Polygons with Prescribed Angles in 2D and 3D⋆

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Abstract. We consider the construction of a polygon P with n vertices whose turning angles at the vertices are given by a sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$, $\alpha_i \in (-\pi, \pi)$, for $i \in \{0, \ldots, n-1\}$. The problem of realizing A by a polygon can be seen as that of constructing a straight-line drawing of a graph with prescribed angles at vertices, and hence, it is a special case of the well studied problem of constructing an *angle graph*. In 2D, we characterize sequences A for which every generic polygon $P \subset \mathbb{R}^2$ realizing A has at least c crossings, for every $c \in \mathbb{N}$, and describe an efficient algorithm that constructs, for a given sequence A, a generic polygon $P \subset \mathbb{R}^2$ that realizes A with the minimum number of crossings. In 3D, we describe an efficient algorithm that tests whether a given sequence A can be realized by a (not necessarily generic) polygon $P \subset \mathbb{R}^3$, and for every realizable sequence the algorithm finds a realization.

Keywords: crossing number · polygon · spherical polygon · angle graph

1 Introduction

Straight-line realizations of graphs with given metric properties have been one of the earliest applications of graph theory. Rigidity theory, for example, studies realizations of graphs with prescribed edge lengths, but also considers a mixed model where the edges have prescribed lengths or directions [4,13,14,15,21]. In this paper, we extend research on the so-called *angle graphs*, introduced by Vijayan [27] in the 1980s, which are geometric graphs with prescribed angles between adjacent edges. Angle graphs found applications in mesh flattening [29], and computation of conformal transformations [8,22] with applications in the theory of minimal surfaces and fluid dynamics.

Viyajan [27] characterized planar angle graphs under various constraints, including the case when the graph is a cycle [27, Theorem 2] and when the graph is 2-connected [27, Theorem 3]. In both cases, the characterization leads to an efficient algorithm to find a planar straight-line drawing or report that none

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exists. Di Battista and Vismara [6] showed that for 3-connected angle graphs (e.g., a triangulation), planarity testing reduces to solving a system of linear equations and inequalities in linear time. Garg [10] proved that planarity testing for angle graphs is NP-hard, disproving a conjecture by Viyajan. Bekos et al. [2] showed that the problem remains NP-hard even if all angles are multiples of $\pi/4$.

The problem of computing (straight-line) realizations of angle graphs can be seen as the problem of reconstructing a drawing of a graph from the given partial information. The research problems to decide if the given data uniquely determine the realization or its parameters of interest are already interesting for cycles, and were previously considered in the areas of conformal transformations [22] and visibility graphs [7].

In 2D, we are concerned with realizations of angle cycles as polygons minimizing the number of crossings which, as we shall see, depends only on the sum of the turning angles. It follows from the seminal work of Tutte [26] and Thomassen [25] that every positive instance of a 3-connected planar angle graph admits a crossing-free realization if the prescribed angles yield convex faces. Convexity will also play a crucial role in our proofs.

In 3D, we would like to determine whether a given angle cycle can be realized by a (not necessarily generic) polygon. Somewhat counter-intuitively, self-intersections cannot be always avoided in a polygon realizing the given angle cycle in 3D; we present examples below. Di Battista et al. [5] characterized oriented polygons that can be realized in $\mathbb{R}^3$ without self-intersections with axis-parallel edges of given directions. Patrignani [20] showed that recognizing crossing-free realizibility is NP-hard for graphs of maximum degree 6 in this setting.

Throughout the paper we assume modulo n arithmetic on the indices, and use $\langle ., . \rangle$ scalar product notation.

Angle sequences in 2-space. In the plane, an *angle sequence* A is a sequence $(\alpha_0, \ldots, \alpha_{n-1})$ of real numbers such that $\alpha_i \in (-\pi, \pi)$ for all $i \in \{0, \ldots, n-1\}$. Let $P \subset \mathbb{R}^2$ be an oriented polygon with n vertices $v_0, \ldots, v_{n-1}$ that appear in the given order along P, which is consistent with the given orientation of P. The *turning angle* of P at v_i is the angle in $(-\pi, \pi)$ between the vector $v_i - v_{i-1}$ and $v_{i+1} - v_i$. The sign of the angle is positive if rotating the plane, so that the vector $v_i - v_{i-1}$ points in the positive direction of the x-axis, makes the y-coordinate of $v_{i+1} - v_i$ positive. Otherwise, the angle nonpositive; see Fig. 1.

The oriented polygon P *realizes* the angle sequence A if the turning angle of P at v_i is equal to α_i, for $i = 0, \ldots, n-1$. A polygon P is *generic* if all its self-intersections are transversal (that is, proper crossings), vertices of P are distinct points, and no vertex of P is contained in a relative interior of an edge of P. Following the terminology of Viyajan [27], an *angle sequence* is *consistent* if there exists a generic closed polygon P with n vertices realizing A. For a polygon P that realizes an angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ in the plane, the *total curvature* of P is $\mathrm{TC}(P) = \sum_{i=0}^{n-1} \alpha_i$, and the *turning number* (also known as *rotation number*) of P is $\mathrm{tn}(P) = \mathrm{TC}(P)/(2\pi)$; it is known that $\mathrm{tn}(P) \in \mathbb{Z}$ in the plane [24].

