Singular radial solutions for the Keller-Segel equation in high dimension

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Abstract

We study singular radially symmetric solution of the stationary Keller-Segel equation, that is, an elliptic equation with exponential nonlinearity, which is super-critical in dimension $N \geq 3$. The solutions are unbounded at the origin and we show that they describe the asymptotics of bifurcation branches of regular solutions. It is shown that for any ball and any $k \geq 0$, there is a singular solution that satisfies Neumann boundary condition and oscillates at least k times around the constant equilibrium. Moreover, we prove that in dimension $3 \leq N \leq 9$ there are regular solutions satisfying Neumann boundary conditions that are close to singular ones when the value at the origin is close to infinity. Hence, it follows that there exist regular solutions on any ball with arbitrarily fast oscillations. For generic radii, we show that the bifurcation branches of regular solutions oscillate in the bifurcation plane when $4 \leq N \leq 9$ and approach to a singular solution. In dimension $N > 10$, we show that the Morse index of the singular solution is finite, and therefore the existence of regular solutions

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with fast oscillations is not expected.

Résumé

Nous étudions les solutions singulières radiales de l'équation de Keller-Segel stationnaire, qui est une équation elliptique avec une non-linéarité exponentielle, surcritique en dimension $N \geq 3$. Ces solutions ne sont pas bornées à l'origine et nous montrons qu'elles décrivent les asymptotiques des branches de bifurcation des solutions régulières. Nous prouvons également que pour toutes boules et tout $k \geq 0$, il existe une solution singulière qui satisfait une condition de Neumann homogène au bord et qui oscille exactement k fois autours de l'équilibre constant. De plus, nous montrons qu'en dimension $3 \leq N \leq 9$, il existe des solutions régulières satisfaisant la condition de Neumann homogène au bord qui sont proches de nos solutions singulières. Par conséquent, dans toute boule, il existe des solutions régulières qui oscillent arbitrairement rapidement. Pour des rayons génériques, nous montrons que les branches de bifurcation des solutions régulières oscillent dans le plan de bifurcation quand $4 \leq N \leq 9$. En dimension $N > 10$, nous prouvons que l'indice de Morse des solutions singulières est fini. Par conséquent, nous ne nous attendons pas à trouver des solutions régulières qui oscillent rapidement dans ce cas.

Keywords: Bifurcation branches, radial Keller-Segel equation, oscillations, singular solutions.

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1. Introduction

The goal of the present paper is to investigate singular, radial solutions of the so-called Keller-Segel equation

$$\begin{cases} -\Delta v + v = \lambda e^v & \text{in } B_R \setminus \{0\}\,, \\ v > 0 & \text{in } B_R \setminus \{0\}\,, \\ \partial_\nu v = 0 & \text{on } \partial B_R\,, \end{cases}$$

where $B_R \subset \mathbb{R}^N$, $N \geq 3$ is a ball of radius $R > 0$ centered at the origin. The solutions are assumed to blow-up at the origin with a specific rate (see (**??**) below) which is in some sense minimal so that they are limits of sequences of regular solutions with value at the origin approaching infinity. Then, qualitative properties of singular solutions such as Morse index, yield information about oscillations of the bifurcation branches. We give more details below.

The problem (**??**) is motivated by models of chemotaxis, an omnipresent mechanism in biology that describes the motion of species towards higher (lower) concentration of a chemical substance, for example nutrients or poisons. Sometimes the substance is also secreted by the species themselves, which induces a complicated large scale behavior such as aggregation, scattering, or pattern formation. Mathematically, this phenomenon can be described by a strongly coupled evolution system introduced by Keller and Segel [**?**]

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - D_1 \nabla \cdot (u \nabla \phi(v)) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial v}{\partial t} = D_2 \Delta v - D_3 v + D_4 u & \text{in } \Omega \times (0,T), \end{cases}$$

where $T > 0$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, D_i, $i = 1, \cdots, 4$ are positive constants, and ϕ is a smooth strictly increasing function, which depends on a particular model. Since function v represents the concentration of a chemical substance and u stands for the concentration of the considered organisms, it is natural to suppose

$$u, v \geq 0 \qquad \text{in } \Omega \times (0,T)$$

and to complement the model with no-flux boundary conditions

$$\partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial\Omega \times (0,T),$$

and some non-negative initial conditions. The system (**??**) has attracted a lot of attention these past decades and we refer to surveys [**? ?**], and to references therein for more details on the existence, blow-up, and asymptotic behavior of

solutions.

The analysis of global dynamics of (**??**) crucially depends on the understanding of equilibria, that is, solutions of

$$\nabla \cdot (u\nabla(\log u - D_1\phi(v)) = 0, \qquad D_2\Delta v - D_3 v + D_4 u = 0\,,$$

with boundary conditions (**??**). By a standard reasoning one has $u = Ce^{D_1\phi(v)}$ for some positive constant C. The canonical choices for ϕ are $\phi(v) = v$, which leads to the Keller-Segel equation (**??**) on a domain Ω and $\phi(v) = \ln v$, which after appropriate rescaling, yields Lin-Ni-Takagi equation

$$\begin{cases} -\Delta v + v = v^p & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases}$$

The constants λ and p in (**??**) and (**??**) respectively depend on the parameters D_i of the system. A large amount of literature has been devoted to the Lin-Ni-Takagi equation in the subcritical and critical case, that is, when $N \geq 3$ and $1 < p \leq p_S := \dfrac{N+2}{N-2}$ (see [**? ? ? ?**] and references therein). Much less is known in the super-critical case, $p > p_S$ for (**??**) or $N \geq 3$ for (**??**), see [**? ? ?**].

Clearly, if p increases, the problem (**??**) becomes 'more super-critical', however the role of λ in (**??**) is less obvious, since the character of the nonlinearity remains unchanged as λ varies. To obtain a better insight, notice that (**??**) has two constant equilibria $v \equiv 0$ and $v \equiv 1$ which are in particular independent of p. On the other hand if $\lambda < 1/e$, then (**??**) has two constant solutions $\underline{u}_\lambda < 1 < \bar{u}_\lambda$ satisfying

$$\lambda e^\mu = \mu$$

and if $\lambda > 1/e$ there is no constant solution. Furthermore, $\bar{u}_\lambda \to \infty$ and $\underline{u}_\lambda \to 0$ as $\lambda \to 0^+$. To reveal the analogy between (**??**) and (**??**), denote $\mu = \bar{u}_\lambda$ and

$$u := \frac{v}{\bar{u}_\lambda} = \frac{v}{\mu}\,.$$

Then, u satisfies

$$\begin{cases} -\Delta u + u = e^{\mu(u-1)} & \text{in } B_R \\ u > 0 & \text{in } B_R, \\ \partial_\nu u = 0 & \text{on } \partial B_R\,, \end{cases}$$

and constant solutions of (**??**) are $u \equiv 1$ and $u \equiv \underline{u}_\mu$, with

$$\underline{u}_\mu = e^{\mu(\underline{u}_\mu - 1)} = (e^{\underline{u}_\mu - 1})^\mu, \qquad \underline{u}_\mu < 1\,.$$

Then, the nonlinearity is 'more critical' (has larger exponent) if μ is large, which is equivalent to λ being small.

The following bifurcation result for (**??**) with parameter μ was obtained in [**?**], see [**?**] for an analogous result for (**??**). Note that for fixed parameters the radial solutions of the second order equations are uniquely determined by the value of the function at 0 (since $u'(0) = 0$), therefore it suffices to investigate bifurcation diagrams in $\mathbb{R}^2$ with coordinates corresponding to μ and $u(0)$. Specifically, by (μ_0, A) we denote a pair (μ_0, u), where u is the solution of (**??**) with $\mu = \mu_0$ and $A = u(0)$. Here and below, λ_i^{rad} denotes the i-th eigenvalue of the operator $-\Delta + Id$ in the ball $B_R := \{x \in \mathbb{R}^N : |x| < R\}$ with Neumann boundary conditions, restricted to the space of radial functions.

Theorem 1.1. *For every $i \geq 2$, the trivial branch $(\mu, 1)$ of problem (**??**) has a bifurcation point at $(\lambda_i^{rad}, 1)$. Let $\mathcal{B}_i \subset \mathbb{R}^2$ be the continuum that branches out of $(\lambda_i^{rad}, 1)$. The following holds*

(i) the branches $\mathcal{B}_i$ are unbounded and do not intersect, and furthermore close to $(\lambda_i^{rad}, 1)$, $\mathcal{B}_i$ is a C^1-curve;

(ii) if $(\mu, A) \in \mathcal{B}_i$, then the corresponding solution u_μ satisfies $u_\mu > 0$ in B_R;

(iii) each branch consists of two connected components $\mathcal{B}_i^- := \mathcal{B}_i \cap \{(\mu, A) : A < 1\}$ and $\mathcal{B}_i^+ := \mathcal{B}_i \cap \{(\mu, A) : A > 1\}$;

(iv) if $(\mu, A) \in \mathcal{B}_i$ then the corresponding $u_\mu - 1$ has exactly $i - 1$ zeros, u'_μ has exactly $i - 2$ zeros;

(v) the functions satisfying $u_\mu(0) < 1$ are uniformly bounded in the C^1-norm.

The above theorem guarantees that $\mathcal{B}_i^-$ is a subset $\mathbb{R} \times (0,1)$ and it is unbounded. Since there are no non-trivial solutions for $\mu \leq 0$, we obtain that for each $i \geq 2$ the curve $\mathcal{B}_i^-$ is unbounded from above in the μ coordinate. We refer an interested reader to [**?** **?** **?** **?**] for the construction of solutions that we expect to be on the lower branches $\mathcal{B}_i^-$ (the solutions lie in the half plane $\{u(0) < 1\}$, but it is not known whether they are connected with the trivial solution). Note that all the references above except [**?**] deal with radial solutions and analogous results to Theorem **??** for (**??**) were proved in [**?**]. We also refer to [**?** **?**] for related problems involving the p-Laplace operator.

Properties of the upper branches $\mathcal{B}_i^+$ are more delicate, since the corresponding solutions are not a priori uniformly bounded. Although our interest is in dimension $N \geq 3$, we first recall known results in two dimensions.

If $N = 2$, then we call the problem 'critical' since the exponential nonlinearity is critical. It is proved in [**?**] that the branches $\mathcal{B}_i^+$ are unbounded and they exist for all values of $\mu \geq \lambda_i^{rad}$. Since $\lambda \to 0$ as $\mu \to \infty$, this means that in $(\lambda, u(0))$ plane, $\mathcal{B}_i^+$ approaches arbitrary close to the line $\lambda = 0$. Moreover, for $N = 2$ del Pino and Wei [**?**] constructed a class of radial solutions $(u_\lambda)_{\lambda \ll 1}$ of (**??**) such that

$$u_\lambda(x) \to 8\pi \mathcal{G}(x,0) \qquad \text{as } \lambda \to 0^+$$

uniformly on compact subsets of $B_R \setminus \{0\}$, where $\mathcal{G}$ is the Green's function, that is, for any $y \in \overline{B}_R$, $x \mapsto \mathcal{G}(x,y)$ solves

$$-\Delta_x \mathcal{G} + \mathcal{G} = \delta_y \text{ in } B_R, \quad \frac{\partial \mathcal{G}}{\partial \nu_x} = 0 \text{ on } \partial B_R$$

and δ_y is the Dirac measure supported at y. We remark that in [**?**], a result for non-radial solutions on general domains is also proved. Since one can check that $w_\lambda(0) = u_\lambda(0)/\overline{u}_\lambda > 1$, the functions $(w_\lambda)_{\lambda > 0}$ belong to solutions in the upper half plane, and their oscillation properties indicate that $\lambda \mapsto w_\lambda$ corresponds to the asymptotic part of the first upper branch $\mathcal{B}_1^+$. The results of [**?**] were extended, by the first two authors in collaboration with Román in [**?**],

to solutions concentrating on the boundary and/or on an interior sphere and blowing-up at the origin. Even more generally, under suitable non-degeneracy assumptions, it is possible to show the existence of solutions $(v_\lambda)_{\lambda>0}$ such that $v_\lambda(0) \to \infty$ as $\lambda \to 0^+$ and for every $M \geq 0$ there is $(r_j)_{j=1}^M \subset (0,R)$ such that $v_\lambda(r_j) \to \infty$ as $\lambda \to 0^+$. These non-degeneracy conditions are conjectured to hold, and it is believed that the solutions that concentrate on i spheres form the asymptote of the upper branch $\mathcal{B}_i^+$. We remark that in the 'asymptotically critical' case, $p \approx p_S$ for Lin-Ni-Takagi equation with $N = 3$, Rey and Wei [**?**] constructed a class of solutions that are believed to form the asymptote of $\mathcal{B}_1^+$.

Our main aim is to describe the purely supercritical upper branches of (**??**), a problem that recently attracted a lot of attention especially with Dirichlet boundary conditions

$$\begin{cases} U_{rr} + \dfrac{N-1}{r} U_r + \lambda g(U) = 0, & 0 < r < 1, \\ U > 0, & 0 < r < 1, \\ U(1) = 0, \end{cases}$$

see [**? ? ? ? ? ? ?**]. In [**?**], see also [**?** , Chapter 2] for a recent survey, Joseph and Lundgren considered $g(U) = e^U$ and proved that the set of positive solutions to (**??**) forms a curve γ emanating from the trivial solution $U \equiv 0$, $\lambda = 0$. When $3 \leq N \leq 9$, γ has infinitely many turning points around $\lambda^* = 2(N-2)$ and blows up at λ^*. The case $N = 3$ was treated earlier by Gel'fand [**?**]. When $N \geq 10$, the branch consists of minimal solutions for $0 < \lambda < \lambda^*$ with an asymptote at $\lambda = \lambda^*$. If $g(U) = (1+U)^p$, then in [**?**] a special exponent p_{JL} was found, namely

$$p_{JL} = \begin{cases} 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}}, & \text{when } N \geq 11, \\ \infty, & \text{when } 2 \leq N \leq 10, \end{cases}$$

and it was proved that when $p_S < p < p_{JL}$, the branch emanating from $(U, \lambda) = (0,0)$ has infinitely many turning points around $\lambda^* = \theta(N-2-\theta)$, $\theta = \dfrac{2}{p-1}$ and blows up at λ^* (the singular solution is given by $U^* = r^{-\theta} - 1$), whereas if

$p \geq p_{JL}$, the branch exists for all $0 < \lambda < \lambda^*$, does not oscillate and blows up at λ^*. These results were extended to more general nonlinearities, see for instance [**?**], where the author considered nonlinearity of the form

$$g(u) = e^u + h(u),$$

with h being a smooth lower order term.

For analogous Neumann problem we are only aware of [**?**], where the author studied the structure of positive radial solutions u_λ of

$$\begin{cases} -\Delta u_\lambda = \lambda(u_\lambda^p - u_\lambda) & \text{in } B_1, \\ \partial_\nu u_\lambda = 0 & \text{on } \partial B_1, \end{cases}$$

that bifurcate from the trivial solution 1. The exponent $p > \frac{N+2}{N-2}$ is fixed here. Problem (**??**) as well as (**??**) possesses a crucial scaling, that allows for exchange of the parameter λ for the size of the domain. More precisely, if $u_\lambda(\cdot)$ solves an appropriate problem on B_R with parameter λ, then $u_{\alpha^2\lambda}(\alpha\cdot)$ solves the same problem on $B_{R/\alpha}$. This property allows for a construction of explicit singular solutions as well as proofs of various important non-degeneracy properties.

Neumann problems even with scale invariance are more complicated than Dirichlet ones since there might be several bifurcation branches that contain positive solutions. In fact, we show below that there are infinitely many such branches. Also, radial eigenfunctions of Laplacian with Neumann boundary conditions correspond to large eigenvalues.

In our problem (**??**) due to the presence of the zero order term, we cannot rely on any scaling or transformation that removes the parameter λ from the equation. Moreover, the constant equilibria depend on λ and after appropriate normalization (cf. (**??**)) the parameter appears in the exponent of the nonlinearity, which introduces a novel parameter dependent problem.

To study the behavior of radial solutions for fixed parameter $\lambda > 0$, we first show that as the value of a solution at the origin increases, it converges to a solution U_λ^* satisfying the same problem with an explicit singularity at the origin. The existence and uniqueness of U_λ^* is shown on $(0, \infty)$, and in order

to prove the existence of singular solution on a finite interval with appropriate boundary conditions we first show that U_λ^* has infinitely many critical points. In other words, we show that for fixed λ, the restriction of U_λ^* satisfies Neumann problem on infinitely many balls. More precisely, we prove that U_λ^* oscillates around $\bar{u}_\lambda$.

Before we formulate our first result, let us recall that $\bar{u}_\lambda$ is the largest solution of $u = \lambda e^u$ and define $O(r^s)$ for $s \geq 0$ be a function such that $O(r^s)/r^s \to 0$ as $r \to 0^+$.

Theorem 1.2. *For any $N \geq 3$ and $\lambda > 0$, there exists $U_\lambda^* = U^* > 0$ satisfying, for each $\delta \in (0,1)$,*

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' + u = \lambda e^u & \text{on } \mathbb{R}^+ \\ u(r) = -2\ln r + \ln \dfrac{2(N-2)}{\lambda} + O(r^{2\delta}) & \text{when } r \to 0. \end{cases}$$

Moreover, a solution satisfying the equation in (??) with the condition at the origin

$$u(r) = -2\ln r + \ln \frac{2(N-2)}{\lambda} + O(1)$$

is unique. In addition, if

$$\lambda < \lambda_N^* := \begin{cases} 0.16 & N = 3\,, \\ 0.35 & N = 4\,, \\ 0.36 & N = 5\,, \\ \frac{1}{e} & N > 5\,, \end{cases}$$

then U^* attains infinitely many times the value $\bar{u}_\lambda$. Furthermore, for any sequences $(\gamma_n)_{n=1}^\infty$ and $(\lambda_n)_{n=1}^\infty$ with $\gamma_n \to \infty$ and $\lambda_n \to \lambda_\infty \in (0,\infty)$, one has $u_n \to U^*$ in $C^1_{loc}((0,\infty))$, where u_n is the solution to

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' + u = \lambda_n e^u & \text{on } \mathbb{R}^+\,, \\ u(0) = \gamma_n\,, \qquad u'(0) = 0 \end{cases}$$

and U^* satisfies (??) and (??) with $\lambda = \lambda_\infty$.

We require the restriction $\lambda < 1/e \approx 0.37$ to guarantee that the nonlinearity $\lambda e^u - u$ changes sign, since otherwise due to compatibility condition, the solution U^*_λ cannot have critical points. Also, if $\lambda < 1/e$, then there are two solutions of $u = \lambda e^u$, or equivalently, two constant equilibria of (**??**). We believe that the additional restriction on λ in lower dimensions is technical (see Lemma **??** below) and the result should hold without it. However, since we are interested in the asymptotes of bifurcation branches, that is, in small λ, this assumption does not cause any problems below. The oscillation result in Theorem **??** implies that there exists an increasing, unbounded sequence of positive real numbers $(R^i_\lambda)_{i=1}^\infty$ depending on N and $\lambda > 0$ such that $(U^*_\lambda)'(R^i_\lambda) = 0$, that is, U^*_λ satisfies Neumann boundary conditions on $\partial B_{R^i_\lambda}$. Consequently, U^*_λ is a singular radial solution to (**??**) in the ball of radius R^i_λ, $i \in \mathbb{N}$.

Our next main result states that if the radius R and any large integer $i > 1$ are fixed, we can choose $\lambda > 0$ such that $R^i_\lambda = R$, that is, U^*_λ has prescribed number of intersections with $\bar{u}_\lambda$ on B_R. Note that this result does not follow from a rescaling of the domain, since our equation is not scaling invariant. Clearly, such singular solutions have exactly i critical points (including the one on the boundary).

Theorem 1.3. *Assume $N \geq 3$ and let $R > 0$. Fix any $\tilde{\lambda} \in (0, \lambda^*_N)$ (cf. Theorem **??**) and let $U^*_{\tilde{\lambda}}$ be the function constructed in Theorem **??**. Denote by $(R^i_{\tilde{\lambda}})_{i\in\mathbb{N}}$ the increasing sequence such that $(U^*_{\tilde{\lambda}})'(R^i_{\tilde{\lambda}}) = 0$ and let i^* be the smallest integer such that $R^{i^*}_{\tilde{\lambda}} > R$. Then, for any $i \geq i^*$, there exists $\lambda^i > 0$ such that*

$$R^i_{\lambda^i} = R.$$

*In particular, for any $i \geq i^*$, there exists $\lambda^i > 0$ such that the equation (**??**) admits a singular radial solution $U^*_{\lambda^i}$ satisfying*

$$\sharp\{r \in (0,1) | U^*_{\lambda^i}(r) = \overline{u}_{\lambda^i}\} = i.$$

Once the existence of singular solutions on bounded domains is established, we turn our attention to the character of bifurcation branches parametrized by the value of solutions at the origin.

First we claim that the branch $\mathcal{B}_i^+$ (see Theorem **??**) is bounded in μ, that is, $\mathcal{B}_i^+ \subset (0, C_i) \times (1, \infty)$, where C_i depends only on i. Indeed, by testing (**??**) with v we see that there is no positive solution if $\lambda \leq 0$ and therefore by (**??**) no solution of (**??**) if $\mu \leq 0$. Next, let u be a solution to (**??**) such that $u(0) > 1$. Then setting $\tilde{u} = u - 1$, we see that

$$-\Delta \tilde{u} + \tilde{u} = e^{\mu \tilde{u}} - 1 \geq \mu \tilde{u}.$$

Hence, the Sturm-Picone comparison theorem implies that $\tilde{u} - 1$ has arbitrary large number of zeros if μ is large. However, since number of zeros is constant along $\mathcal{B}_i^+$ (cf. Theorem **??**), the claim follows.

However, the branch $\mathcal{B}_i^+$ (see Theorem **??**) is unbounded, and therefore by Theorem **??**, $\mathcal{B}_i^+$ (up to sub-sequence) converge to singular solutions. Next, we turn our attention to asymptotic behavior of $\mathcal{B}_i^+$.

To formulate the next result, for given $\lambda, \gamma > 0$ we denote by $(r^i_{\lambda,\gamma})_i$ the increasing sequence satisfying $u'(r^i_{\lambda,\gamma}, \gamma) = 0$, where $u(\cdot, \gamma)$ is the unique solution to (**??**). Note that if $u(\cdot, \gamma)$ is non-constant, its critical points are necessarily discrete and the sequence $(r^i_{\lambda,\gamma})_i$ is either finite or countable.

The following theorem gives a strong indication that for each $i \geq 1$, the branch $\mathcal{B}_i^+$ oscillates around λ^i when $3 \leq N \leq 9$. Below we show that the oscillations of $\mathcal{B}_i^+$ indeed take place for a generic radius.

Theorem 1.4. *Fix $3 \leq N \leq 9$, $R > 0$, $i \geq i^*$ (see Theorem **??**), and let $\lambda^i > 0$ be the positive real number given in Theorem **??**. Then, there exists a sequence of initial data $(\gamma_n)_n$ with $\gamma_n \to \infty$ and a sequence of positive integer $(j_n)_n$ such that $r^{j_n}_{\lambda^i,\gamma_n} = R$.*

Another evidence that the branch $\mathcal{B}_i^+$ oscillates around λ^i infinitely many times if $3 \leq N \leq 9$ and finitely many times if $N > 10$ is provided by the Morse index of the singular solution. We leave open the border line case $N = 10$. Recall that the Morse index of v satisfying (**??**), denoted $m(v)$, in the space of radial functions is the number of negative eigenvalues α (counting multiplicities)

of the eigenvalue problem

$$\begin{cases} -\Delta\phi + \phi - \lambda e^v \phi = \alpha\phi & \text{in } B_R, \\ \partial_\nu \phi = 0 & \text{on } \partial B_R, \\ \phi \text{ is radially symmetric.} \end{cases}$$

Recall that the Morse index of solutions remains constant along a bifurcation branch unless it has a critical point in λ. Thus, each turning point of a bifurcation branch corresponds to a transition of an eigenvalue (of the linearization) across imaginary axis. Since the solutions $u(\cdot,\gamma) \to U^*$ as $\gamma \to \infty$, the Morse index of U^* indicates the total number of turning points of the branch and combined with Theorem **??**, it suggests the number of intersection points of $\mathcal{B}_i^+$ with λ^i.

