TRACE MINMAX FUNCTIONS AND THE RADICAL LAGUERRE-PÓLYA CLASS

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Abstract. We classify functions $f : (a, b) \to \mathbb{R}$ which satisfy the inequality

$$\operatorname{tr} f(A) + f(C) \geq \operatorname{tr} f(B) + f(D)$$

when $A \leq B \leq C$ are self-adjoint matrices, $D = A + C - B$, the so-called *trace minmax functions.* (Here $A \leq B$ if $B - A$ is positive semidefinite, and f is evaluated via the functional calculus.) A function is trace minmax if and only if its derivative analytically continues to a self map of the upper half plane. The negative exponential of a trace minmax function $g = e^{-f}$ satisfies the inequality

$$\det g(A) \det g(C) \leq \det g(B) \det g(D)$$

for A, B, C, D as above. We call such functions *determinant isoperimetric*. We show that determinant isoperimetric functions are in the "radical" of the the Laguerre-Pólya class. We derive an integral representation for such functions which is essentially a continuous version of the Hadamard factorization for functions in the the Laguerre-Pólya class. We apply our results to give some equivalent formulations of the Riemann hypothesis.

1. Introduction

Let $E \subseteq \mathbb{R}$. Let $f : E \to \mathbb{R}$ be a function. Let X be a self-adjoint matrix of size n with spectrum in E. We now briefly recall how to define $f(X)$ via the **matrix functional calculus**. Let X be diagonalized a unitary matrix U. That is,

$$X = U^* \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U.$$

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We define

$$f(X) = U^* \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} U.$$

Therefore, for each $n \in \mathbb{N}$, the function f induces a function on n by n self-adjoint matrices with spectrum in E. Moreover, one can formulate familiar function theoretic notions, such as convexity and monotonicity, in this context.

Given two self-adjoint matrices A and B we say $A \leq B$ if $B - A$ is positive semidefinite. (This is sometimes called the **Löwner order**.)

Say a function is **trace monotone** if $A \leq B$ implies $\operatorname{tr} f(A) \leq \operatorname{tr} f(B)$. If we list the eigenvalues of A as

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n,$$

and those for B as

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n,$$

one can show, for example using the Weyl inequalities[22], that $\mu_i \leq \lambda_i$. Noting that $\operatorname{tr} f(A) = \sum f(\mu_i)$ and $\operatorname{tr} f(B) = \sum f(\lambda_i)$, we see that *$f$ is trace monotone if and only if f is monotone.*

Similarly, we say a function is **trace convex** if $\operatorname{tr} f\left(\frac{A+B}{2}\right) \leq \operatorname{tr} \frac{f(A)+f(B)}{2}$. As happened in the case of monotonicity, *a function f is trace convex if and only if f is convex* [8, 13]. In multivariable settings, the theory of joint trace convexity depends intensely on the expression being analyzed [9, 2, 3].

Say a function is **matrix monotone** if $A \leq B$ implies $f(A) \leq f(B)$. Let Π denote the upper half plane in $\mathbb{C}$. Löwner's theorem states [11, 1] that a function $f : (a, b) \to \mathbb{R}$ is matrix monotone if and only if f analytically continues to Π and $f : \Pi \cup (a, b) \to \overline{\Pi}$. The Nevanlinna representation [18, 16] then says that

$$f(z) = c + dz + \int \frac{1 + tz}{t - z} \mathrm{d}\mu(t)$$

for some $c \in \mathbb{R}, d \in \mathbb{R}^+$ and positive Borel measure μ with support contained in $\mathbb{R} \setminus (a, b)$.

Say a function is **matrix convex** if $f\left(\frac{A+B}{2}\right) \leq \frac{f(A)+f(B)}{2}$. The Kraus theorem states [15, 1] that a matrix convex function $f : (a, b) \to \mathbb{R}$ analytically continues to the upper half plane and possesses an integral representations similar to, but not the same as, the Nevanlinna representation.

In general, the current theory of tracial inequalities is real analytic and the theory of matrix inequalities is complex analytic. We give a class of trace functions that have nice complex analytic properties, which contrasts to existing literature [9, 8, 2, 3, 13].

