WALL-CROSSING FOR NEWTON-OKOUNKOV BODIES AND THE TROPICAL GRASSMANNIAN

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ABSTRACT. Tropical geometry and the theory of Newton-Okounkov bodies are two methods which produce toric degenerations of an irreducible complex projective variety. Kaveh-Manon showed that the two are related. We give geometric maps between the Newton-Okounkov bodies corresponding to two adjacent maximal-dimensional prime cones in the tropicalization of X. Under a technical condition, we produce a natural "algebraic wall-crossing" map on the underlying value semigroups (of the corresponding valuations). In the case of the tropical Grassmannian $Gr(2, m)$, we prove that the algebraic wall-crossing map is the restriction of a geometric map. In an Appendix by Nathan Ilten, he explains how the geometric wall-crossing phenomenon can also be derived from the perspective of complexity-one T-varieties; Ilten also explains the connection to the "combinatorial mutations" studied by Akhtar-Coates-Galkin-Kasprzyk.

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1. INTRODUCTION

Let X be an irreducible complex projective variety of dimension d. To study the geometry of X, we can study the central fiber of a *toric degeneration* $\mathfrak{X}$ of X, where a toric degeneration is a flat family of varieties whose central fiber X_0 is a toric variety; the fact that both X and X_0 appear as fibers of the flat family $\mathfrak{X}$ means that information about X can be read off of X_0. The combinatorial data associated to toric varieties yield powerful tools for computing geometric invariants thereof. Hence, in the presence of a toric degeneration $\mathfrak{X}$, it may be hoped that we can obtain geometric information about X from the combinatorics associated to X_0.

In this paper, we focus on two well-known methods for constructing toric degenerations: tropical geometry, and the theory of Newton-Okounkov bodies. First we briefly recall the tropical geometry picture. Given a variety X as above, realized as $\mathrm{Proj}(A) \cong \mathrm{Proj}(\mathbb{C}[x_1, \dots, x_n]/I)$ where A is its homogeneous coordinate ring and $\mathbb{C}[x_1, \dots, x_n]/I$ a choice of presentation of A, the tropicalization $\mathcal{T}(I)$ is a subset of $\mathbb{R}^n$ consisting of those (weight) vectors whose corresponding initial ideals $\mathrm{in}_w(I)$ contain no monomials (see (2.7)). In fact, $\mathcal{T}(I)$ carries additional combinatorial structure, namely, it is a $(d+1)$-dimensional subfan of the Gröbner fan. A Gröbner degeneration of an ideal I to the initial ideal $\mathrm{in}_w(I)$ yields a toric degeneration when the initial ideal $\mathrm{in}_w(I)$ is prime and binomial. Since primality is impossible if $\mathrm{in}_w(I)$ contains a monomial, the tropicalization $\mathcal{T}(I)$ can be viewed as the set of weight vectors which provide candidates for toric

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degenerations. Now we recall the point of view of Newton-Okounkov bodies. A valuation $\nu : A\setminus\{0\} \to \mathbb{Q}^r$ yields a multiplicative filtration on A and hence an associated graded algebra $gr_\nu(A)$, whose grading is encoded in the value semigroup $S(A,\nu) := \mathrm{image}(\nu)$. When ν is full-rank, then (since $\mathbb{C}$ is algebraically closed) ν has one-dimensional leaves by Abhyankar's inequality (cf. [15, Theorem 2.3], also [11, Theorem 6.6.7]), which implies that the associated graded $gr_\nu(A)$ is a semigroup algebra over the value semigroup $S(A,\nu)$. Hence, when $S(A,\nu)$ is finitely generated, $Proj$ of the associated graded is a (possibly non-normal) toric variety, and the associated degeneration of A to $gr_\nu(A)$ is a toric degeneration [3].

This manuscript was motivated by the results of Kaveh and Manon, who showed in [15] that the two approaches sketched above are related. Let C be a maximal-dimensional cone in $\mathcal{T}(I)$ and let $\mathrm{in}_C(I)$ denote the initial ideal associated to C. Assuming that this $\mathrm{in}_C(I)$ is prime, Kaveh and Manon show that the toric degeneration associated to $\mathrm{in}_C(I)$ can also be obtained from the point of view of Newton-Okounkov bodies. More precisely, they construct – using a set of rational and linearly independent vectors $u_1,\ldots,u_{d+1}$ contained in the cone C – a valuation $\nu : A\setminus\{0\} \to \mathbb{Q}^{d+1}$ with respect to which the associated graded algebra $gr_\nu(A)$ of A is isomorphic to the coordinate ring $\mathbb{C}[x_1,\ldots,x_n]/\mathrm{in}_C(I)$ obtained through Gröbner theory.

We can now sketch the first result of this paper. Suppose that C_1 and C_2 are both maximal-dimensional prime cones in $\mathcal{T}(I)$ and suppose they are adjacent, i.e., they share a codimension-1 face $C = C_1 \cap C_2$. First, we show that there are choices of $\{u_1,u_2,\ldots,u_{d+1}\} \in C_1$ and $\{u'_1,u'_2,\ldots,u'_{d+1}\} \in C_2$ such that the corresponding Newton-Okounkov polytopes $\Delta(A,\nu_1)$ and $\Delta(A,\nu_2)$ project to the same polytope under the linear projection $\mathsf{p}_{[1,d]} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ which forgets the last coordinate. We also show that the fibers are of the same Euclidean length (up to a global constant); we illustrate a very simple example in Figure 2.1. The proof relies on variation of GIT quotients [10]. Once we know that the fiber lengths are equal, it follows that there are two natural piecewise-linear maps $\mathsf{F}_{12} : \Delta(A,\nu_1) \to \Delta(A,\nu_2)$ and $\mathsf{S}_{12} : \Delta(A,\nu_1) \to \Delta(A,\nu_2)$, the "flip" and "shift" maps respectively, which behave as the identity on the first d coordinates. We call these **(geometric) wall-crossing maps**. The precise statement is given in Theorem 2.7.

The geometric wall-crossing phenomenon for Newton-Okounkov bodies, as described above, can also be derived from the theory of complexity-one T-varieties. Specifically, the content of Theorem 2.7 can be obtained by adapting the arguments in [20], which describe Newton-Okounkov bodies for normal complexity-one T-varieties. (More details are in the Appendix.) This was observed by Ilten and Manon already in 2017 although not recorded explicitly in [12]. In the Appendix by Nathan Ilten, this complexity-one perspective is briefly explained; in addition, Ilten explains the connection to the "combinatorial mutations" of polytopes, as studied by Akhtar, Coates, Galkin, and Kasprzyk [1].

We now describe the second set of results in this paper. In addition to the "geometric" wall-crossing maps discussed above, under a certain technical hypothesis (stated precisely in Section 4.2), it is also possible to construct – using a set of standard monomials coming from Gröbner theory – a natural bijection $\Theta : S(A,\nu_1) \to S(A,\nu_2)$ commuting with the projection $\mathsf{p}_{[1,d]}$. We call this the **algebraic** wall-crossing. In general, the map Θ is not straightforward to compute. Since the semigroups $S(A,\nu_i)$ for $i = 1,2$ are subsets of the respective cones $P(A,\nu_i) := \mathrm{cone}(\Delta(A,\nu_i))$ and the maps F_{12} and S_{12} naturally extend to the level of the cones, it is natural to ask whether Θ is simply the restriction to $S(A,\nu_1)$ of either of the geometric wall-crossing maps. In Example 4.5 we show that, in general, the answer is no. However, for the case of the tropical Grassmannian of 2-planes in m-space, we show that the algebraic wall-crossing map Θ is the restriction of the "flip" map F_{12}; this is recorded in Theorem 5.15.

The results of this paper suggest some natural directions for future work; we mention a small sample. First, our Theorem 5.15 motivates the natural question: under what conditions is the algebraic wall-crossing map a restriction of a geometric wall-crossing? Secondly, and as a special case, it seems natural to ask whether our analysis of the algebraic and geometric wall-crossing for $Gr(2,m)$ can be generalized to the tropicalizations of the higher Grassmannians $Gr(k,m)$ for $k > 2$. Recent work of Mohammadi and Shaw [18] on $\mathrm{trop}(Gr(3,m))$ suggest that the case $k = 3$ may be tractable. In addition, it is well-known that the Grassmannian $Gr(2,m)$ is a cluster variety, and in this special case, our algebraic wall-crossing Θ can be seen to be related to cluster mutation. In light of the work of Rietsch and Williams (e.g. [21, Corollary 11.16]) we hope to better understand, in more generality, the connections between (both the geometric and algebraic) wall-crossing maps and clusters.

We now briefly outline the layout of this paper. In Section 2 we establish the notation and setup for the rest of the paper. In particular, we state precisely the result of Kaveh and Manon, on which this paper relies. We then give a statement of our first main result in Theorem 2.7, namely, that the fiber lengths are equal.

In Section 3 we give a proof of half of Theorem 2.7, which we formalize in Theorem 3.4. In Section 4 we prove the second half of Theorem 2.7, namely, we construct the geometric "shift" and "flip" wall-crossing maps; once we know the equality of fiber lengths, this is quite straightforward. Moreover, in Section 4.2 we define, under an additional technical hypothesis, an "algebraic wall-crossing" on the semigroups associated to C_1 and C_2. We also show that, in general, the algebraic wall-crossing need not arise from either of the geometric wall-crossing maps. Section 5 is devoted to the tropical Grassmannian of 2-planes in $\mathbb{C}^m$, and we work out in detail what our results entail for this special case, including a concrete formula for the "flip" geometric wall-crossing map in this case. We prove our main result of this section – that in this case, the algebraic wall-crossing is the restriction of the "flip" geometric map – in Section 5.5. Finally, the Appendix by Nathan Ilten discusses the complexity-one T-variety perspective.

2. Background: Newton-Okounkov bodies and tropical geometry

In this section we briefly recall the background necessary for the statement of our main theorem (Theorem 2.7). Throughout, X is an irreducible complex projective variety of dimension d and A denotes its homogeneous coordinate ring. In particular, A is a finitely generated $\mathbb{C}$-algebra and is positively graded. Moreover, from the assumptions on X it follows that A is a domain and has Krull dimension $d + 1$.

We begin with a brief account of the theory of Newton-Okounkov bodies; see [15] for details. We restrict to the setting above. Let r be an integer, $0 < r \leq d$. Let $\prec$ denote a total order on $\mathbb{Q}^r$ which respects addition.

Definition 2.1. ([15, Definition 2.1]) Consider $(\mathbb{Q}^r, \prec)$ as an abelian group equipped with the total order $\prec$. A function $\nu : A \setminus \{0\} \to \mathbb{Q}^r$ is a **valuation** over $\mathbb{C}$ if

1. for all $0 \neq f, g$ in A with $0 \neq f + g$ we have $\nu(f + g) \succeq \min\{\nu(f), \nu(g)\}$
2. for all $0 \neq f, g$ in A we have $\nu(fg) = \nu(f) + \nu(g)$ and
3. for all $0 \neq f$ and $0 \neq c \in \mathbb{C}$ we have $\nu(cf) = \nu(f)$, or equivalently, $\nu(c) = 0$ for all $0 \neq c \in \mathbb{C}$.

The valuation ν also gives rise to a multiplicative filtration $\mathcal{F}_\nu$ on A as follows. For $a \in \mathbb{Q}^r$ we define

$$F_{\nu \succeq a} := \{f \in A \setminus \{0\} \mid \nu(f) \succeq a\} \cup \{0\} \qquad \text{and} \qquad F_{\nu \succ a} := \{f \in A \setminus \{0\} \mid \nu(f) \succ a\} \cup \{0\}. \tag{2.1}$$

A valuation $\nu : A \setminus \{0\} \to \mathbb{Q}^r$ has **one-dimensional leaves** if for every $a \in \mathbb{Q}^r$ the vector space $F_{\nu \succeq a}/F_{\nu \succ a}$ is at most one-dimensional. The **associated graded algebra** $gr_\nu(A)$ is defined to be

$$gr_\nu(A) = \bigoplus_{a \in \mathbb{Q}^r} F_{\nu \succeq a}/F_{\nu \succ a}. \tag{2.2}$$

The ring structure on $gr_\nu(A)$ is induced from the ring structure on A. By construction, $gr_\nu(A)$ is graded by $S(A, \nu)$ since $F_{\nu \succeq a}/F_{\nu \succ a} \neq 0$ if and only if $a \in S(A, \nu)$. Note that an element $g \in A \setminus \{0\}$ can be mapped to the associated graded $gr_\nu(A)$ by considering its associated equivalence class in the quotient $F_{\nu \succeq a}/F_{\nu \succ a}$, where $a = \nu(g)$. Also, having one-dimensional leaves implies that, given a vector space basis for $gr_\nu(A)$ which is homogeneous with respect to its grading, the map which sends an element of the basis to its degree is a bijection.

We restrict attention to valuations of the following form. For a positively graded algebra $A = \oplus_{k \geq 0} A_k$, we say that a valuation ν is **homogeneous** on A if the following holds: for any $0 \neq f_1 \in A$ and $0 \neq f_2 \in A$, if $\deg(f_1) < \deg(f_2)$ then $\nu(f_1) \succ \nu(f_2)$ (note the switch). Specifically, we always assume we have a valuation $\nu : A \setminus \{0\} \to \mathbb{N} \times \mathbb{Q}^{r-1} \subseteq \mathbb{Q}^r$ such that its first component is the degree, i.e.

$$\nu(f) = (\deg(f), \nu) : A \setminus \{0\} \to \mathbb{N} \times \mathbb{Q}^{r-1}. \tag{2.3}$$

where the total order on $\mathbb{N} \times \mathbb{Q}^{r-1}$ is defined as follows: for $(a, v), (b, w) \in \mathbb{N} \times \mathbb{Q}^{r-1}$,

$$(a, v) \preceq (b, w) \text{ if and only if } (a > b, \text{ or}, (a = b \text{ and } v \preceq_{\mathbb{Q}^{r-1}} w)) \tag{2.4}$$

where the order $\preceq_{\mathbb{Q}^{r-1}}$ on $\mathbb{Q}^{r-1}$ is taken to be the standard lex order. Note this ordering first compares the first coordinates and then breaks ties with the remaining coordinates; moreover, there is a reversal of the ordering on the first coordinate. Clearly, such a valuation is homogeneous. We additionally assume $\nu : A \setminus \{0\} \to \mathbb{Q}^r$ is a discrete [1] valuation. The image $S(A, \nu) := \nu(A \setminus \{0\}) \subseteq \mathbb{Q}^r$ of such a valuation is a

[1] A valuation is **discrete** if the image of the valuation is discrete in the target (in other words, for any $y \in \nu(A \setminus \{0\})$, there exists an open neighborhood U of y in $\mathbb{Q}^r$ such that $U \cap \nu(A \setminus \{0\}) = \{y\}$).

discrete additive semigroup of $\mathbb{Q}^r$ and is called the **the value semigroup (of ν)**. The **rank of the valuation** is the rank of the group generated by its value semigroup.

Definition 2.2. Let A be the homogeneous coordinate ring of a projective variety and ν a discrete homogeneous valuation on A. The **Newton-Okounkov cone** of (A, ν) is the convex set

$$\mathrm{Cone}(S(A,\nu)) := \left\{ \sum_{i=1}^{n} t_i s_i \mid s_i \in S(A,\nu), t_i \in \mathbb{R}_{\geq 0} \right\} \subseteq \mathbb{R}^{d+1},$$

i.e. the non-negative real span of elements of $S(A, \nu)$. The **Newton-Okounkov body** of (A, ν) is the convex set $\Delta(A, \nu) := \{x_1 = 1\} \cap \mathrm{Cone}(S(A, \nu))$.

Following [15] we say that a set $\mathcal{B} \subseteq A \setminus \{0\}$ is a **Khovanskii basis** for (A, ν) if the image of $\mathcal{B}$ in $gr_\nu(A)$ forms a set of algebra generators of $gr_\nu(A)$. Note that existence of a finite Khovanskii basis for (A, ν) implies that the associated value semigroup $S(A, \nu)$ is finitely generated, which in turn means that the Newton-Okounkov body of Definition 2.2 is a convex rational polytope, and thus is a combinatorial object.

We next briefly recall some basic terminology in tropical geometry; for details see [15]. Let A be an algebra as above and suppose $\mathcal{B} = \{b_1, \ldots, b_n\}$ is any finite set of algebra generators of A which we assume to be homogeneous of degree 1. Consider the surjective $\mathbb{C}$-algebra homomorphism

$$\pi : \mathbb{C}[x_1, \ldots, x_n] \to A \tag{2.5}$$

defined by $\pi(x_i) = b_i$ for $1 \leq i \leq n$. This is a map of graded rings provided that we define the grading on the polynomial ring by $\deg(x_i) = 1$ for all i. Let $I := \ker(\pi) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ which is homogeneous since π preserves degrees. Then we have a natural presentation $A \cong \mathbb{C}[x_1, \ldots, x_n]/I$ associated to this choice of generating set $\mathcal{B}$, realizing $Spec(A)$ explicitly as a subvariety of $\mathbb{C}^n$. Note that $Spec(A)$ is the affine cone over $X \cong Proj(A)$ so we use the notation $\tilde{X} := Spec(A)$.

As noted in [15, Introduction], conceptually it is more appropriate to talk about the tropicalization of a subvariety of a torus. Geometrically, this corresponds to looking at the intersection $\tilde{X}^0 := \tilde{X} \cap (\mathbb{C}^*)^n \subseteq \mathbb{A}^n$ of $\tilde{X}$ with the torus $(\mathbb{C}^*)^n$ sitting naturally in $\mathbb{A}^n$. Algebraically, this corresponds to looking at the algebra

$$\mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]/I^L, \quad \text{where } I^L := I \cdot \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]. \tag{2.6}$$

We will consider both I and I^L below. Following [15] we define the **tropicalization** $\mathcal{T}(I)$ (or **tropical variety**) of $\tilde{X}$ corresponding to the choice of presentation $A \cong \mathbb{C}[x_1, \ldots, x_n]/I$ by

$$\mathcal{T}(I) := \{w \in \mathbb{Q}^n \mid \mathrm{in}_w(I) \text{ does not contain any monomials }\}. \tag{2.7}$$

This definition is a priori different from the definition appearing in [16, Section 3.2], but it is not difficult to see that they are in fact equivalent. Thus $\mathcal{T}(I)$ is a polyhedral fan which is pure of dimension $d + 1$ and also is a subfan of the Gröbner fan [16, Proposition 3.2.8, Theorem 3.3.5]. For each cone C in $\mathcal{T}(U)$ there is a unique initial ideal denoted by $\mathrm{in}_C(I)$ associated to this cone, defined to be $\mathrm{in}_\omega(I)$ for any ω in the interior of C.