Proposition 1.5. *If $U^*_{\lambda^i}$ is a solution to (**??**), then $m(U^*_{\lambda^i}) < \infty$ when $N > 10$ while $m(U^*_{\lambda^i}) = \infty$ when $3 \le N \le 9$.*

Finally, we prove the oscillation of the branches $\mathcal{B}_i^+$ in dimension $4 \le N \le 9$ for generic radius. If the scale invariance is available, then one can show that $\mathcal{B}_i^+$ can be parametrized by the value of the solution at the origin, and in particular there are no secondary bifurcations and singular solutions are non-degenerate. In our case the situation is much more complicated and we rely on Sard's theorem which merely yield results for generic domains.

First, we show a generic local uniqueness result for singular solutions, which combined with Theorem **??** yields that $\mathcal{B}_i^+$ (and any other branches) converge to discrete set of functions. More precisely, for generic $R > 0$, if $(U^*_{\lambda^*})'(R) = 0$, then $(U^*_\lambda)'(R) \neq 0$, for λ close but different to λ^*. In other words, if we have a singular solution on B_R for certain λ^*, then we do not have a singular solution for nearby λ, that is, the set (λ^i) (see Theorem **??**) is discrete.

Theorem 1.6. *There exists a set $S^* \subset (0,\infty)$ of Lebesgue measure zero, such that for any radius $R \in (0,\infty) \setminus S^*$ the following holds. If $(U^*_{\lambda^*})'(R) = 0$, then there exists $\delta > 0$ such that for any $\lambda \in (\lambda^* - \delta, \lambda^* + \delta) \setminus \{\lambda^*\}$ one has $(U^*_\lambda)'(R) \neq 0$.*

A direct consequence is the following corollary.

Corollary 1.7. *Let $R \in (0,\infty)\backslash S^*$, where S^* is defined in Theorem* **??**. *Then, there exists $\delta > 0$ such that, for any $\lambda \in (\lambda^i-\delta, \lambda^i+\delta)\backslash\{\lambda^i\}$, there is no singular solution of* (**??**) *satisfying* (**??**).

To formulate a generic uniqueness result for regular solutions, recall $r^i_{\lambda,\gamma}$ defined in Theorem **??**. Then, for any $R \in (0,\infty) \setminus S^*$ and any large γ there exists at most one $\lambda \approx \lambda^i$ such that $r^i_{\lambda,\gamma} = R$.

Theorem 1.8. *Assume $N \geq 4$ and suppose $\lambda < 1/e$ for $N > 4$ and $\lambda < 0.196$ if $N = 4$. Fix λ^i as in Theorem* **??** *and let S^* be the zero measure set as in Theorem* **??**. *Then for any $R \in (0,\infty) \setminus S^*$, there exist $\delta > 0$ and $\Gamma > 0$ such that for each $\gamma \geq \Gamma$ there exists at most one $\lambda \in (\lambda^i - \delta, \lambda^i + \delta)$ such that $r^i_{\lambda,\gamma} = R$.*

As a direct corollary of the two previous theorems and Theorem **??**, we obtain a quite complete picture of the bifurcation diagram in small dimension for generic radius.

Corollary 1.9. *Assume $4 \leq N \leq 9$. Moreover suppose that $\lambda < 0.196$ if $N = 4$. For $R \in (0,\infty)\backslash S^*$, the branches $\mathcal{B}^+_i$ defined in Theorem* **??** *oscillate in the plane $(\mu, u(0))$ around the line $\mu = \lambda^i$, where λ^i is as in Theorem* **??**. *Moreover, no secondary bifurcation occurs for large $u(0)$ and there are no branches bifurcating from infinity. Furthermore, $\mathcal{B}^+_i$ can be parametrized by $u(0)$ for large values of $u(0)$.*

We remark that we expect Theorem **??** and Corollary **??** to hold true also for $N = 3$ and for any $\lambda < 1/e$ (see Lemma **??** and **??**).

Let us briefly describe the main ideas of proofs. We often use the change of variables

$$u(r) = \eta(\zeta) + 2\zeta,$$

where

$$r = \sqrt{\frac{2(N-2)}{\lambda}} e^{-\zeta}$$

which transforms (**??**) to

$$\begin{cases} \eta'' - (N-2)\eta' + 2(N-2)\eta = m^2 e^{-2\eta}(\eta + 2\zeta) - 2(N-2)(e^{\eta} - 1 - \eta) & \eta \in \mathbb{R}, \\ \lim\limits_{\zeta \to \infty} \eta(\zeta) = 0. \end{cases}$$

Note that the zero order term u makes (**??**) non-autonomous and as such we cannot directly use techniques from dynamical systems. However, to gain a better intuition assume that the term $m^2 e^{-2\eta}(\eta + 2\zeta)$, which is exponentially small at infinity, is missing. In that case, we are searching for solutions converging along stable manifold to 0. A standard linear analysis yields that 0 is an unstable focus if $3 \leq N \leq 9$ and unstable node if $N \geq 10$ and as such there is no stable manifold. Thus, if $e^{-2\zeta}(\eta + 2\zeta)$ is missing, then $\eta \equiv 0$ is the only solution of (**??**). This reasoning suggests that solutions of (**??**) are unique and exponentially close to 0 at least for large ζ. The uniqueness yields that solutions of (**??**) are very unstable and are presumably hard to analyze by direct numerical and analytical methods. Thus, to prove the existence and uniqueness of solutions to (**??**) we incorporate the condition at infinity into the choice of functional spaces and use the Banach fixed point theorem.

To analyze the oscillations, we need to understand the behavior of U_λ^* for large r. Since (**??**) admits a Lyapunov functional, intuition (modulo non-autonomous term $\frac{1}{r}u'$ which is small for large r) yields that the function U_λ^* converges as $r \to \infty$ to an equilibrium of the (**??**) viewed as an initial value problem, that is, to the values $\bar{u}_\lambda$ or $\underline{u}_\lambda$ (see (**??**)). Again by ignoring the term u'/r, we can analyze the character of equilibria and obtain that $\underline{u}_\lambda$ is a saddle and $\bar{u}_\lambda$ is a center. Hence, the former does not allow for oscillatory solutions, whereas the latter does. Therefore, an important ingredient of the proof is to show that the singular solution of (**??**) does not converge to $\underline{u}_\lambda$ as $r \to \infty$, see Proposition **??** which is in fact a Pohozaev type identity. This is the only result where we need our technical upper bound $\lambda \leq \lambda^*$ in lower dimensions (cf. Theorem **??**). The final argument is based on Sturm-Picone oscillation theorem and careful estimates of singular solutions. Note that similar ideas were used in [**?**].

The proof of $u(\cdot,\gamma) \to U^*$ as $\gamma \to \infty$ is partly motivated by [**?**] and crucially depends on the uniqueness of the singular solution U^*. Then, it suffices to prove that $u(\cdot,\gamma)$ converges to a function that satisfies both the equation (**??**) and asymptotics at the origin (**??**). Since, $u(\cdot,\gamma)$ and U^* satisfy the same equation (**??**), the convergence of $u(\cdot,\gamma)$ to a solution of (**??**) follows from a priori estimates and standard regularity theory. The asymptotics at the origin is of a different flavor and requires careful estimates in transformed variables.

The proof of Theorem **??** uses an observation that for fixed λ and large u, the term u is negligible compared to e^u, and therefore if γ is large, then close to the origin we can neglect the zero order term which was responsible for the breaking of scaling. Hence, close to the origin u can be approximated by the solution of scale invariant problem

$$\begin{cases} \Delta \bar{u} + \lambda e^{\bar{u}} = 0 & \text{on } (0,\infty)\,, \\ \bar{u}(0,\alpha) = \alpha, \qquad \bar{u}'(0,\alpha) = 0\,. \end{cases}$$

The same reasoning yields that singular solutions of (**??**) can be approximated near the origin by the singular solution of (**??**) which is given by

$$\bar{u}^*(r) = -2\ln r + \ln \frac{2(N-2)}{\lambda}.$$

Using the classical arguments of Joseph and Lundgren [**?**] and scale invariance of (**??**) we conclude that if $3 \le N \le 9$ and α being sufficiently large, the solution $u(\cdot,\alpha)$ of (**??**) intersects arbitrarily many times $\bar{u}^*$ in a small neighborhood of the origin. Using precise estimates we can indeed verify this intuition and conclude that the solution u of (**??**) with $u(0) = \gamma$ intersects arbitrarily many times the singular solution U^* in a small neighborhood of the origin. The rest of the proof of Theorem **??** follows from zero number arguments.

Theorem **??** is a consequence of the continuity of the function $\lambda \to R^i_\lambda$ for all $i \in \mathbb{N}$ and the fact that, for any $i \in \mathbb{N}$,

$$R^i_\lambda \to 0^+, \quad \text{as} \quad \lambda \to 0^+.$$

Although this idea is rather elementary its proof poses the main technical challenge of the paper. In order to prove (**??**) we not only need more precise asymptotics of U_λ^* at the origin, but we require estimates on the *length of the interval* where the asymptotics are valid. In fact, we prove estimates up till r_λ, the first intersection point of U_λ^* with $\bar{u}_\lambda$. The cornerstone of the proofs is an observation that the higher order correction of U_λ^* for small r is negative. Once the first intersection with $\bar{u}_\lambda$ is established, we obtain an estimate on $(U_\lambda^*)'(r_\lambda)$ and finish the proof using careful estimates and Sturm-Piccone theorem. We remark that direct estimates up till the first critical point of U_λ^*, that is, till R_λ^1, seem beyond reach. The continuity of $\lambda \to M_\lambda^i$ is primarily based on the uniqueness of U_λ^*.

The bounds on the Morse index stated in Proposition **??** are based on the asymptotic behavior of U_λ^* when $r \to 0$ combined with Hardy's inequality. The proof of Theorem **??** follows from the fact that the function $\lambda \to R_\lambda^i$ is Lipschitz which allows us to use the Sard's theorem. Lipschitz continuity in turn follows from precise estimates on the modulus of continuity of the function $\lambda \mapsto U_\lambda^*$. The main observation in the proof of Theorem **??** is the fact that the function $\lambda \to r_{\lambda,\gamma}^i$ is bounded in $C^2(I)$, for some compact interval $I \subset (0,\infty)$, by a constant not depending on γ, which follows from precise estimates on the rate of convergence of regular solutions to singular ones.

The paper is organized as follows. In Section **??**, we prove the first part of Theorem **??**, namely, we establish the existence of U_λ^* and prove oscillations around $\overline{u}_\lambda$. We finish the proof of Theorem **??** in Section **??** by showing the convergence of $u_\lambda(r,\gamma)$ to $U_\lambda^*(r)$ as $\gamma \to \infty$. Section **??** is dedicated to the proofs of Theorem **??** and Proposition **??**. Theorem **??** is proved in Section **??**. Finally, generic results, Theorems **??** and **??** are proved in Section **??**. Let us mention that we expect the same results to hold for the Lin-Ni-Takagi equation or all radii, which will be the subject of a forthcoming work.

2. Construction of the positive radial singular solution in the whole space.

Fix $N \geq 3$, $\lambda > 0$ and consider the equation

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' + u = \lambda e^u & \text{in } (0,\infty), \\ u > 0 & \text{in } (0,\infty), \end{cases}$$

where u depends on the radial variable r and the derivatives are with respect to r. The main goal of this section is the proof of the existence and uniqueness of solution of (**??**) with

$$u(r) = -2\ln r + \ln \frac{2(N-2)}{\lambda} + o(1) \qquad \text{as } r \to 0^+ ,$$

where we denote by $o(1)$ the class of functions f such that $\lim_{r\to 0^+} f(r) = 0$. We use the following change of variables

$$u(r) = \eta(\zeta) + 2\zeta,$$

where

$$r = \sqrt{\frac{2(N-2)}{\lambda}} e^{-\zeta}.$$

To simplify notation, we denote $m = \sqrt{\dfrac{2(N-2)}{\lambda}}$ and $\zeta = \ln \dfrac{m}{r}$. A direct computation shows that

$$\frac{du}{dr} = -\frac{1}{r}\frac{d\eta}{d\zeta} - \frac{2}{r},$$

and

$$\frac{d^2u}{dr^2} = \frac{1}{r^2}\frac{d^2\eta}{d\zeta^2} + \frac{1}{r^2}\frac{d\eta}{d\zeta} + \frac{2}{r^2}.$$

In the following, if $f : \mathbb{R} \to \mathbb{R}$ depends only on one variable ρ, usually r or ζ, we denote $f' = \frac{df}{d\rho}$, and analogously for higher order derivatives. Then, (**??**) is equivalent to

$$\begin{aligned} 0 &= -u'' - \frac{N-1}{r}u' + u - \lambda e^u \\ &= \frac{1}{r^2}\left[-\eta'' + (N-2)\eta' + m^2 e^{-2\zeta}(\eta + 2\zeta) - 2(N-2)(e^\eta - 1)\right] , \end{aligned}$$

and consequently

$$\eta''(\zeta) - (N-2)\eta'(\zeta) + 2(N-2)\eta(\zeta) = g(\zeta),$$

where

$$g(\zeta) = m^2 e^{-2\zeta}(\eta(\zeta) + 2\zeta) - 2(N-2)(e^{\eta(\zeta)} - 1 - \eta(\zeta))\,.$$

We also set

$$\phi(\eta) = -2(N-2)(e^{\eta} - 1 - \eta)$$

and note that $\phi(\eta) \approx \frac{1}{2}\eta^2$ for $\eta \approx 0$. The blow up rate (**??**) is equivalent to

$$\lim_{\zeta\to\infty} \eta(\zeta) = 0\,.$$

For any $N \geq 3$ denote

$$\alpha = N-2, \qquad \beta = \sqrt{\frac{(N-2)|N-10|}{4}},$$

and let G_N be the Green's function for the left hand side of (**??**) defined by

$$G_N(z) := \begin{cases} \frac{1}{\beta} e^{-\frac{\alpha}{2}z}\sin(\beta z) & 3 \leq N \leq 9, \\ e^{-\frac{\alpha}{2}z} z & N = 10, \\ \frac{1}{\beta} e^{-\frac{\alpha}{2}z}\sinh(\beta z) & N > 10, \end{cases} \qquad \text{for } z \geq 0, \qquad G_N(z) = 0 \qquad \text{for } z < 0\,.$$

Observe that $G_N \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for any $N \geq 3$. Then, (**??**) is equivalent to

$$\eta(\sigma) = \int_{\mathbb{R}} G_N(\tau - \sigma) g(\tau) d\tau\,.$$

Thus, finding solution of (**??**) satisfying (**??**) reduces to finding a solution of (**??**).

Proposition 2.1. *Let $m > 2\sqrt{2(N-2)}$. The equation* (**??**) *admits a unique solution on $(-\infty, \infty)$ satisfying*

$$\lim_{\zeta\to\infty} \eta(\zeta) = 0.$$

This solution is also unique on any interval (ζ_0, ∞), $\zeta_0 \in \mathbb{R}$.

Remark 2.2. *Clearly, Proposition* **??** *establishes existence and uniqueness of solution* U^* *asserted in Theorem* **??**.

Proof of Proposition **??**. First, we construct a local solution by using the contraction mapping theorem on the Banach space $X = \{\eta \in C^0([\zeta_0; \infty)); |\eta|_\infty < \infty\}$, where ζ_0 is determined below and $C^0([\zeta_0, \infty))$ is the space of continuous function on $[\zeta_0, \infty)$ that decay at infinity, equipped with the supremum norm. Also, for any $\bar{r} \geq 0$ denote $\mathcal{B}_{\bar{r}} = \{\eta \in X; |\eta|_\infty < \bar{r}\}$ and let g be as in (**??**). To avoid confusion, we explicitly indicate the dependence of g on η.

Let G_N be defined by (**??**). For any $\eta \in \mathcal{B}_{\bar{r}}$ and any $\zeta \geq \zeta_0$, denote

$$F(\eta)(\zeta) = \int_{\mathbb{R}} G_N(\sigma - \zeta) g(\eta, \sigma) d\sigma = \int_0^\infty G_N(\sigma) g(\eta, \sigma + \zeta) d\sigma \,.$$

Note that the integrals are well defined since $G_N \in L^1$ and $G_N(z) = 0$ for $z \leq 0$. Since $\eta \in X$, we have that $\eta(\zeta) \to 0$ as $\zeta \to \infty$, and therefore $|g(\eta, \zeta)| \to 0$ as $\zeta \to \infty$. Hence, since $G_N \in L^1$

$$|F(\eta)(\zeta)| \leq C_N \sup_{\sigma \geq 0} |g(\eta, \sigma + \zeta)| \to 0 \qquad \text{as } \zeta \to \infty$$

and in particular $F : X \to X$.

Next, we show that F is a contraction on $\mathcal{B}_{\bar{r}}$. Indeed, for any $\varepsilon > 0$ there is $\bar{r} > 0$ and $\zeta_0 > 0$ such that for every $\eta_1, \eta_2 \in \mathcal{B}_{\bar{r}}$ and $\zeta \geq \zeta_0$ one has

$$|g(\eta_1, \sigma) - g(\eta_2, \sigma)| \leq (m^2 e^{-2\sigma} |\eta_1(\sigma) - \eta_2(\sigma)| + 2(N-2)|e^{\eta_1(\sigma)} - e^{\eta_2(\sigma)} - \eta_1(\sigma) + \eta_2(\sigma)|$$

$$\leq \varepsilon \|\eta_1 - \eta_2\|_\infty \,,$$

where in the last step we used the mean value theorem for the function $m(x) = e^x - x$ and the fact that $|m'(x)| = |e^x - 1|$ is small if x is small, that is, if $|\eta_i(\sigma)| \leq \bar{r} \ll 1$ for $i \in \{1, 2\}$. Then, since $G_N \in L^1$

$$\|F(\eta_1) - F(\eta_2)\|_{L^\infty((\zeta_0, \infty))} \leq \varepsilon \, \|\eta_1 - \eta_2\|_{L^\infty((\zeta_0, \infty))} \, \|G_N\|_{L^1} = C_N \varepsilon \, \|\eta_1 - \eta_2\|_{L^\infty((\zeta_0, \infty))} \,.$$

Fix $\varepsilon > 0$ such that $C_N \varepsilon < \frac{1}{2}$, which in turn fixes small $\bar{r}$ and large ζ_0.

Finally, we show that F maps $\mathcal{B}_{\bar{r}}$ into itself. By increasing ζ_0 if necessary, we can assume that $\zeta e^{-2\zeta} < \varepsilon_0 \bar{r}$ for any $\zeta \geq \zeta_0$, where $0 < \varepsilon_0 < \frac{1}{C_N m^2}$. Then, for any $\zeta \geq \zeta_0$,

$$\|F(0)\|_{L^\infty(\zeta_0,\infty)} = \sup_{\zeta \geq \zeta_0} 2m^2 \left| \int_0^\infty G_N(\sigma) e^{-2(\sigma+\zeta)} (\sigma + \zeta) d\sigma \right| < \varepsilon_0 C_N m^2 \bar{r} < \frac{1}{2} \bar{r} \,.$$

Thus for any $\eta \in \mathcal{B}_{\bar{r}}$ one has

$$\|F(\eta)\|_{L^\infty(\zeta_0,\infty)} \leq \|F(\eta) - F(0)\|_{L^\infty(\zeta_0,\infty)} + \|F(0)\|_{L^\infty(\zeta_0,\infty)} < \frac{1}{2} \|\eta\|_{L^\infty(\zeta_0,\infty)} + \frac{1}{2} \bar{r} \leq \bar{r} \,,$$

and so F is a contraction on $\mathcal{B}_{\bar{r}}$. The existence and uniqueness of solutions on (ζ_0, ∞) follows from the Banach fixed point theorem. To prove the uniqueness in X suppose that there are two solutions η_1 and η_2. Fix $\bar{r}$ as above and by (**??**) we can choose ζ_0 sufficiently large such that $\eta_1, \eta_2 \in \mathcal{B}_{\bar{r}}$. By the already proved uniqueness we obtain that $\eta_1 = \eta_2$ on (ζ_0, ∞). The fact that $\eta_1 \equiv \eta_2$ follows from the uniqueness of the initial value problems.

Let us prove that η can be extended to the whole real line. We proceed by showing that a solution u of (**??**) defined on the interval $(0, r_0)$ can be extended to the interval $(0, \infty)$. Indeed, let $(0, R_0)$ be the maximal existence interval of the solution and assume $R_0 < \infty$. Since the nonlinearity is Lipschitz it suffices to show that u is bounded on the interval $I_0 = (R_0/2, R_0)$. Next, observe that the functional

$$V(r) = \frac{(u'(r))^2 - u^2(r)}{2} + \lambda e^{u(r)}$$

is a Lyapunov functional for the flow, that is, $r \mapsto V(r)$ is decreasing on $r \in (0, R_0)$. Hence,

$$V(r) \leq V(R_0/2) = C^* \qquad \text{for any } r \in I_0 \,,$$

that is, V is bounded from above on I_0. To prove that u is bounded, note that (**??**) yields $(u'(r))^2 - u^2(r) \leq C^*$, and therefore

$$(u^2)' = 2uu' \leq u^2 + (u')^2 \leq C^* + 2u^2 \,.$$

The Gronwall inequality yields that $u^2(r) \le Ce^{2r}$ for $r \in I_0$, where C depends on C^*, R_0, and $u(R_0/2)$. Thus, u is bounded on I_0, and therefore can be continued beyond R_0, a contradiction. □

Next we obtain more precise asymptotics on η at infinity, which in turn transforms into more precise asymptotics of u at the origin.

Lemma 2.3. *If η is a solution of* (**??**), (**??**), *then for any $\delta > 0$,*

$$\lim_{\zeta\to\infty} e^{(2-\delta)\zeta}\eta(\zeta) = \lim_{\zeta\to\infty} e^{(2-\delta)\zeta}\eta'(\zeta) = 0\,.$$

Proof. By applying Young convolution inequality to (**??**), we have

$$\int_\zeta^\infty |\eta(\sigma)|d\sigma \le \|G_N\|_{L^1}\int_\zeta^\infty |g(\sigma)|d\sigma = C_N\int_\zeta^\infty |g(\sigma)|d\sigma\,.$$

Since, for every $\varepsilon > 0$, one has $2(N-2)|e^a - 1 - a| \le \varepsilon|a|$ for any sufficiently small $|a|$, and since $\eta(\sigma) \to 0$ as $\sigma \to \infty$, we deduce that for any $\delta > 0$, there exists large ζ_0 such that, for any $\zeta \ge \zeta_0$,

$$|g(\zeta)| \le \varepsilon|\eta(\zeta)| + 2m^2\zeta e^{-2\zeta} + m^2e^{-2\zeta}|\eta(\zeta)| \le 2\varepsilon(e^{-2(1-\delta/2)\zeta} + |\eta(\zeta)|).$$

This implies that, for $\varepsilon = \frac{1}{4}$ and any $\zeta \ge \zeta_0$

$$\int_\zeta^\infty |\eta(\sigma)|d\sigma \le Ce^{-2(1-\delta/2)\zeta}\,,$$

where C depends on δ and N. Substituting this estimate and (**??**) with $\varepsilon = \frac{1}{4}$ to (**??**), we obtain that

$$|\eta(\zeta)| \le \frac{1}{2}\int_\sigma^\infty |G_N(\tau-\sigma)|(e^{-2(1-\delta)\sigma} + |\eta(\sigma)|d\tau \le Ce^{-2(1-\delta/2)\zeta}$$

and the first assertion follows.

Finally, since

$$\eta'(\sigma) = \int_{\mathbb{R}} G_N'(\tau-\sigma)g(\tau)d\tau$$

and $G_N' \in L^1$, we can proceed as in (**??**) by replacing G_N by G_N' and conclude the proof. □

Next, we show that $U^* \in H^1_{loc}(\mathbb{R}^N)$ where U^* is defined in Theorem **??**.

