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Decay estimates of solutions to the compressible micropolar fluids system in $\mathbb{R}^3$ ☆

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Abstract

This work is concerned with the compressible micropolar fluids system in three–dimensional space. We consider the asymptotic behavior of the solution to the Cauchy problem near the constant equilibrium state provided that the initial perturbation is sufficiently small. Under some assumptions of the initial data, we show that the solution of the Cauchy problem converges to its constant equilibrium state at the exact same L^2–decay rates as the linearized equations, which shows the convergence rates are optimal. The proof is based on the spectral analysis of the semigroup generated by the linearized equations and the nonlinear energy estimates.

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Keywords: Lower and upper bound decay rates; Spectral analysis; Energy method

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1. Introduction

The micropolar fluids describe a class of microstructure related fluids, for example, animal blood, polymeric suspensions, and liquid crystals. The model has many potential applications in both mathematics and engineering [19,26]. The theory of micropolar fluids is introduced in [4,18]. The motion of the compressible micropolar fluids can be described by the following equations [18]:

$$
\begin{cases}
\partial_t \tilde{\rho} + \operatorname{div}(\tilde{\rho}\tilde{u}) = 0, \\
\partial_t(\tilde{\rho}\tilde{u}) + \operatorname{div}(\tilde{\rho}\tilde{u} \otimes \tilde{u}) + \nabla P(\tilde{\rho}) - (\mu+\alpha)\Delta\tilde{u} - (\mu+\lambda-\alpha)\nabla\operatorname{div}\tilde{u} - 2\alpha\nabla\times\tilde{w} = 0, \\
\partial_t(\tilde{\rho}\tilde{w}) + \operatorname{div}(\tilde{\rho}\tilde{u} \otimes \tilde{w}) + 4\alpha\tilde{w} - \mu'\Delta\tilde{w} - (\mu'+\lambda')\nabla\operatorname{div}\tilde{w} - 2\alpha\nabla\times\tilde{u} = 0, \\
(\tilde{\rho}, \tilde{u}, \tilde{w})^T|_{t=0} = (\tilde{\rho}_0, \tilde{u}_0, \tilde{w}_0)^T,
\end{cases}
\tag{1.1}
$$

for $(x,t) \in \mathbb{R}^3 \times [0,\infty)$. Here the unknowns $\tilde{\rho} = \tilde{\rho}(x,t)$, $\tilde{u} = \tilde{u}(x,t)$, $\tilde{w} = \tilde{w}(x,t)$ denote the density, velocity and the micro–rotational velocity, respectively. The pressure $P(\tilde{\rho})$ is a C^1–function satisfying $P'(\rho_\infty) > 0$ with some constant $\rho_\infty > 0$. The quantity $\alpha > 0$ means dynamics microrotation viscosity. The constant coefficients $\mu,\ \lambda$ are the shear and bulk viscosity coefficients of the flow and satisfy the physical restrictions

$$\mu > 0,\ 2\mu + 3\lambda - 4\alpha \geq 0.$$

μ' and λ' are the angular viscosity coefficients satisfying

$$\mu' > 0,\ 2\mu' + 3\lambda' \geq 0.$$

Before stating the main result, we first review the known related work. Concerning the micro–rotational velocity $\tilde{w} = 0$, the equations (1.1) will become the three dimensional compressible Navier–Stokes equations, which have been extensively studied. Many works were dedicated to proving the existence and the time decay rate for the nonlinear system. The local existence and uniqueness of classical solutions to Cauchy problem were considered by [6,8,38,41]. The decay rates of solutions for the Cauchy problem have been investigated extensively since the first global existence of small solutions obtained by Matsumura–Nishida [36,37]. If the initial data belongs to L^p $(1 \leq p < 2)$, the decay rate was established in [23,29,36] and the reference therein by combining the linear decay rate of spectral analysis and the energy method, and in [9,10,20,31] and the reference therein through the energy method.

Although there have been many results mentioned above about the compressible Navier–Stokes system, rare studies were devoted to the micropolar fluids. In fact, the micropolar fluids system is more complicated because of the complex coupling structure containing curl. There are some results about the incompressible viscous micropolar fluids. Lukaszewicz in [25] established the existence of local weak solutions by using linearization and a fixed point theorem. The existence of global weak solutions with initial density not necessarily strictly positive was established in [45,46] by a semi–Galerkin method. The existence and uniqueness of strong solution were considered in [2] by using the spectral semi-Galerkin method and compactness arguments and in [1,40] by an iterative approach. We can see [17,28] and the reference therein for the regularity criteria.

The problems about the compressible micropolar fluids have also received considerable attentions in the last few years. One can see [3,5] for the blowup criterion and [44] for the limit problem. The initial boundary value problems of this model in the one–dimensional case were described by [7,32–34,48]. One can refer to [32–34] for the local or global existence of the strong or generalized solutions and [7,48] for the asymptotic behavior. For the $3D$ case, this model with heat conduction was analyzed in relation to existence, uniqueness and stability of the spherically symmetric solutions to initial–boundary value problems. Dražić and Mujaković [12] used the Faedo–Galerkin method to prove the local existence of generalized spherically symmetric solutions assuming that the initial functions were also spherically symmetric, and then proved the uniqueness of the solution in [35]. Based on these two results, by the combination of the local existence [12], the uniqueness theorem [35] and the extension principle, Dražić and Mujaković [13] proved the global existence of the spherically symmetric generalized solution. We can refer to [21,22] for the large time behavior of the global symmetric generalized solution. For the Cauchy problems in $1D$, we can refer to [24,39] and the reference therein for the well-posedness problem. For the $3D$ case, results are much less. Liu and Zhang [30] obtained the optimal time decay rates if the initial perturbation was small in $H^N \cap L^1$ $(N \geq 4)$. However, there is no result showing that the solution to the compressible micropolar fluids equations has the exactly same decay rates as the linearized problem. Our aim in this work is investigating the lower and upper decay rates of the compressible micropolar fluids system.

It is worth mentioning that the optimal convergence rate is an important topic in the study of the fluid dynamics for the purpose of the computation. It was mentioned in [23] by an example that the solution of the nonlinear system with constant viscosity terms B_{jk}

$$u_t + f_j(u)_{x_j} = B_{jk} u_{x_j x_k},$$

had a sharp L^2 decay rate of $(1+t)^{-\frac{n}{4}}$ with dimension $n \geq 2$. The lower and upper decay rates of the solutions for incompressible Navier–Stokes equations were considered in [42,43]. When the average of the initial data $\int_{\mathbb{R}^3} u_0(x)dx \neq 0$, it was shown in [42] that

$$(1+t)^{-\frac{3}{4}} \lesssim \|u\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}}.$$

The case $\int_{\mathbb{R}^3} u_0(x)dx = 0$ was investigated in [43], it was proved that the lower bound decay rate depended on the order of the zero of the initial data. Later, [27] considered the compressible Navier–Stokes–Poisson system, and shown the sharp decay rate of the solution to Cauchy problem with small initial data. Motivated by the work of [27,42,43], in this work, we use a similar method as [27] to derive the large time behavior of the global classical solution of the compressible micropolar fluids equations in $\mathbb{R}^3$, provided the prescribed initial data is close to the constant steady state $(\rho_\infty, 0, 0)^T$ with $\rho_\infty > 0$. We will show that the behavior of the perturbation is asymptotically equivalent to that of the linearized problem. To establish this, much effort will be spent on the spectral analysis of the linearized system. To this end, we now linearize (1.1) near the constant state $(\rho_\infty, 0, 0)^T$. Without loss of generality, the constant ρ_∞ is taken to be 1. We define

$$\tilde{m} = \tilde{\rho}\tilde{u}, \ \tilde{W} = \tilde{\rho}\tilde{w},$$

and want to rewrite problem (1.1) in a symmetric form. By changing the unknown functions and denoting the perturbations by

$$\rho = \tilde{\rho} - 1,\ m = \frac{1}{\sqrt{P'(1)}}(\tilde{m} - 0),\ W = \frac{1}{\sqrt{P'(1)}}\left(\tilde{W} - 0\right),$$

problem (1.1) is reduced to

$$\begin{cases} \partial_t \rho + \sqrt{P'(1)}\mathrm{div} m = 0, \\ \partial_t m + \sqrt{P'(1)}\nabla\rho - (\mu+\alpha)\Delta m - (\mu+\lambda-\alpha)\nabla\mathrm{div} m - 2\alpha\nabla\times W = \mathbb{N}_1, \\ \partial_t W + 4\alpha W - \mu'\Delta W - (\mu'+\lambda')\nabla\mathrm{div} W - 2\alpha\nabla\times m = \mathbb{N}_2, \end{cases} \tag{1.2}$$

here

$$\begin{aligned} \mathbb{N}_1 = &-\sqrt{P'(1)}\mathrm{div}\left(\frac{m\otimes m}{1+\rho}\right) - \frac{1}{\sqrt{P'(1)}}\nabla\left[P(1+\rho) - P(1) - P'(1)\rho\right] \\ &- (\mu+\alpha)\Delta\left(\frac{\rho m}{1+\rho}\right) - (\mu+\lambda-\alpha)\nabla\mathrm{div}\left(\frac{\rho m}{1+\rho}\right) - 2\alpha\nabla\times\left(\frac{\rho W}{1+\rho}\right), \\ \mathbb{N}_2 = &-\sqrt{P'(1)}\mathrm{div}\left(\frac{m\otimes W}{1+\rho}\right) + 4\alpha\frac{\rho W}{1+\rho} - \mu'\Delta\left(\frac{\rho W}{1+\rho}\right) - (\mu'+\lambda')\nabla\mathrm{div}\left(\frac{\rho W}{1+\rho}\right) \\ &- 2\alpha\nabla\times\left(\frac{\rho m}{1+\rho}\right). \end{aligned}$$

Initial data of the system (1.2) is expressed as

$$\begin{aligned} (\rho, m, W)^T|_{t=0} &= \left(\tilde{\rho}_0 - 1, \frac{1}{\sqrt{P'(1)}}(\tilde{m}_0 - 0), \frac{1}{\sqrt{P'(1)}}\left(\tilde{W}_0 - 0\right)\right)^T \\ &= (\rho_0, m_0, W_0)^T. \end{aligned} \tag{1.3}$$

Unfortunately, the method in [27] doesn't work directly in the micropolar fluids system because of its complex linearized system. Here this problem is overcome as in [11,30] by introducing the decomposition such that the solution to (1.2)–(1.3) can be decomposed into two parts in the form of

$$\begin{pmatrix} \rho \\ m \\ W \end{pmatrix} = \begin{pmatrix} \rho \\ m_{\shortparallel} \\ W_{\shortparallel} \end{pmatrix} + \begin{pmatrix} 0 \\ m_\perp \\ W_\perp \end{pmatrix}, \tag{1.4}$$

where

$$m_{\shortparallel} = \Delta^{-1}\nabla\mathrm{div} m,\ m_\perp = -\Delta^{-1}\nabla\times(\nabla\times m),$$

and likewise for $W_{\shortparallel},\ W_\perp$. For brevity, the first part on the right–hand side of (1.4) is called the fluid part and the second part is called the electromagnetic part, and we also denote

$$U_{\shortparallel} = (\rho, m_{\shortparallel}, W_{\shortparallel})^T, \; U_{\perp} = (m_{\perp}, W_{\perp})^T.$$

The above decomposition is of great help in dealing with complex linearized system containing curl. We refer to [11,14,30] for the detailed spectrum analysis with the use of a similar decomposition. In the aid of the above decomposition, we give explicit representations of solutions to the two eigenvalue problems.

