\theta), \mathbf{u} \cdot \nabla \theta) = 0. \quad (6.5)$$

Therefore, the functionals F_k are conservation laws for (1.1). Fix n and functions $(f_k)_{k=1}^n$ on \mathbb{R} such that

- (i) There is a constant C independent of k such that $|f^{(p)}(z)| \leq C$ for $p \in \{0, \dots, 4\}$, that is, the sequence (f_k) has uniformly (in k) bounded derivatives up to fourth order. Note that the bound can depend on n .
- (ii) $f_k(0) = 0$ for each k .
- (iii) $f_1 \geq 0$ and $f_1 > 0$ on $(-\delta^*, \delta^*) \setminus \{0\}$ for some $\delta^* > 0$.
- (iv) If for some $v \in \mathbb{R}^n$ and some continuous function $m : \mathbb{T}^2 \rightarrow \mathbb{R}$ with zero mean one has

$$\sum_{i=1}^n v_i f'_i(m(x)) = \text{Const}, \quad \text{for all } x \in \mathbb{T}^2,$$

then $v_i = 0$ for each i , or $m \equiv 0$. (6.6)

For any $n \geq 1$, such $(f_k)_{k=1}^n$ indeed exists. For example let f_k be smooth functions, compactly supported on $[-2, 2]$ and $f_k(z) = z^{k+1}$ on $(-1, 1)$. In addition, we assume that $f_1 \geq 0$. Clearly (i)–(iii) holds and it remains to verify (iv). Fix any zero mean continuous function $m \not\equiv 0$. Then the image of m contains the interval $(-\delta, \delta)$ for some $\delta > 0$ and, consequently,

$$\sum_{i=1}^n c_i f'_i(z) = \text{Const} \quad \text{for all } z \in (-\delta, \delta).$$

Since f'_k are non-constant polynomials on $(-1, 1)$ one obtains that $v_i = 0$ for each i as desired.

Since the second order derivatives of f_k are bounded, then for any solution θ of (1.1) we can use Theorem B.1 and (6.5) to obtain

$$\begin{aligned} \mathbb{E}F_k(\theta(t)) + \alpha \mathbb{E} \int_0^t (f'_k(\theta), \Delta^2 \theta - \nabla \cdot (|\nabla \theta|^2 \nabla \theta)) ds \\ = \mathbb{E}F_k(\theta_0) + \frac{\alpha}{2} \sum_{m=1}^{\infty} d_m^2 \mathbb{E} \int_0^t (f''_k(\theta; e_m, e_m)) ds. \end{aligned}$$

Next, with the use of a summation convention,

$$\begin{aligned} (f'_k(\theta), \partial_{ii}^2 \partial_{jj}^2 \theta) &= -(f''_k(\theta), \partial_i \theta \partial_i \partial_{jj}^2 \theta) \\ &= (f''_k(\theta), \partial_{ii}^2 \theta \partial_{jj}^2 \theta) + (f_k^{(3)}(\theta), (\partial_i \theta)^2 \partial_{jj}^2 \theta) =: (1) + (2). \end{aligned}$$

On the other hand,

$$\begin{aligned} (f'_k(\theta), \partial_{ii}^2 \partial_{jj}^2 \theta) &= -(f''_k(\theta), \partial_i \theta \partial_i \partial_{jj}^2 \theta) \\ &= (f''_k(\theta), (\partial_{ij}^2 \theta)^2) + (f_k^{(3)}(\theta), \partial_i \theta \partial_j \theta \partial_{ij}^2 \theta) =: (3) + (4). \end{aligned}$$

Furthermore,

$$(2) = -(f_k^{(4)}(\theta), (\partial_i \theta)^2 (\partial_j \theta)^2) - 2(f_k^{(3)}(\theta), \partial_i \theta \partial_j \theta \partial_{ij}^2 \theta).$$