Definition 2.3. We say a cone C in $\mathcal{T}(I) = \mathrm{trop}(X^0)$ is a **prime cone** if the corresponding initial ideal $\mathrm{in}_C(I)$ is a prime ideal. A **maximal-dimensional prime cone** is a prime cone C with maximal dimension, i.e., $\dim_{\mathbb{R}}(C) = d + 1$. (In [15] they use the terminology "maximal prime cone" instead.)

To state the result of Kaveh and Manon which relates Newton-Okounkov theory to tropicalizations, we need the notion of a quasivaluation, which is nearly identical to that of a valuation (cf. Definition 2.1) except that we allow for superadditivity in the multiplication.

Definition 2.4. ([15, Definition 2.26]) Consider $(\mathbb{Q}^r, \prec)$ as an abelian group equipped with the total order $\prec$. Let A be a $\mathbb{C}$-algebra. A function $\nu : A \setminus \{0\} \to \mathbb{Q}^r \cup \{\infty\}$ is a **quasivaluation** over k if

(1) For all $0 \neq f, g, f + g$ we have $\nu(f + g) \succeq \min\{\nu(f), \nu(g)\}$.
(2) For all $0 \neq f, g \in A$ we have $\nu(fg) \succeq \nu(f) + \nu(g)$.
(3) For all $0 \neq f \in A$ and $0 \neq c \in \mathbb{C}$ we have $\nu(cf) = \nu(f)$.

As in the case of valuations, a quasivaluation gives rise to a filtration of the original algebra, as well as an associated graded algebra, by using the same formulas (2.1) and (2.2). Conversely, one can construct a quasivaluation from a decreasing algebra filtration $\mathcal{F} = \{F_a\}_{a \in \mathbb{Q}^r}$ of A by $\mathbb{C}$-subspaces by defining, for any $0 \neq f \in A$,

$$\nu_{\mathcal{F}}(f) := \max\{a \in \mathbb{Q}^r : f \in F_a\}. \tag{2.8}$$

(If the max is not attained, we define $\nu_{\mathcal{F}}(f) := \infty$.) The quasivaluations which are central to Kaveh and Manon (and also for this paper) all arise in this manner via a pushforward filtration, as we now describe.

Let $M \in \mathbb{Q}^{r \times n}$ be a matrix. For $p = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ we define the $\mathbb{Q}^r$-valued **weight valuation** $\tilde{\nu}_M : \mathbb{C}[x_1, \ldots, x_n] \setminus \{0\} \to \mathbb{Q}^r$ associated to M by

$$\tilde{\nu}_M(p) := \min\{M\alpha : c_\alpha \neq 0\} \in \mathbb{Q}^r \tag{2.9}$$

where $M\alpha$ is the usual matrix multiplication, the exponent vector α is treated as a column vector, and the minimum is taken with respect to the fixed total ordering on $\mathbb{Q}^r$. [2] In Gröbner theory one frequently takes a maximum, but in this paper we take the "minimum" convention. Similarly we define the initial form of p with respect to M by

$$\mathrm{in}_M(p) = \sum_{\beta : M\beta = \tilde{\nu}_M(p)} c_\beta x^\beta$$

so we take only those terms with minimal value of $M\beta$. We define the initial ideal of I with respect to M, denoted $\mathrm{in}_M(I)$, to be the ideal generated by all $\mathrm{in}_M(p)$ for $p \in I$. Note that $\mathrm{in}_M(I)$ is M-homogeneous in the sense that $h = \mathrm{in}_M(h)$ for all $h \in \mathrm{in}_M(I)$. Next, let $\tilde{\mathcal{F}}_M$ denote the (decreasing) filtration on $\mathbb{C}[x_1, \ldots, x_n]$ obtained from $\tilde{\nu}_M$. We define the **weight filtration** of A (associated to the surjection π and the matrix M) to be the pushforward filtration $\pi(\tilde{\mathcal{F}}_M)$ on A given by the surjection in (2.5). The **weight quasivaluation** ν_M is the quasivaluation on A associated to this weight filtration as defined by (2.8). In general, it need not be a valuation.

We say that a $\mathbb{C}$-vector space basis $\mathbb{B}$ for A is an **adapted basis** for (A, ν) if the image of $\mathbb{B}$ in $gr_\nu(A)$ forms a vector space basis for $gr_\nu(A)$. In the case of a valuation of the form ν_M for some M as above, we will see below (Theorem 2.5) that an adapted basis can be obtained through Gröbner theory. Recall that the maximal cones of the Gröbner fan are indexed by monomial orders $<$; let $C_<$ denote the maximal cone corresponding to $<$. Now suppose C is a cone of the tropicalization $\mathcal{T}(I)$ which is also a face of $C_<$. Let $\mathcal{S}(<, I) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ denote the set of **standard monomials with respect to** I **and** $<$, i.e. the monomials not contained in $\mathrm{in}_<(I)$. It is well-known that the projection onto $\mathbb{C}[x_1, \ldots, x_n]/I$ of the monomials $\mathcal{S}(<, I)$ form a vector space basis for $\mathbb{C}[x_1, \ldots, x_n]/I$. Finally, recall that the **Gröbner region** $GR(I) \subseteq \mathbb{R}^n$ of an ideal I is the set of $u \in \mathbb{Q}^n$ such that there exists a monomial order $<$ such that $\mathrm{in}_<(\mathrm{in}_u(I)) = \mathrm{in}_<(I)$. We have the following theorem, which motivates the current manuscript.

Theorem 2.5. *([15, Propositions 4.2 and 4.8]) Following the notation in this section, let $C \subset \mathcal{T}(I)$ be a maximal-dimensional prime cone. Let $\{u_1, \ldots, u_{d+1}\} \subset C$ be a collection of rational vectors which span a real vector space of dimension $d + 1 = dim(C)$. Let $M \in \mathrm{Mat}((d + 1) \times n, \mathbb{Q})$ be the $(d + 1) \times n$ matrix whose row vectors are $u_1, \ldots, u_{d+1}$. Let $\nu_M : A \setminus \{0\} \to \mathbb{Q}^{d+1}$ denote the corresponding weight quasivaluation. Then ν_M is a valuation, and the following hold:*

1. *$gr_{\nu_M}(A) \cong \mathbb{C}[x_1, \ldots, x_n]/\mathrm{in}_M(I)$ as $\mathbb{Q}^{d+1}$-graded algebras,*
2. *If C lies in the Gröbner region of I, the valuation ν_M has an adapted basis which can be taken to be the projection via π of the standard monomial basis $\mathcal{S}(<, I)$ for a maximal cone $C_<$ in the Gröbner fan of I containing C.*

Remark 2.6. *By an argument similar to [23, Proposition 1.12], if I is a homogeneous ideal then its Gröbner region equals $\mathbb{Q}^n$, so in our case, the hypothesis in item (2) above always holds.*

From item (1) of Theorem 2.5 it follows from basic tropical theory that the value semigroup $S(A, \nu_M)$ (which is the semigroup of the toric variety corresponding to $\mathbb{C}[x_1, \ldots, x_n]/\mathrm{in}_C(I)$) is generated by the

[2] Note that if $M \in \mathbb{Q}^{1 \times n}$ is a single row vector, then $M\alpha$ is just the usual inner product pairing of a "rank-1 weight vector" against the exponent vector α, and the above rule recovers the usual Gröbner theory.

column vectors of the matrix M, and also that the associated Newton-Okounkov body $\Delta(A, \nu_M)$ can be explicitly computed as

$$\Delta(A, \nu_M) = \text{ convex hull of the columns of } M. \tag{2.10}$$

The results above suggest that there should be a straightforward relationship between the Newton-Okounkov bodies associated to two maximal-dimensional prime cones C_1 and C_2 in $\mathcal{T}(I)$ if they are *adjacent* in $\mathcal{T}(I)$, i.e., they share a codimension-1 face $C := C_1 \cap C_2$. The goal of this manuscript is to describe such a "wall-crossing phenomenon" for Newton-Okounkov bodies and to work out the case of the Grassmannians $Gr(2, m)$. The first main result is Theorem 2.7 below. To state the theorem, we need some preparation. For C, C_1 and C_2 as above, fix, once and for all, a linearly independent set $\{u_1, u_2, \ldots, u_d\}$ of integral vectors contained in C. In particular, $\{u_1, \ldots, u_d\}$ span a real vector space of dimension $d = \dim_{\mathbb{R}}(C)$. We also fix a total order $\prec$ satisfying (2.4). We may assume that u_1 is chosen to be the vector $(1, 1, \ldots, 1)$ (this is possible because the ideal I is homogeneous); this ensures that the corresponding weight valuation is homogeneous. We also fix integral vectors $w_1 \in C_1$ and $w_2 \in C_2$ such that $w_1 + \sum_j u_j$ (respectively $w_2 + \sum_j u_j$) lies in the interior of C_1 (respectively C_2). Let M be the $d \times n$ matrix whose j-th row is the vector u_j chosen above, and let M_1 (respectively M_2) denote the $(d+1) \times n$ matrix whose top d rows are the same as those in M and whose bottom $(d+1)$-st row is equal to w_1 (respectively w_2).

Let ν_{M_1}, ν_{M_2} and ν_M be the corresponding weight quasivaluations on A. Theorem 2.5 implies that ν_{M_1}, ν_{M_2} are valuations. Although we remarked above that ν_M for arbitrary M need not be a valuation, for M chosen as in our setting, we will prove in Lemma 3.3 that $\nu_M = \mathsf{p}_{[1,d]} \circ \nu_{M_i}$ for $i = 1, 2$. It can be deduced that ν_M is also a valuation from the fact that ν_{M_i} are valuations and $\mathsf{p}_{[1,d]}$ is a linear projection to the first d coordinates.

Theorem 2.7. *Let $A = \oplus_k A_k$ be a positively graded algebra over $\mathbb{C}$, and assume A is an integral domain and has Krull dimension $d+1$. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a subset of A_1 (the homogeneous degree 1 elements of A) which generate A as an algebra. Let I be the homogeneous ideal such that the presentation induced by $\mathcal{B}$ is $A \cong \mathbb{C}[x_1, \ldots, x_n]/I$ (as in (2.5)), and let $\mathcal{T}(I)$ denote its tropicalization. Suppose that C_1 and C_2 are two maximal-dimensional prime cones in $\mathcal{T}(I)$ that share a codimension-1 face C. Let M_1, M_2, and M be the matrices described above and ν_{M_1}, ν_{M_2} and ν_M the corresponding weight valuations on A. Let $\Delta(A, \nu_{M_1}) \subseteq \{1\} \times \mathbb{R}^d$, $\Delta(A, \nu_{M_2}) \subseteq \{1\} \times \mathbb{R}^d$ and $\Delta(A, \nu_M) \subseteq \{1\} \times \mathbb{R}^{d-1}$ denote the corresponding Newton-Okounkov bodies. Let $\mathsf{p}_{[1,d]} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ denote the linear projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$ obtained by deleting the last coordinate. Then*

$$\mathsf{p}_{[1,d]}(\Delta(A, \nu_{M_1})) = \mathsf{p}_{[1,d]}(\Delta(A, \nu_{M_2})) = \Delta(A, \nu_M)$$

and for any $\xi \in \Delta(A, \nu_M)$, the Euclidean lengths of the fibers $\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(A, \nu_{M_1})$ and $\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(A, \nu_{M_2})$ are equal, up to a global constant which is independent of ξ. Moreover, there exist two piece-wise linear identifications $\mathsf{S}_{12} : \Delta_{M_1} \to \Delta_{M_2}$ and $\mathsf{F}_{12} : \Delta_{M_1} \to \Delta_{M_2}$, called the "shift map" and the "flip map" respectively, which have the following properties: for $\Phi_{12} \in \{\mathsf{S}_{12}, \mathsf{F}_{12}\}$, we have that the diagram

$$\begin{array}{ccc} \Delta(A, \nu_{M_1}) & \xrightarrow{\quad \Phi_{12} \quad} & \Delta(A, \nu_{M_2}) \\ & {\scriptstyle \mathsf{p}_{[1,d]}} \searrow \quad \swarrow {\scriptstyle \mathsf{p}_{[1,d]}} & \\ & \Delta(A, \nu_M) & \end{array}$$

commutes, and Φ_{12} preserves the Euclidean lengths of the fibers of $\mathsf{p}_{[1,d]}$.

Remark 2.8. *The global constant appearing in Theorem 2.7 above depends only on the choices of the matrices M_1, M_2 and M which represent the cones C_1, C_2 and C respectively, which is why the constant is independent of the choice of basepoint $\xi \in \Delta(A, \nu_M)$.*

Example 2.9. We illustrate Theorem 2.7 in an example which is explained in detail in Section 4. In this example we can see explicitly that the lengths of the fibers under p_1 and p_2 are the same length; see Figure 2.1.

The next two sections are devoted to a proof of Theorem 2.7.

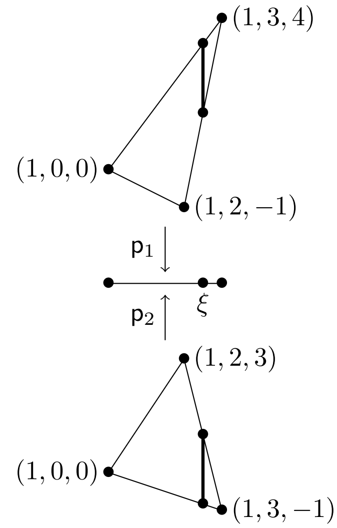

FIGURE 2.1. Two polytopes projecting onto a common interval, and their fibers under the projection maps.

3. THE FIBER LENGTHS ARE EQUAL

The purpose of this section is to prove the first half of Theorem 2.7. Specifically, we show in Lemma 3.1 that we have a diagram

(3.1)
$$\Delta(A,\nu_{M_1}) \xrightarrow{\mathsf{p}_{[1,d]}} \Delta(A,\nu_M) \xleftarrow{\mathsf{p}_{[1,d]}} \Delta(A,\nu_{M_2})$$

relating the 3 polytopes; then in Theorem 3.4 we show the second assertion of Theorem 2.7, namely, that the fiber lengths are equal (up to a global constant – cf. Remark 2.8). Theorem 3.4 is the substantive assertion of Theorem 2.7, and our argument uses a variation of GIT quotients. In addition to the projection $\mathsf{p}_{[1,d]}$, we will also use $\mathsf{p}_{[1]} : \mathbb{R}^{d+1} \to \mathbb{R}$, the projection which maps onto the first coordinate with respect to the standard basis.

Lemma 3.1. *Following the notation in this section, the images under the projection $\mathsf{p}_{[1,d]} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ of $\Delta(A,\nu_{M_1})$ and $\Delta(A,\nu_{M_2})$ are the same and are equal to the Newton-Okounkov body associated to ν_M, i.e.*

$$\mathsf{p}_{[1,d]}(\Delta(A,\nu_{M_1})) = \mathsf{p}_{[1,d]}(\Delta(A,\nu_{M_2})) = \Delta(A,\nu_M).$$

For the proof of Lemma 3.1 the following is useful (see [15, Lemma 3.2] and remarks following). In analogy to the classical Gröbner theory, we say the **(rank r) Gröbner region** $GR^r(I) \subseteq \mathbb{R}^{r\times n}$ is the set of M such that there exists a monomial order $<$ with $\mathrm{in}_<(\mathrm{in}_M(I)) = \mathrm{in}_<(I)$.

Lemma 3.2. *Following the above notation, for any $f \in \mathbb{C}[x_1,\ldots,x_n]/I$ we have*

(3.2)
$$\nu_M(f) = \max\{\tilde{\nu}_M(\tilde{f}) \mid \tilde{f} \in \mathbb{C}[x_1,\ldots,x_n] \text{ and } \pi(\tilde{f}) = f\}.$$

Moreover, in our setting, maximum on the RHS of the above equation is always attained.

Proof. The first claim is [15, Lemma 3.2]. The second claim follows from [15, Proposition 3.3, Lemma 8.7] and the remarks following [15, Definition 2.27]. □

Using the above, we can explicitly compute ν_M as follows.

Lemma 3.3. *Following the notation above,* $\mathsf{p}_{[1,d]} \circ \nu_{M_1} = \mathsf{p}_{[1,d]} \circ \nu_{M_2} = \nu_M$.

Proof. The valuation $\tilde{\nu}_M$ is by definition a minimum, i.e. $\tilde{\nu}_M(\tilde{f}) = \min\{M\alpha \mid c_\alpha \neq 0\}$ for $\tilde{f} = \sum_\alpha c_\alpha x^\alpha$ and similarly for $\tilde{\nu}_{M_1}$ and $\tilde{\nu}_{M_2}$. Therefore, the formula (3.2) is in fact a max-min formula. Moreover, by our assumption on M_1, M_2 and M we know that $\mathsf{p}_{[1,d]}(M_1\alpha) = \mathsf{p}_{[1,d]}(M_2\alpha) = M\alpha$. To prove the lemma we first prove that

$$\mathsf{p}_{[1,d]}(\min T) = \min \mathsf{p}_{[1,d]}(T) \quad \text{and} \quad \mathsf{p}_{[1,d]}(\max T) = \max \mathsf{p}_{[1,d]}(T) \tag{3.3}$$

for any $T \subset \mathbb{Z}_{\geq 0}^{d+1}$ such that both $\min T$ and $\max T$ exist. Indeed, from the definition of the total order (2.4) we have that $a \preceq b$ implies $\mathsf{p}_{[1,d]}(a) \preceq \mathsf{p}_{[1,d]}(b)$ for any $a, b \in \mathbb{Z}_{\geq 0}^{d+1}$. Then it readily follows that if T achieves its min (respectively max) then the left (respectively right) equation of (3.3) holds. Now suppose $\tilde{f} = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$. For any $N \in \{M, M_1, M_2\}$ we define $T_{\tilde{f},N} := \{N\alpha \mid c_\alpha \neq 0\}$. Since $T_{\tilde{f},N}$ is finite, it achieves both its minimum and maximum, and by definition $\tilde{\nu}_N(\tilde{f}) := \min T_{\tilde{f},N}$. For $f \in \mathbb{C}[x_1, \ldots, x_n]/I$ and $i = 1, 2$, we define

$$T_{f,M_i} = \{\min T_{\tilde{f},M_i} \mid \tilde{f} \in \mathbb{C}[x_1, \ldots, x_n] \text{ and } \pi(\tilde{f}) = f\}.$$

By the last claim of Lemma 3.2 we know that the maximum of T_{f,M_i} is achieved for $i = 1, 2$, and therefore $\mathsf{p}_{[1,d]}(\max T_{f,N}) = \max \mathsf{p}_{[1,d]}(T_{f,N})$. As observed above, $T_{\tilde{f},N}$ also achieves its minimum, so that $\mathsf{p}_{[1,d]}(\min T_{\tilde{f},n}) = \min \mathsf{p}_{[1,d]}(T_{\tilde{f},N})$. From the above we can compute that for $i = 1, 2$

$$\begin{aligned}
\mathsf{p}_{[1,d]}(\nu_{M_i}(f)) &= \mathsf{p}_{[1,d]}(\max T_{f,M_i}) \quad \text{by (3.2)} \\
&= \max \mathsf{p}_{[1,d]}(T_{f,M_i}) \quad \text{since the max of } T_{f,M_i} \text{ is achieved} \\
&= \max\{\mathsf{p}_{[1,d]}(\min T_{\tilde{f},M_i}) \mid \tilde{f} \in \mathbb{C}[x_1, \ldots, x_n] \text{ and } \pi(\tilde{f}) = f\} \quad \text{by definition of } T_{f,M_i} \\
&= \max\{\min \mathsf{p}_{[1,d]}(T_{\tilde{f},M_i}) \mid \tilde{f} \in \mathbb{C}[x_1, \ldots, x_n] \text{ and } \pi(\tilde{f}) = f\} \quad \text{since } T_{\tilde{f},M_i} \text{ is finite} \\
&= \max\{\min T_{\tilde{f},M} \mid \tilde{f} \in \mathbb{C}[x_1, \ldots, x_n] \text{ and } \pi(\tilde{f}) = f\} \quad \text{since } \mathsf{p}_{[1,d]}(M_i\alpha) = M\alpha \text{ for all } \alpha \\
&= \max\{\tilde{\nu}_M(\tilde{f}) \mid \tilde{f} \in \mathbb{C}[x_1, \ldots, x_n] \text{ and } \pi(\tilde{f}) = f\} \quad \text{by definition of } \tilde{\nu}_M \\
&= \nu_M(f)
\end{aligned}$$

as desired. □

We can now prove Lemma 3.1.