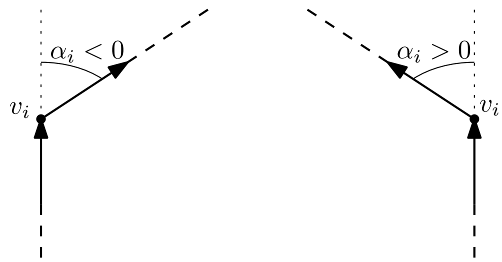

Fig. 1. A negative, or right, (on the left) and a positive, or left, (on the right) turning angle α_i at the vertex v_i of an oriented polygon.

The *crossing number*, denoted by $\mathrm{cr}(P)$, of a generic polygon is the number of self-crossings of P. The *crossing number* of a consistent angle sequence A is the minimum integer c, denoted by $\mathrm{cr}(A)$, such that there exists a generic polygon $P \in \mathbb{R}^2$ realizing A with $\mathrm{cr}(P) = c$. Our first main results is the following theorem.

Theorem 1. *For a consistent angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ in the plane, we have*

$$\mathrm{cr}(A) = \begin{cases} 1 & \text{if } \sum_{i=0}^{n-1} \alpha_i = 0, \\ |k| - 1 & \text{if } \sum_{i=0}^{n-1} \alpha_i = 2k\pi \text{ and } k \neq 0. \end{cases}$$

The proof of Theorem 1 can be easily converted into a weakly linear-time algorithm that constructs, for a given consistent sequence A, a generic polygon $P \subset \mathbb{R}^2$ that realizes A with the minimum number of crossings.

Angle sequences in 3-space and spherical polygonal linkages. In $\mathbb{R}^d$, $d \geq 3$, the sign of a turning angle no longer plays a role: The *turning angle* of an oriented polygon P at v_i is in $(0, \pi)$, and an angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ is in $(0, \pi)^n$. The unit-length direction vectors of the edges of P determine a spherical polygon P' in $\mathbb{S}^{d-1}$. Note that the turning angles of P correspond to the spherical lengths of the segments of P'. It is not hard to see that this observation reduces the problem of realizability of A by a polygon in $\mathbb{R}^d$ to the problem of realizability of A by a spherical polygon in $\mathbb{S}^{d-1}$, in the sense defined below, that additionally contains the origin $\mathbf{0}$ in the interior of its convex hull.

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit 2-sphere. A *great circle* $C \subset \mathbb{S}^2$ is the intersection of $\mathbb{S}^2$ with a 2-dimensional hyperplane in $\mathbb{R}^3$ containing $\mathbf{0}$. A *spherical line segment* is a connected subset of a great circle that does not contain a pair of antipodal points of $\mathbb{S}^2$. The *length* of a spherical line segment ab equals the measure of the central angle subtended by ab. A *spherical polygon* $P \subset \mathbb{S}^2$ is a closed curve consisting of finitely many spherical segments; and a spherical polygon $P = (\mathbf{u}_0, \ldots, \mathbf{u}_{n-1})$, $\mathbf{u}_i \in \mathbb{S}^2$, realizes an angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ if the spherical segment $(\mathbf{u}_{i-1}, \mathbf{u}_i)$ has (spherical) length α_i, for every i. As usual, the *turning angle* of P at $\mathbf{u}_i$ is the angle in $[0, \pi]$ between the tangents to $\mathbb{S}^2$ at $\mathbf{u}_i$ that are co-planar with the great circles containing $(\mathbf{u}_i, \mathbf{u}_{i+1})$ and $(\mathbf{u}_i, \mathbf{u}_{i-1})$. Unlike for polygons in $\mathbb{R}^2$ and $\mathbb{R}^3$, we do not put any constraints on turning angles of spherical polygons (i.e., angles 0 and π are allowed).

Regarding realizations of A by spherical polygons, we prove the following.

Theorem 2. *Let $A = (\alpha_0, \dots, \alpha_{n-1})$, $n \geq 3$, be an angle sequence. There exists a generic polygon $P \subset \mathbb{R}^3$ realizing A if and only if $\sum_{i=0}^{n-1} \alpha_i \geq 2\pi$ and there exists a spherical polygon $P' \subset \mathbb{S}^2$ realizing A. Furthermore, P can be constructed efficiently if P' is given.*

Theorem 3. *There exists a constructive weakly polynomial-time algorithm to test whether a given angle sequence $A = (\alpha_0, \dots, \alpha_{n-1})$ can be realized by a spherical polygon $P' \subset \mathbb{S}^2$.*

A simple exponential-time algorithm for realizability of angle sequences by spherical polygons follows from a known characterization [3, Theorem 2.5], which also implies that the order of angles in A does not matter for the spherical realizability. The topology of the configuration spaces of spherical polygonal linkages have also been studied [16]. Independently, Streinu et al. [19,23] showed that the configuration space of *noncrossing* spherical linkages is connected if $\sum_{i=0}^{n-1} \alpha_i \leq 2\pi$. However, these results do not seem to help prove Theorem 3.

The combination of Theorems 2 and 3 yields our second main result.

Theorem 4. *There exists a constructive weakly polynomial-time algorithm to test whether a given angle sequence $A = (\alpha_0, \dots, \alpha_{n-1})$ can be realized by a polygon $P \subset \mathbb{R}^3$.*

Organization. We prove Theorem 1 in Section 2 and Theorems 2, 3, and 4 in Section 3. We finish with concluding remarks in Section 4.

2 Crossing Minimization in the Plane

The first part of the following lemma gives a folklore necessary condition for the consistency of an angle sequence A in the plane. The condition is also sufficient except when $k = 0$. The second part follows from a result of Grünbaum and Shepard [11, Theorem 6], using a decomposition due to Wiener [28]. We provide a proof for the sake of completeness.

