

Classifying the Complexity of the Possible Winner Problem on Partial Chains

Vishal Chakraborty

University of California Santa Cruz
vchakrab@ucsc.edu

Phokion G. Kolaitis

University of California Santa Cruz and IBM Research
kolaitis@ucsc.edu

ABSTRACT

The POSSIBLE WINNER (PW) problem, a fundamental algorithmic problem in computational social choice, concerns elections where voters express only partial preferences between candidates. A sequence of investigations led to a complete classification of the complexity of this problem for all pure positional scoring rules: the PW problem is in P for the plurality and veto rules, and NP-complete for all other such rules. The PW problem has also been studied on classes of restricted partial orders, such as partitioned partial orders and truncated partial orders; one of the findings is that there are positional scoring rules for which the complexity of the PW problem drops from NP-complete to P on such restricted partial orders. Here, we investigate the PW problem on partial chains, i.e., partial orders that consist of a total order on a subset of their domains. Such orders arise naturally in a variety of settings, including rankings of movies or restaurants. We classify the complexity of the PW problem on partial chains by establishing that, perhaps surprisingly, this restriction does not change the complexity of the problem. Specifically, we show that the PW problem on partial chains is NP-complete for all pure positional scoring rules other than the plurality rule and the veto rule, while, of course, for the latter two rules this problem remains in P.

KEYWORDS

Elections; incomplete votes; possible winner; partial chains; computational complexity; NP-completeness

ACM Reference Format:

Vishal Chakraborty and Phokion G. Kolaitis. 2021. Classifying the Complexity of the Possible Winner Problem on Partial Chains. In *Proc. of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2021)*, Online, May 3–7, 2021, IFAAMAS, 9 pages.

1 INTRODUCTION

Determining the winners in an election under various voting rules has been a mainstream topic of research in computational social choice. Ideally, each voter has a clear ranking among the candidates, from the most preferred one to the least preferred one. In reality, however, a voter may have only limited information about the candidates. When the set of candidates is extremely large, voters may have information only about a subset of the candidates. For example, consider the large collection of movies on Netflix; it is natural that each viewer has watched only a subset of the movies. The set of candidates can also change. Continuing the previous example, newly released movies may be added on Netflix. Other

similar scenarios include hiring faculty, where new candidates may be added to the pool and candidates who have accepted offers elsewhere withdraw. These scenarios translate to a voter providing only a partial order among the candidates that reflects the voter's incomplete preferences (for a survey, see [6]).

The problem of voting under partial information was introduced by Konczak and Lang in [19], and was formalised via the notions of *possible winners* and *necessary winners*, where a candidate is a possible (necessary) winner if that candidate is a winner in at least one collection (respectively, in all collections) of linear orders that extend the partial orders in the collection provided by the voters. A thorough study of the complexity of the associated decision problems POSSIBLE WINNER (PW) and NECESSARY WINNER (NW) led to the classification of the complexity for all *pure positional scoring rules* [3, 4, 19, 22]. Specifically, the NW problem is in P with respect to all pure positional scoring rules, while the PW problem is in P with respect to the plurality and the veto rules, but it is NP-complete with respect to all other such rules.

More recently, the PW problem was studied on classes of restricted partial orders that arise in natural settings (for a survey, see [20]). These include *doubly-truncated* partial orders, where each voter linearly orders some top and bottom candidates, but expresses no preference for the ones in the middle; special cases of these partial orders are the *top-truncated* and the *bottom-truncated* partial orders. While no complete classification has been obtained for the PW problem under these restrictions, it was shown in [1, 5, 11] that there are pure positional scoring rules, such as the 2-approval rule, for which the complexity of PW drops from NP-complete to P on doubly-truncated partial orders, while for others rules, such as the Borda count, PW remains NP-complete. The complexity of PW on a generalisation of doubly-truncated partial orders, called *partitioned* partial orders, was studied in [18]. A partial order is *partitioned* if its elements can be partitioned into disjoint sets with a linear order between the disjoint sets, but no preference between elements in each set. In [18], it was shown that, for all 2-valued rules (which contain t -approval, for every $t \geq 2$, as a special case) and the rule given by the scoring vector $(2, 1, \dots, 1, 0)$ the complexity of PW on partitioned partial orders drops from NP-complete to P, but remains NP-complete for the Borda count.

The PW problem on a different restriction of partial orders was studied in [8] under the name the POSSIBLE COWINNER WITH NEW CANDIDATES problem. This models the setting of an election in which one or more candidates enter the race late; at that point, a complete ranking of the original candidates is available, but no new candidate has been ranked yet. The question is to tell whether a given original candidate is a possible winner when all (original and new) candidates are considered. In [8], it was shown that this problem is in P for the 2-approval rule and for the Borda count, but it is NP-complete for the t -approval rule, for each $t \geq 3$.

Proc. of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2021), U. Endriss, A. Nowé, F. Dignum, A. Lomuscio (eds.), May 3–7, 2021, Online. © 2021 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org). All rights reserved.

Summary of results. In this paper, we investigate the PW problem on a special kind of incomplete preferences, which we call *partial chains*. By definition, a *partial chain* is a partial order that consists of a total order on a non-empty subset of its domain. Partial chains arise naturally when the number of candidates is large and so each voter can rank only a subset of the candidates [7]. Consider, for example, the set of movies released in 2020. Most viewers have seen only a subset of these movies and, therefore, can only rank the movies they have seen. Moreover, the subset of movies may vary from viewer to viewer. Partial chains are the most fitting model for this type of scenario. Indeed, it might be the case that a voter will like a movie they have not seen so far more (or less) than any of the movies they have already seen. This state of affairs can be modelled by partial chains, but not by partitioned, doubly-truncated, top-truncated, or bottom-truncated partial orders.

Our main results is a complete classification of the complexity of the PW problem on partial chains: the problem is in P for plurality and veto, and it is NP-complete for all other positional scoring rules. This should be contrasted with the results about the restrictions of partial orders discussed earlier. Unlike partitioned partial orders, the complexity of PW for 2-approval does not drop to P when restricted to partial chains; also, unlike the partial orders in the POSSIBLE COWINNER WITH NEW CANDIDATES problem, the complexity of PW for the Borda count does not drop to P when restricted to partial chains. Note that our classification result implies the classification of the PW problem on arbitrary partial orders, but not the other way around. The proof of our classification theorem is rather compact and, in essence, uses only three reductions, all of which are from the 3-DIMENSIONAL MATCHING problem.

We also discuss the connection between the PW problem on partial chains and the POSSIBLE COWINNER WITH NEW CANDIDATES problem studied in [8]. By definition, a collection of partial chains is said to be *uniform* if all partial chains in the collection consist of a total order on the *same* non-empty subset of their domains. This is precisely the restriction obeyed by the collections of partial orders in [8]. Thus, the PW problem on uniform collections of partial chains coincides with the POSSIBLE COWINNER WITH NEW CANDIDATES problem (and when the number of the new candidates is part of the input). We note that, while some complexity results have been established, the complexity of the POSSIBLE COWINNER WITH NEW CANDIDATES problem remains unsettled for a large number of positional scoring rules.

2 PRELIMINARIES AND EARLIER WORK

A (strict) *partial order* on a set C is a binary relation $>$ on C that is irreflexive (i.e., $a \not> a$, for every $a \in C$) and transitive (i.e., $a > b$ and $b > c$ imply $a > c$, for all $a, b, c \in C$). A *total order* on C is a partial order $>$ on C such that $a = b$ or $a > b$ or $b > a$, for all $a, b \in C$.

Let $C = \{c_1, \dots, c_m\}$ be a set of *candidates* and let $V = \{v_1, \dots, v_n\}$ be a set of *voters*. A *vote* is a partial order on the set of candidates. A (complete) *voting profile* is a tuple $T = (T_1, \dots, T_n)$ of total orders on elements of C , where each T_i represents the ranking (preference) of voter v_i on the candidates in C . Similarly, a *partial voting profile* is a tuple $P = (P_1, \dots, P_n)$ of partial orders on C , where each P_i represents the partial preferences of voter v_i on the candidates in C . A *completion* of a partial voting profile $P = (P_1, \dots, P_n)$ is a complete

voting profile $T = (T_1, \dots, T_n)$ such that each T_i is a completion of the partial order P_i , i.e., T_i is a total order that extends P_i . Note that a partial voting profile may have exponentially many completions.