1.1. **Trace minmax functions.** Say a function f is **trace minmax** if

$$\operatorname{tr} f(A) + \operatorname{tr} f(C) \geq \operatorname{tr} f(B) + \operatorname{tr} f(D)$$

whenever $A \leq B \leq C$ are like-sized matrices with spectrum in the domain of f and $D = A + C - B$. We use the term "minmax" because when $A \leq C$, we can increase $\operatorname{tr} f(A) + \operatorname{tr} f(C)$ by increasing their difference, where the term minmax is taken from the naïve practice in tabletop gaming of maximizing certain statistics of a player character at the expense of others to make them apparently more powerful.

Theorem 1.1. *Let $f : (a, b) \to \mathbb{R}$. The following are equivalent:*

(1) f is trace minmax,
(2) f' is matrix monotone on (a, b),
(3) f analytically continues to the upper half plane Π and $f' : \Pi \cup (a, b) \to \overline{\Pi}$.
(4) For each $c \in (a, b)$, there exist unique $\alpha, \beta \in \mathbb{R}$ and a unique finite measure μ on $[\frac{1}{a-c}, \frac{1}{b-c}]$ such that

$$f(z) = \alpha + \beta z + \int_{[\frac{1}{a-c}, \frac{1}{b-c}]} \frac{-\log(1 - t(z - c)) - t(z - c)}{t^2} \mathrm{d}\mu.$$

Here we interpret $\frac{-\log(1-t(z-c))-t(z-c)}{t^2}|_{t=0} = z^2$.

Theorem 1.1 is proven in Section 6 based on results gathered from Section 2 through Section 5.

Somewhat surprisingly, trace minmax functions are also matrix convex, for the sole reason that $\log x$ is matrix concave on $(0, \infty)$. (See e.g. [1, Exercise V.2.11] which states that x^r is matrix convex for $-1 \leq r \leq 0$ and therefore $\log x = \lim_{r \to 0^-} \frac{x^r - 1}{r}$ is matrix concave.) That is, by Theorem 1.1, a trace minmax function on (a, b) is a convex combination of matrix convex functions on (a, b) of the form $\alpha, \beta z$ and positive multiples of $-\log(1 - t(z - c)) - t(z - c)$.

Corollary 1.2. *If $f : (a,b) \to \mathbb{R}$ is trace minmax, then f is matrix convex.*

1.2. **The radical Laguerre-Pólya class.** Recall that

$$\det e^A = e^{\operatorname{tr} A}. \tag{1.1}$$

That is, if the eigenvalues of A are $\mu_1, \ldots, \mu_n$, then the eigenvalues of e^A are $e^\mu_1, \ldots, e^\mu_n$, and therefore,

$$\det e^A = \prod e^{\mu_i} = e^{\sum \mu_i} = e^{\operatorname{tr} A}.$$

We say $f : (a,b) \to \mathbb{R}^{\geq 0}$ is **determinant isoperimetric** whenever

$$\det f(A) \det f(C) \leq \det f(B) \det f(D)$$

for $A \leq B \leq C$ with spectrum in (a,b) and $D = A + C - B$. We use the term "isoperimetric" because when $A \leq C$, we can increase the quantity $\det f(A) \det f(C)$ by decreasing the difference between A and C. Note that f is trace minmax if and only if e^{-f} is determinant isoperimetric. That is, if f satisfies the inequality

$$\operatorname{tr} f(A) + \operatorname{tr} f(C) \geq \operatorname{tr} f(B) + \operatorname{tr} f(D)$$

then

$$\det e^{-f(A)} \det e^{-f(C)} \leq \det e^{-f(B)} \det e^{-f(D)}$$

by Equation (1.1) coupled with monotonicity of the exponential function.

Theorem 1.1 implies that $-\log(1-tx)$, x^2 and $\pm x$ are trace minmax on intervals containing zero where they are well-defined. Specifically, for $-\log(1-tx)$ one chooses the measure to be a point mass at t with weight t^2, α to be 0, c to be 0 and β to be t. For z^2 one chooses the measure to be a point mass at zero with weight one, with constants α, β, c chosen to be 0. For $\pm x$, we choose the measure and α to be 0 and β to be ± 1. As these arise from point masses, these are in some sense the most "extreme" trace minmax functions. (This can be made formal in terms of Choquet theory.) Therefore, $1 - tx$, e^{-x^2}, $e^{\pm x}$ and constant functions are determinant isoperimetric. Thus, we obtain the following system of inequalities.