Lemma 1. *If an angle sequence $A = (\alpha_0, \dots, \alpha_{n-1})$ is consistent, then $\sum_{i=0}^{n-1} \alpha_i = 2k\pi$ for some $k \in \mathbb{Z}$. Furthermore, if $k \neq 0$ then $\mathrm{cr}(A) \geq |k| - 1$.*

Proof. Let P be a polygon such that $\mathrm{cr}(A) = \mathrm{cr}(P)$. We prove that $\mathrm{cr}(A) \geq |k| - 1 = |\mathrm{tn}(P)| - 1$, by induction on $\mathrm{cr}(P)$.

We consider the base case, where $\mathrm{cr}(P) = 0$. By the Jordan-Schönflies curve theorem, P bounds a compact region homeomorphic to a disk. By a well-known

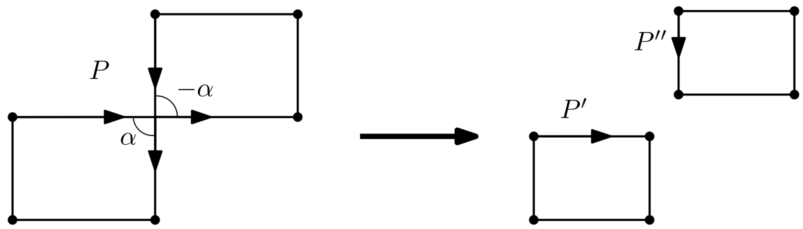

Fig. 2. Splitting an oriented closed polygon P at a self-crossing point into 2 oriented closed polygons P' and P'' such that $\mathrm{tn}(P) = \mathrm{tn}(P') + \mathrm{tn}(P'')$.

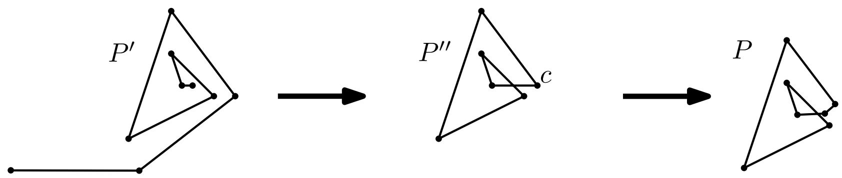

Fig. 3. Constructing a polygon P with $|\mathrm{tn}(P)| - 1$ crossings.

fact, the internal angles at the vertices of P sum up to $(n-2)\pi$. Since A is consistent, $\sum_{i=0}^{n-1} \alpha_i = 2k\pi$, and thus, $(n-2)\pi = \sum_{i=0}^{n-1}(\pi - \alpha_i) = (n-2k)\pi$ or $(n-2)\pi = \sum_{i=0}^{n-1}(\pi + \alpha_i) = (n+2k)\pi$, depending on the orientation of the polygon. The claim follows since $|\mathrm{tn}(P)| = |k| = 1$ in this case.

Refer to Fig. 2. In the inductive step, we have $\mathrm{cr}(P) \geq 1$. By splitting P into two closed parts P' and P'' at a self-crossing, we obtain a pair of closed polygons such that $\mathrm{tn}(P) = \mathrm{tn}(P') + \mathrm{tn}(P'')$. We have $\mathrm{cr}(P) \geq 1 + \mathrm{cr}(P') + \mathrm{cr}(P'') \geq 1 + |\mathrm{tn}(P')| - 1 + |\mathrm{tn}(P'')| - 1 \geq |\mathrm{tn}(P)| - 1$. Thus, the induction goes through, since both $\mathrm{cr}(P')$ and $\mathrm{cr}(P'')$ are smaller than $\mathrm{cr}(P)$. □

The following lemma shows that the lower bound in Lemma 1 is tight when $\alpha_i > 0$ for all $i \in \{0, \ldots, n-1\}$.

Lemma 2. *If $A = (\alpha_0, \ldots, \alpha_{n-1})$ is an angle sequence such that $\sum_{i=0}^{n-1} \alpha_i = 2k\pi$, $k \neq 0$, and $\alpha_i > 0$, for all i, then $\mathrm{cr}(A) \leq |k| - 1$.*

Proof. Refer to Fig. 3. In three steps, we construct a polygon P realizing A with $|\mathrm{tn}(P)| - 1$ self-crossings thereby proving $\mathrm{cr}(A) \leq |k| - 1 = |\mathrm{tn}(P)| - 1$. In the first step, we construct an oriented self-crossing-free polygonal line P' with $n+2$ vertices, whose first and last (directed) edges are parallel to the positive x-axis, and whose internal vertices have turning angles $\alpha_0, \ldots, \alpha_{n-1}$ in this order. We construct P' incrementally: The first edge has unit length starting from the origin; and every successive edge lies on a ray emanating from the endpoint of the previous edge. If the ray intersects neither the x-axis nor previous edges, then the next edge has unit length, otherwise its length is chosen to avoid any such intersection. In the second step, we prolong the last edge of P' until it creates the last self-intersection/crossing c and denote by P'' the resulting closed polygon composed of the part of P' from c to c via the prolonged part. By making the differences between the lengths of the edges of P' sufficiently large a prolongation of the last edge of P' has to eventually create at least one desired self-intersection. Hence, P'' is well-defined. Finally, we construct P realizing A from P'' by an appropriate modification of P'' in a small neighborhood of c without creating additional self-crossings. The number of self-crossings of P follows by the winding number of P with respect to the point just a bit north from the end vertex of P', which is k or $-k$. □

