



Eigensystem Multiscale Analysis for the Anderson Model via the Wegner Estimate

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Abstract. We present a new approach to the eigensystem multiscale analysis (EMSA) for random Schrödinger operators that relies on the Wegner estimate. The EMSA treats all energies of the finite volume operator in an energy interval at the same time, simultaneously establishing localization of all eigenfunctions with eigenvalues in the energy interval with high probability. It implies all the usual manifestations of localization (pure point spectrum with exponentially decaying eigenfunctions, dynamical localization). The new method removes the restrictive level spacing hypothesis used in the previous versions of the EMSA. The method is presented in the context of the Anderson model, allowing for single-site probability distributions that are Hölder continuous of order $\alpha \in (0, 1]$.

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Introduction

In [16, 17] we developed an eigensystem multiscale analysis (EMSA) for proving localization (pure point spectrum with exponentially decaying eigenfunctions, dynamical localization) for random Schrödinger operators. The EMSA treats all energies of the finite volume operator in an energy interval at the same time, simultaneously establishing localization of all eigenfunctions with eigenvalues in the energy interval with high probability. The analysis in [16, 17] (and its bootstrap enhancement in [30]) relies on a probability estimate for level spacing. For the Anderson model with a Hölder continuous single-site probability distribution of order $\alpha \in (\frac{1}{2}, 1]$ such an estimate is provided by [29, Lemma 2], where it is derived from Minami's estimate [31]. (This is the level spacing probability estimate used in [16, 17, 30].) A weaker level spacing estimate is proven for the continuous Anderson model in [13, Theorem 2.2]; it requires a covering condition for the random potential, it holds only in a certain interval at the bottom of the spectrum, it requires the single-site probability distribution to be absolutely continuous with a density that is uniformly Lipschitz continuous and bounded below on its support, and provides weak probability estimates. The fact that level spacing probability estimates are not widely known, and where known require extra hypotheses, imposes a strong limitation on the applicability of the EMSA.

The well-known methods previously developed for proving localization for random Schrödinger operators are the multiscale analysis (MSA) (see [7, 9, 14, 15, 19–23, 25, 27, 32]) and the fractional moment method (FMM) (see [1–5]). As opposed to the EMSA, these methods are based on the study of finite volume Green's functions, and the analysis is performed either at a fixed energy in a single box, or for all energies in an interval at once but with two boxes with an 'either or' statement for each energy. Green's functions-based methods do not rely on level spacing. Rather, they use either explicitly (MSA) or implicitly (FMM) a more widely available bound, the Wegner estimate (e.g., [8–10, 26, 28, 33]). This estimate is proven for a large family of both lattice and continuum random Schrödinger operators, making it possible to establish localization in these contexts.

Unfortunately, the Green's function quickly becomes an inadequate tool in the study of many-body localization, rendering the traditional approaches to localization ineffective. The EMSA approach to localization shows more flexibility in this regard: In a forthcoming paper, [18], we use the EMSA to establish many-body localization results in the context of random XXZ spin quantum chains. However, as we already mentioned, the previously available version of the method uses the level spacing hypothesis, which (although expected) has never been proven for many-body systems so far. The main innovation of the present work is the removal of this restrictive condition, replacing it by an argument based on the Wegner estimate. More precisely, the new approach uses Wegner estimates between boxes, as in [15, 22, 23, 27]. To illustrate the method, we consider here its application to a single particle lattice Anderson model. In this context it applies when the single-site probability distribution is Hölder

continuous of order $\alpha \in (0, 1]$, in contrast to the EMSA with level spacing of [16, 17] that requires $\alpha \in (\frac{1}{2}, 1]$. Moreover, this version of EMSA is expected to admit extensions to random Schrödinger operators where a suitable Wegner estimate is available, such as the continuum Anderson model.

1. Definitions and Results

A discrete Schrödinger operator is an operator of the form $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where Δ is the (centered) discrete Laplacian:

$$(\Delta\varphi)(x) := \sum_{\substack{y \in \mathbb{Z}^d \\ |y-x|=1}} \varphi(y) \quad \text{for } \varphi \in \ell^2(\mathbb{Z}^d), \quad (1.1)$$

and V is a bounded potential.

Definition 1.1. The Anderson model is the random discrete Schrödinger operator

$$H_\omega := -\Delta + V_\omega \quad \text{on } \ell^2(\mathbb{Z}^d), \quad (1.2)$$

where V_ω is a random potential: $V_\omega(x) = \omega_x$ for $x \in \mathbb{Z}^d$, where $\omega = \{\omega_x\}_{x \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables, whose common probability distribution μ has bounded support and is assumed to be Hölder continuous of order $\alpha \in (0, 1]$:

$$S_\mu(t) \leq Kt^\alpha \quad \text{for all } t \in [0, 1], \quad (1.3)$$

where K is a constant and $S_\mu(t) := \sup_{a \in \mathbb{R}} \mu\{[a, a+t]\}$ is the concentration function of the measure μ .

To formulate our main result, we need to introduce some additional notation. Given $\Theta \subset \mathbb{Z}^d$, we let H_Θ be the restriction of $\chi_\Theta H \chi_\Theta$ to $\ell^2(\Theta)$. We write $\|\varphi\| = \|\varphi\|_{\ell^2(\Theta)}$ for $\varphi \in \ell^2(\Theta)$. We call (φ, λ) an eigenpair for H_Θ if $\varphi \in \ell^2(\Theta)$ with $\|\varphi\| = 1$, $\lambda \in \mathbb{R}$, and $H_\Theta \varphi = \lambda \varphi$. (In other words, λ is an eigenvalue for H_Θ and φ is a corresponding normalized eigenfunction.) A collection $\{(\varphi_j, \lambda_j)\}_{j \in J}$ of eigenpairs for H_Θ will be called an *eigensystem* for H_Θ if $\{\varphi_j\}_{j \in J}$ is an orthonormal basis for $\ell^2(\Theta)$. If $\Theta \subset \mathbb{Z}^d$ is finite, we let $\tilde{\sigma}(H_\Theta)$ denote the eigenvalues of H_Θ repeated according to multiplicity (and thought of as different points in $\tilde{\sigma}(H_\Theta)$), so an eigensystem for H_Θ can be rewritten as $\{(\varphi_\lambda, \lambda)\}_{\lambda \in \tilde{\sigma}(H_\Theta)}$, i.e., it can be labeled by $\tilde{\sigma}(H_\Theta)$.

If $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we set $|x| = |x|_2 = \left(\sum_{j=1}^d x_j^2\right)^{\frac{1}{2}}$ and $\|x\| = |x|_\infty = \max_{j=1,2,\dots,d} |x_j|$. We consider boxes in $\mathbb{Z}^d$ centered at points of $\mathbb{R}^d$. The box in $\mathbb{Z}^d$ of side $L > 0$ centered at $x \in \mathbb{R}^d$ is given by

$$\Lambda_L(x) = \Lambda_L^\mathbb{R}(x) \cap \mathbb{Z}^d, \quad \text{where } \Lambda_L^\mathbb{R}(x) = \{y \in \mathbb{R}^d; \|y - x\| \leq \frac{L}{2}\}. \quad (1.4)$$

By a box Λ_L we will mean a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$. We have

$$(L-2)^d < |\Lambda_L(x)| \leq (L+1)^d \quad \text{for all } L \geq 2 \quad \text{and } x \in \mathbb{R}^d. \quad (1.5)$$

The EMSA is based on the study of localized eigensystems. The relevant definitions are stated in terms of exponents $\tau, \kappa' \in (0, 1)$ that will be chosen later. We use the notation $L_\tau = \lfloor L^\tau \rfloor$ for $L \geq 1$.

Definition 1.2. Let Λ_L be a box, $x \in \Lambda_L$, and $m \geq 0$. Then $\varphi \in \ell^2(\Lambda_L)$ is said to be (x, m) -localized if $\|\varphi\| = 1$ and

$$|\varphi(y)| \leq e^{-m\|y-x\|} \quad \text{for all } y \in \Lambda_L \quad \text{with} \quad \|y-x\| \geq L_\tau. \quad (1.6)$$

We consider energy intervals $I(E, A) = (E - A, E + A)$ with center $E \in \mathbb{R}$ and radius $A > 0$. (When we write $I(E, A)$ it will be implicit that $E \in \mathbb{R}$ and $A > 0$.) Given an interval $I = I(E, A)$, we set

$$h_I(t) = h\left(\frac{t-E}{A}\right) \quad \text{for } t \in \mathbb{R}, \quad \text{with} \quad h(s) = \begin{cases} 1-s^2 & \text{if } s \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}. \quad (1.7)$$

Note that $h_I(t) > 0 \iff t \in I$, which implies $h_I = \chi_I h_I$.

Definition 1.3. Given an energy interval $I = I(E, A)$, a box Λ_L will be called (m, I) -localizing for H if

$$L^{-\kappa'} \leq m \leq \frac{1}{2} \log\left(1 + \frac{A}{4d}\right), \quad (1.8)$$

and there exists an (m, I) -localized eigensystem for H_{Λ_L} , that is, an eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \tilde{\sigma}(H_{\Lambda_L})}$ for H_{Λ_L} such that for all $\nu \in \tilde{\sigma}(H_{\Lambda_L})$ there is $x_\nu \in \Lambda_L$ so φ_ν is $(x_\nu, m h_I(\nu))$ -localized.

Given a box $\Lambda_\ell \subset \Theta$, a crucial step in our analysis shows that if (ψ, λ) is an eigenpair for H_Θ , with $\lambda \in I$ not too close to the eigenvalues of H_{Λ_ℓ} , and the box Λ_ℓ is (m, I) -localizing for H , then ψ is exponentially small deep inside Λ_ℓ (see Lemma 2.2.). This is proven by expanding the values of ψ inside Λ_ℓ in terms of an (m, I) -localizing eigensystem for H_{Λ_ℓ} . The problem is we only know decay for the eigenfunctions with eigenvalues in I ; we have no information whatsoever concerning eigenfunctions with eigenvalues that lie outside the interval I . As in [17], the decay of the term containing the latter eigenfunctions comes from the distance from the eigenvalue λ to the complement of the interval I , and consequently the decay rate for the localization of an eigenfunction goes to zero as the corresponding eigenvalue approaches the edges of the interval I . The introduction of the modulating function h_I in the decay rate models this phenomenon.

The control of the term containing eigenfunctions corresponding to eigenvalues that lie outside the interval I is given by [17, Lemma 3.2(ii)], which requires the upper bound in (1.8). The lower bound in (1.8) is a requirement for the multiscale analysis, as in [20, 23, 25, 27].

Our main result pertaining to the eigensystem multiscale analysis in an energy interval is given in the following theorem. To state the theorem, given exponents $0 < \xi < \zeta < 1$, we choose the exponents $\tau, \kappa' \in (0, 1)$ that appear in Definitions 1.2 and 1.3, as well as exponents $\beta, \kappa, \varrho \in (0, 1)$ and $\gamma > 1$, satisfying the relations described in Appendix A. In what follows, once the exponents $0 < \xi < \zeta < 1$ are fixed, we always assume we choose and fix the other exponents as in Appendix A.

Theorem 1.4. *Let H_ω be an Anderson model. Given $0 < \xi < \zeta < 1$, there exists a finite scale $\mathcal{L} = \mathcal{L}(d, \xi, \zeta)$ and a constant $C_d = C_{d, \xi, \zeta} > 0$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}$ and interval $I_0 = I(E, A_0)$ we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_{L_0}(x) \text{ is } (m_0, I_0)\text{-localizing for } H_\omega \} \geq 1 - e^{-L_0^\xi}. \quad (1.9)$$

Then for all $L \geq L_0^\gamma$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_L(x) \text{ is } (m_\infty, I_\infty^L)\text{-localizing for } H_\omega \} \geq 1 - e^{-L^\xi}, \quad (1.10)$$

where

$$\begin{aligned} I_\infty^L &= I_\infty^L(L_0) = I(E, A_\infty(1 - L^{-\kappa})^{-1}), \\ A_\infty &= A_\infty(L_0) = A_0 \prod_{k=0}^{\infty} \left(1 - L_0^{-\kappa \gamma^k} \right), \\ L_0^{-\gamma \kappa'} &\leq m_\infty = m_\infty(L_0) = m_0 \prod_{k=0}^{\infty} \left(1 - C_d L_0^{-\varrho \gamma^k} \right) < \frac{1}{2} \log \left(1 + \frac{A_\infty}{4d} \right). \end{aligned} \quad (1.11)$$

In particular, $\lim_{L_0 \rightarrow \infty} A_\infty(L_0) = A_0$ and $\lim_{L_0 \rightarrow \infty} m_\infty(L_0) = m_0$.