We focus on *positional scoring rules*, a widely studied class of voting rules. A positional scoring rule r on a set of m candidates is specified by a scoring vector $s = (s_1, \dots, s_m)$ of non-negative integers, called the *score values*, such that $s_1 \geq s_2 \geq \dots \geq s_m$ and $s_1 > s_m$. Suppose that $T = (T_1, \dots, T_n)$ is a total voting profile. The score $s(T_i, c)$ of a candidate c on T_i is the score value s_k where k is the position of candidate c in T_i . The *score* of c under the positional scoring rule r on the total profile T is the sum $s(T, c) = \sum_{i=1}^n s(T_i, c)$. A candidate c is a *winner* if c 's score is greater than or equal to the scores of all other candidates; similarly, c is a *unique winner* if c 's score is greater than the scores of all other candidates. The set of all winners is denoted by $W(r, T)$.

We consider positional scoring rules that are defined for every number m of candidates. Thus, a *positional scoring rule* is an infinite sequence $s_1, s_2, \dots, s_m, \dots$ of scoring vectors such that each s_m is a scoring vector of length m . Alternatively, a positional scoring rule is a function r that takes as argument a pair (m, s) of positive integers with $s \leq m$ and returns as value a non-negative integer $r(m, s)$ such that $r(m, 1) \geq r(m, 2) \geq \dots \geq r(m, m)$ and $r(m, 1) > r(m, m)$. We assume that the function r is computable in time polynomial in m , hence the winners can be computed in polynomial time. Such a rule is *pure* if the scoring vector s_{m+1} of length $(m+1)$ is obtained from the scoring vector s_m of length m by inserting a score value in some position of s_m , provided that the non-increasing order of score values is maintained. The plurality rule $(1, 0, \dots, 0)$, the veto rule $(1, \dots, 1, 0)$, the t -approval rule $(\underbrace{1, \dots, 1}_t, 0, \dots, 0)$ with a fixed $t \geq 2$,

for $m > 2$, and the Borda count $(m-1, m-2, \dots, 1, 0)$ are prominent pure positional scoring rules. We assume that each scoring vector s_m is normalised, i.e., the score values are eventually 0 and the gcd of the non-zero score values is 1; this is not a restriction (see [17]).

Let r be a voting rule and P a partial voting profile. The following notions were introduced by Konczak and Lang [19].

- The set $PW(r, P)$ of the *possible winners* w.r.t. r and P is the union of the sets $W(r, T)$, where T varies over all completions of P . Thus, a candidate c is a *possible winner* w.r.t. r and P , if c is in the set $W(r, T)$ of winners, for at least one completion T of P .

The POSSIBLE WINNER PROBLEM (PW) w.r.t. r asks: given a set of candidates C , a partial profile P , and a distinguished candidate $c \in C$, is $c \in PW(r, P)$?

- The set $NW(r, P)$ of the *necessary winners* w.r.t. r and P is the intersection of the sets $W(r, T)$, where T varies over all completions of P . Thus, a candidate c is a *necessary winner* w.r.t. r and P , if c is in the set $W(r, T)$ of winners, for every completion T of P .

The NECESSARY WINNER PROBLEM (NW) w.r.t. r asks: given a set of candidates C , a partial profile P , and a distinguished candidate $c \in C$, is $c \in NW(r, P)$?

The notions of *necessary unique winners* and *possible unique winners* are defined in an analogous manner.

In [19], it was shown that if r is an arbitrary pure positional scoring rule, then the necessary winner problem NW w.r.t. r is in P. The following classification theorem concerning the computational

complexity of the possible winner problem PW was established through a sequence of investigations.

THEOREM 1. [Classification Theorem [3, 4, 19, 22]] *The possible winner problem PW w.r.t. the plurality rule and the veto rule is in P. For all other pure positional scoring rules r , the possible winner problem PW is NP-complete. The same classification holds for the possible unique winner problem.*

The proof of the above classification is rather involved; furthermore, the NP-hardness proofs of PW for various scoring rules use reductions from several different known NP-complete problems, including the problems 3-DIMENSIONAL MATCHING, 3-SAT, EXACT 3-COVER, HITTING SET, and MULTICOLOURED CLIQUES.

3 PW ON PARTIAL CHAINS

In this section, we present the main result of the paper.

Definition 1. A partial order on a set C is said to be a *partial chain* if it consists of a linear order on a non-empty subset C' of C .

Let $C = \{a, b, c, d, e\}$ be a set of candidates. Clearly, every total order on C is a partial chain. Two other examples of partial chains on C are $a > d > c$ and $d > a > c > b$.

Definition 2. We write PW-PC to denote the restriction of the PW problem to partial chains. More precisely, the PW-PC problem with respect to a positional scoring rule r asks: given a set of candidates C , a partial profile P in which every partial order P_l , $1 \leq l \leq n$, is a partial chain, and a distinguished candidate $c \in C$, is $c \in PW(r, P)$?

Since PW-PC is a special case of PW, Theorem 1 implies that if r is the plurality rule or the veto rule, then the PW-PC problem with respect to r is in P. The main result of this paper asserts that these are the only tractable cases, and thus it yields a classification of the PW-PC problem.

THEOREM 2. *Let r be a pure positional scoring rule other than the plurality and the veto rules. Then the PW-PC problem with respect to r is NP-complete.*

Note that our Theorem 2 implies Theorem 1, but not the other way around.

3.1 Proof outline of Theorem 2

This section contains an outline of the proof of Theorem 2.

NP-complete problem used. As mentioned earlier, the NP-completeness of PW for rules other than plurality and veto in Theorem 1 was established via reductions from a variety of well known NP-complete problems. Furthermore, none of these reductions used partial chains in the PW-instances constructed. Here, we will establish the NP-hardness of PW-PC for rules other than plurality and veto via reductions from a *single* well known NP-complete problem, namely, the 3-DIMENSIONAL MATCHING (3DM) Problem (Problem [SP1] in [16]). The problem asks: given three disjoint sets $X = \{x_1, \dots, x_q\}$, $Y = \{y_1, \dots, y_q\}$, $Z = \{z_1, \dots, z_q\}$ of the same size, and a set $\mathcal{S} = \{S_i \in X \times Y \times Z \mid 1 \leq i \leq \tau\}$, is there a subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $|\mathcal{S}'| = q$ and $\mathcal{S}'$ does not contain two different triples that agree in at least one of their coordinates?

Grouping of pure positional scoring rules. In Theorem 1, the NP-hardness of PW with respect to rules other than plurality

and veto was established by considering either groups of rules with similar characteristics [22] or individual rules, such as the rule with scoring vectors of the form $(2, 1, \dots, 1, 0)$ [3]. Here, we will establish the NP-hardness of PW-PC with respect to pure positional scoring rules other than plurality and veto by grouping these rules into *bounded* rules and *unbounded* rules.

Definition 3. Let r be a pure positional scoring rule.

- We say that r is *p-valued*, where p is a positive integer greater than 1, if there exists a positive integer n_0 such that for all $m \geq n_0$, the scoring vector s_m of r contains exactly p distinct values.
- We say that r is *bounded* if r is p -valued, for some $p > 1$; otherwise, r is *unbounded*.

Clearly, the plurality rule, the veto rule, and the t -approval rule for each fixed $t \geq 2$, are 2-valued rules. Other examples include rules of the form $(1, \dots, 1, 0, \dots, 0)$, where the number of positions with score value 1 is not fixed across all scoring vectors. Furthermore, the rule with scoring vectors of the form $(2, 1, \dots, 1, 0)$ is 3-valued, while the Borda count $(m-1, m-2, \dots, 0)$ is an unbounded rule. Unlike the Borda count, an unbounded scoring rule may have score values that are not decreasing at the same rate or may have arbitrarily long repeating score values.