To prove the upper bound in Theorem 1, it remains to consider the case that $A = (\alpha_0, \ldots, \alpha_{n-1})$ contains both positive and negative angles. The crucial notion in the proof is that of an (essential) sign change of A which we define next. Let $\beta_i = \sum_{j=0}^{i} \alpha_j \mod 2\pi$. Let $\mathbf{v}_i \in \mathbb{R}^2$ denote the unit vector $(\cos \beta_i, \sin \beta_i)$. Hence, $\mathbf{v}_i$ is the direction vector of the $(i+1)$-st edge of an oriented polygon P realizing A if the direction vector of the first edge of P is $(1, 0) \in \mathbb{R}^2$. As observed by Garg [10, Section 6], the consistency of A implies that $\mathbf{0}$ is a strictly positive convex combination of vectors $\mathbf{v}_i$, that is, there exist scalars $\lambda_0, \ldots, \lambda_{n-1} > 0$ such that $\sum_{i=0}^{n-1} \lambda \mathbf{v}_i = \mathbf{0}$ and $\sum_{i=0}^{n-1} \lambda_i = 1$.

The *sign change* of A is an index i such that $\alpha_i < 0$ and $\alpha_{i+1} > 0$, or vice versa, $\alpha_i > 0$ and $\alpha_{i+1} < 0$. Let $\mathrm{sc}(A)$ denote the number of sign changes of A. Note that the number of sign changes of A is even. A sign change i of a consistent angle sequence A is *essential* if $\mathbf{0}$ is not a strictly positive convex combination of $\{\mathbf{v}_0, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_{n-1}\}$.

Lemma 3. *If $A = (\alpha_0, \ldots, \alpha_{n-1})$ is a consistent angle sequence, where $\sum_{i=0}^{n-1} \alpha_i = 2k\pi$, $k \in \mathbb{Z}$, and all sign changes are essential, then* $\mathrm{cr}(A) \leq \big||k| - 1\big|$.

Proof. We distinguish between two cases depending on whether $\sum_{i=0}^{n-1} \alpha_i = 0$.

Case 1: $\sum_{i=0}^{n-1} \alpha_i = 0$. Since $\sum_{i=0}^{n-1} \alpha_i = 0$, we have $\mathrm{sc}(A) \geq 2$. Since all sign changes are essential, for any two distinct sign changes $i \neq j$, we have $\mathbf{v}_i \neq \mathbf{v}_j$, therefore counting different vectors $\mathbf{v}_i$, where i is a sign change, is equivalent to counting essential sign changes. We show next that $\mathrm{sc}(A) = 2$.

Suppose, to the contrary, that $\mathrm{sc}(A) > 2$. Note that $\mathrm{sc}(A)$ is even, since the number of sign changes in a cyclic sequence of signs is even. Thus, we have $\mathrm{sc}(A) \geq 4$. Note that if $\mathbf{v}_i$ corresponds to an essential sign change i, then there exists an open halfplane bounded by a line through the origin that contains only $\mathbf{v}_i$ in $\{\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}\}$. Thus, if i and i' are distinct essential sign changes, for any other essential sign change j we have that $\mathbf{v}_j$ is contained in a closed convex cone bounded by $-\mathbf{v}_i$ and $-\mathbf{v}_{i'}$ unless $-\mathbf{v}_i = \mathbf{v}_{i'}$. Hence, the only possibility for having 4 essential sign changes i, i', j, and j' is if they satisfy $\mathbf{v}_i = -\mathbf{v}_{i'}$, $\mathbf{v}_j = -\mathbf{v}_{j'}$ and $\mathbf{v}_i \neq \pm\mathbf{v}_j$. Since all i, i', j, and j' are sign changes, there exists a fifth vector $\mathbf{v}_k$, which implies that one of i, i', j, and j' is not essential (contradiction).

Assume w.l.o.g. that j and $n-1$ are the only two essential sign changes. We have that $\mathbf{v}_j \neq -\mathbf{v}_{n-1}$. Indeed, since the sign changes j and $n-1$ are essential, all the other vectors $\mathbf{v}_i$, other than $\mathbf{v}_j$ and $\mathbf{v}_{n-1}$, either must be contained in the same open half-plane defined by a line through $\mathbf{v}_j$ and $-\mathbf{v}_{n-1}$, which is impossible due to the consistency of A, or must be orthogonal to $\mathbf{v}_j$ and $\mathbf{v}_{n-1}$. Then due to the consistency of A, there exists a pair $\{i, i'\}$ such that $\mathbf{v}_i = -\mathbf{v}_{i'}$. However, j and $n-1$ are the only sign changes by assumption, and thus there exists some index ℓ such that $\mathbf{v}_\ell \neq \pm\mathbf{v}_i$ (contradiction).

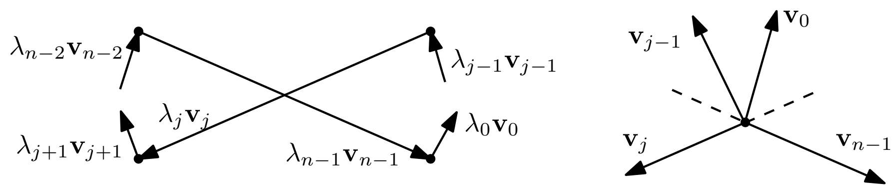

Fig. 4. The case of exactly 2 sign changes $n-1$ and j, both of which are essential, when $\sum_{i=0}^{n-1} \alpha_i = 0$. Both missing parts of the polygon on the left are convex chains.