Corollary 1.3. *Let $A, B, C \in M_n(\mathbb{C})$ such that $A \leq B \leq C$. Let $D = A + C - B$. The following are true:*

(1) $\det e^A \det e^C = \det e^B \det e^D$,
(2) $\det e^{B^2} \det e^{D^2} \leq \det e^{A^2} \det e^{C^2}$, *and thus,*

$$\|B\|_F^2 + \|D\|_F^2 \leq \|A\|_F^2 + \|C\|_F^2,$$

(3) for all $t \in \left(-\frac{1}{\|A\|}, \frac{1}{\|C\|}\right)$,

$$\det(1-tA)\det(1-tC) \leq \det(1-tB)\det(1-tD).$$

In principle, these generate (under the operations of products, n-th roots, and taking limits) all inequalities of the form

$$\prod f(\alpha_i) \prod f(\gamma_i) \leq \prod f(\beta_i) \prod f(\delta_i)$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are the eigenvalues of A, B, C, D respectively, where $A \leq B \leq C$ and $D = A+C-B$. One wonders if there is a classification of all eigenvalue inequalities satisfied by D such that $D = A + C - B$ where $A \leq B \leq C$ along the lines of Horn's conjecture [10] and the Knutson-Tao theorem [14].

The function $-\log x$ is trace minmax on $(0, \infty)$, and therefore x is determinant isoperimetric there, yielding a more memorable inequality along the lines of the characteristic polynomials inequality in item 3 in Corollary 1.3.

Corollary 1.4 (Isoperimetric inequality)**.** *Let* $A, B, C \in M_n(\mathbb{C})$ *such that* $0 \leq A \leq B \leq C$. *Let* $D = A + C - B$. *Then,*

$$\det A \det C \leq \det B \det D.$$

The set of determinant isoperimetric functions is closed under multiplication and pointwise convergent limits. Moreover, as $1 - tx$, e^{-x^2}, $e^{\pm x}$ and constant functions are determinant isoperimetric, we see that any Hadamard product of the form

$$f(x) = x^k e^{-a-bx-cx^2} \prod (1 - x/\rho_i) e^{x/\rho_i} \tag{1.2}$$

where $b \in \mathbb{R}$, $c \geq 0$ is determinant isoperimetric on open intervals in $\mathbb{R}$ where f takes nonnegative values. The **Laguerre-Pólya class** is the set of entire functions which are the locally uniform limits of real-rooted polynomials. Laguerre-Pólya class functions are important in various contexts, [4, 5, 21, 20, 12]. The Laguerre-Pólya class is exactly the set of functions of the form (1.2) and the zero function.

Define the **radical Laguerre-Pólya class of** (a, b) to be the set of functions on (a, b) which are the pointwise limits of real n-th roots of functions in the Laguerre-Pólya class which are on nonnegative (a, b). Note that every function in the Laguerre-Pólya class which is nonnegative on (a, b) has $f|_{(a,b)}$ in the radical Laguerre-Pólya class of (a, b). Every function in the radical Laguerre-Pólya class which does not vanish on (a, b) and is positive there is determinant isoperimetric, essentially by Equation (1.2).

For example, assume $a = 1, b = -1$. Assume f^n is in the Laguerre-Pólya class and does not vanish on (a, b). Equation (1.2) implies

$$f(x)^n = e^{-n\alpha - n\beta x - n\gamma x^2} \prod (1 - x/\rho_i) e^{x/\rho_i}$$

for some sequence of ρ_i not in the the interval $(-1, 1)$, and therefore

$$-\log f(x) = \alpha + \beta x + \gamma x^2 + \frac{1}{n} \sum - \log(1 - x/\rho_i) - x/\rho_i.$$

We can further rewrite the formula in a contrived way into the language of integration as

$$-\log f(x) = \alpha + \beta x + \int \frac{-\log(1 - tx) - tx}{t^2} \mathrm{d}\mu$$

where $\mu = \gamma\delta_0 + \sum \frac{1}{n\rho_i^2} \delta_{1/\rho_i}$ which of the form in Theorem 1.1. So, $-\log f$ is trace minmax, and, therefore, f is determinant isoperimetric. Moreover, by varying the collection of zeros ρ_i and n we can weakly approximate any finite measure μ and work the construction backwards.