Lemma 2.4. *If U^* is as in Theorem* **??**, *then*

$$\lim_{r\to 0}(U^*)'(r)+\frac{2}{r}=0.$$

Moreover, $U^*\in H^1(B_{R_1})$ for any $R_1>0$.

Proof. By Proposition **??**, U^* exists on $(0,\infty)$. Relation (**??**) implies that

$$(U^*)'(r)=-\frac{1}{r}\eta'(\zeta)-\frac{2}{r}=-\frac{1}{m}e^{\zeta}\eta'(\zeta)-\frac{2}{r}$$

and the first assertion follows from Lemma **??**. Next, recall that $U^*(r)=\eta(\zeta)-2\ln\frac{r}{m}$. So, using Lemma **??**, we deduce that

$$\begin{aligned}\|U^*\|^2_{H^1(B_{R_1})}&=\omega_N\int_0^{R_1}(|(U^*)'|^2+|U^*|^2)r^{N-1}dr\\&\le C\int_0^{R_1}(r^{-2}+(\ln r)^2+1)r^{N-1}dr<\infty\end{aligned}$$

for $N\ge 3$. This establishes the lemma. □

Next, we focus on the behavior of U^* for large r. As a preliminary we prove the following lemma which is based on Pohozaev-type identity.

Lemma 2.5. *Fix $N\ge 3$ and $\lambda\in(0,\lambda^*_N)$, where λ^*_N is as in Theorem* **??**. *If U^* is the unique solution of* (**??**), *then $U^*>\underline{u}_\lambda$ and*

$$\liminf_{r\to\infty}U^*(r)>\underline{u}_\lambda\,,$$

where $\underline{u}_\lambda<1$ is the smaller solution of $u=\lambda e^u$.

Proof. If $v=U^*-\underline{u}_\lambda$, then v satisfies

$$-\Delta v+v=\underline{u}_\lambda(e^v-1)$$

with

$$\begin{aligned}v(r)&=-2\ln r+\ln\frac{2(N-2)}{\lambda}-\underline{u}_\lambda+O(1)=-2\ln r+\ln\frac{2(N-2)}{\lambda e^{\underline{u}_\lambda}}+O(1)\\&=-2\ln r+\ln\frac{2(N-2)}{\underline{u}_\lambda}+O(1)\,.\end{aligned}$$

For a contradiction, assume that either there exists the smallest R_0 such that $v(R_0) = 0$, or $v > 0$ and $\liminf_{r\to\infty} v(r) = 0$. In the latter case we set $R_0 = \infty$. Denote $\varepsilon_0 = 1 - \underline{u}_\lambda \in (0, 1)$.

We claim that $R_0 = \infty$ implies $\lim_{r\to\infty} v(r) = 0$. Indeed, if not then there exist $v_0 > 0$ and a sequence $r_n \to \infty$ as $n \to \infty$ such that $v(r_n) \geq v_0 > 0$. Since $\liminf_{r\to\infty} v(r) = 0$, by the mean value theorem, there is a local minimizer r^* of v. In particular, $v'(r^*) = 0$ and thus $(U^*)'(r^*) = 0$. Since the Lyapunov functional V defined by (**??**) is decreasing, we obtain that $V(r^*) > V(r)$ for any $r > r^*$. This implies that there is no $r > r^*$ such that $U^*(r) = U^*(r^*)$ and since r^* is a local minimum $U^*(r) \geq U^*(r^*) > 0$ for any $r \geq r^*$. This contradicts $\liminf_{r\to\infty} U^*(r) = 0$, and the claim follows.

If $R_0 = \infty$, then since $v \searrow 0$, we can fix $R > 0$ such that $|e^{v(r)} - 1| \leq (1 + \varepsilon_0)|v(r)|$ for each $r \geq R$. Consequently,

$$-\Delta v + v \leq \underline{u}_\lambda(1 + \varepsilon_0)v = (1 - \varepsilon_0^2)v \qquad \text{in } \mathbb{R}^N \setminus B_R(0)\,.$$

Define $\psi(r) = C_1 e^{-\frac{\varepsilon_0}{2}(r-R)}$, for some $C_1 > 0$ specified below. It is easy to see after increasing R if necessary, that we have

$$-\Delta\psi + \psi = \left(1 - \frac{\varepsilon_0^2}{4}\right)\psi + \varepsilon_0\frac{(N-1)}{2r}\psi \geq \left(1 - \varepsilon_0^2\right)\psi \qquad \text{in } \mathbb{R}^N\setminus B_R(0).$$

Fix R and choose C_1 such that $C_1 > v(R)$. Then, $\psi(R) - v(R) \leq 0$ and $\lim_{r\to\infty}(v(r) - \psi(r)) = 0$. Then, a comparison principle yields $v(r) \leq \psi(r)$, for all $r \geq R$. Also, elliptic regularity theory implies that v' decays exponentially at infinity.

Fix any $R \in (0, R_0)$ multiplying (**??**) by $r^N v'$ and integrating, we find, for any $0 < \rho < R$,

$$\frac{N-2}{2}\int_\rho^R |v'|^2 r^{N-1}dr + \left[\frac{r^N(v'(r))^2}{2}\right]_\rho^R - \left[\frac{r^N v^2}{2}\right]_\rho^R + \frac{N}{2}\int_\rho^R v^2 r^{N-1}dr$$
$$+ \underline{u}_\lambda\left[r^N(e^v - 1 - v)\right]_\rho^R = N\underline{u}_\lambda\int_\rho^R (e^v - 1 - v)r^{N-1}dr.$$

On the other hand, multiplying (**??**) by vr^{N-1} and integrating, we have

$$\int_\rho^R |v'|^2 r^{N-1}dr - [r^{N-1}vv']_\rho^R + \int_\rho^R v^2 r^{N-1}dr = \underline{u}_\lambda\int_\rho^R v(e^v - 1)r^{N-1}dr.$$

Since for small ρ one has $|v(\rho)| \leq C|\ln \rho|$ by (**??**), and

$$|v'(\rho)| = |(U^*)'(\rho)| \leq \frac{4}{\rho},$$

by Lemma **??**, we have that the lower boundary terms converge to 0 as $\rho \to 0$. Also, if $R_0 = \infty$ since $v(R)$ and $v'(R)$ decay exponentially as $R \to \infty$, the upper boundary terms decay to 0 as $R \to \infty$. If $R_0 < \infty$, one has $v(R_0) = 0$ and clearly $R_0^N (v'(R_0))^2 \geq 0$. This implies that

$$\int_0^R v^2 r^{N-1} dr + o_R(1) \leq \underline{u}_\lambda \left(N \int_0^R (e^v - 1 - v) r^{N-1} dr - \frac{N-2}{2} \int_0^R v(e^v - 1) r^{N-1} dr \right).$$

where $o_R(1) \to 0$ as $R \to \infty$ if $R_0 = \infty$ and $o_R(1) = 0$ if $R_0 < \infty$. Let us denote

$$f(x) = x^2 - \underline{u}_\lambda \left(N(e^x - 1 - x) - \frac{N-2}{2} x(e^x - 1) \right).$$

We will obtain a contradiction to (**??**) for sufficiently large R if we prove that $f(x) > 0$ for any $x > 0$. Since $f(0) = f'(0) = 0$, it suffices to show that $f''(x) > 0$ for any $x > 0$. A simple computation shows that $f''(x) = 2 - \underline{u}_\lambda \left(2e^x - \frac{N-2}{2} x e^x \right)$ and that

$$\min_{x \geq 0} f''(x) = \begin{cases} f'' \left(\frac{6-N}{N-2} \right), & \text{if } N < 6, \\ f''(0) = 2(1 - \underline{u}_\lambda), & \text{if } N \geq 6. \end{cases}$$

Since by definition $\underline{u}_\lambda < 1$, we have $f''(x) > 0$ for $N \geq 6$, a contradiction. Also, we obtain a contradiction if

$$\underline{u}_\lambda < \frac{4}{N-2} e^{-\frac{6-N}{N-2}} \approx \begin{cases} 0.20, & \text{if } N = 3, \\ 0.74, & \text{if } N = 4, \\ 0.96, & \text{if } N = 5. \end{cases}$$

One can check that the previous values corresponds to

$$\lambda = \begin{cases} 0.16, & \text{if } N = 3, \\ 0.35, & \text{if } N = 4, \\ 0.36, & \text{if } N = 5, \end{cases}$$

that is, $\lambda < \lambda_N^*$. □

Remark 2.6. *Under the assumptions of Lemma* **??** *one has* $U^* > 0$.

Next, we prove that U^* oscillates around $\overline{u}_\lambda$ as claimed in Theorem **??**.

Lemma 2.7. *Let* $N \geq 3$ *and suppose that* $\lambda \in (0, \lambda_N^*)$, *where* λ_N^* *is as in the statement of Theorem* **??**. *If* U^* *is as in Theorem* **??** *(for the existence and uniqueness see Proposition* **??***), then there exists a sequence* $0 < R_\lambda^1 < \ldots < R_\lambda^k < \ldots \to \infty$ *such that* $U^*(R_\lambda^k) = \bar{u}_\lambda$. *In particular, there is a sequence* (R_λ^{*k}) *such that* $(U^*)'(R_\lambda^{*k}) = 0$.

Proof. By Lemma **??** one has $M := \inf U^* > \underline{u}_\lambda$. If we denote $w(r) = r^{\frac{N-1}{2}}(U^*(r) - \bar{u}_\lambda)$, then standard calculations yield that w satisfies

$$w'' = \left(\frac{U^* - \lambda e^{U^*}}{U^* - \bar{u}_\lambda} + \frac{(N-1)(N-3)}{4r^2} \right) w =: m(r)w.$$

Set

$$F(x) = \frac{x - \lambda e^x}{x - \bar{u}_\lambda} \quad x \neq \bar{u}_\lambda, \qquad F(\bar{u}_\lambda) = 1 - \bar{u}_\lambda \, .$$

It is easy to see that F is continuous and $F \to -\infty$ as $x \to \infty$. Furthermore, the numerator is positive if and only if $x \in (\underline{u}_\lambda, \bar{u}_\lambda)$, whereas the denominator is positive if and only if $x > \bar{u}_\lambda$. Thus, $F < 0$ on $(\underline{u}_\lambda, \infty)$, and consequently $F \leq -2\varepsilon_1 < 0$ on $[M, \infty)$. Choose R_2 large such that

$$\frac{(N-1)(N-3)}{4r^2} < \varepsilon_1$$

and we obtain $m(r) \leq -\varepsilon_1$ for $r \geq R_2$. By the Sturm-Picone comparison theorem we obtain that w has infinitely many zeros on (R, ∞), which in particular implies that U^* intersects $\bar{u}_\lambda$ infinitely many times. □

3. Convergence to the singular solution.

In this section, we finish the proof of Theorem **??**, that is, for any fixed $\lambda > 0$ we show that the solution u_n of (**??**) converges to the solution U^* of (**??**) in $C^1_{loc}(0, \infty)$ as $n \to \infty$. Although, the framework originates from [**?**], our setting is different due to breaking of scaling, and dependence of λ on n. For

clarity of notation, we often drop the subscript n of functions if the dependence is clear from the context.

If $\hat{u}_n(\rho) = u_n(r, \gamma_n) - \gamma_n$ with $\rho = e^{\frac{\gamma_n}{2}} r$, then $\hat{u}(\cdot, \gamma)$ satisfies

$$\begin{cases} \hat{u}'' + \dfrac{N-1}{r}\hat{u}' + \lambda_n e^{\hat{u}} - e^{-\gamma_n}(\hat{u} + \gamma_n) = 0 & \text{in } (0, \infty)\,, \\ \hat{u}(0) = \hat{u}'(0) = 0. \end{cases}$$

Next, let $\bar{u}(r, \bar{\gamma})$ be the unique radial solution of

$$\begin{cases} \bar{u}'' + \dfrac{N-1}{r}\bar{u}' + \lambda_\infty e^{\bar{u}} = 0 & \text{in } (0, \infty)\,, \\ \bar{u}(0) = 0, \quad \bar{u}'(0) = 0\,. \end{cases}$$

The existence of global solutions of (**??**) and (**??**) is established in the proof of the following lemma.

Lemma 3.1. *For any $n > 0$ there exist unique solutions $\hat{u}_n$ and $\bar{u}$ of* (**??**) *and* (**??**) *respectively. Moreover,*

$$\hat{u}_n \to \bar{u} \text{ in } C^1_{loc}([0, \infty)) \text{ as } \gamma \to \infty.$$

Proof. Since the non-linearities are locally Lipschitz, local existence and uniqueness of solution to (**??**) and (**??**) follow from standard arguments for radial solutions. Also, if the solutions exist, then they are necessarily unique. Next, define

$$E_n(\rho) = \frac{(\hat{u}'(\rho))^2}{2} - e^{-\gamma_n}\frac{\hat{u}^2(\rho)}{2} + \lambda_n e^{\hat{u}(\rho)} - e^{-\gamma_n}\gamma_n \hat{u}(\rho)\,.$$

It is easy to check that E_n is decreasing and $E_n(0) = \lambda_n$ and since (λ_n) converges, $|E_n(0)| \le C$. Thus, since $\gamma > 0$ and $\gamma \mapsto e^{-\gamma}\gamma$ is bounded on $(0, \infty)$, Young inequality yields

$$((\hat{u})^2)'(\rho) \le \hat{u}^2(\rho) + (\hat{u}')^2(\rho) \le E_n(\rho) + \hat{u}^2(\rho)\left(1 + \frac{e^{-\gamma_n}}{2}\right) + e^{-\gamma_n}\gamma_n \hat{u}(\rho) \le \lambda_n + C(\hat{u}^2(\rho) + 1)\,,$$

where C is a universal constant. Then, Gronwall inequality implies

$$|\hat{u}(\rho)| \le C_1 e^{C_2 \rho}\,,$$

where C_1, C_2 are universal constants. Thus, $\hat{u}$ is a priori bounded, and therefore it can be uniquely extended to $[0,\infty)$. Also, since all coefficients are bounded by elliptic regularity, $\hat{u}$ has bounded first, second, and third order derivatives locally on $[0,\infty)$, uniformly in n. Then, by Arzelà-Ascoli theorem $\hat{u}_n \to \hat{u}_\infty$ in $C^2_{\rm loc}[0,\infty)$. Furthermore, $e^{-\gamma_n}(\hat{u}(\rho)+\gamma_n) \to 0$ and $\lambda_n e^{\hat{u}(\rho)} \to \lambda_\infty e^{\hat{u}_\infty(\rho)}$ as $n\to\infty$ locally uniformly in ρ. Thus $(\hat{u}_n)$ converges (up to sub-sequence) locally uniformly in $C^2([0,\infty))$ to a solution of (**??**), and since such solution is unique, we obtain that $\bar{u} = \hat{u}_\infty$ is globally defined. Convergence (**??**) follows. □

As in (**??**), we define $\zeta = \ln m - \ln r$ with $m = \sqrt{\frac{2(N-2)}{\lambda_\infty}}$ and we let $\eta(\zeta) = u(r) - 2\zeta$. Then, η satisfies (cf. (**??**))

$$\begin{cases} \eta'' - (N-2)\eta' + 2(N-2)(\frac{\lambda_n}{\lambda_\infty}e^{\eta} - 1) - m^2 e^{-2\zeta}(\eta + 2\zeta) = 0, \qquad -\infty < \zeta < \infty\,, \\ \lim_{\zeta\to\infty}(\eta(\zeta) + 2\zeta) = \gamma_n, \\ \lim_{\zeta\to\infty} e^{\zeta}(\eta'(\zeta) + 2) = 0. \end{cases}$$

For $\rho = e^{\frac{\gamma_n}{2}} r$ we set $\tau = \zeta - \gamma_n/2 = \ln m - \ln\rho$ and $\hat{\eta}(\tau) := \eta(\zeta)$. Observe that $\hat{\eta}(\tau) = u(r) - 2\zeta = \hat{u}(\rho) + \gamma_n - 2\zeta = \hat{u}(\rho) - 2\tau$ is a transformed function corresponding to $\hat{u}$ solving (**??**). Standard computations show that

$$\begin{cases} \hat{\eta}'' - (N-2)\hat{\eta}' + 2(N-2)(\frac{\lambda_n}{\lambda_\infty}e^{\hat{\eta}} - 1) - m^2 e^{-2\tau-\gamma_n}(\hat{\eta} + 2\tau + \gamma_n) = 0, \qquad -\infty < \tau < \infty\,, \\ \lim_{\tau\to\infty}(\hat{\eta}(\tau) + 2\tau) = 0, \\ \lim_{\tau\to\infty} e^{\tau}(\hat{\eta}'(\tau) + 2) = 0. \end{cases}$$

We also define $\bar{\eta}(\tau) = \bar{u}(\rho,\gamma) - 2\tau$, a transformed function of $\bar{u}$. Then $\bar{\eta}$ satisfies

$$\begin{cases} \bar{\eta}'' - (N-2)\bar{\eta}' + 2(N-2)(e^{\bar{\eta}} - 1) = 0, \ -\infty < \tau < \infty\,, \\ \lim_{\tau\to\infty}(\bar{\eta}(\tau) + 2\tau) = 0, \\ \lim_{\tau\to\infty} e^{\tau}(\bar{\eta}(\tau)' - 2) = 0. \end{cases}$$

In the transformed variables, Lemma **??** rewrites as

Corollary 3.2. *We have*

$$\hat{\eta}_n(\cdot) \to \bar{\eta}(\cdot), \ \textit{in } C^1_{loc}((-\infty,\infty)) \ \ \textit{as} \ \ n\to\infty.$$

Proof. For any compact $A \subset (-\infty, \infty)$, denote $B = \{\rho : \ln m - \ln \rho \in A\}$ and observe that $B \subset (0, \infty)$ is bounded, compact, and independent of γ. Then, Lemma **??** implies that

$$\sup_{\tau\in A} |\hat{\eta}_n'(\tau) - \bar{\eta}'(\tau)| = \sup_{\tau\in A} |(\hat{\eta}_n(\tau)+2\tau)' - (\bar{\eta}(\tau)+2\tau)'| = \sup_{\rho\in B} |\rho(\hat{u}_n'(\rho) - \bar{u}'(\rho))| \to 0 \text{ as } n \to \infty\,.$$

Analogously, we obtain $\sup_{\tau\in A} |\hat{\eta}_n(\tau) - \bar{\eta}(\tau)| \to 0$ as $n \to \infty$ and the assertion follows. □

Next, a standard calculation yields that

$$\bar{E}(\tau) = \frac{(\bar{\eta}'(\tau))^2}{2} + 2(N-2)(e^{\bar{\eta}(\tau)} - \bar{\eta}(\tau) - 1)$$

is non-decreasing, and strictly increasing unless $\bar{\eta}'(\tau) = 0$. Also, since $e^x - x - 1 \geq 0$, we obtain $\bar{E} \geq 0$. A standard theory of Lyapunov functions implies that $\bar{\eta}$ converges to a set of equilibria as $\tau \to -\infty$. Since 0 is the only equilibrium, we have $(\bar{\eta}(\tau), \bar{\eta}'(\tau)) \to (0,0)$ as $\tau \to -\infty$.

Fix any τ_0 and recall that $\eta_n(\zeta) = \hat{\eta}_n(\tau)$ with $\tau = \zeta - \frac{\gamma_n}{2}$. Then, Corollary **??** implies

$$\lim_{n\to\infty} \left(\eta_n\left(\tau_0 + \frac{\gamma_n}{2}\right), \eta_n'\left(\tau_0 + \frac{\gamma_n}{2}\right)\right) = \lim_{n\to\infty} (\hat{\eta}_n(\tau_0), \hat{\eta}_n(\tau_0)) = (\bar{\eta}(\tau_0), \bar{\eta}(\tau_0))\,.$$

Since $(\bar{\eta}(\tau), \bar{\eta}'(\tau)) \to (0,0)$ as $\tau \to -\infty$, we have that the right hand side of (**??**) is arbitrary close to $(0,0)$ if τ_0 is large negative.

In the following result we implicitly assume as above that the functions depend on n. Denote $z(\zeta) = \eta'(\zeta)$. Next, we show that there is $\zeta^* > 0$ independent of n such that if $(\eta(\bar{\zeta}), z(\bar{\zeta}))$ is close to $(0,0)$ for some $\bar{\zeta} > \zeta^*$, then $(\eta(\zeta), \eta'(\zeta))$ is close to $(0,0)$ for any $\zeta \in (\zeta^*, \bar{\zeta})$. Note that by (**??**), $\bar{\zeta}$ is indeed large, since τ_0 is fixed and γ_n is large.

Lemma 3.3. *For any $n > 0$ and $\varepsilon > 0$ denote $\Gamma_\varepsilon^n = \{(\eta, z) \in \mathbb{R}^2 : 2(N-2)(\frac{\lambda_n}{\lambda_\infty} e^\eta - 1 - \eta) + \frac{1}{2} z^2 \leq \varepsilon\}$ and fix $\varepsilon_0 > 0$ such that $\Gamma_{2\varepsilon_0}^n \subset \{(\eta, z) \in \mathbb{R}^2 : |\eta| < 1\}$. Note that since $\lambda_n \to \lambda_\infty$, ε_0 can be chosen independent to n. Fix*

$\varepsilon \in (0, \varepsilon_0)$ and let $\zeta^* \geq 2$ depending on $\varepsilon > 0$, but independent of n be so large that

$$m^2 \frac{e^{-2\zeta}}{2}(1+2\zeta)^2 \leq \frac{\varepsilon}{2} \qquad \text{for any } \zeta > \zeta^* .$$

If there are $\bar\zeta > \zeta^*$ and $n > 0$ such that $(\eta(\bar\zeta), \eta'(\bar\zeta)) \in \Gamma^n_\varepsilon$, then $(\eta(\zeta), \eta'(\zeta)) \in \Gamma^n_{2\varepsilon}$, for any $\zeta \in (\zeta^*, \bar\zeta)$.

Proof. Fix any $n > 0$. We set

$$\tilde E_n(\eta, z, \zeta) = \frac{z^2}{2} + 2(N-2)\left(\frac{\lambda_n}{\lambda_\infty} e^\eta - 1 - \eta\right) - \frac{m^2 e^{-2\zeta}}{2}(\eta + 2\zeta)^2.$$

Since η satisfies (**??**), it is easy to check that

$$\begin{aligned}\frac{d\tilde E_n(\eta(\zeta), \eta'(\zeta), \zeta)}{d\zeta} &= \eta'\left(\eta'' + 2(N-2)\left(\frac{\lambda_n}{\lambda_\infty} e^\eta - 1\right) - m^2 e^{-2\zeta}(\eta + 2\zeta)\right) \\ &\quad - 2m^2 e^{-2\zeta}(\eta + 2\zeta) + m^2 e^{-2\zeta}(\eta + 2\zeta)^2 \\ &= (N-2)(\eta')^2 + m^2 e^{-2\zeta}(\eta + 2\zeta)(\eta + 2\zeta - 2).\end{aligned}$$

Fix $\varepsilon \in (0, \varepsilon_0)$ and let $\bar\zeta > \zeta^*$ be as in the statement of the lemma. Since $\varepsilon < \varepsilon_0$, then $\Gamma^n_{2\varepsilon} \subset \{(\eta, z) \in \mathbb{R}^2 : |\eta| < 1\}$ and, by continuity, $(\eta(\zeta), z(\zeta)) \in \Gamma^n_{2\varepsilon}$ for any ζ close to $\bar\zeta$. By contradiction assume that there is $T > \zeta^*$ such that

$$(\eta(\zeta), z(\zeta)) \in \Gamma^n_{2\varepsilon}, \text{ for } \zeta \in (T, \bar\zeta) \text{ and } (\eta(T), z(T)) \notin \Gamma^n_{2\varepsilon}.$$

Integrating (**??**) between T and $\bar\zeta$ and recalling that $|\eta(\zeta)| \leq 1$, for $\zeta \in (T, \bar\zeta)$ and $T > \zeta^* \geq 2$, we find

$$\tilde E_n(\eta(\bar\zeta), z(\bar\zeta), \bar\zeta) - \tilde E_n(\eta(T), z(T), T) \geq m^2 \int_T^{\bar\zeta} e^{-2\zeta}(\eta(\zeta) + 2\zeta)(\eta(\zeta) + 2\zeta - 2) d\zeta \geq 0 .$$

Then, recalling that $(\eta(\bar\zeta), z(\bar\zeta)) \in \Gamma_\varepsilon$, we deduce from the previous line and (**??**) that

$$\begin{aligned}\frac{(z(T))^2}{2} + 2(N-2)(e^{\eta(T)} - 1 - \eta(T)) &\leq \frac{(z(\bar\zeta))^2}{2} + 2(N-2)(e^{\eta(\bar\zeta)} - 1 - \eta(\bar\zeta) + m^2 \frac{e^{-2T}}{2}(\eta(T) + 2T)^2 \\ &\leq \frac{3}{2}\varepsilon ,\end{aligned}$$

a contradiction to the definition of T. □

Now, we prove the convergence of u_n to U^* when $n \to \infty$, which completes the proof of Theorem **??**.