Main Steps. The technical cornerstones of the proof of Theorem 2 are three polynomial-time reductions from the 3DM problem to the PW-PC problem w.r.t. the following types of scoring rules:

- 2-approval (extended to all 2-valued rules other than plurality and veto);
- 3-valued rules (extended to all p -valued rules with $p > 3$);
- unbounded scoring rules.

In each reduction, we are given a 3DM instance $(X, Y, Z, \mathcal{S})$ where $\mathcal{S} = \{S_1, \dots, S_\tau\} \subseteq X \times Y \times Z$ such that $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, for $1 \leq i \leq \tau$. The reduction produces a PW-PC instance consisting of set C of candidates, a partial profile, and a distinguished candidate $c \in C$. The partial profile we construct from the 3DM instance has two parts. The first part is a set of partial chains (which are not total orders) that encode the given instance of the 3DM problem. It is worth pointing out that these partial chains have *at most two* candidates “missing”. The high-level idea of the construction is as follows. In order for the candidate c to win in some completion of this partial profile, some other candidates have to lose points. Suppose c' is one such candidate. To lose points, c' has to be in a higher position. Whenever c' is in a higher position, a few other candidates are “pushed up” to lower positions, and they gain points. The score of these candidates are set in such a way that they can be “pushed up” only once. We set the specific scores for every candidate using the second part of the partial profile, which consists of a total profile. These votes, which fulfil certain properties, can be constructed in time polynomial in the number of candidates using a result similar to Lemma 4.2 in [2]; we note that variants of that lemma have been used in the literature [12–15, 18]. Here, we prove the following variant of Lemma 4.2 in [2] and then use it in all our reductions of 3DM to PW-PC.

In what follows, for a scoring vector $(s_1, \dots, s_m)$ and for every j with $1 \leq j \leq m-1$, we define $\delta_j = s_j - s_{j+1}$. For two profiles $P = (P_1, \dots, P_{n_1})$ and $Q = (Q_1, \dots, Q_{n_2})$, we write $P \cup Q$ to denote the profile $(P_1, \dots, P_{n_1}, Q_1, \dots, Q_{n_2})$.

Table 1: Values of R for Theorem 3.

Components of R	Candidate
$R_{c'} = 1 - (s(P', c') - \lambda_{P'})$	for all $c' \in X \cup Y$
$R_{z_i} = -1 - (s(P', z_i) - \lambda_{P'})$	for $1 \leq i \leq q$
$R_{d_1} = -q - (s(P', d_1) - \lambda_{P'})$	d_1

LEMMA 1. Given a set $C = \{c_1, \dots, c_m\}$ and a singleton $D = \{d\}$ of candidates, a scoring vector s of length $m+1$, and for every c_i , a list of integers $\eta_{i,1}, \dots, \eta_{i,m}$ with $\sum_{j=1}^m |\eta_{i,j}| \leq O(m^4)$, one can construct, in time polynomial in m , a total voting profile Q and a $\lambda_Q \in \mathbb{N}$ such that, for $1 \leq i \leq m$, the score $s(Q, c_i) = \lambda_Q + R_i$, where $R_i = \sum_{j=1}^m \eta_{i,j} \delta_j$ and $s(Q, d) < \lambda_Q$. In particular, the number of votes in the profile Q is polynomial in m .

In all the reductions from 3DM to PW-PC, for each $c_i \in C$, the value R_i will be of the form $R_i = \sum_{k=1}^m l_k \delta_k + \sum_{k=1}^{m+1} h_k s_k$, where $\sum_{k=1}^m l_k \leq O(m)$, and each $\sum_{k=1}^{m+1} h_k \leq \tau \leq O(m^3)$ with $\tau = |\mathcal{S}|$ in the 3DM instance. Since, for $1 \leq k \leq m$, the score value $s_k = (\delta_k + \dots + \delta_m)$, and $s_m = 0$, we have that $R_i = \sum_{k=1}^m l_k \delta_k + \sum_{k=1}^m h_k \left(\sum_{l=k}^m \delta_l \right)$. From this, it follows that $R_i = \sum_{j=1}^m \eta_{i,j} \delta_j$, where each $\eta_{i,j}$ is the sum of suitable l_k 's and h_k 's.

In our reductions, the candidates corresponding to the elements of the sets in 3DM are called *element candidates*. For a set S , we write $\vec{S}$ to denote an arbitrary total order on S . For $c_i, c_j \in S$, we write $c_j > c_i$ in $\vec{S}$ to denote that c_i is in a higher position than c_j in the total order.

3.2 Hardness of PW-PC w.r.t. 2-approval

We reduce 3DM to PW-PC w.r.t. 2-approval. The reduction can be easily generalised for all two-valued scoring rules.

THEOREM 3. PW-PC w.r.t. 2-approval is NP-complete.

PROOF. We reduce a 3DM-instance $(X, Y, Z, \mathcal{S})$ to a PW-PC-instance. The set of candidates is $C = X \cup Y \cup Z \cup \{c, d_1, w\}$, where the sets X, Y , and Z consist of candidates corresponding to the elements of the sets $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$. Let $m = |C| = 3q + 3$ and c be the distinguished candidate. We construct the partial profile in two parts. For each $S_i \in \mathcal{S}$ where $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, let $C'_i = C \setminus (\{x_{i_1}, y_{i_2}, z_{i_3}\} \cup \{d_1\})$ and let $\vec{C}'_i$ be such that $c > w$ in $\vec{C}'_i$. Define the total orders p'_i and the partial chains p_i , where

$$p'_i = x_{i_1} > y_{i_2} > z_{i_3} > d_1 > \vec{C}'_i$$

$$p_i = x_{i_1} > y_{i_2} > \vec{C}'_i.$$

Let $P = \bigcup_{i=1}^r p_i$ and $P' = \bigcup_{i=1}^r p'_i$. Observe that each p'_i extends p_i . Let $s(P', c) = \lambda_{P'} = 0$. Since w is in a position greater c in all the votes of P' , we have $s(P', w) = \lambda_{P'}$. Let $\{w\}$ be the set D required in Lemma 1 and R be as in Table 1. Recall that $R_c = 0$. Let $\lambda_{P'} + \lambda_Q = \lambda$. By Lemma 1, there exist a $\lambda_Q \in \mathbb{N}$ and a total profile Q which can be constructed in time polynomial in m and such that the scores of the candidates in the profile $P' \cup Q$ are as in Table 2. Let $P \cup Q$ be the partial profile of the PW-UPC instance. This completes the reduction.

Table 2: Score values of the candidates in Theorem 3.

Candidate	Score
$\forall c' \in X \cup Y,$ $s(P' \cup Q, c')$	$s(P', c') + s(Q, c')$ $= \lambda_{P'} + s(P', c') - \lambda_{P'} + \lambda_Q + R_{c'}$ $= \lambda + 1.$
$\forall z \in Z,$ $s(P' \cup Q, z)$	$s(P', z) + s(Q, z)$ $= \lambda_{P'} + s(P', z) - \lambda_{P'} + \lambda_Q + R_z$ $= \lambda - 1.$
$s(P' \cup Q, c)$	$s(P', c) + s(Q, c) = \lambda_{P'} + \lambda_Q = \lambda.$
$s(P' \cup Q, d_1)$	$s(P', d_1) + s(Q, d_1)$ $= \lambda_{P'} + s(P', d_1) - \lambda_{P'} + \lambda_Q + R_{d_1}$ $= \lambda - q.$
$s(P' \cup Q, w)$	$s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda.$

($\Leftarrow$) Assume that the PW-PC instance is positive. Therefore, there exists a total profile $P^* = \bigcup_{i=1}^r p_i^*$ such that: for all $1 \leq i \leq r$, we have p_i^* extends p_i ; c is a possible winner and has score λ . When we say that a candidate “gains” or “loses” points, it is in relation to the complete profile P' in the reduction. For $1 \leq i \leq q$, each element candidate x_i in X , has to lose at least one point. Since a candidate can lose at most one point in any vote, let $p_{k_i}^*$ be the vote in which the element candidate x_i loses a point, where $1 \leq i \leq q$. Let $K = \{k_i | 1 \leq i \leq q\}$. Observe that in all the q votes in K , both d_1 and an element candidate from Z must be in the top two positions. Without loss of generality, assume that, in these q votes, candidate d_1 is in the first position and the element candidate from Z is in the second position. Therefore, candidate d_1 gains a total of q points. Since each $z \in Z$ can gain at most a point, the element candidate of Z in the second position in each of the above q votes must be distinct, i.e., no two votes in K have the same element candidate of Z in the second position. By construction, candidates d_1 and z cannot gain any more points. Since c is a possible winner, it must be the case that each of the q element candidates in Y also lost at least a point each in the q votes in K . Therefore, the element candidates of Y in the q votes in K must be distinct. It follows that the set $\{x_{i_1}, y_{i_2}, z_{i_3} | i \in K\}$ must form a cover of $X \cup Y \cup Z$.