Proof of Lemma 3.1. By Definition 2.2 we know $\Delta(A, \nu_{M_i}) = \mathrm{Cone}(S(A, \nu_{M_i})) \cap (\{1\} \times \mathbb{R}^d)$ for $i = 1, 2$ and similarly for $\Delta(A, \nu_M)$. Since $\mathsf{p}_{[1,d]}$ is a linear map, $\mathsf{p}_{[1,d]}(\mathrm{Cone}(S(A, \nu_{M_i}))) = \mathrm{Cone}(\mathsf{p}_{[1,d]}(S(A, \nu_{M_i})))$. Now by Lemma 3.3 we know that $\mathsf{p}_{[1,d]}(S(A, \nu_{M_i})) = S(A, \nu_M)$ for $i = 1, 2$. Hence, $\mathrm{Cone}(\mathsf{p}_{[1,d]}(S(A, \nu_{M_i}))) = \mathrm{Cone}(S(A, \nu_M))$ for $i = 1, 2$. The projection $\mathsf{p}_{[1,d]}$ preserves the first coordinate, so taking the level-1 slice commutes with $\mathsf{p}_{[1,d]}$ and the statement follows. □

We now wish to deduce a relationship between the fibers on the corresponding polytopes

$$\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(A, \nu_{M_1}) \quad \text{and} \quad \mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(A, \nu_{M_2}).$$

for $\xi \in \Delta(A, \nu_M)$. An example was illustrated in Figure 2.1. To facilitate this, we define functions $\mathcal{L}_1$ and $\mathcal{L}_2$ which record the lengths of these fibers, i.e.,

$$\mathcal{L}_i : \Delta(A, \nu_M) \to \mathbb{R}, \quad \xi \mapsto \mathrm{len}(\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(A, \nu_{M_i})) \tag{3.4}$$

for $i = 1, 2$, where len denotes the standard Euclidean length in $\mathbb{R}^{d+1}$ with respect to which each standard basis vector ε_i, $1 \leq i \leq d+1$, has length 1. Since any polytope is an intersection of finitely many affine half-spaces which are defined by linear inequalities, it is clear that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are piecewise-linear.[3] With this notation in place, we can state the following.

Theorem 3.4. *Let $\xi \in \Delta(A, \nu_M)$. Then the Euclidean lengths of $\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(A, \nu_M)$ and $\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(A, \nu_{M_2})$ are equal, up to a global constant which is independent of ξ. Equivalently, there exists a global constant $\kappa > 0$ such that $\kappa\mathcal{L}_1 = \mathcal{L}_2$ as piecewise linear functions on $\Delta(A, \nu_M)$.*

[3] A real-valued function on a polytope Δ is *piecewise linear* if Δ can be written as a finite union of polytopes, on each of which f is an affine function, i.e., it is a linear function plus a global translation.

To prove this, we start with some preliminary observations. First, since the $\mathcal{L}_i, i = 1,2$ are piecewise linear, it is straightforward that there exists a regular subdivision of $\Delta(A, \nu_M)$ such that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are affine on each cell. With this in mind, the following lemma shows that to prove that $\kappa\mathcal{L}_1 = \mathcal{L}_2$ it suffices to check equality on a suitable subset of points in $\Delta(A, \nu_M)$.

Lemma 3.5. *Let Δ be an m-dimensional polytope, and let $f, g : \Delta \to \mathbb{R}$ be piecewise-linear functions on Δ. Suppose there exist $Q_j \subseteq \Delta$ for $1 \leq j \leq N$ for some positive integer N such that $\Delta = \cup_{j=1}^N Q_j$, where each Q_j is a polytope and both f and g are affine on Q_j for each $j = 1, \dots, N$. Suppose that, for each j, $1 \leq j \leq N$, there exist a set of $m+1$ points $\{x_{j,1}, x_{j,2}, \cdots, x_{j,m+1}\} \subseteq Q_j$ whose convex hull is an m-simplex, and such that $f(x_{j,k}) = g(x_{j,k})$ for all k, $1 \leq k \leq m+1$. Then $f = g$ on Δ. In particular, to check equality of f and g above, it suffices to check, for each Q_j, the equality $f(x) = g(x)$ for x in a dense subset of any open m-ball of positive radius contained Q_j.*

Proof. For the first statement, it suffices to check equality on each Q_j where $\mathcal{L}_1, \mathcal{L}_2$ are affine. Choose a j, $1 \leq j \leq N$. Since Q_j is m-dimensional, an affine function on Q_j is determined by its values on $m+1$ affinely independent vectors in Q_j. Since a set of $m+1$ points whose convex hull is an m-simplex must be affinely independent, the result follows. For the last statement, note that any open ball contains an m-simplex, as long as the simplex is small enough, and it is clear that the vertices can be arranged to lie in the dense subset. □

For the rest of the section we use the notation $S_i := S(A, \nu_{M_i})$ and $S := S(A, \nu_M)$. By assumption on the M_i and M, the semigroups S_i and S are contained in $\mathbb{Z}^{d+1}$ and $\mathbb{Z}^d$ respectively. We will also use $\Delta(S)$ (resp. $\Delta(S_i)$) to denote $\Delta(A, \nu_M)$ (resp. $\Delta(A, \nu_{M_i})$). Denote by $G(S)$ (resp. $G(S_i)$) the group generated by S (resp. S_i). The starting point of our argument is to observe that for appropriately chosen ξ, the Euclidean lengths of the fibers $\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(S_i)$ have a geometric interpretation; this is the content of Lemma 3.7 below. We need some preparation. Let w_1, w_2 be the integral vectors which were chosen before the statement of Theorem 2.7.

Lemma 3.6. $\operatorname{in}_{w_i}(\operatorname{in}_M(I)) = \operatorname{in}_{M_i}(I)$ *for* $i = 1, 2$.

Proof. This is immediate from [15, Lemma 8.8]. □

Since the cones C_i are prime and maximal-dimensional by assumption, the corresponding initial ideals $\operatorname{in}_{M_i}(I)$ are toric ideals. Let X_i for $i = 1,2$ denote the corresponding Gröbner toric degenerations. Note that Lemma 3.6 says that we may also realize X_i as a Gröbner toric degeneration of $Y := Proj(\mathbb{C}[x_1, \dots, x_n]/\operatorname{in}_M(I))$. By construction, and also by the assumptions in the special case under consideration, we know that $\operatorname{in}_M(I)$ is homogeneous with respect to a $\mathbb{Z}^d$-grading; thus, Y is equipped with the action of a codimension-1 torus T, and this torus still acts on the toric degeneration X_i. More specifically, the full-dimensional torus acting on X_i (with respect to which X_i is a toric variety) contains T as a subtorus. We have the following.

Lemma 3.7. *Let $\xi \in \Delta(S) \cap \mathbb{Q}^d$ be a rational point in the relative interior of $\Delta(S)$. Let $m \in \mathbb{Z}$, $m > 0$ such that $m\xi \in \mathbb{Z}^d$. Let $i = 1$ or $i = 2$. Then there exists a real positive constant κ_i, independent of ξ, such that the length $\operatorname{len}(\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(S_i))$ is equal to κ_i/m times the degree of the GIT quotient $X_i//_{m\xi}T$.*

Proof. Let $i = 1$ or $i = 2$. We know X_i is a toric variety and the moment map of the codimension-1 subtorus is obtained by projection of $\Delta(S_i)$ to $\Delta(S)$ via $\mathsf{p}_{[1,d]}$ [6, Section 28.3]. For m chosen as in the statement of the lemma, we may consider $m\xi$ as a point in $m\Delta(S)$, i.e. the m-scalar multiple of $\Delta(S)$. Note that since T is codimension 1, the GIT (equivalently, symplectic) quotient by T will be complex 1-dimensional and real 2-dimensional (cf. [17], [6, Theorem 23.1]). The degree of the GIT quotient $X_i//_{m\xi}T$ is also the symplectic volume of the symplectic quotient of X_i at $m\xi$ with respect to the m-scalar multiple of the original T-moment map ([7, Theorem 13.4.1], [6, Section 30.1]). The symplectic (GIT) quotient $X_i//_{m\xi}T$ is equipped with a residual S^1-action ($\mathbb{C}^*$-action) whose moment map image is precisely the fiber $\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(S_i)$ (multiplied by m) [6, Section 24.3]. It follows that the symplectic volume of the symplectic quotient is m times a normalized Euclidean length of $\mathsf{p}_{[1,d]}^{-1}(\xi) \cap \Delta(S_i)$ [6, Section 30.1]. Here the normalization factor κ_i depends on the index of $G(S_i) \cap \{x_1 = \cdots = x_d = 0\}$ in $\mathbb{Z}$ and is hence independent of ξ, as claimed. □

The above lemma indicates that in order to prove Theorem 3.4, it suffices to show that the degrees of the two GIT quotients $X_1//_{m\xi}T$ and $X_2//_{m\xi}T$ are equal. This is where we use a variation of GIT. We have the following.

Lemma 3.8. $\deg(X_1//_{m\xi}T) = \deg(X_2//_{m\xi}T)$.

Proof. We observed above that both X_1 and X_2 are Gröbner toric degenerations of Y, since $\mathrm{in}_{w_i}(\mathrm{in}_M(I)) = \mathrm{in}_{M_i}(I)$. This means that there exist flat families $\mathfrak{X}_1$ and $\mathfrak{X}_2$ over $\mathbb{A}^1$ such that the generic fibers are isomorphic to Y for both $\mathfrak{X}_1$ and $\mathfrak{X}_2$, and the special fiber is isomorphic to X_1 and X_2 respectively. We also saw that there is an action of a codimension-1 torus on Y, X_1 and X_2, and it is straightforward to see that this action extends to the families $\mathfrak{X}_1$ and $\mathfrak{X}_2$. By [10, Theorem 2.1.1] we know that, for $i = 1$ or $i = 2$, the global GIT quotient of the entire family by T at $m\xi$ is a flat family $\mathfrak{X}_i//_{m\xi}T$ over $\mathbb{A}^1$ whose generic fiber is $Y//_{m\xi}T$ and whose special fiber is $X_i//_{m\xi}T$. Since the family is flat, we know $\deg(Y//_{m\xi}T) = \deg(X_i//_{m\xi}T)$. Since this equality holds for both $i = 1$ and $i = 2$, we conclude that $\deg(X_1//_{m\xi}T) = \deg(X_2//_{m\xi}T)$, as desired. □

Proof of Theorem 3.4. From Lemma 3.5 it suffices to check the equality of lengths at all rational points in the interior of $\Delta(S)$. Let $\xi \in \Delta(S) \cap \mathbb{Q}^d$ be an interior point and choose $m > 0, m \in \mathbb{Z}$ such that $m\xi \in \mathbb{Z}^d$. By Lemma 3.7 we know that $\mathcal{L}_1(\xi) = \frac{\kappa_1}{m}\deg(X_1//_{m\xi}T)$ and $\mathcal{L}_2(\xi) = \frac{\kappa_2}{m}\deg(X_2//_{m\xi}T)$ where both κ_1, κ_2 are real and positive global constants that are independent of ξ. From Lemma 3.8 we know that the degrees of the two GIT quotients $X_1//_{m\xi}T$ and $X_2//_{m\xi}T$ are equal, so we conclude $\frac{1}{\kappa_1}\mathcal{L}_1(\xi) = \frac{1}{\kappa_2}\mathcal{L}_2(\xi)$. Setting $\kappa = \kappa_2/\kappa_1$ completes the proof. □

4. Wall-crossing formulas for Newton-Okounkov bodies and value semigroups

The main result of this section is the construction of explicit wall-crossing maps S (the "shift map") and F (the "flip map") mentioned in Theorem 2.7, thus completing the proof of Theorem 2.7. This will complete the proof of our main result, Theorem 2.7. Since these maps are defined between the polytopes, we refer to these as the "geometric wall-crossing" formulas. Then, in Section 4.2, we construct a bijective map $\Theta : S_1 \to S_2$ on the semigroups that covers the identity on $S := S(A, \nu_M)$ and behaves well with respect to the generators of the semigroups, in a sense to be described below (see Lemma 4.3). To distinguish the map Θ from the geometric wall-crossing maps, we refer to Θ as the "algebraic wall-crossing map". It should be emphasized that the algebraic wall-crossing map Θ is not necessarily a semigroup homomorphism, and it does not necessarily arise as a restriction of a geometric wall-crossing map to the semigroup. Example 4.5 illustrates these points.

4.1. Geometric wall-crossing for Newton-Okounkov bodies.

The goal of this section is to construct the two piecewise-linear maps F and S between the Newton-Okounkov bodies $\Delta(S_1)$ and $\Delta(S_2)$ in the same setting as Section 3. For the purpose of this discussion we view the polytopes $\Delta(S_1)$ and $\Delta(S_2)$ in the "level-1" affine subspace $\{1\} \times \mathbb{R}^d \subseteq \mathbb{R}^{d+1}$ as in Section 3.

Let $i = 1$ or 2. Since $\Delta(S_i)$ is a polytope and projects to $\Delta(S)$, there exist piecewise-linear functions $\varphi_i : \Delta(S) \to \mathbb{R}$ and $\psi_i : \Delta(S) \to \mathbb{R}$ such that

$$\Delta(S_i) = \{((1,v),z) \in \{1\} \times \mathbb{R}^{d-1} \times \mathbb{R} \mid (1,v) \in \Delta(S), \varphi_i(1,v) \leq z \leq \psi_i(1,v)\} \subseteq \{1\} \times \mathbb{R}^d. \tag{4.1}$$

From Theorem 3.4 we know that for any $(1, v) \in \Delta(S)$ we have

$$\psi_1(1,v) - \varphi_1(1,v) = \mathrm{len}\left(\mathsf{p}^{-1}(1,v) \cap \Delta(S_1)\right) = \frac{1}{\kappa}\mathrm{len}\left(\mathsf{p}^{-1}(1,v) \cap \Delta(S_2)\right) = \frac{1}{\kappa}\left(\psi_2(1,v) - \varphi_2(1,v)\right) \tag{4.2}$$

where $\kappa := |\kappa_1/\kappa_2|$ is the global constant, appearing in Theorem 3.4, which depends on the choices of C_i, M_i. Using this, we define the **shift map** S_{12} by the formula

$$\begin{aligned} \mathsf{S}_{12} : \mathbb{R}^{r+1} &\to \mathbb{R}^{r+1} \\ (1,v,z) &\mapsto (1, v, \kappa(z - \varphi_1(1,v)) + \varphi_2(1,v)) \end{aligned} \tag{4.3}$$

and we define the **flip map** F_{12} as

$$\begin{aligned} \mathsf{F}_{12} : \mathbb{R}^{r+1} &\to \mathbb{R}^{r+1} \\ (1,v,z) &\mapsto (1, v, \kappa(-z + \varphi_1(1,v)) + \psi_2(1,v)). \end{aligned} \tag{4.4}$$

We can now complete the proof of Theorem 2.7.

Remainder of proof of Theorem 2.7. Since we already saw in Lemma 3.1 and Theorem 3.4 that the first claims of Theorem 2.7 hold, it remains to show that the maps S_{12} and F_{12} from $\Delta(S_1)$ to $\Delta(S_2)$ are piecewise-linear, bijective, and that the following diagrams commute:

$$\begin{array}{ccccccc} \Delta(S_1) & \xrightarrow{\mathsf{S}_{12}} & \Delta(S_2) & \qquad & \Delta(S_1) & \xrightarrow{\mathsf{F}_{12}} & \Delta(S_2) \\ & {\scriptstyle \mathsf{p}_{[1,d]}} \searrow \quad \swarrow {\scriptstyle \mathsf{p}_{[1,d]}} & & & & {\scriptstyle \mathsf{p}_{[1,d]}} \searrow \quad \swarrow {\scriptstyle \mathsf{p}_{[1,d]}} & \\ & \Delta(S) & & & & \Delta(S) & \end{array}$$

To do this, we first check that the maps are well-defined, i.e., they take values in $\Delta(S_2)$ as claimed. It is straightforward to check that both maps are injective. Let $(1, v, z) \in \Delta(S_1)$. We have

$$\begin{aligned} \varphi_1(1,v) \le z \le \psi_1(1,v) &\Leftrightarrow 0 \le z - \varphi_1(1,v) \le \psi_1(1,v) - \varphi_1(1,v) \\ &\Leftrightarrow \varphi_2(1,v) \le \kappa(z - \varphi_1(1,v)) + \varphi_2(1,v) \le \kappa(\psi_1(1,v) - \varphi_1(1,v)) + \varphi_2(1,v) \\ &\Leftrightarrow \varphi_2(1,v) \le \kappa(z - \varphi_1(1,v)) + \varphi_2(1,v) \le \psi_2(1,v) - \varphi_2(1,v) + \varphi_2(1,v) \\ &\Leftrightarrow \varphi_2(1,v) \le \kappa(z - \varphi_1(1,v)) + \varphi_2(1,v) \le \psi_2(1,v) \end{aligned} \tag{4.5}$$

where we have used the fact that $\kappa(\psi_1(1,v) - \varphi_1(1,v)) = \psi_2(1,v) - \varphi_2(1,v)$. It follows that S_{12} is well-defined, and the argument for F_{12} is similar. Since $\psi_i, \varphi_i, i = 1, 2$ are piecewise-linear, it follows that both S_{12} and F_{12} are piecewise linear. Similar arguments show that both are bijective, and the diagrams commute by construction. This completes the proof of Theorem 2.7. □

We can extend the definitions of the shift and flip maps to the cones $\mathrm{Cone}(S_1), \mathrm{Cone}(S_2)$. This is useful when we consider the relationship between the geometric wall-crossing maps S_{12} and F_{12} with the algebraic wall-crossing map to be defined in the next section.