We now derive the equations of $U_{\shortparallel}$ and $U_{\perp}$ respectively. Taking the divergence of the last two equations of (1.2), and then applying $\Delta^{-1}\nabla$, it follows that

$$\begin{cases} \partial_t \rho + \sqrt{P'(1)}\mathrm{div} m_{\shortparallel} = 0, \\ \partial_t m_{\shortparallel} + \sqrt{P'(1)}\nabla\rho - (2\mu+\lambda)\Delta m_{\shortparallel} = \Delta^{-1}\nabla\mathrm{div}\mathbb{N}_1 = F_1, \\ \partial_t W_{\shortparallel} + 4\alpha W_{\shortparallel} - (2u'+\lambda')\Delta W_{\shortparallel} = \Delta^{-1}\nabla\mathrm{div}\mathbb{N}_2 = F_2, \\ (\rho, m_{\shortparallel}, W_{\shortparallel})^T|_{t=0} = (\rho_0, m_{\shortparallel 0}, W_{\shortparallel 0})^T, \end{cases} \tag{1.5}$$

with

$$\begin{aligned} F_1 \sim & -\sqrt{P'(1)}\mathrm{div}\left(\frac{m\otimes m}{1+\rho}\right) - \frac{1}{\sqrt{P'(1)}}\nabla\left[P(1+\rho) - P(1) - P'(1)\rho\right] \\ & - (\mu+\alpha)\Delta\left(\frac{\rho m}{1+\rho}\right) - (\mu+\lambda-\alpha)\nabla\mathrm{div}\left(\frac{\rho m}{1+\rho}\right), \end{aligned} \tag{1.6}$$

and

$$F_2 \sim -\sqrt{P'(1)}\mathrm{div}\left(\frac{m\otimes W}{1+\rho}\right) + 4\alpha\frac{\rho W}{1+\rho} - \mu'\Delta\left(\frac{\rho W}{1+\rho}\right) - (\mu'+\lambda')\nabla\mathrm{div}\left(\frac{\rho W}{1+\rho}\right). \tag{1.7}$$

Taking the curl of the last two equations of (1.2) and then applying Δ^{-1}curl, one deduces

$$\begin{cases} \partial_t m_{\perp} - (\mu+\alpha)\Delta m_{\perp} - 2\alpha\nabla\times W_{\perp} = N_1, \\ \partial_t W_{\perp} + 4\alpha W_{\perp} - \mu'\Delta W_{\perp} - 2\alpha\nabla\times m_{\perp} = N_2, \\ (m_{\perp}, W_{\perp})^T|_{t=0} = (m_{\perp 0}, W_{\perp 0})^T, \end{cases} \tag{1.8}$$

with

$$N_1 \sim -(\mu+\alpha)\Delta\left(\frac{\rho m}{1+\rho}\right) - 2\alpha\nabla\times\left(\frac{\rho W}{1+\rho}\right), \tag{1.9}$$

and

$$N_2 \sim 4\alpha\frac{\rho W}{1+\rho} - \mu'\Delta\left(\frac{\rho W}{1+\rho}\right) - 2\alpha\nabla\times\left(\frac{\rho m}{1+\rho}\right), \tag{1.10}$$

where we have replaced $-\nabla\times\nabla\times W$ by $\Delta W - \nabla\mathrm{div}W$, likewise for $-\nabla\times\nabla\times m$.

We now record the following existence and uniqueness of the solution to (1.2)–(1.3). The proof has been described in [47], for convenience, we just state the result in the following theorem.

Theorem 1.1. *Assume that $\|(\rho_0, u_0, w_0)\|_{H^3} \le \delta$ for some sufficiently small δ. The Cauchy problem (1.2)–(1.3) admits a unique solution $(\rho, u, w)^T$ satisfying*

$$\|(\rho, u, w)(t)\|_{H^3}^2 + \int_0^t \left(\|\nabla \rho(\tau)\|_{H^2}^2 + \|\nabla (u, w)(\tau)\|_{H^3}^2 \right) d\tau \le C \|(\rho_0, u_0, w_0)\|_{H^3}^2 . \tag{1.11}$$

To avoid confusion, we should point out that the notation $(\rho, u, w)^T$ in Theorem 1.1 has the following relation with that used in our paper:

$$\rho = \tilde{\rho} - 1,\ u = \tilde{u} - 0 = \frac{\widetilde{m}}{\widetilde{\rho}} = \frac{\sqrt{P'(1)}m}{1+\rho},\ w = \tilde{w} - 0 = \frac{\tilde{W}}{\widetilde{\rho}} = \frac{\sqrt{P'(1)}W}{1+\rho}.$$

Based on the stability result above, the main purpose of the paper is to investigate the following theorem concerning the convergence rates of the solution of Eqs. (1.2)–(1.3).

Theorem 1.2. *Under the assumptions of*

$$\|(\rho_0, m_0, W_0)\|_{H^3 \cap L^1} \lesssim \delta; \tag{1.12}$$

$$\int_{\mathbb{R}^3} \rho_0(x) dx \neq 0; \quad \int_{\mathbb{R}^3} m_{\perp 0}(x) dx \neq 0, \tag{1.13}$$

and

$$\int_{\mathbb{R}^3} |x \rho_0(x)|\, dx < \infty; \quad \int_{\mathbb{R}^3} \left|x m_{\shortparallel 0}(x)\right| dx < \infty; \quad \int_{\mathbb{R}^3} |x m_{\perp 0}(x)|\, dx < \infty, \tag{1.14}$$

for $t > t_0$ with some t_0 suitably large, we have

$$(1+t)^{-\frac{3}{4}} \lesssim \|(\rho, m)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}}, \tag{1.15}$$

$$(1+t)^{-\frac{5}{4}} \lesssim \|W\|_{L^2} \lesssim (1+t)^{-\frac{5}{4}}, \tag{1.16}$$

and

$$(1+t)^{-\frac{5}{4}} \lesssim \|\nabla (\rho, m)\|_{L^2} \lesssim (1+t)^{-\frac{5}{4}}. \tag{1.17}$$

In order to obtain the optimal decay rates (1.15)–(1.17), based on the method in [27], we also need to introduce the decomposition (1.4) to overcome the difficulty caused by the curl term. We consider the linearized systems (2.1) and (3.1) near a constant equilibrium state and investigate the spectrum of the semigroup in terms of the decomposition of wave modes at the lower frequency and higher frequency, respectively. Under the conditions (1.12)–(1.14), we obtain the lower and upper decay rates of the linearized cases. Moreover, in Section 4, in virtue of the careful analysis on the semigroup, and the local energy estimates, we show that the difference $(\rho_h, m_h, W_h)^T = (\rho - \bar{\rho}, m - \bar{m}, W - \bar{W})^T$ has a faster decay rate than the linearized one obtained in Proposition 3.2. This implies the solution $(\rho, m, W)^T$ has the same decay rate as $(\bar{\rho}, \bar{m}, \bar{W})^T$.

Remark 1.3. About the global existence in Theorem 1.1, we only assume that the H^3 norms of the initial data are small, while their higher order Sobolev norms could be arbitrarily large. Under the smallness assumption of $\|(\rho_0, m_0, W_0)\|_{H^3\cap L^1}$, we can obtain the upper decay rates of $\|(\rho, m, W)\|_{L^2}$. This maybe is an improved factor since it requires that the $H^N(N \geq 4)$ norms of the initial data are small in [30]. What's more, under the further assumptions of the initial data (1.13)–(1.14), we investigate the lower decay rates of the linearized equations and prove that the solution of the nonlinear system also has the same decay property as the linearized one, in this case, we show the decay rates are optimal.

Remark 1.4. Notice that the micro–rotational velocity decays faster than the density $\tilde{\rho}$ and the velocity $\tilde{u}$. Here the damping term $4\alpha\tilde{w}$ plays a crucial role in obtaining the faster decay rate for $\tilde{w}$.

Remark 1.5. If the initial data $\|(\rho_0, m_0, W_0)\|_{H^N\cap L^1}$ $(N \geq 3)$ is suitably small, by combining the linear estimates and the Fourier splitting method, we can derive the following decay results

$$\left\|\nabla^k (\rho, m)\right\|_{L^2} \leq C(1+t)^{-\left(\frac{3}{4}+\frac{k}{2}\right)}, \ for\ k = 0, 1, \ldots, N-1, \tag{1.18}$$

and

$$\left\|\nabla^l W\right\|_{L^2} \leq C(1+t)^{-\left(\frac{5}{4}+\frac{l}{2}\right)}, \ for\ l = 0, 1, \ldots, N-2. \tag{1.19}$$

In fact, we can give the proof by induction. We have already proved in Theorem 1.2 that when $k, l = 0$, (1.18)–(1.19) hold true. Now, we assume that (1.18) holds for $k = \ell - 1$ with $\ell = 1, 2, \cdots, N-1$. From [47], we know

$$\frac{d}{dt}\left\|\nabla^\ell (\rho, u, w)\right\|^2_{H^{N-\ell}} + C\left\|\nabla^{\ell+1}\rho\right\|^2_{H^{N-\ell-1}} + C\left\|\nabla^{\ell+1}(u, w)\right\|^2_{H^{N-\ell}} \leq 0. \tag{1.20}$$

We define

$$S(t) = \left\{\xi \in \mathbb{R}^3 : |\xi| \leq \sqrt{\frac{a}{1+t}}\right\},$$

with $a = \frac{\ell+2}{C}$. Then

$$\left\|\nabla^{\ell+1}\rho\right\|^2_{L^2} = \int_{\mathbb{R}^3} |\xi|^{2(\ell+1)} \left|\hat{\rho}\right|^2 d\xi \geq \int_{\mathbb{R}^3/S(t)} |\xi|^{2(\ell+1)} \left|\hat{\rho}\right|^2 d\xi$$

$$
\begin{aligned}
&\geq \frac{a}{1+t} \int\limits_{\mathbb{R}^3/S(t)} |\xi|^{2\ell} \left|\hat{\rho}\right|^2 d\xi \\
&\geq \frac{a}{1+t} \int\limits_{\mathbb{R}^3} |\xi|^{2\ell} \left|\hat{\rho}\right|^2 d\xi - \frac{a^2}{(1+t)^2} \int\limits_{S(t)} |\xi|^{2(\ell-1)} \left|\hat{\rho}\right|^2 d\xi \\
&\geq \frac{a}{1+t} \int\limits_{\mathbb{R}^3} |\xi|^{2\ell} \left|\hat{\rho}\right|^2 d\xi - \frac{a^2}{(1+t)^2} \int\limits_{\mathbb{R}^3} |\xi|^{2(\ell-1)} \left|\hat{\rho}\right|^2 d\xi.
\end{aligned}
$$

It is clearly that

$$
\left\|\nabla^{\ell+1}(\rho,u,w)\right\|^2_{H^{N-\ell}} \geq \frac{a}{1+t}\left\|\nabla^{\ell}(\rho,u,w)\right\|^2_{H^{N-\ell}} - \frac{a^2}{(1+t)^2}\left\|\nabla^{\ell-1}(\rho,u,w)\right\|^2_{H^{N-\ell}}. \tag{1.21}
$$

It follows from (1.20) and (1.21) that

$$
\frac{d}{dt}\left\|\nabla^{\ell}(\rho,u,w)\right\|^2_{H^{N-\ell}} + \frac{Ca}{1+t}\left\|\nabla^{\ell}(\rho,u,w)\right\|^2_{H^{N-\ell}} \leq C(1+t)^{-\left(\frac{5+2\ell}{2}\right)}. \tag{1.22}
$$

Multiplying both sides by $(1+t)^{\ell+2}$ and solving the inequality directly, we can prove (1.18).

Energy estimates on $(1.2)_3$ yield

$$
\begin{aligned}
\frac{d}{dt}\left\|\nabla^l W\right\|^2_{L^2} + C\left\|\nabla^l W\right\|^2_{L^2} &\leq \frac{d}{dt}\left\|\nabla^l W\right\|^2_{L^2} + C\left\|\nabla^l W\right\|^2_{L^2} + C\left\|\nabla^{l+1} W\right\|^2_{L^2} \\
&\leq C\left\|\nabla^{l+1}(\rho,u)\right\|^2_{L^2} \leq C(1+t)^{-\frac{5+2l}{2}}.
\end{aligned} \tag{1.23}
$$

By Gronwall inequality, we prove (1.19).

We should also note that, the decay rates (1.18)–(1.19) are optimal with $k=0,1,\cdots,N-2$ and $l=0,1,\cdots,N-3$, which can be proved in the same way as Theorem 1.2.

Notation. Through out the paper, we use $f \lesssim g$ to denote $f \leq Cg$ and $f \gtrsim g$ to denote $f \geq Cg$, where $C>0$ is some positive constant. $f \sim g$ means $f \gtrsim g$ and $f \lesssim g$. For simplicity, the notation $\|(f,g)\|_X$ means $\|f\|_X + \|g\|_X$ with $f,g \in X$.

The rest of the paper is structured as follows. In Section 2, we will analyze the property of the solution semigroup $e^{t\mathbb{A}}$ and obtain the decay property of the solution to the linearized fluid part. In Section 3, we use the similar method as in Section 2 to explore the decay rates of the solution to the linearized electromagnetic part system. In Section 4, by using the linear estimates and some energy methods, we will prove that the difference $(\rho_h, m_{h_{\shortparallel}}, m_{h\perp}, W_{h_{\shortparallel}}, W_{h\perp})^T$ has a faster decay rate than $(\bar{\rho}, \bar{m}_{\shortparallel}, \bar{m}_{\perp}, \bar{W}_{\shortparallel}, \bar{W}_{\perp})^T$, which implies the optimal time decay rate of the solution to the micropolar fluids system (1.2)–(1.3) and gives the proof of Theorem 1.2.

2. L^2 decay rates for the solution of the linearized fluid part

In this section, we will give some analysis on the semigroup generated by the linearized fluid part and obtain the upper and lower bound decay rates for the solution of the linearized one.