So, evidently, negative exponentials of trace minmax functions are exactly nonvanishing members the radical Laguerre-Pólya class of (a, b) which are, in turn, determinant isoperimetric.

Theorem 1.5. *Let $f : (a, b) \to \mathbb{R}$. The following are equivalent:*

(1) f is trace minmax,
(2) e^{-f} is determinant isoperimetric,
(3) e^{-f} is in the radical Laguerre-Pólya class.

2. Preliminaries

2.1. **Derivatives in the functional calculus.** We adopt the following notation for derivatives taken in the functional calculus,

$$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t},$$

$$D^2 f(X)[H, K] = \lim_{t} \frac{Df(X + tK)[H] - Df(X)[H]}{t},$$

where X, H, K are like-sized self-adjoint matrices.

Lemma 2.1. *If f is analytic, trace minmaxity is equivalent to saying that $D^2 f(X)[H,K] \geq 0$ whenever $H, K \geq 0$.*

Proof. First, suppose f is trace minmax. Let X be a self-adjoint matrix and let $H, K \geq 0$. Note $X \leq X + tH \leq X + tH + sK$. So, $f(X + tH + sK) + f(X) \leq f(X + sK) + f(X + tH)$. Therefore,

$$\frac{f(X + tH + sK) + f(X) - f(X + sK) - f(X + tH)}{st} \geq 0.$$

Taking the limit as $t \to 0$, we see that

$$\frac{Df(X + sK)[H] - Df(X)[H]}{s} \geq 0.$$

Now taking $s \to 0$, $D^2 f(X)[H,K] \geq 0$.

To see the converse, let $A \leq B \leq C$. Let $H = B - A, K = C - B$. Now, $Df(A + tH + sK)[H,K] \geq 0$. Next,

$$\begin{aligned} 0 &\leq \int_0^1 Df(A + tH + sK)[H,K] \mathrm{d}t \\ &= Df(B + sK)[K] - Df(A + sK)[K]. \end{aligned}$$

Finally,

$$\begin{aligned} 0 &\leq \int_0^1 Df(B + sK)[K] - Df(A + sK)[K] \mathrm{d}s \\ &= f(A) + f(C) - f(B) - f(A + C - B). \end{aligned}$$

□

2.2. **Nevanlinna's solutions to moment problems.** In 1922, Nevanlinna considered the question of when a sequence ρ_n is a sequence of moments for some finite positive Borel measure. The problem is intimately connected to the theory of self maps of the upper half plane.

Theorem 2.2 ([18])**.** *Let ρ_n be a sequence of real numbers. Let $a, b > 0$ The following are equivalent:*

(1) *There exists a positive Borel measure μ on $[\frac{-1}{a}, \frac{1}{b}]$ such that $\rho_n = \int t^n \mathrm{d}\mu$,*
(2) *The moment generating function $f(z) = \sum_{n=0}^{\infty} a_n z^{n+1}$ analytically continues to $\Pi \cup (a,b)$ and $f : \Pi \cup (a,b) \to \overline{\Pi}$.*

There is also a nice Hankel matrix type condition. (In fact, this is used in conjunction with a GNS-type construction to prove the prior theorem.)

Theorem 2.3 ([18]). *Let ρ_n be a sequence of real numbers. The following are equivalent:*

(1) There exists a positive Borel measure μ on $\mathbb{R}$ such that $\rho_n = \int t^n \mathrm{d}\mu$,
(2) The infinite Hankel matrix

$$\begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots \\ \rho_1 & \rho_2 & \rho_3 & \cdots \\ \rho_2 & \rho_3 & \rho_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is positive semidefinite.

3. Trace duality

We now endeavor to show that

$$\operatorname{tr} Df(X)[H] = \operatorname{tr} Hf'(X),$$

which we will use later.

For example, consider $f(x) = x^3$. The derivative is given by

$$Df(X)[H] = HX^2 + XHX + X^2H.$$

Note,

$$\operatorname{tr} Df(X)[H] = \operatorname{tr} H3X^2 = \operatorname{tr} Hf'(X).$$

It is clear that an inductive argument would prove this for polynomials. However, for general functions, matters are a bit more delicate. Our approach uses algebraic manipulation in the functional calculus. It is also likely there is a somewhat involved argument using Stone-Weierstrauss.

