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On the Born–Infeld equation for electrostatic fields with a superposition of point charges

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Abstract

In this paper, we study the static Born–Infeld equation

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)=\sum_{k=1}^{n} a_k\delta_{x_k} \quad \text{in } \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x)=0,$$

where $N \geq 3$, $a_k \in \mathbb{R}$ for all $k = 1, \dots, n$, $x_k \in \mathbb{R}^N$ are the positions of the point charges, possibly non-symmetrically distributed, and δ_{x_k} is the Dirac delta distribution centered at x_k. For this problem, we give explicit quantitative sufficient conditions on a_k and x_k to guarantee that the minimizer of the energy functional associated with the problem solves the associated Euler–Lagrange equation. Furthermore, we provide a more rigorous proof of some previous results on the nature of the singularities of the minimizer at the points x_k's depending on the sign of charges a_k's. For every $m \in \mathbb{N}$, we also consider the approximated problem

$$-\sum_{h=1}^{m} \alpha_h \Delta_{2h} u=\sum_{k=1}^{n} a_k\delta_{x_k} \quad \text{in } \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x)=0$$

where the differential operator is replaced by its Taylor expansion of order $2m$ (see (2.1)). It is known that each of these problems has a unique solution. We study the regularity of the approximating solution, the nature of its singularities, and the asymptotic behavior of the solution and of its gradient near the singularities.

Keywords Born–Infeld equation · Nonlinear electromagnetism · Mean curvature operator in the Lorentz–Minkowski space · Inhomogeneous quasilinear equation

Mathematics Subject Classification 35B40 · 35B65 · 35J62 · 35Q60 · 78A30

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1 Introduction

The classical electrostatic Maxwell equations in the vacuum lead to the following relations for the electric field:

$$\mathbf{E} = -\nabla u, \qquad -\Delta u = \varrho, \tag{1.1}$$

where ϱ is the charge density, u the electric potential, and $\mathbf{E}$ the electric field. However, in physically relevant cases when ρ is only an L^1-function, or in the case of point charges, the model violates the principle of finiteness of the energy; see [13,14] for a counterexample. In [6], Born and Infeld proposed a nonlinear theory of electromagnetism by modifying Maxwell's equation mimicking Einstein's special relativity. They introduced a parameter $b \gg 1$, whose inverse is proportional to the radius of the electron, and replaced the Maxwellian Lagrangian density $\mathcal{L}_M := \frac{1}{2}|\mathbf{E}|^2$ by

$$\mathcal{L}_{BI} := b^2\left(1 - \sqrt{1 - \frac{|\mathbf{E}|^2}{b^2}}\right) \quad \text{for } |\mathbf{E}| \leq b,$$

so that $\mathcal{L}_M$ is a first-order approximation of $\mathcal{L}_{BI}$ as $|\mathbf{E}|/b \to 0$. In the presence of a charge density ϱ, this new Lagrangian leads, at least formally, to replace Poisson's equation in (1.1) by the nonlinear equation

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2/b^2}}\right) = \varrho,$$

which agrees with the finiteness of the energy even when ϱ is a point charge or an L^1-density. After scaling u/b and ϱ/b, we get

$$-Qu := -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right) = \varrho. \tag{1.2}$$

It is interesting to notice that the nonlinear operator in (1.2) has also a geometric interpretation (see [3,12]). Indeed, Q is the so-called mean curvature operator in the Lorentz–Minkowski space and (1.2) can be seen as the equation for hypersurfaces in Minkowski space with prescribed mean curvature ρ. In particular, when ϱ is a superposition of point charges, (1.2) is the equation for area maximizing hypersurfaces in Minkowski space having isolated singularities (cf. [12]). Since the density ρ is not smooth, we look for weak solutions in the space

$$\mathcal{X} := \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \cap \left\{u \in C^{0,1}\left(\mathbb{R}^N\right) \,:\, \|\nabla u\|_\infty \leq 1\right\} \tag{1.3}$$

endowed with the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^{1/2}.$$

Definition 1.1 A *weak solution* of (1.2) coupled with the decay condition

$$\lim_{|x|\to\infty} u(x) = 0$$

is a function $u \in \mathcal{X}$ such that

$$\int_{\mathbb{R}^N} \frac{\nabla u \cdot \nabla v}{\sqrt{1 - |\nabla u|^2}} \mathrm{d}x = \langle \varrho, v \rangle \quad \text{for all } v \in \mathcal{X}.$$

We recall that $\mathcal{D}^{1,2}(\mathbb{R}^N) := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|}$, that is, $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the closure of the space of smooth compactly supported functions with respect to the norm $\|\cdot\|$. Mathematically, (1.2) has a variational structure, since it can be (at least formally) seen as the Euler–Lagrange equation of the energy functional $I_\varrho : \mathcal{X} \to \mathbb{R}$ defined by

$$I_\varrho(u) := \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla u|^2}\right) \mathrm{d}x - \langle \varrho, u \rangle \quad \text{for all } u \in \mathcal{X}. \tag{1.4}$$

We also denote the dual space of $\mathcal{X}$ by $\mathcal{X}^*$ with respect to $L^2(\mathbb{R}^N)$ inner product, and we write $\langle \cdot, \cdot \rangle$ for the dual pairing between $\mathcal{X}^*$ and $\mathcal{X}$. It is known that I_ϱ has a unique minimizer u_ϱ for all $\varrho \in \mathcal{X}^*$ (cf. [4] and Sect. 2). However, due to the lack of regularity of I_ϱ on functions u such that $|\nabla u(x)| = 1$ for some points $x \in \mathbb{R}^N$, the justification that minimizers of (1.4) are also weak solutions of (1.2) presents many difficulties, which will be partly addressed in the present paper. We remark that some variational problems with a gradient constraint present similar difficulties (see, e.g., [7–9,24]). In those papers, the main idea is to remove the constraint on the gradient by defining an appropriate obstacle problem. We believe that some ideas from those papers could be useful in our context, but we do not push further those ideas here.

To address the lack of smoothness, Bonheure et al. [4] used classical methods from non-smooth analysis and weakened the definition of critical point of I_ϱ, using the notion critical points in the *weak sense* (see [23]). Also, they proved the existence and uniqueness of a critical point of I_ϱ in the weak sense and showed that the PDE is weakly satisfied in the sense of Definition 1.1 for radially symmetric or locally bounded ϱ's.

Fortunato et al. [14] studied (1.2) in $\mathbb{R}^3$ and its second-order approximation (by taking the Taylor expansion of the Lagrangian density). In the same spirit, in [4,17] the authors performed higher-order expansions of the Lagrangian density, so that, in the limit, the operator Q can be formally seen as the series of $2h$-Laplacians

$$-Qu = -\sum_{h=1}^{\infty} \alpha_h \Delta_{2h} u, \tag{1.5}$$

where we refer to Sect. 2 for the precise expression of the coefficients and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. This expansion allows to approximate Q with the operators sum

$$-\sum_{h=1}^{m} \alpha_h \Delta_{2h} \tag{1.6}$$

and (1.2) with the quasilinear equations

$$-\sum_{h=1}^{m} \alpha_h \Delta_{2h} \phi = \varrho \qquad \text{for } m \in \mathbb{N}.$$

Each of such equations, complemented with the condition $\lim_{|x|\to\infty} u(x) = 0$, has a unique solution u_m. In [17], respectively [4], it is further proved that the approximating solutions u_m's weakly converge to the minimizer u_ϱ of (1.2) when ρ is a superposition of point charges, respectively for any $\varrho \in \mathcal{X}^*$.

It is worth noting that $\mathcal{X}^*$ contains Radon measures, and in particular, superpositions of point charges and L^1-densities, which are in turn dense in the space of Radon measures. Due to these reasons, we will assume that ρ is a finite superposition of charges without any symmetry conditions, that is, we consider

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \displaystyle\sum_{k=1}^{n} a_k \delta_{x_k} \quad \text{in } \mathbb{R}^N, \\ \displaystyle\lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.7}$$

where $N \geq 3$ and δ_{x_k} is the Dirac delta function centered at x_k, $a_k \in \mathbb{R}$ and $x_k \in \mathbb{R}^N$ for $k = 1, \ldots, n$. This situation is general enough to cover most of the phenomena, yet simple enough that it can be analyzed explicitly. The energy functional associated with (1.7) has the form

$$I(u) = \int_{\mathbb{R}^N} \left(1 - \sqrt{1-|\nabla u|^2}\right) \mathrm{d}x - \sum_{k=1}^{n} a_k u(x_k) \quad \text{for all } u \in \mathcal{X}. \tag{1.8}$$

Problem (1.7) has been first studied in [4,17], see also Sect. 2, where we report some recalls.

Our first goal is to provide a rigorous proof concerning the nature of the singularities x_k's for the minimizer u_ϱ of I, depending on the sign of the charges a_k's (see [17] and Theorem 3.5). More precisely, in Theorem 3.5, we show that if the charge a_k is positive (resp. negative), then the point charge x_k is a relative strict maximizer (resp. minimizer) for u_ϱ. Our proof uses geometric results proved by Ecker [12], and the comparison principle in bounded domains proved in Lemma 3.4. This result is far from obvious, since u_ϱ is globally bounded, and in particular, it does not diverge at x_k; rather, ∇u_ϱ is discontinuous at the location of the charges. Of course since the problem is not linear, it cannot be decomposed into several problems, each with just one point charge. However, this is not the only obstacle; if one replaces our curvature operator with Laplacian in one dimension, then the Green's function for the charge located at x_k has the form $|x - x_k|$, and in particular, it is bounded in the neighborhood of x_k. But, adding several Green's functions one obtains that the solution is a piecewise linear function, which might not have local extrema at x_k. Although the singularity is of the same nature as one for Laplacian in one dimension, it is crucial that the solution vanishes at infinity, which introduces a non-local argument into the proofs.

We immediately show an application of these results in the question whether the minimizer u_ϱ of (1.8) is a weak solution of (1.7). To our best knowledge, this problem has not been completely solved yet. Some results in this direction can be found in [17], but the main arguments in that paper need to be adjusted (see Discussion in [4, Sect. 4]). As far as we know, the case of a generic ϱ is still open. In [4,17], the authors proved that u_ϱ solves the equation in (1.7), in $\mathbb{R}^N \setminus \Gamma$, where $\Gamma := \bigcup_{k\neq j} \overline{x_k x_j}$ and $\overline{x_k x_j}$ denotes the line segment with endpoints x_k and x_j. Furthermore, it is proved in [4] that if the charges are sufficiently small or far apart, u_ϱ solves the equation in $\mathbb{R}^N \setminus \{x_1, \ldots, x_n\}$. In particular, in [17] it is showed that if two point charges x_k, x_j have the same sign $a_k \cdot a_j > 0$, then u_ϱ solves the equation also along the open line segment $\mathrm{Int}(\overline{x_k x_j})$.