Proposition 3.4. *Let U^* be the singular solution given by Theorem* **??** *(cf. Proposition* **??***). Then,*

$$u_n \to U^* \qquad as\ \gamma \to \infty \quad in\ \ C^1_{loc}((0,\infty)).$$

Proof. Fix sequences $(\gamma_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ with $\gamma_n \nearrow \infty$ and $\lambda_n \to \lambda_\infty \in (0,\infty)$ as $n \to \infty$ and let $z_n = \eta_n'$ (see (**??**)). Fix any small $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 > 0$ and ζ^* are as in Lemma **??**. Also, recall the definition of Γ_ε from Lemma **??** and denote

$$\Gamma^*_\varepsilon = \bigcap_{n\geq 0} \Gamma^n_\varepsilon, \qquad \Gamma'_\varepsilon = \bigcup_{n\geq 0} \Gamma^n_\varepsilon.$$

Since $\lambda_n \to \lambda_\infty$, Γ^*_ε and Γ'_ε are non-empty bounded sets that approach to $\{(0,0)\}$ as $\varepsilon \to 0^+$.

By (**??**), there exists $\tau_0 < 0$ such that, for any sufficiently large n, one has $\zeta_n := \tau_0 + \frac{\gamma_n}{2} > \zeta^*$ and $(\eta_n(\zeta_n), z_n(\zeta_n)) \in \Gamma^*_\varepsilon$. Then, Lemma **??** implies that $(\eta_n(\zeta), z_n(\zeta)) \in \Gamma'_{2\varepsilon}$, for any $\zeta \in (\zeta^*, \zeta_n]$.

Since η satisfies (**??**), we deduce that $\eta \in C^2((\zeta^*, \zeta_n])$ and, after differentiating (**??**) with respect to ζ, we obtain $\eta \in C^3((\zeta^*, \zeta_n])$. Since ζ^* is independent of n and $\zeta_n \to \infty$ as $n \to \infty$, we have, by Arzelà-Ascoli's theorem, that $\lambda_n \to \lambda_\infty$ and a standard diagonal argument shows that (η, z) converges (up to sub-sequence) to (η_*, z_*) in $(C^1_{loc}(T,\infty))^2$, where $(\eta_*(\zeta), z_*(\zeta))$ satisfies

$$\eta_*'' - (N-2)\eta_*' + 2(N-2)\eta_* = m^2 e^{-2\zeta}(\eta_* + 2\zeta) - 2(N-2)(e^{\eta_*} - 1 - \eta_*),\ \zeta \in \mathbb{R}.$$

In view of the uniqueness property established in Proposition **??**, to finish the proof, we only need to show that

$$\eta_*(\zeta) \to 0 \text{ when } \zeta \to \infty.$$

Suppose for contradiction that there exists a sequence $(\zeta'_k)_{k\in\mathbb{N}}$ such that $\zeta'_k \to$

∞, as $k \to \infty$ and a constant $\delta > 0$ such that

$$(\eta_*(\zeta_k'), z_*(\zeta_k')) \notin \Gamma_\delta', \text{ for any } k \geq 1\,.$$

By decreasing ε if necessary, we can suppose that $\varepsilon \leq \delta/4$. Choose sufficiently large k such that $\zeta_k' > \zeta^*$, $\tau_0 < 0$ and sufficiently large n such that $\zeta_n > \zeta_k'$ and $(\eta(\zeta_n, \gamma_n), z(\zeta_n, \gamma_n)) \in \Gamma_\varepsilon^*$ (cf. (**??**)). Then, by Lemma **??** one has $(\eta(\zeta, \gamma_n), z(\zeta, \gamma_n)) \in \Gamma_{2\varepsilon}' \subset \Gamma_\delta'$ for any $\zeta \in (\zeta^*, \zeta_n)$, a contradiction.

Overall, we proved that $\eta(\cdot, \gamma) \to \eta_\infty$ in $C^1_{\rm loc}(\mathbb{R})$, where $\eta(\cdot, \gamma)$ solves (**??**) and η_∞ satisfies (**??**) with (**??**). Finally, fix any open set A with $\bar{A} \subset (0, \infty)$ and let $B := \{\zeta \in \mathbb{R} : \ln m - \ln r \in A\}$. Since B is open and bounded, one has, for some constant C_A depending on A,

$$\begin{aligned}\|u(\cdot, \gamma) - U^*(\cdot)\|_{C^1(A)} &= \|(u(\cdot, \gamma) - 2(\ln m - \ln \cdot)) - (U^*(\cdot) - 2(\ln m - \ln \cdot))\|_{C^1(A)} \\ &\leq C_A \|\eta(\cdot, \gamma) - \eta_\infty(\cdot)\|_{C^1(B)} \to 0 \qquad \text{as } \gamma \to \infty\,,\end{aligned}$$

as desired. □

4. Oscillation of the branch and Morse index: proof of Theorem ?? and Proposition ??.

To prove Theorem **??**, we first recall a result of Joseph and Lundgren [**?**]. Let

$$\bar{u}^*(r) = -2 \log r + k, \qquad k = \log \frac{2(N-2)}{\lambda}$$

be the singular solution of (**??**), that is, it satisfies the equation in (**??**) and blows-up at the origin.

Proposition 4.1. *For any $\alpha \geq 0$, let $\bar{u}(\cdot, \alpha)$ and $\bar{u}^*$ be defined in* (**??**) (**??**) *respectively. Then,*

$$Z_{[0,\infty)}[\bar{u}(\cdot, \alpha) - \bar{u}^*(\cdot)] = \begin{cases} \infty & \text{if } 3 \leq N \leq 9 \\ 0 & \text{if } N \geq 10, \end{cases}$$

where $Z_I(u) = \sharp\{r \in I | u(r) = 0\}$ and $\sharp A$ is the cardinality of the set A.

For any given $\gamma > 0$, let $(r^i_{\lambda,\gamma})$ be an increasing (finite or infinite) sequence of positive real numbers such that $u'(r^i_{\lambda,\gamma}, \gamma) = 0$, where $u(\cdot, \gamma) = u_\lambda(\cdot, \gamma)$ is the unique solution of (**??**). We show that if $3 \le N \le 9$, then $r^i_{\lambda,\gamma}$ oscillates around R^i_λ (recall that $(U^*)'(R^i_\lambda) = 0$) infinitely many times as $\gamma \to \infty$.

The following main result of this section is partly motivated by [**?** , Lemma 5], where a problem with Dirichlet boundary conditions is considered (see also [**?**] for a related problem with Neumann boundary condition). However, in the works above, it is assumed that the parameter can be removed from the equation by rescaling of the domain. Our situation is different and we have to work directly with parameter dependent equation. We also have to appropriately modify the zero number argument to treat Neumann boundary conditions.

Lemma 4.2. *Assume $3 \le N \le 9$ and fix $R > 0$. If λ^i be the positive real number given in Theorem **??**, then there exist a sequence of initial data $(\gamma_n)_n$ with $\gamma_n \to \infty$ and a sequence positive integer $(j_n)_n$ such that $r^{j_n}_{\lambda^i,\gamma_n} = R$. In other words, $u(\cdot, \gamma_n)$ satisfies Neumann boundary data on ∂B_R.*

Proof. First, for any $\lambda > 0$ we show that, for any $A > 0$ and $I = (0, A)$, one has

$$Z_I[u(\cdot, \gamma) - U^*(\cdot)] \to \infty \text{ as } \gamma \to \infty\,.$$

Recall that, by Lemma **??**, we have

$$\hat{u}(\rho, \gamma) \to \bar{u}(\rho, 0) \text{ in } C^1_{loc}([0, \infty)) \text{ when } \gamma \to \infty\,,$$

where $\rho = e^{\frac{\gamma}{2}} r$, $\bar{u}(\rho, 0) = \bar{u}(r, \gamma) - \gamma$ satisfies (**??**) and $\hat{u}(\rho, \gamma) = u(r, \gamma) - \gamma$ satisfies (**??**). Set $\hat{U}^*(\rho, \gamma) = U^*(r) - \gamma$ and $k = \ln \frac{2(N-2)}{\lambda}$. Then (**??**) yields that, for any $r_0 > 0$, there is $C = C(r_0)$ such that

$$|U^*(r) + 2 \ln r - k| \le Cr \qquad \text{for any } r \le r_0\,,$$

and therefore

$$|\hat{U}^*(\rho, \gamma) + 2 \ln \rho - k| \le C\rho e^{-\frac{\gamma}{2}} \qquad \text{for any } \rho \le r_0 e^{\gamma/2}\,.$$

Consequently,

$$\hat{U}^*(\cdot,\gamma) \to \bar{u}^* \text{ in } C_{loc}((0,\infty)) \text{ as } \gamma \to \infty,$$

where $\bar{u}^*$ is defined in (**??**). Fix any $M > 0$. Then, by Proposition **??**, there exists a bounded interval $I_M \subset (0,\infty)$ such that

$$Z_{I_M}[\bar{u}(\cdot,1) - \bar{u}^*(\cdot)] \geq M.$$

By scale invariance of the equation, one has $\bar{u}^*(r) = \bar{u}^*(e^{\alpha/2}r) + \alpha$ and $\bar{u}(r, 1+\alpha) = \bar{u}(e^{\alpha/2}r, 1) + \alpha$, and therefore, for any $\gamma \geq 1$,

$$Z_{I_M}[\bar{u}(\cdot,\gamma) - \bar{u}^*(\cdot)] = Z_{e^{\frac{\gamma-1}{2}} I_M}[\bar{u}(\cdot,1) - \bar{u}^*(\cdot)] \geq M.$$

Then, thanks to (**??**) and (**??**), we have

$$Z_{I_M}[\hat{u}(\cdot,\gamma) - \hat{U}^*(\cdot,\gamma)] \geq Z_{I_M}[\bar{u}(\cdot,\gamma) - \bar{u}^*(\cdot)] \geq M\,.$$

Finally, for given I and sufficiently large γ one has $I_M \subset e^{\frac{\gamma}{2}} I$, and consequently

$$Z_I[u(\cdot,\gamma) - U^*(\cdot,\gamma)] = Z_{e^{\frac{\gamma}{2}} I}[\hat{u}(\cdot,\gamma) - \hat{U}^*(\cdot,\gamma)] \geq M\,.$$

Since M was arbitrary, the claim (**??**) follows.

For $\lambda := \lambda^i$, let U^* be the solution of (**??**) and notice that $(U^*)'(R) = 0$. Also, for the same λ, let $u(\cdot,\gamma)$ be the solution of (**??**). Observe that $u(\cdot,\gamma)$ does not necessarily satisfy Neumann boundary condition at R. Since $w_\gamma := u(\cdot,\gamma) - U^*$ satisfies a linear differential equation, it follows from the uniqueness of initial value problem that every zero of w_γ is simple.

Observe that, for every $\gamma > 0$, $Z_{[0,1]}(w_\gamma) < \infty$ since otherwise by continuity, the accumulation point would be a degenerate zero. Also, since w_λ has only finitely many simple zeros, continuous dependence on parameters yields that zeros of w_γ depend continuously on γ. For each $\gamma > 0$, let $m_\gamma := Z_{[0,1]}(w_\gamma)$ and let $(z_j^\gamma)_{j=1}^{m_\gamma} \subset [0,R]$ be the increasing sequence of zeros of w_γ. Since $w_\gamma(0) = -\infty$, we have that $z_1^\gamma > 0$ for each $\gamma > 0$, and moreover $w_\gamma'(z_1^\gamma) > 0$. By induction it is easy to prove that $w_\gamma'(z_i^\gamma) > 0$ if i is odd and $w_\gamma'(z_i^\gamma) < 0$ if i is even.

Since the zeros of w_γ are non-degenerate a new zero of w_γ cannot be created in the interior of $[0, R]$. Furthermore, $w_\gamma(0) = -\infty$, and therefore a new zero cannot enter $[0, R]$ through 0. Hence, (**??**) yields that there exists a sequence (γ_k) with $\gamma_k \to \infty$ as $k \to \infty$ such that $w_{\gamma_k}(R) = 0$. Since $w'_{\gamma_k}(R) > 0$ if k is odd and $w'_{\gamma_k}(R) < 0$ if k is even, by the continuous dependence on parameters, we obtain that there exists $\gamma_k^* \in (\gamma_k, \gamma_{k+1})$ such that $w'_{\gamma_k^*}(R) = 0$. Since $(U^*)'(R) = 0$, we infer that $u'(R, \gamma_k^*) = 0$ and the lemma follows. □

Next, we prove that the Morse index of the singular solution U_λ^* is finite when $N > 10$ and infinite when $3 \le N \le 9$.

Proof of Proposition **??**. Assume $3 \le N \le 9$. In order to prove that $U^*_{\lambda^i}$ has infinite Morse index, by variational characterization of eigenvalues, it suffices to prove that there are infinitely many linearly independent functions $f : (0, 1) \to \mathbb{R}$ such that

$$\mathcal{J}(f) = \int_0^1 \left(|f'|^2 + (1 - \lambda^i e^{U^*_{\lambda^i}}) f^2 \right) r^{N-1} dr < 0 \,.$$

By the boundary conditions (**??**), we see that, for any $\varepsilon > 0$, there exists r_0 such that, for all $r \in (0, r_0)$,

$$\lambda^i e^{U^*_{\lambda^i}(r)} - 1 \ge \frac{2(N-2)}{r^2}(1 - \varepsilon).$$

Then, it follows that if $3 \le N \le 9$, we have, for some small $\varepsilon_0 > 0$,

$$\lambda^i e^{U^*_{\lambda^i}(r)} - 1 \ge \left(\frac{(N-2)^2}{4} + \varepsilon_0^2 \right) \frac{1}{r^2}.$$

Next, we define $f_j(r) = f(r)\tilde{\chi}_j(r)$, where

$$\tilde{\chi}_j(r) = \begin{cases} 1, & \text{if } r \in [r_{j+1}, r_j], \\ 0, & \text{elsewhere }, \end{cases} \qquad r_j = e^{-2\pi j/\varepsilon_0}$$

and $f(r) = r^{-(N-2)/2} \sin(\varepsilon_0 \log r/2)$. Notice that f_j and f_k have disjoint supports for $j \ne k$, and therefore they are linearly independent. Moreover, f_j is a

solution of

$$-f_j'' - \frac{N-1}{r} f_j' - \left(\frac{(N-2)^2}{4} + \frac{\varepsilon_0^2}{4} \right) \frac{1}{r^2} f_j = 0, \quad r \in (r_{j+1}, r_j).$$

Since $f_j(r_j) = f_j(r_{j+1}) = 0$ we have that $f_j \in W^{1,2}((0,\infty))$ and by (**??**)

$$\mathcal{J}(f_j) = \int_{r_{j+1}}^{r_j} \left(|f_j'|^2 - \left(\frac{(N-2)^2}{4} + \varepsilon_0^2 \right) \frac{1}{r^2} f_j^2 \right) r^{N-1} dr = -\frac{3}{4} \varepsilon_0^2 \int_{r_{j+1}}^{r_j} \frac{1}{r^2} f_j^2 dx < 0\,.$$

Thus the Morse index of $U^*_{\lambda_i}$ is infinite.

Next, let us consider the case $N > 10$. We show that there are at most finitely many linearly independent functions satisfying (**??**). Recall that $(U^*_{\lambda_i})'(R) = 0$. Again, by using asymptotics of $U^*_{\lambda^i}$ at the origin, we have that, there is $\varepsilon > 0$ and $r_0 \in (0,1)$ such that, for any $r \in (0, r_0)$,

$$\lambda^i e^{U^*_{\lambda^i}(r)} - 1 \leq \frac{2(N-2)}{r^2}(1+\varepsilon) \leq \frac{(N-2)^2}{4r^2}(1-\varepsilon)\,,$$

where the last inequality holds for $N > 10$. Next, choose $\chi_0 \in C^1(\mathbb{R}^N)$ such that

$$\chi_0(r) = \begin{cases} 1, & \text{if } r \in (0, r_0/2), \\ 0, & \text{if } r > r_0, \end{cases}$$

and set $\chi_1 = 1 - \chi_0$. For $\phi \in H^1_{rad}(B_1(0))$ with $\phi'(R) = 0$, the Hardy inequality [**?**] and (**??**) imply

$$\begin{aligned} \mathcal{J}(\phi) &= \int_0^1 (|\phi'|^2 - (\chi_0 + \chi_1)(\lambda^i e^{U^*_{\lambda^i}(r)} - 1)\phi^2) r^{N-1} dr \\ &\geq (1-\varepsilon) \int_0^1 \left(|\phi'|^2 - \chi_0 \frac{(N-2)^2}{4r^2} \phi^2 \right) r^{N-1} dr + \int_0^1 (\varepsilon |\phi'|^2 - \chi_1 (\lambda^i e^{U^*_{\lambda^i}(r)} - 1)\phi^2) r^{N-1} dr \\ &\geq \int_0^1 (\varepsilon |\phi'|^2 - \chi_1 (\lambda^i e^{U^*_{\lambda^i}(r)} - 1)\phi^2) r^{N-1} dr \end{aligned}$$

Since $|(\lambda^i e^{U^*_{\lambda^i}(r)} - 1)| \leq C_{\lambda^i}$, for $r \in (r_0/2, 1)$, the operator $-\varepsilon\Delta - \chi_1(\lambda^i e^{U^*_{\lambda^i}(r)} - 1)$ on $B_1(0)$ with Neumann boundary condition has finitely may negative eigenvalues, and therefore

$$\int_0^1 (\varepsilon |\phi'|^2 - \chi_1 (\lambda^i e^{U^*_{\lambda^i}(r)} - 1)\phi^2) r^{N-1} dr < 0$$

has only finitely many linearly independent solutions. Thus, the Morse index of $U^*_{\lambda^i}$ is finite, as desired. □

5. Proof of Theorem ??

In this section, we prove Theorem **??**. Let $(R^i_\lambda)_{i=1}^\infty$, be an increasing, unbounded sequence of positive real numbers depending on N and λ such that $(U^*_\lambda)'(R^i_\lambda) = 0$ (see Lemma **??**), where U^*_λ is the solution to (**??**). To prove Theorem **??**, we need two ingredients. First we show that, for any $i \in \mathbb{N}$,

$$R^i_\lambda \to 0, \text{ as } \lambda \to 0^+$$

and obtain necessary bounds on solutions. Then, we show that, for any $i \in \mathbb{N}$, the map $\lambda \to R^i_\lambda$ is continuous.

Proposition 5.1. *For each $\lambda > 0$, let U^*_λ be the unique solution to* (**??**) *and denote by $(R^i_\lambda)_{i=1}^\infty$, the increasing sequence of all positive real numbers such that $(U^*_\lambda)'(R^i_\lambda) = 0$. Then, for any fixed $i \in \mathbb{N}$, we have*

$$R^i_\lambda \to 0, \text{ as } \lambda \to 0^+.$$

Proof. The proof is divided into several steps. We begin by giving some notations. Many constants and functions in the proof depend on λ and

$$m = \sqrt{\frac{2(N-2)}{\lambda}}\,.$$

However for the clarity of the notation, this dependence is not explicitly indicated, but the needed asymptotic is explained. If a constant depends only on the dimension N, we usually denote it by C_N, c_N, etc. Note that such constant can change from line to line. First, we define

$$f(\zeta) = \frac{m^2}{2(N-1)} e^{-2\zeta}\left(\zeta + \frac{N-2}{4(N-1)}\right).$$

and let $\zeta \mapsto \eta$ be the unique solution of (**??**) (see Proposition **??**). Setting $\tilde\eta(\zeta) = \eta(\zeta) - f(\zeta)$, we see that η satisfies

$$\tilde\eta'' - (N-2)\tilde\eta' + 2(N-2)\tilde\eta = m^2 e^{-2\zeta}\eta(\zeta) + \phi(\eta(\zeta)) =: \tilde g(\zeta),$$

where ϕ is as in (**??**).

Define α, β, and G_N as in (**??**) and (**??**) and recall that $G_N \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for any $N \geq 3$. Hence,

$$\tilde{\eta}(\zeta) = \int_\zeta^\infty G_N(\sigma - \zeta)\tilde{g}(\sigma)\, d\sigma = G_N * \tilde{g}(\zeta).$$

If $v := U^*/\overline{u}_\lambda$, then v satisfies (see (**??**))

$$-v'' - \frac{N-1}{r}v' + v = e^{\overline{u}_\lambda(v-1)}$$

and $w(r) = r^{\frac{N-1}{2}}(v(r) - 1)$ satisfies (see the proof of Lemma **??**)

$$w'' + \left(\frac{e^{\overline{u}_\lambda(v-1)} - v}{v-1} - \frac{(N-1)(N-3)}{4r^2} \right) w = 0.$$

For any $\lambda \in (0, 1/e)$, we recall that $\overline{u}_\lambda > 1$ is the solution of the equation $u = \lambda e^u$. Let r_λ be the smallest r such that $U^*(r) = \overline{u}_\lambda$, or equivalently the smallest point such that $v(r) = 1$ or $w(r) = 0$.

Step 1. For any $\delta > 0$ there is $\lambda_\delta > 0$ such that

$$-(1-\delta)\ln\lambda \leq \overline{u}_\lambda,$$

for any $\lambda \in (0, \lambda_\delta)$.

Proof of Step 1. By taking the logarithm of the equality $\overline{u}_\lambda = \lambda e^{\overline{u}_\lambda}$, we obtain

$$\ln \overline{u}_\lambda - \overline{u}_\lambda - \ln\lambda = 0\,.$$

For $v_1 = -(1-\delta)\ln\lambda$, we have

$$\ln v_1 - v_1 - \ln\lambda = \ln((1-\delta)\ln\lambda^{-1}) - \delta\ln\lambda > 0,$$

for any $\lambda \in (0, \lambda_\delta)$, where λ_δ is sufficiently small. On the other hand, for any fixed λ and sufficiently large v, one has

$$\ln v - v - \ln\lambda < 0\,.$$

In particular there is a solution of $u = \lambda e^u$ which is bigger than $v_1 = -(1-\delta)\ln\lambda$. Finally, since $\overline{u}_\lambda$ is the biggest solution, $\bar{u}_\lambda \geq v_1$ and the claim follows. □

Remark 5.2. *For any $\delta > 0$, one can prove more the precise bound*

$$-\ln\lambda + \ln(-\ln\lambda) < \overline{u}_\lambda \leq -(1+\delta)\ln\lambda\,,$$

for any sufficiently small λ depending on δ.

Step 2. Recall that r_λ is the smallest r such that $U^*(r) = \bar{u}_\lambda$. Then, there exists $K_N > 0$ such that $r_\lambda^2 < \frac{K_N}{\overline{u}_\lambda}$, for any small $\lambda > 0$.