2.1. Spectral representation

The solution $\bar{U}_{\shortparallel} = (\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel})^T$ of the linearized fluid part satisfies

$$\begin{cases} \partial_t \bar{\rho} + \sqrt{P'(1)}\mathrm{div}\bar{m}_{\shortparallel} = 0, \\ \partial_t \bar{m}_{\shortparallel} + \sqrt{P'(1)}\nabla\bar{\rho} - (2\mu+\lambda)\Delta\bar{m}_{\shortparallel} = 0, \\ \partial_t \bar{W}_{\shortparallel} + 4\alpha\bar{W}_{\shortparallel} - (2u'+\lambda')\Delta\bar{W}_{\shortparallel} = 0, \\ (\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel})^T|_{t=0} = (\rho_0, m_{\shortparallel 0}, W_{\shortparallel 0})^T. \end{cases} \tag{2.1}$$

If we denote $\bar{U}_{1\shortparallel} = (\bar{\rho}, \bar{m}_{\shortparallel})^T$ to be the solution of the first two equations of (2.1), taking the Fourier transform to system (2.1) with respect to the space variable, by the semigroup theory for evolution equations, the solution $\bar{U}_{\shortparallel} = (\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel})^T$ can be expressed as

$$\begin{cases} \hat{\bar{U}}_{1\shortparallel}(\xi) = e^{t\mathbb{A}(\xi)}\hat{\bar{U}}_{1\shortparallel 0}(\xi), \;\; \hat{\bar{U}}_{1\shortparallel 0} = (\hat{\rho}_0, \hat{m}_{\shortparallel 0})^T, \\ \hat{\bar{W}}_{\shortparallel}(\xi) = e^{-\left[4\alpha+(2\mu'+\lambda')|\xi|^2\right]t}\hat{W}_{\shortparallel 0}, \end{cases} \tag{2.2}$$

where $\mathbb{A}(\xi)$ is defined as

$$\begin{pmatrix} 0 & -\sqrt{P'(1)}i\xi^T \\ -\sqrt{P'(1)}i\xi & -(2\mu+\lambda)|\xi|^2\mathbb{I}_{3\times 3} \end{pmatrix}.$$

The characteristic polynomial of $\mathbb{A}(\xi)$ is

$$\det(\mathbb{A}(\xi) - \bar{\lambda}\mathbb{I}) = \left(\bar{\lambda} + (2\mu+\lambda)|\xi|^2\right)^2\left(\bar{\lambda}^2 + (2\mu+\lambda)|\xi|^2\bar{\lambda} + P'(1)|\xi|^2\right) = 0, \tag{2.3}$$

which implies the eigenvalues of (2.3)

$$\begin{aligned} &\lambda_0 = -(2\mu+\lambda)|\xi|^2 \; (double), \\ &\lambda_1 = -(\mu+\lambda/2)|\xi|^2 + \frac{i}{2}\sqrt{4P'(1)|\xi|^2 - (2\mu+\lambda)^2|\xi|^4}, \\ &\lambda_2 = -(\mu+\lambda/2)|\xi|^2 - \frac{i}{2}\sqrt{4P'(1)|\xi|^2 - (2\mu+\lambda)^2|\xi|^4}. \end{aligned}$$

The matrix exponential $e^{t\mathbb{A}}$ has the spectral resolution

$$e^{t\mathbb{A}} = e^{\lambda_0 t}P_0 + e^{\lambda_1 t}P_1 + e^{\lambda_2 t}P_2,$$

where the project operators P_0, P_1 and P_2 can be computed as

$$P_i = \prod_{j\neq i} \frac{\mathbb{A}(\xi) - \lambda_j \mathbb{I}}{\lambda_i - \lambda_j}.$$

Note the fact that

$$\nabla \mathrm{div} m_{\shortparallel} = \nabla \mathrm{div} m = \Delta m_{\shortparallel},$$

by a direct computation, we can obtain the exact expression of the semigroup

$$\begin{aligned} e^{t\mathbb{A}} &= e^{\lambda_0 t} P_0 + e^{\lambda_1 t} P_1 + e^{\lambda_2 t} P_2 \\ &= \begin{pmatrix} \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} & -\frac{i\xi^T \sqrt{P'(1)}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)}{\lambda_1 - \lambda_2} \\ -\frac{i\xi \sqrt{P'(1)}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)}{\lambda_1 - \lambda_2} & \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \mathbb{I}_{3\times 3} \end{pmatrix}. \end{aligned} \tag{2.4}$$

Now we turn to deal with the terms of the matrix exponential $e^{t\mathbb{A}}$. We need to verify the approximation of the semigroup $e^{t\mathbb{A}}$ for both lower frequency and high frequency. In terms of the definition of the eigenvalues, we are able to obtain that it holds for $|\xi| \leq \eta$ that

$$\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} = e^{-\left(\mu + \frac{\lambda}{2}\right)|\xi|^2 t} \left[\cos(bt) + \left(\mu + \frac{\lambda}{2}\right) \frac{\sin(bt)}{b} |\xi|^2\right], \tag{2.5}$$

$$\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = e^{-\left(\mu + \frac{\lambda}{2}\right)|\xi|^2 t} \left[\cos(bt) - \left(\mu + \frac{\lambda}{2}\right) \frac{\sin(bt)}{b} |\xi|^2\right], \tag{2.6}$$

$$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = \frac{\sin(bt)}{b} e^{-\left(\mu + \frac{\lambda}{2}\right)|\xi|^2 t}, \tag{2.7}$$

where

$$b = \frac{1}{2}\sqrt{4P'(1)|\xi|^2 - (2\mu + \lambda)^2 |\xi|^4} \sim |\xi| + O(|\xi|^3), \tag{2.8}$$

and

$$\eta = \frac{2\sqrt{P'(1)}}{2\mu + \lambda}.$$

By direct computation, we have the leading orders of the eigenvalues for $\xi \gg 1$ as

$$\begin{aligned} \lambda_1 &\sim -(2\mu + \lambda)|\xi|^2 + \frac{P'(1)}{2\mu + \lambda} + O(|\xi|^{-1}), \\ \lambda_2 &\sim -\frac{P'(1)}{2\mu + \lambda} + O(|\xi|^{-1}). \end{aligned}$$

This approximation implies

$$\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}, \ \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \ \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \lesssim e^{-R_0 t}, \ |\xi| \geq \eta. \tag{2.9}$$

2.2. The L^2–decay rates of $(\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel})^T$

In aid of the matrix exponential $e^{t\mathbb{A}}$ and the analysis of the lower and high frequency (2.5)–(2.9), we can deduce the following upper bound decay rates of $(\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel})^T$.

Lemma 2.1. *Under the assumption that $(\rho_0, m_0, W_0)^T \in H^3 \cap L^1$, we can deduce for $k = 0, 1, 2, 3$*

$$\left\| \nabla^k \left(\bar{\rho}, \bar{m}_{\shortparallel}\right) \right\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}} \left(\left\| (\rho_0, m_{\shortparallel 0}) \right\|_{L^1} + \left\| \nabla^k (\rho_0, m_{\shortparallel 0}) \right\|_{L^2} \right) \tag{2.10}$$

and

$$\left\| \nabla^k \bar{W}_{\shortparallel} \right\|_{L^2} \sim e^{-ct} \left\| W_{\shortparallel 0} \right\|_{H^k}. \tag{2.11}$$

Proof. Owing to the formula (2.2) and (2.4), it follows

$$\hat{\bar{\rho}}(\xi, t) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} \hat{\rho}_0 - \frac{i\xi^T \sqrt{P'(1)} \left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)}{\lambda_1 - \lambda_2} \hat{m}_{\shortparallel 0}$$

and

$$\hat{\bar{m}}_{\shortparallel}(\xi, t) = -\frac{i\xi \sqrt{P'(1)} \left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)}{\lambda_1 - \lambda_2} \hat{\rho}_0 + \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \hat{m}_{\shortparallel 0}. \tag{2.12}$$

It's clearly that the lower and high frequency analysis on the semigroup can imply

$$\hat{\bar{\rho}}(\xi, t) \lesssim \begin{cases} e^{-\left(\mu + \frac{\lambda}{2}\right)|\xi|^2 t} \left(|\hat{\rho}_0| + |\hat{m}_{\shortparallel 0}|\right), \ |\xi| \leq \eta, \\ e^{-R_0 t} \left(|\hat{\rho}_0| + |\hat{m}_{\shortparallel 0}|\right), \ |\xi| \geq \eta, \end{cases} \tag{2.13}$$

and

$$\hat{\bar{m}}_{\shortparallel}(\xi, t) \lesssim \begin{cases} e^{-\left(\mu + \frac{\lambda}{2}\right)|\xi|^2 t} \left(|\hat{\rho}_0| + |\hat{m}_{\shortparallel 0}|\right), \ |\xi| \leq \eta, \\ e^{-R_0 t} \left(|\hat{\rho}_0| + |\hat{m}_{\shortparallel 0}|\right), \ |\xi| \geq \eta. \end{cases} \tag{2.14}$$

By Plancherel's Theorem, one has

$$\left\| \nabla^k (\bar{\rho}, \bar{m}_{\shortparallel}) \right\|_{L^2}^2 = \left\| \left(\widehat{\nabla^k \bar{\rho}}, \widehat{\nabla^k \bar{m}_{\shortparallel}} \right) \right\|_{L^2}^2.$$

Further, it follows from (2.13)–(2.14)

$$
\begin{aligned}
\left\|\left(\nabla^k\bar{\rho},\nabla^k\bar{m}_{\shortparallel}\right)\right\|_{L^2}^2 &= \int\limits_{|\xi|\le\eta}|\xi|^{2k}\left(\left|\hat{\bar{\rho}}\right|^2+\left|\hat{\bar{m}}_{\shortparallel}\right|^2\right)d\xi+\int\limits_{|\xi|\ge\eta}|\xi|^{2k}\left(\left|\hat{\bar{\rho}}\right|^2+\left|\hat{\bar{m}}_{\shortparallel}\right|^2\right)d\xi \\
&\lesssim \int\limits_{|\xi|\le\eta}e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2t}|\xi|^{2k}\left(|\hat{\rho}_0|^2+|\hat{m}_{\shortparallel 0}|^2\right)d\xi+\int\limits_{|\xi|\ge\eta}e^{-2R_0t}|\xi|^{2k}\left(|\hat{\rho}_0|^2+|\hat{m}_{\shortparallel 0}|^2\right)d\xi \\
&\lesssim \left\|(\hat{\rho}_0,\hat{m}_{\shortparallel 0})\right\|_{L^\infty}^2\int\limits_{|\xi|\le\eta}|\xi|^{2k}e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2t}d\xi+e^{-2R_0t}\int\limits_{|\xi|\ge\eta}|\xi|^{2k}\left(|\hat{\rho}_0|^2+|\hat{m}_{\shortparallel 0}|^2\right)d\xi \\
&\lesssim (1+t)^{-\frac{3}{2}-k}\left(\left\|(\rho_0,m_{\shortparallel 0})\right\|_{L^1}^2+\left\|\nabla^k(\rho_0,m_{\shortparallel 0})\right\|_{L^2}^2\right),
\end{aligned}
$$

which proves (2.10).

Notice that

$$
\hat{\bar{W}}_{\shortparallel}=e^{-\left[4\alpha+(2\mu'+\lambda')|\xi|^2\right]t}\hat{W}_{\shortparallel 0}=\begin{cases} O(1)e^{-ct}\hat{W}_{\shortparallel 0}, & |\xi|\le\eta, \\ O(1)e^{-R_0t}\hat{W}_{\shortparallel 0}, & |\xi|\ge\eta, \end{cases}
$$

with the constants $R_0\ge c>0$, it follows that

$$
\begin{aligned}
\left\|\widehat{\nabla^k\bar{W}_{\shortparallel}}\right\|_{L^2}^2 &= \int\limits_{|\xi|\le\eta}|\xi|^{2k}\left|\hat{\bar{W}}_{\shortparallel}\right|^2d\xi+\int\limits_{|\xi|\ge\eta}|\xi|^{2k}\left|\hat{\bar{W}}_{\shortparallel}\right|^2d\xi \\
&\sim e^{-2ct}\left\|\nabla^kW_{\shortparallel 0}\right\|_{L^2}^2+e^{-2R_0t}\left\|\nabla^kW_{\shortparallel 0}\right\|_{L^2}^2 \\
&\sim e^{-2ct}\left\|\nabla^kW_{\shortparallel 0}\right\|_{L^2}^2,
\end{aligned}
$$

therefor, we prove (2.11). Hence, we complete the proof of Lemma 2.1. □

Before obtaining the lower decay rates for the solution $(\bar{\rho},\bar{m}_{\shortparallel})^T$, let's present some properties of $(\hat{\rho}_0(\xi),\hat{m}_{\shortparallel 0}(\xi))^T$ as well as $\hat{m}_{\perp 0}(\xi)$. For simplicity, we let $Y=\hat{\rho}_0(0)$ or $\hat{m}_{\perp 0}(0)$, $X=\hat{\rho}_0(\xi)$, $\hat{m}_{\shortparallel 0}(\xi)$ or $\hat{m}_{\perp 0}(\xi)$.