It follows that $\mathbf{v}_j$ and $\mathbf{v}_{n-1}$ are not collinear, and we have that the remaining $\mathbf{v}_i$'s belong to the closed convex cone bounded by $-\mathbf{v}_j$ and $-\mathbf{v}_{n-1}$; refer to Fig. 4. Thus, we may assume that (i) $\beta_{n-1} = 0$, (ii) the sign changes of A are j and $n-1$, and (iii) $0 < \beta_0 < \ldots < \beta_j$ and $\beta_j > \beta_{j+1} > \ldots > \beta_{n-1} = 0$. Now, realizing A by a generic polygon with exactly 1 crossing between the line segments in the direction of $\mathbf{v}_j$ and $\mathbf{v}_{n-1}$ is a simple exercise.

Case 2: $\sum_{i=0}^{n-1} \alpha_i \neq 0$. We show that, unlike in the first case, none of the sign changes of A can be essential. Indeed, suppose j is an essential sign change, and let $A' = (\alpha'_0, \ldots, \alpha'_{n-2}) = (\alpha_0, \ldots, \alpha_{j-1}, \alpha_j + \alpha_{j+1}, \ldots, \alpha_{n-1})$ and $\beta'_i = \sum_{j=0}^{i} \alpha'_j \mod 2\pi$. Consider the unit vectors $\mathbf{v}'_0, \ldots, \mathbf{v}'_{n-2}$, where $\mathbf{v}'_i = (\cos\beta'_i, \sin\beta'_i)$. Since j is an essential sign change, there exists a nonzero vector $\mathbf{v}$ such that $\langle \mathbf{v}, \mathbf{v}_j \rangle > 0$ and $\langle \mathbf{v}, \mathbf{v}'_i \rangle \leq 0$ for all i. Hence, by symmetry, we may assume that $0 \leq \beta'_i \leq \pi$, for all i. Since j is a sign change, we have $-\pi < \alpha'_i < \pi$ for all i, consequently $\beta'_j = \sum_{i=0}^{j} \alpha'_i \mod 2\pi = \sum_{i=0}^{j} \alpha'_i$, which in turn implies, by Lemma 1, that $0 = \beta'_{n-2} = \sum_{i=0}^{n-2} \alpha'_i = \sum_{i=0}^{n-1} \alpha_i$ (contradiction).

We have shown that A has no sign changes. By Lemma 2, we have $\mathrm{cr}(A) \leq |k| - 1$, which concludes the proof. □

Theorem 1. *For a consistent angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ in the plane, we have*

$$\mathrm{cr}(A) = \begin{cases} 1 & \text{if } \sum_{i=0}^{n-1} \alpha_i = 0, \\ |k| - 1 & \text{if } \sum_{i=0}^{n-1} \alpha_i = 2k\pi \text{ and } k \neq 0. \end{cases}$$

Proof. The claimed lower bound $\mathrm{cr}(A) \geq \big||k| - 1\big|$ on the crossing number of A follows by Lemma 1, in the case when $k \neq 0$, and the result of Viyajan [27, Theorem 2] in the case when $k = 0$. It remains to prove the upper bound $\mathrm{cr}(A) \leq \big||k| - 1\big|$.

We proceed by induction on n. In the base case, we have $n = 3$. Then P is a triangle, $\sum_{i=0}^{2} \alpha_i = \pm 2\pi$, and $\mathrm{cr}(A) = 0$, as required. In the inductive step, assume $n \geq 4$, and that the claim holds for all shorter angle sequences. Let $A = (\alpha_0, \ldots, \alpha_{n-1})$ be an angle sequence with $\sum_{i=0}^{n-1} \alpha_i = 2k\pi$.

If A has no sign changes or if all sign changes are essential, then Lemma 2 or Lemma 3 completes the proof. Otherwise, we have at least one nonessential sign change s. Let $A' = (\alpha'_0, \ldots, \alpha'_{n-2}) = (\alpha_0, \ldots, \alpha_{s-1}, \alpha_s + \alpha_{s+1}, \ldots, \alpha_{n-1})$. Note that $\sum_{i=0}^{n-2} \alpha'_i = 2k\pi$. Since the sign change s is nonessential, $\mathbf{0}$ is a strictly positive convex combination of the β'_i's, where $\beta'_i = \sum_{j=0}^{i} \alpha'_j \mod 2\pi$. Indeed, this follows from $\beta'_i = \beta_i$, for $i < s$, and $\beta'_i = \beta_{i+1}$, for $i \geq s$.

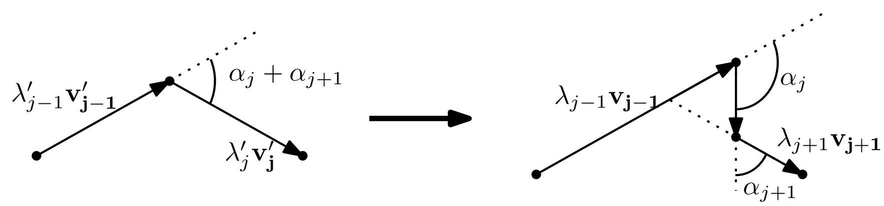

Fig. 5. Re-introducing the j-th vertex to a polygon realizing A' in order to obtain a polygon realizing A.