Remark 4.1. *Let $(s, v) \in \mathrm{Cone}(S_1)$ for $s \neq 0$. By rescaling, we obtain that that $(1, \frac{1}{s}v) \in \Delta(S_1)$, since $\mathrm{Cone}(S_1)$ is the cone over $\Delta(S_1)$. Then $\mathsf{F}_{12}(1, \frac{1}{s}v) \in \Delta(S_2)$ and therefore $s \cdot \mathsf{F}_{12}(1, \frac{1}{s}v) \in \mathrm{Cone}(S_2)$. A similar formula holds for S_{12}. Thus we can extend the shift map (4.3) and the flip map (4.4) to $\mathrm{Cone}(S_1)$ as follows:*

$$\begin{aligned} \mathsf{F}_{12} : \mathrm{Cone}(S_1) &\to \mathrm{Cone}(S_2) \\ (s, v) &\mapsto s \cdot \mathsf{F}_{12}(1, v/s). \end{aligned}$$

The same holds for S_{12}.

4.2. **Wall-crossing for value semigroups.** In the previous section, we constructed maps between the Newton-Okounkov polytopes $\Delta(S_1)$ and $\Delta(S_2)$ associated to the maximal-dimensional prime cones C_1 and C_2. In this section, we turn our attention to the underlying semigroups S_1 and S_2 and ask whether there exists a natural bijection $\Theta : S_1 \to S_2$ between them which would cover the identity on S, i.e., so that the diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{\Theta} & S_2 \\ & {\scriptstyle \mathsf{p}_{[1,d]}} \searrow \quad \swarrow {\scriptstyle \mathsf{p}_{[1,d]}} & \\ & S & \end{array} \tag{4.6}$$

commutes. The answer, which is the content of this section, is that there does exist such a natural map, at least under the hypothesis that the two cones C_1 and C_2 are both faces of a single maximal cone $C_<$ of the Gröbner fan of I. Let $\mathcal{S}(<, I) \subset \mathbb{C}[x_1, \ldots, x_n]$ denote the set of standard monomials with respect to I and the monomial order $<$ and let $b_\alpha := \pi(x^\alpha)$ denote the projection to A of $x^\alpha \in \mathcal{S}(<, I)$. The following is known.

Proposition 4.2. *([15, Proposition 3.3]) Given C as above, let M be an $r \times n$ matrix with j-th row equal to u_j for linearly independent vectors $\{u_1, \ldots, u_r\} \subset C$. Then the set $\mathbb{B} := \{b_\alpha\}$ is an adapted basis of A with respect to ν_M. Moreover, we have $\mathrm{in}_<(\mathrm{in}_M(I)) = \mathrm{in}_<(I)$.*

The point of the above proposition is that, if C_1 and C_2 are both faces of the same maximal cone $C_<$ in the Gröbner fan, then the same set $\mathcal{S}(<, I)$ of standard monomials with respect to $<$ projects to give an adapted basis of A for both ν_{M_1} and ν_{M_2}. This fact allows us to produce a function $S_1 \to S_2$ as follows. Applying Proposition 4.2 to M_i for $i = 1$ and 2, we conclude that $\mathbb{B}$ is adapted to both ν_{M_1} and ν_{M_2}. Since both ν_{M_1} and ν_{M_2} have one-dimensional leaves, we can conclude that the valuations ν_{M_i} for $i = 1$ and 2 induce bijections

$$\theta_1 : \mathbb{B} \to S_1 \text{ defined by } b_\alpha \mapsto \nu_{M_1}(b_\alpha)$$

for each $b_\alpha \in \mathbb{B}$, and similarly

$$\theta_2 : \mathbb{B} \to S_2 \text{ defined by } b_\alpha \mapsto \nu_{M_2}(b_\alpha).$$

Then the function on semigroups may be defined by

$$\Theta := \theta_2 \circ \theta_1^{-1} : S_1 \to S_2. \tag{4.7}$$

We refer to Θ as the **algebraic wall-crossing** map. Moreover, the above argument shows that this is well-defined and a bijection.

The following, which is a straightforward consequence of [15, Lemma 2.32], will be computationally useful.

Lemma 4.3. *Let $\Theta : S_1 \to S_2$ be the map defined above and let $x^\alpha \in \mathcal{S}(<, I)$. Then $\Theta(M_1\alpha) = M_2\alpha$.*

We now show that the diagram (4.6) commutes. Recall that the projection map $\mathsf{p}_{[1,d]} : S_i \to S$ forgets the last coordinate.

Lemma 4.4. *The map Θ covers the identity on S, i.e., for all $u \in S_1$, we have $\mathsf{p}_{[1,d]}(u) = \mathsf{p}_{[1,d]}(\Theta(u))$.*

Proof. Since θ_1 and θ_2 are bijections, we know that any element in S_1 (respectively S_2) can be written as $M_1\alpha$ (respectively $M_2\alpha$) for some $x^\alpha \in \mathcal{S}(<, I)$. Lemma 4.3 implies that it suffices to show that, for all $x^\alpha \in \mathcal{S}(<, I)$, we have $\mathsf{p}_{[1,d]}(M_1\alpha) = \mathsf{p}_{[1,d]}(M_2\alpha)$. This follows immediately from the fact that M_1 and M_2 are equal except on the bottom row. □

Since the map Θ defined above is a map between semigroups, it is natural to ask whether Θ is in fact a semigroup homomorphism. Moreover, since the shift and flip maps of Section 4.1 can be defined on all of $\mathrm{Cone}(S_1)$ and the semigroup S_1 lies in $\mathrm{Cone}(S_1)$, we can ask whether the restriction of either of the "geometric" wall-crossing maps – i.e. the shift or the flip map – to the subset S_1 is equal to Θ. It turns out that, in general, Θ need not be a semigroup homomorphism, and Θ is not necessarily obtained by restriction of S_{12} or F_{12}. We give an example to illustrate this.

Example 4.5. First we illustrate that the algebraic wall-crossing map need not be the restriction of either of the geometric wall-crossing maps.

Let $f = x_2^{11} - x_1^6x_3^4x_4 - x_1^7x_3x_4^3 \in \mathbb{C}[x_1, x_2, x_3, x_4]$ and let $I = \langle f \rangle$ be the principal ideal generated by f. The tropical hypersurface $\mathcal{T}(\langle f \rangle)$ is defined to be the set of $(u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ such that $\mathrm{in}_u(f)$ is not monomial. It is not hard to see that two of the maximal (3-dimensional) cones of $\mathcal{T}(\langle f \rangle)$ are given by

$$C_1 = \mathrm{Cone}\{(0, 0, -1, 4), \pm(1, 1, 1, 1), \pm(0, 1, 2, 3)\} \text{ and}$$
$$C_2 = \mathrm{Cone}\{(0, 0, 3, -1), \pm(1, 1, 1, 1), \pm(0, 1, 2, 3)\}.$$

(There is another maximal cone C_3 which we do not need to consider, since it is not prime.) The initial terms of f corresponding to the cones C_1 and C_2 above are

$$\mathrm{in}_{C_1}(f) = x_2^{11} - x_1^6x_3^4x_4 \text{ and}$$
$$\mathrm{in}_{C_2}(f) = x_2^{11} - x_1^7x_3x_4^3.$$

We claim that both $\mathrm{in}_{C_1}(f)$ and $\mathrm{in}_{C_2}(f)$ are irreducible, and thus that C_1 and C_2 are maximal-dimensional prime cones in $\mathcal{T}(\langle f \rangle)$. It is clear from the above that C_1 and C_2 share a codimension-1 face, so this means we are in the situation being discussed in this manuscript. Since the arguments for irreducibility of $\mathrm{in}_{C_1}(f)$ and $\mathrm{in}_{C_2}(f)$ are similar, we sketch the argument only for $\mathrm{in}_{C_1}(f)$. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 4 \end{pmatrix}.$$

Then the kernel of A, considered as a linear transformation $\mathbb{C}^4 \to \mathbb{C}^4$, is spanned by the vector $(-6, 11, -4, -1)$. Define a map $\varphi : \mathbb{C}[x_1, x_2, x_3, x_4] \to \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$ by $x_1 \mapsto t_1, x_2 \mapsto t_1t_2, x_3 \mapsto t_1t_2^2t_3^{-1}$ and $x_4 \mapsto t_1t_2^3t_3^4$. Let $I_A := \ker\varphi$. This is a prime ideal since the image of φ is a domain (being the subring of a domain); it is also called the toric ideal of A. By [9, Exercise 3.2, Section 3.1], I_A is principal since A has rank 3. It is straightforward to check that $\mathrm{in}_{C_1}(f)$ is contained in $I_A = \ker\varphi$. We now claim that $\mathrm{in}_{C_1}(f)$ is a minimal generator of I_A. We need some notation. For a vector $\alpha \in \mathbb{Z}^4$ we define α_+ and α_- by the formulas

$$(\alpha_+)_i := \begin{cases} \alpha_i \text{ if } \alpha_i \geq 0 \\ 0 \text{ if } \alpha_i < 0 \end{cases} \qquad (\alpha_-)_i := \begin{cases} 0 \text{ if } \alpha_i > 0 \\ -\alpha_i \text{ if } \alpha_i \leq 0. \end{cases}$$

By [9, Theorem 3.2], there exists a binomial $f_\alpha = \underline{x}^{\alpha_+} - \underline{x}^{\alpha_-}$ with $\alpha \in \ker(A) \cap \mathbb{Z}^4$ such that f_α generates I_A and divides $\mathrm{in}_{C_1}(f)$. Since the kernel of A is spanned by $(-6, 11, -4, -1)$, there must exist a constant $c \in \mathbb{C}$ such that $c(-6, 11, -4, -1) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Since $\alpha \in \mathbb{Z}^4$, we conclude c must be an integer. If $c \neq \pm 1$, then the total degree of f_α is greater than 11, so we conclude $c = \pm 1$. This then implies $f_\alpha = \pm\,\mathrm{in}_{C_1}(f)$. Thus, $\mathrm{in}_{C_1}(f)$ is a minimal generator of I_A, and since I_A is prime, $\mathrm{in}_{C_1}(f)$ is irreducible. A similar argument shows $\mathrm{in}_{C_2}(f)$ is irreducible. We conclude that C_1 and C_2 are both maximal-dimensional prime cones.

For the cones C_1 and C_2 we may choose the corresponding matrices M_1 and M_2 as follows

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 4 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & -1 \end{pmatrix}.$$

We illustrated the pair of polytopes associated to these matrices in Figure 2.1.

Notice that both C_1 and C_2 lie in the maximal cone of the Gröbner fan corresponding to $\mathrm{in}_<(I) = \langle x_2^{11} \rangle$. The standard monomials $\mathcal{S}(<, I)$ for I with respect to (a choice of such) a monomial order $<$ for this Gröbner cone is the set of all monomials not divisible by x_2^{11}. Since x_1, x_2, x_3, x_4 are all standard monomials, the algebraic wall-crossing map Θ sends the j-th column of M_1 to the j-th column of M_2 for $1 \leq j \leq 4$. In particular, $\Theta(1,1,0) = (1,1,0)$, since the second column goes to the second column. The Newton-Okounkov bodies in question are the convex hulls of the columns of M_i, which we now view as polygons in $\mathbb{R}^2 \cong \{1\} \times \mathbb{R}^2$. Thus the point $(1,1,0)$ considered above is now identified with the point $(1,0)$ in $\mathbb{R}^2$, and this point is contained in the interior of both $\Delta(A, \nu_{M_1})$ and $\Delta(A, \nu_{M_2})$, see Figure 4.1. Moreover, we have just seen that this interior point $(1,1,0)$ in $\Delta(A, \nu_{M_1})$ must be sent by Θ to the point $(1,1,0)$ in $\Delta(A, \nu_{M_2})$. It is an easy exercise to check that neither the geometric "flip" map F_{12} nor the geometric "shift" map S_{12} can accomplish this. Therefore, the algebraic wall-crossing map Θ does not arise as the restriction of a geometric wall-crossing in this case.

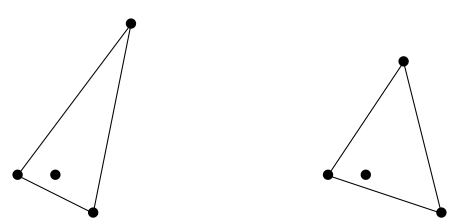

FIGURE 4.1. The Newton-Okounkov bodies for the matrices M_1 and M_2 in Example 4.5.

Secondly, we show that for this example the algebraic wall-crossing Θ is also not a semigroup homomorphism. We follow the notation above. As already noted, the standard monomials of $\mathrm{in}_{C_1}(I)$ with respect to $<$ are all monomials not divisible by x_2^{11}. We have also already seen that $\Theta(1,1,0) = (1,1,0)$. If Θ were a semigroup map, then we must have $\Theta(11,11,0) = 11 \cdot \Theta(1,1,0)$. However, since x_2^{11} is not a standard monomial, in order to compute Θ of $(11,11,0) = 11(1,1,0) = M_1 \cdot (0,11,0,0)$ we must first find a standard monomial $x^\alpha \in \mathcal{S}(<, I)$ such that $M_1 \cdot (0,11,0,0) = M_1\alpha$. Notice that $x_1^6 x_3^4 x_4$ accomplishes this. Therefore,

$$\Theta(11,11,0) = \Theta(M_1 \cdot (6,0,4,1)^T) = M_2(6,0,4,1)^T = (11,11,11) \neq 11 \cdot \Theta(1,1,0)$$

where by slight abuse of notation we have denoted vectors occasionally as rows and at other times as columns. Hence we conclude that Θ is not a semigroup map.

5. EXAMPLE: THE GRASSMANNIAN OF 2-PLANES IN m-SPACE

In this section, we illustrate the wall-crossing phenomena developed above for the tropical Grassmannian $\mathrm{trop}(Gr(2,m))$. In addition, although we saw in Example 4.5 that the algebraic wall-crossing map is not necessarily the restriction of a geometric wall-crossing, we show in Theorem 5.15 that in the case of $\mathrm{trop}(Gr(2,m))$, the algebraic crossing Θ is the restriction of the geometric "flip" map.

5.1. **Background on the tropical Grassmannians.** To begin, we briefly establish some notation. Let $Gr(2,m)$ denote the Grassmannian of 2-planes in $\mathbb{C}^m$ embedded in $\mathbb{P}(\Lambda^2(\mathbb{C}^m))$ via the **Plücker embedding**. For each subset $J \subseteq [m] := \{1,2,\ldots,m\}$ of cardinality 2 we associate a variable p_J. It is well-known that the homogeneous coordinate ring A of $Gr(2,m)$ with respect to the Plücker embedding satisfies $A \cong \mathbb{C}[p_J : J \subset [m], |J| = 2]/I_{2,m}$ where $I_{2,m}$ is the **Plücker ideal**

$$I_{2,m} = \langle p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \mid 1 \leq i < j < k < l \leq m \rangle \tag{5.1}$$

(see e.g. [16, Proposition 2.2.10]). Let us now briefly summarize some facts about the tropical Grassmannian $\mathcal{T}(I_{2,m}) = \mathrm{trop}(\tilde{Gr}^0(2,m))$; see [16, 22]. We need some terminology. A **phylogenetic tree** on $[m]$ is a tree with m labelled leaves and no vertices of degree 2. The m edges which are adjacent to the leaves of the tree are called **pendant edges** and the others are called **interior edges**. Given a phylogenetic tree τ on $[m]$, a **tree distance** is a vector $\underline{d} = (d_{ij}) \in \mathbb{R}^{\binom{m}{2}}$ constructed as follows. Assign a length $\ell_\varepsilon \in \mathbb{R}$ to each edge ε in τ (note we do not assume the lengths are positive). Since τ is a tree, there is a unique path connecting any two leaves i and j; let d_{ij} be the sum of the lengths ℓ_ε of all the edges in this path. The set of all tree distances in $\mathbb{R}^{\binom{m}{2}}$ is called the **space of phylogenetic trees**.

Theorem 5.1. *([16, Theorem 4.3.5]) The negative $-\mathcal{T}(I_{2,m}) \subseteq \mathbb{R}^{\binom{m}{2}}$ of the tropical Grassmannian is equal to the space of phylogenetic trees with m labelled leaves.*

We now briefly describe the fan structure of $\mathcal{T}(I_{2,m}) \subseteq \mathbb{R}^{\binom{m}{2}}$. For details and proofs see [16]. The maximal cones of $\mathcal{T}(I_{2,m})$ are in bijective correspondence with the set of trivalent trees on $[m]$, where a tree is **trivalent** if all the interior vertices are incident to exactly three edges. We label the coordinates in $\mathbb{R}^{\binom{m}{2}}$ by the subsets J of $[m]$ of cardinality 2, corresponding naturally to the Plücker coordinates p_J. For such a subset J, let e_J denote the standard basis ("indicator") vector with a 1 in the coordinate labelled by J and 0's elsewhere. The lineality space [4] L is given by

$$L = \mathrm{span}\left(\sum_{J : i \in J} e_J \mid 1 \leq i \leq m \right)$$

and this m-dimensional subspace is contained in all cones of $\mathcal{T}(I_{2,m})$. The ideal $I_{2,m}$ is a homogeneous ideal with respect to the usual $\mathbb{Z}$-grading where $\deg(p_J) = 1$ for each Plücker coordinate p_J, so we also note that L contains the vector $\mathbb{1} := (1,1,1,\ldots,1) \in \mathbb{R}^{\binom{m}{2}}$. Next, let τ be a trivalent tree and ε be an edge of τ. The choice of ε naturally yields a partition of the m leaves into two subsets J_ε and J_ε^c, given by the decomposition of the vertices obtained by removing ε. We can define a corresponding tree distance

$$\underline{d}_\varepsilon := \sum_{i \in J_\varepsilon, j \in J_\varepsilon^c} e_{ij}, \tag{5.2}$$

obtained by assigning length 1 to the edge ε. By Theorem 5.1, $-\underline{d}_\varepsilon \in \mathcal{T}(I_{2,m})$. However, we would like to keep the entries positive. In the case in which ϵ is a pendant edge, since $-\underline{d}_\varepsilon \in L$ we will use $\underline{d}_\varepsilon$ instead. In the case in which ϵ is an interior edge, since $\mathbb{1} \in L$ we will use the vector $\mathbb{1} - \underline{d}_\varepsilon \in \mathcal{T}(I_{2,m})$ instead. The maximal cone C_τ corresponding to such a tree τ is isomorphic to $\mathbb{R}_{\geq 0}^{m-3} \times \mathbb{R}^m$ and can be described explicitly as

$$C_\tau = \mathrm{Cone}\left\{ \mathbb{1} - \sum_{i \in J_\varepsilon, j \in J_\varepsilon^c} e_{ij} \,\middle|\, \varepsilon \text{ an interior edge} \right\} \times \mathrm{span}\left\{ \sum_{J : i \in J} e_J \,\middle|\, 1 \leq i \leq m \right\} \cong \mathbb{R}_{\geq 0}^{m-3} \times \mathbb{R}^m$$

where Cone denotes the non-negative span of the given set of vectors, and span denotes the usual $\mathbb{R}$-span [16, Proposition 4.3.10].