Lemma 2.2. *Let $|\xi|\in[0,\eta]$, under the conditions of (1.12)–(1.14), we know that*

$$
Y\neq 0, \tag{2.15}
$$

and there exist some $\bar{\xi}_i\in(0,\xi)$, with $i=1,2,3$, such that

$$
X(\xi)=X(0)+\frac{\partial X(\bar{\xi}_i)}{\partial\xi}\xi. \tag{2.16}
$$

Proof. We just prove the case $Y = \hat{\rho}_0(0)$ and $X = \hat{\rho}_0(\xi)$, the other terms can be proved in the same way. From (1.13), we can easily obtain

$$\hat{\rho}_0(0) = \int_{\mathbb{R}^3} e^{-i\xi x}\rho_0(x)dx|_{\xi=0} = \int_{\mathbb{R}^3} \rho_0(x)dx \neq 0. \tag{2.17}$$

Since $\rho_0 \in L^1$, we can derive

$$\left|\int_{\mathbb{R}^3} e^{-i\xi x}\rho_0(x)dx\right| \lesssim \int_{\mathbb{R}^3} |\rho_0(x)|dx < \infty. \tag{2.18}$$

The condition (1.14) implies

$$\left|\int_{\mathbb{R}^3} \frac{\partial\left(e^{-i\xi x}\rho_0(x)\right)}{\partial\xi}dx\right| = \left|\int_{\mathbb{R}^3} ixe^{-i\xi x}\rho_0(x)dx\right| \lesssim \int_{\mathbb{R}^3} |x\rho_0(x)|dx < \infty. \tag{2.19}$$

It concludes from (2.18)–(2.19) that $\hat{\rho}_0(\xi)$ is continuous in $[-\eta,\eta]$, and has the derivative of order one. Thus, there exists $\bar{\xi}_1 \in (0,\xi)$ such that

$$\hat{\rho}_0(\xi) = \hat{\rho}_0(0) + \frac{\partial\hat{\rho}_0(\bar{\xi}_1)}{\partial\xi}\xi. \quad \square$$

Having completed this preparatory work, we now go to the proof of the lower bound time decay rates of the solution $(\bar{\rho}, \bar{m}_{\shortparallel})^T$.

Lemma 2.3. *Under the conditions of* (1.12)–(1.14)*, we obtain*

$$\left\|(\bar{\rho}, \bar{m}_{\shortparallel})\right\|_{L^2} \geq C_0(1+t)^{-\frac{3}{4}}, \tag{2.20}$$

and

$$\left\|\nabla(\bar{\rho}, \bar{m}_{\shortparallel})\right\|_{L^2} \geq C_0(1+t)^{-\frac{5}{4}}. \tag{2.21}$$

Proof. In terms of the formula (2.2) and the frequency analysis (2.5)–(2.8), we can see when $|\xi| \leq \eta$,

$$\begin{aligned}
\hat{\bar{\rho}} &= e^{-\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\left[\cos(bt) + \left(\mu+\frac{\lambda}{2}\right)\frac{\sin(bt)}{b}|\xi|^2\right]\hat{\rho}_0 - i\xi^T\sqrt{P'(1)}\frac{\sin(bt)}{b}e^{-\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\hat{m}_{\shortparallel 0} \\
&= e^{-\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\cos(bt)\hat{\rho}_0 + \left(\mu+\frac{\lambda}{2}\right)e^{-\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\frac{\sin(bt)}{b}|\xi|^2\hat{\rho}_0 \\
&\quad - i\xi^T\sqrt{P'(1)}\frac{\sin(bt)}{b}e^{-\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\hat{m}_{\shortparallel 0}.
\end{aligned}$$

By Plancherel's Theorem, it is easy to verify that

$$
\begin{aligned}
\|\bar{\rho}\|_{L^2}^2 &= \int\limits_{|\xi|\le\eta} \left|\hat{\bar{\rho}}\right|^2 d\xi + \int\limits_{|\xi|\ge\eta} \left|\hat{\bar{\rho}}\right|^2 d\xi \\
&\gtrsim \int\limits_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\left[\cos(bt)\hat{\rho}_0 - \frac{i\xi^T\sqrt{P'(1)}\sin(bt)}{b}\hat{m}_{\shortparallel 0}\right]^2 d\xi \\
&\quad - \int\limits_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\frac{\sin^2(bt)}{|b|^2}|\xi|^4|\hat{\rho}_0|^2 d\xi - \int\limits_{|\xi|\ge\eta} e^{-2R_0 t}\left(|\hat{\rho}_0|^2 + |\hat{m}_{\shortparallel 0}|^2\right) d\xi \\
&\gtrsim \int\limits_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\left[\cos(|\xi|t)\hat{\rho}_0(0) - \sin(|\xi|t)\hat{m}_{\shortparallel 0}(0)\right]^2 d\xi \\
&\quad - \int\limits_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}|\xi|^6 t^2\left(\left|\hat{\rho}_0(0)\right|^2 + \left|\hat{m}_{\shortparallel 0}(0)\right|^2\right) d\xi \\
&\quad - \int\limits_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}|\xi|^2\left(\left|\frac{\partial\hat{\rho}_0(\bar{\xi}_1)}{\partial\xi}\right|^2 + \left|\frac{\partial\hat{m}_{\shortparallel 0}(\bar{\xi}_2)}{\partial\xi}\right|^2\right) d\xi \\
&\quad - \int\limits_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\frac{\sin^2(bt)}{|b|^2}|\xi|^4|\hat{\rho}_0|^2 d\xi - \int\limits_{|\xi|\ge\eta} e^{-2R_0 t}\left(|\hat{\rho}_0|^2 + |\hat{m}_{\shortparallel 0}|^2\right) d\xi \\
&= J_1 + J_2 + J_3 + J_4 + J_5,
\end{aligned} \tag{2.22}
$$

where we have used (2.16) and the fact that

$$
\left|\cos\left(|\xi|t + |\xi|^3 t\right) - \sin\left(|\xi|t + |\xi|^3 t\right)\right| \gtrsim |\cos(|\xi|t) - \sin(|\xi|t)| - |\xi|^3 t.
$$

J_5 is estimated by

$$
|J_5| = e^{-2R_0 t}\int\limits_{|\xi|\ge\eta}\left(|\hat{\rho}_0|^2 + |\hat{m}_{\shortparallel 0}|^2\right) d\xi \le C e^{-2R_0 t}\left\|(\rho_0, m_{\shortparallel 0})\right\|_{L^2}^2. \tag{2.23}
$$

In terms of the Taylor expansion (2.8), J_4 can be bounded as

$$
\begin{aligned}
|J_4| &\lesssim \int\limits_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}|\xi|^2|\hat{\rho}_0|^2 d\xi \\
&\lesssim \|\hat{\rho}_0\|_{L^\infty}^2 \int\limits_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}|\xi|^2 d\xi \\
&\lesssim (1+t)^{-\frac{5}{2}}\|\rho_0\|_{L^1}^2.
\end{aligned} \tag{2.24}
$$

Like (2.24), we deduce

$$|J_2| + |J_3| \lesssim (1+t)^{-\frac{5}{2}}. \tag{2.25}$$

To estimate the term J_1 on the right-hand side of (2.22), we make the change of variables $y = |\xi|\sqrt{t}$, to deduce for sufficiently large time t

$$\begin{aligned}
J_1 &\gtrsim t^{-\frac{3}{2}} \int\limits_{y\le\eta\sqrt{t}} e^{-2\left(\mu+\frac{\lambda}{2}\right)y^2} \left[\cos\left(y\sqrt{t}\right)\hat{\rho}_0(0) - \sin\left(y\sqrt{t}\right)\hat{m}_{\shortparallel 0}(0)\right]^2 dy \\
&\ge C_0 t^{-\frac{3}{2}} \sum_{k=0}^{[\eta t/\pi]-1} \int\limits_{\frac{k\pi}{\sqrt{t}}}^{\frac{k\pi+\frac{\pi}{12}}{\sqrt{t}}} e^{-2\left(\mu+\frac{\lambda}{2}\right)y^2} \left[\cos\left(y\sqrt{t}\right) - \sin\left(y\sqrt{t}\right)\right]^2 dy \\
&\ge C_0 t^{-\frac{3}{2}} \sum_{k=0}^{[\eta t/\pi]-1} \int\limits_{\frac{k\pi}{\sqrt{t}}}^{\frac{k\pi+\frac{\pi}{12}}{\sqrt{t}}} e^{-2\left(\mu+\frac{\lambda}{2}\right)y^2} \cos^2\left(y\sqrt{t}+\pi/4\right) dy \\
&\ge C_0 t^{-\frac{3}{2}} \sum_{k=0}^{[\eta t/\pi]-1} \int\limits_{\frac{k\pi}{\sqrt{t}}}^{\frac{k\pi+\frac{\pi}{12}}{\sqrt{t}}} e^{-2\left(\mu+\frac{\lambda}{2}\right)y^2} dy \\
&\ge C_0(1+t)^{-\frac{3}{2}},
\end{aligned} \tag{2.26}$$

with some positive constant C_0 depending on $\hat{\rho}_0(0)$ and $\hat{m}_{\shortparallel 0}(0)$.

In terms of (2.23)–(2.26), we know that

$$\|\bar{\rho}\|_{L^2}^2 \ge C_0(1+t)^{-\frac{3}{2}} - C(1+t)^{-\frac{5}{2}} \ge C_0(1+t)^{-\frac{3}{2}},$$

for t large enough. Likewise, one can obtain

$$\left\|\bar{m}_{\shortparallel}\right\|_{L^2}^2 \ge C_0(1+t)^{-\frac{3}{2}}. \tag{2.27}$$

Now we deal with the decay rates for the first–order derivative of $(\bar{\rho}, \bar{m}_{\shortparallel})^T$. We employ the Plancherel's Theorem and use the same method as proving (2.20) to imply

$$\begin{aligned}
\|\nabla\bar{\rho}\|_{L^2}^2 &= \int\limits_{|\xi|\le\eta} |\xi|^2 \left|\hat{\bar{\rho}}\right|^2 d\xi + \int\limits_{|\xi|\ge\eta} |\xi|^2 \left|\hat{\bar{\rho}}\right|^2 d\xi \\
&\gtrsim \int\limits_{|\xi|\le\eta} |\xi|^2 e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t} \left[\cos(bt)\hat{\rho}_0 - \frac{i\xi^T\sqrt{P'(1)}\sin(bt)}{b}\hat{m}_{\shortparallel 0}\right]^2 d\xi
\end{aligned}$$

$$
\begin{aligned}
&-\int_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\frac{\sin^2(bt)}{|b|^2}|\xi|^6|\hat{\rho}_0|^2 d\xi-\int_{|\xi|\ge\eta} e^{-2R_0 t}|\xi|^2\left(|\hat{\rho}_0|^2+|\hat{m}_{\shortparallel 0}|^2\right)d\xi\\
&\ge C_0(1+t)^{-\frac{5}{2}}-C(1+t)^{-\frac{7}{2}}\ge C_0(1+t)^{-\frac{5}{2}}.
\end{aligned}
\tag{2.28}
$$

Similarly,

$$
\left\|\nabla\bar{m}_{\shortparallel}\right\|_{L^2}^2\ge C_0(1+t)^{-\frac{5}{2}}.
\tag{2.29}
$$

To this end, we have proved (2.21) of Lemma 2.3. □

3. Decay rates for the solution of the linearized electromagnetic part

In this section, we should deal with the electromagnetic part. First of all, we give the spectral analysis on the semigroup generated by the linearized electromagnetic part.

3.1. Spectral representation for electromagnetic part

Recall that the electromagnetic part $\bar{U}_\perp=(\bar{m}_\perp,\bar{W}_\perp)^T$ satisfies the following equations

$$
\begin{cases}
\partial_t\bar{m}_\perp-(\mu+\alpha)\Delta\bar{m}_\perp-2\alpha\nabla\times\bar{W}_\perp=0,\\
\partial_t\bar{W}_\perp+4\alpha\bar{W}_\perp-\mu'\Delta\bar{W}_\perp-2\alpha\nabla\times\bar{m}_\perp=0,\\
(\bar{m}_\perp,\bar{W}_\perp)^T|_{t=0}=(m_{\perp 0},W_{\perp 0})^T.
\end{cases}
\tag{3.1}
$$

From [30], we know that

$$
\begin{cases}
\hat{\bar{m}}_\perp=\left(\frac{k_1e^{k_2t}-k_2e^{k_1t}}{k_1-k_2}-(\mu+\alpha)|\xi|^2\frac{e^{k_1t}-e^{k_2t}}{k_1-k_2}\right)\hat{m}_{\perp 0}+\frac{e^{k_1t}-e^{k_2t}}{k_1-k_2}2\alpha i\xi\times\hat{W}_{\perp 0},\\
\hat{\bar{W}}_\perp=\frac{e^{k_1t}-e^{k_2t}}{k_1-k_2}2\alpha i\xi\times\hat{m}_{\perp 0}+\left(\frac{k_1e^{k_2t}-k_2e^{k_1t}}{k_1-k_2}-(\mu'|\xi|^2+4\alpha)\frac{e^{k_1t}-e^{k_2t}}{k_1-k_2}\right)\hat{W}_{\perp 0},
\end{cases}
\tag{3.2}
$$

with the real character roots

$$
\begin{cases}
k_1=\frac{-\left[(\mu+\alpha+\mu')|\xi|^2+4\alpha\right]+\sqrt{(\mu+\alpha-\mu')^2|\xi|^4+16\alpha^2+8\alpha(\mu'+\alpha-\mu)|\xi|^2}}{2},\\
k_2=\frac{-\left[(\mu+\alpha+\mu')|\xi|^2+4\alpha\right]-\sqrt{(\mu+\alpha-\mu')^2|\xi|^4+16\alpha^2+8\alpha(\mu'+\alpha-\mu)|\xi|^2}}{2}.
\end{cases}
\tag{3.3}
$$