Lemma 3.1. *Let $f : (a, b) \to \mathbb{R}$ be a function. Let U be a unitary. Then,*

$$f(U^*XU) = U^*f(X)U.$$

Proof. Suppose the unitary V diagonalizes X.

$$f(X) = V^* \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} V.$$

Now, VU diagonalizes U^*XU, and so

$$\begin{aligned} f(U^*XU) &= U^*V \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} VU \\ &= U^*f(X)U \end{aligned}$$

□

Lemma 3.2. *Let $f : (a, b) \to \mathbb{R}$ be a function. Let U be a unitary. Then,*

$$Df(U^*XU)[U^*HU] = U^*Df(X)[H]U.$$

Proof. Calculating using Lemma 3.1

$$\begin{aligned} Df(U^*XU)[U^*HU] &= \lim_{t\to 0} \frac{f(U^*XU + tU^*HU) - f(U^*XU)}{t} \\ &= \lim_{t\to 0} \frac{f(U^*(X + tH)U) - f(U^*XU)}{t} \\ &= \lim_{t\to 0} \frac{U^*f(X + tH)U - U^*f(X)U}{t} \\ &= \lim_{t\to 0} \frac{U^*(f(X + tH) - f(X))U}{t} \\ &= U^* \left(\lim_{t\to 0} \frac{f(X + tH) - f(X)}{t} \right) U \\ &= U^*Df(X)[H]U. \end{aligned}$$

□

Theorem 3.3. *Let $f : (a, b) \to \mathbb{R}$ be a C^1 function. Then,*

$$\operatorname{tr} Df(X)[H] = \operatorname{tr} Hf'(X).$$

Proof. Because f is C^1, for each self-adjoint matrix X with spectrum in (a, b), $\operatorname{tr} Df(X)[H]$ is linear map from $n \times n$ matrices to $n \times n$ matrices as a function of H and there is a unique quantity $g(X)$ such that $\operatorname{tr} f(X)[H] = \operatorname{tr} Hg(X)$. We will show that:

(1) $g(U^*XU) = U^*g(X)U$ for all unitaries U,

(2) $g(X_1 \oplus X_2) = g(X_1) \oplus g(X_2)$,
(3) $g(x) = f'(x)$ whenever x is a real number in (a, b).

To see (1), note that by Lemma 3.2

$$Df(U^*XU)[H] = U^*Df(X)[UHU^*]U.$$

Therefore,

$$\begin{aligned}
\operatorname{tr} Hg(U^*XU) &= \operatorname{tr} Df(U^*XU)[H] \\
&= \operatorname{tr} U^*Df(X)[UHU^*]U \\
&= \operatorname{tr} Df(X)[UHU^*] \\
&= \operatorname{tr} UHU^*g(X) \\
&= \operatorname{tr} HU^*g(X)U
\end{aligned}$$

So, $g(U^*XU) = U^*g(X)U$.

To see (2), first write

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}.$$

Note that $f(X_1 \oplus X_2) = f(X_1) \oplus f(X_2)$, therefore $Df(X_1 \oplus X_2)[H_{11} \oplus H_{22}] = Df(X_1)[H_{11}] \oplus Df(X_2)[H_{22}]$. Translating the relation to g, one sees that $g(X_1 \oplus X_2)$ is of the form:

$$g\begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix} = \begin{bmatrix} g(X_1) & A(X_1, X_2) \\ A(X_2, X_1) & g(X_2) \end{bmatrix}$$

for some unknown quantities $A(X_1, X_2), A(X_2, X_1)$. Now by (1),

$$\begin{aligned}
g\begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix} &= g\left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g\begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\
&= \begin{bmatrix} g(X_1) & -A(X_1, X_2) \\ -A(X_2, X_1) & g(X_2) \end{bmatrix},
\end{aligned}$$

and therefore $A(X_1, X_2), A(X_2, X_1)$ both equal 0. Thus, $g(X_1 \oplus X_2) = g(X_1) \oplus g(X_2)$.

Now to see (3), let x be a real number. Note

$$\operatorname{tr} Df(x)[h] = Df(x)[h] = hf'(x) = \operatorname{tr} hf'(x),$$

and therefore $g(x) = f'(x)$.