Proof of Step 2. Set $K_N = \max\{(N-1)(N-3), 2(16\pi)^2\}$. For a contradiction, assume that there exists a sequence $\lambda_n \to 0$ as $n \to \infty$ such that $r_n^2 \geq K_N/\overline{u}_n$, where $r_n := r_{\lambda_n}$ and $\overline{u}_n := \overline{u}_{\lambda_n}$. Then $U^*_{\lambda_n} > \bar{u}_n$ on $I_n := [A_n, 2A_n]$ with $A_n = \sqrt{K_N/(16\overline{u}_n)}$, and consequently $w_n := w_{\lambda_n} > 0$ and $v_n := v_{\lambda_n} > 1$ (solutions of (**??**) and (**??**) with $\lambda = \lambda_n$) on I_n. Since for any $x \geq 0$ one has $e^x \geq x + 1$ and $v_n > 1$, we have for any $r \in I_n$

$$\frac{e^{\overline{u}_n(v-1)} - v}{v-1} - \frac{(N-1)(N-3)}{4r^2} \geq \frac{\overline{u}_n(v-1) + 1 - v}{v-1} - \frac{\overline{u}_n(N-1)(N-3)}{4K_N} \geq \frac{3}{4}\overline{u}_n - 1\,,$$

where the last inequality holds by the definition of K_N. Furthermore, by Step 1, $\bar{u}_\lambda \to \infty$, and therefore it is possible to choose n large enough such that

$$\frac{3}{4}\overline{u}_n - 1 > \frac{1}{2}\overline{u}_n \geq \frac{(16\pi)^2\overline{u}_n}{K_N} \geq \frac{(4\pi)^2}{A_n^2}.$$

Then, w_n satisfies

$$w_n'' + q_n(r)w_n = 0, \text{ on } [A_n, 2A_n],$$

with $q_n > (4\pi)^2/A_n^2$ for any sufficiently large n. However, the equation

$$m'' + \frac{(4\pi)^2}{A_n^2}m = 0$$

has a solution $m(r) = \sin(4\pi r/A_n)$ which has zeros at $A_n + \frac{k}{4}A_n \in [A_n, 2A_n]$ for any $k \in \{0, 1, \cdots, 4\}$. By the Sturm-Piccone comparison theorem, w_n has also a zero on I_n, contradicting the fact that $w_n > 0$ on I_n. □

Let r_λ be as in Step 3. and let ζ_λ be defined by (see (**??**))

$$r_\lambda = \sqrt{\frac{2(N-2)}{\lambda}}e^{-\zeta_\lambda}.$$

Step 3. There exists a constant C_N such that for any sufficiently small $\lambda > 0$, one has $f(\zeta_\lambda) \le C_N$.

Proof of Step 3. Step 1 and Step 2 with $\delta = \frac{1}{2}$ imply for any small $\lambda > 0$

$$r_\lambda^2 \le \frac{K_N}{\overline{u}_\lambda} \le \frac{2K_N}{-\ln \lambda}\,,$$

which is equivalent to

$$e^{-2\zeta_\lambda} \le \frac{K_N}{N-2}\frac{\lambda}{-\ln\lambda}.$$

The previous inequality can be rewritten as

$$\zeta_\lambda \ge -\frac{1}{2}\left(\ln\left(\frac{K_N}{N-2}\lambda\right) - \ln(-\ln\lambda)\right)\,.$$

In particular, we see that $\zeta_\lambda \to \infty$ as $\lambda \to 0$. Since the function $x \mapsto xe^{-x}$ is decreasing on $(0,\infty)$, for any sufficiently small $\lambda > 0$ one has

$$f(\zeta_\lambda) = \frac{N-2}{N-1}\frac{e^{-2\zeta_\lambda}}{\lambda}\left(\zeta_\lambda + \frac{N-2}{4(N-1)}\right) \le -\frac{K_N}{2N-2}\frac{\left(\ln(\frac{K_N}{N-2}\lambda) - \ln(-\ln\lambda)\right)\frac{\lambda}{-\ln\lambda}}{\lambda} \le C_N.$$

This proves Step 3. □

Remark 5.3. *For clarity let us indicate explicitly the dependence of f on λ (or equivalently on m). Fix any $M > 0$ and for each $\lambda > 0$ choose $\bar{\zeta}_\lambda \ge 0$ such that $f_\lambda(\bar{\zeta}_\lambda) \le M$. Since $\inf_K f \to \infty$ as $m \to \infty$ on any compact set $K \subset (0,\infty)$, one has*

$$\bar{\zeta}_\lambda \to \infty, \qquad \text{as } \lambda \to 0 \quad \text{or equivalently if} \quad m \to \infty\,.$$

We frequently use this observation below, often without further reference.

Next, we derive estimates on $\tilde{\eta}$ solving (**??**) . We consider two cases: $f(\zeta_\lambda) \le 1.1$ and $f(\zeta_\lambda) \ge 1.1$, where ζ_λ is given by (**??**).

Step 4. There exists a constant $\tilde{C}_N$ such that if, for sufficiently small $\lambda > 0$, one has $1.1 \le f(\zeta_\lambda)$, then $|\tilde{\eta}(\zeta_\lambda)| \le \tilde{C}_N$.

Proof. First, the assumption and Step 3 yield $1 \le f(\zeta_\lambda) \le C_N$, and therefore by Remark **??**, $\zeta_\lambda \to \infty$ as $\lambda \to \infty$. Hence, there exist two constants $c_N < 1 < C_N$ such that, for any sufficiently small λ,

$$c_N \lambda \le e^{-2\zeta_\lambda} 2\zeta_\lambda \le C_N \lambda.$$

Using that $\lambda = \overline{u}_\lambda e^{-\overline{u}_\lambda}$ and $\overline{u}_\lambda \to \infty$ as $\lambda \to 0$ (see Step 1), we obtain that for small λ

$$C_N \lambda = \overline{u}_\lambda e^{-\overline{u}_\lambda + \ln C_N} \le (\overline{u}_\lambda - 2\ln C_N) e^{-\overline{u}_\lambda + 2\ln C_N},$$

and

$$c_N \lambda = \overline{u}_\lambda e^{-\overline{u}_\lambda + \ln c_N} \ge (\overline{u}_\lambda - 2\ln c_N) e^{-\overline{u}_\lambda + 2\ln c_N}.$$

Consequently

$$(\overline{u}_\lambda - 2\ln c_N) e^{-\overline{u}_\lambda + 2\ln c_N} \le c_N \lambda \le e^{-2\zeta_\lambda} 2\zeta_\lambda \le C_N \lambda \le (\overline{u}_\lambda - 2\ln C_N) e^{-\overline{u}_\lambda + 2\ln C_N}.$$

Since the function $x \mapsto xe^{-x}$ is decreasing on $(0, \infty)$, we have

$$\overline{u}_\lambda - 2\ln c_N \ge 2\zeta_\lambda \ge \overline{u}_\lambda - 2\ln C_N.$$

By (**??**),

$$\overline{u}_\lambda = u(r_\lambda) = f(\zeta_\lambda) + \tilde{\eta}(\zeta_\lambda) + 2\zeta_\lambda,$$

we deduce that

$$-2\ln C_N \le f(\zeta_\lambda) + \tilde{\eta}(\zeta_\lambda) \le -2\ln c_N.$$

Since $1 \le f(\zeta_\lambda) \le C_N$, we obtain the desired result. □

Before proceeding let us introduce some additional notation. Define

$$\Gamma = 1.1,$$

and denote ζ_1^* the largest solution of $f(\zeta) = \Gamma$, where of course ζ_1^* depends on λ and by Remark **??**, $\zeta_1^* \to \infty$ as $\lambda \to \infty$. We remark that instead of 1.1, we can take any number bigger than 1, sufficiently close to 1 (see (**??**)).

Fix any $\varepsilon_0 > 0$ and set

$$\zeta_2^* := \inf\{\zeta \geq \zeta_1^* : |\tilde{\eta}(z)| \leq (1+\varepsilon_0) P_N f(z) \text{ for any } z \geq \zeta\},$$

with $\inf \emptyset = \infty$, where

$$P_N := \frac{|\phi(\Gamma)|}{\Gamma} \tilde{P}_N := \frac{|\phi(\Gamma)|}{\Gamma} \times \begin{cases} (1+e^{-\frac{(\alpha+8)\pi}{2\beta}}) \frac{4}{(\alpha+8)^2+4\beta^2} & \text{if } 3 \leq N \leq 9, \\ \frac{4}{(\alpha+8)^2} & \text{if } N = 10, \\ \frac{4}{(\alpha+8)^2-4\beta^2} & \text{if } N > 10\,, \end{cases}$$

and ϕ is defined in (**??**). Clearly, P_N and $\tilde{P}_N$ are constants depending only on N and ζ_2^* depends on the solution η. Since $\zeta_2^* \geq \zeta_1^*$, one has

$$\zeta_2^* \to \infty \qquad \text{as } m \to \infty\,.$$

Moreover,

$$P_N = \frac{e^\Gamma - \Gamma - 1}{\Gamma} 2(N-2)\tilde{P}_N < \frac{e^\Gamma - \Gamma - 1}{3\Gamma} < \frac{1}{3}\,,$$

where in the first inequality, after standard manipulations, we used that $N \mapsto 2(N-2)\tilde{P}_N$ is increasing and $\tilde{P}_N \to 1/3$ as $N \to \infty$. Denote $P_{N,\varepsilon_0} := P_N(1+\varepsilon_0)$ and by the definition of ζ_2^* one has $|\tilde{\eta}(\sigma)| \leq P_{N,\varepsilon_0} f(\sigma)$ for each $\sigma \in (\zeta_2^*, \infty)$.

Next, in the following three steps we obtain estimates on $\tilde{\zeta}$ on the interval $[\zeta_1^*, \infty)$ and in particular we prove that the assertion of Step 4 remains valid if $f(\zeta_\lambda) \leq 1.1$.

Step 5: For any $m > 0$ and $\varepsilon_0 > 0$, one has $\zeta_2^* < \infty$.

Proof of Step 5. We proceed as in the proof of Lemma **??**. Using the representation formula (**??**) and Young's inequality for convolutions, we obtain

$$\int_\zeta^\infty |\tilde{\eta}(\sigma)| d\sigma \leq C_N \int_\zeta^\infty |\tilde{g}(\sigma)| d\sigma\,,$$

where $C_N = \|G_N\|_{L^1}$. Since $\eta(\zeta) \to 0$ as $\zeta \to \infty$, for any $\varepsilon > 0$ there is $\zeta_0 > 0$ depending on N, η, and ε such that for any $\zeta \geq \zeta_0$ one has

$$|\phi(\eta)| = 2(N-2)\,|e^\eta - 1 - \eta| \leq \frac{\varepsilon}{2}|\eta|\,, \qquad \text{where } \eta = \eta(\zeta)\,.$$

By the definition of $\tilde{g}$ and (**??**), one has, for $\sigma \geq \zeta_0$,

$$|\tilde{g}(\sigma)| \leq m^2 e^{-2\sigma}(|f(\sigma)| + |\tilde{\eta}(\sigma)|) + \frac{\varepsilon}{2}(|\tilde{\eta}(\sigma)| + |f(\sigma)|)\,.$$

Fix $\overline{\zeta}_0 > 0$ such that $m^2 e^{-2\overline{\zeta}_0} \leq \frac{\varepsilon}{2}$, and set $\zeta_0^* = \max\{\zeta_0, \overline{\zeta}_0\}$. Then we have, for $\sigma \geq \zeta_0^*$,

$$|\tilde{g}(\sigma)| \leq \varepsilon(|f(\sigma)| + |\tilde{\eta}(\sigma)|)\,.$$

Substituting (**??**) into (**??**) and requiring that $\varepsilon \in (0, 1/(2C_N))$ we obtain, for $\zeta \geq \zeta_0^* > 0$,

$$\int_\zeta^\infty |\tilde{\eta}(\sigma)|\,d\sigma \leq \frac{1}{2}\int_\zeta^\infty |f(\sigma)|\,d\sigma + \frac{1}{2}\int_\zeta^\infty |\tilde{\eta}(\sigma)|\,d\sigma\,,$$

and consequently

$$\int_\zeta^\infty |\tilde{\eta}(\sigma)|\,d\sigma \leq \int_\zeta^\infty |f(\sigma)|\,d\sigma\,.$$

Using (**??**) and (**??**), we obtain for $\zeta \geq \zeta_0^*$,

$$\begin{aligned}|\tilde{\eta}(\zeta)| &\leq \|G_N\|_{L^\infty}\int_\zeta^\infty |\tilde{g}(\sigma)|\,d\sigma \leq \varepsilon C_N \int_\zeta^\infty f(\sigma) + |\tilde{\eta}(\sigma)|\,d\sigma \\ &\leq \varepsilon C_N \int_\zeta^\infty f(\sigma)\,d\sigma \leq \varepsilon C_N f(\zeta)\,.\end{aligned}$$

By making $\varepsilon > 0$ smaller if necessary such that $\varepsilon \leq P_N(1+\varepsilon_0)/C_N$ we obtain $\zeta_2^* \leq \zeta_0^*$, and the claim follows. □

Step 6: For any small $\varepsilon_0 > 0$, there exists $m_0 > 0$ such that for each $m \geq m_0$ we have $\tilde{\eta} \leq 0$ on $[\zeta_2^*, \infty)$, where ζ_2^* is defined in (**??**).

Proof of Step 6. Suppose first that $3 \leq N \leq 9$. Then, we rewrite (**??**) as

$$\tilde{\eta}(\zeta) = \int_\zeta^\infty G_N(\sigma - \zeta)\tilde{g}(\sigma)d\sigma =: \int_\zeta^\infty F(\zeta, \sigma)d\sigma$$

and

$$\begin{aligned}\int_\zeta^\infty F(\zeta,\sigma)d\sigma &= \sum_{k=0}^\infty \int_{\zeta+\frac{2k\pi}{\beta}}^{\zeta+\frac{(2k+1)\pi}{\beta}} F(\zeta,\sigma)d\sigma + \int_{\zeta+\frac{(2k+1)\pi}{\beta}}^{\zeta+\frac{(2k+2)\pi}{\beta}} F(\zeta,\sigma)d\sigma \\ &= \sum_{k=0}^\infty \int_{\zeta+\frac{2k\pi}{\beta}}^{\zeta+\frac{(2k+1)\pi}{\beta}} F(\zeta,\sigma) + F\left(\zeta, \sigma + \frac{\pi}{\beta}\right)d\sigma\,,\end{aligned}$$

where

$$F(\zeta,\sigma)+F\left(\zeta,\sigma+\frac{\pi}{\beta}\right)=G_N(\sigma-\zeta)\left(\tilde g(\sigma)-e^{-\frac{\alpha\pi}{2\beta}}\tilde g\left(\sigma+\frac{\pi}{\beta}\right)\right).$$

Recall, for any $\sigma\geq\zeta_2^*$ we have $|\tilde\eta(\sigma)|\leq P_{N,\varepsilon_0}f(\sigma)$ and by (**??**), $1>P_{N,\varepsilon_0}$ for any sufficiently small $\varepsilon_0>0$. In the following, we use the notation $O(m^{-1})$ for quantities converging to zero as $m\to\infty$. Then, since ϕ is decreasing on $(0,\infty)$ and $f\pm\tilde\eta\geq 0$ on $[\zeta_2^*,\infty)$, one has

$$\begin{aligned}\phi((f+\tilde\eta)(\sigma))-e^{-\frac{\alpha\pi}{2\beta}}\phi\left((f+\tilde\eta)\left(\sigma+\frac{\pi}{\beta}\right)\right)&\leq\phi((f-|\tilde\eta|)(\sigma))-e^{-\frac{\alpha\pi}{2\beta}}\phi\left((f+|\tilde\eta|)\left(\sigma+\frac{\pi}{\beta}\right)\right)\\&\leq\phi((1-P_{N,\varepsilon_0})f(\sigma))-e^{-\frac{\alpha\pi}{2\beta}}\phi\left((1+P_{N,\varepsilon_0})f\left(\sigma+\frac{\pi}{\beta}\right)\right)\\&\leq\phi((1-P_{N,\varepsilon_0})f(\sigma))-e^{-\frac{\alpha\pi}{2\beta}}\phi\left((1+P_{N,\varepsilon_0})(e^{-\frac{2\pi}{\beta}}+O(m^{-1}))f(\sigma)\right),\end{aligned}$$

where in the last step we used that, for $\sigma\geq\zeta_2^*$,

$$\begin{aligned}f(\sigma+\pi/\beta)&=e^{-\frac{2\pi}{\beta}}f(\sigma)+\frac{\pi m^2}{2\beta(N-1)}e^{-\frac{2\pi}{\beta}}e^{-2\sigma}\leq e^{-\frac{2\pi}{\beta}}f(\sigma)+\frac{\pi}{\beta\zeta_1}f(\sigma)\\&\leq(e^{-\frac{2\pi}{\beta}}+O(m^{-1}))f(\sigma).\end{aligned}$$

We claim that for any sufficiently small $\varepsilon_0,\varepsilon_1>0$ and any sufficiently large m, one has

$$\phi((1-P_{N,\varepsilon_0})z)\leq e^{-\frac{\alpha\pi}{2\beta}}\phi\left((1+P_{N,\varepsilon_0})(e^{-\frac{2\pi}{\beta}}+O(m^{-1}))z\right)-\varepsilon_1 z^2,\qquad\text{for any } z\in[0,\Gamma].$$

Indeed, for any $\kappa>1$ sufficiently close to one (see below), define

$$\psi_\kappa(z)=\phi((1-P_N)z)-e^{-\frac{\alpha\pi}{2\beta}}\phi\left((1+P_N)\kappa e^{-\frac{2\pi}{\beta}}z\right)$$

and note that $\psi_\kappa(0)=\psi_\kappa'(0)=0$. Moreover, using that $\phi''(z)=-2(N-2)e^z$ and $P_N<1/3$, we have, for $\kappa=1$ and any $z\in[0,\Gamma]$,

$$\begin{aligned}\psi_1''(z)&=-2(N-2)\left((1-P_N)^2e^{(1-P_N)z}-e^{-(\frac{\alpha}{2}+4)\frac{\pi}{\beta}}(1+P_N)^2e^{(1+P_N)e^{-\frac{2\pi}{\beta}}z}\right)\\&\leq-2(N-2)e^{(1-P_N)z}\left((1-P_N)^2-e^{-(\frac{\alpha}{2}+4)\frac{\pi}{\beta}}(1+P_N)^2e^{2P_Nz}\right)\\&\leq-2(N-2)e^{(1-P_N)z}\left((1-P_N)^2-e^{-(\frac{\alpha}{2}+4)\frac{\pi}{\beta}}(1+P_N)^2e^{2P_N\Gamma}\right)\\&\leq-2(N-2)e^{(1-P_N)z}\left(\frac{4}{9}-e^{-\pi}\frac{16}{9}e^{2/3\cdot1.1}\right)=-2c_N<0.\end{aligned}$$

Then by the continuity $\psi''_\kappa(z) < -c_N < 0$ for any $\kappa > 1$ sufficiently close to 1 and any $z \in [0, \Gamma]$. Fix such $\kappa_0 > 1$. Thus $\psi_\kappa(z) < -c_N z^2$ on $[0, \Gamma]$, and we obtain that (**??**) holds true for any $\varepsilon_1 < c_N$ and for any sufficiently small $\varepsilon_0 > 0$ and large m. The claim follows.

In addition, using that f is decreasing and that $|\tilde{\eta}(\sigma)| \le P_{N,\varepsilon_0} f(\sigma)$, we have, for $\sigma \ge \zeta_2^* \ge \zeta_1^*$,

$$m^2 e^{-2\sigma} \left((\tilde{\eta} + f)(\sigma) - e^{-\frac{\pi}{2\beta}(4+\alpha)} (\tilde{\eta} + f) \left(\sigma + \frac{\pi}{\beta} \right) \right) \le 2(1 + P_{N,\varepsilon_0}) m^2 e^{-2\sigma} f(\sigma)$$
$$\le \frac{C_N}{\zeta_1^*} f^2(\sigma) \le \varepsilon_1 f^2(\sigma)\,,$$

where we used that by Remark **??**, $\zeta_1^* \to \infty$ as $m \to \infty$. Therefore, after recalling that $\tilde{g}(\zeta) = \phi(\eta(\zeta) + m^2 e^{-2\zeta} \eta(\zeta)$, (**??**) yields

$$\tilde{g}(\sigma) - e^{-\frac{\alpha\pi}{2\beta}} \tilde{g} \left(\sigma + \frac{\pi}{\beta} \right) \le 0.$$

Since for any integer $k \ge 0$, one has $G_N(\sigma - \zeta) \ge 0$ on the interval $\left(\zeta + \frac{2k\pi}{\beta}, \zeta + \frac{(2k+1)\pi}{\beta} \right)$, we obtain from (**??**)

$$F(\zeta, \sigma) + F\left(\zeta, \sigma + \frac{\pi}{\beta} \right) \le 0,$$

and Step 6 follows for $3 \le N \le 9$.

Next, assume $N \ge 10$ and notice that $G_N \ge 0$ in this case. Also, since $|\tilde{\eta}(\sigma)| \le P_{N,\varepsilon_0} f(\sigma)$ on $[\zeta_2^*, \infty)$ and $P_{N,\varepsilon_0} < 1$ for any sufficiently small ε_0, we obtain that $\eta = f + \tilde{\eta} \ge 0$ on $[\zeta_2^*, \infty)$. Since $e^x - 1 - x \ge \frac{1}{2} x^2$ for $x \ge 0$, then for any $\zeta \ge \zeta_2^*$,

$$\tilde{\eta}(\zeta) \le \int_\zeta^\infty G_N(\sigma - \zeta) \Big(m^2 e^{-2\sigma} \eta(\sigma) - (N-2) \eta^2(\sigma) \Big) d\sigma.$$

Also, since $\eta \ge 0$ we have

$$m^2 e^{-2\sigma} \eta(\sigma) - (N-2) \eta^2(\sigma) \le \eta(\sigma) \left(c_N \frac{f(\sigma)}{\sigma} - (f(\sigma) - |\tilde{\eta}(\sigma)|) \right)$$
$$\le \eta(\sigma) f(\sigma) \left(\frac{c_N}{\zeta_1^*} - (1 - P_{N,\varepsilon_0}) \right) \le 0\,,$$

where c_N depends only on N and the last inequality follows for any sufficiently large ζ_1^*, that is, for sufficiently large m. Thus $\eta(\zeta) \le 0$ for each $\tilde{\eta} \ge \zeta_2^*$ as desired. □

Step 7: For any sufficiently small $\varepsilon_0 > 0$, there exists m_0 such that for each $m \geq m_0$ we have $\zeta_2^* = \zeta_1^*$, where ζ_2^* is defined in (**??**) and ζ_1^* is the largest solution to $f(\zeta) = \Gamma$. In particular, $|\tilde{\eta}(\zeta_\lambda)| \leq \tilde{C}_N$.

Proof of Step 7. In Step 6, we proved that $\tilde{\eta} \leq 0$ on (ζ_2^*, ∞). In order to obtain an estimate on $|\tilde{\eta}|$, we need a lower bound on $\tilde{\eta}$.