We obtain

$$\begin{aligned} (f'_k(\theta), \partial_{ii}^2 \partial_{jj}^2 \theta) &= \frac{1}{3}[(1) + (2)] + \frac{2}{3}[(3) + (4)] \\ &= \frac{1}{3} \left((f''_k(\theta), \partial_{ii}^2 \theta \partial_{jj}^2 \theta + 2(\partial_{ij}^2 \theta)^2) - (f_k^{(4)}(\theta), (\partial_i \theta)^2 (\partial_j \theta)^2) \right), \end{aligned}$$

and consequently,

$$\begin{aligned} (f'_k(\theta), \Delta^2 \theta - \nabla \cdot (|\nabla \theta|^2 \nabla \theta)) \\ = \frac{1}{3} \left((f''_k(\theta), (\Delta \theta)^2 + 2(D^2 \theta)^2) + (f''_k(\theta) - f_k^{(4)}(\theta), |\nabla \theta|^4) \right) \\ =: A_k(\theta). \end{aligned} \tag{6.7}$$

For any positive integer n , denote

$$V_n(\theta) = \begin{pmatrix} F_1(\theta) \\ F_2(\theta) \\ \vdots \\ F_n(\theta) \end{pmatrix}. \tag{6.8}$$

By the Itô's formula and (6.5),

$$V_n(\theta) = V_n(\theta_0) + \int_0^t x_s ds + \sqrt{\alpha} \sum_m \int_0^t y_m(s) dW_m(s),$$

and with A_k defined as in (6.7), one has

$$\begin{aligned} x_s &= -\alpha \begin{pmatrix} A_1(\theta) \\ A_2(\theta) \\ \vdots \\ A_n(\theta) \end{pmatrix} + \frac{\alpha}{2} \sum_m a_m^2 \begin{pmatrix} \nabla_\theta^2 F_1(\theta; e_m, e_m) \\ \nabla_\theta^2 F_2(\theta; e_m, e_m) \\ \vdots \\ \nabla_\theta^2 F_n(\theta; e_m, e_m) \end{pmatrix} \\ &= -\alpha \begin{pmatrix} A_1(\theta) \\ A_2(\theta) \\ \vdots \\ A_n(\theta) \end{pmatrix} + \frac{\alpha}{2} \sum_m a_m^2 \begin{pmatrix} (f_1''(\theta), e_m^2) \\ (f_2''(\theta), e_m^2) \\ \vdots \\ (f_n''(\theta), e_m^2) \end{pmatrix} =: -\alpha A(\theta) + \frac{\alpha}{2} B(\theta), \\ y_m &= a_m \begin{pmatrix} (f_1'(\theta), e_m) \\ (f_2'(\theta), e_m) \\ \vdots \\ (f_n'(\theta), e_m) \end{pmatrix}. \end{aligned}$$

Let $y_m^i = a_m(f_i'(\theta), e_m)$ be the i th component of y_m . Denote by M the $n \times n$ matrix with entries

$$M_{i,j} = \sum_m y_m^i y_m^j = \sum_m a_m^2 (f_i'(\theta), e_m) (f_j'(\theta), e_m),$$

and note that M depends on t , but is independent of x .

Since f_k' , f_k'' , and $f_k^{(4)}$ are bounded, by (6.7) and (4.4) one has for any $\alpha \in (0, 1)$

$$\begin{aligned} &\mathbb{E} \int_0^t |x_s| + \alpha \sum_{m=1}^\infty |y_m(s)|^2 ds \\ &\leq C\alpha \mathbb{E} \sum_{k=1}^n \int_0^t \|\theta\|_{H^2}^2 + \|\nabla \theta\|_{L^4}^4 + 1 + \sum_{m=1}^\infty a_m^2 ds \leq C, \end{aligned} \quad (6.9)$$

where C is independent of α . Then, [43, Theorem 7.9.1] and (6.9) (bound on $|x_s|$) imply for any bounded measurable function g the Krylov's estimate

$$\mathbb{E}_{\mu_\alpha} \int_0^1 (\det M)^{1/n} g(V_n) dt \leq C_n \|g\|_{L^n} \mathbb{E}_{\mu_\alpha} \int_0^1 |x_s| \leq C_n \|g\|_{L^n}.$$