It is obvious that (3.2) can be rewritten as

$$
\begin{pmatrix}\hat{\bar{m}}_\perp\\ \hat{\bar{W}}_\perp\end{pmatrix}=\hat{G}\begin{pmatrix}\hat{m}_{\perp 0}\\ \hat{W}_{\perp 0}\end{pmatrix},
\tag{3.4}
$$

where

$$\hat{G} = \begin{pmatrix} \frac{k_1 e^{k_2 t} - k_2 e^{k_1 t}}{k_1 - k_2} - (\mu + \alpha)|\xi|^2 \frac{e^{k_1 t} - e^{k_2 t}}{k_1 - k_2} & \frac{e^{k_1 t} - e^{k_2 t}}{k_1 - k_2} 2\alpha i \xi \times \\ \frac{e^{k_1 t} - e^{k_2 t}}{k_1 - k_2} 2\alpha i \xi \times & \frac{k_1 e^{k_2 t} - k_2 e^{k_1 t}}{k_1 - k_2} - (\mu' |\xi|^2 + 4\alpha) \frac{e^{k_1 t} - e^{k_2 t}}{k_1 - k_2} \end{pmatrix}.$$

Here we use the notations $\hat{G}_{11}$, $\hat{G}_{12}$, $\hat{G}_{21}$ and $\hat{G}_{22}$ to denote the four elements of $\hat{G}$ for simplicity. Due to the definition of $k_{1,2}$ in (3.3), in terms of Taylor's expansion, we can easily find that when $|\xi| \ll 1$,

$$\begin{aligned} k_1 &= -\mu|\xi|^2 + O(|\xi|^4), \\ k_2 &= -4\alpha - (\mu' + \alpha)|\xi|^2 + O(|\xi|^4), \\ k_1 - k_2 &= 4\alpha + (\mu' + \alpha - \mu)|\xi|^2 + O(|\xi|^4), \end{aligned} \tag{3.5}$$

which implies

$$\begin{aligned} \hat{G}_{11} &= \frac{\alpha|\xi|^2 + O(|\xi|^4)}{4\alpha + (\mu' + \alpha - \mu)|\xi|^2 + O(|\xi|^4)} e^{-4\alpha t - (\mu' + \alpha)|\xi|^2 t + O(|\xi|^4) t} \\ &\quad - \frac{-4\alpha - (\mu' - \mu)|\xi|^2 + O(|\xi|^4)}{4\alpha + (\mu' + \alpha - \mu)|\xi|^2 + O(|\xi|^4)} e^{-\mu|\xi|^2 t + O(|\xi|^4) t} \\ &\sim |\xi|^2 e^{-ct} + e^{-\mu|\xi|^2 t}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \hat{G}_{12} = \hat{G}_{21} &= \frac{2\alpha i \xi}{4\alpha + (\mu' + \alpha - \mu)|\xi|^2 + O(|\xi|^4)} e^{k_1 t} \left[1 - e^{-(4\alpha + (\mu' + \alpha - \mu)|\xi|^2 + O(|\xi|^4))t} \right] \\ &\sim |\xi| e^{-\mu|\xi|^2 t}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \hat{G}_{22} &= \frac{(\mu' - \mu)|\xi|^2 + 4\alpha + O(|\xi|^4)}{4\alpha + (\mu' + \alpha - \mu)|\xi|^2 + O(|\xi|^4)} e^{-4\alpha t - (\mu' + \alpha)|\xi|^2 t + O(|\xi|^4) t} \\ &\quad - \frac{-\alpha|\xi|^2 + O(|\xi|^4)}{4\alpha + (\mu' + \alpha - \mu)|\xi|^2 + O(|\xi|^4)} e^{-\mu|\xi|^2 t + O(|\xi|^4) t} \\ &\sim e^{-ct} + |\xi|^2 e^{-\mu|\xi|^2 t}. \end{aligned} \tag{3.8}$$

When $|\xi| \gg 1$, from [30], we know

$$|\hat{G}_{11}| + |\hat{G}_{12}| + |\hat{G}_{21}| + |\hat{G}_{22}| \le C e^{-R_0 t}. \tag{3.9}$$

Now we estimate the decay rates of $(\hat{\bar{m}}_\perp, \hat{\bar{W}}_\perp)^T$ according to the analysis on the terms of $\hat{G}$.

3.2. The L^2 decay rates of $(\bar{m}_\perp, \bar{W}_\perp)^T$

With the help of the analysis on the terms of $\hat{G}$, we show the following results.

Lemma 3.1. *It holds that for $k = 0, 1, 2, 3,$*

$$\left\|\nabla^k \bar{m}_\perp\right\|_{L^2}^2 \lesssim (1+t)^{-\frac{3}{2}-k}\left(\|(m_{\perp 0}, W_{\perp 0})\|_{L^1}^2 + \left\|\nabla^k (m_{\perp 0}, W_{\perp 0})\right\|_{L^2}^2\right), \tag{3.10}$$

$$\left\|\nabla^k \bar{W}_\perp\right\|_{L^2}^2 \lesssim (1+t)^{-\frac{5}{2}-k}\left(\|(m_{\perp 0}, W_{\perp 0})\|_{L^1}^2 + \left\|\nabla^k (m_{\perp 0}, W_{\perp 0})\right\|_{L^2}^2\right), \tag{3.11}$$

and for $l = 0, 1,$

$$\left\|\nabla^l \bar{m}_\perp\right\|_{L^2}^2 \geq C_0(1+t)^{-\frac{3}{2}-l}, \tag{3.12}$$

$$\|\bar{W}_\perp\|_{L^2}^2 \geq C_0(1+t)^{-\frac{5}{2}}. \tag{3.13}$$

Proof. Since

$$\hat{\bar{m}}_\perp = \hat{G}_{11}\hat{m}_{\perp 0} + \hat{G}_{12}\hat{W}_{\perp 0}, \tag{3.14}$$

based on Plancherel's Theorem, (3.6)–(3.7) and (3.9), one can deduce

$$\begin{aligned}
\left\|\nabla^k \bar{m}_\perp\right\|_{L^2}^2 = \left\|\widehat{\nabla^k \bar{m}}_\perp\right\|_{L^2}^2 &\lesssim \int\limits_{|\xi|\leq\eta} |\xi|^{2k}\left|\hat{G}_{11}\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}\left|\hat{G}_{12}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + \int\limits_{|\xi|\geq\eta} |\xi|^{2k}\left|\hat{G}_{11}\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}\left|\hat{G}_{12}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\lesssim \int\limits_{|\xi|\leq\eta} |\xi|^{2k}e^{-2\mu|\xi|^2 t}\left|\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k+2}e^{-2\mu|\xi|^2 t}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + \int\limits_{|\xi|\leq\eta} |\xi|^{2k+4}e^{-2ct}\left|\hat{m}_{\perp 0}\right|^2 d\xi \\
&\quad + \int\limits_{|\xi|\geq\eta} |\xi|^{2k}e^{-2R_0 t}\left|\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}e^{-2R_0 t}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\lesssim \left\|\left(\hat{m}_{\perp 0}, \hat{W}_{\perp 0}\right)\right\|_{L^\infty}^2 \int\limits_{|\xi|\leq\eta} |\xi|^{2k}e^{-2\mu|\xi|^2 t}d\xi + e^{-2ct}\int\limits_{|\xi|\leq\eta} |\xi|^{2k}\left|\hat{m}_{\perp 0}\right|^2 d\xi \\
&\quad + e^{-2R_0 t}\int\limits_{|\xi|\geq\eta} |\xi|^{2k}\left(|\hat{m}_{\perp 0}|^2 + |\hat{W}_{\perp 0}|^2\right) d\xi \\
&\lesssim (1+t)^{-\frac{3}{2}-k}\left(\|(m_{\perp 0}, W_{\perp 0})\|_{L^1}^2 + \left\|\nabla^k (m_{\perp 0}, W_{\perp 0})\right\|_{L^2}^2\right).
\end{aligned} \tag{3.15}$$

Further more, we just estimate like (3.15) and it derives from (3.7)–(3.9)

$$
\begin{aligned}
\left\|\nabla^k \bar{W}_\perp\right\|_{L^2}^2 = \left\|\widehat{\nabla^k \bar{W}_\perp}\right\|_{L^2}^2 &\lesssim \int\limits_{|\xi|\le\eta} |\xi|^{2k}\left|\hat{G}_{21}\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}\left|\hat{G}_{22}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + \int\limits_{|\xi|\ge\eta} |\xi|^{2k}\left|\hat{G}_{21}\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}\left|\hat{G}_{22}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\lesssim \int\limits_{|\xi|\le\eta} |\xi|^{2k+2}e^{-2\mu|\xi|^2 t}\left|\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k+4}e^{-2\mu|\xi|^2 t}\left|\hat{W}_{\perp 0}\right|^2 + |\xi|^{2k}e^{-2ct}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + \int\limits_{|\xi|\ge\eta} |\xi|^{2k}e^{-2R_0 t}\left|\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}e^{-2R_0 t}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\lesssim \left\|\left(\hat{m}_{\perp 0}, \hat{W}_{\perp 0}\right)\right\|_{L^\infty}^2 \int\limits_{|\xi|\le\eta} |\xi|^{2k+2}e^{-2\mu|\xi|^2 t} d\xi + e^{-2ct}\int\limits_{|\xi|\le\eta} |\xi|^{2k}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + e^{-2R_0 t}\int\limits_{|\xi|\ge\eta} |\xi|^{2k}\left(|\hat{m}_{\perp 0}|^2 + |\hat{W}_{\perp 0}|^2\right) d\xi \\
&\lesssim (1+t)^{-\frac{5}{2}-k}\left(\|(m_{\perp 0}, W_{\perp 0})\|_{L^1}^2 + \left\|\nabla^k (m_{\perp 0}, W_{\perp 0})\right\|_{L^2}^2\right).
\end{aligned}
$$

On the other hand, together with (2.16), it stems from (3.6)–(3.7) and (3.9) again that

$$
\begin{aligned}
\left\|\nabla^l \hat{\bar{m}}_\perp\right\|_{L^2}^2 = &\int\limits_{|\xi|\le\eta} |\xi|^{2l}\left|\hat{G}_{11}\hat{m}_{\perp 0} + \hat{G}_{12}\hat{W}_{\perp 0}\right|^2 d\xi \\
&+ \int\limits_{|\xi|\ge\eta} |\xi|^{2l}\left|\hat{G}_{11}\hat{m}_{\perp 0} + \hat{G}_{12}\hat{W}_{\perp 0}\right|^2 d\xi \\
\gtrsim &\int\limits_{|\xi|\le\eta} |\xi|^{2l}e^{-2\mu|\xi|^2 t}\left|\hat{m}_{\perp 0}\right|^2 d\xi - \int\limits_{|\xi|\le\eta} |\xi|^{2l+4}e^{-2ct}\left|\hat{m}_{\perp 0}\right|^2 d\xi \\
&- \int\limits_{|\xi|\le\eta} |\xi|^{2l+2}e^{-2\mu|\xi|^2 t}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&- \int\limits_{|\xi|\ge\eta} e^{-2R_0 t}|\xi|^{2l}\left(\left|\hat{m}_{\perp 0}\right|^2 + \left|\hat{W}_{\perp 0}\right|^2\right) d\xi \\
\gtrsim &\int\limits_{|\xi|\le\eta} |\xi|^{2l}e^{-2\mu|\xi|^2 t} d\xi \left|\hat{m}_{\perp 0}(0)\right|^2 - \int\limits_{|\xi|\le\eta} |\xi|^{2l+2}e^{-2\mu|\xi|^2 t}\left|\frac{\partial \hat{m}_{\perp 0}(\bar{\xi}_3)}{\partial \xi}\right|^2 d\xi
\end{aligned}
$$

$$
\begin{aligned}
&- e^{-2ct} \int\limits_{|\xi|\le\eta} |\xi|^{2l+4} \left|\hat{m}_{\perp 0}\right|^2 d\xi - \int\limits_{|\xi|\le\eta} |\xi|^{2l+2} e^{-2\mu|\xi|^2 t} \left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&- e^{-2R_0 t} \int\limits_{|\xi|\ge\eta} |\xi|^{2l} \left(|\hat{m}_{\perp 0}|^2 + |\hat{W}_{\perp 0}|^2\right) d\xi \\
\ge\ & C_0(1+t)^{-\frac{3}{2}-l},
\end{aligned}
$$

with a positive constant C_0 depending on $\hat{m}_{\perp 0}(0)$. With this method once again, it gives

$$
\begin{aligned}
\left\|\hat{\bar{W}}_\perp\right\|_{L^2}^2 &= \int\limits_{|\xi|\le\eta} \left|\hat{G}_{21}\hat{m}_{\perp 0} + \hat{G}_{22}\hat{W}_{\perp 0}\right|^2 d\xi + \int\limits_{|\xi|\ge\eta} \left|\hat{G}_{21}\hat{m}_{\perp 0} + \hat{G}_{22}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\gtrsim \int\limits_{|\xi|\le\eta} |\xi|^2 e^{-2\mu|\xi|^2 t} \left|\hat{m}_{\perp 0}\right|^2 d\xi - \int\limits_{|\xi|\le\eta} |\xi|^4 e^{-2\mu|\xi|^2 t} \left|\hat{W}_{\perp 0}\right|^2 d\xi - \int\limits_{|\xi|\le\eta} e^{-2ct} \left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad - \int\limits_{|\xi|\ge\eta} e^{-2R_0 t} \left(\left|\hat{m}_{\perp 0}\right|^2 + \left|\hat{W}_{\perp 0}\right|^2\right) d\xi \\
&\gtrsim \int\limits_{|\xi|\le\eta} |\xi|^2 e^{-2\mu|\xi|^2 t} d\xi \left|\hat{m}_{\perp 0}(0)\right|^2 - \int\limits_{|\xi|\le\eta} |\xi|^4 e^{-2\mu|\xi|^2 t} \left|\frac{\partial \hat{m}_{\perp 0}(\bar{\xi}_3)}{\partial \xi}\right|^2 d\xi \\
&\quad - \int\limits_{|\xi|\le\eta} |\xi|^4 e^{-2\mu|\xi|^2 t} \left|\hat{W}_{\perp 0}\right|^2 d\xi - e^{-2ct} \int\limits_{|\xi|\le\eta} \left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad - e^{-2R_0 t} \int\limits_{|\xi|\ge\eta} \left(|\hat{m}_{\perp 0}|^2 + |\hat{W}_{\perp 0}|^2\right) d\xi \\
&\ge C_0(1+t)^{-\frac{5}{2}}.
\end{aligned}
\tag{3.16}
$$

This completes the proof of Lemma 3.1. □

It should be noted that the solution $(\bar{\rho}, \bar{m}, \bar{W})$ of the linearized micropolar fluids system has an optimal decay rate. Indeed, from Lemma 2.1, Lemma 2.3 and Lemma 3.1, we can present the following important proposition.