[4] The lineality space of $\mathcal{T}(I)$ for an ideal I is the subspace of $w \in \mathbb{R}^n$ such that $\mathrm{in}_w(I) = I$.

5.2. **Newton-Okounkov bodies of adjacent maximal-dimensional prime cones in $\mathcal{T}(I_{2,m})$.** We now describe the Newton-Okounkov bodies and value semigroups associated to adjacent maximal-dimensional prime cones in $\mathcal{T}(I_{2,m})$. To begin, we need to know the set of maximal-dimensional prime cones in $\mathcal{T}(I_{2,m})$. The following is known.

Lemma 5.2. *([16, Remark 4.3.11]) Let τ be a trivalent tree on $[m]$ and let C_τ be the associated cone in $\mathcal{T}(I_{2,m})$. Then the initial ideal $\mathrm{in}_{C_\tau}(I_{2,m})$ corresponding to C_τ is a prime ideal. Equivalently, all the maximal cones of $\mathcal{T}(I_{2,m})$ are prime in the sense of Definition 2.3.*

From Lemma 5.2 it follows that we can apply Theorem 2.5 to any maximal cone C_τ in $\mathcal{T}(I_{2,m})$. Doing so involves an explicit choice of linearly independent vectors in the relevant cones. We wish to describe the Newton-Okounkov bodies concretely and also to compare the Newton-Okounkov bodies of adjacent maximal-dimensional prime cones, so to facilitate our computations, we will make a systematic choice of these vectors.

We begin by characterizing adjacency of the maximal-dimensional prime cones. It is known that two maximal-dimensional prime cones C_{τ_1} and C_{τ_2} are adjacent exactly if there exists an interior edge in τ_1 and an interior edge in τ_2, such that we obtain the same tree after contracting these edges in their corresponding tree. Figure 5.1 shows what this looks like locally.

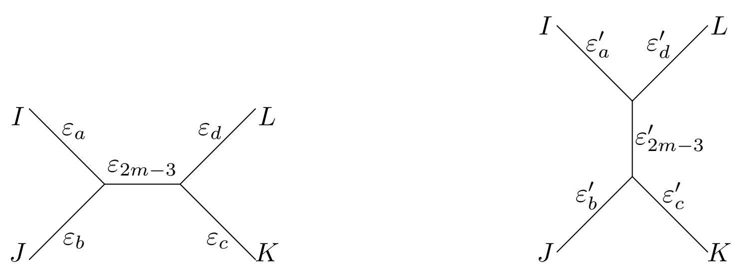

FIGURE 5.1. The figure on the left schematically represents τ_1 and the right figure represents τ_2. If τ_1 and τ_2 are adjacent, then they are identical except on one interior edge; in the figure these are labelled ε_{2m-3} and ε'_{2m-3}. There also exists a decomposition $I \sqcup J \sqcup K \sqcup L = [m]$ of the leaves such that the trees schematically look as above, where the edge ε_i leading to I indicates that the vertices to which ε_i leads lie precisely in $I \subseteq [m]$, and similarly for the others. It is understood that τ_1 and τ_2 are identical except near the edges ε_{2m-3} and ε'_{2m-3}.

Now suppose C_{τ_1} and C_{τ_2} are adjacent maximal-dimensional prime cones. Fix $\tau \in \{\tau_1, \tau_2\}$. We choose linearly independent vectors $u_1, u_2, \ldots, u_{2m-3} \in C_\tau$ as follows. For the purposes of this discussion, we assume that the edges of τ are labelled $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2m-3}\}$ where the first m edges $\varepsilon_1, \ldots, \varepsilon_m$ are the pendant edges incident to the leaves labelled $1, 2, \ldots, m$ respectively, the last $m-3$ edges $\varepsilon_{m+1}, \ldots, \varepsilon_{2m-3}$ are the interior edges, and moreover, the very last interior edge ε_{2m-3} (for both C_{τ_1} and for C_{τ_2}) corresponds to "the" edge by which the two trees differ, as in Figure 5.1 above. As discussed in Section 3 we always choose $u_1 = \mathbb{1} := (1, 1, 1, \ldots, 1)$ so that the corresponding weight valuation is homogeneous with respect to the (usual) degree. Next, for $2 \leq i \leq m$ we choose u_i to be the tree distance $\underline{d}_{\varepsilon_i} = \sum_{i \in J} e_J$. Finally, for $m+1 \leq a \leq 2m-3$, we let $u_a = \mathbb{1} - \underline{d}_{\varepsilon_a}$. We then obtain a $(2m-3) \times \binom{m}{2}$ matrix M_τ whose i-th row is the vector u_i. By construction, M_{τ_1} and M_{τ_2} are identical except on the last (bottom) row.

Example 5.3. Let $m = 4$. In this case the Plücker coordinates for $Gr(2,4)$ are given by the $6 = \binom{4}{2}$ coordinates p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}; throughout this discussion we assume that these 6 Plücker coordinates are ordered as in the list just given. Let τ_1 (resp. τ_2) be the trivalent tree on the LHS (resp. RHS) in Figure 5.2.

The interior edge of τ_1 partitions the set $[4]$ into the subsets $J = \{1,2\}$ and $J^c = \{3,4\}$, so $u_5 = \mathbb{1} - \sum_{i \in J, j \in J^c} e_{ij} = \mathbb{1} - (e_{13} + e_{14} + e_{23} + e_{24}) = (1, 0, 0, 0, 0, 1)$. For τ_2 the partition is $J = \{1,4\}$ and $J^c = \{2,3\}$. The matrices can be computed to be

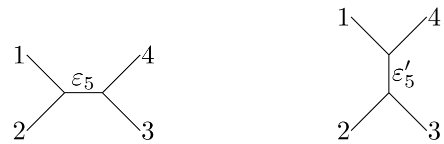

FIGURE 5.2. Two trivalent trees for $\mathcal{T}(I_{2,4})$.

$$M_{\tau_1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_{\tau_2} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

so they are identical except on the last row.

By Theorem 2.5 and (2.10) we know that for any τ we have

$$P_\tau := P(A, \nu_{M_\tau}) = \mathbb{R}_{\geq 0}\text{-span of the columns of } M_\tau,$$
$$\Delta_\tau := \Delta(A, \nu_{M_\tau}) = \text{ convex hull of the columns of } M_\tau.$$

Note that $\Delta_\tau = P_\tau \cap \{x_1 = 1\}$. Furthermore, denoting by $M_{\tau_1\tau_2}$ the matrix resulting from deleting the bottom row of M_{τ_1} we also have

$$P_{\tau_1\tau_2} := P(A, \nu_{M_{\tau_1\tau_2}}) = \mathbb{R}_{\geq 0}\text{-span of the columns of } M_{\tau_1\tau_2},$$
$$\Delta_{\tau_1\tau_2} := \Delta(A, \nu_{M_{\tau_1\tau_2}}) = \text{ convex hull of the columns of } M_{\tau_1\tau_2}.$$

5.3. **The geometric wall-crossing maps for $Gr(2,m)$.** In this section, we describe the geometric wall-crossing maps for $Gr(2,m)$ for two adjacent maximal-dimensional prime cones C_1 and C_2 corresponding to trivalent trees τ_1 and τ_2. Let M_{τ_1} and M_{τ_2} denote the corresponding choices of matrices described in the previous section.

To proceed, it will be convenient to first give the inequalities which cut out the cone P_τ for a given trivalent tree τ. In order to do so, we make a change of coordinates $\gamma : \mathbb{R}^{2m-3} \to \mathbb{R}^{2m-3}$ which transforms P_τ to a cone $\widetilde{P}_\tau$. It will turn out that $\widetilde{P}_\tau$ is more compatible with the combinatorics of phylogenetic trees, and moreover, the inequalities defining $\widetilde{P}_\tau$ are known from the work of Nohara and Ueda [19].

We begin by explicitly defining the polytope $\widetilde{P}_\tau$; from this we can deduce the transformation γ. For a trivalent tree τ we define a $(2m-3) \times \binom{m}{2}$ matrix $\widetilde{M}_\tau$ by taking its a-th row to be the tree distance $\underline{d}_{\varepsilon_a}$ obtained by assigning 1 to edge ε_a and 0 elsewhere. Labelling columns of $\widetilde{M}_\tau$ by pairs of leaves $\{i,j\}$ and rows by a, the matrix entries c_a^{ij} of $\widetilde{M}_\tau$ can then be seen to satisfy

$$c_a^{ij} = \begin{cases} 1, & \text{if the (unique) path from } i \text{ to } j \text{ contains edge } \varepsilon_a \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

We now define

$$\widetilde{P}_\tau := \mathrm{Cone}\{c^{ij} \mid 1 \leq i < j \leq m\} \subseteq \mathbb{R}^{2m-3}, \tag{5.4}$$

i.e. $\widetilde{P}_\tau$ is the cone spanned in $\mathbb{R}^{2m-3}$ by the columns of $\widetilde{M}_\tau$. Similarly we define

$$\widetilde{\Delta}_\tau := \text{ convex hull of the columns of } \widetilde{M}_\tau \tag{5.5}$$

and

$$\widetilde{S}_\tau := \text{semigroup generated by the columns of } \widetilde{M}_\tau. \tag{5.6}$$

We have the following.

Lemma 5.4. *The linear map $\gamma : \mathbb{R}^{2m-3} \to \mathbb{R}^{2m-3}$ defined by*

$$\gamma : \mathbb{R}^{2m-3} \to \mathbb{R}^{2m-3} \tag{5.7}$$

$$(z_1, \dots, z_{2m-3}) \mapsto \left(\frac{1}{2}(z_1 + \cdots + z_m), z_2, \dots, z_m, \frac{1}{2}(z_1 + \cdots + z_m) - z_{m+1}, \dots, \frac{1}{2}(z_1 + \cdots + z_m) - z_{2m-3}\right)$$

is a linear isomorphism and maps the ij-th column of $\widetilde{M}_\tau$ to the ij-th column of M_τ. In particular, γ restricts to bijections $\widetilde{S}_\tau \to S_\tau$ and $\widetilde{P}_\tau \to P_\tau$.

Proof. Recall that our convention is to order the edges so that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ are the pendant edges, with edge ε_i adjacent to leaf i, and the edges $\varepsilon_{m+1}, \cdots, \varepsilon_{2m-3}$ are the interior edges. In order to show that γ takes c^{ij} to the corresponding ij-th column of M_τ, we check each coordinate of $\gamma(c^{ij})$.

Fix a column c^{ij}. First, we consider the first coordinate. By definition, each column in M_τ has first entry equal to 1. Therefore, to show that $\gamma(c^{ij})$ agrees with the corresponding column in M_τ, we must show that the function $\frac{1}{2}(z_1 + \cdots + z_m)$ (here the z_k denote the standard coordinate functions in $\mathbb{R}^{2m-3}$) evaluates to 1 on c^{ij}. By (5.3) we see that the i-th and j-th coordinates of c^{ij}, corresponding to the pendant edges ε_i and ε_j respectively, are equal to 1, since these edges are contained in the path connecting i and j. Moreover, no other pendant edge is contained in this path, so all the other coordinates corresponding to pendant edges are equal to 0. Therefore, $z_1 + z_2 + \cdots + z_m = 2$ on c^{ij} and hence $\frac{1}{2}(z_1 + \cdots + z_m) = 1$, as desired.

Second, we consider the coordinates corresponding to the pendant edges ε_a for $2 \leq a \leq m$. By definition, the a-th row of M_τ is the tree distance $\underline{d}_{\varepsilon_a}$, which is equal to the a-th row of $\widetilde{M}_\tau$. Hence the entries are in fact equal, so the identity map on those coordinates, namely $z_2, \dots, z_m$, takes the corresponding entries of c^{ij} to those of the columns of M_τ as desired.

Finally, consider the coordinates corresponding to interior edges, i.e. the a-th coordinates for $m+1 \leq a \leq 2m-3$. From the construction of M_τ we know that the a-th entry of the ij-th column of M_τ is $1 - c^{ij}_a$. Therefore we need to show that $\frac{1}{2}(z_1 + z_2 + \cdots + z_m) - z_a$ evaluates on c^{ij} to $1 - c^{ij}_a$. But we already saw above that $\frac{1}{2}(z_1 + z_2 + \cdots + z_m) = 1$ on c^{ij}, so the claim follows.

This shows that γ takes the columns of $\widetilde{M}_\tau$ to the corresponding columns of M_τ, as desired. The second claim of the lemma follows immediately from the definitions of $\widetilde{P}_\tau$ and P_τ. □

Remark 5.5. *The proof of the lemma above shows also that $\widetilde{\Delta}_\tau$ is the intersection of $\widetilde{P}_\tau$ with the hyperplane*

$$\left\{\frac{1}{2}(z_1 + z_2 + \cdots + z_m) = 1\right\} = \gamma^{-1}(\{z_1 = 1\}).$$

It follows that γ also restricts to a bijection $\gamma : \widetilde{\Delta}_\tau \to \Delta_\tau$ which can be written explicitly as $(z_1, \dots, z_{2m-3}) \mapsto (1, z_2, \dots, z_m, 1 - z_{m+1}, \dots, 1 - z_{2m-3})$.

In order to give the inequality description of P_τ it now suffices to give an inequality description of $\widetilde{P}_\tau$ and then to translate this back to P_τ using the coordinate change γ. In fact, the half-spaces defining $\widetilde{P}_\tau$ were given by Nohara and Ueda. We have the following, which follows from [19, Theorem 4.9].

Theorem 5.6. *The polytope $\widetilde{\Delta}_\tau$ is the intersection of the half-spaces defined by the inequalities*

$$z_1 + \cdots + z_m = 2 \qquad \text{and} \tag{5.8}$$

$$|z_b - z_c| \leq z_a \leq z_b + z_c, \tag{5.9}$$

where $\varepsilon_a, \varepsilon_b, \varepsilon_c$ are incident to a single interior vertex of τ, and these inequalities run over all interior vertices of τ. The cone $\widetilde{P}_\tau$ is defined as the intersection of the inequalities of (5.9).

Proof. As mentioned above, the statement of the theorem is essentially that of [19, Theorem 4.9]. However, a change of coordinates is required to deduce the above statement from [19] so we explain this briefly here. For details we refer the reader to [19]. In [19, Section 6], the authors give a set of lattice points in $\mathbb{R}^{2m-3}$ whose convex hull is a polytope which they denote as Δ_Γ. In [19, Section 4], the authors perform a change of coordinates [19, Equation (4.2)], and it is not hard to see that, under this change of coordinates, the lattice points whose convex hull is Δ_Γ get mapped to $m/2$ times the columns of our matrix $\widetilde{M}_\tau$. Therefore, under the change of coordinates [19, Equation (4.2)], the polytope Δ_Γ of Nohara and Ueda is mapped to $\frac{m}{2}\widetilde{\Delta}_\tau$. The equations of [19, Theorem 4.9] describe the inequalities of $\frac{m}{2}\widetilde{\Delta}_\tau$ as a subset of the hyperplane

$z_1 + \cdots + z_{2m-3} = m/2$ and therefore the cone $\widetilde{P}_\tau$ is the intersection of the inequalities of (5.9). The claim about $\widetilde{\Delta}_\tau$ now follows straightforwardly. □

We can now give explicit formulas for the geometric wall-crossing maps as in Section 4 for $Gr(2,m)$. Let τ_1 and τ_2 be two trivalent trees corresponding to two adjacent maximal-dimensional prime cones. The trees τ_1 and τ_2 agree everywhere except near one edge which we may take to be labelled as ε_{2m-3}, and that locally near ε_{2m-3} the trees τ_1 and τ_2 are of the form given in Figure 5.1. More specifically, we assume that τ_1 looks locally near ε_{2m-3} like the figure on the left in Figure 5.1 and τ_2 is the figure on the right. Let $\widetilde{P}_{\tau_1\tau_2}$ denote the projection of $\widetilde{P}_{\tau_1}$ (equivalently $\widetilde{P}_{\tau_2}$) to $\mathbb{R}^{2m-4}$, obtained by forgetting the last coordinate. Then, as in (4.1), we may express $\widetilde{P}_{\tau_i}$ for $i = 1, 2$ as follows:

$$\widetilde{P}_{\tau_i} = \{(v, z_{2m-3}) \,|\, v \in \widetilde{P}_{\tau_1\tau_2},\ \widetilde{\varphi}_i(v) \leq z_{2m-3} \leq \widetilde{\psi}_i(v)\}$$

for certain affine functions $\widetilde{\varphi}_i$ and $\widetilde{\psi}_i$. We have the following.

Lemma 5.7. *In the setting above, we have*

$$\widetilde{\varphi}_1(v) = \max\{|z_a - z_b|, |z_c - z_d|\}, \qquad \widetilde{\psi}_1(v) = \min\{z_a + z_b, z_c + z_d\},$$
$$\widetilde{\varphi}_2(v) = \max\{|z_a - z_d|, |z_b - z_c|\}, \qquad \widetilde{\psi}_2(v) = \min\{z_a + z_d, z_b + z_c\}.$$

Proof. We prove the formulas for $\widetilde{\varphi}_1$ and $\widetilde{\psi}_1$. The proof for $i = 2$ is similar. It may be helpful to refer to Figure 5.1. From Theorem 5.6 we know $|z_a - z_b| \leq z_{2m+3}$, and similarly $|z_d - z_c| \leq z_{2m+3}$. We conclude $z_{2m+3} \geq \max\{|z_a - z_b|, |z_d - z_c|\}$. The other inequality in Theorem 5.6 immediately imply $z_{2m+3} \leq \min\{z_a + z_b, z_d + z_c\}$. This yields the desired formulas. □

We can now deduce that the lengths of the fibers are equal.