Refer to Fig. 5. Hence, by applying the induction hypothesis we obtain a realization of A' as a generic polygon P' with $\big||k|-1\big|$ crossing. A generic polygon realizing A is then obtained by modifying P in a small neighborhood of one of its vertices without introducing any additional crossing, similarly as in the paper by Guibas et al. [12]. $\square$

3 Realizing Angle Sequences in 3-Space

In this section, we describe a polynomial-time algorithm to decide whether an angle sequence $A = (\alpha_0, \ldots \alpha_{n-1})$, where $0 < \alpha_i < \pi$ for all i, can be realized as a polygon in $\mathbb{R}^3$.

We remark that our problem can be expressed as solving a system of polynomial equations, where $3n$ variables describe the coordinates of the n vertices of P, and each of n equations is obtained by the cosine theorem applied for a vertex and two incident edges of P. However, it is not clear to us how to solve such a system efficiently.

By Fenchel's theorem in differential geometry [9], the total curvature of any smooth curve in $\mathbb{R}^d$ is at least 2π. Fenchel's theorem has been adapted to closed polygons [24, Theorem 2.4], and it gives the following a necessary condition for an angle sequence A to have a realization in $\mathbb{R}^d$, for all $d \geq 2$:

$$\sum_{i=0}^{n-1} \alpha_i \geq 2\pi. \tag{1}$$

We show that a slightly stronger condition is both necessary and sufficient, hence it characterizes realizable angle sequences in $\mathbb{R}^3$.

Lemma 4. *Let $A = (\alpha_0, \ldots, \alpha_{n-1})$, $n \geq 3$, be an angle sequence. There exists a polygon $P \subset \mathbb{R}^3$ realizing A if and only if there exists a spherical polygon $P' \subset \mathbb{S}^2$ realizing A such that $\mathbf{0} \in \mathrm{relint}(\mathrm{conv}(P'))$ (relative interior of $\mathrm{conv}(P')$). Furthermore, P can be constructed efficiently if P' is given.*

Proof. Assume that an oriented polygon $P = (v_0, \ldots, v_{n-1})$ realizes A in $\mathbb{R}^3$. Let $\mathbf{u}_i = (v_{i+1} - v_i)/\|v_{i+1} - v_i\| \in \mathbb{S}^2$ be the unit direction vector of the edge $v_i v_{i+1}$ of P according to its orientation. Then $P' = (\mathbf{u}_0, \ldots, \mathbf{u}_{n-1})$ is a spherical polygon that realizes A. Suppose, for the sake of contradiction, that $\mathbf{0}$ is not in the relative interior of $\mathrm{conv}(P')$. Then there is a plane H that separates $\mathbf{0}$ and P', that is, if $\mathbf{n}$ is the normal vector of H, then $\langle \mathbf{n}, \mathbf{u}_i \rangle > 0$ for all $i \in \{0, \ldots, n-1\}$. This implies $\langle \mathbf{n}, (v_{i+1} - v_i) \rangle > 0$ for all i, hence $\langle \mathbf{n}, \sum_{i=1}^{n-1} (v_{i+1} - v_i) \rangle > 0$, which contradicts the fact that $\sum_{i=1}^{n-1} (v_{i+1} - v_i) = \mathbf{0}$, and $\langle \mathbf{n}, \mathbf{0} \rangle = 0$.

Conversely, assume that a spherical polygon P' realizes A, with edge lengths $\alpha_0, \ldots, \alpha_{n-1} > 0$. If all the vertices of P' lie on a common great circle, then $\mathbf{0} \in \mathrm{relint}(\mathrm{conv}(P'))$ implies $\sum_{i=0}^{n-1} \pm\alpha_i = 0 \mod 2\pi$, where the sign is determined by the direction (cw. or ccw.) in which a particular segment of P' traverses the common great circle according to its orientation. As observed by Garg [10, Section 6], the signed angle sequence is consistent in this case due to the assumption that $\mathbf{0} \in \mathrm{relint}(\mathrm{conv}(P'))$. Thus, we obtain a realization of A that is contained in a plane.

Otherwise we may assume that $\mathbf{0} \in \mathrm{int}(\mathrm{conv}(P'))$. By Carathéodory's theorem [17, Thereom 1.2.3], P' has 4 vertices whose convex combination is the origin $\mathbf{0}$. Then we can express $\mathbf{0}$ as a strictly positive convex combination of *all* vertices of P'. The coefficients in the convex combination encode the lengths of the edges of a polygon P realizing A, which concludes the proof in this case.

We now show how to compute strictly positive coefficients in strongly polynomial time. Let $\mathbf{c} = \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{u}_i$ be the centroid of the vertices of P'. If $\mathbf{c} = \mathbf{0}$, we are done. Otherwise, we can find a tetrahedron $T = \mathrm{conv}\{\mathbf{u}_{i_0}, \ldots, \mathbf{u}_{i_3}\}$ such that $\mathbf{0} \in T$ and such that the ray from $\mathbf{0}$ in the direction $-\mathbf{c}$ intersects $\mathrm{int}(T)$, by solving an LP feasibility problem in $\mathbb{R}^3$. By computing the intersection of the ray with the faces of T, we find the maximum $\mu > 0$ such that $-\mu\mathbf{c} \in \partial T$ (the boundary of T). We have $-\mu\mathbf{c} = \sum_{j=0}^{3} \lambda_j \mathbf{u}_{i_j}$ and $\sum_{j=0}^{3} \lambda_j = 1$ for suitable coefficients $\lambda_j \geq 0$. Now $\mathbf{0} = \mu\mathbf{c} - \mu\mathbf{c} = \frac{\mu}{n} \sum_{i=0}^{n-1} \mathbf{u}_i + \sum_{j=0}^{3} \lambda_j \mathbf{u}_{i_j}$ is a strictly positive convex combination of the vertices of P'. $\square$

It is easy to find an angle sequence A that satisfies (1) but does not correspond to a spherical polygon P'. Consider, for example, $A = (\pi - \varepsilon, \pi - \varepsilon, \pi - \varepsilon, \varepsilon)$, for some small $\varepsilon > 0$. Points in $\mathbb{S}^2$ at (spherical) distance $\pi - \varepsilon$ are nearly antipodal. Hence, the endpoints of a polygonal chain $(\pi - \varepsilon, \pi - \varepsilon, \pi - \varepsilon)$ are nearly antipodal as well, and cannot be connected by an edge of (spherical) length ε. Thus a spherical polygon cannot realize A.