Proposition 3.2. *Under the conditions of* (1.12)–(1.14)*, it holds for* $l = 0,\ 1$

$$
C_0(1+t)^{-\frac{3}{4}-\frac{l}{2}} \le \left\|\nabla^l(\bar{\rho}, \bar{m})\right\|_{L^2} \le C\, \|(\rho_0, m_0, W_0)\|_{H^3\cap L^1} (1+t)^{-\frac{3}{4}-\frac{l}{2}}, \tag{3.17}
$$

and

$$
C_0(1+t)^{-\frac{5}{4}} \le \|\bar{W}\|_{L^2} \le C\, \|(\rho_0, m_0, W_0)\|_{H^3\cap L^1} (1+t)^{-\frac{5}{4}}, \tag{3.18}
$$

with $C_0 > 0$ *depending on* $\hat{\rho}_0(0)$, $\hat{m}_{\shortparallel 0}(0)$, $\hat{m}_{\perp 0}(0)$ *suitably smaller than* δ *given in Theorem 1.1.*

Proof. The upper decay rates hold clearly. For the lower decay rates, in view of (3.13) and (2.11), we can obtain

$$\|\bar{W}\|_{L^2} = \|\bar{W}_{\shortparallel} + \bar{W}_{\perp}\|_{L^2} \geq \|\bar{W}_{\perp}\|_{L^2} - \|\bar{W}_{\shortparallel}\|_{L^2} \geq C_0(1+t)^{-\frac{5}{4}} - Ce^{-ct} \geq C_0(1+t)^{-\frac{5}{4}}.$$

For the term $\bar{m}$, from (2.12) and (3.14), and use the same method as proving (2.20)–(2.21), we get

$$\left\|\nabla^l \bar{m}\right\|_{L^2}^2 = \left\|\nabla^l \bar{m}_{\shortparallel} + \nabla^l \bar{m}_{\perp}\right\|_{L^2}^2 \geq C_0(1+t)^{-\frac{3}{2}-l}. \quad \square$$

4. Estimates on the nonlinear equations

In this section, we want to deduce the L^2 decay rates for the nonlinear system. Define

$$\rho_h = \rho - \bar{\rho},\ m_{h\shortparallel} = m_{\shortparallel} - \bar{m}_{\shortparallel},\ W_{h\shortparallel} = W_{\shortparallel} - \bar{W}_{\shortparallel},\ m_{h\perp} = m_{\perp} - \bar{m}_{\perp},\ W_{h\perp} = W_{\perp} - \bar{W}_{\perp}.$$

The nonlinear system (1.2)–(1.3) is reformulated as follows:

$$\begin{cases} \partial_t \rho_h + \sqrt{P'(1)}\mathrm{div} m_{h\shortparallel} = 0, \\ \partial_t m_{h\shortparallel} + \sqrt{P'(1)}\nabla \rho_h - (2\mu+\lambda)\Delta m_{h\shortparallel} = \Delta^{-1}\nabla\mathrm{div}\mathbb{N}_1 = F_1, \\ \partial_t W_{h\shortparallel} + 4\alpha W_{h\shortparallel} - (2u'+\lambda')\Delta W_{h\shortparallel} = \Delta^{-1}\nabla\mathrm{div}\mathbb{N}_2 = F_2, \\ (\rho_h, m_{h\shortparallel}, W_{h\shortparallel})^T|_{t=0} = (\rho_{h0}, m_{h\shortparallel 0}, W_{h\shortparallel 0})^T = (0,0,0)^T, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \partial_t m_{h\perp} - (\mu+\alpha)\Delta m_{h\perp} - 2\alpha\nabla\times W_{h\perp} = N_1, \\ \partial_t W_{h\perp} + 4\alpha W_{h\perp} - \mu'\Delta W_{h\perp} - 2\alpha\nabla\times m_{h\perp} = N_2, \\ (m_{h\perp}, W_{h\perp})^T|_{t=0} = (m_{h\perp 0}, W_{h\perp 0})^T = (0,0)^T, \end{cases} \tag{4.2}$$

where F_1, F_2, N_1, N_2 are defined as (1.6)–(1.7) and (1.9)–(1.10). For the sake of convenience, we denote

$$F_1 = \nabla f_1 + \mathrm{div} f_2, \quad N_1 = \nabla f,$$

with

$$f_1 \sim \rho^2 + \mathrm{div}\left(\frac{\rho m}{1+\rho}\right),\ f_2 \sim \frac{m\otimes m}{1+\rho} + \nabla\left(\frac{\rho m}{1+\rho}\right),\ f \sim \nabla\left(\frac{\rho m}{1+\rho}\right) + \frac{\rho W}{1+\rho}.$$

In order to deduce the L^2 decay rates for the nonlinear system, we just need to prove that the solutions $(\rho_h, m_{h\shortparallel}, W_{h\shortparallel})^T$ and $(m_{h\perp}, W_{h\perp})^T$ to (4.1) and (4.2) have a faster decay rates than $(\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel})^T$ and $(\bar{m}_{\perp}, \bar{W}_{\perp})^T$.

We can represent the solution in terms of the Duhamel principle

$$\begin{cases} \hat{U}_{1h_{\shortparallel}} = (\hat{\rho}_h, \hat{m}_{h_{\shortparallel}})^T = \int_0^t e^{(t-s)\mathbb{A}}(0, \hat{F}_1)^T ds, \\ \hat{W}_{h_{\shortparallel}} = \int_0^t e^{-\left[4\alpha+(2\mu'+\lambda')|\xi|^2\right](t-s)} \hat{F}_2(s) ds, \\ \hat{U}_{h\perp} = (\hat{m}_{h\perp}, \hat{W}_{h\perp})^T = \int_0^t \hat{G}(\xi, t-s)(\hat{N}_1, \hat{N}_2)^T(s) ds. \end{cases} \tag{4.3}$$

In the following, we should assume

$$\begin{aligned} N(t) = \sup_{0\le\tau\le t} \Big\{ & \left\| (\rho_h, m_{h_{\shortparallel}}, W_{h_{\shortparallel}}, m_{h\perp}, W_{h\perp}) \right\|_{H^2} (1+\tau)^{\frac{3}{4}+\frac{\epsilon}{2}} \\ & + \left\| \nabla^3 (\rho_h, m_{h_{\shortparallel}}, W_{h_{\shortparallel}}, m_{h\perp}, W_{h\perp}) \right\|_{L^2} \Big\}. \end{aligned} \tag{4.4}$$

We claim that for $0 \le \epsilon \le 1$,

$$N(t) \le C\delta, \tag{4.5}$$

with δ defined in Theorem 1.1.

Lemma 4.1. *Under the assumption of* (4.4) *and Proposition 3.2, we have*

$$\left\| (\rho_h, m_{h_{\shortparallel}}, W_{h_{\shortparallel}}, m_{h\perp}, W_{h\perp}) \right\|_{L^2} \lesssim (1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)} \left(\delta^2 + N^2(t)\right). \tag{4.6}$$

Proof. From (4.3) and (2.4), we know

$$\hat{\rho}_h(\xi, t) = -\int_0^t \frac{i\xi^T \sqrt{P'(1)} \left(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}\right)}{\lambda_1 - \lambda_2} \hat{F}_1(s) ds,$$

and

$$\hat{m}_{h_{\shortparallel}}(\xi, t) = \int_0^t \frac{\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)}}{\lambda_1 - \lambda_2} \hat{F}_1(s) ds.$$

It derives from (2.5)–(2.9) that

$$
\begin{aligned}
&\left\| \frac{i\xi^T \left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)}{\lambda_1 - \lambda_2} \hat{F}_1 \right\|_{L^2}^2 + \left\| \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \hat{F}_1 \right\|_{L^2}^2 \\
&\le \int_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t} \left|\hat{F}_1\right|^2 d\xi + \int_{\xi\ge\eta} e^{-2R_0 t} \left|\hat{F}_1\right|^2 d\xi \\
&\le \int_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t} |\xi|^2 \left|\left(\hat{f}_1, \hat{f}_2\right)\right|^2 d\xi + e^{-2R_0 t} \int_{\xi\ge\eta} \left|\hat{F}_1\right|^2 d\xi \\
&\le C(1+t)^{-\frac{5}{2}} \left(\|(f_1, f_2)\|_{L^1}^2 + \|F_1\|_{L^2}^2 \right).
\end{aligned}
$$

Thus, one has

$$
\left\|(\rho_h, m_{h_{11}})\right\|_{L^2} \le C \int_0^t (1+t-s)^{-\frac{5}{4}} \left(\|f_1\|_{L^1} + \|f_2\|_{L^1} + \|F_1(s)\|_{L^2} \right) ds, \tag{4.7}
$$

and

$$
\left\| W_{h_{11}} \right\|_{L^2} \le C \int_0^t e^{-c(t-s)} \|F_2(s)\|_{L^2} \, ds. \tag{4.8}
$$

In light of (3.6)–(3.9), it can be computed directly

$$
\begin{aligned}
\left\| \hat{G}_{11} \hat{N}_1(s) \right\|_{L^2}^2 &\lesssim \int_{|\xi|\le\eta} |\xi|^4 e^{-2ct} \left|\hat{N}_1\right|^2 + e^{-2\mu|\xi|^2 t} \left|\hat{N}_1\right|^2 d\xi + \int_{\xi\ge\eta} e^{-2R_0 t} \left|\hat{N}_1\right|^2 d\xi \\
&\lesssim \int_{|\xi|\le\eta} |\xi|^4 e^{-2ct} \left|\hat{N}_1\right|^2 d\xi + \int_{|\xi|\le\eta} e^{-2\mu|\xi|^2 t} |\xi|^2 \left|\hat{f}\right|^2 d\xi + \int_{\xi\ge\eta} e^{-2R_0 t} \left|\hat{N}_1\right|^2 d\xi \\
&\lesssim (1+t)^{-\frac{5}{2}} \left(\|f\|_{L^1}^2 + \|N_1\|_{L^2}^2 \right),
\end{aligned}
$$

and

$$
\begin{aligned}
\left\| \hat{G}_{12} \hat{N}_2(s) \right\|_{L^2}^2 &\lesssim \int_{|\xi|\le\eta} |\xi|^2 e^{-2\mu|\xi|^2 t} \left|\hat{N}_2\right|^2 d\xi + \int_{\xi\ge\eta} e^{-2R_0 t} \left|\hat{N}_2\right|^2 d\xi \\
&\lesssim (1+t)^{-\frac{5}{2}} \left(\|N_2\|_{L^1}^2 + \|N_2\|_{L^2}^2 \right).
\end{aligned}
$$