We now claim $f'(X) = g(X)$. Write

$$X = U^* \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U.$$

Now,

$$\begin{aligned} f'(X) &= U^* \begin{pmatrix} f'(\lambda_1) & & \\ & \ddots & \\ & & f'(\lambda_n) \end{pmatrix} U \\ &= U^* \begin{pmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & g(\lambda_n) \end{pmatrix} U \\ &= U^* g \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U \\ &= g \left(U^* \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U \right) \\ &= g\left(X\right). \end{aligned}$$

□

4. Derivatives of trace minmax functions are matrix monotone

Lemma 4.1. *Let $f : (a, b) \to \mathbb{R}$ be C^1. The function f is trace minmax if and only if f' is matrix monotone on (a, b).*

Proof. Let $A \le B \le C$. One can rewrite the defining inequality for trace minmaxity

$$\operatorname{tr} f(A) + f(C) \ge \operatorname{tr} f(B) + f(A + C - B)$$

as

$$\operatorname{tr} f(C) - f(B) \ge \operatorname{tr} f(A + C - B) - f(A)$$

Let $C = B + tH$. Now

$$\operatorname{tr} f(B + tH) - f(B) \geq \operatorname{tr} f(A + tH) - f(A).$$

Dividing by t and taking the limit as $t \to 0$ gives

$$\operatorname{tr} Df(B)[H] \geq \operatorname{tr} Df(A)[H].$$

Applying trace duality established in Theorem 3.3, we see that

$$\operatorname{tr} Hf'(B) \geq \operatorname{tr} Hf'(A).$$

Now, $\operatorname{tr} H(f'(B) - f'(A)) \geq 0$ for an arbitrary positive semidefinite matrix H and therefore $f'(B) - f'(A)$ is positive semidefinite. Therefore $f'(A) \leq f'(B)$ and so f' is matrix monotone. □

Theorem 4.2. *Let $f : (a, b) \to \mathbb{R}$. The function f is trace minmax if and only if f' is matrix monotone on (a, b).*

Proof. Without loss of generality $a = -1$ and $b = 1$ First observe that as a function on $(-1, 1)$, f is convex, and therefore continuous. Fix φ a positive smooth function such that $\int_{\mathbb{R}} \varphi = 1$ with support contained in $(-1, 1)$. Write $\varphi_t(x) = \varphi(x/t)/t$. Write $f_t = f * \varphi_t$. Note f_t is trace minmax on $(-1 + t, 1 - t)$. Therefore, by Lemma 4.1, f_t' is matrix monotone on $(-1 + t, 1 - t)$. As $f_t \to f$ as $t \to 0$ because f is continuous, and a pointwise limit of matrix monotone functions is matrix monotone, we are done.

To see the converse, note that, if f' is matrix monotone and H, K are positive semidefinite,

$$\operatorname{tr} D^2 f(X)[H, K] = \operatorname{tr} HDf'(X)[K] \geq 0,$$

so we are done by Lemma 2.1.

□

5. Trace minmax representation theorems

We now prove our representation theorem for trace minmax functions.

Proposition 5.1. *Let $f : (a, b) \to \mathbb{R}$. If f is trace minmax then for each $c \in (a, b)$, there exists a unique measure $\alpha, \beta \in \mathbb{R}$ and a unique finite measure μ on $[\frac{1}{a-c}, \frac{1}{b-c}]$ such that*

$$f(z) = \alpha + \beta z + \int_{[\frac{1}{a-c}, \frac{1}{b-c}]} \frac{-\log(1 - t(z - c)) - t(z - c)}{t^2} \mathrm{d}\mu.$$

Proof. Without loss of generality $c = 0$. Because f is trace minmax, by Theorem 4.2, f' is matrix monotone. Furthermore, by Löwner's theorem, f analytically continues to an analytic function $f : (a, b) \cup \Pi \to \overline{\Pi}$. Write $f(z) = a_n z^n$. As $f'(z)$ is self map of the upper half plane, there is a measure μ supported on $[\frac{1}{a}, \frac{1}{b}]$ such that $na_n = \int t^{n-2} \mathrm{d}\mu$ by Nevanlinna's solution to the Hamburger moment problem [18], which we gave as Theorem 2.2. Now,

$$\begin{aligned} f(z) &= a_0 + a_1 z + \sum_{n=2}^{\infty} \frac{z^n \int t^{n-2} \mathrm{d}\mu}{n} \\ &= a_0 + a_1 z + z^2 \sum_{n=0}^{\infty} \frac{z^n \int t^n \mathrm{d}\mu}{n+2} \\ &= a_0 + a_1 z + z^2 \sum_{n=0}^{\infty} \int \frac{(zt)^n}{n+2} \mathrm{d}\mu \\ &= a_0 + a_1 z + z^2 \int \frac{-\log(1-tz) - tz}{(zt)^2} \mathrm{d}\mu \\ &= a_0 + a_1 z + \int \frac{-\log(1-tz) - tz}{t^2} \mathrm{d}\mu. \end{aligned}$$