The arguments on the literature are based on the fact that if the minimizer does not satisfy the equation along the segment connecting x_k and x_j, then it must be affine, and since the minimizer is bounded, then one obtains a contradiction. However, the argument is purely qualitative and it does not yield an easily verifiable condition based only on the location and strength of the charges. In this paper, we partly bridge this gap by proving a sufficient *quantitative* condition on the charges and on their mutual distance to guarantee that the minimizer u_ϱ solves (1.7) also along the line segments joining two charges of different signs. Let us denote $\mathcal{K}_+ := \{k : a_k > 0\}$ and $\mathcal{K}_- := \{k : a_k < 0\}$, that is, set of indexes for positive respectively negative charges. Our result reads then as follows.

Theorem 1.2 *If*

$$\left(\frac{N}{\omega_{N-1}}\right)^{\frac{1}{N-1}} \frac{N-1}{N-2}\left[\left(\sum_{k\in\mathcal{K}_+} a_k\right)^{\frac{1}{N-1}} + \left(\sum_{k\in\mathcal{K}_-} |a_k|\right)^{\frac{1}{N-1}}\right] < \min_{\substack{j,l\in\{1,\dots,n\}\\ j\neq l}} |x_j - x_l|, \tag{1.9}$$

where ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$, then

$$u_\varrho \in C^\infty(\mathbb{R}^N \setminus \{x_1,\dots,x_n\}) \cap C(\mathbb{R}^N)$$

and it is a classical solution of (3.4) *in $\mathbb{R}^N \setminus \{x_1,\dots,x_n\}$, with $|\nabla u_\varrho| < 1$.*

Note that the occurrence of the sum of positive and negative charges is natural, since we cannot rule out the situation when these charges are close to each other and they appear as one point charge. The explicit form of the constant on the left-hand side of (1.9) is crucial and observe that is bounded from below independently of N and the number of charges. This allows for passing to the limit in the number of charges, and the formulation of the result is left to the interested reader. We also give in Remark 3.9 a more precise way (although less explicit) to calculate the constant on the left-hand side of (1.9) in the general case, and a yet more optimal one if there are only *two* point charges of different signs in Proposition 3.10.

The proof of this theorem is based both on a new version of comparison principle (Lemma 3.4) and on the explicit expression of the best constant $\bar{C}$ for the inequality

$$\|\nabla u\|^2_{L^2(\mathbb{R}^N)} \geq \bar{C}\|u\|^N_{L^\infty(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{X},$$

proved in Lemmas 3.6 and 3.8, which might be of independent interest. Note that this result has a different flavor than the results for optimal constants for the embeddings since our inequality is inhomogeneous and we have to crucially use that the Lipschitz constant of u is bounded by one.

In Sect. 4, we first turn our attention to the approximating problems

$$\begin{cases} -\sum_{h=1}^{m} \alpha_h \Delta_{2h} u = \sum_{k=1}^{n} a_k \delta_{x_k} & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{1.10}$$

for $m \geq 1$ and study the regularity of the solution u_m: by combining results of Lieberman [19], a linearization, and a bootstrap argument, we prove that the solutions are regular away from the points x_k's.

Proposition 1.3 *Let $2m > \max\{N, 2^*\}$, $2^* := 2N/(N-2)$, and u_m be the solution of* (1.10)*. Then $u_m \in C_0^{0,\beta_m}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{x_1,\dots,x_n\})$, where*

$$C_0^{0,\beta_m}\left(\mathbb{R}^N\right) := \left\{u \in C^{0,\beta_m}\left(\mathbb{R}^N\right) : \lim_{|x|\to\infty} u(x) = 0\right\},$$

with $\beta_m := 1 - \frac{N}{2m}$.

In comparison with full problem (1.7), there is an important difference—we do not have a priori an estimate on $|\nabla u|$, and therefore, the Hölder estimate is not immediate. Note that β_m converges to 1 as $m \to \infty$, in agreement with the fact that the solutions of (1.10) approximate solutions of (1.7). On the other hand, the operator in (1.10) is well defined for any sufficiently smooth function u and the smoothness of solutions can be expected away from x_k's.

We stress that in the proof of Proposition 1.3, we heavily use the fact that, in the operator (1.6), also the Laplacian appears; see Remark 4.1 for further details. Moreover, we also prove that u_m and ∇u_m behave as the fundamental solution (and its gradient) of the $2m$-Laplacian near the singularities x_k's. Intuitively, we could say that the Laplacian, Δ_2, is responsible for the regularity of the approximating solution u_m and the behavior at infinity, while the $2m$-Laplacian (the last one) dictates the local behavior of the solution u_m near the singularities x_k's, in the following sense.

Theorem 1.4 *Let* $2m > \max\{N, 2^*\}$ *and* $k = 1, \ldots, n$. *Then*

$$\lim_{x \to x_k} \frac{u_m(x) - u_m(x_k)}{|x - x_k|^{\frac{2m-N}{2m-1}}} = K_m \tag{1.11}$$

for some $K_m = K_m(a_k, \alpha_m, N) \in \mathbb{R}$ *such that* $K_m \cdot a_k < 0$, *and*

$$\lim_{x \to x_k} \frac{|\nabla u_m(x)|}{|x - x_k|^{\frac{1-N}{2m-1}}} = K_m', \tag{1.12}$$

with $K_m' := \frac{2m-N}{2m-1}|K_m|$. *In particular,* x_k *is a relative strict maximizer (resp. minimizer) of* u_m *if* $a_k > 0$ *(resp.* $a_k < 0$*).*

The same reasons as above make this result non-trivial. The operator is nonlinear; thus, it is not obvious that the local behavior does not depend on the location of all charges as it, for example, does for the Laplacian in one dimension. The asymptotic behavior is a fine interplay between lowest and highest order differential operators in the expansion.

The proof of this theorem is rather technical and relies on a blow-up argument, combined with Riesz potential estimates [2]. Such a usage of blow-up method is quite unusual since the solution is bounded at the blow-up point and we need to rescale the problem in such a way that we keep the boundedness of solution, but remove the lower-order terms.

The fact that the growth rate of u_m near the singularity x_k is of the type $|x - x_k|^{\frac{2m-N}{2m-1}}$, with exponent that goes to 1 as m goes to infinity, shows that the singularities x_k's of u_m approach cone-like singularities for m large, which is coherent with the results found for u_ϱ. In particular, we note that blow-up rate (1.12) of $|\nabla u_m|$ near the singularities and the fact that $\lim_{m\to\infty} K_m' = 1$ (cf. Remark 4.2) suggest that $\lim_{m\to\infty} |\nabla u_m(x)| \approx 1$ as $x \to x_k$, which is the same behavior as $|\nabla u_\varrho|$ (see [17, Theorem 1.4]). Moreover, as an easy consequence of (1.11), we get that the singularity x_k is either a relative strict minimizer or a relative strict maximizer depending on the sign of its coefficient a_k. Altogether, this shows that the approximating solutions u_m's are actually behaving like the minimizer u_ϱ of (1.8), at least qualitatively near the singularities.

Furthermore, it is worth stressing that problem (1.10) is governed by an inhomogeneous operator that behaves like $-\Delta - \Delta_{2m}$ with m large. The interest in inhomogeneous operators of the type sum of a p-Laplacian and a q-Laplacian has recently significantly increased, as shown by the long list of recent papers (see, for instance, [1,2,10,11,20,21] and the references therein).

The paper is organized as follows. In Sect. 2, we collect definitions and known results for problems (1.7) and (1.10) relevant to our proof. Section 3 contains our results concerning the qualitative properties of the minimizer of original problem (1.7) and the sufficient conditions to guarantee that the minimizer u_ϱ of I indeed solves (1.7); in particular, we prove therein Theorem 1.2. Finally, Sect. 4 is devoted to the study of approximating problem (1.10) and the qualitative analysis of the solution and its gradient, namely to the proofs of Proposition 1.3 and Theorem 1.4.

2 Preliminaries

In this section, we summarize used notation and definitions as well as previous results needed in the rest of the paper. We start with properties of functions belonging to the set $\mathcal{X}$ (see (1.3)).

Lemma 2.1 (Lemma 2.1 of [4]). *The following properties hold:*

(i) $\mathcal{X} \hookrightarrow W^{1,p}(\mathbb{R}^N)$ *for all* $p \geq 2^*$*;*
(ii) $\mathcal{X} \hookrightarrow L^\infty(\mathbb{R}^N)$*;*
(iii) *If* $u \in \mathcal{X}$, $\lim_{|x|\to\infty} u(x) = 0$*;*
(iv) $\mathcal{X}$ *is weakly closed;*
(v) *If* $(u_n) \subset \mathcal{X}$ *is bounded, up to a subsequence it converges weakly to a function* $\bar{u} \in \mathcal{X}$*, uniformly on compact sets.*

Throughout the paper, $\overline{xy} := \left\{z \ : \ z = (1-t)x + ty \text{ for } t \in [0,1]\right\}$ denotes the line segment with endpoints x and y and $\mathrm{Int}(\overline{xy})$ the open segment.

Definition 2.2 Let $u \in C^{0,1}(\Omega)$, with $\Omega \subset \mathbb{R}^N$. We say that

(i) u is *weakly spacelike* if $|\nabla u| \leq 1$ a.e. in Ω;
(ii) u is *spacelike* if $|u(x) - u(y)| < |x - y|$ for all $x, \ y \in \Omega$, $x \neq y$, and the line segment $\overline{xy} \subset \Omega$;
(iii) u is *strictly spacelike* if $u \in C^1(\Omega)$, and $|\nabla u| < 1$ in Ω.

Proposition 2.3 (Proposition 2.3 of [4]). *For any* $\varrho \in \mathcal{X}^*$*, there exists a unique* $u_\varrho \in \mathcal{X}$ *that minimizes* I_ϱ *defined by* (1.4)*. If furthermore* $\varrho \neq 0$*, then* $u_\varrho \neq 0$ *and* $I_\varrho(u_\varrho) < 0$*.*

Theorem 2.4 (Theorem 1.6 and Lemma 4.1 of [4]). *Let* $\varrho := \sum_{k=1}^n a_k \delta_{x_k}$ *and* $\Gamma := \bigcup_{k\neq j} \overline{x_k x_j}$*. The minimizer* u_ϱ *of the energy functional* I *given by* (1.8) *is a strong solution of*

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = 0 & \text{in } \mathbb{R}^N \setminus \Gamma, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases}$$

Furthermore,

(i) $u_\varrho \in C^\infty(\mathbb{R}^N \setminus \Gamma) \cap C(\mathbb{R}^N)$*;*
(ii) u_ϱ *is strictly spacelike in* $\mathbb{R}^N \setminus \Gamma$*;*
(iii) *for* $k \neq j$*, either* u_ϱ *is a classical solution on* $\mathrm{Int}(\overline{x_k x_j})$*, or*

$$u_\varrho(tx_k + (1-t)x_j) = tu_\varrho(x_k) + (1-t)u_\varrho(x_j) \quad \text{for all } t \in (0,1).$$

Theorem 2.5 (Corollary 3.2 of [17]). *If* $a_k \cdot a_j > 0$*, then* u_ϱ *is a classical solution on* $\mathrm{Int}(\overline{x_k x_j})$ *provided that no other charges are located on the segment* $\overline{x_k x_j}$*.*

As mentioned in Introduction, in order to overcome the difficulty related to the non-differentiability of I, we consider approximating problems. The idea is to approximate the mean curvature operator Q [for the definition, see (1.5)] by a finite sum of $2h$-Laplacians, by using the Taylor expansion. We note that the operator Q is formally the Fréchet derivative of the functional

$$\int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla u|^2}\right) \mathrm{d}x = \int_{\mathbb{R}^N} \sum_{h=1}^{\infty} \frac{\alpha_h}{2h} |\nabla u|^{2h} \mathrm{d}x, \tag{2.1}$$

where $\alpha_1 := 1$, $\alpha_h := \frac{(2h-3)!!}{(2h-2)!!}$ for $h \geq 2$, and

$$k!! := \prod_{j=0}^{[k/2]-1} (k-2j) \quad \text{for all } k \in \mathbb{N}.$$

The series in the right-hand side of (2.1) converges pointwise, although not uniformly, for all $|\nabla u| \leq 1$. Then, the operator $-Qu = -\text{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)$ can be regarded as the series of $2h$-Laplacians (see (1.5)).