We now state a corollary of Theorem 1.4 that encapsulates the usual forms of Anderson localization (pure point spectrum with exponentially decaying eigenfunctions, dynamical localization, etc.) on the interval $I_\infty = I(E, A_\infty)$, as in [16, 24, 25]. We fix $\nu > \frac{d}{2}$, and given $a \in \mathbb{Z}^d$ we define T_a as the operator on $\ell^2(\mathbb{Z}^d)$ given by multiplication by the function $T_a(x) := \langle x - a \rangle^\nu$, where $\langle x \rangle = \sqrt{1 + \|x\|^2}$. Since $\langle a + b \rangle \leq \sqrt{2} \langle a \rangle \langle b \rangle$, we have $\|T_a T_b^{-1}\| \leq 2^{\frac{\nu}{2}} \langle a - b \rangle^\nu$. A function $\psi: \mathbb{Z}^d \rightarrow \mathbb{C}$ is a ν -generalized eigenfunction for the discrete Schrödinger operator H if ψ is a generalized eigenfunction and $\|T_0^{-1}\psi\| < \infty$. ($\|T_0^{-1}\psi\| < \infty$ if and only if $\|T_a^{-1}\psi\| < \infty$ for all $a \in \mathbb{Z}^d$.) We let $\mathcal{V}(\lambda)$ denote the collection of ν -generalized eigenfunctions for H with generalized eigenvalue $\lambda \in \mathbb{R}$. Given $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{Z}^d$, we set

$$W_\lambda^{(a)}(b) := \begin{cases} \sup_{\psi \in \mathcal{V}(\lambda)} \frac{|\psi(b)|}{\|T_a^{-1}\psi\|} & \text{if } \mathcal{V}(\lambda) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}. \quad (1.12)$$

For all $a, b, c \in \mathbb{Z}^d$ we have

$$W_\lambda^{(a)}(a) \leq 1, \quad W_\lambda^{(a)}(b) \leq \langle b - a \rangle^\nu, \quad \text{and} \quad W_\lambda^{(a)}(c) \leq 2^{\frac{\nu}{2}} \langle b - a \rangle^\nu W_\lambda^{(b)}(c). \quad (1.13)$$

Corollary 1.5. *Suppose the conclusions of Theorem 1.4 hold for an Anderson model H_ω , and let $I = I_\infty$, $m = m_\infty$. There is a finite scale $\mathcal{L} = \mathcal{L}_{d, \nu}$ such that, given $\mathcal{L} \leq L \in 2\mathbb{N}$ and $a \in \mathbb{Z}^d$, there exists an event $\mathcal{Y}_{L, a}$ with the following properties:*

(i) $\mathcal{Y}_{L,a}$ depends only on the random variables $\{\omega_x\}_{x \in \Lambda_{5L}(a)}$ and

$$\mathbb{P}\{\mathcal{Y}_{L,a}\} \geq 1 - Ce^{-L^\varepsilon}. \quad (1.14)$$

(ii) Given $\omega \in \mathcal{Y}_{L,a}$, for all $\lambda \in I$ we have that

$$\max_{b \in \Lambda_{\frac{L}{3}}(a)} W_{\omega,\lambda}^{(a)}(b) > e^{-\frac{1}{4}mh_{IL}(\lambda)L} \implies \max_{y \in A_L(a)} W_{\omega,\lambda}^{(a)}(y) \leq e^{-\frac{7}{132}mh_{IL}(\lambda)\|y-a\|}, \quad (1.15)$$

where

$$A_L(a) := \{y \in \mathbb{Z}^d; \frac{8}{7}L \leq \|y-a\| \leq \frac{33}{14}L\}. \quad (1.16)$$

In particular, for all $\omega \in \mathcal{Y}_{L,a}$ and $\lambda \in I$ we have

$$W_{\omega,\lambda}^{(a)}(a)W_{\omega,\lambda}^{(a)}(y) \leq e^{-\frac{7}{132}mh_{IL}(\lambda)\|y-a\|} \text{ for all } y \in A_L(a). \quad (1.17)$$

Although Corollary 1.5 looks exactly like [17, Theorem 1.7], Theorem 1.4 is not the same as [17, Theorem 1.6] (the definitions of a localizing box are different, the conclusion (1.10) is stated differently from [17, Equation (1.20)]). For this reason the derivation of Corollary 1.5 from Theorem 1.4 has some differences from the derivation of [17, Theorem 1.7] from [17, Theorem 1.6], so it is included in this paper.

The usual forms of localization can be derived from Corollary 1.5 and are stated in the following corollary.

Corollary 1.6. *Suppose the conclusions of Theorem 1.4 hold for an Anderson model H_ω , and let $I = I_\infty$, $m = m_\infty$. Then the following holds with probability one:*

- (i) H_ω has pure point spectrum in the interval I .
- (ii) If ψ_λ is an eigenfunction of H_ω with eigenvalue $\lambda \in I$, then ψ_λ is exponentially localized with rate of decay $\frac{7}{132}mh_I(\lambda)$, more precisely,

$$|\psi_\lambda(x)| \leq C_{\omega,\lambda} \|T_0^{-1}\psi\| e^{-\frac{7}{132}mh_I(\lambda)\|x\|} \text{ for all } x \in \mathbb{R}^d. \quad (1.18)$$

- (iii) If $\lambda \in I$, then for all $x, y \in \mathbb{Z}^d$ we have

$$W_{\omega,\lambda}^{(x)}(x)W_{\omega,\lambda}^{(x)}(y) \leq C_{m,\omega,\nu} (h_I(\lambda))^{-\nu} e^{(\frac{4}{33}+\nu)mh_I(\lambda)(2d \log \langle x \rangle)^{\frac{1}{\xi}}} e^{-\frac{7}{132}mh_I(\lambda)\|y-x\|}. \quad (1.19)$$

- (iv) If $\lambda \in I$, then for $\psi \in \chi_{\{\lambda\}}(H_\omega)$ and all $x, y \in \mathbb{Z}^d$ we have

$$\begin{aligned} & |\psi(x)| |\psi(y)| \\ & \leq C_{m,\omega,\nu} (h_I(\lambda))^{-\nu} \|T_x^{-1}\psi\|^2 e^{(\frac{4}{33}+\nu)mh_I(\lambda)(2d \log \langle x \rangle)^{\frac{1}{\xi}}} e^{-\frac{7}{132}mh_I(\lambda)\|y-x\|} \\ & \leq 2^\nu C_{m,\omega,\nu} (h_I(\lambda))^{-\nu} \|T_0^{-1}\psi\|^2 \langle x \rangle^{2\nu} e^{(\frac{4}{33}+\nu)mh_I(\lambda)(2d \log \langle x \rangle)^{\frac{1}{\xi}}} e^{-\frac{7}{132}mh_I(\lambda)\|y-x\|}. \end{aligned} \quad (1.20)$$

(v) If $\lambda \in I$, then there exists $x_\lambda = x_{\omega, \lambda} \in \mathbb{Z}^d$, such that for $\psi \in \chi_{\{\lambda\}}(H_\omega)$ and all $x \in \mathbb{Z}^d$ we have

$$\begin{aligned} |\psi(x)| &\leq C_{m, \omega, \nu} (h_I(\lambda))^{-\nu} \|T_{x_\lambda}^{-1} \psi\| e^{(\frac{4}{33} + \nu) m h_I(\lambda) (2d \log \langle x_\lambda \rangle)^{\frac{1}{2}}} e^{-\frac{7}{132} m h_I(\lambda) \|x - x_\lambda\|} \\ &\leq 2^{\frac{\nu}{2}} C_{m, \omega, \nu} (h_I(\lambda))^{-\nu} \|T_0^{-1} \psi\| \langle x_\lambda \rangle^\nu e^{(\frac{4}{33} + \nu) m h_I(\lambda) (2d \log \langle x_\lambda \rangle)^{\frac{1}{2}}} e^{-\frac{7}{132} m h_I(\lambda) \|x - x_\lambda\|}. \end{aligned} \quad (1.21)$$

In Corollary 1.6, (i) and (ii) are statements of Anderson localization, (iii) and (iv) are statements of dynamical localization ((iv) is called SUDEC (summable uniform decay of eigenfunction correlations) in [24]), and (v) is SULE (semi-uniformly localized eigenfunctions; see [11, 12, 24]). Statements of localization in expectation can also be derived, as in [24, 25].

The proof of Corollary 1.6 from Corollary 1.5 is the same as the proof of [17, Corollary 1.8] from [17, Theorem 1.7], with some obvious modifications, so we refer to [17].

Theorem 1.4 also implies localization at the bottom of the spectrum as in [17, Section 2].

The conclusions of Theorem 1.4 are equivalent to the conclusions of the energy interval multiscale analysis [15, 22, 23, 27]; this can be seen proceeding as in [17, Section 6]. Finally, we stress that the theorem holds for Anderson models whose single-site probability distributions satisfy (1.3).

In the remainder of this paper, we fix $0 < \xi < \zeta < 1$ and the corresponding exponents $\tau, \beta, \kappa, \kappa', \varrho \in (0, 1)$ and $\gamma > 1$, as in Appendix A. The deterministic lemmas for the EMSA are introduced in Sect. 2. The probability estimates based on Wegner estimates are presented in Sect. 3. Theorem 1.4 is proven in Sect. 4. The proof of Corollary 1.5 is given in Sect. 5.

2. Lemmas for the Eigensystem Multiscale Analysis

In this section we introduce notation and deterministic lemmas that will play an important role in the eigensystem multiscale analysis. By H we always denote a discrete Schrödinger operator $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$. We also fix an interval $I = I(E, A)$.

2.1. Preliminaries

Let $\Phi \subset \Theta \subset \mathbb{Z}^d$. We define the boundary, exterior boundary, and interior boundary of Φ relative to Θ , respectively, by

$$\begin{aligned} \partial^\Theta \Phi &= \{(u, v) \in \Phi \times (\Theta \setminus \Phi) ; |u - v| = 1\}, \\ \partial_{\text{ex}}^\Theta \Phi &= \left\{ v \in (\Theta \setminus \Phi) ; (u, v) \in \partial^\Theta \Phi \text{ for some } u \in \Phi \right\}, \\ \partial_{\text{in}}^\Theta \Phi &= \left\{ u \in \Phi ; (u, v) \in \partial^\Theta \Phi \text{ for some } v \in \Theta \setminus \Phi \right\}. \end{aligned} \quad (2.1)$$

If $t \geq 1$, we let

$$\Phi^{\Theta, t} = \{y \in \Phi ; \text{dist}(y, \Theta \setminus \Phi) > [t]\} \quad \text{and} \quad \partial_{\text{in}}^{\Theta, t} \Phi = \Phi \setminus \Phi^{\Theta, t}. \quad (2.2)$$

We use the notation

$$R_\Theta(y) = \text{dist}(y, \partial_{\text{in}}^\Theta \Phi) \quad \text{for } y \in \Phi. \quad (2.3)$$

For a box $\Lambda_L \subset \Theta \subset \mathbb{Z}^d$ we write $\Lambda_L^{\Theta, t}(x) = (\Lambda_L(x))^{\Theta, t}$. For $L \geq 2$ we have

$$|\partial_{\text{in}}^\Theta \Lambda_L| \leq |\partial_{\text{ex}}^\Theta \Lambda_L| = |\partial^\Theta \Lambda_L| \leq s_d L^{d-1}, \quad \text{where } s_d = 2^d d. \quad (2.4)$$

For $v \in \Theta$ we let $\hat{v} \in \partial_{\text{in}}^\Theta \Lambda_L$ be the unique $u \in \partial_{\text{in}}^\Theta \Lambda_L$ such that $(u, v) \in \partial^\Theta \Lambda_L$ if $v \in \partial_{\text{ex}}^\Theta \Lambda_L$, and set $\hat{v} = 0$ otherwise.

If $\Phi \subset \Theta \subset \mathbb{Z}^d$, we consider $\ell^2(\Phi) \subset \ell^2(\Theta)$ by extending functions on Φ to functions on Θ that are identically 0 on $\Theta \setminus \Phi$. We have

$$\begin{aligned} H_\Theta &= H_\Phi \oplus H_{\Theta \setminus \Phi} + \Gamma_{\partial^\Theta \Phi} \quad \text{on } \ell^2(\Theta) = \ell^2(\Phi) \oplus \ell^2(\Theta \setminus \Phi), \\ \text{where } \Gamma_{\partial^\Theta \Phi}(u, v) &= \begin{cases} -1 & \text{if either } (u, v) \text{ or } (v, u) \in \partial^\Theta \Phi \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \quad (2.5)$$

Given $J \subset \mathbb{R}$, we set $\sigma_J(H_\Theta) = \sigma(H_\Theta) \cap J$ and $\tilde{\sigma}_J(H_\Theta) = \tilde{\sigma}(H_\Theta) \cap J$.

A function $\psi: \Theta \rightarrow \mathbb{C}$ is called a generalized eigenfunction for H_Θ with generalized eigenvalue $\lambda \in \mathbb{R}$, and (ψ, λ) is called a generalized eigenpair for H_Θ , if ψ is not identically zero and

$$\langle (H_\Theta - \lambda)\varphi, \psi \rangle = 0 \quad \text{for all } \varphi \in \ell^2(\Theta) \text{ with finite support.} \quad (2.6)$$

Lemma 2.1. *Let $\Theta \subset \mathbb{Z}^d$ and let (ψ, λ) be a generalized eigenpair for H_Θ . Let $\Phi \subset \Theta$ finite, $\eta > 0$, and suppose*

$$\text{dist}(\lambda, \sigma(H_\Phi)) \geq \eta. \quad (2.7)$$

Then for all $y \in \Phi$ we have

$$|\psi(y)| \leq 2d\eta^{-1} |\partial_{\text{ex}}^\Theta \Phi|^{\frac{1}{2}} |\psi(y_1)| \quad \text{for some } y_1 \in \partial_{\text{ex}}^\Theta \Phi. \quad (2.8)$$

The estimate (2.8) also holds (trivially) for $y \in \partial_{\text{ex}}^\Theta \Phi$ if $2d\eta^{-1} \geq 1$.