($\Rightarrow$) Assume that there exists a cover $\mathcal{S}'$ of the 3DM instance where $|\mathcal{S}'| = q$. Let $P^* = \bigcup_{i=1}^r p_i^*$ where

$$p_i^* : d_1 > z_{i_3} > x_{i_1} > y_{i_2} > \vec{C}'_i \text{ if } S_i \in \mathcal{S}'$$

$$p_i^* : x_{i_1} > y_{i_2} > z_{i_3} > d_1 > \vec{C}'_i \text{ if } S_i \notin \mathcal{S}'.$$

The scores of the candidates in the profile $P^* \cup Q$ are as follows:

- $s(P^* \cup Q, x) = s(P^*, x) + s(Q, x) = s(P', x) - 1 + s(Q, x) = (\lambda_{P'} + s(P', x) - \lambda_{P'}) - 1 + (\lambda_Q + R_x) = \lambda$, for all $x \in X$;
- $s(P^* \cup Q, y) = s(P^*, y) + s(Q, y) = s(P', y) - 1 + s(Q, y) = (\lambda_{P'} + s(P', y) - \lambda_{P'}) - 1 + (\lambda_Q + R_y) = \lambda$, for all $y \in Y$;
- $s(P^* \cup Q, z) = s(P^*, z) + s(Q, z) = s(P', z) + 1 + s(Q, z) = (\lambda_{P'} + s(P', z) - \lambda_{P'}) + 1 + (\lambda_Q + R_z) = \lambda$, for all $z \in Z$;
- $s(P^* \cup Q, c) = s(P^*, c) + s(Q, c) = s(P', c) + s(Q, c) = \lambda$;
- $s(P^* \cup Q, d_1) = s(P^*, d_1) + s(Q, d_1) = s(P', d_1) + q + s(Q, d_1) = (\lambda_{P'} + s(P', d_1) - \lambda_{P'}) + q + (\lambda_Q + R_{d_1}) = \lambda$;
- $s(P^* \cup Q, w) = s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda$.

Therefore, c is a possible winner. $\square$

We contrast the above hardness result with Theorem 2 in [9] which tells that, when restricted to uniform collections of partial chains, PW w.r.t. 2-approval is in P. This is not a contradiction since uniform collections of partial chains are a restriction of partial chains (see Section 4 for details).

The hardness of PW-PC w.r.t. t -approval, $t \geq 3$, follows from the results in [9]. There are, however, 2-valued rules that are different from t -approval, for every fixed t . As an example, consider the rule with scoring vectors $s_{2m} = (\underbrace{1, \dots, 1}_m, \underbrace{0, \dots, 0}_m)$ and $s_{2m+1} = (\underbrace{1, \dots, 1}_{m+1}, \underbrace{0, \dots, 0}_m)$, where $m \geq 1$. Notice that in this rule the number of positions with score value 1 is not fixed (unlike t -approval). Furthermore, an arbitrary 2-valued scoring rule can have score values other than one and zero. Our reduction for 2-approval can be easily generalised to cover all such rules.

THEOREM 4. *If r is a 2-valued rule, then PW-PC w.r.t. r is NP-complete.*

PROOF. (Hint) Let the scoring vector be $(\underbrace{a_1, \dots, a_1}_t, 0, 0, \dots, 0)$.

We reduce a 3DM-instance $(X, Y, Z, \mathcal{S})$ to a PW-PC-instance. The set C of candidates is $C = X \cup Y \cup Z \cup \{c, d_1, w\}$, where the sets X, Y , and Z consist of candidates corresponding to the elements of the sets X, Y , and Z . Let $m = |C| = 3q + 3$ and let c be the distinguished candidate. We construct the partial profile in two parts. For each S_i , define the total orders p'_i such that candidates x_{i_1} and y_{i_2} are in positions $t-1$ and t respectively, while candidates z_{i_3} and d are in positions $t+1$ and $t+2$, respectively, and the partial chains p_i as follows

$$p'_i = \vec{C}_i^1 > x_{i_1} > y_{i_2} > z_{i_3} > d_1 > \vec{C}_i^2,$$

$$p_i = \vec{C}_i^1 > x_{i_1} > y_{i_2} > \vec{C}_i^2,$$

where C_i^1 and C_i^2 are partitions of $C \setminus \{x_{i_1}, y_{i_2}, z_{i_3}, d_1\}$ such that $|C_i^1| = t-2$ and $C_i^2 = C \setminus (C_i^1 \cup \{x_{i_1}, y_{i_2}, z_{i_3}, d_1\})$. Since $2 \leq t \leq (m-2)$, the positions $(t-1), t, (t+1)$, and $(t+2)$ are well defined. Let $P = \bigcup_{i=1}^{\tau} p_i$ and $P' = \bigcup_{i=1}^{\tau} p'_i$. Observe that, for $1 \leq i \leq \tau$, each p'_i extends p_i .

Consider the set $C = X \cup Y \cup Z \cup \{c, d_1\} \cup \{w\}$. Let $\{w\}$ be the set D required in Lemma 1 and R be as follows.

- $R_{x_i} = a_1 - (s(P', x_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
- $R_{y_i} = a_1 - (s(P', y_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
- $R_{z_i} = -a_1 - (s(P', z_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
- $R_{d_1} = -qa_1 - (s(P', d_1) - \lambda_{P'})$.
- $R_c = 0$.

By the lemma, there exists a $\lambda_Q \in \mathbb{N}$ and a total profile Q that can be constructed in time polynomial in m in which the score of a candidate $c' \in C$ is $\lambda_Q + R_{c'}$. Let $P \cup Q$ be the partial profile of the PW-PC instance. This completes the reduction. Due to limited space, we defer the proof of correctness. $\square$

3.3 Hardness of p -valued rules, for $p \geq 3$

In this section, we prove NP-completeness of p -valued positional scoring rules, for $p \geq 3$. Consider a p -valued rule, where $p \geq 3$,

with a size m scoring vector containing the values $a_1 > a_2 > \dots > a_p$. We define, for $1 \leq j \leq p$, a function $\ell(m, j)$ that returns the number of times the score value a_j repeats in the scoring vector. Schematically, a scoring vector of a p -valued rule, where $p \geq 3$, can be represented as $(\underbrace{a_1, \dots, a_1}_{\ell(m,1)}, \underbrace{a_2, \dots, a_2}_{\ell(m,2)}, \dots, \underbrace{a_p, \dots, a_p}_{\ell(m,p)})$.

The following proposition follows from the purity of the scoring rules considered in this paper.

PROPOSITION 1. *Let r be a p -valued scoring rule. For all positive integers γ , there exists a length $m \leq \gamma p$ such that, in the scoring vector s_m , there exists $1 \leq u \leq p$ such that $\ell(m, u) = \gamma$.*

THEOREM 5. *PW-PC w.r.t. p -valued rules, for $p \geq 3$, is NP-complete.*

PROOF. (Outline) Let $\mathcal{I} = (X, Y, Z, \mathcal{S})$ be a 3DM instance where $\mathcal{S} = \{S_1, \dots, S_\tau\} \subseteq X \times Y \times Z$ such that $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, for $1 \leq i \leq \tau$, and $|X| = |Y| = |Z| = q$. Let r be the p -valued scoring rule which has scoring vectors with blocks of repeating score values. More precisely, in the scoring vector of length m , the score value a_j repeats $\ell(m, j)$ times, for $1 \leq j \leq p$. Let $\delta_j = a_j - a_{j+1}$, for $1 \leq j < p$. Let $\gamma = 3q$. By Proposition 1, there is a number $m \leq 3qp$ such that in the scoring vector s_m , there exists $1 \leq u \leq p$ such that the block of repeating score value a_u has length $\ell(m, u) = 3q$. We consider the following three cases:

Case 1. $u = 1$;

Case 2. $u = p$;

Case 3. $1 < u < p$.