Let $\mathcal{B} \subset \mathbb{R}^n$ be a Borel set and denote $g = \chi_{\mathcal{B}}$ the indicator function of \mathcal{B} . Then, since μ_α is an invariant measure,

$$\begin{aligned} \int (\det M(\bar{\theta}))^{1/n} \chi_{\mathcal{B}}(V_n(\bar{\theta})) d\mu_\alpha(\bar{\theta}) &= \mathbb{E}_{\mu_\alpha} \int_0^1 (\det M(\theta(s)))^{1/n} \chi_{\mathcal{B}}(V_n(\theta(s))) ds \\ &\leq C_n A_0(\ell_n(\mathcal{B}))^{\frac{1}{n}}. \end{aligned} \quad (6.10)$$

For any integer $k > 0$ denote $\mathcal{B}_k = \mathcal{B} \cap B_k^c$, where $B_k \subset \mathbb{R}^n$ is a ball of radius $\frac{1}{k}$ centred at the origin, and B_k^c is its complement. Note that $\mathcal{B} = \{0\} \cup \bigcup_k \mathcal{B}_k$, and by (6.10), for any $k > 0$,

$$\int (\det M(\bar{\theta}))^{1/n} \chi_{\mathcal{B}_k}(V_n(\bar{\theta})) d\mu_\alpha(\bar{\theta}) \leq C_n A_0(\ell_n(\mathcal{B}))^{\frac{1}{n}}. \quad (6.11)$$

Estimate on det M . The matrix M is clearly a non-negative symmetric $n \times n$ matrix. We show that for any $\epsilon \geq 0$, M is positively bounded from below outside the ball in H^1 of radius $\epsilon \geq 0$ centered at the origin. Observe that M is an infinite sum of non-negative matrices M^m with coefficients

$$M_{i,j}^m = a_m^2 (f'_i(\theta), e_m)(f'_j(\theta), e_m).$$

Then, for any vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, we have

$$\begin{aligned} (v, Mv) &= \sum_{m \geq 0} (v, M^m v) = \sum_{m \geq 0} \sum_{1 \leq i, j \leq n} M_{i,j}^m v_i v_j \\ &= \sum_{m \geq 0} a_m^2 \sum_{1 \leq i, j \leq n} v_i v_j (f'_i(\theta(x)), e_m(x))(f'_j(\theta(x)), e_m(x)) \\ &= \sum_{m \geq 0} a_m^2 \left(\sum_{j=1}^n v_j (f'_j(\theta(x)), e_m(x)) \right)^2 \\ &= \sum_{m \geq 0} a_m^2 \left(\sum_{j=1}^n v_j f'_j(\theta(x)), e_m \right)^2. \end{aligned}$$

Suppose that $(v, Mv) = 0$ for some $v \neq 0$, since $a_m \neq 0$, for all m ,

$$\left(\sum_{j=1}^n v_j f'_j(\theta), e_m \right) = 0 \quad \text{for all } m \geq 0.$$

Hence, the function $x \mapsto \sum_{j=1}^n v_j f'_j(\theta(x))$ is constant, that is, there is C such that

$$\sum_{j=1}^n v_j f'_j(\theta(x)) = C \quad \text{for all } x \in \mathbb{T}^2.$$

By the independence property (6.6), either $v_k = 0$ for each k or $\theta \equiv C$. Since $v \neq 0$, the latter property holds and since θ has zero mean we have $\theta \equiv 0$.