For every natural number $m \geq 1$, we define the space $\mathcal{X}_{2m}$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{X}_{2m}} := \left[\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \left(\int_{\mathbb{R}^N} |\nabla u|^{2m} \mathrm{d}x\right)^{1/m}\right]^{1/2}.$$

Let $\varrho \in \mathcal{X}_{2m}^*$ for some $m \geq 1$. We study the approximating problem

$$\begin{cases} -\sum_{h=1}^{m} \alpha_h \Delta_{2h} u = \varrho & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{2.2}$$

and we denote by $I_m : \mathcal{X}_{2m} \to \mathbb{R}$ the energy functional associated with (2.2)

$$I_m(u) := \sum_{h=1}^{m} \frac{\alpha_h}{2h} \int_{\mathbb{R}^N} |\nabla u|^{2h} \mathrm{d}x - \langle \varrho, u \rangle_{\mathcal{X}_{2m}},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{X}_{2m}}$ denotes the duality pairing between $\mathcal{X}_{2m}^*$ and $\mathcal{X}_{2m}$. The functional I_m is of class C^1 and is the mth-order approximation of I.

Definition 2.6 A weak solution of (2.2) is a function $u_m \in \mathcal{X}_{2m}$ such that

$$\sum_{h=1}^{m} \alpha_h \int_{\mathbb{R}^N} |\nabla u_m|^{2h-2} \nabla u_m \nabla v \mathrm{d}x = \langle \varrho, v \rangle_{\mathcal{X}_{2m}} \quad \text{for all } v \in C_c^\infty\left(\mathbb{R}^N\right).$$

Clearly, a function is a weak solution of (2.2) if and only if it is a critical point of I_m.

Proposition 2.7 (Proposition 5.1 of [4]). *Let $\varrho \in \mathcal{X}_{2m_0}^*$ for some $m_0 \geq 1$. Then, for all $m \geq m_0$, the functional $I_m : \mathcal{X}_{2m} \to \mathbb{R}$ has one and only one critical point u_m. Furthermore, u_m minimizes I_m.*

Theorem 2.8 (Theorem 5.2 of [4]). *Let $\varrho \in \mathcal{X}_{2m_0}^*$ for some $m_0 \geq 1$. Then $u_m \rightharpoonup u_\varrho$ in $\mathcal{X}_{2\bar{m}}$ for all $\bar{m} \geq m_0$ and uniformly on compact sets.*

3 Born–Infeld problem

In this section, we study the nature of the singularities of the minimizer of energy functional (1.8) and sufficient conditions guaranteeing that the minimizer is a solution of (1.7) on $\mathbb{R}^N \setminus \{x_1, \ldots, x_n\}$. To this aim, we isolate one singularity, and we investigate (1.7) on bounded domains. We start with definitions and preliminary results.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\varphi : \partial\Omega \to \mathbb{R}$ a bounded function and $\varrho_\Omega \in \mathcal{X}_\Omega^*$, where $\mathcal{X}_\Omega^*$ is the dual space of $\mathcal{X}_\Omega := \{u \in C^{0,1}(\Omega) : |\nabla u| \le 1 \text{ a.e. in } \Omega\}$. We consider the variational problem

$$\min_{u \in \mathcal{C}(\varphi,\Omega)} I_{\Omega,\varrho}(u), \tag{3.1}$$

where

$$I_{\Omega,\varrho}(u) := \int_\Omega \left(1 - \sqrt{1 - |\nabla u|^2}\right) \mathrm{d}x - \langle \varrho_\Omega, u \rangle_{\mathcal{X}_\Omega} \quad \text{for all } u \in \mathcal{X}_\Omega$$

and

$$\mathcal{C}(\varphi, \Omega) := \{v \in \mathcal{X}_\Omega : v = \varphi \text{ on } \partial\Omega\}.$$

Lemma 3.1 *Problem* (3.1) *has at most one solution.*

Proof Although the argument is similar to [3, Proposition 1.1], we include it here for completeness. Let $u_1, u_2 \in \mathcal{X}_\Omega$ be two solutions of (3.1) and consider $u_t := (1-t)u_1 + tu_2$ for any $t \in (0,1)$. By the convexity of $1 - \sqrt{1 - |x|^2}$, we have

$$\begin{aligned} I_{\Omega,\varrho}(u_t) &\le (1-t) \int_\Omega \left(1 - \sqrt{1 - |\nabla u_1|^2}\right) \mathrm{d}x + t \int_\Omega \left(1 - \sqrt{1 - |\nabla u_2|^2}\right) \mathrm{d}x \\ &\quad - (1-t)\langle \varrho_\Omega, u_1 \rangle_{\mathcal{X}_\Omega} - t \langle \varrho_\Omega, u_2 \rangle_{\mathcal{X}_\Omega} \\ &= (1-t) I_{\Omega,\varrho}(u_1) + t I_{\Omega,\varrho}(u_2) = I_{\Omega,\varrho}(u_1), \end{aligned} \tag{3.2}$$

where we used $I_{\Omega,\varrho}(u_1) = I_{\Omega,\varrho}(u_2) = \min I_{\Omega,\varrho}$. By the minimality of $I_{\Omega,\varrho}(u_1)$, we have $I(u_t) = I(u_1)$, and so the equality must hold in (3.2). Now, being $x \mapsto 1 - \sqrt{1 - |x|^2}$ strictly convex, we have $\nabla u_1 = \nabla u_2$ a.e. in Ω. Since $u_1 = u_2$ on $\partial\Omega$, $u_1 - u_2$ can be extended to a Lipschitz function on $\mathbb{R}^N$ that vanishes in $\mathbb{R}^N \setminus \Omega$ (cf. [3]). Thus, being $\nabla(u_1 - u_2) = 0$ a.e. in Ω, we have $u_1 = u_2$ and the proof is concluded. □

Remark 3.2 Concerning existence of a minimizer for (3.1), we observe that in the case under consideration $\varrho = \sum_{k=1}^n a_k \delta_{x_k}$, it is immediate to see that for every $\Omega \subset \mathbb{R}^N \setminus \{x_1, \ldots, x_n\}$, $u_\varrho|_\Omega$ minimizes I_Ω over $\mathcal{C}(u_\rho, \Omega)$, where we recall that u_ϱ denotes the unique minimizer of I_ϱ in all of $\mathbb{R}^N$ (cf. Proposition 2.3). Indeed, let $v \in \mathcal{C}(u_\rho, \Omega)$ and denote $\psi := v - u_\rho \in \mathcal{C}(0, \Omega)$ and $\tilde{\psi}$ Lipschitz continuation of ψ that vanishes outside of Ω. Then $u_\rho + \tilde{\psi} \in \mathcal{X}$ and the minimality of u_ϱ yields

$$\begin{aligned} I(u_\varrho + \tilde{\psi}) &= \int_\Omega \left(1 - \sqrt{1 - |\nabla(u_\varrho + \tilde{\psi})|^2}\right) \mathrm{d}x + \int_{\mathbb{R}^N \setminus \Omega} \left(1 - \sqrt{1 - |\nabla u_\varrho|^2}\right) \mathrm{d}x \\ &\quad - \sum_{k=1}^n a_k u_\varrho(x_k) \\ &\ge I(u_\varrho) = \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla u_\varrho|^2}\right) \mathrm{d}x - \sum_{j=1}^n a_k u_\varrho(x_k). \end{aligned}$$

Hence,

$$\int_\Omega \left(1 - \sqrt{1 - |\nabla(u_\varrho|_\Omega + \psi)|^2}\right) \mathrm{d}x \ge \int_\Omega \left(1 - \sqrt{1 - |\nabla u_\varrho|_\Omega|^2}\right) \mathrm{d}x$$

or equivalently

$$I_\Omega(v) = I_\Omega(u_\varrho|_\Omega + \psi) \ge I_\Omega(u_\varrho|_\Omega),$$

which proves the claim by the arbitrariness of $v \in \mathcal{C}(u_\varrho, \Omega)$.

Definition 3.3 Let $\varrho_1, \varrho_2 \in \mathcal{X}_\Omega^*$. We say that $\varrho_1 \le \varrho_2$ if $\langle \varrho_1, v\rangle_{\mathcal{X}_\Omega} \le \langle \varrho_2, v\rangle_{\mathcal{X}_\Omega}$ for all $v \in \mathcal{X}_\Omega$ with $v \ge 0$.

Lemma 3.4 *Let $\varrho_1, \varrho_2 \in \mathcal{X}_\Omega^*$, $\varphi_1, \varphi_2 : \partial\Omega \to \mathbb{R}$ be two bounded functions, $u_1 \in \mathcal{C}(\varphi_1, \Omega)$ be the minimizer of I_{Ω,ϱ_1}, and $u_2 \in \mathcal{C}(\varphi_2, \Omega)$ be the minimizer of I_{Ω,ϱ_2}. If $\varrho_2 \le \varrho_1$, then*

$$u_2(x) \le u_1(x) + \sup_{\partial\Omega}(\varphi_2 - \varphi_1) \quad \textit{for all } x \in \Omega.$$

Proof Throughout this proof, we use the following simplified notation

$$I_1 := I_{\Omega,\varrho_1}, \quad I_2 := I_{\Omega,\varrho_2}, \quad \langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\mathcal{X}_\Omega}, \quad \mathcal{Q}(u) := \int_\Omega (1 - \sqrt{1 - |\nabla u|^2})\mathrm{d}x.$$

Let $\alpha := \sup_{\partial\Omega}(\varphi_2 - \varphi_1)$ and $\tilde{u}_1 := u_1 + \alpha$. We claim that $\tilde{u}_1$ minimizes I_1 in $\mathcal{C}(\varphi_1 + \alpha, \Omega)$. Indeed, since u_1 minimizes I_1 in $\mathcal{C}(\varphi_1, \Omega)$, for all $u \in \mathcal{C}(\varphi_1, \Omega)$ we have

$$I_1(\tilde{u}_1) = \mathcal{Q}(u_1) - \langle \varrho_1, u_1\rangle - \langle \varrho_1, \alpha\rangle \le I_1(u) - \langle \varrho_1, \alpha\rangle = I_1(u + \alpha).$$

Since $\mathcal{C}(\varphi_1 + \alpha, \Omega) = \mathcal{C}(\varphi_1, \Omega) + \alpha$, the claim is proved.