We outline the construction for Case 1. The set of candidates is $C = X \cup Y \cup Z \cup \{c, w\} \cup H$, where c is the distinguished candidate, the sets X, Y , and Z comprise of candidates corresponding to the elements of the sets X, Y and Z . The set H consists of dummy candidates such that $|H| = m - 3q - 2$. We construct a partial profile in two parts. For each $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, let $C'_i = C \setminus (\{x_{i_1}, y_{i_2}, z_{i_3}\} \cup H)$. Let $\vec{C}_i^1$ be such that candidate c is ranked lower than w , i.e., we have $c > w$. Let $H_1 \subseteq H$ such that $|H_1| = \ell(m, 2) - 1$ and $H' = H \setminus H_1$. Define the total orders p'_i and the partial chains p_i , where

$$p'_i = \vec{C}_i^1 > x_{i_1} > y_{i_2} > \vec{H}_1 > z_{i_3} > \vec{H}'$$

$$p_i = \vec{C}_i^1 > y_{i_2} > \vec{H}_1 > z_{i_3} > \vec{H}'.$$

Let $P = \bigcup_{i=1}^{\tau} p_i$ and $P' = \bigcup_{i=1}^{\tau} p'_i$. Observe that, for $1 \leq i \leq \tau$, each p'_i extends p_i . Let $s(P', c) = \lambda_{P'}$. Since w is placed at a position greater c in all the votes of P' , we have $s(P', w) < \lambda_{P'}$.

Consider $C = X \cup Y \cup Z \cup \{c\} \cup H \cup \{w\}$. Let $\{w\}$ be the set D required in Lemma 1 and R be as follows.

- $R_{x_i} = \delta_1 + \delta_2 - (s(P', x_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
- $R_{y_i} = -\delta_1 - (s(P', y_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
- $R_{z_i} = -\delta_2 - (s(P', z_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
- $R_c = 0$.
- $R_h = 0 - (s(P', h) - \lambda_{P'})$, for all $h \in H$.

By Lemma 1, there exists a $\lambda_Q \in \mathbb{N}$ and a total profile Q which can be constructed in time polynomial in m' such that the scores of the candidates in the profile $P' \cup Q$ are as follows. Let $\lambda_{P'} + \lambda_Q = \lambda$.

- For all $x \in X$, we have $s(P' \cup Q, x) = s(P', x) + s(Q, x)$
 $= (\lambda_{P'} + s(P', x) - \lambda_{P'}) + (\lambda_Q + R_x) = \lambda + \delta_2 + \delta_1.$
- For all $y \in Y$, we have $s(P' \cup Q, y) = s(P', y) + s(Q, y)$
 $= (\lambda_{P'} + s(P', y) - \lambda_{P'}) + (\lambda_Q + R_y) = \lambda - \delta_1.$
- For all $z \in Z$, we have $s(P' \cup Q, z) = s(P', z) + s(Q, z)$
 $= (\lambda_{P'} + s(P', z) - \lambda_{P'}) + (\lambda_Q + R_z) = \lambda - \delta_2.$
- $s(P' \cup Q, c) = s(P', c) + s(Q, c) = \lambda_{P'} + \lambda_Q = \lambda.$
- For all $h \in H$, we have
 $s(P' \cup Q, h) = s(P', h) + s(Q, h) = (\lambda_{P'} + s(P', h) - \lambda_{P'}) + (\lambda_Q + R_h) = \lambda.$
- $s(P' \cup Q, w) = s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda.$

Let $P \cup Q$ be the partial profile of the PW-PC instance. This completes the reduction.

($\implies$) Let $(X, \mathcal{Y}, \mathcal{Z}, \mathcal{S})$ be a positive instance of 3DM. Let $\mathcal{S}' \subseteq \mathcal{S}$ be the cover. Recall that $|\mathcal{S}'| = q$. Let $P^* = \bigcup_{i=1}^{\tau} p_i^*$, where each p_i^* , an extension of p_i , is defined as follows

$$p_i^* = \vec{C}_i' > y_{i_2} > \vec{H}_1 > z_{i_3} > x_{i_1} > \vec{H}' \text{ if } S_i \in \mathcal{S}'$$

$$p_i^* = \vec{C}_i' > x_{i_1} > y_{i_2} > \vec{H}_1 > z_{i_3} > \vec{H}' \text{ if } S_i \notin \mathcal{S}'$$

The scores of the candidates in the profile $P^* \cup Q$ are as follows:

- $s(P^* \cup Q, x) = s(P^*, x) + s(Q, x) =$
 $s(P', x) - (\delta_2 + \delta_1) + s(Q, x) = (\lambda_{P'} + s(P', x) - \lambda_{P'}) - (\delta_2 + \delta_1) + (\lambda_Q + R_x) = \lambda, \text{ for all } x \in X;$
- $s(P^* \cup Q, y) = s(P^*, y) + s(Q, y) = s(P', y) + \delta_1 + s(Q, y) =$
 $(\lambda_{P'} + s(P', y) - \lambda_{P'}) + \delta_1 + (\lambda_Q + R_y) = \lambda, \text{ for all } y \in Y;$
- $s(P^* \cup Q, z) = s(P^*, z) + s(Q, z) = s(P', z) + \delta_2 + s(Q, z) =$
 $(\lambda_{P'} + s(P', z) - \lambda_{P'}) + \delta_2 + (\lambda_Q + R_z) = \lambda, \text{ for all } z \in Z;$
- $s(P^* \cup Q, c) = s(P^*, c) + s(Q, c) = s(P', c) + s(Q, c) = \lambda;$
- $s(P^* \cup Q, c') = s(P^*, c') + s(Q, c') = s(P', c') + 0 + s(Q, c') =$
 $(\lambda_{P'} + s(P', c') - \lambda_{P'}) + 0 + (\lambda_Q + R_{c'}) = \lambda, \text{ for all } c' \in H;$
- $s(P^* \cup Q, w) = s(P^*, w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda.$

Therefore, c is a possible winner.

($\impliedby$) We outline the proof of correctness of the reduction for Case 1. Given a 3DM-instance $(X, \mathcal{Y}, \mathcal{Z}, \mathcal{S})$, construct a PW-PC-instance where C is the set of candidates, $P \cup Q$ is the partial profile, and c is the distinguished candidate according to the above reduction. Assume that the PW-PC-instance $(C, P \cup Q, c)$ is a positive one. Thus, there exists a total profile $P^* = \bigcup_{i=1}^{\tau} p_i^*$ such that for all $1 \leq i \leq \tau$, the vote p_i^* extends p_i and c is a possible winner. Observe that the score of c is λ in all extensions of the partial orders.

When we say that a candidate “gains” or “loses” points, it is in relation to the complete profile P' in the reduction. For $1 \leq i \leq q$, each element candidate x_i in X , has to lose at least $(\delta_2 + \delta_1)$ points. Therefore, it has to be in a position greater than $\ell(m, 1)$ in at least one vote. Assume, for now, that each x_i loses at least $(\delta_2 + \delta_1)$ points in one vote, i.e., it is in a position greater than or equal to $\ell(m, 1) + \ell(m, 2)$. Let these q votes be $p_{k_1}, \dots, p_{k_q}$ where $1 \leq k_i \leq b$ and $K = \{k_i | 1 \leq i \leq q\}$. For each $i \in K$, in the completion p_i^* , the element candidate x_{i_1} loses the points (and, therefore, is in position greater than $\ell(m, 1) + \ell(m, 2)$), candidates z_{i_3} and y_{i_2} gain δ_1 and δ_2 points respectively.

By construction, each element candidate of Y can gain at most δ_1 points, and each element candidate of Z can gain at most δ_2 points.