First assume $3 \leq N \leq 9$. Since $G_N(\sigma - \zeta) \leq 0$ on the interval $\left(\zeta + \frac{(2k+1)\pi}{\beta}, \zeta + \frac{(2k+2)\pi}{\beta}\right)$, (**??**) and (**??**) yield

$$F(\zeta, \sigma) + F\left(\zeta, \sigma + \frac{\pi}{\beta}\right) \geq 0.$$

Consequently by using that ϕ is decreasing and $\tilde{\eta} \leq 0$, we obtain, for any $\zeta \geq \zeta_2^*$,

$$\tilde{\eta}(\zeta) \geq \int_\zeta^{\zeta + \frac{\pi}{\beta}} G_N(\sigma - \zeta)\phi((f + \tilde{\eta})(\sigma))d\sigma + m^2 \int_\zeta^\infty G_N(\sigma - \zeta)e^{-2\sigma}(\tilde{\eta} + f)(\sigma)d\sigma$$

$$\geq \int_\zeta^{\zeta + \frac{\pi}{\beta}} G_N(\sigma - \zeta)\phi(f(\sigma))d\sigma - m^2(1 + P_{N,\varepsilon_0}) \int_\zeta^\infty |G_N(\sigma - \zeta)|e^{-2\sigma} f(\sigma)d\sigma.$$

In order to estimate $\phi(f(\sigma))$ we use that, for any $y \geq x > 0$, one has

$$\frac{\phi(x)}{\phi(y)} \leq \frac{x^2}{y^2}\,.$$

Indeed, this inequality is equivalent to

$$\frac{e^x - x - 1}{x^2} \leq \frac{e^y - y - 1}{y^2}$$

which is true since the function $x \mapsto (e^x - x - 1)/x^2$ is increasing on $(0, \infty)$. Hence, since $\phi < 0$ on $(0, \infty)$ we have for $\sigma \geq \zeta$

$$\phi(f(\sigma)) \geq \phi(f(\zeta)) \left(\frac{f(\sigma)}{f(\zeta)}\right)^2 = \phi(f(\zeta))e^{-4(\sigma - \zeta)} \left(\frac{\sigma + c_N}{\zeta + c_N}\right)^2\,.$$

Using that $\sigma \in (\zeta, \zeta + \pi/\beta)$ and $\zeta \geq \zeta_1^* \to \infty$ as $m \to \infty$, we have

$$\phi(f(\sigma)) \geq \phi(f(\zeta))e^{-4(\sigma - \zeta)}(1 + O(m^{-1}))\,,$$

and therefore

$$m^2(1+P_{N,\varepsilon_0})\int_\zeta^\infty |G_N(\sigma-\zeta)|e^{-2\sigma}f(\sigma)d\sigma \le c_N m^2 f(\zeta)e^{-2\zeta} \le c_N\frac{f^2(\zeta)}{\zeta} = O(m^{-1})f^2(\zeta)\,.$$

Thus, for any $\zeta \ge \zeta_2^*$

$$\begin{aligned}\tilde\eta(\zeta) &\ge \frac{\phi(f(\zeta))}{\beta}\int_\zeta^{\zeta+\frac{\pi}{\beta}} e^{-(\frac{\alpha}{2}+4)(\sigma-\zeta)}\sin(\beta(\sigma-\zeta))d\sigma - O(m^{-1})f^2(\zeta)\\ &= \frac{4\phi(f(\zeta))}{(\alpha+8)^2+4\beta^2}(1+e^{-\frac{(\alpha+8)\pi}{2\beta}}) - O(m^{-1})f^2(\zeta)\,.\end{aligned}$$

Using that $\tilde\eta \le 0$ and $x \mapsto \phi(x)/x$, is decreasing and the definition of P_N we obtain for any $\zeta \ge \zeta_2^* \ge \zeta_1^*$, that is, $f(\zeta) \in (0,\Gamma]$ and sufficiently large m

$$\begin{aligned}|\tilde\eta(\zeta)| &\le \frac{4\phi(f(\zeta))}{((\alpha+8)^2+4\beta^2)f(\zeta)}(1+e^{-\frac{(\alpha+8)\pi}{2\beta}})f(\zeta) + O(m^{-1})f^2(\zeta)\\ &\le \left(P_N + O(m^{-1})\right)f(\zeta) < \left(1+\frac{\varepsilon_0}{2}\right)P_N f(\zeta)\,.\end{aligned}$$

If $\zeta_2^* > \zeta_1^*$, then, by continuity, $|\tilde\eta(\zeta)| \le (1+\varepsilon_0)P_N|f(\zeta)|$ holds for any $\zeta_1^* \le \zeta \le \zeta_2^*$ sufficiently close to ζ_2^*, a contradiction to the definition of ζ_2^*. Thus $\zeta_1^* = \zeta_2^*$ as desired.

If $N \ge 10$, using $G_N \ge 0$, the monotonicity of ϕ, and $\tilde\eta \le 0$ as above, we obtain, for any $\zeta \ge \zeta_1^*$,

$$\begin{aligned}\eta(\zeta) &\ge \int_\zeta^\infty G_N(\sigma-\zeta)(m^2e^{-2\sigma}\tilde\eta(\sigma)+\phi(f(\sigma)))d\sigma\\ &\ge \frac{\phi(f(\zeta))}{(\zeta+c_N)^2}\int_\zeta^\infty G_N(\sigma-\zeta)e^{-4(\sigma-\zeta)}(\sigma+c_N)^2d\sigma - O(m^{-1})f^2(\zeta)\,,\end{aligned}$$

and note that we could not use (**??**) since $\sigma-\zeta$ is unbounded. Then, if $N > 10$, one has

$$\tilde\eta(\zeta) \ge \frac{\phi(f(\zeta))}{((\alpha/2+4)^2-\beta^2)}\left(1-\frac{\tilde c_N}{\zeta+c_N}\right) - O(m^{-1})f^2(\zeta)$$

and using again that $|\phi(f(\zeta))| \le \bar c_N f^2(\zeta)$ for any $\zeta \ge \zeta_1^*$ and $\zeta \to \infty$ as $m \to \infty$, we have

$$\tilde\eta(\zeta) \ge \frac{\phi(f(\zeta))}{((\alpha/2+4)^2-\beta^2)} - O(m^{-1})f^2(\zeta)\,.$$

If $N = 10$, one similarly has

$$\tilde\eta(\zeta) \ge \frac{\phi(f(\zeta))}{(\alpha/2+4)^2} - O(m^{-1})f^2(\zeta)\,.$$

The rest of the proof is the same as in the case $3 \le N \le 9$. □

Remark 5.4. *In Steps 5-7 we proved that*

$$0 \ge \tilde{\eta}(\zeta) \ge -f(\zeta) \ge -\Gamma \qquad \text{for any } \zeta \ge \zeta_1^*$$

which in turn implies

$$0 \le \eta \le f(\zeta) \qquad \text{for any } \zeta \ge \zeta_1^* .$$

In the original variables, for $U_\lambda^*(r) = \eta(\zeta) + 2\zeta$ we have

$$-2\ln r + \ln \frac{2(N-2)}{\lambda} \le U_\lambda^*(r) \le -2\ln r + \ln \frac{2(N-2)}{\lambda} + c_N r^2(1-\ln r) \qquad \text{for any } r \le \tilde{c}_N .$$

The importance of this bound is in the estimate on U_λ^* on an interval which is independent of λ. An interested reader can calculate explicitly constants c_N and $\tilde{c}_N$.

Remark 5.5. *From Remark* **??** *we can also obtain an estimates on $(U_\lambda^*)'$ as follows. By* (**??**)

$$\tilde{\eta}'(\zeta) = -\int_\zeta^\infty G_N'(\sigma - \zeta)\tilde{g}(\sigma)\, d\sigma .$$

Since G_N' is a bounded integrable function, using Remark **??** *and analogous estimates as in* (**??**) *we have*

$$|\tilde{\eta}'(\zeta)| \le C_N f(\zeta) \qquad \text{for any } \zeta \ge \zeta_1^* .$$

In the original variables the last bound translates into

$$\left| (U_\lambda^*)'(r) + \frac{2}{r} + r c_N^* - \frac{r}{N-1}\left(\ln \sqrt{\frac{2(N-2)}{\lambda}} - \ln r \right) \right| = \frac{1}{r}|\tilde{\eta}'(\zeta)|$$
$$\le C_N \frac{1}{r} f(\zeta) \le C_{N,\lambda} r(1 + |\ln r|) \qquad \text{for any } r \le c_N ,$$

where $C_{N,\lambda}$ is bounded in λ uniformly on compact subsets of $(0,\infty)$.

Step 8: Proof of Proposition **??**.

Proof of Step 8. Recall that r_λ is the smallest solution of $U^*(r) = \overline{u}_\lambda$ and ζ_λ is the corresponding transformed variable, see (**??**). Denote $z_\lambda := M_\lambda^1$, that is, z_λ

is first critical point of U^* and let ρ_λ be transformed z_λ, see (**??**) with r_λ and ζ_λ replaced respectively by z_λ and ρ_λ.

First, we show that $z_\lambda \geq r_\lambda$. Indeed, otherwise $z_\lambda < r_\lambda$ and as in the proof of Lemma **??**, we have that the function V defined by (**??**) is decreasing in r. Then, as in the proof of Lemma **??** we obtain that $U^*(r) \geq U^*(z_\lambda) > \bar{u}_\lambda$ a contradiction to Lemma **??**. Hence for the rest of the proof we assume that $z_\lambda \geq r_\lambda$.

By Steps 5-7 (cf. Remark **??**) one has $|\tilde{\eta}(\zeta)| \leq C_N$ for any $\zeta \geq \zeta_1^*$. In particular, $|\tilde{\eta}(\zeta_\lambda)| \leq C_N$ if $\zeta_\lambda \geq \zeta^*$, that is, if $f(\zeta_\lambda) \leq \Gamma$. But $|\tilde{\eta}(\zeta_\lambda)| \leq C_N$ holds also by Step 4 if $f(\zeta_\lambda) \geq \Gamma$. Overall, we have $|\tilde{\eta}(\zeta_\lambda)| \leq C_N$.

We claim that $|\eta(\zeta)| \leq C_N$ holds in fact for all $\zeta \geq \zeta_\lambda$ and any sufficiently small $\lambda > 0$. Indeed, if $\zeta_\lambda \geq \zeta_1^*$, then the statement is already proved in Steps 5-7. If $\zeta_\lambda < \zeta_1^*$, assume that there exists $\tilde{\zeta} \in (\zeta_\lambda, \zeta_1^*]$ such that $\tilde{\eta}'(\tilde{\zeta}) = 0$. Without loss of generality let $\tilde{\zeta}$ be the largest such number. Since $\tilde{\zeta} < \zeta_1^*$, then $f(\tilde{\zeta}) \geq \Gamma = 1.1$, and consequently for any large m (or small λ), one has $f'(\tilde{\zeta}) < -\frac{2}{\Gamma} f(\tilde{\zeta}) \leq -2$. If $\tilde{r}$ corresponds to $\tilde{\zeta}$, see (**??**), then

$$(U^*)'(\tilde{r}) = -\frac{1}{\tilde{r}}(f'(\tilde{\zeta}) + \tilde{\eta}'(\tilde{\zeta}) + 2) = -\frac{1}{\tilde{r}}(f'(\tilde{\zeta}) + 2) > 0.$$

Since $(U^*)'(r) < 0$ for r sufficiently close to 0, we obtain $\tilde{r} \geq z_\lambda$, and therefore $\tilde{\zeta} \leq \zeta_\lambda$, a contradiction to $\zeta_\lambda < \tilde{\zeta}$. Thus, no such $\tilde{\zeta}$ exists, and therefore $\tilde{\eta}$ is increasing on $(\zeta_\lambda, \zeta_1^*)$. Since $\max\{|\eta(\tilde{\eta}_\lambda)|, |\tilde{\eta}(\zeta_1^*)|\} \leq C_N$, we deduce that $|\tilde{\eta}(\zeta)| \leq C_N$, for all $\zeta \geq \zeta_\lambda$ and the claim follows.

Now, $|\tilde{\eta}| \leq C_N$ implies that $|\tilde{g}| \leq C_N$ (defined in (**??**)) on (ζ_λ, ∞). Differentiating (**??**) and using that G_N' is integrable, we find

$$|\tilde{\eta}'(\zeta_\lambda)| \leq \int_{\zeta_\lambda}^{\infty} |G_N'(\sigma - \zeta_\lambda)||\tilde{g}(\sigma)| d\sigma \leq C_N.$$

Then, using (**??**) and $|f'(\zeta)| \leq 3f(\zeta)$, for any $|\zeta|$ large enough, we have

$$|(U^*)'(r_\lambda)| \leq \frac{|f'(\zeta_\lambda)| + |\tilde{\eta}'(\zeta_\lambda)| + 2}{r_\lambda} \leq \frac{C_N}{r_\lambda} = C_N \sqrt{\lambda e^{2\zeta_\lambda}}\,.$$

Furthermore, recalling that $U^*(r) = 2\zeta + f(\zeta) + \tilde{\eta}(\zeta)$, we deduce from Step 3, and $|\tilde{\eta}(\zeta_\lambda)| \leq C_N$ that $2\zeta_\lambda - c_N \leq U^*(r_\lambda) = \overline{u}_\lambda \leq 2\zeta_\lambda + c_N$. Consequently,

$\overline{u}_\lambda = \lambda e^{\overline{u}_\lambda}$ yields

$$|(U^*)'(r_\lambda)| \le C_N \sqrt{\lambda e^{\overline{u}_\lambda}} = C_N \sqrt{\overline{u}_\lambda}.$$

Recalling that $v := U^*/\overline{u}_\lambda$ solves (**??**), we obtain

$$|v'(r_\lambda)| \le C_N \frac{1}{\sqrt{\overline{u}_\lambda}} \qquad \text{and} \quad v(r_\lambda) = 1\,.$$

Furthermore, since the function

$$r \mapsto E(r) = \frac{(v'(r))^2}{2} + \frac{e^{\overline{u}_\lambda (v(r)-1)}}{\overline{u}_\lambda} - \frac{v^2(r)}{2}$$

is non-increasing, any $r \ge r_\lambda$ one has

$$\frac{C_N}{\overline{u}_\lambda} - \frac{1}{2} \ge E(r_\lambda) \ge E(r) \ge \frac{e^{\overline{u}_\lambda (v(r)-1)}}{\overline{u}_\lambda} - \frac{v^2(r)}{2} \ge -\frac{v^2(r)}{2}\,.$$

Thus, for sufficiently small $\lambda > 0$,

$$v^2(r) \ge 1 - \frac{C_N}{\overline{u}_\lambda} \qquad \text{for any} \quad r \ge r_\lambda\,,$$

and consequently

$$\sup_{r \ge r_\lambda} (1 - v)_+ \to 0 \qquad \text{as} \quad \lambda \to 0\,,$$

where $g_+ = \max\{g, 0\}$ denotes a positive part of a function g. Recall that $w(r) = r^{\frac{N-1}{2}}(v(r) - 1)$ satisfies (**??**) and clearly

$$\frac{e^{\overline{u}_\lambda (v-1)} - v}{v - 1} = \frac{e^{\overline{u}_\lambda (v-1)} - 1}{v - 1} - 1\,.$$

Fix any $\mu > 0$, $a > 0$ and denote $I_a := \left[\frac{a}{4}, a\right]$. Choose any $r \in I_a$. If $\overline{u}_\lambda (v(r) - 1) \ge -1$, then using that $x \mapsto (e^x - 1)/x$ is increasing we have for sufficiently small λ (or large $\bar{u}_\lambda$ by Step 1)

$$\frac{e^{\overline{u}_\lambda (v(r)-1)} - 1}{v(r) - 1} = \frac{e^{\overline{u}_\lambda (v(r)-1)} - 1}{\overline{u}_\lambda (v(r) - 1)} \overline{u}_\lambda \ge \overline{u}_\lambda \left(1 - \frac{1}{e}\right) \ge \mu\,.$$

On the other hand if $\overline{u}_\lambda (v(r) - 1) < -1$, then $v(r) < 1$ and

$$\frac{e^{\overline{u}_\lambda (v(r)-1)} - 1}{v(r) - 1} \ge \frac{e^{-1} - 1}{v(r) - 1} \ge \frac{1 - e^{-1}}{\sup_{\rho \ge r_\lambda} (1 - v(\rho))_+} > \mu\,,$$

for sufficiently small λ, where we used (**??**) in the last inequality. Hence, for any $\mu > 0$ and $a > 0$ one has for sufficiently small $\lambda > 0$ that

$$\frac{e^{\overline{u}_\lambda (v-1)} - v}{v-1} - \frac{(N-1)(N-3)}{4r^2} \geq \mu - C_{N,a} \qquad \text{for any} \quad r \in I_a := \left[\frac{a}{4}, a\right].$$

Thus, given $a > 0$ and integer $i > 0$, there is large μ, such that a solution of the equation $z'' + (\mu - C_{N,a})z = 0$ has at least $i+2$ zeros on I_a, and by Sturm-Picone oscillation theorem for any sufficiently small $\lambda > 0$, the function w has at least $i+1$ zeros on I_a. Consequently, $U^*(r) = \overline{u}_\lambda$ has at least $i+1$ solutions on I_a, and therefore U^* has at least i critical points on I_a. In a different notation for any $j \in \{1, \cdots, i\}$ and any $a > 0$ one has $R^j_\lambda < a$ for any sufficiently small $\lambda > 0$. □

Proposition 5.6. *For any $i \in \mathbb{N}$, the function $\lambda \to R^i_\lambda$ is continuous.*

Proof. Fix $\lambda^* > 0$ and an open interval $I_0 = (A, B)$ with $0 < A < B < \infty$. Without loss of generality assume $A \leq c_N$, where c_N is as in Remark **??**. Then by Remark **??** there is $\delta > 0$ such that for any $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$ one has

$$|U^*_\lambda(A)| \leq C_N \,.$$

If for some $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$ the function U^*_λ is decreasing on (A, B), then non-negativity of U^* yields $|U^*_\lambda| \leq C_N$ on (A, B). If for some $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$, there is a smallest $z_\lambda < B$ such that $(U^*_\lambda)'(z_\lambda) = 0$, then since V defined by (**??**) is non-decreasing and by the Step 8 in the proof of Proposition **??** one has $U^*_\lambda(z_\lambda) < \bar{u}_\lambda$, then, for any $r \geq z_\lambda$,

$$\bar{u}_\lambda \geq \lambda e^{U^*_\lambda(z_\lambda)} \geq \lambda e^{U^*_\lambda(z_\lambda)} - \frac{1}{2}(U^*_\lambda(z_\lambda))^2 = V(r_\lambda) \geq V(r) \geq \lambda e^{U^*_\lambda(r)} - \frac{1}{2}(U^*_\lambda(r))^2.$$

Since $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$, we obtain that the left hand side is bounded by a constant independent of λ, and consequently U^*_λ is bounded on (z_λ, ∞), by a constant independent of $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. Overall, we have

$$\sup_{\lambda \in (\lambda^* - \delta, \lambda^* + \delta)} \sup_{(A,B)} U^*_\lambda \leq C(A) \,.$$

Then, the elliptic regularity implies that, for any $q > 1$,

$$\|U^*_\lambda\|_{W^{3,q}(I_0)} \leq C(N, q, A, B - A, \lambda^*, \delta) \qquad \text{for any } \lambda \in (\lambda^* - \delta, \lambda^* + \delta)\,.$$

For $\alpha_0 \in (0, 1)$, we choose $q_0 > 0$ large enough such that $W^{3,q_0}(I_0) \hookrightarrow C^{2+\alpha_0}(I_0)$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence such that $\lambda_n \to \lambda^*$ when $n \to \infty$. Thanks to (**??**), using Arzelà-Ascoli's theorem, there exists a sub-sequence, still denoted (λ_n), such that $U^*_{\lambda_n} \to w$, as $n \to \infty$, in $C^2(I_0)$. Noticing that

$$|\lambda_n e^{U^*_{\lambda_n}(s)} - \lambda^* e^{w(s)}| \leq \lambda_n |e^{U^*_{\lambda_n}(s)} - e^{w(s)}| + |\lambda_n - \lambda^*| e^{w(s)} \to 0 \qquad \text{as } n \to \infty,$$

we deduce that w satisfies the equation

$$-\Delta w + w = \lambda^* e^w \qquad \text{in } I_0\,.$$

Since I_0 is an arbitrary compact interval, proceeding as above and using standard diagonal arguments, we obtain the existence of a sub-sequence $(\lambda_{k_n})_n$, $\lambda_n \in (\lambda^* - \delta, \lambda^* + \delta)$, for all $n \in \mathbb{N}$, such that $U^*_{\lambda_{k_n}} \to w$, as $n \to \infty$, in $C^2_{loc}((0, \infty))$, for some function w satisfying

$$-\Delta w + w = \lambda^* e^w \qquad \text{in } (0, \infty)\,.$$

Next, we claim that w is in fact $U^*_{\lambda^*}$. Using the uniqueness of solution proved in Proposition **??**, it is sufficient to show that

$$\lim_{r\to 0^+} w(r) + 2\ln r = A_{\lambda^*, N},$$

where $A_{\lambda,N} = \ln \dfrac{2(N-2)}{\lambda}$. However, by Remark **??** there is $r_0(\varepsilon)$ independent of $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$ such that

$$A_{\lambda,N} \leq U^*_\lambda(r) + 2\ln r \leq A_{\lambda,N} + \varepsilon \qquad \text{for all} \quad r \in (0, r_0(\varepsilon))\,.$$

Clearly $A_{\lambda_n,N} \to A_{\lambda^*,N}$ when $n \to \infty$ and using that $U^*_{\lambda_n} \to w$ in $C^2_{\rm loc}((0, r_0(\varepsilon)))$, we obtain

$$A_{\lambda^*,N} \leq w(r) + 2\ln r \leq A_{\lambda^*,N} + \varepsilon, \qquad \text{for all} \quad r \in (0, r_0(\varepsilon))\,.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (**??**) holds, and therefore $w = U^*_{\lambda^*}$ by the uniqueness. Hence,

$$U^*_\lambda \to U^*_{\lambda^*}, \text{ as } \lambda \to \lambda^*, \text{ in } C^2_{loc}((0,\infty)).$$

Finally, we prove the continuity of the function $\lambda \to R^i_\lambda$. In the following, we assume that R^i_λ is a local minimum of U^*_λ, the case of local maximum follows analogously. Note that $U^*(R^i_\lambda) \neq \bar{u}_\lambda$, otherwise $U^* \equiv \bar{u}_\lambda$, and we have a contradiction to the uniqueness of the initial value problem. Thus, for any sufficiently small $\bar{\varepsilon} > 0$, we obtain

$$U^*_{\lambda^*}(R^i_{\lambda^*} - \bar{\varepsilon}) > U^*_{\lambda^*}(R^i_{\lambda^*}) \qquad \text{and} \qquad U^*_{\lambda^*}(R^i_{\lambda^*} + \bar{\varepsilon}) > U^*_{\lambda^*}(R^i_{\lambda^*})\,.$$

Then (**??**), yields that for $\lambda > 0$ sufficiently close to λ^* that (**??**) holds true with $U^*_{\lambda^*}$ replaced by U^*_λ. Thus, there exists a local minimizer q_λ of U^*_λ in any small neighborhood of $R^i_{\lambda^*}$, or equivalently for every $\lambda > 0$ there is q_λ with $(U^*_\lambda)'(q_\lambda) = 0$ such that

$$\lim_{\lambda \to \lambda^*} q_\lambda = R^i_{\lambda^*}\,.$$

On the other hand assume that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n \to \lambda^*$ and $(q_{\lambda_n})_{n\in\mathbb{N}}$ converging to q^*. Then by (**??**) one has $(U^*_{\lambda^*})'(q^*) = 0$.

Finally assume that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n \to \lambda^*$ and two sequences $(q_{\lambda_n})_{n\in\mathbb{N}}$, $(q'_{\lambda_n})_{n\in\mathbb{N}}$ that converge to q^*. Then by the mean value theorem, there exists s_{λ_n} between q_{λ_n} and q'_{λ_n} such that $(U^*)''(s_{\lambda_n}) = 0$. By passing to the limit, one has $(U^*_{\lambda^*})'(q^*) = (U^*_{\lambda^*})''(q^*) = 0$, a contradiction to the fact that every critical point is either strict minimizer or strict maximizer.