Therefore if $\theta \neq 0$, then $\det(M) > 0$. Next, denote the set $I_\epsilon = \{\|\theta\| \geq \epsilon, \|\theta\|_{H^1} \leq \frac{1}{\epsilon}\}$, and note that I_ϵ is compact in L^2 . Indeed, if $(\theta_j)_j \subset I_\epsilon$, then $\|\theta_j\|_{H^1} \leq \frac{1}{\epsilon}$, and therefore there exists subsequence, still denoted $(\theta_j)_j$, converging to θ_∞ weakly in H^1 and strongly in L^2 . Weak lower semi-continuity, and strong continuity of norms yield $\theta \in I_\epsilon$ as desired.

By smoothness of f_k , uniform boundedness of f' , and $A_0 < \infty$, the map $\theta \mapsto M(\theta) : H^1 \rightarrow \mathbb{R}^{n \times n}$ is continuous, and consequently $\theta \mapsto \det M(\theta) : H^1 \rightarrow [0, \infty)$ is continuous as well. Since $\det M > 0$ on the compact set I_ϵ , $\det M(\theta) \geq c_\epsilon > 0$ on I_ϵ .

Conclusion. For the Borel set \mathcal{B}_k fixed in (6.11), and V_n defined in (6.8) one has, for any k ,

$$\begin{aligned} \mu_\alpha(\{V_n(\theta) \in \mathcal{B}_k\}) &\leq \mu_\alpha(\{V_n(\theta) \in \mathcal{B}_k\} \cap \{\theta \in I_\epsilon\}) \\ &\quad + \mu_\alpha(\{V_n(\theta) \in \mathcal{B}_k\} \cap \{\theta \in I_\epsilon^c\}) = I + II. \end{aligned}$$

Since $\det M(\theta) \geq c_\epsilon$, on I_ϵ , (6.11) yields

$$I \leq A_0 C_n c_\epsilon^{-\frac{1}{n}} (\ell_n(\mathcal{B}))^{\frac{1}{n}}.$$

Next, since (f_k) are uniformly, globally Lipschitz and $f_k(0) = 0$, there exists $L \geq 0$ such that $|f_k(z)| \leq L|z|$ for any k and any $z \in \mathbb{R}$. Then, for any θ with $V_n(\theta) \in \mathcal{B}_k$ one has $|V_n(\theta)| \geq \frac{1}{k}$, and therefore,

$$\frac{1}{k} \leq C \max_j |F_j(\theta)| \leq C \max_j \int_{\mathbb{T}^2} |f_j(\theta(x))| dx \leq C \|\theta\|,$$

where we used Jensen's inequality in the last estimate. Without loss of generality assume $C \geq 1$. Thus, by (4.4) and the Chebyshev inequality, if $\epsilon < \frac{1}{Ck}$, then

$$\begin{aligned} II &\leq \mu_\alpha \left(\{V_n(\theta) \in \mathcal{B}_k\} \cap \left\{ \|\theta\|_{H^1} \geq \frac{1}{\epsilon} \right\} \right) \\ &\quad + \underbrace{\mu_\alpha \left(\{V_n(\theta) \in \mathcal{B}_k\} \cap \{\|\theta\| \leq \epsilon\} \right)}_{=0} \\ &\leq \mu_\alpha \left(\left\{ \|\theta\|_{H^1} \geq \frac{1}{\epsilon} \right\} \right) \leq \frac{A_0}{2} \epsilon^2. \end{aligned}$$

Gathering these estimates, we arrive at

$$\mu_\alpha(\{V_n(\theta) \in \mathcal{B}_k\}) \leq \frac{A_0}{2} \epsilon^2 + C_n c_\epsilon^{-\frac{1}{n}} (\ell_n(\mathcal{B}))^{\frac{1}{n}},$$

and by the Portmanteau theorem, (6) is valid with μ_α replaced by the limiting measure μ . If $\ell_n(\mathcal{B}) = 0$, then since $\epsilon > 0$ is arbitrary, we obtain $\mu(\{V_n(\theta) \in \mathcal{B}_k\}) = 0$ for any $k > 0$. Taking the countable union in integer $k > 0$, we arrive to $\mu(\{V_n(\theta) \in \mathcal{B} \setminus \{0\}\}) = 0$.