Table 3: Values of R for Theorem 6.

Components of R	Candidate
$R_{c'} = -1 - (s(P', c') - \lambda_{P'})$	for all $c' \in X \cup Y \cup Z$
$R_g = 4q - (s(P', g) - \lambda_{P'})$	g
$R_d = -q - (s(P', d) - \lambda_{P'})$	d

Moreover, there are no votes where these element candidates can lose points. Therefore, the element candidates of Y and the element candidates of Z , which gain points in the q votes in K must be distinct. We had assumed that each element candidate of X loses at least $(\delta_2 + \delta_1)$ points in one vote. Observe that whenever $x \in X$ is in a position greater than $\ell(m, 1)$, an element candidate of Y gains the maximum points it can without defeating c , i.e., δ_1 points. Since there are q element candidates in X and q element candidates in Y , every time an element candidate of Y gains δ_1 points, an element candidate of X must lose at least $(\delta_1 + \delta_2)$ points. The remaining partial votes in P (p_i for $1 \leq i \leq \tau$ and $i \notin K$), must have the same completion as in P' . Therefore, the set $\{x_{i_1}, y_{i_2}, z_{i_3} | i \in K\}$ must form a cover for $X \cup Y \cup Z$. $\square$

3.4 Hardness of PW-PC w.r.t. Borda count

It remains to examine the group of unbounded rules. We focus on Borda count. The PW problem w.r.t. Borda count on arbitrary partial orders is NP-complete [22]. It continues to be hard when the partial orders are restricted to bottom- and top-truncated, doubly truncated, and partitioned partial orders [2]. However, PW w.r.t. Borda count on uniform collections of partial chains drops to P [9]. In what follows, we prove that PW w.r.t. Borda count on partial chains is NP-complete.

THEOREM 6. PW-PC w.r.t. Borda count is NP-complete.

PROOF. (Outline) Given a 3DM-instance $(X, \mathcal{Y}, \mathcal{Z}, \mathcal{S})$, we construct an instance of PW-PC. The set of candidates is $C = X \cup Y \cup Z \cup \{c, g, d, w\}$ where X , Y , and Z contains candidates corresponding to the elements in X , $\mathcal{Y}$, and $\mathcal{Z}$ respectively. Let $m = |C| = 3q + 4$ and c be the distinguished candidate. Since the scoring vector is $(3q + 3, \dots, 1, 0)$, with m distinct values. We construct the profile in two parts. For each $S_i \in \mathcal{S}$ where $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, let $C_i' = C \setminus (\{x_{i_1}, y_{i_2}, z_{i_3}\} \cup \{g, d\})$ and $\vec{C}_i'$ be such that $c > w$. Define the total orders p_i' and the partial chains p_i , where

$$p_i' = \vec{C}_i' > g > d > x_{i_1} > y_{i_2} > z_{i_3}$$

$$p_i = \vec{C}_i' > d > x_{i_1} > y_{i_2} > z_{i_3}.$$

Let $P = \bigcup_{i=1}^{\tau} p_i$ and $P' = \bigcup_{i=1}^{\tau} p_i'$. Observe that, for $1 \leq i \leq \tau$, each p_i' extends p_i . Let $s(P', c) = \lambda_{P'}$. Moreover, $s(P', w) < \lambda_{P'}$ since w is in a position greater c in all $\vec{C}_i'$, for $1 \leq i \leq \tau$. Let $\{w\}$ be the set D required in Lemma 1 and R is given in Table 3. Recall that $R_c = 0$.

Let $\lambda_{P'} + \lambda_Q = \lambda$. By Lemma 1, there exist a $\lambda_Q \in \mathbb{N}$ and a total profile Q which can be constructed in time polynomial in m such that the scores of the candidates in $P' \cup Q$ are as in Table 4. Let $P \cup Q$ be the partial profile of the PW-PC instance. This completes the reduction.

Table 4: Score values of the candidates in Theorem 6.

Candidate	Score
For all $x \in X$	$s(P', x) + s(Q, x)$
$s(P' \cup Q, x)$	$= (\lambda_{P'} + s(P', x) - \lambda_{P'}) + (\lambda_Q + R_x)$ $= \lambda - 1.$
For all $y \in Y$	$s(P', y) + s(Q, y)$
$s(P' \cup Q, y)$	$= (\lambda_{P'} + s(P', y) - \lambda_{P'}) + (\lambda_Q + R_y)$ $= \lambda - 1.$
For all $z \in Z$	$s(P', z) + s(Q, z)$
$s(P' \cup Q, z)$	$= (\lambda_{P'} + s(P', z) - \lambda_{P'}) + (\lambda_Q + R_z)$ $= \lambda - 1.$
$s(P' \cup Q, c)$	$s(P', c) + s(Q, c) = \lambda_{P'} + \lambda_Q = \lambda.$
$s(P' \cup Q, g)$	$s(P', g) + s(Q, g)$ $= (\lambda_{P'} + s(P', g) - \lambda_{P'}) + (\lambda_Q + R_g)$ $= \lambda + 4q.$
$s(P' \cup Q, d)$	$s(P', d) + s(Q, d)$ $= (\lambda_{P'} + s(P', d) - \lambda_{P'}) + (\lambda_Q + R_d)$ $= \lambda - q.$
For all $h \in H$,	$s(P', h) + s(Q, h)$
$s(P' \cup Q, h)$	$= (\lambda_{P'} + s(P', h) - \lambda_{P'}) + (\lambda_Q + R_h) = \lambda.$
$s(P' \cup Q, w)$	$s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda.$

($\Leftarrow$) Assume that the PW-PC instance is positive. Therefore, there exists a total profile $P^* = \bigcup_{i=1}^{\tau} p_i^*$ such that for all $1 \leq i \leq \tau$, the vote p_i^* extends p_i and c is a possible winner with score λ . In the following, when one says that a candidate “gains” or “loses” points, it is in relation to the complete profile P' . Candidate g must lose at least $4q$ points for c to be a possible winner. Therefore, it must be in a position greater than $3q$ at least q times. Whenever g is in a position greater than $3q$, candidate d gains 1 point. Since d cannot gain more than q points, there are at most q votes where g is in position greater than $3q$. Let these votes be $p_{k_1}^*, \dots, p_{k_q}^*$ where each $1 \leq k_j \leq b$ and $K = \{k_j | 1 \leq j \leq q\}$. Note that candidate g has to lose at least $4q$ points in these q votes. This is possible if and only if it is in position $3q + 4$. Furthermore, whenever g is in position $3q + 4$ in a vote p_i^* , candidates x_{i_1} , y_{i_2} and z_{i_3} gain one point each, for $i \in K$. Since $|X| = |Y| = |Z| = q$, and each $x \in X$, each $y \in Y$, and each $z \in Z$ can gain at most one point each, it must be the case that the element candidates of Y and Z which gained points in the q votes in K are distinct. Since no other candidate can gain any more points, the remaining partial votes in P (p_i for $1 \leq i \leq \tau$ and $i \notin K$), must have the same completion as in P' . Therefore, the set $\{x_{i_1}, y_{i_2}, z_{i_3} | i \in K\}$ must form a cover for $X \cup Y \cup Z$. ($\Rightarrow$) Assume that there exists a cover $\mathcal{S}'$ of the 3DM instance. Let $P^* = \bigcup_{i=1}^{\tau} p_i^*$ where each p_i is extended as given below.

$$p_i^* = \vec{C}_i^* > d > x_{i_1} > y_{i_2} > z_{i_3} > g \text{ if } S_i \in \mathcal{S}'$$

$$p_i^* = \vec{C}_i^* > g > d > x_{i_1} > y_{i_2} > z_{i_3} \text{ if } S_i \notin \mathcal{S}'$$

All the candidates have score λ in $P^* \cup Q$. Therefore, candidate c is a possible winner. $\square$

In what follows, we prove an observation analogous to Proposition 1 and then prove hardness for arbitrary unbounded scoring rules. As noted earlier, unbounded scoring rules may have score

values that repeat in blocks. Moreover, unlike Borda count, the score values can be non-uniformly decreasing. Recall, that for a scoring vector of length m , with m' distinct score values, the function $\ell(m, j)$ returns the number of times the distinct score value a_j repeats in a block, for $1 \leq j \leq m'$. Schematically, such a scoring vector can be represented as $(\underbrace{a_1, \dots, a_1}_{\ell(m,1)}, \underbrace{a_2, \dots, a_2}_{\ell(m,2)}, \dots, \underbrace{a_{m'}, \dots, a_{m'}}_{\ell(m,m')})$.