Overall, we proved that in each neighborhood of $R^i_{\lambda^*}$, for sufficiently small λ, there exists exactly one critical point of U^*_λ and the proof is finished. □

Proof of Theorem **??**. By the definition of i^*, for any $i \geq \tilde{i}$ we have $R^i_{\tilde{\lambda}} > R$. On the other hand, by Proposition **??**, we know that, for any $i > 0$, $\lim_{\lambda\to 0} R^i_\lambda < R$. Since for any $i \in \mathbb{N}$ the function $\lambda \to R^i_\lambda$ is continuous by Proposition **??**,

we deduce that there exists $\lambda^i > \tilde{\lambda}$ such that $R^i_{\lambda^i} = R$. This concludes the proof. □

6. Oscillation of the branches for generic radius

In this section, we prove two generic uniqueness results, one for singular and one for regular solutions i.e. we prove Theorems **??** and **??**. More precisely we show that, for generic $R > 0$, if $(U^*_{\lambda^*})'(R) = 0$ for some $\lambda^* > 0$, then $(U^*_\lambda)'(R) \neq 0$, for any $\lambda \approx \lambda^*$, $\lambda \neq \lambda^*$ and that there exists at most one λ such that $r^i_{\lambda,\gamma} = R$, if $\lambda \approx \lambda^*$ and γ is large enough. The proof of Theorem **??** relies on the Sard's theorem applied to the function $\lambda \to R^i_\lambda$, and therefore our first goal is to show that this function is Lipschitz. We start with the following lemma.

Lemma 6.1. *For any $\lambda^* > 0$ and any compact $I \subset (0, \infty)$, there exists $C_{\lambda^*,I} > 0$ locally bounded in λ^* such that*

$$\|U^*_{\lambda^*} - U^*_\lambda\|_{C^2(I)} \leq C_{\lambda^*,I}|\lambda - \lambda^*|,$$

for any λ sufficiently close to λ^*.

Proof. Fix any $\lambda > 0$ and denote $V_\lambda = U^*_\lambda + \delta^*$, where $\delta^* = \ln(\lambda/\lambda^*)$. Then, V_λ satisfies

$$-V''_\lambda - \frac{N-1}{r}V'_\lambda + V_\lambda = \lambda^* e^{V_\lambda} + \delta^*$$

and by Remark **??**

$$V_\lambda(r) = -2\ln r + \ln\left(\frac{2(N-2)}{\lambda^*}\right) + O(r^2(1 - \ln r))\,.$$

Denote $W = U^*_{\lambda^*} - V_\lambda$. We see that W satisfies

$$-W'' - \frac{N-1}{r}W' + W = \lambda^*(e^{U^*_{\lambda^*}} - e^{V_\lambda}) - \delta^* = \lambda^* e^{U^*_{\lambda^*}}(1 - e^{-W}) - \delta^*$$

and after simple algebraic manipulations, we end up with

$$-W'' - \frac{N-1}{r}W' + W - \frac{2(N-2)}{r^2}W = \left(\lambda^* e^{U^*_{\lambda^*}}\frac{1 - e^{-W} - W}{W} + \lambda^* e^{U^*_{\lambda^*}} - \frac{2(N-2)}{r^2}\right)W - \delta^*\,.$$

Furthermore,

$$W(r) = O(r^2(1 - \ln r)),$$

where $O(r^{2-\delta})$ in general depends on λ and λ^*. By Remark **??** one has that $|W| \leq 1$ on $(0, c_N)$ and combined with (**??**) one has $|W| \leq 1$ on I for any λ sufficiently close to λ^*. Hence,

$$\left| \frac{1 - e^{-W} - W}{W} \right| \leq CW \qquad \text{on } I,$$

and then by Remark **??** for $\delta < 1$, we have

$$\left| \lambda^* e^{U^*_{\lambda^*}} \frac{1 - e^{-W} - W}{W} \right| \leq c_N \frac{1}{r^2} r^2 |1 - \ln r| \leq c_N (1 + |\ln r|) \qquad \text{on } I.$$

Also, by Remark **??**, we infer that

$$\left| \lambda^* e^{U^*_{\lambda^*}} - \frac{2(N-2)}{r^2} \right| = \frac{2(N-2)}{r^2} \left| e^{q(r)} - 1 \right|,$$

where $|q(r)| \leq c_N r^2 (1 + |\ln r|)$. Thus, if we set $\bar{W} = W/\delta^*$, we obtain

$$-\bar{W}'' - \frac{N-1}{r} \bar{W}' - \frac{2(N-2)}{r^2} \bar{W} = m(r) \bar{W} - 1$$

with $|m(r)| \leq c_N (1 + |\ln r|)$. Furthermore, Remark **??** implies $\bar{W}'(0) = 0$ and this condition is fulfilled continuously. Finally, denote $Z_1(r) = \bar{W}(r)$, $Z_2(r) = r\bar{W}'(r)$ and $Z = (Z_1, Z_2)$. Then

$$Z' = \frac{1}{r} JZ - rm(r) \begin{pmatrix} 0 \\ Z_1 \end{pmatrix} + \begin{pmatrix} 0 \\ r \end{pmatrix}, \qquad Z(0) = 0,$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -2(N-2) & 2-N \end{pmatrix}.$$

Since the eigenvalues μ_1, μ_2 of J have negative real parts, proceeding as in Lemma 2.3 of [**?**], one can show that Z_1 is bounded on I.

Consequently, $\bar{W}$ is bounded, and therefore $|W| \leq C|\delta^*|$ on I. Since $|\delta^*| \leq C_{\lambda^*} |\lambda - \lambda^*|$ for any λ sufficiently close to λ^*, the assertion of the lemma follows from standard regularity theory since I is bounded away from the origin, and therefore the coefficients of the equation are bounded, uniformly in $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. This also implies that $\lambda^* \mapsto C_{\lambda^*}$ is bounded locally uniformly on $(0, \infty)$. □

Lemma 6.2. *The function $\lambda \mapsto R^i_\lambda$ is Lipschitz.*

Proof. Fix $\lambda^* > 0$ and $i \in \mathbb{N}^+$. Without loss of generality, we assume that $R^i_{\lambda^*}$ is a local minimizer of $U^*_{\lambda^*}$. Thus, from the equation in (**??**), we infer that $(U^*_{\lambda^*})''(R^i_{\lambda^*}) = M > 0$. By continuity, there is $\varepsilon_0 > 0$ such that

$$(U^*_{\lambda^*})''(r) \geq \frac{M}{2} \qquad r \in I_0 := [R^i_{\lambda^*} - \varepsilon_0, R^i_{\lambda^*} + \varepsilon_0]\,.$$

Now for fixed $\lambda > 0$ sufficiently close to λ^*, we estimate R^i_λ. Without loss of generality assume $(U^*_\lambda)'(R^i_{\lambda^*}) \leq 0$, the other case is analogous. For any $r \in I_0^+ := [R^i_{\lambda^*}, R^i_{\lambda^*} + \varepsilon_0]$, the mean value theorem implies

$$(U^*_\lambda)'(r) - (U^*_\lambda)'(R^i_{\lambda^*}) = (U^*_\lambda)''(\zeta)(r - R^i_{\lambda^*}) = (U^*_{\lambda^*})''(\zeta)(r - R^i_{\lambda^*}) + (U^*_\lambda - U^*_{\lambda^*})''(\zeta)(r - R^i_{\lambda^*})\,.$$

for some $\zeta \in I_0$ depending on λ. Setting $r = R^i_{\lambda^*} + \varepsilon_0$ in (**??**), one has by Lemma **??** combined with (**??**) and the fact that $(U^*_{\lambda^*})'(R^i_{\lambda^*}) = 0$ that

$$(U^*_\lambda)'(R^i_{\lambda^*} + \varepsilon_0) \geq \frac{M}{2}\varepsilon_0 + (U^*_\lambda - U^*_{\lambda^*})''(\zeta)\varepsilon_0 \geq \frac{M}{2}\varepsilon_0 - C_N|\lambda - \lambda^*|\,.$$

Consequently, for λ sufficiently close to λ^*, we deduce that $(U^*_\lambda)'(R^i_{\lambda^*}) \leq 0 \leq (U^*_\lambda)'(R^i_{\lambda^*} + \varepsilon_0)$ and by the intermediate value theorem, there is j such that $R^j_\lambda \in [R^i_{\lambda^*}, R^i_{\lambda^*} + \varepsilon_0]$. Since critical points of $U^*_{\lambda^*}$ are non-degenerate, it is standard to see that $j = i$. Therefore, by setting $r = R^i_\lambda$ in (**??**), one obtains

$$0 \leq \frac{M}{2}(R^i_\lambda - R^i_{\lambda^*}) \leq (U^*_{\lambda^*})''(\zeta)(R^i_\lambda - R^i_{\lambda^*}) \leq |(U^*_\lambda - U^*_{\lambda^*})'(R^i_{\lambda^*})| + |(U^*_\lambda - U^*_{\lambda^*})''(\zeta)(r - R^i_{\lambda^*})| \leq C_{\lambda^*}|\lambda - \lambda^*|,$$

where we used the fact that $(U^*_{\lambda^*})'(R^i_{\lambda^*}) = (U^*_\lambda)'(R^i_\lambda) = 0$ in the last inequality. This concludes the proof. □

We are now in position to prove Theorem **??**.

Proof of Theorem **??**. Fix $i \in \mathbb{N}^+$. Since by Lemma **??** the function $\lambda \mapsto F_i(\lambda) := R^i_\lambda$ is Lipschitz, Rademacher's theorem implies that F_i is differentiable for any $\lambda \in (0, \infty) \setminus S_i$, where S_i has Lebesgue measure zero. Denote by $E_i = \{\lambda \in (0, \infty) \setminus S_i : F_i'(\lambda) = 0\}$. Then, by Sard's theorem for Lipschitz

functions (see [**?**]), one has that $F_i(E_i)$ is a set of measure zero. Moreover, since F_i is Lipschitz with locally uniformly bounded Lipschitz constant, one has that $F_i(S_i)$ has also measure zero.

Overall, $S_i^* := F_i(S_i \cup E_i)$ is a set of zero measure, and therefore $S^* = \bigcup_i S_i^*$ has measure zero as well. Thus for any radius $R \in (0,\infty) \setminus S^*$, any i and any λ^* such that $R^i_{\lambda^*} = R$, the function $\lambda \mapsto R^i_\lambda$ is differentiable at λ^* with nonzero derivative. We claim that for any λ sufficiently close to λ^* one has $(U^*_\lambda)'(R) \neq 0$. Indeed, otherwise there is a sequence $\lambda_n \to \lambda^*$ such that $(U^*_{\lambda_n})'(R) = 0$, or equivalently, $R^{i_n}_{\lambda_n} = R$ for any $n \geq 1$. Since critical points of $U^*_{\lambda^*}$ are non-degenerate, $i_n = i$ for any sufficiently large n. Then, by the definition of the derivative $F_i'(\lambda^*) = \partial_\lambda R^i_{\lambda^*} = 0$, a contradiction. □

Next, we turn to the proof of Theorem **??**. The main ingredient is the proof of the fact that for some compact interval $I \subset (0,\infty)$, the function $\lambda \to r^i_{\lambda,\gamma}$ is bounded in $C^2(I)$ by a constant that does not depend on γ. To show this, we will prove that $\partial_\lambda u(\cdot,\gamma,\lambda)$ and $\partial^2_\lambda u(\cdot,\gamma,\lambda)$ are uniformly bounded in γ.

Lemma 6.3. *Assume $4 \leq N \leq 9$. Moreover, suppose that $\lambda < 0.196$ if $N = 4$. For any compact interval $I \subset (0,\infty)$, the function $\lambda \mapsto u(\cdot,\gamma,\lambda)$ is a locally C^2 map from $(0,\infty)$ to $C^2(I)$, where $u(\cdot,\gamma,\lambda)$ is the solution to (**??**). Moreover, for any $\delta > 0$ and sufficiently large γ (depending on δ), there holds*

$$-\frac{2}{\lambda-\delta} \leq \partial_\lambda u(r,\lambda) < 0,$$

for any $r > 0$.

Proof. Due to the smooth dependence on data, the function $\lambda \mapsto u(\cdot,\gamma,\lambda)$ is smooth, so the main challenge is to prove uniformity in γ. Fix $\gamma > 0$ and $\lambda > 0$ and denote $w(r) = \partial_\lambda u(r,\lambda)$.

Fix $M > 0$. Then, by Remark **??** there exists $\varepsilon > 0$ such that $\lambda e^{U_s} > M$ on $(0,\varepsilon)$. By Theorem **??**, for sufficiently large γ, one has $\lambda e^{u(\varepsilon)} > M$. Also, decreasing $\varepsilon > 0$ if necessary, we can assume that $u' \leq 0$ on $(0,\varepsilon)$, for any sufficiently large γ.

By the smooth dependence on parameters, w is a smooth function satisfying the problem

$$-w'' - \frac{N-1}{r}w' + w = \lambda e^u w + e^u, \qquad w(0) = w'(0) = 0\,.$$

Define the energy

$$E(r) = \frac{(w')^2}{2} + \lambda e^u \frac{w^2}{2} - \frac{w^2}{2}.$$

A standard calculation yields

$$E'(r) = -(N-1)\frac{(w')^2}{r} + \lambda e^u u' \frac{w^2}{2} - e^u w'\,.$$

Fix $\varepsilon_0 > 0$ as in Corollary **??**. Then, for any $r \in (0, \varepsilon_0)$, one has

$$E'(r) \le u' \frac{(w')^2}{2} + \lambda e^u u' \frac{w^2}{2} - e^u w' - \delta_0 \frac{(w')^2}{r} \le u' E(r) - e^u w' - \delta_0 \frac{(w')^2}{r}\,,$$

where we used that $u' < 0$ on $(0, \varepsilon_0)$. Equivalently, we have

$$(E(r)e^{-u})' \le -w' - \delta_0 \frac{(w')^2}{r} e^{-u}.$$

So, using that $w(0) = E(0) = 0$, we obtain

$$\delta_0 \int_0^r \frac{(w')^2}{\rho} e^{-u} d\rho + e^{-u(r)} E(r) \le -w(r)\,.$$

For fixed δ, by choosing M sufficiently large, we have

$$e^{-u(r)} E(r) \ge \frac{\lambda - \delta}{2} w^2(r) + \frac{(w'(r))^2}{2} e^{-u(r)}.$$

Combining the two previous inequalities, we see that $0 \ge w \ge -\frac{2}{\lambda - \delta}$, for any $r \in (0, \varepsilon_0)$. Moreover,

$$\int_0^r \frac{(w')^2}{\rho} e^{-u} d\rho \le C \qquad |w'(r)| \le C e^{\frac{u(r)}{2}}\,.$$

Next, we extend the bound to the desired interval $I = (a, A)$ with $a > 0$. Without loss of generality assume $a \le \varepsilon$ and recall that $\varepsilon > 0$ is independent of γ. Theorem **??** implies that by choosing sufficiently large γ, one has $|u - U| < 1$ on $(a/2, A)$, where U is the singular solution corresponding to λ. In addition $|U|$

is bounded on $(a/2,\infty)$, and therefore $|u|<C$ on $(a/2,A)$ with C independent of large γ. Then since $|\bar{w}|\le C$ on $(\varepsilon/2,\varepsilon)$, we obtain that there is a point $s_0\in(a/2,a)$ possibly depending on γ such that $|w(s)|,|w'(s)|\le C$. Since w solves on $(a/2,\infty)$ an equation with coefficients bounded independently of γ, we obtain that $|\bar{w}|\le C$ on (a,A) as desired.

Furthermore, by the standard elliptic regularity theory and above bounds on coefficients, one obtains

$$\|\partial_\lambda u\|_{C^2(a,A)}=\|w\|_{C^2(a,A)}\le C_{\lambda,a,A}\,.$$

Next, we prove that the second derivative of u with respect to λ is bounded uniformly in γ. First, we obtain the estimate for small r, and then we extend it to I as above. Again, by the smooth dependence on the data, $\lambda\mapsto u(\cdot,\lambda)$ is a smooth map and therefore

$$\partial_\lambda^2 u(\cdot,\lambda)=\partial_\lambda w(\cdot,\lambda)=:Z(\cdot,\lambda)\,,$$

is a well defined smooth function. Also, Z solves

$$-Z''-\frac{N-1}{r}Z'+Z=\lambda e^u Z+2e^u w+\lambda e^u w^2,\qquad Z(0)=Z'(0)=0\,.$$

Next, let us estimate $|Z'|$ near the origin. If we define the energy

$$\hat{E}(r)=\frac{(Z')^2}{2}+\lambda e^u\frac{Z^2}{2}-\frac{Z^2}{2},$$

then, as above, we have $\hat{E}(0)=0$ and

$$\begin{aligned}\hat{E}'(r)&\le u'\lambda e^u\frac{Z^2}{2}-e^u(2w+\lambda w^2)Z'-(N-1)\frac{(Z')^2}{r}\\&\le u'\hat{E}(r)-\left(\frac{1}{\lambda}(\lambda w+1)^2-\frac{1}{\lambda^2}\right)Z'e^u-\delta_0\frac{(Z')^2}{r}\,,\end{aligned}$$

which can be rewritten as

$$(e^{-u}\hat{E})'\le-\left(\frac{1}{\lambda}(\lambda w+1)^2-\frac{1}{\lambda^2}\right)Z'-\delta_0\frac{(Z')^2}{r}e^{-u}\,.$$

Note that we cannot proceed as in the previous case, since the coefficient of Z' is not constant. Rather after integrating and using that $\hat{E}(0)=0=Z(0)$, we

obtain

$$\frac{\lambda-\delta}{2}(Z(r))^2 \le -\frac{1}{\lambda}\int_0^r (\lambda w+1)^2 Z' d\rho + \frac{1}{\lambda^2}Z(r) - \delta_0 \int_0^r \frac{(Z')^2}{\rho} e^{-u} d\rho\,.$$

Then, Young's inequality yields

$$\left|\frac{1}{\lambda}\int_0^r (\lambda w+1)^2 Z' d\rho\right| \le \delta_0 \int_0^r \frac{(Z')^2}{\rho} e^{-u} d\rho + C\int_0^r \rho(\lambda w+1)^4 e^u d\rho\,.$$

We conclude the proof by showing that the last integral on the right hand side is bounded. Observing that

$$(\lambda w+1)^4 = (\lambda w+1)(\lambda^3 w^3 + 3\lambda^2 w^2 + 3\lambda w + 1),$$

the assertion follows once we show for any $p \ge 1$

$$\left|\int_0^r \rho e^u (\lambda w+1) w^{p-1} d\rho\right| \le C\,.$$

We need more estimates on w. Since w solves (**??**)

$$\begin{aligned} -(w^p)'' - \frac{N-1}{r}(w^p)' &= p w^{p-1}\left(-w'' - \frac{N-1}{r}w'\right) - p(p-1)w^{p-2}(w')^2 \\ &= p w^{p-1}(\lambda e^u w + e^u - w) - p(p-1)w^{p-2}(w')^2\,. \end{aligned}$$

So, using the representation formula (**??**), we find

$$w^p(r) = -\frac{1}{N-2}\int_0^r \rho\left(1-\left(\frac{\rho}{r}\right)^{N-2}\right) p[(\lambda e^u w + e^u - w)w^{p-1} - (p-1)w^{p-2}(w')^2] d\rho$$

and

$$(w^p)'(r) = p w^{p-1} w' = -\frac{1}{r}\int_0^r \rho\left(\frac{\rho}{r}\right)^{N-2} p[(\lambda e^u w + e^u - w)w^{p-1} - (p-1)w^{p-2}(w')^2] d\rho\,.$$

Since $|w| \le C$ and $u \le U_s + C$ (see Remark **??**), then (**??**) yields

$$\int_0^r \rho (w')^2 w^{p-1} d\rho \le C\int_0^r \rho (w')^2 e^u e^{-u} d\rho \le C\int_0^r \frac{1}{\rho}(w')^2 e^{-u} d\rho \le C\,.$$

Then, (**??**), (**??**), (**??**), and $|w| \le C$ imply

$$\left|\int_0^r \rho\left(1-\left(\frac{\rho}{r}\right)^{N-2}\right)(\lambda e^u w + e^u - w) w^{p-1} d\rho\right| \le C,$$

$$\left|\int_0^r \rho\left(\frac{\rho}{r}\right)^{N-2}(\lambda e^u w + e^u - w) w^{p-1} d\rho\right| \le C\,,$$

and consequently

$$\left| \int_0^r \rho e^u (\lambda w + 1) w^{p-1} d\rho \right| \leq C \, .$$

This established (**??**). Overall (see (**??**) and (**??**)), we have proved that

$$\frac{\lambda}{2} (Z(r))^2 \leq C - \frac{1}{\lambda^2} Z(r)$$

which yields $|Z| \leq C$. The boundedness in $C^2(a, A)$ for $a > 0$ follows from the elliptic regularity as above. □

Lemma 6.4. *Assume $4 \leq N \leq 9$. Moreover suppose that $\lambda < 0.196$ if $N = 4$. For any sufficiently large $\gamma > 0$, the function $\lambda \mapsto r^i_{\lambda,\gamma}$ belongs to $C^2_{loc}(0, \infty)$, where $r^i_{\lambda,\gamma}$ is defined as in Theorem **??**. Furthermore, for any compact interval $I \subset (0, \infty)$, we have*

$$\| \lambda \mapsto r^i_{\lambda,\gamma} \|_{C^2(I)} \leq C_I \, ,$$

where C_I is in particular independent of γ.

Proof. Denote $w = \partial_\lambda u$, where u satisfies (**??**). Then, as in Lemma **??**, one has that w satisfies (**??**). By the definition of $r^i_{\lambda,\gamma}$, we have $u'(r^i_{\lambda,\gamma}, \lambda) = 0$ and, after differentiation with respect to λ, one obtains

$$\partial_\lambda r^i_{\lambda,\gamma} = - \frac{w'(r^i_{\lambda,\gamma}, \lambda)}{u''(r^i_{\lambda,\gamma}, \lambda)} = \frac{w'(r^i_{\lambda,\gamma}, \lambda)}{\lambda \exp(u(r^i_{\lambda,\gamma}, \lambda)) - u(r^i_{\lambda,\gamma}, \lambda)} \, ,$$

where in the last equality we used the equation for u. By Theorem **??**, for large γ one has that $r^i_{\lambda,\gamma}$ is close to R^i_λ, a critical point of U^*. Thus for large γ, Lemma **??** yields $|w'(r^i_{\lambda,\gamma}, \lambda)| \leq C$. Also, Theorem **??** implies that $u(r^i_{\lambda,\gamma}, \lambda)$ is close to $U^*(R^i_\lambda)$, and in particular the denominator of (**??**) is bounded away from 0 uniformly in large γ. A combination of these observations yields $|\partial_\lambda r^i_{\lambda,\gamma}| \leq C$.

Analogously, by using full power of Lemma **??**, we obtain the boundedness of the second derivatives, and the assertion follows. □

We are now in position to prove Theorem **??**.

*Proof of Theorem **??**.* For a contradiction, assume that there exist sequences $\gamma_n \to \infty$ and $\lambda_n, \lambda'_n \to \lambda^i$ such that $r^i_{\lambda_n,\gamma_n} = r^i_{\lambda'_n,\gamma_n} = R$. Then, by the mean

value theorem, there exists λ_n^* between λ_n and λ_n' such that $\partial_\lambda r^i_{\lambda_n^*,\gamma_n} = 0$. Since $\lambda_n^* \to \lambda^i$ and $\lambda \mapsto \partial_\lambda r^i_{\lambda,\gamma_n}$ has bounded (uniformly in γ and locally in λ) second derivative, one has that $\partial_\lambda r^i_{\lambda^i,\gamma_n} \to 0$ as $n \to \infty$.

Furthermore, by Lemma **??**, the sequence of functions $(\lambda \mapsto r^i_{\lambda_n,\gamma_n})_n$ is uniformly bounded in $C^2_{\rm loc}$, and therefore by Arzelà-Ascoli's theorem, it converges in $C^1_{\rm loc}$. In addition $(\lambda \mapsto r^i_{\lambda_n,\gamma_n})_n$ converges point-wise to $R^i_{\lambda^i}$, and therefore it converges to $R^i_{\lambda^i}$ in $C^1_{\rm loc}$.