Lemma 5.8. *For all* $v \in \widetilde{\Delta}_{\tau_1\tau_2}$

$$\widetilde{\psi}_1(v) - \widetilde{\varphi}_1(v) = \widetilde{\psi}_2(v) - \widetilde{\varphi}_2(v)$$

and therefore

$$\textit{length of } \left(\mathsf{p}^{-1}(v) \cap \widetilde{\Delta}_{\tau_1}\right) = \textit{length of } \left(\mathsf{p}^{-1}(v) \cap \widetilde{\Delta}_{\tau_2}\right).$$

Proof. A computation verifies that for $\alpha, \beta, \gamma, \delta$ real numbers, we have

$$\begin{aligned}\min(\alpha+\beta, \gamma+\delta) &- \max(|\alpha-\beta|, |\gamma-\delta|) \\ &= \min(2\alpha, 2\beta, 2\gamma, 2\delta, \alpha+\beta+\gamma-\delta, \alpha+\beta-\gamma+\delta, \alpha-\beta+\gamma+\delta, -\alpha+\beta+\gamma+\delta).\end{aligned}$$

Applying the above formula to both $\min(z_a + z_b, z_c + z_d) - \max(|z_a - z_b|, |z_c - z_d|)$ and to $\min(z_a + z_d, z_b + z_c) - \max(|z_a - z_d|, |z_b - z_c|)$ yields the result. □

In particular, the above shows that, in this case of $Gr(2,m)$, the constant κ appearing in Theorem 3.4 is equal to 1. Following (4.3), we can now compute that the shift map in this case is

$$\begin{aligned}&\widetilde{\mathsf{S}}_{12} : \mathbb{R}^{2m-3} \to \mathbb{R}^{2m-3} \\ &(z_1, \ldots, z_{2m-3}) \mapsto (z_1, \ldots, z_{2m-2}, z_{2m-3} + \max(|z_a - z_d|, |z_b - z_c|) - \max(|z_a - z_b|, |z_c - z_d|)),\end{aligned}$$

and by (4.4) the flip map is

$$\begin{aligned}&\widetilde{\mathsf{F}}_{12} : \mathbb{R}^{2m-3} \to \mathbb{R}^{2m-3} \\ &(z_1, \ldots, z_{2m-3}) \mapsto (z_1, \ldots, z_{2m-2}, -z_{2m-3} + \min(z_a + z_d, z_b + z_c) + \max(|z_a - z_b|, |z_c - z_d|).\end{aligned} \tag{5.10}$$

In fact, it is not hard to see that the same formulas extend to give maps on the cones $\widetilde{P}_{\tau_1} \to \widetilde{P}_{\tau_2}$.

We can now describe the flip and shift maps on the original polytopes (respectively cones) $\Delta_{\tau_1}, \Delta_{\tau_2}$ (respectively P_{τ_1}, P_{τ_2}) by translating via the change of coordinates γ. Specifically, the flip map $\mathsf{F}_{12} : P_{\tau_1} \to P_{\tau_2}$ and the shift map $\mathsf{S}_{12} : P_{\tau_1} \to P_{\tau_2}$ are given by the formulas

$$\mathsf{F}_{12} := \gamma \circ \widetilde{\mathsf{F}}_{12} \circ \gamma^{-1} \qquad \text{and} \qquad \mathsf{S}_{12} := \gamma \circ \widetilde{\mathsf{S}}_{12} \circ \gamma^{-1} \tag{5.11}$$

such that the following diagram commutes:

$$\begin{array}{ccc} P_{\tau_1} & \overset{\mathsf{F}_{12}}{\dashrightarrow} & P_{\tau_1} \\ {\scriptstyle \gamma^{-1}}\downarrow & & \uparrow{\scriptstyle \gamma} \\ \widetilde{P}_{\tau_1} & \xrightarrow{\widetilde{\mathsf{F}}_{12}} & \widetilde{P}_{\tau_2} \end{array}$$

and an analogous diagram commutes for S_{12}.

Remark 5.9. *In [19, Proposition 3.5] the authors describe a wall-crossing formula for $\widetilde{\Delta}_\tau$. This map agrees with the shift map $\widetilde{\mathsf{S}}_{12}$.*

Example 5.10. Let $m = 4$ and τ_1 be the tree of Figure 5.2. The corresponding matrix is

$$\widetilde{M}_{\tau_1} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The inequality description of $\widetilde{\Delta}_{\tau_1}$ is

$$2 = z_1 + z_2 + z_3 + z_4,$$
$$z_1 \leq z_2 + z_5, \quad z_2 \leq z_1 + z_5, \quad z_5 \leq z_1 + z_2,$$
$$z_3 \leq z_4 + z_5, \quad z_4 \leq z_3 + z_5, \quad z_5 \leq z_3 + z_4.$$

Using the map γ we can now obtain the inequality description of Δ_{τ_1}. Specifically, since $2 = z_1 + z_2 + z_3 + z_4$ we obtain $y_1 = \frac{1}{2}(z_1 + z_2 + z_3 + z_4) = 1$; we also have $y_i = z_i$ for $i = 2, 3, 4$ and $y_5 = y_1 - z_5 = 1 - z_5$. Making appropriate substitutions in the above inequalities we obtain the inequalities for Δ_{τ_1}:

$$1 = y_1,$$
$$y_5 + 1 \leq 2y_2 + y_3 + y_4, \quad 2y_2 + y_3 + y_4 + y_5 \leq 3, \quad y_3 + y_4 \leq 1 + y_5$$
$$y_3 + y_5 \leq 1 + y_4, \quad y_4 + y_5 \leq 1 + y_3, \quad 1 \leq y_3 + y_4 + y_5.$$

Now let τ_2 be the other tree of Figure 5.2. The shift map is

$$\widetilde{\mathsf{S}}_{12} : \widetilde{\Delta}_{\tau_1} \to \widetilde{\Delta}_{\tau_2}$$
$$(z_1, \ldots, z_5) \mapsto (z_1, \ldots, z_4, z_5 + \max(|z_1 - z_4|, |z_2 - z_3|) - \max(|z_1 - z_2|, |z_3 - z_4|)),$$

and the flip map is

$$\widetilde{\mathsf{F}}_{12} : \widetilde{\Delta}_{\tau_1} \to \widetilde{\Delta}_{\tau_2}$$
$$(z_1, \ldots, z_5) \mapsto (z_1, \ldots, z_4, -z_5 + \min(z_1 + z_4, z_2 + z_3) + \max(|z_1 - z_2|, |z_3 - z_4|).$$

Similarly, one can give an explicit description for the shift and flip maps for $\Delta_{\tau_1} \to \Delta_{\tau_2}$.

Remark 5.11. *Our flip maps are related to cluster mutations in the case of $Gr(2, m)$. Recall that the homogeneous coordinate ring of the Grassmannian is a cluster algebra with the Plücker coordinates as its cluster variables [8] and this cluster structure gives rise to an atlas of complex tori on (an open dense subset inside) $Gr(2, m)$. The transition maps between adjacent tori are called (cluster) mutations. In this case the tropicalized mutation coincides with our "flip" wall-crossing. Let us exhibit this in the example above. Starting with the seed $\{p_{12}, p_{23}, p_{34}, p_{14}, p_{13}\}$ and mutating at p_{13} replaces this coordinate with*

$$\frac{p_{14}p_{23} + p_{12}p_{34}}{p_{13}} = p_{24}$$

yielding the seed $\{p_{12}, p_{23}, p_{34}, p_{14}, p_{24}\}$. The tropicalization of this Laurent polynomial with respect to the maximum convention is $-p_{13} + \max(p_{14} + p_{23}, p_{12} + p_{34})$. By identifying the variables as follows

$$z_1 = p_{12}, \qquad z_2 = p_{23}, \qquad z_3 = p_{34}, \qquad z_4 = p_{14}, \qquad z_5 = p_{13}$$

and using the identity

$$\min(z_a + z_d, z_b + z_c) + \max(|z_a - z_b|, |z_c - z_d|) = \max(z_a + z_b, z_c + z_d)$$

we obtain $\widetilde{\mathsf{F}}_{12}$. In [21] Rietsch and Williams obtain piecewise linear maps for Newton-Okounkov bodies for $Gr(2,m)$, and more generally $Gr(k,m)$, by tropicalizing cluster mutation. See also [4, 5] for related discussion.

5.4. **The algebraic wall-crossing map for $Gr(2,m)$.** In this section, we give a description of the algebraic wall-crossing map described in Section 4.2 for the case of $Gr(2,m)$, or more precisely $\mathcal{T}(I_{2,m})$. In the next section we will prove that the algebraic wall-crossing obtained below is also the restriction (to the semi-group) of the geometric "flip" map. For the definition of the algebraic wall-crossing, as explained in Section 4.2 we restrict to the case when the two adjacent maximal-dimensional prime cones of $\mathcal{T}(I_{2,m})$ lie in a certain maximal cone of the Gröbner fan. This may appear to be a restrictive condition. While not strictly logically necessary, we illustrate in the first few lemmas below that, up to the symmetry of $\mathfrak{S}_m$, this is true for any pair of adjacent maximal-dimensional prime cones of $\mathcal{T}(I_{2,m})$.

Recall that the semigroups

$$S_{\tau_1} := S(A, \nu_{M_{\tau_1}}) \qquad \text{and} \qquad S_{\tau_2} := S(A, \nu_{M_{\tau_2}})$$

are generated by the columns of M_{τ_1} and M_{τ_1}, respectively. As explained in Section 4.2, we will use an adapted basis to construct the algebraic wall-crossing map $\Theta : S_{\tau_1} \to S_{\tau_2}$. To describe Θ more concretely we need some preliminaries. By [24, §3.7] we know there exists a total order $\prec$ on $\mathbb{C}[p_I \mid I \subset [m], |I| = k]$ such that for any quadruple $\{i,j,k,\ell\}$ of indices in $[m]$ with $i<j<k<l$ we have that

$$\mathrm{in}_{\prec}(p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}) = -p_{ik}p_{jl}. \tag{5.12}$$

The next lemma is essentially [16, Second proof of $\supset$ in Theorem 4.3.5] and will be useful in what follows, so we briefly recall the idea of the argument. Note that the symmetric group $\mathfrak{S}_m$ naturally acts on the variables p_{ij} by permuting the indices.

Lemma 5.12. *Up to this $\mathfrak{S}_m$ symmetry, all the maximal cones in $\mathcal{T}(I_{2,m})$ are contained in the maximal cone of the Gröbner fan of $I_{2,m}$ corresponding to the monomial order $\prec$ above.*

Proof. Let τ be a trivalent tree with m leaves. Fix a planar embedding of the graph where the m leaves are arranged in a circle. We can act by $\mathfrak{S}_m$ to relabel the leaves so that they appear $1,2,\ldots,m$, in order, counterclockwise. We claim that, in this situation, the cone C_τ lies in the maximal cone of the Gröbner fan corresponding to the monomial order $\prec$ above. To see this, it suffices to check that for any choice of four leaves $1 \le i < j < k < \ell \le m$ of τ, the initial term of the corresponding Plücker relation $\mathrm{in}_{C_\tau}(p_{ij}p_{k\ell} - p_{ik}p_{j\ell} + p_{i\ell}p_{jk})$ contains the monomial $-p_{ik}p_{j\ell}$, since this implies that $\prec$ refines the weight order corresponding to C_τ. Recall that the cone C_τ is spanned by the tree distances of the form $\mathbb{1} - d_\varepsilon$ for interior edges ε, and also the lineality space. By definition, the lineality space does not affect the Plücker relations so it suffices to consider the interior edges. Let ε be an interior edge. Due to the counterclockwise ordering of the vertices, it is not hard to see that if ε has the property that $|\{i,j,k,\ell\}| \cap J_\varepsilon| = 2$, then we have

$$\{i,j,k,\ell\} \cap J_\varepsilon = \{a,b\} \quad \text{and} \quad \{i,j,k,\ell\} \cap J_\varepsilon^c = \{c,d\} \tag{5.13}$$

where either $\{a,b\} = \{i,j\}$ or $\{a,b\} = \{i,\ell\}$. It can then be checked that the initial term $\mathrm{in}_{\mathbb{1}-d_\varepsilon}(p_{ij}p_{k\ell} - p_{ik}p_{j\ell} + p_{i\ell}p_{jk})$ contains the monomial $-p_{ik}p_{j\ell}$ in either case. For any other internal edge ε', since τ is a tree we can see that either $|\{i,j,k,\ell\}| \cap J_{\varepsilon'}| \neq 2$ or, the decomposition $\{i,j,k,\ell\} = (\{i,j,k,\ell\} \cap J_\varepsilon) \sqcup (\{i,j,k,\ell\} \cap J_\varepsilon^c)$ is the same as that for ε in (5.13). In the former case,

$$\mathrm{in}_{\mathbb{1}-d_{\varepsilon'}}(p_{ij}p_{k\ell} - p_{ik}p_{j\ell} + p_{i\ell}p_{jk}) = p_{ij}p_{k\ell} - p_{ik}p_{j\ell} + p_{i\ell}p_{jk}$$

and in the latter case, the initial term is the same as that for ε. It follows that $\mathrm{in}_{C_\tau}(p_{ij}p_{k\ell} - p_{ik}p_{j\ell} + p_{i\ell}p_{jk}) = -p_{ac}p_{bd} + p_{ad}p_{bc}$, as desired. □

To describe the algebraic wall-crossing map, we first need to act by $\mathfrak{S}_n$ to simultaneously take two adjacent maximal-dimensional prime cones in $\mathcal{T}(I_{2,n})$ to the same maximal cone in the Gröbner fan. This is the content of the next lemma.

Lemma 5.13. *Let τ_1 and τ_2 correspond to two adjacent maximal cones in $\mathcal{T}(I_{2,n})$. Then there exists an element of $\mathfrak{S}_n$ which takes both C_{τ_1} and C_{τ_2} to the maximal cone in the Gröbner fan corresponding to $\prec$.*

Proof. Let $\varepsilon_{2m-3} \in \tau_1$ and $\varepsilon'_{2m-3} \in \tau_2$ be the edges by which the two trees differ, as in Figure 5.1. There exist $I,J,K,L \subset [m]$ such that ε_1 corresponds to the partition $I \cup J, K \cup L$ and ε_2 corresponds to the partition $I \cup L, J \cup K$. There are planar realizations of τ_1 and τ_2 with the leaves arranged in a circle so that I,J,K,L are arranged in counterclockwise order. □

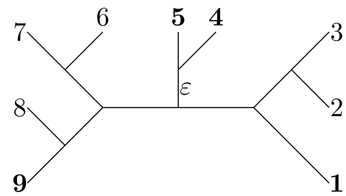

FIGURE 5.3. The initial form of the Plücker relation for $1, 4, 5, 9$ with respect to the tree above is $p_{14}p_{59} - p_{15}p_{49}$.

The above discussion shows that the assumption that two adjacent maximal-dimensional prime cones are both contained in the same maximal cone of a Gröbner fan is not very restrictive. With this in mind, we now describe the wall-crossing Θ for value semigroups, under the assumption that both C_{τ_1} and C_{τ_2} lie in the maximal Gröbner cone corresponding to the above monomial order $\prec$. By Theorem 2.5 (2), we may take the projection onto $\mathbb{C}[p_J : J \subset [m], |J| = 2]/I_{2,m}$ of the standard monomial basis for $I_{2,m}$ with respect to $\prec$ as an adapted basis $\mathbb{B}$ for $(A, \nu_{M_{\tau_k}})$. Since the Plücker relations are a Gröbner basis for $\prec$ [16, Second proof of $\supseteq$ of Theorem 4.3.5] we have

$$\mathrm{in}_\prec(I_{2,m}) = \langle p_{ik}p_{jl} \mid 1 \le i < j < k < l \le m \rangle$$

then $\mathbb{B}$ is the image of $\mathcal{S}(\prec, I_{2,m})$ under the projection $\pi : \mathbb{C}[p_I] \to \mathbb{C}[p_I]/I_{2,m}$, where

$$\mathcal{S}(\prec, I_{2,m}) = \left\{ \prod_{i<j} p_{ij}^{\alpha_{ij}} \mid \alpha_{ik}\alpha_{jl} = 0 \quad \text{for} \quad 1 \le i < j < k < l \le m \right\}. \tag{5.14}$$

As before, the maps

$$\theta_k : \mathbb{B} \to S(A, \nu_{M_{\tau_k}})$$
$$b \mapsto \nu_{M_{\tau_k}}(b)$$

are bijections for $k = 1, 2$. We then obtain the bijection $\Theta : S_{\tau_1} \to S_{\tau_2}$ defined by $\Theta = \theta_2 \circ \theta_1^{-1}$ as in Section 4.2. We work out a concrete example below.

Example 5.14. Let $m = 4$ and consider $\mathcal{T}(I_{2,4})$. Let τ_1 and τ_2 be as in Example 5.10 and let $e_{12}, \ldots, e_{34}$ (respectively $f_{12}, \ldots, f_{34}$) denote the columns of M_{τ_1} (respectively M_{τ_2}), labelled by the same Plücker indices as the columns themselves. We claim that the algebraic map is given by the concrete formula

$$\Theta(M_{\tau_1}\alpha) = M_{\tau_2} \begin{pmatrix} \alpha_{12} \\ \max(0, \alpha_{13} - \alpha_{24}) \\ \alpha_{14} + \min(\alpha_{13}, \alpha_{24}) \\ \alpha_{23} + \min(\alpha_{13}, \alpha_{24}) \\ \max(0, \alpha_{24} - \alpha_{13}) \\ \alpha_{34} \end{pmatrix}. \tag{5.15}$$

To see this, first let $M_{\tau_1}\alpha \in S(A, \nu_{M_{\tau_1}})$ for an arbitrary $\alpha \in \mathbb{Z}_{\ge 0}^{\binom{4}{2}}$. We check (5.15) by cases. Note that the only set of indices satisfying the condition $1 \le i < j < k < \ell \le 4$ is $i = 1, j = 2, k = 3, \ell = 4$. Hence if $\alpha_{13}\alpha_{24} = 0$, then by (5.14) it follows that $p^\alpha \in \mathcal{S}(\prec, I_{2,m})$ and therefore $\Theta(M_{\tau_1}\alpha) = M_{\tau_2}\alpha$, which agrees with (5.15). On the other hand, if $\alpha_{13}\alpha_{24} \neq 0$, then from the definition of Θ we must first find $\beta \in \mathbb{Z}_{\ge 0}^{\binom{4}{2}}$ such that $p^\beta \in \mathcal{S}(\prec, I_{2,m})$ and such that $\nu_{M_{\tau_1}}(\pi(p^\alpha)) = \nu_{M_{\tau_1}}(\pi(p^\beta))$. This would imply that $\Theta(M_{\tau_1}\alpha) = M_{\tau_2}\beta$. To achieve this, we can use the relation $p_{13}p_{24} = p_{14}p_{23}$ in $\mathbb{C}[p_I]/\,\mathrm{in}_{C_{\tau_1}}(I_{2,4})$ to see that

$$p_{13}^{\alpha_{13}}p_{24}^{\alpha_{24}} = \begin{cases} p_{14}^{\alpha_{13}}p_{23}^{\alpha_{13}}p_{24}^{\alpha_{24}-\alpha_{13}} & \text{if} \quad \alpha_{24} \ge \alpha_{13} \\ p_{13}^{\alpha_{13}-\alpha_{24}}p_{14}^{\alpha_{13}}p_{23}^{\alpha_{13}} & \text{if} \quad \alpha_{13} \ge \alpha_{24} \end{cases}$$

in $\mathbb{C}[p_I]/\,\mathrm{in}_{C_{\tau_1}}(I_{2,4})$. The vector β can be found by using the above substitution in p^α. Then (5.15) follows from combining the cases.