Proof. Let $\{(\varphi_\nu, \nu)\}_{\nu \in \tilde{\sigma}(H_\Phi)}$ be an eigensystem for H_Φ . If $\nu \in \tilde{\sigma}(H_\Phi)$, we have $|\lambda - \nu| \geq \eta$ by (2.7). Since Φ is finite, using (2.6) and (2.5) we get

$$\begin{aligned} \langle \varphi_\nu, \psi \rangle &= (\lambda - \nu)^{-1} \langle (H_\Theta - \nu)\varphi_\nu, \psi \rangle = (\lambda - \nu)^{-1} \langle (H_\Theta - H_\Phi)\varphi_\nu, \psi \rangle \\ &= (\lambda - \nu)^{-1} \langle \varphi_\nu, \Gamma_{\partial^\Theta \Phi} \psi \rangle. \end{aligned} \quad (2.9)$$

It follows that for $y \in \Phi$ we have

$$\begin{aligned} \psi(y) &= \langle \delta_y, \psi \rangle = \left\langle \delta_y, \sum_{\nu \in \tilde{\sigma}(H_\Phi)} \langle \varphi_\nu, \psi \rangle \varphi_\nu \right\rangle \\ &= \left\langle \delta_y, \sum_{\nu \in \tilde{\sigma}(H_\Phi)} (\lambda - \nu)^{-1} \langle \varphi_\nu, \Gamma_{\partial^\Theta \Phi} \psi \rangle \varphi_\nu \right\rangle \\ &= \left\langle \delta_y, \sum_{\nu \in \tilde{\sigma}(H_\Phi)} (\lambda - \nu)^{-1} \langle \varphi_\nu, \chi_\Phi \Gamma_{\partial^\Theta \Phi} \psi \rangle \varphi_\nu \right\rangle \end{aligned}$$

$$\begin{aligned}
 &= \left\langle \delta_y, (\lambda - H_\Phi)^{-1} \sum_{\nu \in \tilde{\sigma}(H_\Phi)} \langle \varphi_\nu, \chi_\Phi \Gamma_{\partial^\Theta \Phi} \psi \rangle \varphi_\nu \right\rangle \\
 &= \left\langle \delta_y, (\lambda - H_\Phi)^{-1} \chi_\Phi \Gamma_{\partial^\Theta \Phi} \psi \right\rangle.
 \end{aligned} \tag{2.10}$$

Using (2.7), we get

$$\begin{aligned}
 |\psi(y)| &\leq \eta^{-1} \|\chi_\Phi \Gamma_{\partial^\Theta \Phi} \psi\| = \eta^{-1} \|\chi_\Phi \Gamma_{\partial^\Theta \Phi} \chi_{\partial_{\text{ex}}^\Theta \Phi} \psi\| \leq 2d\eta^{-1} \|\chi_{\partial_{\text{ex}}^\Theta \Phi} \psi\| \\
 &\leq 2d\eta^{-1} |\partial_{\text{ex}}^\Theta \Phi|^{\frac{1}{2}} |\psi(y_1)| \quad \text{for some } y_1 \in \partial_{\text{ex}}^\Theta \Phi.
 \end{aligned} \tag{2.11}$$

□

For the interval $I = I(E, A)$ and $L > 1$, we set

$$I_L = I(E, A(1 - L^{-\kappa})) \subsetneq I = I(E, A) \subsetneq I^L = (E, A(1 - L^{-\kappa})^{-1}). \tag{2.12}$$

We write $I_L^{L'} = (I_L)^{L'} = \left(I^{L'}\right)_L$, and observe that $I_L^L = I$. Note that

$$h_I(t) \geq 1 - (1 - L^{-\kappa})^2 \geq L^{-\kappa} \quad \text{for all } t \in I_L, \quad \text{so } h_I \chi_{I_L} \geq L^{-\kappa} \chi_{I_L}. \tag{2.13}$$

2.2. Localizing Boxes

The following lemma plays a crucial role in the multiscale analysis. It says that given an eigenpair (ψ, λ) for H_Θ and a box $\Lambda_\ell \subset \Theta$ with $\lambda \in I_\ell$ not too close to the eigenvalues of H_{Λ_ℓ} , then ψ is exponentially small deep inside Λ_ℓ if the box Λ_ℓ is (m, I) -localizing for H .

If Λ_ℓ is an (m, I) -localizing box, $\{(\varphi_\nu, \nu)\}_{\nu \in \tilde{\sigma}(H_{\Lambda_\ell})}$ will denote an (m, I) -localized eigensystem for H_{Λ_ℓ} . If $\Lambda_\ell \subset \Theta \subset \mathbb{Z}^d$, $J \subset I$ and $t > 0$, we set

$$\tilde{\sigma}_J^{\Theta, t}(H_{\Lambda_\ell}) = \left\{ \nu \in \tilde{\sigma}_J(H_{\Lambda_\ell}); x_\nu \in \Lambda_\ell^{\Theta, t} \right\}. \tag{2.14}$$

Given a scale $\ell \geq 1$, we set $L = \ell^\gamma$. The exponent $\tilde{\tau}$ is defined in (A.3). We use the notation $L_\tau = \lfloor L^\tau \rfloor$ and $L_{\tilde{\tau}} = \lfloor L^{\tilde{\tau}} \rfloor$.

Lemma 2.2. *Let $\psi: \Theta \subset \mathbb{Z}^d \rightarrow \mathbb{C}$ be a generalized eigenfunction for H_Θ with generalized eigenvalue $\lambda \in I_\ell$. Consider a box $\Lambda_\ell \subset \Theta$ such that Λ_ℓ is (m, I) -localizing for H . Suppose*

$$\text{dist}(\lambda, \sigma_I(H_{\Lambda_\ell})) \geq \frac{1}{2} e^{-L^\beta}. \tag{2.15}$$

Then, if ℓ is sufficiently large, for all $y \in \Lambda_\ell^{\Theta, \ell^{\tilde{\tau}}}$ we have

$$|\psi(y)| \leq e^{-m_3 h_I(\lambda) R_\Theta(y)} |\psi(v)| \quad \text{for some } v \in \partial_{\text{ex}}^\Theta \Lambda_\ell, \tag{2.16}$$

where

$$m_3 = m_3(\ell) \geq m \left(1 - C_d \ell^{-\frac{1-\tau}{2}} \right). \tag{2.17}$$

Lemma 2.2 resembles [17, Lemma 3.4], but the hypothesis (2.15) is stronger than the corresponding hypothesis [17, Eq. (3.24)], so the proof is slightly easier, and the conclusions are slightly stronger. The main issue in the proof is the same: the hypothesis that the box $\Lambda_\ell \subset \Theta$ is (m, I) -localizing only gives decay for eigenfunctions with eigenvalues in I . To compensate, we take $\lambda \in I_\ell$, and use [17, Lemmas 3.2 and 3.3].

Proof (Proof of Lemma 2.2).

Given $y \in \Lambda_\ell$ and $t > 0$, it follows from [17, Lemma 3.2(i)] that

$$\psi(y) = \left\langle e^{-t((H_{\Lambda_\ell} - E)^2 - (\lambda - E)^2)} \delta_y, \psi \right\rangle - \left\langle F_{t, \lambda - E}(H_{\Lambda_\ell} - E) \delta_y, \Gamma_{\partial \Theta_{\Lambda_\ell}} \psi \right\rangle, \quad (2.18)$$

where $\Gamma_{\partial \Theta_\Phi}$ is defined in (2.5) and $F_{t, \lambda}(z)$ is the entire function given by

$$F_{t, \lambda}(z) = \frac{1 - e^{-t(z^2 - \lambda^2)}}{z - \lambda} \quad \text{for } z \in \mathbb{C} \setminus \{\lambda\} \quad \text{and} \quad F_{t, \lambda}(\lambda) = 2t\lambda. \quad (2.19)$$

We take $E = 0$ by replacing the potential V by $V - E$. Setting $P_I = \chi_I(H_{\Lambda_\ell})$ and $\bar{P}_I = 1 - P_I$, we have

$$\left\langle e^{-t(H_{\Lambda_\ell}^2 - \lambda^2)} \delta_y, \psi \right\rangle = \left\langle e^{-t(H_{\Lambda_\ell}^2 - \lambda^2)} P_I \delta_y, \psi \right\rangle + \left\langle e^{-t(H_{\Lambda_\ell}^2 - \lambda^2)} \bar{P}_I \delta_y, \psi \right\rangle. \quad (2.20)$$

It follows from [17, Lemma 3.3] that

$$\begin{aligned} \left| \left\langle e^{-t(H_{\Lambda_\ell}^2 - \lambda^2)} \bar{P}_I \delta_y, \psi \right\rangle \right| &\leq \|\chi_{\Lambda_\ell} \psi\| \left\| e^{-t(H_{\Lambda_\ell}^2 - \lambda^2)} \bar{P}_I \right\| \\ &\leq (\ell + 1)^{\frac{d}{2}} e^{-tA^2 h_I(\lambda)} |\psi(v)|, \end{aligned} \quad (2.21)$$

for some $v \in \Lambda_\ell$. Estimating $|\psi(v)|$ by Lemma 2.1, we get

$$\begin{aligned} \left| \left\langle e^{-t(H_{\Lambda_\ell}^2 - \lambda^2)} \bar{P}_I \delta_y, \psi \right\rangle \right| &\leq 4d (s_d \ell^{d-1})^{\frac{1}{2}} (\ell + 1)^{\frac{d}{2}} e^{L^\beta} e^{-tA^2 h_I(\lambda)} |\psi(v_0)| \\ &\leq e^{2L^\beta} e^{-tA^2 h_I(\lambda)} |\psi(v_0)|, \quad \text{for some } v_0 \in \partial_{\text{ex}}^\Theta \Lambda_\ell. \end{aligned} \quad (2.22)$$

We now use the fact that Λ_ℓ is (m, I) -localizing for H , so it has an (m, I) -localized eigensystem $\{\varphi_\nu, \nu\}_{\nu \in \tilde{\sigma}(H_{\Lambda_\ell})}$, and write

$$\left\langle e^{-t(H_{\Lambda_\ell}^2 - \lambda^2)} P_I \delta_y, \psi \right\rangle = \sum_{\nu \in \tilde{\sigma}_I(\mathcal{H}_{\Lambda_\ell})} e^{-t(\nu^2 - \lambda^2)} \varphi_\nu(y) \langle \varphi_\nu, \psi \rangle. \quad (2.23)$$

If $\nu \in \tilde{\sigma}_I(H_{\Lambda_\ell})$, we have $|\lambda - \nu| \geq \frac{1}{2} e^{-L^\beta}$ by (2.15). Since Λ_ℓ is finite, (2.6) gives

$$\langle \varphi_\nu, \psi \rangle = (\lambda - \nu)^{-1} \langle (H_\Theta - \nu) \varphi_\nu, \psi \rangle. \quad (2.24)$$

It follows from [16, Eq. (3.12) in Lemma 3.2] that

$$|\varphi_\nu(y) \langle \varphi_\nu, \psi \rangle| \leq 2e^{L^\beta} \sum_{v \in \partial_{\text{ex}}^\Theta \Lambda_\ell} |\varphi_\nu(y) \varphi_\nu(\hat{v})| |\psi(v)|. \quad (2.25)$$

We now assume $y \in \Lambda_\ell^{\Theta, \ell\tilde{\tau}}$, so $R_\Theta(y) \geq \ell\tilde{\tau}$. For $\nu \in \tilde{\sigma}_I^{\Theta, \ell\tau}(H_{\Lambda_\ell})$ and $v' \in \partial_{\text{in}}^\Theta \Lambda_\ell$, we have, as in [16, Eq. (3.41)],

$$|\varphi_\nu(y)\varphi_\nu(v')| \leq e^{-m'_1 h_I(\nu)R_\Theta(y)} \quad \text{with} \quad m'_1 \geq m(1 - 2\ell^{\frac{\tau-1}{2}}), \quad (2.26)$$

so, as in [16, Eq. (3.44)], for $\nu \in \tilde{\sigma}_I^{\Theta, \ell\tau}(H_{\Lambda_\ell})$ we have

$$\begin{aligned} |\varphi_\nu(y) \langle \varphi_\nu, \psi \rangle| &\leq 2e^{L^\beta} s_d \ell^{d-1} e^{-m'_1 h_I(\nu)R_\Theta(y)} |\psi(v_1)| \\ &\leq e^{2L^\beta} e^{-m'_1 h_I(\nu)R_\Theta(y)} |\psi(v_1)|, \end{aligned} \quad (2.27)$$

for some $v_1 \in \partial_{\text{ex}}^\Theta \Lambda_\ell$. If $\nu \in \tilde{\sigma}_I(H_{\Lambda_\ell})$ with $x_\nu \in \partial_{\text{in}}^{\Theta, \ell\tau} \Lambda_\ell$, we have

$$\|x_\nu - y\| \geq R_\Theta(y) - \ell_\tau \geq R_\Theta(y) \left(1 - 2\ell^{\tau-\tilde{\tau}}\right) = R_\Theta(y) \left(1 - 2\ell^{\frac{\tau-1}{2}}\right), \quad (2.28)$$

so

$$\begin{aligned} |\varphi_\nu(y) \langle \varphi_\nu, \psi \rangle| &\leq e^{-mh_I(\nu)\|x_\nu - y\|} \|\chi_\Lambda \psi\| \\ &\leq e^{-mh_I(\nu)R_\Theta(y) \left(1 - 2\ell^{\frac{\tau-1}{2}}\right)} (\ell + 1)^{\frac{d}{2}} |\psi(v_2)| \leq (\ell + 1)^{\frac{d}{2}} e^{-m'_1 h_I(\nu)R_\Theta(y)} |\psi(v_2)|, \end{aligned} \quad (2.29)$$

for some $v_2 \in \Lambda_\ell$, where m'_1 is given in (2.26). It follows that for all $\nu \in \tilde{\sigma}_I(H_{\Lambda_\ell})$ we have

$$e^{-t(\nu^2 - \lambda^2)} |\varphi_\nu(y) \langle \varphi_\nu, \psi \rangle| \leq e^{2L^\beta} e^{-t(\nu^2 - \lambda^2)} e^{-m'_1 h_I(\nu)R_\Theta(y)} |\psi(v)|, \quad (2.30)$$

for some $v \in \Lambda_\ell \cup \partial_{\text{ex}}^\Theta \Lambda_\ell$.