Likely, we have

$$
\left\| \hat{G}_{21} \hat{N}_1(s) \right\|_{L^2}^2 \lesssim \int_{|\xi|\le\eta} |\xi|^2 e^{-2\mu|\xi|^2 t} \left|\hat{N}_1\right|^2 d\xi + \int_{\xi\ge\eta} e^{-2R_0 t} \left|\hat{N}_1\right|^2 d\xi
$$

$$\lesssim \int\limits_{|\xi|\leq\eta} e^{-2\mu|\xi|^2 t}|\xi|^4 \left|\hat{f}\right|^2 d\xi + \int\limits_{\xi\geq\eta} e^{-2R_0 t}\left|\hat{N}_1\right|^2 d\xi$$
$$\lesssim (1+t)^{-\frac{7}{2}}\left(\|f\|_{L^1}^2 + \|N_1\|_{L^2}^2\right),$$

and

$$\left\|\hat{G}_{22}\hat{N}_2(s)\right\|_{L^2}^2 \lesssim \int\limits_{|\xi|\leq\eta} e^{-2ct}\left|\hat{N}_2\right|^2 d\xi + \int\limits_{|\xi|\leq\eta} |\xi|^4 e^{-2\mu|\xi|^2 t}\left|\hat{N}_2\right|^2 d\xi + \int\limits_{\xi\geq\eta} e^{-2R_0 t}\left|\hat{N}_2\right|^2 d\xi$$
$$\lesssim e^{-2ct}\|N_2\|_{L^2}^2 + (1+t)^{-\frac{7}{2}}\left(\|N_2\|_{L^1}^2 + \|N_2\|_{L^2}^2\right)$$
$$\lesssim (1+t)^{-\frac{7}{2}}\left(\|N_2\|_{L^1}^2 + \|N_2\|_{L^2}^2\right).$$

Therefore, we end up with

$$\|m_{h\perp}\|_{L^2} \leq \int_0^t (1+t-s)^{-\frac{5}{4}}\left(\|f(s)\|_{L^1} + \|(N_1,N_2)(s)\|_{L^1\cap L^2}\right) ds, \tag{4.9}$$

and

$$\|W_{h\perp}\|_{L^2} \leq \int_0^t (1+t-s)^{-\frac{7}{4}}\left(\|f(s)\|_{L^1} + \|(N_1,N_2)(s)\|_{L^1\cap L^2}\right) ds. \tag{4.10}$$

Now, it turns to estimate the nonlinear terms f_1, f_2, f, F_1, F_2, N_1, N_2. Note that we denote

$$m_h = m - \bar{m} = m_{\shortparallel} + m_\perp - \bar{m}_{\shortparallel} - \bar{m}_\perp = m_{h\shortparallel} + m_{h\perp},$$
$$W_h = W - \bar{W} = W_{\shortparallel} + W_\perp - \bar{W}_{\shortparallel} - \bar{W}_\perp = W_{h\shortparallel} + W_{h\perp}.$$

It is easy to verify from (4.4), Lemma 2.1, Lemma 3.1 and Lemma A.2 that

$$\left\|\frac{m\otimes m}{1+\rho}\right\|_{L^1} \lesssim \|\bar{m}\|_{L^2}^2 + \|\bar{m}\|_{L^2}\|m_h\|_{L^2} + \|m_h\|_{L^2}^2 \lesssim (1+t)^{-\frac{3}{2}}(\delta^2 + N^2(t)), \tag{4.11}$$

likewise, one has

$$\|\nabla\rho\cdot\nabla m\|_{L^1} + \left\|\rho\nabla^2 m\right\|_{L^1} + \|\rho\nabla m\|_{L^1} \lesssim (1+t)^{-\left(\frac{3}{2}+\frac{\epsilon}{2}\right)}(\delta^2 + N^2(t)). \tag{4.12}$$

We can estimate the remaining terms as (4.11) and (4.12), hence

$$\|(f_1, f_2, f, N_1, N_2)\|_{L^1} \lesssim (1+t)^{-\frac{3}{2}}\left(\delta^2 + N^2(t)\right). \tag{4.13}$$

It derives from the assumption (4.4), Lemma 2.1, Lemma 3.1 and Lemma A.2 once again that

$$\begin{aligned}\|m\cdot\nabla m\|_{L^2} &= \|\bar m\|_{L^\infty}\|\nabla\bar m\|_{L^2}+\|\bar m\|_{L^\infty}\|\nabla m_h\|_{L^2}+\|\nabla\bar m\|_{L^2}\|m_h\|_{L^\infty}+\|m_h\|_{L^\infty}\|\nabla m_h\|_{L^2}\\ &\lesssim (1+t)^{-2\left(\frac34+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)),\end{aligned}\tag{4.14}$$

and likely

$$\|\rho W\|_{L^2}+\left\|\rho\nabla^2 m\right\|_{L^2}+\|\nabla m\cdot\nabla\rho\|_{L^2}\lesssim(1+t)^{-\left(\frac32+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)).\tag{4.15}$$

We can estimate like (4.14)–(4.15) to imply

$$\|(F_1,F_2,N_1,N_2)\|_{L^2}\lesssim(1+t)^{-\left(\frac32+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)).\tag{4.16}$$

In light of (4.7)–(4.10), (4.13) and (4.16) together with (A.1) of Lemma A.1, we have

$$\begin{aligned}\left\|((\rho_h,m_{h\shortparallel},m_{h\perp})\right\|_{L^2}&\le C\int_0^t(1+t-s)^{-\frac54}(1+s)^{-\frac32}ds(\delta^2+N^2(t))\le C(1+t)^{-\frac54}(\delta^2+N^2)\\ &\le C(1+t)^{-\left(\frac34+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t));\\ \left\|W_{h\shortparallel}\right\|_{L^2}&\lesssim\int_0^t e^{-c(t-s)}(1+s)^{-\frac32}ds(\delta^2+N^2(t))\lesssim(1+t)^{-\frac32}(\delta^2+N^2(t))\end{aligned}$$

and

$$\|W_{h\perp}\|_{L^2}\le C\int_0^t(1+t-s)^{-\frac74}(1+s)^{-\frac32}ds(\delta^2+N^2(t))\le C(1+t)^{-\frac32}(\delta^2+N^2(t)),$$

which prove (4.6). $\square$

In the following lemma, we employ the energy method to obtain the decay rates of the derivatives of $(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})^T$.

Lemma 4.2. *Under the same conditions of Lemma 4.1, it holds that*

$$\left\|\nabla(\rho_h,m_{h\shortparallel},W_{h\shortparallel},m_{h\perp},W_{h\perp})\right\|_{H^1}\le C(1+t)^{-\left(\frac34+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)).\tag{4.17}$$

Proof. Applying ∇^k with $k=0,1,2$ to (4.1) and multiplying by $\nabla^k\rho_h$, $\nabla^k m_{h\shortparallel}$, $\nabla^k W_{h\shortparallel}$, respectively, we have

$$
\begin{aligned}
&\frac{d}{dt}\left\|\nabla^k(\rho_h, m_{h\shortparallel}, W_{h\shortparallel})\right\|_{L^2}^2 + \left\|\nabla^{k+1}(m_{h\shortparallel}, W_{h\shortparallel})\right\|_{L^2}^2 + \left\|\nabla^k W_{h\shortparallel}\right\|_{L^2}^2 \\
&\quad \lesssim \int_{\mathbb{R}^3} \nabla^k F_1 \cdot \nabla^k m_{h\shortparallel} dx + \int_{\mathbb{R}^3} \nabla^k F_2 \cdot \nabla^k W_{h\shortparallel} dx. \qquad (4.18)
\end{aligned}
$$

The similar method as (4.18) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla^k(m_{h\perp}, W_{h\perp})\right\|_{L^2}^2 + (\mu+\alpha)\left\|\nabla^{k+1} m_{h\perp}\right\|_{L^2}^2 + \mu'\left\|\nabla^{k+1} W_{h\perp}\right\|_{L^2}^2 + 4\alpha\left\|\nabla^k W_{h\perp}\right\|_{L^2}^2 \\
&\quad \le \int_{\mathbb{R}^3} \nabla^k N_1 \cdot \nabla^k m_{h\perp} dx + \int_{\mathbb{R}^3} \nabla^k N_2 \cdot \nabla^k W_{h\perp} dx + 4\alpha \int_{\mathbb{R}^3} \nabla^k W_{h\perp} \cdot \mathrm{curl} \nabla^k m_{h\perp} dx \\
&\quad \le \int_{\mathbb{R}^3} \nabla^k N_1 \cdot \nabla^k m_{h\perp} dx + \int_{\mathbb{R}^3} \nabla^k N_2 \cdot \nabla^k W_{h\perp} dx + 4\alpha\left\|\nabla^k W_{h\perp}\right\|_{L^2}^2 + \alpha\left\|\nabla^{k+1} m_{h\perp}\right\|_{L^2}^2,
\end{aligned}
$$

which from the Cauchy–Schwarz inequality further implies

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla^k(m_{h\perp}, W_{h\perp})\right\|_{L^2}^2 + \mu\left\|\nabla^{k+1} m_{h\perp}\right\|_{L^2}^2 + \mu'\left\|\nabla^{k+1} W_{h\perp}\right\|_{L^2}^2 \\
&\quad \le \int_{\mathbb{R}^3} \nabla^k N_1 \cdot \nabla^k m_{h\perp} dx + \int_{\mathbb{R}^3} \nabla^k N_2 \cdot \nabla^k W_{h\perp} dx. \qquad (4.19)
\end{aligned}
$$

We now estimate the right-hand side of (4.18)–(4.19). For $k = 0$, by Hölder's inequality, we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^3} \frac{\rho W}{1+\rho} W_{h\shortparallel} dx &\le \|\rho\|_{L^3}\|W\|_{L^2}\left\|W_{h\shortparallel}\right\|_{L^6} \\
&\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2 + N^2(t))^2 + \epsilon_1\left\|\nabla W_{h\shortparallel}\right\|_{L^2}^2. \qquad (4.20)
\end{aligned}
$$

By the integration by parts, we know

$$
\begin{aligned}
\int_{\mathbb{R}^3} \mathrm{div}\left(\frac{m\otimes m}{1+\rho}\right)\cdot m_{h\shortparallel} dx &= -\int_{\mathbb{R}^3} \frac{m\otimes m}{1+\rho}\cdot \nabla m_{h\shortparallel} dx \le \|m\|_{L^\infty}\|m\|_{L^2}\left\|\nabla m_{h\shortparallel}\right\|_{L^2} \\
&\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2 + N^2(t))^2 + \epsilon_1\left\|\nabla m_{h\shortparallel}\right\|_{L^2}^2, \qquad (4.21)
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^3} \Delta\left(\frac{\rho m}{1+\rho}\right)\cdot m_{h\shortparallel} dx \le \left\|\nabla\left(\frac{\rho m}{1+\rho}\right)\right\|_{L^2}\left\|\nabla m_{h\shortparallel}\right\|_{L^2}
$$

$$
\begin{aligned}
&\le \|m\nabla\rho\|_{L^2}\left\|\nabla m_{h_{\shortparallel}}\right\|_{L^2}+\|\rho\nabla m\|_{L^2}\left\|\nabla m_{h_{\shortparallel}}\right\|_{L^2}\\
&\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2+\epsilon_1\left\|\nabla m_{h_{\shortparallel}}\right\|_{L^2}^2. \qquad (4.22)
\end{aligned}
$$

We can estimate the remaining terms like (4.20)–(4.22), and it implies

$$
\begin{aligned}
\int\limits_{\mathbb{R}^3} F_1\cdot m_{h_{\shortparallel}}dx+\int\limits_{\mathbb{R}^3} F_2\cdot W_{h_{\shortparallel}}dx &\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2\\
&\quad+\epsilon_1\left\|\left(\nabla m_{h_{\shortparallel}},\nabla W_{h_{\shortparallel}}\right)\right\|_{L^2}^2, \qquad (4.23)
\end{aligned}
$$

and

$$
\begin{aligned}
\int\limits_{\mathbb{R}^3} N_1\cdot m_{h\perp}dx+\int\limits_{\mathbb{R}^3} N_2\cdot W_{h\perp}dx &\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2\\
&\quad+\epsilon_1\|(\nabla m_{h\perp},\nabla W_{h\perp})\|_{L^2}^2. \qquad (4.24)
\end{aligned}
$$

Combining the assumption (4.4), Lemma 2.1, Lemma 3.1 and Lemma A.2, we can estimate

$$
\begin{aligned}
\left\|\nabla m\cdot\nabla^2\rho\right\|_{L^2} &\lesssim \|\nabla m_h\|_{L^\infty}\left\|\nabla^2\rho_h\right\|_{L^2}+\|\nabla\bar{m}\|_{L^\infty}\left\|\nabla^2\rho_h\right\|_{L^2}+\|\nabla m_h\|_{L^6}\left\|\nabla^2\bar{\rho}\right\|_{L^3}\\
&\quad+\|\nabla\bar{m}\|_{L^\infty}\left\|\nabla^2\bar{\rho}\right\|_{L^2}\\
&\lesssim (1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)), \qquad (4.25)
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|m\cdot\nabla^3\rho\right\|_{L^2} &\lesssim \|m_h\|_{L^\infty}\left\|\nabla^3\rho_h\right\|_{L^2}+\|\bar{m}\|_{L^\infty}\left\|\nabla^3\rho_h\right\|_{L^2}+\|m_h\|_{L^\infty}\left\|\nabla^3\bar{\rho}\right\|_{L^2}\\
&\quad+\|\bar{m}\|_{L^\infty}\left\|\nabla^3\bar{\rho}\right\|_{L^2}\\
&\lesssim (1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)). \qquad (4.26)
\end{aligned}
$$

Following the procedure leading to (4.14)–(4.15) and (4.25)–(4.26), we end up with

$$
\|\nabla(F_1,F_2,N_1,N_2)\|_{L^2}\lesssim (1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)). \qquad (4.27)
$$