Now, suppose by contradiction that the set $\Omega^+ := \{x \in \Omega : u_2(x) > \tilde{u}_1(x)\}$ is non-empty. Let $\Omega^- := \Omega \setminus \Omega^+$,

$$\mathcal{Q}^+(u) := \int_{\Omega^+} (1 - \sqrt{1 - |\nabla u|^2})\mathrm{d}x, \quad \mathcal{Q}^-(u) := \int_{\Omega^-} (1 - \sqrt{1 - |\nabla u|^2})\mathrm{d}x\,,$$

$$U := \max\{u_2, \tilde{u}_1\} = \begin{cases} \tilde{u}_1 & \text{in } \Omega^- \\ u_2 & \text{in } \Omega^+, \end{cases} \quad \text{and} \quad V := \min\{u_2, \tilde{u}_1\} = \begin{cases} u_2 & \text{in } \Omega^- \\ \tilde{u}_1 & \text{in } \Omega^+. \end{cases}$$

We observe that, by continuity, $u_2 = \tilde{u}_1$ on $\partial\Omega^+$. Hence, by definition of α, $U \in \mathcal{C}(\varphi_1 + \alpha, \Omega)$ and $V \in \mathcal{C}(\varphi_2, \Omega)$. Furthermore, the following relations hold in the whole of Ω:

$$u_2 - V = U - \tilde{u}_1 \ge 0.$$

Then, by $\varrho_2 \le \varrho_1$, we obtain

$$\begin{aligned} I_1(U) &= \mathcal{Q}(U) - \langle \varrho_1, U - \tilde{u}_1\rangle - \langle \varrho_1, \tilde{u}_1\rangle \\ &\le \mathcal{Q}(U) - \langle \varrho_2, U - \tilde{u}_1\rangle - \langle \varrho_1, \tilde{u}_1\rangle \\ &= \mathcal{Q}^+(u_2) + \mathcal{Q}^-(\tilde{u}_1) - \langle \varrho_2, U - \tilde{u}_1\rangle - \langle \varrho_1, \tilde{u}_1\rangle \\ &= I_1(\tilde{u}_1) - \mathcal{Q}^+(\tilde{u}_1) + \mathcal{Q}^+(u_2) - \langle \varrho_2, U - \tilde{u}_1\rangle \\ &= I_1(\tilde{u}_1) + I_2(u_2) - \mathcal{Q}^-(u_2) + \langle \varrho_2, u_2\rangle - \mathcal{Q}^+(\tilde{u}_1) - \langle \varrho_2, U - \tilde{u}_1\rangle \\ &= I_1(\tilde{u}_1) + I_2(u_2) - \mathcal{Q}(V) + \langle \varrho_2, V\rangle \\ &= I_1(\tilde{u}_1) + I_2(u_2) - I_2(V) \\ &< I_1(\tilde{u}_1), \end{aligned}$$

where in the last step we used the strict minimality of $I_2(u_2)$ over $\mathcal{C}(\varphi_2, \Omega)$ (see Lemma 3.1). This contradicts the fact that $\tilde{u}_1$ minimizes I_1 in $\mathcal{C}(\varphi_1 + \alpha, \Omega)$ and concludes the proof. □

Theorem 3.5 *If u_ϱ is the unique minimizer of problem (3.1), then for every $k = 1, \ldots, n$ one has*

(i) *For every $x \in \mathbb{R}^N$ with $|x| = 1$, there exists* $\lim_{h\to 0^+} \dfrac{u_\varrho(hx + x_k) - u_\varrho(x_k)}{h} = \pm 1$*;*

(ii) *x_k is a relative strict minimizer (resp. maximizer) of u_ϱ if $a_k < 0$ (resp. $a_k > 0$).*

Proof (i) For every $k = 1, \dots, n$, fix $R_k > 0$ such that $B_{R_k}(x_k) \cap \{x_1, \dots, x_n\} = \{x_k\}$, where $B_R(x)$ is an open ball of radius R centered at x. Now, define $u_{\varrho,k}(x) := u_\varrho(x + x_k) - u_\varrho(x_k)$ for every $x \in B_{R_k}(0)$. Since $\nabla u_{\varrho,k}(x) = \nabla u_\varrho(x + x_k)$ and $x \in B_{R_k}(x_k) \setminus \{x_k\}$ iff $x - x_k \in B_{R_k}(0) \setminus \{0\}$, by Remark 3.2 we obtain that for every $\Omega \subset B_{R_k}(0) \setminus \{0\}$, $u_{\varrho,k}|_\Omega$ minimizes the functional $I_\Omega : \mathcal{C}(u_{\varrho,k}|_{\partial\Omega}, \bar{\Omega}) \to \mathbb{R}$ defined by

$$I_\Omega(u) := \int_\Omega \left(1 - \sqrt{1 - |\nabla u|^2}\right) \mathrm{d}x.$$

Hence, the graph of $u_{\varrho,k}|_{B_{R_k}(0)}$ is an area maximizing hypersurface in the Minkowski space having an isolated singularity at 0, in the sense of [12, Definitions 0.2 and 1.1]. By [12, Theorem 1.5], we can conclude that 0 is a light-cone-like singularity in the sense of [12, Definition 1.4]. This implies that, for every $x \in B_{R_k/t}(0)$ with $|x| = 1$,

$$\lim_{h \to 0^+} \frac{u_{\varrho,k}(hx)}{h} \quad \text{exists and} \quad \left| \lim_{h \to 0^+} \frac{u_{\varrho,k}(hx)}{h} \right| = 1.$$

Since $u_{\varrho,k}(0) = 0$, this means that for every direction x, there exists one-sided directional derivative of $u_{\varrho,k}$ along x at 0 and its absolute value is 1, that is,

$$\lim_{h \to 0^+} \frac{u_{\varrho,k}(hx + 0) - u_{\varrho,k}(0)}{h} \quad \text{exists and} \quad \left| \lim_{h \to 0^+} \frac{u_{\varrho,k}(hx + 0) - u_{\varrho,k}(0)}{h} \right| = 1,$$

which concludes the proof of (i).

(ii) Since 0 is a light-cone-like singularity of $u_{\varrho,k}|_{B_{R_k}(0)}$, two cases may occur (cf. [12, Definition 1.4 and Lemma 1.9]) : either

$$u_{\varrho,k} > 0 \quad \text{in } B_R(0) \setminus \{0\}$$

or

$$u_{\varrho,k} < 0 \quad \text{in } B_R(0) \setminus \{0\}$$

for some $0 < R < R_k$. As a consequence, either x_k is a relative strict minimizer of u_ϱ or x_k is a relative strict maximizer of u_ϱ.

Now, in order to detect which situation occurs depending on the sign of a_k, we use the comparison principle proved in Lemma 3.4. If $a_k < 0$, we set $\Omega := B_{R/2}(x_k)$, $\varrho_1 := 0$, $\varphi_1 := 0$, $\varrho_2 := a_k \delta_{x_k}$, and $\varphi_2 := u_\varrho|_{\partial B_{R/2}(x_k)}$. Hence, $u_1 = 0$, $u_2 = u_\varrho|_{B_{R/2}(x_k)}$, and $\varrho_2 \le \varrho_1$. Then, by Lemma 3.4

$$\sup_{B_{R/2}(x_k)} u_\varrho \le \sup_{\partial B_{R/2}(x_k)} u_\varrho. \tag{3.3}$$

Suppose by contradiction that x_k is a relative strict maximizer of u_ϱ in $B_R(x_k)$, then

$$u_\varrho(x_k) = \sup_{B_{R/2}(x_k)} u_\varrho > \max_{\partial B_{R/2}(x_k)} u_\varrho,$$

which contradicts (3.3). Thus, x_k is a relative strict minimizer of u_ϱ. Analogously, it is possible to prove that when $a_k > 0$, x_k is a relative strict maximizer of u_ϱ. □

In what follows, we give an explicit quantitative sufficient condition on the charge values a_k's and on the charge positions x_k's for u_ϱ to be a classical solution of

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right) = 0 \tag{3.4}$$

in some subset of $\mathbb{R}^N \setminus \{x_1, \ldots, x_n\}$. As mentioned in Introduction, our results complement the qualitative ones contained in [4] (see Theorem 2.4 above), stating that if the charges are sufficiently small in absolute value or far away from each other, then the minimizer solves the problem.

First, we prove the following lemma.

Lemma 3.6 *Let $N \geq 3$. There exists a constant $C = C(N) > 0$ such that*

$$\|\nabla u\|^2_{L^2(\mathbb{R}^N)} \geq C\|u\|^N_{L^\infty(\mathbb{R}^N)}, \tag{3.5}$$

for all $u \in \mathcal{X}$. The best constant

$$\bar{C} := \inf_{u\in\mathcal{X}\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{\|u\|^N_{L^\infty(\mathbb{R}^N)}}$$

is achieved by a radial and radially decreasing function.

Proof For all $u \in \mathcal{X} \setminus \{0\}$, we define the ratio

$$\mathcal{R}(u) := \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{\|u\|^N_{L^\infty(\mathbb{R}^N)}}$$

and we observe that for any $t > 0$ it is invariant under the transformation $\phi_t : \mathcal{X} \to \mathcal{X}$, with $\phi_t(v) := tv(\cdot/t)$ for all $v \in \mathcal{X}$.

Furthermore, fix $u \in \mathcal{X} \setminus \{0\}$ and denote by $u^\star$ the symmetric decreasing rearrangement of $|u|$ (see, e.g., [18, Chapter 3]). Then, $\|u\|_{L^\infty(\mathbb{R}^N)} = \|u^\star\|_{L^\infty(\mathbb{R}^N)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^N)} \geq \|\nabla u^\star\|_{L^2(\mathbb{R}^N)}$ by the Polya–Szegő inequality. Hence, $\mathcal{R}(u) \geq \mathcal{R}(u^\star)$. Therefore, if we denote by $\mathcal{X}^{\text{rad}}_-$ the set of $\mathcal{X}$-functions which are radial and radially decreasing, then

$$\bar{C} = \inf_{u\in\mathcal{X}\setminus\{0\}} \mathcal{R}(u) = \inf_{u\in\mathcal{X}^{\text{rad}}_-\setminus\{0\}} \mathcal{R}(u).$$

Finally, we prove the existence of a minimizer of $\mathcal{R}$. Let $(u_n) \subset \mathcal{X}^{\text{rad}}_- \setminus \{0\}$ be a minimizing sequence. Without loss of generality, we may assume that $u_n(0) = \|u_n\|_{L^\infty(\mathbb{R}^N)} = 1$ for all $n \in \mathbb{N}$, otherwise we transform it by an appropriate ϕ_t. Then, $\|\nabla u_n\|^2_{L^2(\mathbb{R}^N)} \to \bar{C}$, and in particular, (u_n) is bounded in $\mathcal{X}$. Hence, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $\mathcal{X}$ and $u_n \to \bar{u}$ uniformly on compact sets of $\mathbb{R}^N$, by Lemma 2.1. In particular, $\bar{u} \in \mathcal{X}^{\text{rad}}_-$, $1 = u_n(0) \to \bar{u}(0)$, and so $\|\bar{u}\|_{L^\infty(\mathbb{R}^N)} = 1$. Therefore, the weak lower semicontinuity of the norm yields

$$\mathcal{R}(\bar{u}) = \int_{\mathbb{R}^N} |\nabla\bar{u}|^2 \mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x = \inf_{u\in\mathcal{X}\setminus\{0\}} \mathcal{R}(u),$$

and so $\bar{u}$ is a minimizer. □

Remark 3.7 The exponent N appearing in the right-hand side of (3.5) naturally arises from the fact that $\mathcal{R}$ is invariant under transformations ϕ_t.