Next, we prove a basic property of scoring vectors of all unbounded rules.

PROPOSITION 2. *Let r be a positional scoring rule and let γ and β be two positive integers greater than 1. Consider the scoring vector s_m of r with length $m = \gamma\beta$. Then either s_m contains at least β distinct values or there exists $1 \leq u \leq \gamma\beta$ such that $\ell(\gamma\beta, u) \geq \gamma$.*

THEOREM 7. *Let r be an unbounded scoring rule. PW-PC w.r.t. r is NP-complete.*

PROOF. (Outline) Assume that $(X, Y, Z, \mathcal{S})$ is a 3DM-instance in which $\mathcal{S} = \{S_1, \dots, S_\tau\} \subseteq X \times Y \times Z$ and $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, for $1 \leq i \leq \tau$. Let s_m be the scoring vector of length $m = (3q + 4)(3q)$. By Proposition 2, we need to consider the following two cases.

Case 1. There exists a u such that $\ell(m, u) = 3q$.

Case 2. There are $m' = 3q + 4$ distinct values.

For Case 1, the reduction mimics the one in Theorem 5 to create a PW-PC instance. For Case 2, the reduction proceeds as follows.

Let $a_1 > a_2 > \dots > a_{m'}$ be the m' distinct values. We define $\delta = (\delta_1, \dots, \delta_{m'-1})$, where $\delta_j = a_j - a_{j+1}$, for $1 \leq j < m'$. The set of candidates is $C = X \cup Y \cup Z \cup \{c, g, d, w\} \cup H$ where X , Y , and Z contains candidates corresponding to the elements in X , Y , and Z respectively. These candidates are called *elements candidates*. The set H contains dummy candidates such that $|H| = m - m'$.

We construct the partial profile P as follows. Let the set H be partitioned into $H_1, \dots, H_{m'}$, such that $|H_j| = \ell(m, j) - 1$, for $1 \leq j \leq m'$. For each $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, let $C_i^* = C \setminus (\{x_{i_1}, y_{i_2}, z_{i_3}\} \cup \{g, d\} \cup \bigcup_{j=m'-4}^{m'} H_j)$ and $\vec{C}_i^*$ be such that the dummy candidates in H_j are in a position with score value a_j , for $1 \leq j \leq m' - 3$ and candidate c is ranked lower than candidate w . Define the total orders p_i^* and the partial chains p_i , where

$$p_i^* = \vec{C}_i^* > g > \vec{H}_{m'-4} > d > \vec{H}_{m'-3}$$

$$> x_{i_1} > \vec{H}_{m'-2} > y_{i_2} > \vec{H}_{m'-1} > z_{i_3} > \vec{H}_{m'}$$

$$p_i = \vec{C}_i^* > \vec{H}_{m'-4} > d > \vec{H}_{m'-3}$$

$$> x_{i_1} > \vec{H}_{m'-2} > y_{i_2} > \vec{H}_{m'-1} > z_{i_3} > \vec{H}_{m'}$$

Let $P = \bigcup_{i=1}^{\tau} p_i$ and $P' = \bigcup_{i=1}^{\tau} p_i^*$. Observe that each p_i^* extends p_i . Let $s(P', c) = \lambda_{P'}$. Moreover, $s(P', w) < \lambda_{P'}$ since w is in a position greater than c in all $\vec{C}_i^*$, for $1 \leq i \leq \tau$.

Consider $C = X \cup Y \cup Z \cup \{c, g, d\} \cup \{w\}$. By Lemma 1, there exists a $\lambda_Q \in \mathbb{N}$ and a total profile Q which can be constructed in time polynomial in m' such that the scores of the candidates in the profile $P' \cup Q$ are as in Table 5. We let C , the profile $P \cup Q$, and c be the input to the PW-PC problem. $\square$

Table 5: Score values of the candidates in Theorem 7.

Candidate	Score
For all $x \in X$, $s(P' \cup Q, x)$	$s(P', x) + s(Q, x)$ $= (\lambda_{P'} + s(P', x) - \lambda_{P'}) + (\lambda_Q + R_x)$ $= \lambda - \delta_{m'-3}$.
For all $y \in Y$, $s(P' \cup Q, y)$	$s(P', y) + s(Q, y)$ $= (\lambda_{P'} + s(P', y) - \lambda_{P'}) + (\lambda_Q + R_y)$ $= \lambda - \delta_{m'-2}$.
For all $z \in Z$, $s(P' \cup Q, z)$	$s(P', z) + s(Q, z)$ $= (\lambda_{P'} + s(P', z) - \lambda_{P'}) + (\lambda_Q + R_z)$ $= \lambda - \delta_{m'-1}$.
$s(P' \cup Q, c)$	$s(P', c) + s(Q, c) = \lambda_{P'} + \lambda_Q = \lambda$.
$s(P' \cup Q, g)$	$s(P', g) + s(Q, g)$ $= (\lambda_{P'} + s(P', g) - \lambda_{P'}) + (\lambda_Q + R_g)$ $= \lambda + q \sum_{j=1}^4 \delta_{m-j}$.
$s(P' \cup Q, d)$	$s(P', d) + s(Q, d)$ $= (\lambda_{P'} + s(P', d) - \lambda_{P'}) + (\lambda_Q + R_d)$ $= \lambda - q(\delta_{m-4})$.
For all $h \in H$, $s(P' \cup Q, h)$	$s(P', h) + s(Q, h)$ $= (\lambda_{P'} + s(P', h) - \lambda_{P'}) + (\lambda_Q + R_h) = \lambda$.
$s(P' \cup Q, w)$	$s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda$.

4 PARTIAL CHAINS AND NEW CANDIDATES

Chevaleyre et al. [8] investigated the POSSIBLE CO-WINNER WITH NEW CANDIDATES (PcWNA) problem, which arises in the following natural scenario: for a given set of candidates, the voters have completely ranked them; new candidates join the election *after* the voters have ranked all the initial candidates. In PcWNA, one asks: is a candidate from amongst the initial set of candidates a possible winner? As we shall see next, the PcWNA problem can be viewed as a special case of the PW problem on partial chains.

Observe that, in the PcWNA problem, the rankings of the voters are total for the initial set of candidates. When both the initial candidates and those who joined late are considered, then we have a collection of partial chains that have a special structure: all of them are total orders on the *same* subset of candidates, namely, the set of initial candidates.

Definition 4. Let C be a set of candidates and $P = (P_1, \dots, P_n)$ be a collection of partial chains on C . We say that P is *uniform* if there exists a set $C' \subseteq C$ such that each $P_i \in P$ consists of a total order on the set C' (no candidates outside C' are comparable).

Definition 5. The PW-UPC problem asks: given a set of candidates C , a uniform collection $P = (P_1, \dots, P_n)$ of partial chains in which every P_i , $1 \leq i \leq n$, consists of a total order on the same set $C' \subseteq C$, and a distinguished candidate $c \in C'$, is $c \in \text{PW}(r, P)$?

In PW-UPC, a candidate $c \in (C \setminus C')$ is trivially a possible winner (each voter ranks c in position one). So the only interesting case is when $c \in C'$. Thus, the PcWNA problem coincides with the PW-UPC problem, which is a special case of the PW-PC problem.

The complexity of the PcWNA problem (PW-UPC in the above terminology) has been investigated in [2, 8, 9]. In these papers, it has been shown that the complexity of the PW-UPC problem w.r.t. 2-approval drops from NP-complete to P. Interestingly, this problem

Table 6: Computational complexity of restrictions of PW.