Algorithms. In the remainder of this section, we show how to find a realization $P \subset \mathbb{R}^3$ or report that none exists, in polynomial time. Our first concern is to decide whether an angle sequence is realizable by a spherical polygon.

Theorem 3. *There exists a constructive weakly polynomial-time algorithm to test whether a given angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ can be realized by a spherical polygon $P' \subset \mathbb{S}^2$.*

Proof. Let $A = (\alpha_0, \ldots, \alpha_{n-1}) \in (0, \pi)^n$ be a given angle sequence. Let $\mathbf{n} = (0, 0, 1) \in \mathbb{S}^2$, that is, $\mathbf{n}$ is the north pole. For $i \in \{0, 1, \ldots, n-1\}$, let $U_i \subseteq \mathbb{S}^2$ be the locus of the end vertices $\mathbf{u}_i$ of all (spherical) polygonal lines $P'_i = (\mathbf{n}, \mathbf{u}_0, \ldots, \mathbf{u}_i)$ with edge lengths $\alpha_0, \ldots, \alpha_{i-1}$. It is clear that A is realizable by a spherical polygon P' if and only if $\mathbf{n} \in U_{n-1}$.

Note that for all $i \in \{0, \ldots, n-1\}$, the set U_i is invariant under rotations about the z-axis, since $\mathbf{n}$ is a fixed point and rotations are isometries. We show how to compute the sets U_i, $i \in \{0, \ldots, n-1\}$, efficiently.

We define a *spherical zone* as a subset of $\mathbb{S}^2$ between two horizontal planes (possibly, a circle, a spherical cap, or a pole). Recall the parameterization of $\mathbb{S}^2$ using spherical coordinates (cf. Figure 6 (left)): for every $\mathbf{v} \in \mathbb{S}^2$, $\mathbf{v}(\psi, \varphi) = (\sin\psi \sin\varphi, \cos\psi \sin\varphi, \cos\varphi)$, with longitude $\psi \in [0, 2\pi)$ and polar angle $\varphi \in [0, \pi]$, where the *polar angle* φ is the angle between $\mathbf{v}$ and $\mathbf{n}$. Using this parameterization, a spherical zone is a Cartesian product $[0, 2\pi) \times I$ for some circular arc $I \subset [0, \pi]$. In the remainder of the proof, we associate each spherical zone with such a circular arc I.

We define additions and subtraction on polar angles $\alpha, \beta \in [0, \pi]$ by

$$\alpha \oplus \beta = \min\{\alpha + \beta, 2\pi - (\alpha + \beta)\},\ \ \alpha \ominus \beta = \max\{\alpha - \beta, \beta - \alpha\};$$

see Figure 6 (right). (This may be interpreted as addition mod 2π, restricted to the quotient space defined by the equivalence relation $\varphi \sim 2\pi - \varphi$.)

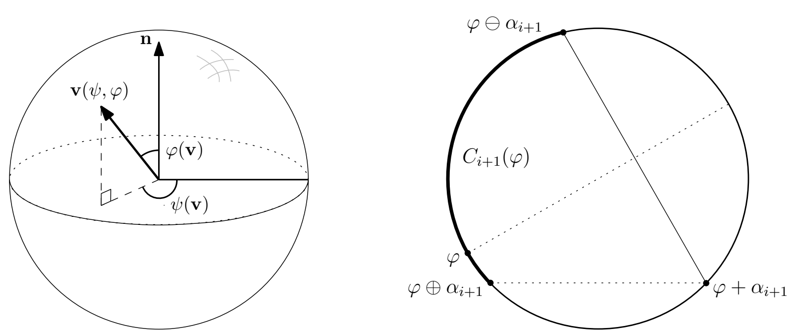

Fig. 6. Parametrization of the unit vectors (left). Circular arc $C_{i+1}(\varphi)$ (right).

We show that U_i is a spherical zone for all $i \in \{0, \ldots, n-1\}$, and show how to compute the intervals $I_i \subset [0, \pi]$ efficiently. First note that U_0 is a circle at (spherical) distance α_0 from $\mathbf{n}$, hence U_0 is a spherical zone with $I_0 = [\alpha_0, \alpha_0]$.