Lemma 3.8 *The best constant for inequality* (3.5) *is given by*

$$\bar{C} = \frac{2}{N}\left(\frac{N-2}{N-1}\right)^{N-1} \omega_{N-1}. \tag{3.6}$$

Proof In order to find the explicit value of $\bar{C}$, we will build by hands a minimizer of $\mathcal{R}$.

Step 1: The minimizer can be found in a smaller function space. We first observe that if $u \in \mathcal{X}$, then $\lambda u \in \mathcal{X}$ if and only if $0 < \lambda \le \|\nabla u\|^{-1}_{L^\infty(\mathbb{R}^N)}$. Moreover, for all $\lambda \in (0, \|\nabla u\|^{-1}_{L^\infty(\mathbb{R}^N)}]$

$$\mathcal{R}(\lambda u) = \lambda^{2-N}\mathcal{R}(u) \ge \frac{1}{\|\nabla u\|^{2-N}_{L^\infty(\mathbb{R}^N)}}\mathcal{R}(u) = \mathcal{R}\left(\frac{u}{\|\nabla u\|_{L^\infty(\mathbb{R}^N)}}\right).$$

Then, set

$$\widetilde{\mathcal{X}} := \{u \in \mathcal{X}^{\mathrm{rad}}_{-} \,:\, u \ge 0 \text{ and } \operatorname{esssup}|u'| = \operatorname{esssup} u = 1\},$$

where with abuse of notation we have written $u(r) := u(x)$ for $r = |x|$. Together with Lemma 3.6, we have

$$\bar{C} = \inf_{u \in \widetilde{\mathcal{X}}\setminus\{0\}} \mathcal{R}(u).$$

Step 2: The minimizer has non-decreasing first derivative. Let $\bar{u} \in \widetilde{X}$ be any minimizer of $\mathcal{R}$ and consider any two (measurable) sets $S_1, S_2 \subset (0,\infty)$ of positive Lebesgue measure such that $\sup S_1 < \inf S_2$. For a contradiction, assume that $\bar{u}' \le B - \delta$ on S_2 and $0 \ge \bar{u}' \ge B + \delta$ on S_1 for some $B \in [-1, 0)$ and $\delta \in (0, -B)$. Note that by making sets S_1, S_2 smaller if necessary (still of positive measure) we can assume that $\operatorname{dist}(S_1, S_2) \ge \varepsilon$ and $S_1 \cup S_2$ is bounded. Since S_1 and S_2 have positive measure, it is standard to see that there exists a translation of S_1, denoted by $S_1 + k$ for some $k \ge \varepsilon$, such that $M_2 := (S_1 + k) \cap S_2$ has positive measure. Denote $M_1 := M_2 - k$ and note that $M_1 \subset S_1$. Of course, M_1 and M_2 are measurable, with positive measure.

Define a new function

$$w'(r) := \begin{cases} \bar{u}'(r+k) & r \in M_1 \\ \bar{u}'(r-k) & r \in M_2 \\ \bar{u}'(r) & \text{otherwise}\,, \end{cases}$$

that is, we exchange the values of $\bar{u}'$ on sets M_1 and M_2. Note that $w' \in L^2((0,\infty))$ and it is the derivative of the function $w(r) = 1 + \int_0^r w'(s)\,\mathrm{d}s$, which is decreasing by Lemma 3.6, belongs to $L^2((0,\infty))$, and has $w(0) = 1$. Observe that $w \equiv \bar{u}$ outside of the convex hull of $S_1 \cup S_2$. Then,

$$\begin{aligned} \|\nabla \bar{u}\|^2_{L^2(\mathbb{R}^N)} - \|\nabla w\|^2_{L^2(\mathbb{R}^N)} &= \int_0^\infty |\bar{u}'|^2 r^{N-1}\,\mathrm{d}r - \int_0^\infty |w'|^2 r^{N-1}\,\mathrm{d}r \\ &= \int_{M_1} \left(|\bar{u}'(r)|^2 - |\bar{u}'(r+k)|^2\right) r^{N-1}\,\mathrm{d}r + \int_{M_2} \left(|\bar{u}'(r)|^2 - |\bar{u}'(r-k)|^2\right) r^{N-1}\,\mathrm{d}r \\ &= \int_{M_1} \left(|\bar{u}'(r+k)|^2 - |\bar{u}'(r)|^2\right)\left[(r+k)^{N-1} - r^{N-1}\right]\mathrm{d}r > 0\,, \end{aligned}$$

a contradiction to $\bar{u}$ being a minimizer. Note that we used that for $r \in M_1$ one has $r + k \in M_2$, and consequently, since $B < 0$, $|\bar{u}'(r+k)|^2 \ge (B-\delta)^2 > (B+\delta)^2 \ge |\bar{u}'(r)|^2$. Moreover, $k \ge \varepsilon > 0$ and the strict inequality follows. By the arbitrariness of $0 < \delta < -B$, we obtain that $\bar{u}'$ is a non-decreasing function.

Step 3: The minimizer is harmonic outside the set of points of -1 derivative. Denote $R := \sup\{r \in (0,1) : \bar{u}'(r) = -1\}$ and set $R = 0$ if $\bar{u}'(r) > -1$ for each $r > 0$. Fix any $\varepsilon > 0$ and note that $B := \bar{u}'(R+\varepsilon) > -1$. Therefore, $\bar{u}'(r) \ge B > -1$ on $(R+\varepsilon, \infty)$.

In order to prove that at points r where $\bar{u}'(r) \neq -1$, $\bar{u}$ is harmonic, fix any smooth $\psi \in C^1_c((R+\varepsilon,\infty))$ and note that for sufficiently small (in absolute value) ξ, one has $(\bar{u}+\xi\psi)' \geq -1$. Then, by the minimality of $\bar{u}$,

$$0 \geq \int_0^\infty |\bar{u}'|^2 r^{N-1}\,\mathrm{d}r - \int_0^\infty |\bar{u}'+\xi\psi'|^2 r^{N-1}\,\mathrm{d}r = -\xi\int_0^\infty \left(2\bar{u}'\psi' + \xi|\psi'|^2\right) r^{N-1}\,\mathrm{d}r\,.$$

Since $|\xi| \ll 1$ is arbitrarily small, positive or negative, we obtain

$$0 = \int_0^\infty \bar{u}'\psi' r^{N-1}\,\mathrm{d}r = -\int_0^\infty \left(\bar{u}' r^{N-1}\right)' \psi\,\mathrm{d}r\,.$$

By the arbitrariness of ψ, this implies that $(\bar{u}' r^{N-1})' = 0$ a.e. in $(R+\varepsilon,\infty)$, which in turn gives that $\bar{u}$ is harmonic in (R,∞), because $\varepsilon > 0$ is arbitrary.

Step 4: The explicit form of a minimizer. Altogether, we have proved that a minimizer $\bar{u}$ of $\mathcal{R}$ can be taken of the form

$$\bar{u}(r) = \begin{cases} 1-r & \text{if } r \in (0,R), \\ c_1 r^{2-N} + c_2 & \text{if } r \in [R,\infty) \end{cases}$$

for suitable constants $c_1, c_2 > 0$ and $R \geq 0$. Since $\lim_{r\to\infty}\bar{u}(r) = 0$, $c_2 = 0$ and since r^{2-N} is unbounded at 0, we have $R > 0$, and clearly, $R \leq 1$. Moreover, $\bar{u}$ is continuous and $|\bar{u}'| \leq 1$, that is

$$c_1 = R^{N-2}(1-R) \quad \text{and} \quad c_1 \leq \frac{R^{N-1}}{N-2}.$$

Consequently, $R \geq \frac{N-2}{N-1}$. Now, we minimize $\|\nabla\bar{u}\|^2_{L^2(\mathbb{R}^N)}$ as a function of R, or equivalently, we minimize

$$E(R) := \int_0^{+\infty} \bar{u}'^2(r) r^{N-1}\mathrm{d}r = \int_0^R r^{N-1}\mathrm{d}r + \int_R^{+\infty} c_1^2 (N-2)^2 r^{1-N}\mathrm{d}r\,.$$

Using the bound on c_1, we have

$$E'(R) = R^{N-1} - c_1^2(N-2)^2 R^{1-N} \geq 0\,, \tag{3.7}$$

and therefore, E is a non-decreasing function. Thus, the minimum is attained at $\bar{R} := \frac{N-2}{N-1}$, and since $\bar{C} = E(\bar{R})\omega_{N-1}$, we obtain the desired assertion. □

We are now ready to prove the Theorem 1.2. Let $\varrho = \sum_{k=1}^n a_k\delta_{x_k}$ and

$$\mathcal{K}_+ := \{k \in \mathbb{N} \,:\, 1 \leq k \leq n \text{ and } a_k > 0\},$$
$$\mathcal{K}_- := \{k \in \mathbb{N} \,:\, 1 \leq k \leq n \text{ and } a_k < 0\}.$$

Proof of Theorem 1.2 Without loss of generality, assume $j \in \mathcal{K}_+$ and $l \in \mathcal{K}_-$. Let $u_\pm \in \mathcal{X}\setminus\{0\}$ be the unique minimizers of

$$I_\pm(u) := \int_{\mathbb{R}^N} \left(1-\sqrt{1-|\nabla u|^2}\right)\mathrm{d}x - \sum_{k\in\mathcal{K}_\pm} a_k u(x_k),$$

respectively. By Proposition 2.3

$$0 > I_\pm(u_\pm) \geq \frac{1}{2}\|\nabla u_\pm\|^2_{L^2(\mathbb{R}^N)} - \left(\sum_{k\in\mathcal{K}_\pm}|a_k|\right)\|u_\pm\|_{L^\infty(\mathbb{R}^N)}, \tag{3.8}$$

where we have used the inequality $\frac{1}{2}t \leq 1 - \sqrt{1-t}$ for $t \in [0, 1]$. On the other hand, by Lemma 3.6, we have

$$\|\nabla u_{\pm}\|^2_{L^2(\mathbb{R}^N)} \geq \bar{C}\|u_{\pm}\|^N_{L^\infty(\mathbb{R}^N)}.$$