Since f_1 is non-negative and $f_1 > 0$ in a punctured neighbourhood of zero, then for any continuous, zero mean function $\theta \neq 0$ one has $F_1(\theta) \neq 0$. Hence,

$$\mu(\{V_n(\theta) \in \mathcal{B}\} \setminus \{0\}) = \mu(\{V_n(\theta) \in \mathcal{B} \setminus \{0\}\}) = 0.$$

By the definition of $\tilde{\mu}$,

$$\tilde{\mu}(\{V_n(\theta) \in \mathcal{B}\}) = \frac{\mu(\{V_n(\theta) \in \mathcal{B}\} \setminus \{0\})}{S} = 0 \quad (6.12)$$

for any \mathcal{B} with $\ell_n(\mathcal{B}) = 0$.

Finally, we prove that $\tilde{\mu}$ is infinite dimensional. Let $K \subset L^2$ be a compact set with finite Hausdorff dimension $\dim_H(K) =: h$ and fix an integer $n > h$. We claim that V_n defined in (6.8) is differentiable on L^2 . Indeed, each component F_k of V_n satisfies

$$|F_k''(\theta; u, v)| = \frac{1}{|\mathbb{T}^2|} \left| \int_{\mathbb{T}^2} f_k''(\theta) uv dx \right| \leq C \int_{\mathbb{T}^2} |uv| dx \leq C \|u\| \|v\|,$$

where we used that f_k'' is bounded. In particular, V_n is locally Lipschitz. Since locally Lipschitz maps do not increase the Hausdorff dimension, $\dim_H(\mathcal{V}) \leq h < n$, where $\mathcal{V} := V_n(K) \in \mathbb{R}^n$, and therefore $\ell_n(\mathcal{V}) = 0$. Then, by (6.12)

$$\tilde{\mu}(K) \leq \tilde{\mu}(\{V_n(\theta) \in \mathcal{V}\}) = 0$$

as desired. \square

Acknowledgements. The research of Juraj Foldes was partly supported by National Science Foundation under the Grant NSF-DMS-1816408. The authors would like to thank the anonymous reviewer for suggestions that considerably helped to improve the manuscript.

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Appendix A: Some Facts on the Fluctuation–Dissipation Approach for Finite-Dimensional Hamiltonian Systems

In this section we elaborate on the question that was raised in the introduction: Do the constructed invariant measure μ for (1.1) concentrates on the equilibria? Although we proved that the support of μ is infinite dimensional, it is also known that the set of equilibria is also infinite dimensional; any solution of the equation

$$(-\Delta)^{\frac{1}{2}} \Phi = F(\Phi)$$

is an equilibrium of (1.1). Since every equilibrium is trivially a global solution, there is a possibility that μ concentrates on the set of equilibria, and we did not construct any new solution. As mentioned above, we don't have a definite answer to this question, however we provide an example of a general system for which the measure arising from fluctuation dissipation method is not supported on equilibria.

Since the SQG equation has a Hamiltonian structure, we will focus only on the Hamiltonian systems. There are several trivial examples in which the equilibria form a discrete set, and therefore are of measure zero, for instance the cubic defocusing Schrödinger equation with only one equilibrium. The example closest to SQG is 2D Euler equation, which has infinite dimensional manifold of equilibria with similar structure. However, whether the invariant measures for 2D Euler equation concentrate on equilibria is an open question, hence regularizing the problem might not help.