□

A consequence of the fact that $f'(z)$ is a Pick function and Theorem 2.3 is a Hankel matrix type test for trace minmaxity.

Observation 5.2. *Let $f(x) = \sum a_n x^n$ be a convergent series on a neighborhood of 0. The function f is trace minmax if and only if the Hankel matrix*

$$\begin{bmatrix} 2a_2 & 3a_3 & 4a_4 & \dots \\ 3a_3 & 4a_4 & 5a_5 & \dots \\ 4a_4 & 5a_5 & 6a_6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is positive semidefinite.

6. Proof of the main result

(1) $\Leftrightarrow$ (2) is Theorem 4.2. (2) $\Leftrightarrow$ (3) is Löwner's theorem. (1) $\Rightarrow$ (4) is Proposition 5.1. (4) $\Rightarrow$ (3) The derivative of such an integral

representation is

$$b + \int_{[\frac{1}{a-c}, \frac{1}{b-c}]} \frac{z}{1-tz} \mathrm{d}\mu.$$

Since each $\frac{z}{1-tz}$ takes the upper half plane to itself, so does whole formula.

7. Examples

We now give some examples.

(1) The function e^z, real-rooted polynomials, and the Gamma function are all determinant isoperimetric by virtue of being in the Laguerre-Pólya class.
(2) The function x^t for $1 \leq t \leq 2$ is trace minmax, because the derivative is a self-map of the upper half plane.
(3) Consider Riemann's original Ξ function. That is, take

$$\xi(z) = \frac{1}{2} z(z-1)\pi^{s/2}\Gamma(z/2)\zeta(z),$$

and define $\Xi(z) = \xi(1/2 + iz)$. The Riemann hypothesis says that the zeros of Ξ are real. Moreover, we know $\Xi(z) = \prod(1 - \frac{z}{\rho_i})e^{z/\rho_i}$ are $1/2 + i\rho_i$ are the nontrivial zeros of the Riemann zeta function. Therefore, if the Riemann hypothesis is true, then Ξ is in the Laguerre-Pólya class. Similarly, the function $\Lambda(z) = \Xi(1/2 + \sqrt{z})$ has a similar factorization and is in the Laguerre-Pólya class if and only if the Riemann hypothesis is true.

Applying our results in tandem, we see the following list of statements which are equivalent to the Riemann hypothesis when applied to Ξ or Λ.

Proposition 7.1. *Suppose $\tilde{\Xi}$ is an entire function of genus at most 1 which is real valued on the real line. (Having genus 1 is equivalent to having a Hadamard factorization of the form:*

$$\tilde{\Xi}(z) = e^{\alpha + \beta z} \prod (1 - \frac{z}{\rho_i}) e^{z/\rho_i}$$

for some constants α, β and zeros ρ_i. Let (a, b) be a nonempty open interval in $\mathbb{R}$ where $\tilde{\Xi}$ is nonvanishing. The following are equivalent:

(a) *The function* $\tilde{\Xi}$ *only has real zeros, (for* Ξ *or* Λ*, this is the Riemann hypothesis,)*
(b) $|\tilde{\Xi}|$ *is in the radical Laguerre-Pólya class of* (a, b)*,*
(c) $\log \tilde{\Xi}(z)$ *has a branch defined on the upper half plane,*
(d) $|\tilde{\Xi}|$ *is determinant isoperimetric on* (a, b)*,*
(e) $-\log|\tilde{\Xi}(z)|$ *is trace minmax on* (a, b)*,*
(f) $-\log|\tilde{\Xi}(z)|$ *is matrix convex on* (a, b)*,*
(g) $-\frac{\mathrm{d}}{\mathrm{d}z}\log|\tilde{\Xi}(z)|$ *is matrix monotone* (a, b)*,*
(h) *Let* $r \in (a, b)$*. If we write* $-\log \tilde{\Xi}(z + r) = \sum a_n z^n$*, then the infinite matrix,*