Together with (3.8), this gives

$$\|u_{\pm}\|_{L^\infty(\mathbb{R}^N)} \leq \left(\frac{2}{\bar{C}} \sum_{k\in\mathcal{K}_\pm} |a_k|\right)^{\frac{1}{N-1}} \tag{3.9}$$

and in particular,

$$\pm\, u_{\pm}(x_j) = |u_{\pm}(x_j)| \leq \left(\frac{2}{\bar{C}} \sum_{k\in\mathcal{K}_\pm} |a_k|\right)^{\frac{1}{N-1}} \quad \text{for all } j \in \{1, \ldots n\}, \tag{3.10}$$

since $u_+ \geq 0$ and $u_- \leq 0$ in all of $\mathbb{R}^N$, by the comparison principle [4, Lemma 2.12]. By the same principle, we also know that

$$u_-(x) \leq u_\varrho(x) \leq u_+(x) \qquad \text{for all } x \in \mathbb{R}^N.$$

Hence, by (3.10), (1.9), and (3.6)

$$\begin{aligned} u_\varrho(x_j) - u_\varrho(x_l) &\leq u_+(x_j) - u_-(x_l) \\ &\leq \left(\frac{2}{\bar{C}} \sum_{k\in\mathcal{K}_+} |a_k|\right)^{\frac{1}{N-1}} + \left(\frac{2}{\bar{C}} \sum_{k\in\mathcal{K}_-} |a_k|\right)^{\frac{1}{N-1}} \\ &< \min_{\substack{h,i\in\{1,\ldots,n\}\\ h\neq i}} |x_h - x_i| \leq |x_j - x_l|. \end{aligned} \tag{3.11}$$

By Theorem 2.4, either u_ϱ is smooth on $\mathrm{Int}(\overline{x_j x_l})$ or

$$u_\varrho(tx_l + (1-t)x_j) = tu_\varrho(x_l) + (1-t)u_\varrho(x_j) \quad \text{for all } t \in (0, 1). \tag{3.12}$$

For a contradiction, assume (3.12). Then, Theorem 3.5 yields that x_j is a strict relative maximizer and

$$\lim_{t\to 0^+} \frac{u_\varrho(t(x_l - x_j) + x_j) - u_\varrho(x_j)}{t|x_l - x_j|} = -1.$$

By (3.12), this gives immediately

$$\frac{u_\varrho(x_l) - u_\varrho(x_j)}{|x_l - x_j|} = -1. \tag{3.13}$$

Whence, together with (3.11), we have

$$-|x_l - x_j| < u_\varrho(x_l) - u_\varrho(x_j) = -|x_l - x_j|,$$

a contradiction. We can now repeat the same argument for all the couples of point charges and conclude the proof. □

Remark 3.9 By (3.11), it is apparent that under the weaker assumption

$$\left(\frac{N}{\omega_{N-1}}\right)^{\frac{1}{N-1}}\frac{N-1}{N-2}\left[\left(\sum_{k\in\mathcal{K}_+}a_k\right)^{\frac{1}{N-1}}+\left(\sum_{k\in\mathcal{K}_-}|a_k|\right)^{\frac{1}{N-1}}\right]<|x_j-x_l|,$$

we get the result (i.e., u_ϱ is a classical solution) only along the line segment $\mathrm{Int}(x_jx_l)$.

Furthermore, it is possible to refine (3.9), and consequently the sufficient condition (1.9), by replacing (3.5) with the following inequality

$$\int_{\mathbb{R}^N}\left(1-\sqrt{1-|\nabla u|^2}\right)\mathrm{d}x\geq\widetilde{C}\|u\|^N_{L^\infty(\mathbb{R}^N)}\quad\text{for all } u\in\mathcal{X}\tag{3.14}$$

and for some $\widetilde{C}=\widetilde{C}(N)\geq\frac{\bar{C}}{2}$. Indeed, suppose we have already proved (3.14). Starting as in the proof of Theorem 1.2, we have

$$0>I_\pm(u_\pm)\geq\int_{\mathbb{R}^N}\left(1-\sqrt{1-\|\nabla u_\pm\|^2}\right)\mathrm{d}x-\left(\sum_{k\in\mathcal{K}_\pm}|a_k|\right)\|u_\pm\|_{L^\infty(\mathbb{R}^N)}$$

that, combined with (3.14), gives

$$\|u_\pm\|_{L^\infty(\mathbb{R}^N)}\leq\left(\frac{1}{\widetilde{C}}\sum_{k\in\mathcal{K}_\pm}|a_k|\right)^{\frac{1}{N-1}}.$$

Hence, it is enough to require

$$\widetilde{C}^{-\frac{1}{N-1}}\left[\left(\sum_{k\in\mathcal{K}_+}a_k\right)^{\frac{1}{N-1}}+\left(\sum_{k\in\mathcal{K}_-}|a_k|\right)^{\frac{1}{N-1}}\right]<|x_j-x_l|\tag{3.15}$$

(which is a weaker assumption than (1.9), since $\widetilde{C}^{-\frac{1}{N-1}}\leq(\bar{C}/2)^{-\frac{1}{N-1}}$) to conclude the statement of Theorem 1.2. As in Lemma 3.6 (see also [5]), we can show that $\widetilde{C}$ is attained by the unique weak solution $\tilde{u}$ of the problem

$$\begin{cases}-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)=a\delta_0 & \text{in } \mathbb{R}^N,\\ \lim_{|x|\to\infty}u(x)=0\end{cases}$$

with $a:=A(N)^{1-N}$ and

$$A(N):=\omega_{N-1}^{-\frac{1}{N-1}}\int_0^{+\infty}\frac{\mathrm{d}s}{\sqrt{s^{2(N-1)}+1}},\tag{3.16}$$

cf. [4, Theorem 1.4]. Such $\tilde{u}$ is radial and radially decreasing, and the previous problem in radial coordinates reads as

$$\begin{cases}\left(r^{N-1}\frac{u'}{\sqrt{1-(u')^2}}\right)'=0 & \text{in } (0,\infty),\\ u(0)=1,\quad \lim_{r\to\infty}u(r)=0,\end{cases}$$

where as usual we have written $u(r) := u(x)$ for $r = |x|$. Therefore,

$$\tilde{u}(r) = \int_r^{+\infty} \frac{a/\omega_{N-1}}{\sqrt{s^{2(N-1)} + (a/\omega_{N-1})^2}} \mathrm{d}s,$$

see below for a similar argument. Hence,

$$\begin{aligned} \widetilde{C} &= \omega_{N-1} \int_0^{\infty} r^{N-1} \left(1 - \sqrt{1 - (\tilde{u}'(r))^2}\right) \mathrm{d}r \\ &= \omega_{N-1} \frac{\displaystyle\int_0^{\infty} r^{N-1} \left(1 - \frac{r^{N-1}}{\sqrt{r^{2(N-1)} + 1}}\right) \mathrm{d}r}{\left(\displaystyle\int_0^{\infty} \frac{1}{\sqrt{r^{2(N-1)} + 1}} \mathrm{d}r\right)^N}. \end{aligned} \tag{3.17}$$

We can numerically check that, for example when $N = 3$,

$$\bar{C} = \frac{\omega_2}{6} \leq 2\widetilde{C} \approx 2 \cdot 0,097\, \omega_2.$$

To end this section, we consider the case of two point charges of different signs, namely

$$\varrho := a_1 \delta_{x_1} + a_2 \delta_{x_2}, \tag{3.18}$$

with $a_1 \cdot a_2 < 0$. In this case, we can give a more precise sufficient condition.

Proposition 3.10 *Let ϱ be as in (3.18). If $a_1 \cdot a_2 < 0$ and*

$$\left(|a_1|^{\frac{1}{N-1}} + |a_2|^{\frac{1}{N-1}}\right) A(N) < |x_1 - x_2|,$$

where $A(N)$ is defined in (3.16), then $u_\varrho \in C^\infty(\mathbb{R}^N \setminus \{x_1, x_2\}) \cap C(\mathbb{R}^N)$, it is a classical solution of (3.4) and it is strictly spacelike in $\mathbb{R}^N \setminus \{x_1, x_2\}$.

Proof It is standard to prove that for $k = 1, 2$ the unique solution $\tilde{u}_k$ of

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right) = a_k \delta_{x_k} \quad \text{in } \mathbb{R}^N, \tag{3.19}$$

with $\lim_{|x|\to\infty} u = 0$, is radial about x_k and satisfies

$$\frac{r^{N-1}\tilde{u}_k'(r)}{\sqrt{1 - \tilde{u}_k'(r)^2}} = C \quad \text{in } \mathbb{R}^N \setminus \{x_k\} \text{ for some } C \in \mathbb{R}, \tag{3.20}$$

where with abuse of notation $\tilde{u}_k(r) = \tilde{u}_k(|x - x_k|)$ and $'$ denotes the derivation with respect to $r := |x - x_k|$. In particular, by (3.20), $\tilde{u}_k'$ never changes sign, and therefore, $\tilde{u}_k$ is monotone in r. Since $\tilde{u}_k$ vanishes at infinity, by (3.20) we obtain

$$\begin{aligned} -C\tilde{u}_k(0) &= C\left(\lim_{r\to+\infty} \tilde{u}_k(r) - \tilde{u}_k(0)\right) = C \int_0^{+\infty} \tilde{u}_k'(r)\mathrm{d}r \\ &= \int_0^{+\infty} \frac{r^{N-1}\tilde{u}_k'(r)^2}{\sqrt{1 - \tilde{u}_k'(r)^2}} \mathrm{d}r = \frac{a_k}{\omega_{N-1}} \tilde{u}_k(0). \end{aligned}$$

Since $\tilde{u}_k$ is monotone in r, $a_k \neq 0$, and $\lim_{r\to\infty} \tilde{u}_k = 0$, we have that $\tilde{u}_k(0) \neq 0$, whence $C = -a_k/\omega_{N-1}$. Furthermore, by solving for $\tilde{u}_k'$ in (3.20) and integrating we have

$$\tilde{u}_k(r) = \int_r^{+\infty} \frac{a_k/\omega_{N-1}}{\sqrt{s^{2(N-1)} + (a_k/\omega_{N-1})^2}} \mathrm{d}s \qquad \text{for } k = 1,\ 2,$$

and in particular,

$$\tilde{u}_k(0) = \mathrm{sign}(a_k)|a_k|^{\frac{1}{N-1}} A(N) \qquad \text{for } k = 1,\ 2. \tag{3.21}$$

Since $a_1 > 0 > a_2$, $a_2\delta_{x_2} \leq \varrho \leq a_1\delta_{x_1}$ (cf. [4, Definition 2.11]). By Comparison Lemma 2.12 of [4], we know that

$$\tilde{u}_2(x) \leq u_\varrho(x) \leq \tilde{u}_1(x) \qquad \text{for all } x \in \mathbb{R}^N.$$

The conclusion follows exactly as in Theorem 1.2. □

4 Approximating problem

In this section, we study some qualitative properties of the approximating solutions u_m of problem (1.10). In particular, we focus on the regularity of u_m in Proposition 1.3 and on their local behavior near the singularities x_k's, proving Theorem 1.4. From these results, it is apparent that u_m's behavior resembles the behavior of the minimizer u_ϱ that we approximate (see also Introduction for more comments).