Scoring Rule	PW	PW-PC	PW-UPC
Plurality & Veto	P	P	P
Non-decreasing rate rules	NP-c	NP-c	P
2-approval	NP-c	NP-c	P
$t \geq 3$ -approval	NP-c	NP-c	NP-c
All other 2-valued rules	NP-c	NP-c	?
$(a_1, a_2, 1, 0, \dots, 0)$ s.t. $a_1 > a_2 > 1$	NP-c	NP-c	NP-c
All remaining rules	NP-c	NP-c	?

continues to be NP-complete w.r.t. t -approval ($t > 2$), unlike the PW problem on other restricted partial orders mentioned earlier, such as doubly-truncated partial orders. The complexity of the PW-UPC w.r.t. Borda count also drops to P; in fact, it drops to P w.r.t. every rule of non-decreasing rate, where a scoring rule r with scoring vector s_m is of *non-decreasing rate* if for all $1 \leq i < m$, we have $s_i - s_{i+1} \leq s_{i+1} - s_{i+2}$. Finally, the PW-UPC problem is NP-complete w.r.t. the rule $(a_1, a_2, 1, 0, \dots, 0)$, where $a_1 > a_2 > 1$.

Table 6 depicts the above results and compares them with the results obtained here.

5 CONCLUDING REMARKS

In this paper, we completely classified the complexity of the PW problem on partial chains w.r.t. to all pure positional scoring rules. This classification yields the earlier classification of the PW problem on arbitrary partial orders as a corollary, but it cannot be derived from that earlier classification.

A complete classification of the complexity of the PW problem on uniform collections of partial chains (equivalently, of the POSSIBLE CO-WINNER WITH NEW CANDIDATES problem) remains an open problem that is worth pursuing.

Our NP-hardness results for the PW problem on partial chains made use of “long” chains, i.e., chains that contained all but a fixed number of candidates. This type of partial chain arises in settings where a new candidate or a small number of new candidates enter the race late and, at that time, the voters do not know how to rank these new candidates. We are currently investigating the exact impact of the length of the chain on the complexity of the PW problem on partial chains. In particular, we are investigating whether, for each positional scoring rule other than plurality and veto, there is a *threshold* on the length of the chain below which the PW problem is in P, while above it becomes NP-complete.

In a different direction, there is a rich body of work on algorithmic problems about manipulation in voting (PW is a special case of one of these problems), where computational hardness is regarded as a feature because it provides an obstacle to such manipulation (see [10] for a survey). Work in this area includes the study of manipulation in voting with incomplete information [15, 21]; in particular, [21] considers such manipulation for top-truncated partial orders. It would be natural to investigate manipulation in voting with partial chains.

Acknowledgements. We thank the reviewers for their valuable feedback. The research of both authors was partially supported by NSF Award No. 1814152.

REFERENCES

- [1] Dorothea Baumeister, Piotr Faliszewski, Jérôme Lang, and Jörg Rothe. 2012. Campaigns for lazy voters: truncated ballots. In *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*. International Foundation for Autonomous Agents and Multiagent Systems, 577–584.
- [2] Dorothea Baumeister, Magnus Roos, and Jörg Rothe. 2011. Computational complexity of two variants of the possible winner problem. In *The 10th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*. International Foundation for Autonomous Agents and Multiagent Systems, 853–860.
- [3] Dorothea Baumeister and Jörg Rothe. 2012. Taking the final step to a full dichotomy of the possible winner problem in pure scoring rules. *Inf. Process. Lett.* 112, 5 (2012), 186–190.
- [4] Nadja Betzler and Britta Dorn. 2010. Towards a dichotomy for the Possible Winner problem in elections based on scoring rules. *J. Comput. Syst. Sci.* 76, 8 (2010), 812–836.
- [5] Nadja Betzler, Rolf Niedermeier, and Gerhard J Woeginger. 2011. Unweighted coalitional manipulation under the Borda rule is NP-hard. In *Twenty-Second International Joint Conference on Artificial Intelligence*.
- [6] Craig Boutilier and Jeffrey S. Rosenschein. 2016. Incomplete Information and Communication in Voting. In *Handbook of Computational Social Choice*. Cambridge University Press, 223–258.
- [7] Vishal Chakraborty, Theo Delemazure, Benny Kimelfeld, Phokion G. Kolaitis, Kunal Relia, and Julia Stoyanovich. 2020. Algorithmic Techniques for Necessary and Possible Winners. [arXiv:2005.06779](https://arxiv.org/abs/2005.06779) [cs.GT]
- [8] Yann Chevaleyre, Jérôme Lang, Nicolas Maudet, and Jérôme Monnot. 2010. Possible winners when new candidates are added: The case of scoring rules. In *Twenty-Fourth AAAI Conference on Artificial Intelligence*.
- [9] Yann Chevaleyre, Jérôme Lang, Nicolas Maudet, Jérôme Monnot, and Lirong Xia. 2011. New Candidates Welcome! Possible Winners with respect to the Addition of New Candidates. [arXiv:1111.3690](https://arxiv.org/abs/1111.3690) [cs] (Nov. 2011). <http://arxiv.org/abs/1111.3690> [arXiv: 1111.3690](https://arxiv.org/abs/1111.3690).
- [10] Vincent Conitzer and Toby Walsh. 2016. Barriers to Manipulation in Voting. In *Handbook of Computational Social Choice*, Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia (Eds.). Cambridge University Press, 127–145.
- [11] Jessica Davies, George Katsirelos, Nina Narodytska, and Toby Walsh. 2011. Complexity of and algorithms for Borda manipulation. In *Twenty-Fifth AAAI Conference on Artificial Intelligence*.
- [12] Palash Dey and Neeldhara Misra. 2017. On the Exact Amount of Missing Information that Makes Finding Possible Winners Hard. In *42nd International Symposium on Mathematical Foundations of Computer Science (MFCS 2017) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 83)*, Kim G. Larsen, Hans L. Bodlaender, and Jean-Francois Raskin (Eds.). Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 57:1–57:14.
- [13] Palash Dey, Neeldhara Misra, and Y Narahari. 2016. Complexity of manipulation with partial information in voting. In *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence*. AAAI Press, 229–235.
- [14] Palash Dey, Neeldhara Misra, and Y Narahari. 2016. Kernelization complexity of possible winner and coalitional manipulation problems in voting. *Theoretical Computer Science* 616 (2016), 111–125.
- [15] Palash Dey, Neeldhara Misra, and Y. Narahari. 2018. Complexity of manipulation with partial information in voting. *Theor. Comput. Sci.* 726 (2018), 78–99. Earlier version in *IJCAI* 2016.
- [16] Michael R Garey and David S Johnson. 1979. A Guide to the Theory of NP-Completeness. *Computers and intractability* (1979), 641–650.
- [17] Edith Hemaspaandra, Lane A Hemaspaandra, and Henning Schnoor. 2014. A control dichotomy for pure scoring rules. In *Twenty-Eighth AAAI Conference on Artificial Intelligence*.
- [18] Batya Kenig. 2019. The Complexity of the Possible Winner Problem with Partitioned Preferences. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS '19, Montreal, QC, Canada, May 13-17, 2019*, Edith Elkind, Manuela Veloso, Noa Agmon, and Matthew E. Taylor (Eds.). International Foundation for Autonomous Agents and Multiagent Systems, 2051–2053.
- [19] Kathrin Konczak and Jérôme Lang. 2005. Voting procedures with incomplete preferences. In *Proc. IJCAI-05 Multidisciplinary Workshop on Advances in Preference Handling*, Vol. 20.
- [20] Jérôme Lang. 2020. Collective Decision Making under Incomplete Knowledge: Possible and Necessary Solutions. In *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI-20*, Christian Bessière (Ed.). International Joint Conferences on Artificial Intelligence Organization, 4885–4891. Survey track.
- [21] Nina Narodytska and Toby Walsh. 2014. The Computational Impact of Partial Votes on Strategic Voting. In *Proceedings of the Twenty-First European Conference on Artificial Intelligence (Prague, Czech Republic) (ECAI'14)*. IOS Press, NLD, 657–662.
- [22] Lirong Xia and Vincent Conitzer. 2011. Determining Possible and Necessary Winners Given Partial Orders. *J. Artif. Intell. Res.* 41 (2011), 25–67.