Let us turn our attention to finite dimensional systems. Consider a $2n$ -dimensional Hamiltonian system

$$\dot{x} = -\partial_y H(x, y), \quad \dot{y} = \partial_x H(x, y), \tag{A.1}$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth Hamiltonian function. It is well known that $f(H)dx dy$ is an invariant measure for the system, for any integrable smooth

function f . We consider now the fluctuation–dissipation model

$$\begin{aligned} dx &= (-\partial_y H(x, y) - \alpha \partial_x H(x, y))dt + \sqrt{2\alpha} d\beta_1, \\ dy &= (\partial_x H(x, y) - \alpha \partial_y H(x, y))dt + \sqrt{2\alpha} d\beta_2, \end{aligned} \quad (\text{A.2})$$

where β_1, β_2 are independent Brownian motions. Then, $e^{-H(x,y)}$ is a density of an invariant measure for (A.2), since $e^{-H(x,y)}$ is solution of the Fokker-Plank equation

$$\mathcal{L}\rho = \alpha \Delta \rho - \nabla \cdot \left[(\partial_y H(x, y) + \alpha \partial_x H(x, y), -\partial_x H(x, y) + \alpha \partial_y H(x, y))^T \rho \right] = 0.$$

Thus $\mu(dx dy) = T^{-1} e^{-H(x,y)} dx dy$ is an invariant probability measure of (A.2), where we denote $T = \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-H(x,y)} dx dy$ to be a partition function (normalization). Note that T is finite if H has appropriate increase at infinity. Observe that μ does not depend on α , thus by passing $\alpha \rightarrow 0$, we see that μ is an invariant measure of (A.1).

If H is constant on the unit ball of $\mathbb{R}^n \times \mathbb{R}^n$, then any point in that ball is an equilibrium of (A.1), and therefore we have an open set of equilibria. On the other hand, μ has positive density everywhere and in particular its support coincides with the whole space. There might be a possibility to apply this reasoning to infinite dimensional systems, but there are serious difficulties with coercivity of the dissipation. We leave this question open.

Appendix B: Itô Formula

For the reader's convenience, we recall Itô's formula in infinite dimensions, which is used several times in the proofs of the main results. We say that the equation (1.2) has the Itô property on the triple (H^{s-1}, H^s, H^{s+1}) if

- (1) for some $T > 0$, (1.2) has a unique solution on $[0, T)$ for any data in H^s ;
- (2) the process $h := -\alpha(\Delta^2 \theta - \nabla(|\nabla \theta|^2 \nabla \theta)) - \mathbf{u} \cdot \nabla \theta$ is \mathcal{F}_t -adapted and

$$\mathbb{P} \left(\int_0^t (\|\theta(r)\|_{s+1}^2 + \|h(r)\|_{s-1}^2) dr < \infty, \quad \forall t > 0 \right) = 1, \quad \sum_{m>0} a_m^2 \lambda_m^s < \infty.$$

We have the following version of Itô's lemma proved in [43, Section A.7]:

Theorem B.1. ([43]) *Let $F \in C^2(H^s, \mathbb{R})$ be a functional which is locally uniformly continuous, together with its first two derivatives, on H^s . Suppose that (1.2) satisfies the Itô property on (H^{s-1}, H^s, H^{s+1}) and that F satisfies the following conditions:*

- (1) *There is a function $K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$|F'(\theta; v)| \leq K(\|\theta\|_s) \|\theta\|_{s+1} \|v\|_{s-1}, \quad \theta \in H^{s+1}, \quad v \in H^{s-1}. \quad (\text{B.1})$$

- (2) *For any sequence $\{w_k\} \subset H^{s+1}$ converging toward $w \in H^{s+1}$ and any $v \in H^{s-1}$, we have*

$$F'(w_k; v) \rightarrow F'(w; v), \quad \text{as } k \rightarrow \infty. \quad (\text{B.2})$$

(3) The solution θ of (1.2) satisfies

$$\sum_m a_m^2 \mathbb{E} \int_0^t |F'(\theta; e_m)|^2 ds < \infty \quad \text{for all } t > 0. \quad (\text{B.3})$$

Then we have

$$\begin{aligned} F(\theta(t)) &= F(\theta(0)) \\ &+ \int_0^t \left(F'(\theta(s); h(s)) + \frac{\alpha}{2} \sum_m a_m^2 F''(\theta(s); e_m, e_m) \right) ds \\ &+ \sqrt{\alpha} \sum_m a_m \int_0^t F'(\theta(s); e_m) dW_m(s). \end{aligned}$$