We now take

$$t = \frac{m'_1 R_\Theta(y)}{A^2} \implies e^{-t(\nu^2 - \lambda^2)} e^{-m'_1 h_I(\nu)R_\Theta(y)} = e^{-m'_1 h_I(\lambda)R_\Theta(y)} \quad \text{for } \nu \in I, \quad (2.31)$$

obtaining

$$\begin{aligned} \left| \left\langle e^{-\frac{m'_1 R_\Theta(y)}{A^2} (H_{\Lambda_\ell}^2 - \lambda^2)} P_I \delta_y, \psi \right\rangle \right| &\leq (\ell + 1)^d e^{2L^\beta} e^{-m'_1 h_I(\lambda)R_\Theta(y)} |\psi(v)| \\ &\leq 4d \left(s_d \ell^{d-1}\right)^{\frac{1}{2}} (\ell + 1)^d e^{3L^\beta} e^{-m'_1 h_I(\lambda)R_\Theta(y)} |\psi(v')| \\ &\leq e^{4L^\beta} e^{-m'_1 h_I(\lambda)R_\Theta(y)} |\psi(v')|, \end{aligned} \quad (2.32)$$

for some $v \in \Lambda_\ell \cup \partial_{\text{ex}}^\Theta \Lambda_\ell$, and then for some $v' \in \partial_{\text{ex}}^\Theta \Lambda_\ell$ using Lemma 2.1.

Combining (2.20), (2.22) and (2.32) yields

$$\left| \left\langle e^{-\frac{m'_1 R_\Theta(y)}{A^2} (H_{\Lambda_\ell}^2 - \lambda^2)} \delta_y, \psi \right\rangle \right| \leq 2e^{4L^\beta} e^{-m'_1 h_I(\lambda)R_\Theta(y)} |\psi(v)|, \quad (2.33)$$

for some $v \in \partial_{\text{ex}}^\Theta \Lambda_\ell$.

We now use [17, Lemma 3.2(ii)] (it follows from (1.8) that $\frac{\ell^{-\kappa'}}{2} < m'_1 \leq m \leq \frac{1}{2} \log(1 + \frac{A}{4d})$), getting

$$\left| \left\langle F_{\frac{m'_1 R_{\Theta}(y)}{A^2}, \lambda} (H_{\Lambda}) \delta_y, \Gamma_{\partial^{\Theta} \Lambda_{\ell}} \psi \right\rangle \right| \leq 70 s_d \ell^{d-1} A^{-1} e^{-m'_1 h_I(\lambda) R_{\Theta}(y)} |\psi(v)|, \quad (2.34)$$

for some $v \in \partial^{\Theta} \Lambda_{\ell}$. We conclude from (2.33) and (2.34) that

$$\begin{aligned} |\psi(y)| &\leq C_d \left(\ell^{d-1+\kappa'} + e^{4L^{\beta}} \right) e^{-m'_1 h_I(\lambda) R_{\Theta}(y)} |\psi(v)| \\ &\leq C'_d e^{4L^{\beta}} e^{-m'_1 h_I(\lambda) R_{\Theta}(y)} |\psi(v)| \\ &\leq e^{-m_3 h_I(\lambda) R_{\Theta}(y)} |\psi(v')| \text{ for some } v \in \partial^{\Theta} \Lambda_{\ell}, \end{aligned} \quad (2.35)$$

where, using $h_I(\lambda) \geq \ell^{-\kappa}$ since $\lambda \in I_{\ell}$, we have

$$m_3 \geq m \left(1 - C_d \ell^{-\min\{\tilde{\tau}-\gamma\beta-\kappa-\kappa', \frac{1-\tau}{2}\}} \right) = m \left(1 - C_d \ell^{-\frac{1-\tau}{2}} \right). \quad (2.36)$$

□

2.3. Buffered Subsets

The probability estimates of a multiscale analysis do not allow all boxes to be localizing, so we must control non-localizing boxes. If a box $\Lambda_{\ell} \subset \Lambda_L$ is not (m, I) -localizing for H , we will add a buffer of (m, I) -localizing boxes and study eigensystems for the enlarged subset.

Definition 2.3. We call $\Upsilon \subset \Lambda_L$ an (m, I) -buffered subset of the box Λ_L if the following holds:

- (i) Υ is a connected set in $\mathbb{Z}^d$ of the form

$$\Upsilon = \bigcup_{j=1}^J \Lambda_{R_j}(a_j) \cap \Lambda_L, \quad (2.37)$$

where $J \in \mathbb{N}$, $a_1, a_2, \dots, a_J \in \Lambda_L^{\mathbb{R}}$, and $\ell \leq R_j \leq L$ for $j = 1, 2, \dots, J$.

- (ii) There exists $\mathcal{G}_{\Upsilon} \subset \Lambda_L^{\mathbb{R}}$ such that:

- (a) $\Lambda_{\ell}(a) \subset \Lambda_L$ for all $a \in \mathcal{G}_{\Upsilon}$ and $\{\Lambda_{\ell}(a)\}_{a \in \mathcal{G}_{\Upsilon}}$ is a collection of (m, I) -localizing boxes for H .
- (b) For all $y \in \partial_{\text{in}}^{\Lambda_L} \Upsilon$ there exists $a_y \in \mathcal{G}_{\Upsilon}$ such that $y \in \Lambda_{\ell}^{\Lambda_L, \ell \tilde{\tau}}(a_y)$.

This definition of a buffered subset has subtle but important differences from [17, Definition 3.6], in addition to not requiring level spacing conditions. Definition 2.3(ii) requires $\Lambda_{\ell}(a) \subset \Lambda_L$ and $y \in \Lambda_{\ell}^{\Lambda_L, \ell \tilde{\tau}}(a_y)$, while the corresponding [17, Definition 3.6](iii) has $\Lambda_{\ell}(a) \subset \Upsilon$ and $y \in \Lambda_{\ell}^{\Upsilon, 2\ell \tau}(a_y)$.

In the multiscale analysis, we control the effect of buffered subsets using the following lemma.

Lemma 2.4. Let $\Lambda_L = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$, and let (ψ, λ) be an eigenpair for H_{Λ_L} with $\lambda \in I_{\ell}$. Let $\Upsilon \subsetneq \Lambda_L$ be an (m, I) -buffered subset, and suppose

$$\text{dist}(\lambda, \sigma_I(H_\Upsilon)) \geq \frac{1}{2}e^{-L^\beta} \quad \text{and} \quad \min_{a \in \mathcal{G}_\Upsilon} \text{dist}(\lambda, \sigma_I(H_{\Lambda_\ell(a)})) \geq \frac{1}{2}e^{-L^\beta}. \quad (2.38)$$

Then for all $y \in \Upsilon$ we have

$$|\psi(y)| \leq e^{-\frac{m_3}{2}h_I(\lambda)\ell_{\tilde{\tau}}} |\psi(y_1)| \quad \text{for some } y_1 \in \bigcup_{a \in \mathcal{G}_\Upsilon} \partial_{\text{ex}}^{\Lambda_L} \Lambda_\ell(a), \quad (2.39)$$

where $m_3 = m_3(\ell)$ is as in (2.17).

Proof. Let $y \in \Upsilon$. In view of (2.38) it follows from Lemma 2.1 that

$$|\psi(y)| \leq 4de^{L^\beta} |\partial_{\text{ex}}^{\Lambda_L} \Upsilon| |\psi(y_1)| \quad \text{for some } y_1 \in \partial_{\text{ex}}^{\Lambda_L} \Upsilon. \quad (2.40)$$

Let $a_1 \in \mathcal{G}_\Upsilon$ be such that $y_1 \in \Lambda_\ell^{\Lambda_L, \ell_{\tilde{\tau}}}(a_1)$. It then follows from (2.38) and (2.16) in Lemma 2.2 that

$$|\psi(y_1)| \leq e^{-m_3h_I(\lambda)\ell_{\tilde{\tau}}} |\psi(y_2)| \quad \text{for some } y_2 \in \partial_{\text{ex}}^{\Lambda_L} \Lambda_\ell(a_1). \quad (2.41)$$

Since $|\Upsilon| \leq |\Lambda_L| \leq (L+1)^d$ and $|\partial_{\text{ex}}^{\Lambda_L} \Upsilon| \leq 2d|\Upsilon| \leq 2d(L+1)^d$, and we have (2.13) as $\lambda \in I_\ell$, we get

$$|\psi(y)| \leq 8d^2(L+1)^d e^{L^\beta} e^{-m_3h_I(\lambda)\ell_{\tilde{\tau}}} |\psi(y_3)| \leq e^{-\frac{m_3}{2}h_I(\lambda)\ell_{\tilde{\tau}}}, \quad (2.42)$$

for some $y_3 \in \bigcup_{a \in \mathcal{G}_\Upsilon} \partial_{\text{ex}}^{\Lambda_L} \Lambda_\ell(a)$, if L is sufficiently large. $\square$

3. Spectral Separation

We recall the Wegner estimate for the Anderson model as in Definition 1.1 (see, e.g., [8, Appendix A]).

Lemma 3.1. *Let H_ω be an Anderson model. Let $\Theta \subset \mathbb{Z}^d$. Then, for all $E \in \mathbb{R}$,*

$$\mathbb{P}\{\text{dist}\{E, \sigma(H_{\Theta, \omega})\} \leq \eta\} \leq \tilde{K}\eta^\alpha |\Theta|, \quad (3.1)$$

where with $\tilde{K} = 2K$ if $\alpha = 1$ and $\tilde{K} = 8 \cdot 2^\alpha K$ if $\alpha \in (0, 1)$.

Definition 3.2. Let $R > 0$. Two finite sets $\Theta, \Theta' \subset \mathbb{Z}^d$ will be called R -separated for H if $\text{dist}\{\sigma(H_\Theta), \sigma(H_{\Theta'})\} \geq e^{-R^\beta}$, i.e., $|\lambda - \lambda'| \geq e^{-R^\beta}$ for all $\lambda \in \sigma(H_\Theta)$ and $\lambda' \in \sigma(H_{\Theta'})$.

Definition 3.3. Let $\Theta \subset \mathbb{Z}^d$ and $R > 0$. A family $\{\Phi_j\}_{j \in J}$ of finite subsets of Θ is called R -separated for H if Φ_j and $\Phi_{j'}$ are R -separated for H for all $j, j' \in J$ such that $\Phi_j \cap \Phi_{j'} = \emptyset$.

Lemma 3.1 implies the Wegner estimate for R -separated sets (see, e.g., [26, Lemma 5.28]).

Lemma 3.4. *Let H_ω be an Anderson model. Let $\Theta, \Theta' \subset \mathbb{Z}^d$ with $\Theta \cap \Theta' = \emptyset$. Then, for all $0 < \eta$,*

$$\mathbb{P}\{\text{dist}\{\sigma(H_\Theta), \sigma(H_{\Theta'})\} \leq \eta\} \leq \tilde{K}\eta^\alpha |\Theta| |\Theta'|. \quad (3.2)$$

In particular,

$$\mathbb{P}\{\Theta, \Theta' \text{ are } R\text{-separated for } H\} \geq 1 - \tilde{K}e^{-\alpha R^\beta} |\Theta| |\Theta'|. \quad (3.3)$$

4. Eigensystem Multiscale Analysis

In this section we fix an Anderson model H_ω and prove Theorem 1.4.

The following is an extension of Definition 1.3.

Definition 4.1. Let $J = I(E, B) \subset I = I(E, A)$ be bounded open intervals with the same center. A box Λ_L will be called (m, J, I) -localizing for H if

$$L^{-\kappa'} \leq m \leq \frac{1}{2} \log \left(1 + \frac{B}{4d} \right), \quad (4.1)$$

and there exists an (m, J, I) -localized eigensystem for H_{Λ_L} , that is, an eigensystem $\{(\varphi_\nu, \nu)\}_{\nu \in \tilde{\sigma}(H_{\Lambda_L})}$ for H_{Λ_L} such that for all $\nu \in \tilde{\sigma}(H_{\Lambda_L})$ there is $x_\nu \in \Lambda_L$ so φ_ν is $(x_\nu, m\chi_J(\nu)h_I(\nu))$ -localized.

Note that (m, I, I) -localizing/localized is the same as (m, I) -localizing/localized. If Λ_L is (m, J, I) -localizing for H it is also (m, J) -localizing for H as $\chi_J h_I \geq h_J$.