Proof of Proposition 1.3 Let us denote

$$\begin{aligned}
A(p) &:= \sum_{h=1}^m \alpha_h |p|^{2h-2} p,\\
a^{ij}(p) &:= \frac{\partial A_i}{\partial p_j} = \sum_{h=1}^m \alpha_h \left[(2h-2)|p|^{2h-4} p_i p_j + |p|^{2h-2}\delta_{ij}\right],\\
F(t) &:= \sum_{h=1}^m \alpha_h t^{2h-2}
\end{aligned}$$

for every $p \in \mathbb{R}^N$ and $t \geq 0$, where δ_{ij} is the Kronecker delta. Then, by straightforward calculations we have for all $p, \ \xi \in \mathbb{R}^N$

$$\begin{aligned}
\sum_{i,j=1}^N a^{ij}(p)\xi_i\xi_j = \left(\sum_{h=1}^m \alpha_h |p|^{2h-2}\right)|\xi|^2 + (p\cdot\xi)^2 \sum_{h=1}^m \alpha_h(2h-2)|p|^{2h-4} \geq F(|p|)|\xi|^2,\\
|a^{ij}(p)| \leq \sum_{h=1}^m \alpha_h |p|^{2h-2} + \sum_{h=1}^m \alpha_h(2h-2)|p|^{2h-2} \leq (2m-1)F(|p|),\\
|A(p)| = \sum_{h=1}^m \alpha_h |p|^{2h-2}|p| = |p|F(|p|).
\end{aligned}$$

Therefore, the operator $-\sum_{h=1}^m \alpha_h \Delta_{2h}$ and the function F satisfy the hypotheses of [19, Lemma 1] with $\Lambda = (2m-1)$. To verify the last assumption in [19, Lemma 1], let u_m be the solution of (1.10). Since $2m > \max\{N, 2^*\}$, one has $\mathcal{X}_{2m} \hookrightarrow C_0^{0,\beta_m}(\mathbb{R}^N)$, and in particular, $u_m \in \mathcal{X}_{2m}$ is bounded. Let B_{4R} be any ball of radius $4R$, such that $x_k \notin B_{4R}$ for any $k = 1, \cdots, n$. Then u_m satisfies

$$-\mathrm{div}\left(\sum_{h=1}^{m} \alpha_h |\nabla u_m|^{2h-2} \nabla u_m\right) = 0 \quad \text{in } B_{4R} \quad \text{in the weak sense,}$$

and since $u_m \in \mathcal{X}_{2m}$,

$$\int_{B_{4R}} F\left(|\nabla u_m|\right)\left(1 + |\nabla u_m|\right)^2 \mathrm{d}x < \infty.$$

Therefore, by [19, Lemma 1], $u_m \in C^{1,\beta}(B_R)$ for some $\beta \in (0, 1)$, and B_R has the same center as B_{4R}. We consider now the linear Dirichlet problem

$$\begin{cases} L_m u := -\,\mathrm{div}\left(\sum_{h=1}^{m} \alpha_h |\nabla u_m|^{2h-2} \nabla u\right) = 0 & \text{in } B_R, \\ u = u_m & \text{on } \partial B_R. \end{cases} \tag{4.1}$$

Clearly, u_m is a weak solution of (4.1). The boundary datum u_m is continuous on ∂B_R, and the operator L_m is strictly elliptic in B_R and has coefficients in $C^{0,\beta}(B_R)$. Hence, by [15, Theorem 6.13], (4.1) has a unique solution in $C(\bar{B}_R) \cap C^{2,\beta}(B_R)$, whence $u_m \in C(\bar{B}_R) \cap C^{2,\beta}(B_R)$. We consider again (4.1). Now we know that the coefficients of L_m are of class $C^{1,\beta}(B_R)$ and that u_m is a C^2-solution of the equation in (4.1). By [15, Theorem 6.17], $u_m \in C^{3,\beta}(B_R)$. By a bootstrap argument, we obtain $u_m \in C^\infty(B_R)$. By the arbitrariness of R and of the center of the ball B_R, $u_m \in C^\infty(\mathbb{R}^N \setminus \{x_1, \ldots, x_n\})$. □

Remark 4.1 The presence of the Laplacian in the operators sum $\sum_{h=1}^{m} \alpha_h \Delta_{2h}$ plays an essential role in the proof of the previous result. Indeed, we observe that, among the hypotheses on F, [19, Lemma 1] requires $F(t) \geq \varepsilon > 0$ for all $t \geq 0$, which is satisfied with $\varepsilon = \alpha_1$ thanks to the presence of the Laplacian.

Next, we study the behavior of the solution u_m of (1.10) and of its gradient, near the point charges x_k's.

Proof of Theorem 1.4 For any $k = 1, \ldots, n$, fix $R_k > 0$ so small that $B_{R_k}(x_k) \cap \{x_1, \ldots, x_n\} = \{x_k\}$. Then, u_m solves

$$\begin{cases} -\sum_{h=1}^{m} \alpha_h \Delta_{2h} u = a_k \delta_{x_k} & \text{in } B_{R_k}(x_k), \\ u = u_m & \text{on } \partial B_{R_k}(x_k) \end{cases} \tag{4.2}$$

for all $k = 1, \ldots, n$. We split the proof into six steps.

Step 1: Translation. For all $\varphi \in C_c^\infty(B_{R_k}(x_k))$

$$\sum_{h=1}^{m} \int_{B_{R_k}(x_k)} \alpha_h |\nabla u_m|^{2h-2} \nabla u_m \cdot \nabla \varphi \mathrm{d}x = a_k \varphi(x_k). \tag{4.3}$$

So, if we define $u_{m,k}(x) := u_m(x + x_k) - u_m(x_k)$ and $\varphi_k(x) := \varphi(x + x_k)$ for all $x \in B_{R_k}(0)$, we get $u_{m,k} \in C^\infty(B_{R_k}(0) \setminus \{0\})$, $\varphi_k \in C_c^\infty(B_{R_k}(0))$ and

$$\sum_{h=1}^{m} \int_{B_{R_k}(0)} \alpha_h |\nabla u_{m,k}|^{2h-2} \nabla u_{m,k} \cdot \nabla \varphi_k \mathrm{d}x = a_k \varphi_k(0). \tag{4.4}$$

Hence, by the arbitrariness of $\varphi \in C_c^\infty(B_{R_k}(x_k))$, $u_{m,k}$ solves weakly

$$\begin{cases} -\sum_{h=1}^m \alpha_h \Delta_{2h} u = a_k \delta_0 & \text{in } B_{R_k}(0), \\ u = u_{m,k} & \text{on } \partial B_{R_k}(0). \end{cases} \tag{4.5}$$

Of course, we have $u_{m,k}(0) = 0$.

Step 2: Potential estimates on $u_{m,k}$. Consider the operator

$$-\sum_{h=1}^{m} \alpha_h \Delta_{2h} u = -\operatorname{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right),$$

with $g(t) := \sum_{h=1}^m \alpha_h t^{2h-1}$ for all $t \geq 0$, and note that

$$1 \leq \frac{g'(t)t}{g(t)} \leq 2m - 1 \quad \text{for all } t > 0.$$

By [2, Theorem 1.2], for every $x_0 \in B_{R_k}(0)$ Lebesgue point of $\nabla u_{m,k}$ and for every ball $B_{2R}(x_0) \subset B_{R_k}(0)$, one has

$$g\left(|\nabla u_{m,k}(x_0)|\right) \leq c\mathbb{I}_1^{|a_k\delta_0|}(x_0, 2R) + cg\left(\fint_{B_R(x_0)} |\nabla u_{m,k}| \mathrm{d}x\right), \tag{4.6}$$

where $c = c(N, m) > 0$ and

$$\mathbb{I}_1^{|a_k\delta_0|}(x_0, R) := \int_0^R \frac{|a_k\delta_0|(B_\rho(x_0))}{\rho^N} d\rho$$

is the truncated linear Riesz potential of the measure $|a_k\delta_0|$. Now, if $0 < |x_0| < R_k - 2R$

$$\mathbb{I}_1^{|a_k\delta_0|}(x_0, 2R) = \int_{|x_0|}^{2R} \frac{|a_k|}{\rho^N} d\rho \leq \frac{|a_k|}{(N-1)|x_0|^{N-1}}. \tag{4.7}$$

If furthermore $R > R_k/4$, it follows for almost every x_0 that

$$\begin{aligned} g\left(\fint_{B_R(x_0)} |\nabla u_{m,k}| \mathrm{d}x\right) &< \sum_{h=1}^m \alpha_h \left(\frac{\|\nabla u_{m,k}\|_{L^1(B_R(x_0))}}{|B_{R_k/4}|}\right)^{2h-1} \\ &\leq \sum_{h=1}^m \alpha_h \left(\frac{\|\nabla u_{m,k}\|_{L^1(B_{R_k}(0))}}{|B_{R_k/4}|}\right)^{2h-1} =: C, \end{aligned} \tag{4.8}$$

where $C = C\left(\|\nabla u_m\|_{L^1(B_{R_k}(x_k))}, N, g\right) > 0$ is independent of the specific x_0 and R considered. We note that if $|x_0| < R_k/4$, then (4.6)–(4.8) hold with any $R \in (R_k/4, 3R_k/8)$. Therefore, by combining (4.8) with (4.6) and (4.7), we obtain for a.e. $x \in B_{R_k/4}(0)$

$$\begin{aligned} |\nabla u_{m,k}(x)| &= \left(\frac{g\left(|\nabla u_{m,k}(x)|\right)}{\alpha_m}\right)^{\frac{1}{2m-1}} \\ &\leq \left\{\frac{c}{\alpha_m |x|^{N-1}}\left[\frac{|a_k|}{N-1} + C\left(\frac{R_k}{4}\right)^{N-1}\right]\right\}^{\frac{1}{2m-1}} =: \frac{C'}{|x|^{\frac{N-1}{2m-1}}}, \end{aligned} \tag{4.9}$$

with $C' = C'(\|\nabla u_m\|_{L^1(B_{R_k}(x_k))}, R_k, |a_k|, N, m, g) > 0$.

Step 3: Scaling. Fix two integers $m > \max\{N/2, 2^*/2\}$ and $k \in \{1, \dots, n\}$. For any $\varepsilon > 0$ and $x \in B_{R_k/\varepsilon}(0) \setminus \{0\}$, define $u_\varepsilon(x) := \varepsilon^{\frac{N-2m}{2m-1}} u_{m,k}(\varepsilon x)$. Then $u_\varepsilon \in C^\infty(B_{R_k/\varepsilon}(0) \setminus \{0\})$, and $\nabla u_\varepsilon(x) = \varepsilon^{\frac{N-1}{2m-1}} \nabla u_{m,k}(\varepsilon x)$ for all $x \in B_{R_k/\varepsilon}(0) \setminus \{0\}$. By substituting into (4.4), we obtain for any $\varphi \in C_c^\infty(B_{R_k/\varepsilon}(0))$

$$\sum_{h=1}^{m} \int_{B_{R_k/\varepsilon}(0)} \varepsilon^{N-2h+\frac{(2m-N)(2h-1)}{2m-1}} \alpha_h |\nabla u_\varepsilon|^{2h-2} \nabla u_\varepsilon \cdot \nabla\varphi \mathrm{d}x = a_k \varphi(0),$$

or in other words u_ε solves weakly

$$-\sum_{h=1}^{m} \varepsilon^{N-2h+\frac{(2m-N)(2h-1)}{2m-1}} \alpha_h \Delta_{2h} u = a_k \delta_0 \quad \text{in } B_{R_k/\varepsilon}(0). \tag{4.10}$$

We note that the exponent of ε is positive for $h < m$ and is zero for $h = m$. Also note that $u_\varepsilon(0) = 0$.