In particular,

$$\begin{aligned} \mathbb{E}F(\theta(t)) &= \mathbb{E}F(\theta(0)) \\ &+ \int_0^t \mathbb{E} \left(F'(\theta(s); h(s)) + \frac{\alpha}{2} \sum_m a_m^2 F''(\theta(s); e_m, e_m) \right) ds. \end{aligned}$$

If one omits (B.3), then we have the formula (B.1) where t is replaced by the stopping time $t \wedge \tau_n$, with

$$\tau_n = \inf\{t \geq 0, \|\theta(t)\|_s > n\}, \quad n \geq 0,$$

with the convention $\inf \emptyset = +\infty$.

Appendix C: Embedding $L^2 H^2 \cap W^{1, \frac{4}{3}} W^{-1, \frac{4}{3}} \hookrightarrow CH^{-\delta}$

Although the parabolic embedding $L^2 H^2 \cap W^{1, \frac{4}{3}} W^{-1, \frac{4}{3}} \hookrightarrow CH^{-\delta}$ follows from standard arguments we were not able to locate the proof in the literature. Hence, we outline the main steps in this appendix.

By [2, Theorem 5.2], we have, for any $\theta > \frac{2}{3}$, that

$$L^2 H^2 \cap W^{1, \frac{4}{3}} W^{-1, \frac{4}{3}} \hookrightarrow C(H^2, W^{-1, \frac{4}{3}})_{\theta, p_\theta},$$

where $(H^2, W^{-1, \frac{4}{3}})_{\theta, p_\theta}$ is the real interpolation space and p_θ satisfies

$$\frac{1}{p_\theta} = \frac{1-\theta}{2} + \frac{\theta}{\frac{4}{3}}.$$

However, by [2, (3.5)], for any $\varepsilon \in (0, 1)$ one has

$$(H^2, W^{-1, \frac{4}{3}})_{\theta, p_\theta} \hookrightarrow (H^{2-\varepsilon}, W^{-1-\varepsilon, \frac{4}{3}})_{\theta, p_\theta} = (B_{2,2}^{2-\varepsilon}, B_{\frac{4}{3}, \frac{4}{3}}^{-1-\varepsilon})_{\theta, p_\theta},$$

where $B_{p,q}^s$ is a Besov space. From [5, Theorem 6.4.5, (3)] and [2, (3.5)] it follows that

$$(B_{2,2}^{2-\varepsilon}, B_{\frac{4}{3},\frac{4}{3}}^{-1-\varepsilon})_{\theta,p_\theta} = B_{\frac{4}{2+\theta},\frac{4}{2+\theta}}^{(-3+\varepsilon)\theta+(2-\varepsilon)} = W^{(-3+\varepsilon)\theta+(2-\varepsilon),\frac{4}{2+\theta}}.$$

Finally, by Sobolev embeddings,

$$W^{(-3+\varepsilon)\theta+(2-\varepsilon),\frac{4}{2+\theta}} \hookrightarrow W^{-\delta,2},$$

where $\delta \geq \frac{\theta}{2} + (-2 + \varepsilon) + (3 - \varepsilon)\theta$. Since $\theta > \frac{2}{3}$ and $\varepsilon > 0$ can be chosen arbitrarily close to $\frac{2}{3}$ and 0, respectively, one obtains that

$$L^2 H^2 \cap W^{1,\frac{4}{3}} W^{-1,\frac{4}{3}} \hookrightarrow C W^{-\delta,2}$$

for any $\delta > \frac{1}{3}$, as desired.

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JURAJ FÖLDES
Department of Mathematics,
University of Virginia,
322 Kerchof Hall,
Charlottesville
VA
22904-4137 USA.
e-mail: foldes@virginia.edu

and

MOUHAMADOU SY
e-mail: ms3wq@virginia.edu

(Received February 25, 2020 / Accepted March 24, 2021)

Published online May 19, 2021

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Nature (2021)