Proposition 4.2. *There exists a finite scale $\mathcal{L} = \mathcal{L}(d)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}$ and interval $I_0 = I(E, A_0)$ we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_{L_0}(x) \text{ is } (m_0, I_0)\text{-localizing for } H_\omega \} \geq 1 - e^{-L_0^\zeta}. \quad (4.2)$$

Set $L_{k+1} = L_k^\gamma$, $A_{k+1} = A_k(1 - L_k^{-\kappa})$, and $I_{k+1} = I(E, A_{k+1})$, for $k = 0, 1, \dots$. Then for all $k = 1, 2, \dots$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_{L_k}(x) \text{ is } (m_k, I_k, I_{k-1})\text{-localizing for } H_\omega \} \geq 1 - e^{-L_k^\zeta}, \quad (4.3)$$

where

$$L_k^{-\kappa'} < m_{k-1} \left(1 - C_d L_{k-1}^{-\varrho} \right) \leq m_k < \frac{1}{2} \log \left(1 + \frac{A_k}{4d} \right). \quad (4.4)$$

The proof of Proposition 4.2 relies on the following lemma, the induction step for the multiscale analysis.

Lemma 4.3. *Let $I = (E, A)$. Suppose for some scale ℓ we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_\ell(x) \text{ is } (m, I)\text{-localizing for } H_\omega \} \geq 1 - e^{-\ell^\zeta}. \quad (4.5)$$

Then, if ℓ is sufficiently large, we have (recall $L = \ell^\gamma$)

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_L(x) \text{ is } (M, I_\ell, I)\text{-localizing for } H_\omega \} \geq 1 - e^{-L^\zeta}, \quad (4.6)$$

where

$$L^{-\kappa'} < m \left(1 - C_d \ell^{-\varrho} \right) \leq M < \frac{1}{2} \log \left(1 + \frac{A(1 - \ell^{-\kappa})}{4d} \right). \quad (4.7)$$

Proof. To prove the lemma we proceed as in [17, Proof of Lemma 4.2], with several modifications.

We assume (4.5) for a scale ℓ . We take $\Lambda_L = \Lambda(x_0)$, where $x_0 \in \mathbb{R}^d$, and let $C_{L,\ell} = C_{L,\ell}(x_0)$ be the suitable ℓ -cover of Λ_L with ς as in (A.7) (see Appendix B). Given $a, b \in \Xi_{L,\ell}$, we will say that the boxes $\Lambda_\ell(a)$ and $\Lambda_\ell(b)$ are

disjoint if and only if $\Lambda_\ell^{\mathbb{R}}(a) \cap \Lambda_\ell^{\mathbb{R}}(b) = \emptyset$, that is, if and only if $\|a - b\| \geq k_\ell \rho \ell^\varsigma$ (see Remark B.3). We take (recall (A.3))

$$N = N_\ell = \left\lfloor \ell^{(\gamma-1)\tilde{\zeta}} \right\rfloor, \quad (4.8)$$

and let $\mathcal{B}_N$ denote the event that there exist at most N disjoint boxes in $\mathcal{C}_{L,\ell}$ that are not (m, I) -localizing for H_ω . For sufficiently large ℓ , we have, using (B.5), (4.5), and the fact that events on disjoint boxes are independent, that

$$\mathbb{P}\{\mathcal{B}_N^c\} \leq \left(\frac{2L}{\ell^\varsigma}\right)^{(N+1)d} e^{-(N+1)\ell^\varsigma} = 2^{(N+1)d\ell^{(\gamma-\varsigma)(N+1)d}} e^{-(N+1)\ell^\varsigma} < \frac{1}{2} e^{-L^\varsigma}. \quad (4.9)$$

We now fix $\omega \in \mathcal{B}_N$. There exists $\mathcal{A}_N = \mathcal{A}_N(\omega) \subset \Xi_{L,\ell} = \Xi_{L,\ell}(x_0)$ such that $|\mathcal{A}_N| \leq N$ and $\|a - b\| \geq k_\ell \rho \ell^\varsigma$ if $a, b \in \mathcal{A}_N$ and $a \neq b$, with the following property: if $a \in \Xi_{L,\ell}$ with $\text{dist}(a, \mathcal{A}_N) \geq k_\ell \rho \ell^\varsigma$, so $\Lambda_\ell^{\mathbb{R}}(a) \cap \Lambda_\ell^{\mathbb{R}}(b) = \emptyset$ for all $b \in \mathcal{A}_N$, the box $\Lambda_\ell(a)$ is (m, I) -localizing for H_ω . In other words,

$$a \in \Xi_{L,\ell} \setminus \bigcup_{b \in \mathcal{A}_N} \Lambda_{2(k_\ell-1)\rho\ell^\varsigma}^{\mathbb{R}}(b) \implies \Lambda_\ell(a) \text{ is } (m, I)\text{-localizing for } H_\omega. \quad (4.10)$$

We want to embed the boxes $\{\Lambda_\ell(b)\}_{b \in \mathcal{A}_N}$ into (m, I) -buffered subsets of Λ_L . To do so, we consider graphs $\mathbb{G}_i = (\Xi_{L,\ell}, \mathbb{E}_i)$, $i = 1, 2$, both having $\Xi_{L,\ell}$ as the set of vertices, with sets of edges given by

$$\begin{aligned} \mathbb{E}_1 &= \left\{ \{a, b\} \in \Xi_{L,\ell}^2; 0 < \|a - b\| \leq (k_\ell - 1)\rho\ell^\varsigma \right\} \\ &= \left\{ \{a, b\} \in \Xi_{L,\ell}^2; a \neq b \text{ and } \Lambda_\ell^{\mathbb{R}}(a) \cap \Lambda_\ell^{\mathbb{R}}(b) \neq \emptyset \right\}, \\ \mathbb{E}_2 &= \left\{ \{a, b\} \in \Xi_{L,\ell}^2; k_\ell \rho \ell^\varsigma \leq \|a - b\| \leq (3k_\ell - 1)\rho\ell^\varsigma \right\} \\ &= \left\{ \{a, b\} \in \Xi_{L,\ell}^2; \Lambda_\ell^{\mathbb{R}}(a) \cap \Lambda_\ell^{\mathbb{R}}(b) = \emptyset \text{ and } \Lambda_{2k_\ell\rho\ell^\varsigma+\ell}^{\mathbb{R}}(a) \cap \Lambda_{2k_\ell\rho\ell^\varsigma+\ell}^{\mathbb{R}}(b) \neq \emptyset \right\}. \end{aligned} \quad (4.11)$$

Given $\Psi \subset \Xi_{L,\ell}$, we let $\overline{\Psi} = \Psi \cup \partial_{\text{ex}}^{\mathbb{G}_1} \Psi$, where $\partial_{\text{ex}}^{\mathbb{G}_1} \Psi$, the exterior boundary of Ψ in the graph $\mathbb{G}_1$, is defined by

$$\begin{aligned} \partial_{\text{ex}}^{\mathbb{G}_1} \Psi &= \{a \in \Xi_{L,\ell} \setminus \Psi; \text{dist}(a, \Psi) \leq (k_\ell - 1)\rho\ell^\varsigma\} \\ &= \{a \in \Xi_{L,\ell} \setminus \Psi; (b, a) \in \mathbb{E}_1 \text{ for some } b \in \Psi\}. \end{aligned} \quad (4.12)$$

Let $\Phi \subset \Xi_{L,\ell}$ be $\mathbb{G}_2$ -connected, so $\text{diam } \Phi \leq (3k_\ell - 1)\rho\ell^\varsigma (|\Phi| - 1)$. (The diameter of a set $\Xi \subset \mathbb{R}^d$ is given by $\text{diam } \Xi = \sup_{x, y \in \Xi} \|y - x\|$.) Then

$$\tilde{\Phi} = \{a \in \Xi_{L,\ell}; \text{dist}(a, \Phi) \leq k_\ell \rho \ell^\varsigma\} \quad (4.13)$$

is a $\mathbb{G}_1$ -connected subset of $\Xi_{L,\ell}$ such that

$$\text{diam } \tilde{\Phi} \leq \text{diam } \Phi + 2k_\ell \rho \ell^\varsigma \leq ((3k_\ell - 1)|\Phi| - (k_\ell - 1))\rho\ell^\varsigma \leq 5\ell|\Phi|. \quad (4.14)$$

We set

$$\Upsilon_\Phi = \bigcup_{a \in \tilde{\Phi}} \Lambda_\ell(a) \quad \text{and} \quad \mathcal{G}_{\Upsilon_\Phi} = \partial_{\text{ex}}^{\mathbb{G}_1} \tilde{\Phi}. \quad (4.15)$$

Let $\{\Phi_r\}_{r=1}^R = \{\Phi_r(\omega)\}_{r=1}^R$ denote the $\mathbb{G}_2$ -connected components of $\mathcal{A}_N$ (i.e., connected in the graph $\mathbb{G}_2$); we have $R \in \{1, 2, \dots, N\}$ and $\sum_{r=1}^R |\Phi_r| = |\mathcal{A}_N| \leq N$. We conclude that $\{\tilde{\Phi}_r\}_{r=1}^R$ is a collection of disjoint, $\mathbb{G}_1$ -connected subsets of $\Xi_{L,\ell}$, such that

$$\text{dist}(\tilde{\Phi}_r, \tilde{\Phi}_s) \geq k_\ell \rho \ell^\varsigma > \ell \quad \text{if } r \neq s. \quad (4.16)$$

Moreover, it follows from (4.10) that

$$a \in \mathcal{G} = \mathcal{G}(\omega) = \Xi_{L,\ell} \setminus \bigcup_{r=1}^R \tilde{\Phi}_r \implies \Lambda_\ell(a) \text{ is } (m, I)\text{-localizing for } H_\omega. \quad (4.17)$$

In particular, we conclude that $\Lambda_\ell(a)$ is (m, I) -localizing for H_ω for all $a \in \partial_{\text{ex}}^{\mathbb{G}_1} \tilde{\Phi}_r$, $r = 1, 2, \dots, R$.

Each $\Upsilon_r = \Upsilon_{\Phi_r}$, $r = 1, 2, \dots, R$, clearly satisfies all the requirements to be an (m, I) -buffered subset of Λ_L with $\mathcal{G}_{\Upsilon_r} = \partial_{\text{ex}}^{\mathbb{G}_1} \tilde{\Phi}_r$ (see Definition 2.3). Moreover the sets $\{\Upsilon_r\}_{r=1}^R$ are disjoint. Note also that it follows from (4.14) that

$$\text{diam } \Upsilon_r \leq \text{diam } \tilde{\Phi}_r + \ell \leq 5\ell |\Phi_r| + \ell \leq 6\ell |\Phi_r|, \quad (4.18)$$

so, using (A.4), we have

$$\sum_{r=1}^R \text{diam } \Upsilon_r \leq 6\ell N \leq 6\ell^{(\gamma-1)\tilde{\zeta}+1} \ll \ell^{\gamma\tau} = L^\tau. \quad (4.19)$$

Let

$$\mathbb{S}_\omega = \{\Lambda_\ell(a)\}_{a \in \mathcal{G}} \cup \{\Upsilon_r\}_{r=1}^R. \quad (4.20)$$

We can arrange for $\mathbb{S}_\omega$ to be an L -separated family of subsets of Λ_L for H as follows. Let

$$\mathcal{F}_N = \bigcup_{r=1}^N \mathcal{F}(r), \text{ where } \mathcal{F}(r) = \{\Phi \subset \Xi_{L,\ell}; \Phi \text{ is } \mathbb{G}_2\text{-connected and } |\Phi| = r\}. \quad (4.21)$$

We set $\tilde{\mathbb{S}}_N = \{\Lambda_\ell(a)\}_{a \in \Xi_{L,\ell}} \cup \{\Upsilon_\Phi\}_{\Phi \in \mathcal{F}_N}$. Given $S_1, S_2 \in \tilde{\mathbb{S}}_N$, $S_1 \cap S_2 = \emptyset$, it follows from Lemma 3.4 that

$$\mathbb{P}\{S_1 \text{ and } S_2 \text{ are not } L\text{-separated for } H_{\varepsilon,\omega}\} \leq \tilde{K} e^{-\alpha L^\beta} (L+1)^{2d} \leq e^{-\frac{\alpha}{2} L^\beta}. \quad (4.22)$$

We have $|\Xi_{L,\ell}| \leq 2^d \ell^{(\gamma-\varsigma)d}$ from (B.5). Setting $\mathcal{F}(r, a) = \{\Phi \in \mathcal{F}(r); a \in \Phi\}$ for $a \in \Xi_{L,\ell}$, and letting $\kappa(a)$ denote the number of nearest neighbors of $a \in \Xi_{L,\ell}$ in the graph $\mathbb{G}_2$, and noting that

$$\begin{aligned} \kappa(a) &\leq (2(3k_\ell - 1) + 1)^d - (2(3k_\ell - 2) + 1)^d \leq d(2(3k_\ell - 1) + 1)^{d-1} \\ &= d(6k_\ell - 1)^{d-1} \leq d20^{d-1} \ell^{(1-\varsigma)(d-1)} \leq \ell^{d-1}, \end{aligned} \quad (4.23)$$

we get

$$\begin{aligned} |\mathcal{F}(r, a)| &\leq (r-1)! \ell^{(d-1)(r-1)} \implies |\mathcal{F}(r)| \leq (L+1)^d (r-1)! \ell^{(d-1)(r-1)} \\ &\implies |\mathcal{F}_N| \leq (L+1)^d N! \ell^{(d-1)(N-1)}. \end{aligned} \quad (4.24)$$

Thus, we get

$$\left| \tilde{\mathbb{S}} \right|_N \leq 2^d \ell^{(\gamma-\zeta)d} + (L+1)^d N! \ell^{(d-1)(N-1)} \leq 2(L+1)^d N! \ell^{(d-1)(N-1)}. \quad (4.25)$$

Letting $\mathcal{S}_N$ denote that the event that $\tilde{\mathbb{S}}_N$ is an L -separated family of subsets of Λ_L for H , and taking $N = N_\ell$ as in (4.8), we get

$$\mathbb{P}\{\mathcal{S}_N^c\} \leq e^{-\frac{\alpha}{2}L^\beta} 2(L+1)^d N_\ell! \ell^{(d-1)(N_\ell-1)} < e^{-\frac{\alpha}{4}L^\beta} < \frac{1}{2}e^{-L^\zeta}, \quad (4.26)$$

for sufficiently large L , since $(\gamma-1)\tilde{\zeta} < (\gamma-1)\beta < \gamma\beta$ and $\zeta < \beta$.