Step 4: Limit as $\varepsilon \to 0$. In terms of u_ε, (4.9) translates for a.e. $x \in B_{R_k/4\varepsilon}(0)$ to a global estimate

$$|\nabla u_\varepsilon(x)| \le C'|x|^{\frac{1-N}{2m-1}}. \tag{4.11}$$

Since $2m > N$, for fixed $\bar{R} \in (0, R_k/4\varepsilon)$, (4.11) yields

$$\int_{B_{\bar{R}}(0)} |\nabla u_\varepsilon|^{2m} \mathrm{d}x \le \frac{2m-1}{2m-N} C'^{2m} \omega_{N-1} \bar{R}^{\frac{2m-N}{2m-1}} =: C'', \tag{4.12}$$

where $C'' = C''(\|\nabla u_m\|_{L^1(B_{R_k}(x_k))}, |a_k|, N, m, g, \bar{R}) > 0$ independent of ε.

Next, we obtain local estimates uniform in ε. Let $A \subset B_{\bar{R}}(0) \setminus \{0\}$ be a compact set. Then, by (4.11) and since $u_\varepsilon(0) = 0$,

$$|u_\varepsilon(x)| \le \int_0^1 |\nabla u_\varepsilon(tx)||x|\mathrm{d}t \le C' \frac{2m-1}{2m-N} \bar{R}^{\frac{2m-N}{2m-1}} \quad \text{for all } x \in A. \tag{4.13}$$

Furthermore, by Proposition 1.3 we have

$$\begin{aligned} |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| &= \varepsilon^{\frac{N-1}{2m-1}} |\nabla u_{m,k}(\varepsilon x) - \nabla u_{m,k}(\varepsilon y)| \\ &\le \varepsilon^{\frac{N-1}{2m-1}+1-\beta_m} |x-y|^{1-\beta_m} \le |x-y|^{1-\beta_m}, \end{aligned}$$

for every $x, y \in A$ and $\varepsilon \le 1$. Since, by (4.11), $|\nabla u_\varepsilon|$ is also uniformly bounded in A, by the Arzelà–Ascoli theorem, there exist a subsequence, still denoted by (u_ε), and a function $\bar{u} \in C^1(A)$ such that $\lim_{\varepsilon\to 0} \nabla u_\varepsilon = \nabla\bar{u}$ in the uniform topology on A. By choosing $\bar{u}(0) = 0$, we obtain that $u_\varepsilon \to \bar{u}$ in $C^1(A)$. By (4.12) and the Fatou lemma, we have that $\|\nabla\bar{u}\|_{L^{2m}(B_{\bar{R}(0)})} \le (C'')^{1/(2m)}$. Hence, for any $\psi \in [L^{2m}(B_{\bar{R}}(0))]^N$

$$\begin{aligned} &\left| \int_{B_{\bar{R}}(0)} \left(|\nabla u_\varepsilon|^{2m-2}\nabla u_\varepsilon - |\nabla\bar{u}|^{2m-2}\nabla\bar{u}\right) \psi \,\mathrm{d}x \right| \\ &\le \int_A \left(\left||\nabla u_\varepsilon|^{2m-2}\nabla u_\varepsilon - |\nabla\bar{u}|^{2m-2}\nabla\bar{u}\right|\right) |\psi| \mathrm{d}x + 2\frac{(C')^{2m-1}}{\bar{R}^{N-1}} \|\,|\psi|\,\|_{L^1(B_{\bar{R}}(0)\setminus A)}. \end{aligned}$$

For any $\delta > 0$, we can take A such that $\|\,|\psi|\,\|_{L^1(B_{\bar{R}}(0)\setminus A)} \le \delta$, and for sufficiently small $\varepsilon > 0$ we have, from the uniform convergence of ∇u_ε on A, that

$$\left| \int_{B_{\bar{R}}(0)} \left(|\nabla u_\varepsilon|^{2m-2}\nabla u_\varepsilon - |\nabla\bar{u}|^{2m-2}\nabla\bar{u}\right) \psi \,\mathrm{d}x \right| \le C\delta$$

for some $C > 0$ independent of ε. Since $\delta > 0$ and $\psi \in [L^{2m}(B_{\bar{R}}(0))]^N$ were arbitrary, we have $|\nabla u_\varepsilon|^{2m-2}\nabla u_\varepsilon \rightharpoonup |\nabla \bar{u}|^{2m-2}\nabla \bar{u}$ in $[L^{(2m)'}(B_{\bar{R}}(0))]^N$. Recalling that u_ε solves weakly (4.10), we have for any $\varphi \in C_c^\infty(B_{\bar{R}}(0))$

$$\sum_{h=1}^{m} \int_{B_{\bar{R}}(0)} \varepsilon^{N-2h+\frac{(2m-N)(2h-1)}{2m-1}} \alpha_h |\nabla u_\varepsilon|^{2h-2}\nabla u_\varepsilon \cdot \nabla\varphi \mathrm{d}x = a_k\varphi(0)$$

and by passing $\varepsilon \to 0$ and using proved weak convergences, we obtain

$$\int_{B_{\bar{R}}(0)} \alpha_m |\nabla \bar{u}|^{2m-2}\nabla \bar{u} \cdot \nabla\varphi \mathrm{d}x = a_k\varphi(0)\,,$$

or equivalently $\bar{u}$ is a weak solution of

$$-\alpha_m \Delta_{2m} u = a_k\delta_0 \quad \text{in } B_{\bar{R}}(0). \tag{4.14}$$

Step 5: Behavior of $\bar{u}$ and its gradient near 0. By (4.14), we know that $\bar{u}$ is $2m$-harmonic in $B_{\bar{R}}(0) \setminus \{0\}$ and $\bar{u}(0) = 0$. As in Step 2, [2, Theorem 1.2] with $g(t) := \alpha_m t^{2m-1}$ yields for a.e. $x \in B_{\bar{R}/4}(0)$

$$\begin{aligned} |\nabla\bar{u}(x)| &\le \left\{ \frac{c}{\alpha_m |x|^{N-1}} \left[\frac{|a_k|}{(N-1)} + \alpha_m \left(\frac{\bar{R}}{4}\right)^{N-1} \left(\frac{\|\nabla\bar{u}\|_{L^1(B_{\bar{R}}(0))}}{|B_{\bar{R}/4}|} \right)^{2m-1} \right] \right\}^{\frac{1}{2m-1}} \\ &=: C_0|x|^{\frac{1-N}{2m-1}} \\ |\bar{u}(x)| &\le \frac{2m-1}{2m-N} C_0 |x|^{\frac{2m-N}{2m-1}}\,, \end{aligned}$$

where the second bound follows as in (4.13). Hence, the isotropy result [16, Remark 1.6] (see also work by Serrin [22]) implies

$$\lim_{x\to 0} \frac{\bar{u}(x)}{\mu(x)} = \gamma \quad \text{and} \quad \lim_{x\to 0} |x|^{\frac{N-1}{2m-1}} \nabla(\bar{u} - \gamma\mu) = 0, \tag{4.15}$$

where $\gamma := \operatorname{sign}(a_k)\left(\frac{|a_k|}{\alpha_m}\right)^{\frac{1}{2m-1}}$, and $\mu(x) := \kappa_m(N)|x|^{\frac{2m-N}{2m-1}}$ with $\kappa_m(N) := -\frac{2m-1}{2m-N}(N|B_1|)^{-\frac{1}{2m-1}}$ is the fundamental solution of the $-\Delta_{2m}$.

Step 6: Behavior of u_m and its gradient near x_k. Since $|x|^{\frac{N-1}{2m-1}}|\nabla\mu| = |\kappa_m|\frac{2m-N}{2m-1}$, from (4.15) follows

$$\lim_{x\to 0} |\nabla\bar{u}(x)||x|^{\frac{N-1}{2m-1}} = \frac{2m-N}{2m-1}|\gamma\kappa_m|.$$

Furthermore, by Step 4 we know in particular that $\nabla u_\varepsilon \to \nabla\bar{u}$ pointwise in $B_{\bar{R}}(0) \setminus \{0\}$. Hence,

$$\lim_{x\to 0}\left(\lim_{\varepsilon\to 0} |\nabla u_\varepsilon(x)||x|^{\frac{N-1}{2m-1}}\right) = \lim_{x\to 0} |\nabla\bar{u}(x)||x|^{\frac{N-1}{2m-1}} = \frac{2m-N}{2m-1}|\gamma\kappa_m|$$

and by the definition of u_ε,

$$\frac{2m-N}{2m-1}|\gamma\kappa_m| = \lim_{x\to 0}\left(\lim_{\varepsilon\to 0} \varepsilon^{\frac{N-1}{2m-1}} |\nabla u_{m,k}(\varepsilon x)||x|^{\frac{N-1}{2m-1}}\right) = \lim_{y\to 0} |\nabla u_{m,k}(y)||y|^{\frac{N-1}{2m-1}}\,.$$

Consequently,

$$|\nabla u_{m,k}(x)| \sim |\gamma\kappa_m| \frac{2m-N}{2m-1}|x|^{\frac{1-N}{2m-1}} \quad \text{as } x \to 0,$$

which in turn implies (1.12) with $K_m' := |\gamma\kappa_m|\frac{2m-N}{2m-1}$. Analogously, by Step 4 we also know that $u_\varepsilon \to \bar{u}$ pointwise in $B_{\bar{R}}(0) \setminus \{0\}$. Therefore, by (4.15)

$$\lim_{x\to 0}\left(\lim_{\varepsilon\to 0}\frac{u_\varepsilon(x)}{\gamma\kappa_m|x|^{\frac{2m-N}{2m-1}}}\right) = 1$$

which in terms of $u_{m,k}$ gives

$$\lim_{x\to 0}\frac{u_{m,k}(x)}{|x|^{\frac{2m-N}{2m-1}}} = \gamma\kappa_m$$

and proves (1.11) with $K_m := \gamma\kappa_m$. In particular, if $a_k > 0$, then $K_m \cdot a_k < 0$, and x_k is a relative strict maximizer of u_m, while if $a_k < 0$ it is a relative strict minimizer of u_m. □

Remark 4.2 Observe that, since $\alpha_m = \frac{(2m-3)!!}{(2m-2)!!}$,

$$\lim_{m\to\infty}|K_m| = \lim_{m\to\infty}\frac{2m-1}{2m-N}\left(\frac{|a_k|}{N|B_1|\alpha_m}\right)^{\frac{1}{2m-1}} = 1.$$

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