In general, the algebraic wall-crossing map is difficult to describe explicitly.

5.5. **The geometric wall-crossing map F_{12} induces the algebraic wall-crossing map for $Gr(2,m)$.** The main result of this section is the following.

Theorem 5.15. *Let $I_{2,m}$ be the Plücker ideal for $Gr(2,m)$ and let C_{τ_1} and C_{τ_2} be two maximal-dimensional prime cones in $\mathcal{T}(I_{2,m})$ which share a codimension-1 face. Assume that C_{τ_1} and C_{τ_2} are contained in the maximal cone of the Gröbner fan corresponding to $\prec$. Extend the flip geometric wall-crossing map F_{12} of (4.4) to a map $\mathsf{F}_{12} : P_{\tau_1} \to P_{\tau_2}$ where P_{τ_i} is the cone spanned by the columns of M_{τ_i}. Then the algebraic wall-crossing map $\Theta : S_{\tau_1} \to S_{\tau_2}$ is the restriction of the flip geometric wall-crossing map F_{12}.*

Our method of proof is to first show the analogous result for the matrices $\widetilde{M}_{\tau_i}$ introduced in Section 5.3. Since the $\widetilde{M}_{\tau_i}$ and M_{τ_i} are related by the change of coordinates γ of (5.7), this gives us the desired result.

First, we define a map

$$\widetilde{\Theta} := \gamma^{-1} \circ \Theta \circ \gamma. \tag{5.16}$$

By Lemma 5.4 we know that γ induces a bijection from $\widetilde{S}_{\tau_i}$ to S_{τ_i}, so it follows immediately that $\widetilde{\Theta}$ restricts to a map

$$\widetilde{\Theta} : \widetilde{S}_{\tau_1} \to \widetilde{S}_{\tau_2}.$$

Since γ takes columns of $\widetilde{M}_{\tau_i}$ to the corresponding columns of M_{τ_i} for $i = 1, 2$, it follows easily that the map $\widetilde{\Theta}$ of (5.16) behaves as follows:

$$\widetilde{\Theta}(\widetilde{M}_{\tau_1}\alpha) = \widetilde{M}_{\tau_2}\alpha$$

for α such that $p^\alpha \in \mathcal{S}(\prec, I_{2,m})$.

The following proposition is the analogue of Theorem 5.15 for the map $\widetilde{\Theta}$. Let $\widetilde{\mathsf{F}}_{12}$ be the flip map defined in Section 5.3.

Proposition 5.16. *The restriction of $\widetilde{\mathsf{F}}_{12}$ to the semigroup $\widetilde{S}_1$ is equal to $\widetilde{\Theta} : \tilde{S}_1 \to \tilde{S}_2$, i.e., for any α such that $p^\alpha \in \mathcal{S}(\prec, I_{2,m})$ we have*

$$\widetilde{\mathsf{F}}_{12}(\widetilde{M}_{\tau_1}\alpha) = \widetilde{\Theta}(\widetilde{M}_{\tau_1}\alpha).$$

The following lemma will be useful to prove the above proposition. Here we use the notation introduced in Figure 5.1.

Lemma 5.17. *Let τ_1 and τ_2 correspond to two adjacent maximal-dimensional prime cones C_{τ_1} and C_{τ_2} in $\mathcal{T}(I_{2,m})$. Assume that C_{τ_1} and C_{τ_2} both lie in the maximal cone of the Gröbner fan corresponding to $\prec$. For $\alpha \in \mathbb{Z}_{\geq 0}^{\binom{m}{2}}$,*

$$(\widetilde{\mathsf{F}}_{12}(\widetilde{M}_{\tau_1}\alpha))_{2m-3} = \alpha_{IJ} + |\alpha_{IK} - \alpha_{JL}| + \alpha_{KL}$$

where the $\alpha_{IJ}, \alpha_{IK}, \ldots$ denote the sums of the form

$$\alpha_{IJ} := \sum_{i \in I, j \in J} \alpha_{ij}$$

and similarly for the others.

Proof. Recall from (5.10) that if $(z_a)_{a=1,\ldots,2m-3}$ denote the coordinate entries of $\widetilde{M}_{\tau_1}\alpha$, then

$$(\widetilde{\mathsf{F}}_{12}(\widetilde{M}_{\tau_1}\alpha))_{2m-3} = -z_{2m-3} + \min(z_a + z_d, z_b + z_c) + \max(|z_a - z_b|, |z_c - z_d|). \tag{5.17}$$

From the formula (5.3) for the matrix entries of $\widetilde{M}_{\tau_1}$, it follows – using the notation of Figure 5.1 – that for any edge ε_h, the coordinate z_h in $\widetilde{M}_{\tau_1}\alpha$ equals the sum of the exponents α_{ij} such that the path from i to j uses ε_h. Applying this to a, b, c, d, and $2m-3$ we conclude

$$z_a = \sum \alpha_{ij} c_a^{ij} = \alpha_{IJ} + \alpha_{IK} + \alpha_{IL}, \qquad z_d = \sum \alpha_{ij} c_d^{ij} = \alpha_{IL} + \alpha_{JL} + \alpha_{KL},$$
$$z_b = \sum \alpha_{ij} c_b^{ij} = \alpha_{IJ} + \alpha_{JK} + \alpha_{JL}, \qquad z_c = \sum \alpha_{ij} c_c^{ij} = \alpha_{IK} + \alpha_{JK} + \alpha_{KL}.$$

We also have

$$z_{2m-3} = \sum \alpha_{ij} c_{2m-3}^{ij} = \alpha_{IK} + \alpha_{IL} + \alpha_{JK} + \alpha_{JL}. \tag{5.18}$$

It is straightforward to compute

$$\min(z_a + z_d, z_b + z_c) = \alpha_{IJ} + \alpha_{IK} + \alpha_{JL} + \alpha_{KL} + 2\min(\alpha_{IL}, \alpha_{JK}), \tag{5.19}$$

and

$$\max(|z_a - z_b|, |z_c - z_d|) = |\alpha_{JK} - \alpha_{IL}| + |\alpha_{JL} - \alpha_{IK}|. \tag{5.20}$$

Since

$$2\min(\alpha_{IL}, \alpha_{JK}) + |\alpha_{JK} - \alpha_{IL}| = \alpha_{IL} + \alpha_{JK},$$

then by combining (5.17), (5.18), (5.19) and (5.20) we obtain

$$(\widetilde{\mathsf{F}}_{12}(\widetilde{M}_{\tau_1}\alpha))_{2m-3} = \alpha_{IJ} + |\alpha_{JL} - \alpha_{IK}| + \alpha_{KL}$$

as desired. □

We now prove Proposition 5.16.

Proof of Proposition 5.16. We first claim that $\widetilde{\Theta}$ and $\widetilde{\mathsf{F}}_{12}$ agree on the first $2m-2$ coordinates. Indeed, note that the flip map F_{12} is the identity on the first $2m-2$ coordinates and thus so is $\widetilde{\mathsf{F}}_{12} = \gamma^{-1} \circ \mathsf{F}_{12} \circ \gamma$. The same is true of $\widetilde{\Theta}$, since the matrices $\widetilde{M}_{\tau_1}$ and $\widetilde{M}_{\tau_2}$ are identical except for the bottom rows. Thus it remains to see that $\widetilde{\Theta}$ and $\widetilde{\mathsf{F}}_{12}$ agree on the last $(2m-3)$-th coordinate.

Let $\alpha \in \mathbb{Z}_{\geq 0}^{\binom{m}{2}}$ such that $p^\alpha \in \mathcal{S}(\prec, I_{2,m})$. In the notation of Figure 5.1, where the right hand figure corresponds to the tree τ_2 and from the formula (5.3) for the entries of $\widetilde{M}_{\tau_2}$ we conclude that the $2m-3$-th coordinate of $\widetilde{\Theta}(\widetilde{M}_{\tau_1}\alpha) = \widetilde{M}_{\tau_2}\alpha$ is

$$(\widetilde{M}_{\tau_2}\alpha)_{2m-3} = \alpha_{IJ} + \alpha_{IK} + \alpha_{JL} + \alpha_{KL}. \tag{5.21}$$

We take cases. If $\alpha_{IK} \neq 0$, then since α corresponds to a standard monomial we know $\alpha_{JL} = 0$. Therefore by (5.21) and Lemma 5.17 we conclude

$$(\widetilde{\mathsf{F}}_{12}(\widetilde{M}_{\tau_1}\alpha))_{2m-3} = \alpha_{IJ} + \alpha_{IK} + \alpha_{KL} = (\widetilde{\Theta}(\widetilde{M}_{\tau_1}\alpha))_{2m-3}.$$

On the other hand, $\alpha_{IK} = 0$ then by (5.21) and Lemma 5.17 we have

$$(\widetilde{\mathsf{F}}_{12}(\widetilde{M}_{\tau_1}\alpha))_{2m-3} = \alpha_{IK} + \alpha_{JK} + \alpha_{KL} = (\widetilde{\Theta}(\widetilde{M}_{\tau_1}\alpha))_{2m-3}$$

which proves the claim. □

Proof of Theorem 5.15. By Remark 4.1 the geometric wall-crossing map F_{12} of (4.4) is a wall-crossing map for the cones which restricts to the flip geometric wall-crossing of the Newton-Okounkov bodies. Let $M_{\tau_1}\alpha \in S_{\tau_1}$. Since γ is an invertible linear map mapping, for all ij, the ij-th column of $\widetilde{M}_{\tau_1}$ to the ij-th column of M_{τ_2} then

$$\gamma(\widetilde{M}_{\tau_1}\alpha) = M_{\tau_1}\alpha \quad \text{and} \quad \gamma^{-1}(M_{\tau_1}\alpha) = \widetilde{M}_{\tau_1}\alpha. \tag{5.22}$$

Let α be such that $p^\alpha \in \mathcal{S}(\prec, I_{2,m})$. We compute

$$\begin{aligned}
\mathsf{F}_{12}(M_{\tau_1}\alpha) &= \gamma \circ \widetilde{\mathsf{F}}_{12} \circ \gamma^{-1}(M_{\tau_1}\alpha), && \text{by 5.11}\\
&= \gamma \circ \widetilde{\mathsf{F}}_{12}(\widetilde{M}_{\tau_1}\alpha), && \text{by (5.22)}\\
&= \gamma(\widetilde{\Theta}(\widetilde{M}_{\tau_1}\alpha)) && \text{by Proposition 5.16}\\
&= \gamma(\widetilde{M}_{\tau_2}\alpha) && \text{by definition of } \widetilde{\Theta}\\
&= M_{\tau_1}\alpha && \text{by (5.22)}\\
&= \Theta(M_{\tau_1}\alpha) && \text{by definition of } \widetilde{\Theta}
\end{aligned}$$

as desired. □

Appendix A. Newton-Okounkov Body Wall-crossing via Complexity-One T-Varieties

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A.1. **Preliminaries.** We continue with notation as in the main paper. In particular, we have a presentation $\mathbb{C}[x_1, \dots, x_n]/I \cong A$ of a $(d+1)$-dimensional positively graded integral domain, and two $(d+1)$-dimensional prime cones C_1 and C_2 in $\mathrm{trop}(I)$ which intersect in a codimension-one face C. In this appendix, we show how the main features of wall-crossing for the Newton-Okounkov bodies as outlined in Theorem 2.7 follow from the theory of complexity-one T-varieties.

The main algebraic objects we will consider are as follows. Let $N = \mathbb{Z}^n \cap \langle C \rangle$, and $R = \mathbb{C}[x_1, \dots, x_n]/\operatorname{in}_C(I)$. The dual lattice to N is $M = (\mathbb{Z}^n)^*/N^\perp = N^*$. Then

(1) The ring R is an M-graded integral domain of dimension $d+1$;
(2) R is finitely generated as a $\mathbb{C}$-algebra;
(3) The degree zero piece R_0 of R is $\mathbb{C}$;
(4) The set of those $v \in M$ with $R_v \neq 0$ generates all of M, which is a rank d lattice.

Rings R satisfying these four properties are exactly the coordinate rings of (potentially non-normal) affine complexity-one T-varieties with a good T-action (where T is the algebraic torus $\operatorname{Spec} \mathbb{C}[M]$).[6] We will thus call rings satisfying these four properties *good complexity-one* M-graded domains.

A.2. **Polyhedral Divisors.** We fix a lattice M and a smooth projective curve Y. Let ω be a full-dimensional cone in $M_\mathbb{R} = M \otimes \mathbb{R}$. A *polyhedral divisor* on Y with weight cone ω is a finite formal sum

$$\mathcal{D} = \sum_{P \in \mathcal{P}} \mathcal{D}_P \otimes P$$

where $\mathcal{P}$ is a finite set of points of Y and the $\mathcal{D}_P$ are piecewise linear concave functions

$$\mathcal{D}_P : \omega \to \mathbb{R}$$

with rational slopes. See [2, §2-3] for details. For any $v \in \omega \cap M$, we obtain a $\mathbb{Q}$-divisor

$$\mathcal{D}(v) := \sum_{P \in \mathcal{P}} \mathcal{D}_P(v) \cdot P$$

on Y. We say $\mathcal{D}$ is a *p-divisor* if $\deg \mathcal{D}(v) > 0$ for $v \in M$ in the interior of ω, and for every $v \in M$ in the boundary of ω, either $\deg \mathcal{D}(v) > 0$, or $\mathcal{D}(v)$ has a principal multiple.

To any p-divisor $\mathcal{D}$ as above, we may associate a *normal* good complexity-one M-graded domain [2, Theorem 3.1]:

$$R(\mathcal{D}) = \bigoplus_{v \in \omega \cap M} H^0\left(Y, \mathcal{O}_Y\left(\mathcal{D}(v)\right)\right) \cdot \chi^v.$$

Furthermore, every normal good complexity-one M-graded domain R arises in this fashion [2, Theorem 3.4]. In geometric terms, there is a bijection between equivariant isomorphism classes of normal affine varieties with good complexity-one torus action, and p-divisors on smooth projective curves modulo a natural equivalence relation, see [2].

A.3. **Newton-Okounkov Bodies.** Consider a good complexity-one M-graded domain R. We now fix a $\mathbb{Z}$-grading on R by considering a projection $\deg : M \to \mathbb{Z}$ satisfying $\deg^{-1}(0) \cap \omega = 0$. Set $\square = \omega \cap \deg^{-1}(1)$. By the discussion of §A.2, there is a p-divisor $\mathcal{D} = \sum_{P \in \mathcal{P}} \mathcal{D}_P \otimes P$ on a curve Y such that the integral closure of R is isomorphic to $R(\mathcal{D})$; we identify R with its image in $R(\mathcal{D})$. In general, R may not be equal to $R(\mathcal{D})$, as R may not be integrally closed.

Fix a total ordering on M. For any point $Q \in Y$, we obtain a valuation

$$\begin{aligned} \mathrm{val}_Q : R(\mathcal{D}) \setminus \{0\} &\to M \times \mathbb{Z} \\ f &\mapsto \min_{v : f_v \neq 0} (v, \mathrm{ord}_Q(f_v)) \end{aligned}$$

where $f = \sum_{v \in M} f_v$ is the decomposition of f into homogeneous pieces, $\mathrm{ord}_Q(f)$ is the order of vanishing of f at Q, and we take the lexicographic ordering on $M \times \mathbb{Z}$. The valuation val_Q restricts to a valuation on R.

[6]Recall that a torus action on an affine variety is *good* if the only invariant regular functions are constants.

Lemma A.1. *The Newton-Okounkov body* $\Delta(R, \mathrm{val}_Q)$ *is equal to*

$$(\dagger) \qquad \Delta(R, \mathrm{val}_Q) = \left\{ (x, y) \in \square \times \mathbb{R} \mid -\mathcal{D}_Q(x) \leq y \leq \sum_{\substack{P \in \mathcal{P} \\ P \neq Q}} \mathcal{D}_P(x) \right\}.$$

For the case $Q \notin \mathcal{P}$*, we use the convention that* $\mathcal{D}_Q = 0$.

Proof. We observe that $\Delta(R, \mathrm{val}_Q) = \Delta(R(\mathcal{D}), \mathrm{val}_Q)$ by e.g. [13, Proposition 2.18], so we reduce to the case $R = R(\mathcal{D})$. In the case $Y = \mathbb{P}^1$, we may now apply [12, Theorem 5.10]. For arbitrary Y, we may apply Petersen's description of Newton-Okounkov bodies for complexity-one T-varieties [20, Proposition 3.13]. For the sake of the reader, we reproduce Petersen's argument below.

Consider any homogeneous element $f \cdot \chi^v \in R(\mathcal{D})$ of degree v. Since f is in $H^0(Y, \mathcal{O}(\mathcal{D}(v)))$, we obtain $0 \leq \mathrm{ord}_Q f + \mathcal{D}_Q(v) \leq \sum_{P \in \mathcal{P}} \mathcal{D}_P(v)$. Since the image of val_Q is determined by valuations of homogeneous elements, it follows that $\Delta(R, \mathrm{val}_Q)$ is contained in the expression on the right hand side of (†).