We now define the event $\mathcal{E}_N = \mathcal{B}_N \cap \mathcal{S}_N$. It follows from (4.9) and (4.26) that

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - e^{-L^\zeta}. \quad (4.27)$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box Λ_L is (M, I_ℓ, I) -localizing for H_ω , where M is given in (4.7).

Let us fix $\omega \in \mathcal{E}_N$. Then we have (4.17), the subsets $\{\Upsilon_r\}_{r=1}^R$ constructed in (4.15) are buffered subsets of Λ_L for H_ω , and the collection $\mathbb{S}_\omega$ is an L -separated family of subsets of Λ_L for H . It follows from (B.4) and Definition 2.3(ii) that

$$\Lambda_L = \left\{ \bigcup_{a \in \mathcal{G}} \Lambda_\ell^{\Lambda_L, \frac{\ell-\ell^\zeta}{2}}(a) \right\} \cup \left\{ \bigcup_{r=1}^R \Upsilon_r \right\}. \quad (4.28)$$

Note that $\Lambda_\ell^{\Lambda_L, \frac{\ell-\ell^\zeta}{2}}(a) \subset \Lambda_\ell^{\Lambda_L, \ell_{\tilde{\tau}}}(a)$.

Let $\{(\psi_\lambda, \lambda)\}_{\lambda \in \tilde{\sigma}(H_{\Lambda_L})}$ be an eigensystem for H_{Λ_L} . (Since ω is fixed, we omit it from the notation.) Given $\lambda \in \tilde{\sigma}_{I_\ell}(H_{\Lambda_L})$, we claim there exists $S_\lambda \in \mathbb{S}_\omega$ such that

$$\text{dist}(\lambda, \sigma(H_{S_\lambda})) \leq \frac{1}{2}e^{-L^\beta}. \quad (4.29)$$

Suppose not, i.e., $\text{dist}(\lambda, \sigma(H_S)) > \frac{1}{2}e^{-L^\beta}$ for all $S \in \mathbb{S}_\omega$. Let $y \in \Lambda_L$. If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell-\ell^\zeta}{2}}(a)$ for some $a \in \mathcal{G}$, we have $R_{\Lambda_L}(y) \geq \lfloor \frac{\ell-\ell^\zeta}{2} \rfloor$, so it follows from (2.16) that

$$|\psi_\lambda(y)| \leq e^{-m_3 h_I(\lambda) \lfloor \frac{\ell-\ell^\zeta}{2} \rfloor} \leq e^{-m_3 \ell^{-\kappa} \lfloor \frac{\ell-\ell^\zeta}{2} \rfloor} \leq e^{-\frac{1}{4}m_3 \ell^{1-\kappa}}. \quad (4.30)$$

If not, it follows from (4.28) that $y \in \Upsilon_r$ for some $r \in \{1, 2, \dots, R\}$. But then it follows from (2.39) in Lemma 2.4 that

$$|\psi_\lambda(y)| \leq e^{-\frac{m_3}{2} h_I(\lambda) \ell_{\tilde{\tau}}} \leq e^{-\frac{m_3}{2} \ell^{-\kappa} \ell_{\tilde{\tau}}} \leq e^{-\frac{1}{4}m_3 \ell^{\tilde{\tau}-\kappa}}. \quad (4.31)$$

We conclude that

$$1 = \|\psi_\lambda\|^2 \leq (L+1)^d e^{-\frac{1}{4}m_3 \ell^{\tilde{\tau}-\kappa}} < 1, \quad (4.32)$$

a contradiction.

We now pick $x_\lambda \in S_\lambda$. We will show that ψ_λ is an $(x_\lambda, Mh_I(\lambda))$ -localized eigenfunction for H_ω , where M is given in (4.7).

Let $\mathbb{S}_\omega^{(\lambda)} = \{S \in \mathbb{S}_\omega; S \cap S_\lambda = \emptyset\}$. If $S \in \mathbb{S}_\omega^{(\lambda)}$, S and S_λ are L -separated, so it follows from (4.29) that

$$\begin{aligned} \text{dist}(\lambda, \sigma(H_S)) &\geq \text{dist}(\sigma(H_S), \sigma(H_{S_\lambda})) - \text{dist}(\lambda, \sigma(H_{S_\lambda})) \\ &\geq e^{-L^\beta} - \frac{1}{2}e^{-L^\beta} = \frac{1}{2}e^{-L^\beta}. \end{aligned} \quad (4.33)$$

We consider two cases:

- (i) Let $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell-\ell^\varsigma}{2}}(a)$, where $\Lambda_\ell(a) \in \mathbb{S}_\omega^{(\lambda)}$. In this case it follows from (2.16) that

$$|\psi_\lambda(y)| \leq e^{-m_3 h_I(\lambda) \lfloor \frac{\ell-\ell^\varsigma}{2} \rfloor} |\psi_\lambda(y_1)| \quad \text{for some } y_1 \in \partial_{\text{ex}}^{\Lambda_L} \Lambda_\ell(a), \quad (4.34)$$

where $m_3 = m_3(\ell)$ is as in (2.17). Moreover, we have

$$\|y - y_1\| \leq \ell + 1 - \lfloor \frac{\ell-\ell^\varsigma}{2} \rfloor \leq \frac{\ell+\ell^\varsigma}{2} + 2 \leq \frac{\ell+2\ell^\varsigma}{2}. \quad (4.35)$$

- (ii) Let $y \in \Upsilon_r$, where $\Upsilon_r \in \mathbb{S}_\omega^{(\lambda)}$ and $\{\Lambda_\ell(a)\}_{a \in \mathcal{G}_{\Upsilon_r}} \subset \mathbb{S}_\omega^{(\lambda)}$. Then it follows from (2.39) in Lemma 2.4 that

$$|\psi_\lambda(y)| \leq e^{-\frac{m_3}{2} h_I(\lambda) \ell \tilde{\tau}} |\psi_\lambda(y_2)| \leq e^{-\frac{m_3}{4} \ell \tilde{\tau} - \kappa} |\psi_\lambda(y_2)| \quad (4.36)$$

for some $y_2 \in \bigcup_{a \in \mathcal{G}_{\Upsilon_r}} \partial_{\text{ex}}^{\Lambda_L} \Lambda_\ell(a)$, where $m_3 = m_3(\ell)$ is as in (2.17). Note that

$$\|y - y_2\| \leq \text{diam } \Upsilon_r + \ell. \quad (4.37)$$

Now let us take $y \in \Lambda_L$ such that $\|y - x_\lambda\| \geq L_\tau$. Suppose $|\psi_\lambda(y)| > 0$, since otherwise there is nothing to prove. We estimate $|\psi_\lambda(y)|$ using either (4.34) or (4.36) repeatedly, as appropriate, stopping when we get too close to x_λ so we are not in one of the two cases described above. (Note that this must happen since $|\psi_\lambda(y)| > 0$.) We accumulate decay only when we use (4.34), and just use $e^{-\frac{m_3}{4} \ell \tilde{\tau} - \kappa} < 1$ when using (4.36). In view of (4.35) and (4.37), this can be done using (4.34) at least S times, as long as

$$\frac{\ell+2\ell^\varsigma}{2} S + \sum_{r=1}^R (\text{diam } \Upsilon_r + \ell) + 2\ell \leq \|y - x_\lambda\|. \quad (4.38)$$

Since $\sum_{r=1}^R (\text{diam } \Upsilon_r + \ell) \leq 7\ell N$ in view of (4.19), this can be guaranteed by requiring

$$\frac{\ell+2\ell^\varsigma}{2} S + 7\ell^{(\gamma-1)\tilde{\zeta}+1} + 2\ell \leq \|y - x_\lambda\|. \quad (4.39)$$

We can thus have

$$\begin{aligned} S &= \left\lfloor \frac{2}{\ell+2\ell^\varsigma} \left(\|y - x_\lambda\| - 7\ell^{(\gamma-1)\tilde{\zeta}+1} - 2\ell \right) \right\rfloor - 1 \\ &\geq \frac{2}{\ell+2\ell^\varsigma} \left(\|y - x_\lambda\| - 7\ell^{(\gamma-1)\tilde{\zeta}+1} - 2\ell \right) - 2 \\ &= \frac{2}{\ell+2\ell^\varsigma} \left(\|y - x_\lambda\| - 7\ell^{(\gamma-1)\tilde{\zeta}+1} - 3\ell - 2\ell^\varsigma \right) \end{aligned}$$

$$\geq \frac{2}{\ell+2\ell^\varsigma} \left(\|y - x_\lambda\| - 8\ell^{(\gamma-1)\tilde{\zeta}+1} \right). \quad (4.40)$$

Thus we conclude that

$$|\psi_\lambda(y)| \leq e^{-m_3 h_I(\lambda) \lfloor \frac{\ell-\ell^\varsigma}{2} \rfloor} \frac{2}{\ell+2\ell^\varsigma} \left(\|y - x_\lambda\| - 8\ell^{(\gamma-1)\tilde{\zeta}+1} \right) \leq e^{-M h_I(\lambda) \|y - x_\lambda\|} \quad (4.41)$$

where

$$\begin{aligned} M &\geq m_3 \left(1 - C_d \ell^{-\min\{1-\varsigma, \gamma\tau - (\gamma-1)\tilde{\zeta}-1\}} \right) \\ &= m_3 \left(1 - C_d \ell^{-(\gamma\tau - (\gamma-1)\tilde{\zeta}-1)} \right) \\ &\geq m \left(1 - C_d \ell^{-\min\{\kappa, \frac{1-\tau}{2}, \gamma\tau - (\gamma-1)\tilde{\zeta}-1\}} \right) = m \left(1 - C_d \ell^{-\varrho} \right), \end{aligned} \quad (4.42)$$

where we used (A.7), (2.17), and (A.6). In particular, M satisfies (4.7) for sufficiently large ℓ .

We conclude that ψ_λ is an $(x_\lambda, M h_I(\lambda))$ -localized eigenfunction for Λ_L , where M satisfies (4.7).

We proved that Λ_L is (M, I_ℓ, I) -localized for H_ω . $\square$

Proof of Proposition 4.2. We assume (4.2) and set $L_{k+1} = L_k^\gamma$, $A_{k+1} = A_k(1 - L_k^{-\kappa})$, and $I_{k+1} = I(E, A_{k+1})$ for $k = 0, 1, \dots$. Since if a box Λ_L is (M, I_ℓ, I) -localizing for H_ω it is also (M, I_ℓ) -localizing, if L_0 is sufficiently large it follows from Lemma 4.3 by an induction argument that we have (4.3) and (4.4) for all $k = 1, 2, \dots$ $\square$

Proposition 4.4. *There exists a finite scale $\mathcal{L} = \mathcal{L}(d)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}$ and interval $I_0 = I(E, A_0)$ we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_{L_0}(x) \text{ is } (m_0, I_0)\text{-localizing for } H_\omega \} \geq 1 - e^{-L_0^\zeta}. \quad (4.43)$$

Set $L_{k+1} = L_k^\gamma$, $A_{k+1} = A_k(1 - L_k^{-\kappa})$, and $I_{k+1} = I(E, A_{k+1})$, for $k = 0, 1, \dots$. Then for all $k = 1, 2, \dots$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_L(x) \text{ is } (m_k, I_k, I_{k-1})\text{-localizing for } H_\omega \} \geq 1 - e^{-L^\xi} \text{ for } L \in [L_k, L_{k+1}), \quad (4.44)$$

where

$$L_k^{-\kappa'} < m_{k-1} \left(1 - C_d L_{k-1}^{-\varrho} \right) \leq m_k < \frac{1}{2} \log \left(1 + \frac{A_k}{4d} \right), \quad (4.45)$$

with C_d as in (4.4).

Proof. We apply Proposition 4.2, which gives a scale $\mathcal{L}$ such that, taking $L_0 \geq \mathcal{L}$ we have the conclusions of Proposition 4.2.