$$\begin{bmatrix} 2a_2 & 3a_3 & 4a_4 & \dots \\ 3a_3 & 4a_4 & 5a_5 & \dots \\ 4a_4 & 5a_5 & 6a_6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is positive semidefinite.
(i) *Let* $r \in (a, b)$*. If we write* $-\log \tilde{\Xi}(z + r) = \sum a_n z^n$*, then the infinite matrix,*

$$\begin{bmatrix} a_2 & a_3 & a_4 & \dots \\ a_3 & a_4 & a_5 & \dots \\ a_4 & a_5 & a_6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is positive semidefinite.
(j) *Let* $r \in (a, b)$*. If we write* $-\log \tilde{\Xi}(z + r) = \sum a_n z^n$*, then there exists a* $k \in \mathbb{N}$ *such that the infinite matrix,*

$$\begin{bmatrix} 2ka_{2k} & (2k+1)a_{2k+1} & (2k+2)a_{2k+2} & \dots \\ (2k+1)a_{2k+1} & (2k+2)a_{2k+2} & (2k+3)a_{2k+3} & \dots \\ (2k+2)a_{2k+2} & (2k+3)a_{2k+3} & (2k+4)a_{2k+4} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is positive semidefinite.

(k) *Let $r \in (a,b)$. If we write $-\log \tilde{\Xi}(z+r) = \sum a_n z^n$, then there exists a $k \in \mathbb{N}$ such that the infinite matrix,*

$$\begin{bmatrix} a_{2k} & a_{2k+1} & a_{2k+2} & \dots \\ a_{2k+1} & a_{2k+2} & a_{2k+3} & \dots \\ a_{2k+2} & a_{2k+3} & a_{2k+4} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is positive semidefinite.

Proof. $(a) \Rightarrow (b)$ If $\tilde{\Xi}$ only has real zeros, then it is in the Laguerre-Pólya class because it is of the form of Equation (1.2), and therefore its absolute value is in the radical Laguerre-Pólya class. (In the case of Ξ or Λ, the equivalence of (a) and (b) is essentially classical and crucial to the Jensen-Pólya approach to the Riemann hypothesis [19].)

$(a) \Leftrightarrow (c)$ $\tilde{\Xi}$ is nonvanishing on the upper half plane if and only if it admits a branch of the logarithm.

$(b) \Leftrightarrow (d) \Leftrightarrow (e)$ is Theorem 1.5.

$(e) \Leftrightarrow (g)$ is part of Theorem 1.1.

$(e) \Rightarrow (f)$ is Corollary 1.2.

$(f) \Rightarrow (c)$ is Kraus theorem [15].

$(e) \Leftrightarrow (h)$ follows from Observation 5.2.

$(h) \Rightarrow (i)$ Note that

$$\begin{bmatrix} 1/2 & 1/3 & 1/4 & \dots \\ 1/3 & 1/4 & 1/5 & \dots \\ 1/4 & 1/5 & 1/6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \geq 0,$$

and, therefore,

$$\begin{bmatrix} 1/2 & 1/3 & 1/4 & \dots \\ 1/3 & 1/4 & 1/5 & \dots \\ 1/4 & 1/5 & 1/6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 2a_2 & 3a_3 & 4a_4 & \dots \\ 3a_3 & 4a_4 & 5a_5 & \dots \\ 4a_4 & 5a_5 & 6a_6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} a_2 & a_3 & a_4 & \dots \\ a_3 & a_4 & a_5 & \dots \\ a_4 & a_5 & a_6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \geq 0.$$

$(h) \Rightarrow (j)$ is trivial.

$(j) \Rightarrow (k)$ has essentially the same proof as $(h) \Rightarrow (i)$.

$(i) \Rightarrow (k)$ is trivial.

$(k) \Rightarrow (c)$ follows from Theorem 2.3 combined with 2.2 applied to the function $\sum_{j=1}^{\infty} a_{2k+j} z^{2k+j+1}$. □

The above formulation of the the Riemann hypothesis evokes a similiarity to approaches using hyperbolicity of Jensen polynomials taken in [19, 6, 7], and a positivity of derivatives approach in Li's criterion [17].

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