Assume that U_i is a spherical zone associated with $I_i \subset [0, \pi]$. Let $\mathbf{u}_i \in U_i$, where $\mathbf{u}_i = \mathbf{v}(\psi, \varphi)$ with $\psi \in [0, 2\pi)$ and $\varphi \in I_i$. By the definition U_i, there exists a polygonal line $(\mathbf{n}, \mathbf{u}_0, \ldots, \mathbf{u}_i)$ with edge lengths $\alpha_0, \ldots, \alpha_i$. The locus of points in $\mathbb{S}^2$ at distance α_{i+1} from u_i is a circle; the polar angles of the points in the circle form an interval $C_{i+1}(\varphi)$. Specifically (see Figure 6 (right)), we have

$$C_{i+1}(\varphi) = [\min\{\varphi \ominus \alpha_{i+1}, \varphi \oplus \alpha_{i+1}\}, \max\{\varphi \ominus \alpha_{i+1}, \varphi \oplus \alpha_{i+1}\}].$$

By rotational symmetry, $U_{i+1} = [0, 2\pi) \times I_{i+1}$, where $I_{i+1} = \bigcup_{\varphi \in I_i} C_{i+1}(\varphi)$. Consequently, $I_{i+1} \subset [0, \pi]$ is connected, and hence, I_{i+1} is an interval. Therefore U_{i+1} is a spherical zone. As $\varphi \oplus \alpha_{i+1}$ and $\varphi \ominus \alpha_{i+1}$ are piecewise linear functions of φ, we can compute I_{i+1} using $O(1)$ arithmetic operations.

We can construct the intervals $I_0, \ldots, I_{n-1} \subset [0, \pi]$ as described above. If $0 \notin I_{n-1}$, then $\mathbf{n} \notin U_{n-1}$ and A is not realizable. Otherwise, we can compute the vertices of a spherical realization $P' \subset \mathbb{S}^2$ by backtracking. Put $\mathbf{u}_{n-1} = \mathbf{n} = (0, 0, 1)$. Given $\mathbf{u}_i = \mathbf{v}(\psi, \varphi)$, we choose $\mathbf{u}_{i-1}$ as follows. Let $\mathbf{u}_{i-1}$ be $\mathbf{v}(\psi, \varphi \oplus \alpha_i)$ or $\mathbf{v}(\psi, \varphi \ominus \alpha_i)$ if either of them is in U_{i-1} (break ties arbitrarily). Else the spherical circle of radius α_i centered at $\mathbf{u}_i$ intersects the boundary of U_{i-1}, and then we choose $\mathbf{u}_{i-1}$ to be an arbitrary such intersection point. The decision algorithm (whether $0 \in I_{n-1}$) and the backtracking both use $O(n)$ arithmetic operations. □

Enclosing the Origin. Theorem 3 provides an efficient algorithm to test whether an angle sequence can be realized by a spherical polygon, however, Lemma 4 requires a spherical polygon P' whose convex hull contains the origin in its relative interior. We show that this is always possible if a realization exists and $\sum_{i=0}^{n-1} \alpha_i \geq 2\pi$. The general strategy in the inductive proof of this claim is to incrementally modify P' by changing the turning angle at one of its vertices to 0 or π. This allows us to reduce the number of vertices of P' and apply induction.

Lemma 5. *Given a spherical polygon P' that realizes an angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$, $n \geq 3$, with $\sum_{i=0}^{n-1} \alpha \geq 2\pi$, we can compute in polynomial time a spherical polygon P'' realizing A such that $\mathbf{0} \in \mathrm{relint}(\mathrm{conv}(P''))$.*

The combination of Theorem 3 with Lemmas 4–5 yields Theorems 2 and 4. The proof of Lemma 5 can be turned into an algorithm with running time polynomial in n if we assume that every arithmetic operation can be carried out in $O(1)$ time. Nevertheless, we get only a weakly polynomial running time, since we are unable to guarantee a polynomial size encoding of the numerical values that are computed in the process of constructing a spherical polygon realizing A that contains $\mathbf{0}$ in its convex hull in the proof of Lemma 5.

4 Conclusion

We devised efficient algorithms to realize a consistent angle cycle with the minimum number of crossings in 2D. In 3D, we can test efficiently whether a given angle sequence is realizable, and find a realization if one exists. However, it remains an open problem to find an efficient algorithms that computes the minimum number of crossings in generic realizations. There exist angle sequences that are realizable in 3D, but every generic realization has crossings. It is not difficult to see that crossings are unavoidable only if every 3D realization of an angle sequence A is contained in a plane, which is the case, for example, when $A = (\pi - \varepsilon, \ldots, \pi - \varepsilon, (n-1)\varepsilon)$, for odd $n \geq 5$ which is the length of A. Thus, an efficient algorithm for this problem would follow by Theorem 1, once one can test efficiently whether A admits a fully 3D realization. The evidence that we have points to the following conjecture, whose "only if" part we can prove.

Conjecture 1. An angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$, where $\alpha_i \in (0, \pi)$ and $n \geq 3$, that can be realized by a polygon in $\mathbb{R}^3$, has a realization by a self-intersection free polygon in $\mathbb{R}^3$ if and only if n is odd or $\sum_{i=0}^{n-1}(\pi - \alpha_i) \neq \pi$.

It can be seen that Conjecture 1 is equivalent to the claim that every realization A in $\mathbb{R}^3$ has a self-intersection if and only if A can be realized in $\mathbb{R}^2$ as a *thrackle*, that is, a polygon where every pair of nonadjacent edges cross each other. Here, we keep all the angles in A positive.

Can our results in $\mathbb{R}^2$ or $\mathbb{R}^3$ be extended to broader interesting classes of graphs? A natural analog of our problem in $\mathbb{R}^3$ would be a construction of triangulated spheres with prescribed dihedral angles, discussed in a recent paper by Amenta and Rojas [1]. For convex polyhedra, Mazzeo and Montcouquiol [18] proved, settling Stoker's conjecture, that dihedral angles determine face angles.

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