Conversely, choose any rational point x in the relative interior of $\square$ and $y \in \mathbb{Q}$ with (x, y) contained in the right hand side of (†). There exists a natural number λ such that $v = \lambda \cdot x \in M$ and $\mathcal{D}_P(v)$ is integral for all $P \in \mathcal{P}$. Since x is in the relative interior of $\square$, the $\mathbb{Z}$-divisor $\mathcal{D}(v)$ has positive degree. By the theorem of Riemann-Roch, there thus exists a sequence of sections $s_i \in H^0(Y, \mathcal{O}(\mathcal{D}(i \cdot v)))$ such that

$$\lim_{i \to \infty} \frac{\mathrm{ord}_Q(s_i)}{i \cdot \lambda} = y.$$

Since $\Delta(R, \mathrm{val}_Q)$ is a closed set, this implies that (x, y), and thus the entire right hand side of (†), is contained in $\Delta(R, \mathrm{val}_Q)$. ∎

A.4. **Wall-Crossing.** We now take R to be as in §A.1, the degeneration of A corresponding to the cone $C = C_1 \cap C_2$. Let $u_1, \ldots, u_d \in \mathbb{Z}^n$ be elements of the relative interior of C which form a lattice basis for N. We assume that $u_1 \in \mathbb{Z}^n$ is the primitive vector giving the $\mathbb{Z}$-grading of the variables x_i; in the standard graded case it is just $(1, \ldots, 1)$. Likewise, for $i = 1, 2$ let $w_i \in \mathbb{Z}^n$ be in the relative interior of C_i such that $u_1, \ldots, u_d, w_i$ form a lattice basis for $\langle C_i \rangle \cap \mathbb{Z}^n$. As in §2, the collection of vectors $u_1, \ldots, u_d, w_i$ gives rise to valuations on both A and R, both of which we denote by val_i.

Theorem A.2. *There exists a rational polytope* $\Delta \subset \{1\} \times \mathbb{R}^{d-1} \subset \mathbb{R}^d$*, and piecewise affine-linear concave functions* Ψ_0, Ψ_1, Ψ_2 *with rational slopes and translation from* Δ *to* $\mathbb{R}$ *satisfying* $\Psi_0 + \Psi_1 + \Psi_2 \geq 0$*, such that*

$$\Delta(A, \mathrm{val}_1) = \{(x, y) \in \Delta \times \mathbb{R} \mid -\Psi_1(x) \leq y \leq \Psi_2(x) + \Psi_0(x)\};$$
$$\Delta(A, \mathrm{val}_2) = \{(x, y) \in \Delta \times \mathbb{R} \mid -\Psi_2(x) \leq y \leq \Psi_1(x) + \Psi_0(x)\}.$$

In particular, under the projection from $\mathbb{R}^d \times \mathbb{R}$ *to* $\mathbb{R}^d$*,* $\Delta(A, \mathrm{val}_1)$ *and* $\Delta(A, \mathrm{val}_2)$ *have the same image* Δ*, with fibers of equal Euclidean lengths.*

Proof. By construction, $\Delta(A, \mathrm{val}_i) = \Delta(R, \mathrm{val}_i)$; we will show that $\Delta(R, \mathrm{val}_i)$ has the desired form. As in §A.3, we identify R with a subring of $R(\mathcal{D})$ for some p-divisor $\mathcal{D}$ on a curve Y. The vectors $u_1, \ldots, u_d$ give an isomorphism $\phi : M \to \mathbb{Z}^d$ by sending $v \in M$ to $(\langle u_1, v \rangle, \ldots, \langle u_d, v \rangle)$. A straightforward adaptation of the arguments of [12, Proposition 5.1] from the case $Y = \mathbb{P}^1$ to arbitrary Y shows that there exists a point $Q_i \in Y$, constant $c_i \in \mathbb{N}$, and linear map $\gamma_i : M \to \mathbb{Z}$ such that the valuation $\mathrm{val}_i : R \setminus \{0\} \to \mathbb{Z}^d \times \mathbb{Z}$ has the form

$$\mathrm{val}_i(f) = \min_{v : f_v \neq 0} (\phi(v), c_i \cdot \mathrm{ord}_{Q_i}(f_v) + \gamma_i(v))$$

for $f = \sum_{v \in M} f_v$. Since $u_1, \ldots, u_d, w_i$ is a basis for $\langle C_i \rangle \cap \mathbb{Z}^n$, the group generated by $\mathrm{val}_i(R \setminus \{0\})$ must be all of $\mathbb{Z}^d \times \mathbb{Z}^1$. This implies that $c_i = 1$.

Both valuations val_i and val_{Q_i} are determined by their behavior on M-homogeneous elements. Together with Lemma A.1, this implies that $\Delta(R, \mathrm{val}_i)$ consists of those $(x, y) \in \phi(\square) \times \mathbb{R}$ satisfying

$$\gamma_i(\phi^{-1}(x)) - \mathcal{D}_{Q_i}(\phi^{-1}(x)) \leq y \leq \gamma_i(\phi^{-1}(x)) + \sum_{\substack{P \in \mathcal{P} \\ P \neq Q_i}} \mathcal{D}_P(\phi^{-1}(x)).$$

By abuse of notation, we are using ϕ and γ_i to also denote their linear extensions to $M_{\mathbb{R}}$. As in Lemma A.1, $\mathcal{D}_Q = 0$ for $Q \notin \mathcal{P}$. The map $\deg : M \to \mathbb{Z}$ giving a $\mathbb{Z}$-grading on R is the map induced by $u_1 \in N$.

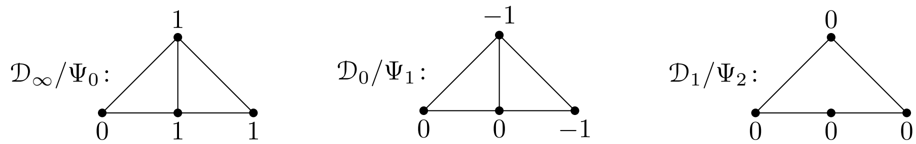

FIGURE A.1. Piecewise linear functions for wall-crossing

We set

$$\Delta = \phi(\square), \quad \Psi_0 = \gamma_1 \circ \phi^{-1} + \gamma_2 \circ \phi^{-1} + \sum_{\substack{P \in \mathcal{P} \\ P \neq Q_1, Q_2}} \mathcal{D}_P \circ \phi^{-1};$$

$$\Psi_i = \mathcal{D}_{Q_i} \circ \phi^{-1} - \gamma_i \circ \phi^{-1} \qquad i = 1, 2$$

to obtain the main claim of the theorem. For the claim regarding fiber lengths, we observe that for both Newton-Okounkov bodies, the length of the fiber over $x \in \Delta$ is exactly $\Psi_0(x) + \Psi_1(x) + \Psi_2(x)$. □

A.5. **Example.** Consider the ideal $I = \langle x_1x_2 - x_3x_4 - x_4^2 - x_5^2 \rangle \subset \mathbb{C}[x_1, \ldots, x_5]$. The tropicalization $\mathrm{trop}(I)$ is the product of its 2-dimensional lineality space with a cone over the complete graph K_4 on four vertices. The initial ideals $I_1 = \langle x_1x_2 - x_3x_4 \rangle$ and $I_2 = \langle x_1x_2 - x_4^2 \rangle$ correspond to prime cones C_1 and C_2; for $C = C_1 \cap C_2$ we have $\mathrm{in}_C(I) = \langle x_1x_2 - x_3x_4 - x_4^2 \rangle$.

We may take the elements u_1, u_2, u_3, w_1, w_2 to be the rows of the matrix

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

Then $\Delta(A, \mathrm{val}_1)$ and $\Delta(A, \mathrm{val}_2)$ are the convex hulls of the columns of this matrix, after removing the fifth and fourth rows, respectively.

We now view this example from the perspective of T-varieties. The lattice N is the rank three lattice generated by the first three rows of the above matrix. This gives an identification of N with $\mathbb{Z}^3$, and we obtain an induced identification of M with $\mathbb{Z}^3$. Under this identification, the image of x_i in $R = \mathbb{C}[x_1, \ldots, x_5]/\mathrm{in}_C(I)$ is homogeneous of degree equal to the first three entries in the ith column of the above matrix.

Let ω be the cone in $\mathbb{R}^3$ generated by $(1, 1, 0)$, $(1, -1, 0)$, and $(1, 0, 1)$. The convex hull Δ of these three vectors is the affine slice of ω on which the first coordinate is equal to 1. We consider the p-divisor $\mathcal{D} = \mathcal{D}_\infty \otimes \{\infty\} + \mathcal{D}_0 \otimes \{0\} + \mathcal{D}_1 \otimes \{1\}$ on $Y = \mathbb{P}^1$, where the piecewise-linear functions $\mathcal{D}_P : \omega \to \mathbb{R}$ induce subdivisions of Δ, and has values on it vertices exactly as pictured in Figure A.1. For the figure, we have projected Δ to the second and third coordinates.

Let $y \in \mathbb{C}(\mathbb{P}^1)$ be such that $\mathrm{div}(y) = \{0\} - \{\infty\}$, that is, y is a rational function vanishing only at the point 0, with a single pole at ∞. We obtain that

$$R(\mathcal{D}) = \mathbb{C}[y\chi^{(1,1,0)}, \chi^{(1,0,0)}, y\chi^{(1,0,0)}, \chi^{(1,-1,0)}, y\chi^{(1,0,1)}].$$

Sending

$$x_1 \mapsto y\chi^{(1,1,0)}, \ x_2 \mapsto \chi^{(1,-1,0)}, \ x_3 \mapsto (1-y)\chi^{(1,0,0)}, \ x_4 \mapsto y\chi^{(1,0,0)}, \ x_5 \mapsto y\chi^{(1,0,1)}$$

induces an isomorphism of R with $R(\mathcal{D})$. This p-divisor $\mathcal{D}$ for R could have been obtained with the method of [2, §11].

Under this identification of R with $R(\mathcal{D})$, the valuation val_1 is just the valuation val_Q for $Q = 0 \in \mathbb{P}^1$; likewise $\mathrm{val}_2 = \mathrm{val}_Q$ for $Q = 1 \in \mathbb{P}^1$. Hence, to obtain Ψ_i as in Theorem A.2, we take Ψ_0, Ψ_1, Ψ_2 to respectively be the restrictions of $\mathcal{D}_\infty$, $\mathcal{D}_0$, and $\mathcal{D}_1$ to Δ. See Figure A.1. One immediately checks that the description of $\Delta(A, \mathrm{val}_i)$ from Theorem A.2 holds.

A.6. **Dual Description and Mutations.** We continue with notation as in §A.4. In particular, Ψ_0, Ψ_1, Ψ_2 and Δ are as in Theorem A.2. Define polyhedra

$$\nabla_i = \{u \in \mathbb{R}^d \mid \langle u, x \rangle \geq \Psi_i(x) \text{ for all } x \in \Delta\}.$$

Using Minkowski addition and the inclusion $\mathbb{R}^d \hookrightarrow \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{d+1}$ sending u to $(u, 0)$, we further define

$$\sigma_1 = \mathrm{Cone}\big(\nabla_1 + e, \nabla_2 + \nabla_0 - e\big) \subset \mathbb{R}^{d+1}$$
$$\sigma_2 = \mathrm{Cone}\big(\nabla_2 + e, \nabla_1 + \nabla_0 - e\big) \subset \mathbb{R}^{d+1}.$$

Here e denotes the $(d+1)$st standard basis vector of $\mathbb{R}^{d+1}$. It is a straightforward exercise in convexity to see that we recover $\Delta(R, \mathrm{val}_i)$ by intersecting the dual cone $\sigma_i^\vee \subset \mathbb{R}^{d+1}$ with $\{1\} \times \mathbb{R}^d$.

Consider a vector η in the interior of Δ with $\Psi_1(\eta), \Psi_2(\eta) > 0$ and $\Psi_0(\eta) = 0$. We obtain polytopes

$$D_i = \{u \in \sigma_i \mid \langle u, (\eta, 0)\rangle = 1\}$$

for $i = 1, 2$, from which we can recover σ_i, and thus $\Delta(R, \mathrm{val}_i)$.

Remark A.3. *In general, such η may not exist. Nonetheless, for any η in the interior of Δ, there exist rational numbers c_0, c_1, c_2 with $c_0 + c_1 + c_2 = 0$ such that the $\Psi_i' = \Psi_i + c_i$ satisfy the desired condition. The polytopes resulting from the Ψ_i' as described in Theorem A.2 will be rational translates of $\Delta(A, \mathrm{val}_1)$ and $\Delta(A, \mathrm{val}_2)$ in the direction of the final coordinate.*

On the other hand, the choice of η is far from unique. However, if $\Delta(A, \mathrm{val}_1)$ (or $\Delta(A, \mathrm{val}_2)$) has a unique interior lattice point (for example, $\Delta(A, \mathrm{val}_1)$ is reflexive), a natural choice for η is the projection to M of this lattice point.

The transition from D_1 to D_2 may be viewed as a generalization of the *combinatorial mutations* considered in [1], as we now explain. To make this connection, we will assume that the point $(\eta, 0) \in \Delta \times \mathbb{R}$ is contained in every facet of the graph of Ψ_0. This is equivalent to requiring that $\langle u, \eta\rangle = 0$ for each vertex u of ∇_0.

For $i = 1, 2$ we define

$$h_i = \max_{u \in \nabla_i} \left(\frac{1}{\langle u, \eta\rangle}\right).$$

Let λ be the smallest natural number such that $\lambda/\langle u, \eta\rangle \in \mathbb{N}$ for all vertices u of $\nabla_0, \nabla_1, \nabla_2$. For any integer ℓ, let $H_\ell = \{u \in \mathbb{R}^d \mid \langle u, \eta\rangle = \ell/\lambda\}$. Set $\tau = (\mathrm{Cone}\Delta)^\vee \cap H_\lambda$ and $F = \nabla_0 \cap H_0$. Then we can rewrite D_1 as

$$D_1 = \mathrm{conv}\left(\tau, \bigcup_{\ell=1}^{\lambda\cdot h_1} \frac{\lambda}{\ell}\left[(\nabla_1 \cap H_\ell) + e\right], \bigcup_{\ell=1}^{\lambda\cdot h_2} \frac{\lambda}{\ell}\left[(\nabla_2 \cap H_\ell) + F - e\right]\right)$$

and D_2 as

$$D_2 = \mathrm{conv}\left(\tau, \bigcup_{\ell=1}^{\lambda\cdot h_2} \frac{\lambda}{\ell}\left[(\nabla_2 \cap H_\ell) + e\right], \bigcup_{\ell=1}^{\lambda\cdot h_1} \frac{\lambda}{\ell}\left[(\nabla_1 \cap H_\ell) + F - e\right]\right).$$

Comparing with [1, Definition 5], we see that up to mirroring the final coordinate, this is a combinatorial mutation, except that we have relaxed the integrality constraints from loc. cit.

A.7. **Mutation Example.** To illustrate the connection to mutations, we present a second example. We keep notation from §A.4 and §A.6. Consider the ideal I of $\mathbb{C}[x_1, \ldots, x_8]$ generated by

$$x_4 + x_5 - x_6, \qquad x_3 - x_6 - x_8, \qquad x_2 - x_5 + x_6 - x_8$$
$$x_1x_7 - x_5x_8, \qquad x_5x_6 - x_6^2 + x_5x_8$$

which is homogeneous with respect to the standard grading. Its tropicalization has three maximal cones, all of which are prime. These three cones intersect in C, the lineality space of $\mathrm{trop}(I)$. The respective initial ideals are

$$I_1 = \langle x_4 - x_6, x_3 - x_8, x_2 - x_8, x_1x_7 - x_5x_8, x_6^2 - x_5x_8\rangle$$
$$I_2 = \langle x_5 - x_6, x_4 - x_8, x_3 - x_6, x_1x_7 - x_6x_8, x_2x_6 - x_8^2\rangle$$
$$I_3 = \langle x_6 + x_8, x_4 + x_5, x_2 - x_5, x_1x_7 - x_5x_8, x_3x_5 - x_8^2\rangle.$$

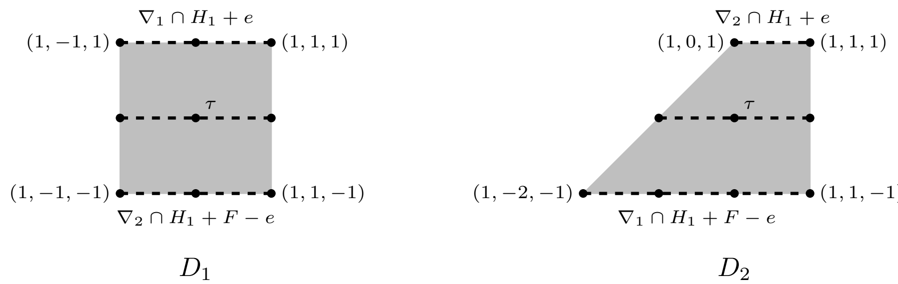

FIGURE A.2. A combinatorial mutation

All three are prime ideals; we focus on the ideals I_1 and I_2. We may take the elements u_1, u_2, w_1, w_2 to be the rows of the matrix

$$\left(\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & -1 & -1 & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 1 & -1 & 0 & -1 & -1 & -1 & 0 \end{array}\right).$$

Then $\Delta(A, \mathrm{val}_1)$ and $\Delta(A, \mathrm{val}_2)$ are the convex hulls of the columns of this matrix, after removing the fourth and third rows, respectively.

In this example, we may identify M with $\mathbb{Z}^2$ via the first two rows of the above matrix. The polytope Δ is exactly the convex hull of $(1,1)$ and $(1,-1)$. The functions Ψ_i from Theorem A.2 are as follows:

$$\Psi_0(x,1) = \begin{cases} 0 & x \leq 0 \\ -x & x \geq 0 \end{cases}$$

$$\Psi_1(x,1) = \begin{cases} x+1 & x \leq 0 \\ -x+1 & x \geq 0 \end{cases}$$

$$\Psi_2(x,1) = \begin{cases} x+1 & x \leq 0 \\ 1 & x \geq 0 \end{cases}.$$

This gives rise to

$$\nabla_0 = \mathrm{conv}\{(0,0),(0,-1)\} + \mathrm{Cone}\{(1,1),(1,-1)\}$$
$$\nabla_1 = \mathrm{conv}\{(1,1),(1,-1)\} + \mathrm{Cone}\{(1,1),(1,-1)\}$$
$$\nabla_2 = \mathrm{conv}\{(1,1),(1,0)\} + \mathrm{Cone}\{(1,1),(1,-1)\}$$

and cones σ_1 and σ_2 generated respectively by the columns of the matrices

$$(\dagger\dagger) \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -2 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

The natural choice of η in this example (see Remark A.3) is $\eta = (1,0)$. With this choice of η, we obtain that D_1 and D_2 are respectively the convex hulls of the columns of the matrices in (††). We also obtain $h_1 = h_2 = \lambda = 1$. Furthermore,

$$\tau = \mathrm{conv}\{(1,-1),(1,1)\} \qquad F = \mathrm{conv}\{(0,0),(0,-1)\}.$$

Considering Figure A.2, we see that D_1 and D_2 are exactly as described at the end of §A.6.

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