Due to the integration by parts and (4.27), it implies

$$
\int\limits_{\mathbb{R}^3}\nabla F_1\cdot\nabla m_{h_{\shortparallel}}dx+\int\limits_{\mathbb{R}^3}\nabla^2 F_1\cdot\nabla^2 m_{h_{\shortparallel}}dx+\int\limits_{\mathbb{R}^3}\nabla F_2\cdot\nabla W_{h_{\shortparallel}}dx+\int\limits_{\mathbb{R}^3}\nabla^2 F_2\cdot\nabla^2 W_{h_{\shortparallel}}dx
$$

$$
\begin{aligned}
&\lesssim \|\nabla(F_1,F_2)\|_{L^2}^2+\epsilon_1\left\|\left(\nabla m_{h\shortparallel},\nabla^3 m_{h\shortparallel},\nabla W_{h\shortparallel},\nabla^3 W_{h\shortparallel}\right)\right\|_{L^2}^2\\
&\lesssim \epsilon_1\left\|\left(\nabla m_{h\shortparallel},\nabla^3 m_{h\shortparallel},\nabla W_{h\shortparallel},\nabla^3 W_{h\shortparallel}\right)\right\|_{L^2}^2+(1+t)^{-2\left(\frac34+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2
\end{aligned}
\tag{4.28}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\nabla N_1\cdot\nabla m_{h\perp}dx+\int_{\mathbb{R}^3}\nabla^2 N_1\cdot\nabla^2 m_{h\perp}dx+\int_{\mathbb{R}^3}\nabla N_2\cdot\nabla W_{h\perp}dx+\int_{\mathbb{R}^3}\nabla^2 N_2\cdot\nabla^2 W_{h\perp}dx\\
&\lesssim \epsilon_1\left\|\left(\nabla m_{h\perp},\nabla^3 m_{h\perp},\nabla W_{h\perp},\nabla^3 W_{h\perp}\right)\right\|_{L^2}^2+(1+t)^{-2\left(\frac34+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2.
\end{aligned}
\tag{4.29}
$$

It follows by plugging (4.23)–(4.24) and (4.28)–(4.29) into (4.18)–(4.19)

$$
\begin{aligned}
&\frac{d}{dt}\left\|(\rho_h,m_{h\shortparallel},W_{h\shortparallel},m_{h\perp},W_{h\perp})\right\|_{H^2}^2+\left\|\nabla(m_{h\shortparallel},m_{h\perp},W_{h\perp})\right\|_{H^2}^2+\left\|W_{h\shortparallel}\right\|_{H^3}^2\\
&\lesssim (1+t)^{-2\left(\frac34+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2.
\end{aligned}
\tag{4.30}
$$

Applying ∇^l with $l=0,1$ to $(4.1)_2$ and multiplying by $\nabla^l\nabla\rho_h$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^l m_{h\shortparallel}\cdot\nabla^{l+1}\rho_h dx+\left\|\nabla^{l+1}\rho_h\right\|_{L^2}^2\\
&\lesssim \int_{\mathbb{R}^3}\nabla^l F_1\cdot\nabla^{l+1}\rho_h dx+\left\|\nabla^{l+1}m_{h\shortparallel}\right\|_{L^2}^2+\left\|\nabla^{l+2}m_{h\shortparallel}\right\|_{L^2}^2.
\end{aligned}
\tag{4.31}
$$

In view of (4.16) and (4.27), one can deduce

$$
\begin{aligned}
&\int_{\mathbb{R}^3}F_1\cdot\nabla\rho_h dx+\int_{\mathbb{R}^3}\nabla F_1\cdot\nabla^2\rho_h dx\\
&\lesssim \|F_1\|_{L^2}^2+\|\nabla F_1\|_{L^2}^2+\epsilon_1\left\|\left(\nabla\rho_h,\nabla^2\rho_h\right)\right\|_{L^2}^2\\
&\lesssim (1+t)^{-2\left(\frac34+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2+\epsilon_1\left\|\left(\nabla\rho_h,\nabla^2\rho_h\right)\right\|_{L^2}^2.
\end{aligned}
\tag{4.32}
$$

Plugging (4.32) into (4.31), we get

$$
\frac{d}{dt}\sum_{l=0}^{1}\int_{\mathbb{R}^3}\nabla^l m_{h\shortparallel}\cdot\nabla^{l+1}\rho_h dx+\|\nabla\rho_h\|_{H^1}^2\lesssim (1+t)^{-2\left(\frac34+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2+\left\|\nabla m_{h\shortparallel}\right\|_{H^2}^2.
\tag{4.33}
$$

Multiplying (4.33) by η_1 for suitably small η_1 and adding it to (4.30), then there exist some positive constant a, such that

$$\frac{d}{dt}M(t)+aM(t)\lesssim \left\|(\rho_h, m_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{L^2}^2+(1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2,$$

where

$$\begin{aligned}M&=\left\|(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{H^2}^2+\sum_{l=0}^{1}\int_{\mathbb{R}^3}\nabla^l m_{h\shortparallel}\cdot\nabla^{l+1}\rho_h dx\\ &\sim\left\|(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{H^2}^2.\end{aligned}$$

By Gronwall inequality, and (4.6) of Lemma 4.1, we obtain

$$M\lesssim\int_0^t e^{-a(t-s)}(1+s)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}ds(\delta^2+N^2)^2\lesssim(1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2.$$

Therefore, we have proved (4.17). $\square$

To enclose the a priori estimates, we need to prove that

$$\left\|\nabla^3(\rho_h, m_h, W_h)\right\|_{L^2}\leq C.$$

In fact, note that

$$(\tilde{\rho}, \tilde{u}, \tilde{w})=\left(1+\rho, \frac{\sqrt{P'(1)}m}{1+\rho}, \frac{\sqrt{P'(1)}W}{1+\rho}\right)=(1+\rho, u, w).$$

By the existence Theorem 1.1, we obtain

$$\|(\rho, m, W)\|_{H^3}^2\lesssim\|(\rho, u, w)\|_{H^3}^2\lesssim\|(\rho_0, u_0, w_0)\|_{H^3}^2\lesssim\|(\rho_0, m_0, W_0)\|_{H^3}^2\lesssim\delta^2.$$

Thus

$$\begin{aligned}\|(\rho_h, m_h, W_h)\|_{H^3}&\lesssim\|(\rho, m, W)\|_{H^3}+\left\|(\bar{\rho}, \bar{m}, \bar{W})\right\|_{H^3}\lesssim\|(\rho_0, u_0, w_0)\|_{H^3}\\ &\lesssim\|(\rho_0, m_0, W_0)\|_{H^3}.\end{aligned}\tag{4.34}$$

The combination of (4.6), (4.17) and (4.34) leads to

$$N(t)\lesssim\delta^2+N^2(t)+\delta,\tag{4.35}$$

which together with the smallness of $\delta>0$ leads to the estimates (4.5). Hence, we obtain the optimal decay rates for $(\rho, m, W)^T$ and prove (1.15) and (1.16) of Theorem 1.2.

In the following, we want to prove (1.17) of Theorem 1.2. Assume

$$N_1(t) = \sup_{0\le s\le t}\left\{ \left\|\nabla(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{H^1}(1+s)^{\frac{5}{4}+\epsilon'} + \left\|\nabla^3(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{L^2}\right\}. \tag{4.36}$$

We claim that for $0<\epsilon'\le 1/4$,

$$N_1(t)\le\delta. \tag{4.37}$$

By the assumption (4.36), Lemma 2.1 and Lemma 3.1, estimating in the same way as proving (4.16) in Lemma 4.1, we obtain

$$\|(F_1, F_2, N_1, N_2)\|_{L^2} + \|\nabla(F_1, F_2, N_1, N_2)\|_{L^2} \lesssim (1+t)^{-\left(\frac{5}{4}+\epsilon'\right)}(\delta^2+N_1^2(t)). \tag{4.38}$$

Like (4.7)–(4.8) and (4.9)–(4.10), we know

$$\begin{aligned}
&\left\|\nabla(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{L^2}\\
&\le C\int_0^t (1+t-s)^{-\frac{7}{4}}\left(\|(f_1, f_2, f, N_1, N_2)(s)\|_{L^1} + \|\nabla(F_1, F_2, N_1, N_2)(s)\|_{L^2}\right)ds\\
&\le C\int_0^t (1+t-s)^{-\frac{7}{4}}(1+s)^{-\left(\frac{5}{4}+\epsilon_1\right)}ds(\delta^2+N_1^2)\\
&\le C(1+t)^{-\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2+N_1^2).
\end{aligned} \tag{4.39}$$

Taking (4.18) and (4.19) with $k=1,2$ and (4.31) with $l=1$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{H^1}^2 + \epsilon_1\int_{\mathbb{R}^3}\nabla m_{h\shortparallel}\cdot\nabla^2\rho_h dx\right)\\
&\quad + \left\|\nabla^2\left(m_{h\shortparallel}, m_{h\perp}, W_{h\shortparallel}, W_{h\perp}\right)\right\|_{H^1}^2 + \left\|\nabla^2\rho_h\right\|_{L^2}^2\\
&\lesssim \int_{\mathbb{R}^3} F_1\cdot\nabla^2 m_{h\shortparallel}dx + \int_{\mathbb{R}^3}\nabla F_1\cdot\nabla^3 m_{h\shortparallel}dx + \int_{\mathbb{R}^3} N_1\cdot\nabla^2 m_{h\perp}dx\\
&\quad + \int_{\mathbb{R}^3}\nabla N_1\cdot\nabla^3 m_{h\perp}dx + \int_{\mathbb{R}^3} N_2\cdot\nabla^2 W_{h\perp}dx + \int_{\mathbb{R}^3}\nabla N_2\cdot\nabla^3 W_{h\perp}dx\\
&\quad + \int_{\mathbb{R}^3} F_2\cdot\nabla^2 W_{h\shortparallel}dx + \int_{\mathbb{R}^3}\nabla F_2\cdot\nabla^3 W_{h\shortparallel}dx + \epsilon_1\int_{\mathbb{R}^3}\nabla F_1\cdot\nabla^2\rho_h dx
\end{aligned}$$

$$\lesssim \epsilon \left\| (\nabla^2\rho_h, \nabla^2 m_{h\shortparallel}, \nabla^3 m_{h\shortparallel}, \nabla^2 m_{h\perp}, \nabla^3 m_{h\perp}, \nabla^2 W_{h\shortparallel}, \nabla^3 W_{h\shortparallel}, \nabla^2 W_{h\perp}, \nabla^3 W_{h\perp}) \right\|_{L^2}^2$$
$$+ \|(F_1, F_2, N_1, N_2)\|_{L^2}^2 + \|\nabla(F_1, F_2, N_1, N_2)\|_{L^2}^2 . \tag{4.40}$$

From (4.40) and (4.38), we have

$$\frac{d}{dt}M_1(t) + bM_1(t) \le C(1+t)^{-2\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2 + N_1^2)^2$$
$$+ \left\|\nabla(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{L^2}^2 , \tag{4.41}$$

where

$$M_1(t) = \left(\left\|\nabla(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{H^1}^2 + \epsilon_1 \int_{\mathbb{R}^3} \nabla m_{h\shortparallel} \cdot \nabla^2 \rho_h dx \right)$$
$$\sim \left\|\nabla(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{H^1}^2 .$$

By Gronwall inequality, we have

$$M_1(t) \le C \int_0^t e^{-b(t-s)}(1+s)^{-2\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2 + N_1^2)^2 ds$$
$$\le C(1+t)^{-2\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2 + N_1^2)^2. \tag{4.42}$$

That is,

$$\left\|\nabla(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp})\right\|_{H^1} \le C(1+t)^{-\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2 + N_1^2). \tag{4.43}$$

It holds from (4.34) and (4.43) that

$$N_1(t) \lesssim \delta^2 + N_1^2(t) + \delta, \tag{4.44}$$

which together with the smallness of $\delta > 0$ leads to the estimates (4.37). Hence, we obtain the optimal decay rate for $(\nabla\rho, \nabla m)^T$ and prove (1.17) of Theorem 1.2.

Appendix A. Analytic tools

Lemma A.1. *Let $r_1 > 1$, $0 \le r_2 \le r_1$, then it holds that*

$$\int_0^t (1+t-s)^{-r_1}(1+s)^{-r_2} ds \le C(r_1, r_2)(1+t)^{-r_2} \tag{A.1}$$

where $C(r_1, r_2)$ is defined as

$$C(r_1, r_2) = \frac{2^{r_2+1}}{r_1 - 1}.$$

Proof. The proof can be seen in [16]. □

Lemma A.2. *Let $l \geq 0$ be an integer, it holds*

$$\left\|\nabla^l (gh)\right\|_{L^{p_0}} \lesssim \|g\|_{L^{p_1}} \left\|\nabla^l h\right\|_{L^{p_2}} + \left\|\nabla^l g\right\|_{L^{p_3}} \|h\|_{L^{p_4}} . \tag{A.2}$$

In the above, $p_0, p_1, p_2, p_3, p_4 \in [1, +\infty]$ *such that*

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} .$$

Proof. The proof can be seen in [15]. □

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