Combining the previous observations, we obtain that $\partial_\lambda R^i_{\lambda^i} = 0$, a contradiction to the definition of the set S^*. □

7. Appendix

Recall that $\hat{u}$ defined in (**??**) satisfies

$$-\hat{u}'' - (N-1)\frac{\hat{u}'}{r} + e^{-\gamma}(\hat{u}+\gamma) = \lambda e^{\hat{u}} \qquad \text{in } (0,\infty),$$

with $\hat{u}(0) = \hat{u}'(0) = 0$. Also, $U_s = -2\ln r + \ln\frac{2(N-2)}{\lambda}$ solves

$$-\Delta U_s = \lambda e^{U_s}.$$

Define $\bar{u}$ as a solution of

$$-\Delta \bar{u} = \lambda e^{\bar{u}}, \qquad \bar{u}(0) = \bar{u}'(0) = 0$$

and, by Lemma **??**, $\hat{u} \to \bar{u}$ in $C^1_{\rm loc}[0,\infty)$ as $\gamma \to \infty$. First we obtain some preliminary estimates on $\bar{u}$.

Lemma 7.1. *The function $\bar{u}$ is decreasing and $\bar{u}(r) \geq -\frac{1}{2}\lambda r^2$. Furthermore, $\bar{u}(r) < U_s(r)$ for $r \in [0, \sqrt{2(N-2)/\lambda}]$.*

Proof. Note that $-\Delta\bar{u} > 0$, which can be rewritten as

$$-(r^{N-1}\bar{u}') > 0.$$

Integrating, we deduce that $\bar{u}' < 0$, for $r > 0$. In particular, we see that $\bar{u} \leq 0$. Then, using that $\bar{u}, \bar{u}' \leq 0$, we have

$$\bar{u}'' = -\frac{N-1}{r}\bar{u}' - \lambda e^{\bar{u}} \geq -\lambda e^{\bar{u}} \geq -\lambda\,.$$

Integrating twice the previous inequality and using the initial conditions, one obtains that $\bar{u}(r) \geq -\frac{1}{2}\lambda r^2$. Finally, the last assertion follows since $\bar{u} \leq 0$ and $U_s \geq 0$ on the desired interval. □

Remark 7.2. *Since $\hat{u} \to \bar{u}$ in $C^1_{loc}[0,\infty)$ as $\gamma \to \infty$, analogous estimates as in Lemma ?? holds for $\hat{u}$ up to an error term of order $o_\gamma(1)$. Note that $o_\gamma(1) \to 0$ as $\gamma \to \infty$.*

We show that on small intervals, we have an uniform (with respect to γ) upper bound for $\hat{u} - U_s$. Before proceeding, we formulate the following Grönwall lemma proved in [? , Theorem 15].

Theorem 7.3. *Let u be a non-negative function satisfying, for $s > s_0$,*

$$x(s) \leq c + \int_{s_0}^{s} k(s,r)x(r)\,dr,\ c > 0,$$

where $k(s,r)$ is a continuously differentiable function in s and continuous in r with $k(s,t) \geq 0$, for $s \geq r \geq s_0$. Then

$$x(s) \leq c \exp\left(\int_{s_0}^{s}\left(k(r,r) + \int_{s_0}^{r}\frac{\partial k}{\partial r}(r,t)dt\right)dr\right).$$

Thanks to this inequality, we have the following result.

Lemma 7.4. *Assume $4 \leq N \leq 9$. Let $w = \hat{u} - U_s$. Then, for any $\delta, \tilde{\delta} > 0$, there are $\varepsilon_0, \gamma_0 > 0$ such that for any $\gamma \geq \gamma_0$, and any $s \leq \varepsilon_0 e^{\gamma/2}$, one has*

$$w(s) \leq \frac{2 + (2+\tilde{\delta})\tilde{s}_0^{2-N})}{N-2} + \delta =: K,$$

where $\tilde{s}_0 = \sqrt{\frac{2(N-2)}{\lambda}}$.

Remark 7.5. *Lemma ?? immediately implies*

$$\hat{u}(s) \leq U_s(s) + K \qquad s \leq \varepsilon_0 e^{\gamma/2}\,,$$

which in original variables yields for large γ

$$u(r) \leq U_s(s) + K + \gamma = U_s(r) + K \qquad r \leq \varepsilon_0\,.$$

Proof of Lemma **??**. First observe that $w_1(s) = 1$ and $w_2(s) = s^{2-N}$ are linearly independent radial solutions of the problem $\Delta w = 0$. We define the Wronskian $W(w_1, w_2) = w_1 w_2' - w_2 w_1' = (2-N)s^{1-N}$. Moreover, if w is a radial solution of $\Delta w = f$, then, for any fixed $a \in \mathbb{R}^+$, the variation of parameters yields

$$w(s) = c_1 w_1(s) + c_2 w_2(s) - w_1(s) \int_a^s \frac{w_2(r) f(r)}{W(w_1, w_2)(r)} dr + w_2(s) \int_a^s \frac{w_1(r) f(r)}{W(w_1, w_2)(r)} dr,$$

where $c_1,\ c_2$ are constants depending on $w(a)$ and $w'(a)$. For $\hat{u}$, we have

$$\hat{u}(s) = c_1 w_1(s) + c_2 w_2(s) - \frac{1}{N-2} \int_1^s r \left(1 - \left(\frac{r}{s}\right)^{N-2}\right) (\lambda e^{\hat{u}} - e^{-\gamma}(\hat{u} + \gamma)) dr\,.$$

Since, by definition, $\hat{u}'(0) = 0$, we obtain

$$c_2 = -\frac{1}{N-2} \int_1^0 r^{N-1} (\lambda e^{\hat{u}} - e^{-\gamma}(\hat{u} + \gamma)) dr\,,$$

and then $\hat{u}(0) = 0$ implies

$$c_1 = \frac{1}{N-2} \int_1^0 r (\lambda e^{\hat{u}} - e^{-\gamma}(\hat{u} + \gamma)) dr\,.$$

By Remark **??**, one has $|\hat{u}| \leq o_\gamma(1)$

$$o_\gamma(1) \geq c_1 \geq -\frac{\lambda}{2(N-2)} + o_\gamma(1) \qquad o_\gamma(1) \leq c_2 \leq \frac{\lambda}{N(N-2)}$$

and in particular $c_1,\ c_2$ are bounded, independently of large $\gamma > 0$. On the other hand, since $\lambda e^{U_s(r)} = 2(N-2)/r^2$, a direct calculation yields

$$U_s(s) = \left(\ln \frac{2(N-2)}{\lambda} - \frac{2}{N-2}\right) w_1(s) + \frac{2 w_2(s)}{N-2} - \frac{1}{N-2} \int_1^s r \left(1 - \left(\frac{r}{s}\right)^{N-2}\right) \lambda e^{U_s} dr\,.$$

By definition of $w = \hat{u} - U_s$, we infer that

$$\begin{aligned} w(s) = &\left(c_1 + \frac{2}{N-2} - \ln \frac{2(N-2)}{\lambda}\right) + \left(c_2 - \frac{2}{N-2}\right) w_2(s) \\ &- \frac{1}{N-2} \int_1^s r \left(1 - \left(\frac{r}{s}\right)^{N-2}\right) (\lambda(e^{\hat{u}} - e^{U_s}) - e^{-\gamma}(\hat{u} + \gamma)) dr\,. \end{aligned}$$

Using again that $\lambda e^{U_s(r)} = 2(N-2)/r^2$, we obtain

$$\lambda(e^{\hat{u}} - e^{U_s}) = \frac{2(N-2)}{r^2}(e^w - 1)\,.$$

By Remark **??**, $w(s) < 0$ on $(0,1]$ for any large γ. We set $s_0 \in (1, \varepsilon e^{\gamma/2})$ to be the smallest real number such that $w(s_0) = 0$. If no such s_0 exists, then the assertion follows. Thus, on the interval $(0, s_0)$, one has

$$-1 \le e^w - 1 \le 0\,.$$

Moreover, by definition, observe that $-\gamma \le \hat{u} \le 0$. Thus, for $s \in [1, s_0]$, the integrand in (**??**) is negative and therefore we have for $s \in [1, s_0]$,

$$w(s) \ge \left(c_1 + \frac{2}{N-2} - \ln \frac{2(N-2)}{\lambda}\right) + \left(c_2 - \frac{2}{N-2}\right) w_2(s) \ge C\,,$$

where C is a constant independent of γ. Since $w(s) \le 0$, for $0 \le s \le s_0$, and $w(s_0) = 0$, one has $w'(s_0) \ge 0$. On the other hand, differentiating (**??**), we obtain

$$\begin{aligned} &w'(s) \\ &= \frac{2-N}{s}\left[\left(c_2 - \frac{2}{N-2}\right) w_2(s) + \frac{1}{N-2}\int_1^s r\left(\frac{r}{s}\right)^{N-2}\left(\frac{2(N-2)}{r^2}(e^w - 1) - e^{-\gamma}(\hat{u} + \gamma)\right) dr\right]. \end{aligned}$$

Since $w \le 0$ is bounded on $[1, s_0]$ and $s_0 \le \varepsilon e^{\frac{\gamma}{2}}$, we have for any $r \in [1, s_0]$,

$$|\hat{u}(r) + \gamma| = \left|w + \gamma - 2\ln r + \ln \frac{2(N-2)}{\lambda}\right| \le C + \gamma - 2\ln r,$$

and

$$|e^w - 1| \le 1\,.$$

Thus, for any $s \in [1, s_0]$, we obtain that

$$w'(s) \le C_\lambda \frac{w_2(s)}{s} + \frac{1}{s^{N-1}}\int_1^s r^{N-1}\left(\frac{2(N-2)}{r^2} + e^{-\gamma}(C + \gamma - 2\ln r)\right) dr\,,$$

where by (**??**)

$$C_\lambda = 2 - (N-2)c_2 \le 2 + o_\gamma(1)\,.$$

Recalling that, for $m \ne -1$,

$$\int s^m \ln s\, ds = s^{m+1}\frac{(m+1)\ln s - 1}{(m+1)^2} + C,$$

we obtain for $m = N-1$, that

$$\int_1^s r^{N-1}(C+\gamma-2\ln r)\,dr \le \left(-2s^N\frac{N\ln s-1}{N^2}+\frac{s^N}{N}(C+\gamma)\right)$$
$$= s^{N-1}\left(-2s\frac{N\ln s-1}{N^2}+\frac{s}{N}(C+\gamma)\right),$$

where we dropped the lower limit term since it is negative. Since $s_0 < \varepsilon e^{\gamma/2}$, for small $\varepsilon > 0$ the expression in the parentheses on the right had side of (**??**) is increasing on $(0, \varepsilon e^{\frac{\gamma}{2}})$, we replace s by $\varepsilon e^{\frac{\gamma}{2}}$ and obtain

$$\int_1^s r^{N-1}(C+\gamma-2\ln r)\,dr \le s^{N-1}\varepsilon e^{\frac{\gamma}{2}}\left(-2\frac{N\ln\varepsilon-1}{N^2}+\frac{C}{N}\right).$$

Therefore, we have for small $\varepsilon > 0$ independent of γ

$$\frac{1}{s^{N-1}}\int_1^s r^{N-1}\left(\frac{2(N-2)}{r^2}+e^{-\gamma}(C+\gamma-2\ln r)\right)dr \le \left(\frac{2(N-2)}{(N-2)s}+e^{-\gamma/2}\varepsilon|\ln\varepsilon|\right) \le \frac{1}{s}\left(2+\varepsilon^2|\ln\varepsilon|\right).$$

Thus, combining (**??**) and (**??**)

$$w'(s) \le \frac{C_\lambda}{s^{N-1}}+\frac{1}{s}\left(2+\varepsilon^2|\ln\varepsilon|\right) \le \frac{2+\varepsilon+C_\lambda s^{2-N}}{s}.$$

Next, we rewrite the representation formula (**??**) for w with different starting point. Namely, we have

$$w(s) = \bar{c}_1+\bar{c}_2 w_2(s)-\frac{1}{N-2}\int_{s_0}^s r\left(1-\left(\frac{r}{s}\right)^{N-2}\right)(\lambda(e^{\hat{u}}-e^{U_s})-e^{-\gamma}(\hat{u}+\gamma))dr,$$

for some constants $\bar{c}_1,\ \bar{c}_2$. We have $w(s_0) = 0$ and by Remark **??** for sufficiently large γ one has $s_0 \ge \sqrt{2(N-2)/\lambda} = \tilde{s}_0$. Hence, (**??**) and (**??**) yields

$$0 \le w'(s_0) \le \frac{1}{s_0}\left(2+\varepsilon+C_\lambda s_0^{2-N}\right) \le \frac{1}{s_0}\left[2+\varepsilon+C_\lambda\tilde{s}_0^{2-N}\right]$$

Then, by using (**??**) and similar expression for w' as well as its derivative, we obtain

$$\bar{c}_1+\bar{c}_2 s_0^{2-N} = 0, \qquad 0 \le \bar{c}_2(2-N)s_0^{1-N} \le \frac{2+\varepsilon+C_\lambda\tilde{s}_0^{2-N}}{s_0}.$$

Consequently,

$$0 \ge \bar{c}_2 w_2(s_0) = \bar{c}_2 s_0^{2-N} \ge -\frac{2+\varepsilon+C_\lambda\tilde{s}_0^{2-N}}{N-2} \quad \text{and} \quad 0 \le c_1 \le \frac{2+\varepsilon+C_\lambda\tilde{s}_0^{2-N}}{N-2}.$$

Denote $s_1 \le \varepsilon e^{\gamma/2}$ the smallest number larger than s_0 such that $w(s_1) = 0$. If such number does not exist, then we set $s_1 = \varepsilon e^{\gamma/2}$. So, for $s \in [s_0, s_1]$, we have $e^{w(s)} - 1 \ge 0$, and therefore

$$w(s) \le \frac{2+\varepsilon+C_\lambda \tilde{s}_0^{2-N}}{N-2} - \frac{1}{N-2}\int_{s_0}^{s} r\left(1-\left(\frac{r}{s}\right)^{N-2}\right)\left(\frac{2(N-2)}{r^2}(e^w-1) - e^{-\gamma}(\hat{u}+\gamma)\right)dr$$

$$\le \frac{2+\varepsilon+C_\lambda \tilde{s}_0^{2-N}}{N-2} + \frac{1}{N-2}\int_{s_0}^{s} r\left(1-\left(\frac{r}{s}\right)^{N-2}\right) e^{-\gamma}(\hat{u}+\gamma)dr$$
$$= \frac{2+\varepsilon+C_\lambda \tilde{s}_0^{2-N}}{N-2} + \frac{1}{N-2}\int_{s_0}^{s} r\left(1-\left(\frac{r}{s}\right)^{N-2}\right) e^{-\gamma}(w + C - 2\ln r + \gamma)dr.$$

Using (**??**) with $N = 2$ and $s_1 \le \varepsilon e^{\gamma/2}$ we have for small $\varepsilon > 0$

$$\int_{s_0}^{s} r\left(1-\left(\frac{r}{s}\right)^{N-2}\right) e^{-\gamma}(C-2\ln r+\gamma)dr \le \int_{s_0}^{s} re^{-\gamma}(C-2\ln r+\gamma)dr \le C\varepsilon e^{-\frac{\gamma}{2}} s|\ln\varepsilon| \le C\varepsilon^2|\ln\varepsilon|,$$

for some constant C not depending on γ. Therefore,

$$w(s) \le \frac{2+\tilde{\varepsilon}+C_\lambda \tilde{s}_0^{2-N}}{N-2} + \frac{1}{N-2}\int_{s_0}^{s} r\left(1-\left(\frac{r}{s}\right)^{N-2}\right) e^{-\gamma} w dr\,.$$

Then, by Theorem **??**, we obtain, for any $s \in (s_0, s_1)$,

$$\begin{aligned} w(s) &\le \frac{2+\tilde{\varepsilon}+C_\lambda \tilde{s}_0^{2-N}}{N-2} \exp\left(e^{-\gamma}\int_{s_0}^{s}\int_{s_0}^{r}\left(\frac{t}{r}\right)^{N-1} dtdr\right) \\ &\le \frac{2+\tilde{\varepsilon}+C_\lambda \tilde{s}_0^{2-N}}{N-2} \exp\left(Ce^{-\gamma}s^2\right) \\ &\le \frac{2+\tilde{\varepsilon}+C_\lambda \tilde{s}_0^{2-N}}{N-2} + o_\gamma(1)\,, \end{aligned}$$

where in the last inequality we used that $s \le s_1 \le \varepsilon e^{\gamma/2}$.

Finally, let us treat the interval $(s_1, \varepsilon e^{\frac{\gamma}{2}})$ if $s_1 < \varepsilon e^{\frac{\gamma}{2}}$. Note, that we cannot merely reiterate the approach above, since we used that $w < 0$ on $(0, s_0)$ to obtain the estimate for $w'(s_0)$. If $s_1 < \varepsilon e^{\frac{\gamma}{2}}$ denote $s^* \in (s_0, s_1)$ the first local maximum of w and the above arguments yield $w(s^*) \le \frac{2+\tilde{\varepsilon}+C_\lambda \tilde{s}_0^{2-N}}{(N-2)} + o_\gamma(1)$. Using $\lambda e^{U_s} = 2(N-2)/r^2$, we obtain that w satisfies

$$-w''-(N-1)\frac{w'}{r} = \lambda(e^{\hat{u}}-e^{U_S})-e^{-\gamma}(\hat{u}+\gamma) = \frac{2(N-2)}{r^2}(e^w-1)-e^{-\gamma}(w+\gamma+C_\lambda-2\ln r)\,.$$

Define the energy

$$E(r) = \frac{(w')^2}{2} + \frac{2(N-2)(e^w - w)}{r^2} - e^{-\gamma}\frac{w^2}{2}$$

and observe that

$$E'(r) = -(N-1)\frac{(w')^2}{r} - \frac{4(N-2)(e^w - w)}{r^3} + e^{-\gamma}(\gamma + C_\lambda - 2\ln r)w' \,.$$

Then, Young's inequality implies for $N > 3$

$$e^{-\gamma}(\gamma - C_\lambda - 2\ln r)|w'| \le (N-3)\frac{(w')^2}{r} + Cre^{-2\gamma}(\gamma + C_\lambda - 2\ln r)^2 \,,$$

and consequently, since $N \ge 4$,

$$E'(r) \le -2\frac{(w')^2}{r} - \frac{4(N-2)(e^w - w)}{r^3} + Cre^{-2\gamma}(\gamma + C_\lambda - 2\ln r)^2 \le -\frac{2}{r}E(r) + Cre^{-2\gamma}(\gamma + C_\lambda - 2\ln r)^2 \,.$$

Hence,

$$(r^2 E(r))' \le Cr^3 e^{-2\gamma}(\gamma + C_\lambda - 2\ln r)^2 \,.$$

Using the substitution $p = te^{-\frac{\gamma}{2}}$ and $r \le \varepsilon e^{-\frac{\gamma}{2}}$, we obtain

$$\int_s^r t^3 e^{-2\gamma}(\gamma + C_\lambda - 2\ln t)^2 dt = \int_{se^{-\frac{\gamma}{2}}}^{re^{-\frac{\gamma}{2}}} p^3(\gamma + C_\lambda - 2\ln(e^{\frac{\gamma}{2}}p))^2 dp \le \int_0^\varepsilon p^3(C_\lambda - 2\ln p)^2 dp = O(\varepsilon) \,.$$

Finally, we have, for $r > s^*$,

$$\begin{aligned} 2(N-2)(e^{w(r)} - w(r)) - \frac{e^{-\gamma}r^2}{2}w^2(r) &\le r^2 E(r) \\ &\le (s^*)^2 E(s^*) + O(\varepsilon) \le 2(N-2)(e^{w(s^*)} - w(s^*)) - \frac{e^{-\gamma}(s^*)^2}{2}w^2(s^*) + O(\varepsilon) \,, \end{aligned}$$

and therefore, for $s^* < r \le \varepsilon e^{-\frac{\gamma}{2}}$,

$$e^{w(r)} - w(r) - \frac{\varepsilon^2}{2}w^2(r) \le e^{w(s^*)} - w(s^*) + O(\varepsilon) \,.$$

If $\varepsilon = 0$, the monotonicity of function $x \mapsto e^x - x$ on $(0, \infty)$ yields that $w(r) \le w(s^*) \le \frac{2 + C_\lambda \tilde{s}_0^{2-N}}{N-2} + O(\varepsilon)$. Finally, by continuity and monotonicity, we have that, for small $\varepsilon > 0$, $w(r) \le w(s^*) \le \frac{2 + C_\lambda \tilde{s}_0^{2-N}}{N-2} + O(\varepsilon)$ as desired. □

Lemma 7.6. *Assume $4 \le N \le 9$. Let $w = \hat{u} - U_s$ and $\tilde{s}_0 = \sqrt{\frac{2(N-2)}{\lambda}}$. For any $\delta, \tilde{\delta} > 0$, there exists $\gamma_0, \varepsilon_0 > 0$ such that, for any $\gamma \ge \gamma_0$ and any $s \le \varepsilon_0 e^{\gamma/2}$, there holds*

$$w'(s) \ge -\frac{2(e^K - 1)}{s},$$

where K is defined in Lemma **??**.

Proof. Using (**??**) and the fact that $\hat{u} \le \gamma$, we have

$$w'(s) \ge \frac{2-N}{s}\left[\left(c_2 - \frac{2}{N-2}\right) w_2(s) + \frac{1}{N-2}\int_1^s r\left(\frac{r}{s}\right)^{N-2} \frac{2(N-2)}{r^2}(e^w - 1)dr\right].$$

By Lemma **??**

$$w'(s) \ge \frac{2-N}{s}\left[\left(c_2 - \frac{2}{N-2}\right) w_2(s) + \frac{1}{N-2}\int_1^s r\left(\frac{r}{s}\right)^{N-2} \frac{2(N-2)}{r^2}(e^K - 1)dr\right]$$

$$\ge \frac{2-N}{s}\left[\left(c_2 - \frac{2}{N-2}\right) w_2(s) + \frac{2}{N-2}\left(e^K - 1\right)\right].$$

By (**??**) and $\lambda < 1/e$

$$c_2 - \frac{2}{N-2} < \frac{\lambda}{N(N-2)} - \frac{2}{N-2} < 0$$

and the assertion follows. □

Corollary 7.7. *Assume $4 \le N \le 9$. Moreover, suppose that $\lambda < 0.196$ if $N = 4$. For any $\delta > 0$, there exists $\gamma_0, \varepsilon_0 > 0$ such that, for any $\gamma \ge \gamma_0$ and any $r \le \varepsilon_0$, there holds*

$$u'(r) \ge -\frac{2(N-1) + \delta}{r}.$$

Proof. We set $w = \hat{u} - U_s$. Choose $\varepsilon_0 > 0$ as in Lemma **??** (and recall the definition of K there) and observe that, for $s \le \varepsilon_0 e^{\frac{\gamma}{2}}$,

$$\hat{u}'(s) = U_s'(s) + w'(s) \ge -\frac{2}{s} - \frac{2(e^K - 1)}{s} = -\frac{2e^K}{s}.$$

Furthermore, by the definition of $\hat{u}$ (cf. (**??**) and note that $s = e^{\frac{\gamma}{2}} r$)

$$\frac{d}{dr}u(r) = \frac{d}{ds}\hat{u}(s)\frac{ds}{dr} = \frac{d}{ds}\hat{u}(s)e^{\frac{\gamma}{2}} \ge -\frac{2e^K}{s}e^{\frac{\gamma}{2}} = -\frac{2e^K}{r}.$$

We finish the proof, once we show that $e^{K_0} \geq (N-1)$, where $K_0 = \frac{2+2\tilde{s}_0^{2-N}}{N-2}$ (compare with the definition of K). Solving for λ, yields

$$\lambda \leq 2(N-2)\left(\frac{\ln(N-1) - \frac{2}{N-2}}{2}\right)^{2/(N-2)}.$$

which is indeed true since the right hand side is bigger than $1/e$ if $N \geq 5$ and bigger than 0.197 if $N = 4$. □

Remark 7.8. *Numerical simulations suggest that the value of $\tilde{s}_0$ (namely the small positive real number such that $w(\tilde{s}_0) = 0$) can be improved in order for the previous corollary to hold true for any $\lambda < 1/e$ when $N = 4$.*

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