Given a scale $L \geq L_1$, let $k = k(L) \in \{1, 2, \dots\}$ be defined by $L_k \leq L < L_{k+1}$. We have $L_k = L_{k-1}^\gamma \leq L < L_{k+1} = L_{k-1}^{\gamma^2}$, so $L = L_{k-1}^{\gamma'}$ with $\gamma \leq \gamma' < \gamma^2$. We proceed as in Lemma 4.3. We take $\Lambda_L = \Lambda_L(x_0)$, where $x_0 \in \mathbb{R}^d$, let $\{(\psi_\lambda, \lambda)\}_{\lambda \in \tilde{\sigma}(H_{\Lambda_L})}$ be an eigensystem for H_{Λ_L} , and let $\mathcal{C}_{L, L_{k-1}} = \mathcal{C}_{L, L_{k-1}}(x_0)$ be the suitable L_{k-1} -cover of Λ_L . We let $\mathcal{B}_0$ denote the event that all boxes

in $\mathcal{C}_{L, L_{k-1}}$ are (m_{k-1}, I_{k-1}) -localizing for H_ω . It follows from (B.5) and (4.3) that

$$\mathbb{P}\{\mathcal{B}_0^c\} \leq \left(\frac{2L}{L_{k-1}^\xi}\right)^d e^{-L\xi_{k-1}} = 2^d L_{k-1}^{(\gamma' - \xi)d} e^{-L\xi_{k-1}} \leq 2^d L^{(1 - \frac{\gamma}{\gamma'})d} e^{-L\frac{\xi}{\gamma'}} < \frac{1}{2} e^{-L^\xi}, \quad (4.46)$$

if L_0 is sufficiently large, since $\xi\gamma' < \xi\gamma^2 < \xi$. Moreover, given $\Lambda_1, \Lambda_2 \in \mathcal{C}_{L, L_{k-1}}$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, it follows from Lemma 3.4 that

$$\mathbb{P}\{\Lambda_1 \text{ and } \Lambda_2 \text{ are not } L\text{-separated for } H_\omega\} \leq \tilde{K} e^{-\alpha L^\beta} (L_{k-1} + 1)^{2d} \leq e^{-\frac{\alpha}{2} L^\beta}. \quad (4.47)$$

Thus, letting $\mathcal{S}_0$ denote the event that $\mathcal{C}_{L, L_{k-1}}$ is an L -separated family of subsets of Λ_L for H , it follows from (B.5) that

$$\mathbb{P}\{\mathcal{S}_0^c\} \leq \left(\frac{2L}{L_{k-1}^\xi}\right)^{2d} e^{-\frac{\alpha}{2} L^\beta} \leq \frac{1}{2} e^{-L^\xi}, \quad (4.48)$$

if L_0 is sufficiently large, since $\xi < \beta$. Thus, letting $\mathcal{E}_0 = \mathcal{B}_0 \cap \mathcal{S}_0$, we have

$$\mathbb{P}\{\mathcal{E}_0\} \geq 1 - e^{-L^\xi}. \quad (4.49)$$

It only remains to prove that Λ_L is (m_k, I_k, I_{k-1}) -localizing for H_ω for all $\omega \in \mathcal{E}_0$. To do so, we fix $\omega \in \mathcal{E}_0$ and proceed as in the proof of Lemma 4.3. Since $\omega \in \mathcal{B}_0$, $\Lambda_{L_{k-1}}(a)$ is (m_{k-1}, I_{k-1}) -localizing for H_ω for all $a \in \mathcal{G} = \Xi_{L, L_{k-1}}$. Since ω is now fixed, we omit them from the notation. ,

Let $\lambda \in \tilde{\sigma}_{I_k}(H_{\Lambda_L})$ (note $(I_{k-1})_{L_{k-1}} = I_k$). To finish the proof we need to show that ψ_λ is (m_k, I_k, I_{k-1}) -localized. Since $\mathcal{C}_{L, L_{k-1}}$ is an L -separated family of subsets of Λ_L for H , there must exist $a_\lambda \in \mathcal{G} = \Xi_{L, L_{k-1}}$ such that, setting $\Lambda_\lambda = \Lambda_{L_{k-1}}(a_\lambda)$, we have (as in the proof of Lemma 4.3)

$$\text{dist}(\lambda, \sigma(H_{\Lambda_\lambda})) \leq \frac{1}{2} e^{-L^\beta}, \quad (4.50)$$

and if $a \in \mathcal{G}_\lambda = \{b \in \mathcal{G}; \Lambda_{L_{k-1}}(b) \cap \Lambda_\lambda = \emptyset\}$,

$$\text{dist}(\lambda, \sigma(H_\Lambda)) \geq \frac{1}{2} e^{-L^\beta}. \quad (4.51)$$

If $y \in \Lambda_L$ and $\|y - a_\lambda\| \geq 2L_{k-1}$, it follows from (B.4) that $y \in \Lambda_{L_{k-1}}^{\Lambda_L, \frac{L_{k-1} - L_{k-1}^\xi}{2}}(a)$ for some $a \in \mathcal{G}_\lambda$, so it follows from (2.16) that

$$|\psi_\lambda(y)| \leq e^{-m_{k-1,3} h_{I_{k-1}}(\lambda) \left\lfloor \frac{L_{k-1} - L_{k-1}^\xi}{2} \right\rfloor} |\psi_\lambda(y_1)|, \quad (4.52)$$

for some $y_1 \in \partial^{\Lambda_L, 2(L_{k-1})^\tau} \Lambda_{L_{k-1}}(a)$, where we need

$$m_{k-1,3} = m_{k-1,3}(L_{k-1}) \geq m_{k-1} \left(1 - C_d L_{k-1}^{-\left(\frac{1-\tau}{2}\right)}\right), \quad (4.53)$$

and we have

$$\|y - y_1\| \leq \frac{L_{k-1} + 2L_{k-1}^\xi}{2}, \quad (4.54)$$

as in (4.35).

Now consider $y \in \Lambda_L$ such that $\|y - a_\lambda\| \geq L_\tau$. Suppose $|\psi_\lambda(y)| > 0$, since otherwise there is nothing to prove. We estimate $|\psi_\lambda(y)|$ using either

(4.52) repeatedly, as appropriate, stopping when we get within $2L_{k-1}$ of a_λ . In view of (4.54), we can use (4.52) S times, as long as

$$\frac{L_{k-1} + 2L_{k-1}^\zeta}{2} S + 2L_{k-1} \leq \|y - a_\lambda\|. \quad (4.55)$$

We can thus have

$$\begin{aligned} S &= \left\lfloor \frac{2}{L_{k-1} + 2L_{k-1}^\zeta} (\|y - a_\lambda\| - 2L_{k-1}) \right\rfloor - 1 \\ &\geq \frac{2}{L_{k-1} + 2L_{k-1}^\zeta} (\|y - a_\lambda\| - 2L_{k-1}) - 2 \\ &\geq \frac{2}{L_{k-1} + 2L_{k-1}^\zeta} (\|y - a_\lambda\| - 3L_{k-1} - 2L_{k-1}^\zeta) \\ &\geq \frac{2}{L_{k-1} + 2L_{k-1}^\zeta} (\|y - a_\lambda\| - 4L_{k-1}). \end{aligned} \quad (4.56)$$

Thus we conclude that

$$\begin{aligned} |\psi_\lambda(y)| &\leq e^{-m_{k-1}, 3h_{I_{k-1}}(\lambda) \left\lfloor \frac{L_{k-1} - L_{k-1}^\zeta}{2} \right\rfloor} \frac{2}{L_{k-1} + 2L_{k-1}^\zeta} (\|y - a_\lambda\| - 4L_{k-1}) \\ &\leq e^{-m_k h_{I_{k-1}}(\lambda) \|y - a_\lambda\|} \end{aligned} \quad (4.57)$$

where m_k can be taken to satisfy (4.4).

We conclude that ψ_λ is an (m_k, I_k, I_{k-1}) -localized eigenfunction, where m_k satisfies (4.4).

We proved that the box Λ_L is (m_k, I_k, I_{k-1}) -localizing for H_ω . $\square$

Proof of Theorem 1.4. Let $L_{k+1} = L_k^\gamma$, $A_{k+1} = A_k(1 - L_k^{-\kappa})$, $I_{k+1} = I(E, A_{k+1})$, and $m_{k+1} = m_k(1 - C_d L_k^{-\varrho})$ for $k = 0, 1, \dots$. Given $L \geq L_0^\gamma = L_1$, let $k = k(L) \in \{1, 2, \dots\}$ be defined by $L_k \leq L < L_{k+1}$. Let $A_\infty, I_\infty, m_\infty$ be defined by (1.11). Since

$$A_k = A_\infty \prod_{j=k}^{\infty} (1 - L_j^{-\kappa})^{-1} \quad \text{for } k = 0, 1, \dots, \quad (4.58)$$

we have

$$A_\infty (1 - L^{-\kappa})^{-1} \leq A_\infty (1 - L_k^{-\kappa})^{-1} < A_k, \quad (4.59)$$

and hence $I_\infty^L \subset I_k$. Since $m_\infty \leq m_k$, we conclude that (1.10) follows from (4.44). $\square$

5. Localization

In this section we prove Theorem 1.5 for an Anderson model H_ω .

Lemma 5.1. *Let $I = (E, A)$. There exists a finite scale $\mathcal{L}_{d,\nu}$ such that for all $L \geq \mathcal{L}_{d,\nu}$ and $a \in \mathbb{Z}^d$, given an (m, I^L) -localizing box $\Lambda_L(a)$ for the discrete Schrödinger operator H , then for all $\lambda \in I$,*

$$\max_{b \in \Lambda_{\frac{L}{3}}(a)} W_\lambda^{(a)}(b) > e^{-\frac{1}{4} m h_{I^L}(\lambda) L} \implies \min_{\theta \in \sigma_{I^L}(H_{\Lambda_L(a)})} |\lambda - \theta| < \frac{1}{2} e^{-L^{\gamma\beta}}. \quad (5.1)$$

Proof. Let $\lambda \in I = (I^L)_L$, and suppose $|\lambda - \theta| \geq \frac{1}{2}e^{-L^{\gamma\beta}}$ for all $\theta \in \sigma_{I^L}(H_{\Lambda_L(a)})$. Let $\psi \in \mathcal{V}(\lambda)$. Then it follows from Lemma 2.2 that for large L and $b \in \Lambda_{\frac{L}{3}}(a)$ we have

$$|\psi(b)| \leq e^{-m_3(L)h_{I^L}(\lambda)(\frac{L}{3}-1)} \|T_a^{-1}\psi\| \langle \frac{L}{2} + 1 \rangle^\nu \leq e^{-\frac{1}{4}mh_{I^L}(\lambda)L} \|T_a^{-1}\psi\|. \quad (5.2)$$

□

Proof of Theorem 1.5. Assume Theorem 1.4 holds for some L_0 , and let $I = I_\infty$, $m = m_\infty$. Consider $L_0^\gamma \leq L \in 2\mathbb{N}$ and $a \in \mathbb{Z}^d$. We have

$$\Lambda_{5L}(a) = \bigcup_{b \in \{a + \frac{1}{2}L\mathbb{Z}^d\}, \|b-a\| \leq 2L} \Lambda_L(b). \quad (5.3)$$

Let $\mathcal{Y}_{L,a}$ denote the event that $\{\Lambda_L(b)\}_{b \in \{a + \frac{1}{2}L\mathbb{Z}^d\}, \|b-a\| \leq 2L}$ is an L^γ -separated family of (m, I^L) -localizing boxes for H . It follows from (1.10) and Lemma 3.4 that

$$\mathbb{P}\{\mathcal{Y}_{L,a}^c\} \leq 9^d e^{-L^\xi} + \tilde{K} 9^{2d} (L+1)^{2d} e^{-\alpha L^{\gamma\beta}} \leq C_\mu e^{-L^\xi}. \quad (5.4)$$

Suppose $\omega \in \mathcal{Y}_{L,a}$, $\lambda \in I$, and $\max_{b \in \Lambda_{\frac{L}{3}}(a)} W_{\omega,\lambda}^{(a)}(b) > e^{-\frac{1}{4}mh_{I^L}(\lambda)L}$. It follows from Lemma 5.1 that $\min_{\theta \in \sigma_{I^L}(H_{\Lambda_L(a)})} |\lambda - \theta| < \frac{1}{2}e^{-L^{\gamma\beta}}$. Since the family of boxes is L^γ -separated family for $\tilde{H}_\omega$, we conclude that

$$\min_{\theta \in \sigma_{I^L}(H_{\Lambda_L(b)})} |\lambda - \theta| \geq \frac{1}{2}e^{-L^{\gamma\beta}} \quad (5.5)$$

for all $b \in \{a + \frac{1}{2}L\mathbb{Z}^d\}$ with $\frac{3}{2}L \leq \|b-a\| \leq 2L$. Since

$$A_L(a) \subset \bigcup_{b \in \{a + \frac{1}{2}L\mathbb{Z}^d\}, \frac{3}{2}L \leq \|b-a\| \leq 2L} \Lambda_L^{\frac{L}{2}}(b), \quad (5.6)$$

it follows from Lemma 2.2 that for all $y \in A_L(a)$ we have, given $\psi \in \mathcal{V}_\omega(\lambda)$,

$$\begin{aligned} |\psi(y)| &\leq e^{-m_3(L)h_{I^L}(\lambda)(\frac{L}{7}-2)} \|T_a^{-1}\psi\| \langle \frac{5}{2}L + 1 \rangle^\nu \leq e^{-mh_{I^L}(\lambda)\frac{L}{8}} \|T_a^{-1}\psi\| \\ &\leq e^{-\frac{7}{132}mh_{I^L}(\lambda)\|y-a\|} \|T_a^{-1}\psi\|, \end{aligned} \quad (5.7)$$

so we get

$$W_{\omega,\lambda}^{(a)}(y) \leq e^{-\frac{7}{132}mh_{I^L}(\lambda)\|y-a\|} \quad \text{for all } y \in A_L(a). \quad (5.8)$$

Since we have (1.13), we conclude that for $\omega \in \mathcal{Y}_{L,a}$ we always have

$$\begin{aligned} W_{\omega,\lambda}^{(a)}(a)W_{\omega,\lambda}^{(a)}(y) &\leq \max \left\{ e^{-\frac{7}{66}mh_{I^L}(\lambda)\|y-a\|} \langle y-a \rangle^\nu, e^{-\frac{7}{132}mh_{I^L}(\lambda)\|y-a\|} \right\} \\ &\leq e^{-\frac{7}{132}mh_{I^L}(\lambda)\|y-a\|} \quad \text{for all } y \in A_L(a). \end{aligned} \quad (5.9)$$

□

Appendix A: Exponents

Given $0 < \xi < \zeta < 1$, we consider $\beta, \tau \in (0, 1)$ and $\gamma > 1$ such that

$$0 < \xi < \zeta < \beta < \frac{1}{\gamma} < 1 < \gamma < \sqrt{\frac{\zeta}{\xi}} \quad \text{and} \quad \max \left\{ \gamma\beta, \frac{(\gamma-1)\beta+1}{\gamma} \right\} < \tau < 1; \quad (\text{A.1})$$

it follows that

$$0 < \xi < \xi\gamma^2 < \zeta < \beta < \frac{\tau}{\gamma} < \frac{1}{\gamma} < \tau < 1 < \frac{1-\beta}{\tau-\beta} < \gamma < \frac{\tau}{\beta}. \quad (\text{A.2})$$

We set

$$\tilde{\zeta} = \frac{\zeta + \beta}{2} \in (\zeta, \beta) \quad \text{and} \quad \tilde{\tau} = \frac{1 + \tau}{2} \in (\tau, 1), \quad (\text{A.3})$$

so

$$(\gamma - 1)\tilde{\zeta} + 1 < (\gamma - 1)\beta + 1 < \gamma\tau. \quad (\text{A.4})$$

We take $\kappa \in (0, 1)$ and $\kappa' \in [0, 1)$ such that

$$\kappa + \kappa' < \tau - \gamma\beta. \quad (\text{A.5})$$

We let

$$\varrho = \min \left\{ \kappa, \frac{1-\tau}{2}, \gamma\tau - (\gamma - 1)\tilde{\zeta} - 1 \right\}, \quad \text{note} \quad 0 < \kappa \leq \varrho < 1, \quad (\text{A.6})$$

and choose

$$\varsigma \in (0, 1 - \varrho], \quad \text{so} \quad \varrho < 1 - \varsigma. \quad (\text{A.7})$$

We select exponents satisfying (A.1)–(A.7) and fix these exponents.

Appendix B: Suitable Covers of a Box

To perform the multiscale analysis in an efficient way we use suitable covers of a box as in [17, Section 3.4], an adaptation of [25, Definition 3.12]. We state the definition and properties for the reader's convenience.

Definition B.1. Fix $\varsigma \in (0, 1)$. Let $\Lambda_L = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$ be a box in $\mathbb{Z}^d$, and let $\ell < L$. A suitable ℓ -cover of Λ_L is the collection of boxes

$$\mathcal{C}_{L,\ell} = \mathcal{C}_{L,\ell}(x_0) = \{\Lambda_\ell(a)\}_{a \in \Xi_{L,\ell}}, \quad (\text{B.1})$$

where

$$\Xi_{L,\ell} = \Xi_{L,\ell}(x_0) := \{x_0 + \rho\ell^\varsigma \mathbb{Z}^d\} \cap \Lambda_L^{\mathbb{R}} \quad \text{with} \quad \rho \in \left[\frac{1}{2}, 1\right] \cap \left\{ \frac{L-\ell}{2\ell^\varsigma k}; k \in \mathbb{N} \right\}. \quad (\text{B.2})$$

We call $\mathcal{C}_{L,\ell}$ the suitable ℓ -cover of Λ_L if $\rho = \rho_{L,\ell} := \max \left[\frac{1}{2}, 1 \right] \cap \left\{ \frac{L-\ell}{2\ell^\varsigma k}; k \in \mathbb{N} \right\}$.

Lemma B.2 ([25, Lemma 3.13], [17, Lemma 3.10]) *Let $\ell \leq \frac{L}{2}$. Then for every box $\Lambda_L = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$, a suitable ℓ -cover $\mathcal{C}_{L,\ell} = \mathcal{C}_{L,\ell}(x_0)$ satisfies*

$$\Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell(a); \quad (\text{B.3})$$

for all $b \in \Lambda_L$ there is $\Lambda_\ell^{(b)} \in \mathcal{C}_{L,\ell}$ such that $b \in \left(\Lambda_\ell^{(b)}\right)^{\Lambda_L, \frac{\ell-\ell^\varsigma}{2}}$,

$$\text{i.e., } \Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell^{\Lambda_L, \frac{\ell-\ell^\varsigma}{2}}(a); \quad (\text{B.4})$$

$$\#\Xi_{L,\ell} = \left(\frac{L-\ell}{\rho\ell^\varsigma} + 1\right)^d \leq \left(\frac{2L}{\ell^\varsigma}\right)^d. \quad (\text{B.5})$$

Moreover, given $a \in x_0 + \rho\ell^\varsigma\mathbb{Z}^d$ and $k \in \mathbb{N}$, it follows that

$$\Lambda_{(2k\rho\ell^\varsigma+\ell)}(a) = \bigcup_{b \in \{x_0 + \rho\ell^\varsigma\mathbb{Z}^d\} \cap \Lambda_{(2k\rho\ell^\varsigma+\ell)}^\mathbb{R}(a)} \Lambda_\ell(b), \quad (\text{B.6})$$

and $\{\Lambda_\ell(b)\}_{b \in \{x_0 + \rho\ell^\varsigma\mathbb{Z}^d\} \cap \Lambda_{(2k\rho\ell^\varsigma+\ell)}^\mathbb{R}(a)}$ is a suitable ℓ -cover of the box $\Lambda_{(2k\rho\ell^\varsigma+\ell)}(a)$.

Note that $\Lambda_\ell^{(b)}$ does not denote a box centered at b , just some box in $\mathcal{C}_{L,\ell}(x_0)$ satisfying (B.4). By $\Lambda_\ell^{(b)}$ we will always mean such a box. We will use

$$\text{dist}\left(b, \partial_{\text{in}}^{\Lambda_L} \Lambda_\ell^{(b)}\right) \geq \frac{\ell-\ell^\varsigma}{2} - 1 \quad \text{for all } b \in \Lambda_L. \quad (\text{B.7})$$

Note also that $\rho \leq 1$ yields (B.4). We specified $\rho = \rho_{L,\ell}$ in for the suitable ℓ -cover for convenience, so there is no ambiguity in the definition of $\mathcal{C}_{L,\ell}(x_0)$.

Suitable covers are convenient for the construction of buffered subsets (see Definition 2.3) in the multiscale analysis, where we will assume $\varsigma \in (0, 1)$ is as in (A.7). We will use the following observation:

Remark B.3. Let $\mathcal{C}_{L,\ell}$ be a suitable ℓ -cover for the box Λ_L , and set $k_\ell = k_{L,\ell} = \lfloor \rho^{-1}\ell^{1-\varsigma} \rfloor + 1$. Then for all $a, b \in \mathcal{C}_{L,\ell}$ we have

$$\Lambda_\ell^\mathbb{R}(a) \cap \Lambda_\ell^\mathbb{R}(b) = \emptyset \quad \Longleftrightarrow \quad \|a - b\| \geq k_\ell \rho \ell^\varsigma. \quad (\text{B.8})$$

References

- [1] Aizenman, M.: Localization at weak disorder: some elementary bounds. *Rev. Math. Phys.* **6**, 1163–1182 (1994)
- [2] Aizenman, M., Schenker, J., Friedrich, R., Hundertmark, D.: Finite volume fractional-moment criteria for Anderson localization. *Commun. Math. Phys.* **224**, 219–253 (2001)
- [3] Aizenman, M., Elgart, A., Naboko, S., Schenker, J., Stolz, G.: Moment analysis for localization in random Schrödinger operators. *Inv. Math.* **163**, 343–413 (2006)
- [4] Aizenman, M., Molchanov, S.: Localization at large disorder and extreme energies: an elementary derivation. *Commun. Math. Phys.* **157**, 245–278 (1993)

- [5] Aizenman, M., Warzel, S.: Random operators. Disorder effects on quantum spectra and dynamics. Graduate Studies in Mathematics, vol. 168. American Mathematical Society, Providence, RI (2015)
- [6] Anderson, P.: Absence of diffusion in certain random lattices. *Phys. Rev.* **109**, 1492–1505 (1958)
- [7] Bourgain, J., Kenig, C.: On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.* **161**, 389–426 (2005)
- [8] Combes, J.M., Germinet, F., Klein, A.: Generalized eigenvalue-counting estimates for the Anderson model. *J. Stat. Phys.* **135**, 201–216 (2009). <https://doi.org/10.1007/s10955-009-9731-3>
- [9] Combes, J.M., Hislop, P.D.: Localization for some continuous, random Hamiltonians in d-dimension. *J. Funct. Anal.* **124**, 149–180 (1994)
- [10] Combes, J.M., Hislop, P.D., Klopp, F.: Optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators. *Duke Math. J.* **140**, 469–498 (2007)
- [11] Del Rio, R., Jitomirskaya, S., Last, Y., Simon, B.: What is Localization? *Phys. Rev. Lett.* **75**, 117–119 (1995)
- [12] Del Rio, R., Jitomirskaya, S., Last, Y., Simon, B.: Operators with singular continuous spectrum IV: Hausdorff dimensions, rank one perturbations and localization. *J. d'Analyse Math.* **69**, 153–200 (1996)
- [13] Dietlein, A., Elgart, A.: Level spacing for continuum random Schrödinger operators with applications, [arXiv:1712.03925](https://arxiv.org/abs/1712.03925)
- [14] von Dreifus, H.: On the effects of randomness in ferromagnetic models and Schrödinger operators. Ph.D. thesis, New York University (1987)
- [15] von Dreifus, H., Klein, A.: A new proof of localization in the Anderson tight binding model. *Commun. Math. Phys.* **124**, 285–299 (1989)
- [16] Elgart, A., Klein, A.: An eigensystem approach to Anderson localization. *J. Funct. Anal.* **271**, 3465–3512 (2016). <https://doi.org/10.1016/j.jfa.2016.09.008>
- [17] Elgart, A., Klein, A.: Eigensystem multiscale analysis for Anderson localization in energy intervals. *J. Spectr. Theory* **9**, 711–765 (2019). <https://doi.org/10.4171/JST/261>
- [18] Elgart, A., Klein, A.: In preparation
- [19] Figotin, A., Klein, A.: Localization phenomenon in gaps of the spectrum of random lattice operators. *J. Statist. Phys.* **75**, 997–1021 (1994). <https://doi.org/10.1007/BF02186755>
- [20] Figotin, A., Klein, A.: Localization of classical waves I: Acoustic waves. *Commun. Math. Phys.* **180**, 439–482 (1996)
- [21] Fröhlich, J., Spencer, T.: Absence of diffusion with Anderson tight binding model for large disorder or low energy. *Commun. Math. Phys.* **88**, 151–184 (1983)
- [22] Fröhlich, J., Martinelli, F., Scoppola, E., Spencer, T.: Constructive proof of localization in the Anderson tight binding model. *Commun. Math. Phys.* **101**, 21–46 (1985)
- [23] Germinet, F., Klein, A.: Bootstrap multiscale analysis and localization in random media. *Commun. Math. Phys.* **222**, 415–448 (2001). <https://doi.org/10.1007/s002200100518>

- [24] Germinet, F., Klein, A.: New characterizations of the region of complete localization for random Schrödinger operators. *J. Stat. Phys.* **122**, 73–94 (2006). <https://doi.org/10.1007/s10955-005-8068-9>
- [25] Germinet, F., Klein, A.: A comprehensive proof of localization for continuous Anderson models with singular random potentials. *J. Eur. Math. Soc.* **15**, 53–143 (2013). <https://doi.org/10.4171/JEMS/356>
- [26] Kirsch, W.: An invitation to random Schrödinger operators. In *Random Schrödinger-Operators. Panoramas et Synthèses*, vol. 25. Societe Mathematique de France, Paris, pp. 1–119 (2008)
- [27] Klein, A.: Multiscale analysis and localization of random operators. *Random Schrödinger Operators. Panoramas et Synthèses* 25, pp. 121–159. Société Mathématique de France, Paris (2008)
- [28] Klein, A.: Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators. *Comm. Math. Phys.* **323**, 1229–1246 (2013). <https://doi.org/10.1007/s00220-013-1795-x>
- [29] Klein, A., Molchanov, S.: Simplicity of eigenvalues in the Anderson model. *J. Stat. Phys.* **122**, 95–99 (2006). <https://doi.org/10.1007/s10955-005-8009-7>
- [30] Klein, A., Tsang, C.S.S.: Eigensystem bootstrap multiscale analysis for the Anderson model. *J. Spectr. Theory* **8**, 1149–1197 (2018). <https://doi.org/10.4171/JST/224>
- [31] Minami, N.: Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. *Commun. Math. Phys.* **177**, 709–725 (1996)
- [32] Spencer, T.: Localization for random and quasiperiodic potentials. *J. Stat. Phys.* **51**, 1009–1019 (1988)
- [33] Wegner, F.: Bounds on the density of states in disordered systems. *Z. Phys. B* **44**, 9–15 (1981)

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