Introducing a Concise Formulation of the Jacobian Matrix for Newton-Raphson Power Flow Solution in the Engineering Curriculum

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***Abstract*-- The power flow computer program is fundamentally important for power system analysis and design. Many textbooks teach the Newton-Raphson method of power flow solution. The typical formulation of the Jacobian matrix in the NR method is cumbersome, inelegant, and laborious to program. Recent papers have introduced a method for calculating the Jacobian matrix that is concise, elegant, and simple to program. The concise formulation of the Jacobian matrix makes writing a power flow program more accessible to students. However, its derivation in the research literature involves advanced manipulations using higher dimensional derivatives, which are challenging for dual level students. This paper presents alternative derivations of the concise formulation that are suitable for undergraduate students, where some cases can be presented in lecture while other cases are assigned as exercises. These derivations have been successfully taught in a dual level course on computational methods for power systems for about ten years.**

I. Introduction

The power flow computer program computes the voltage magnitude and angle at each bus in a balanced three-phase power system and is fundamentally important for power system analysis and design. This program yields information about the real and reactive power flows in equipment such as transmission lines and transformers as well as equipment losses [1]. Engineers use power flow solutions to plan how new generation and transmission can meet projected load growth.

Many textbooks on power systems analysis and design and papers on power flow program improvements teach or use the Newton-Raphson method of power flow solution [1], [2], [3], [4], [5], [6]. The iterative Newton-Raphson method is necessary because the power-flow problem requires solving a set of nonlinear algebraic equations. Starting with an initial guess for the unknown voltage magnitudes and phase angles, the power flows are calculated as,

$$P_i = \sum_n |Y_{in} V_i V_n| \cos(\theta_{in} + \delta_n - \delta_i) \quad (1)$$

$$Q_i = - \sum_n |Y_{in} V_i V_n| \sin(\theta_{in} + \delta_n - \delta_i) \quad (2)$$

and compared with their scheduled values. The independent voltage magnitudes and phase angles are then updated using Newton's method which involves a first-order approximation to the nonlinear equations. The first-order approximation requires calculating a Jacobian matrix and the associated linear equations can be solved by Gauss elimination and back substitution. The procedure is iterated until the mismatches between the power flows and their scheduled values are acceptably small.

The Jacobian matrix in [1] is calculated using eight equations derived from the partial derivatives of the real and reactive power with respect to δ_j and $|V_j|$. A simple variant described in [2] involves multiplying derivatives of power quantities with respect to voltage magnitude by the voltage magnitude. Otherwise, the structure of the solution in [2] is essentially the same as in [1]. Texts [3], [5], and [6] use a formulation similar to [1] while [4] uses a formulation similar to [2].

Current textbook formulations of the Jacobian matrix for power flow solutions are cumbersome, inelegant, and laborious to program [1], [2], [3], [4], [5], [6]. Recent papers have introduced a method for calculating the Jacobian matrix that is concise, elegant, and simple to program [7], [8], [9]. The concise formulation of the Jacobian matrix makes writing a power flow program more accessible to students. However, its derivation in the research literature involves advanced manipulations using higher dimensional derivatives, which are challenging for undergraduate students in engineering. This paper presents alternative derivations of the concise formulation of the Jacobian that are suitable for undergraduate students. The derivations presented in this paper use the real and reactive power flows in terms of trigonometric functions and only require knowledge of partial derivatives as taught in a standard undergraduate course on vector calculus.

The derivation of the concise formulation presented in this paper includes several cases that are similar in structure but have different details. Therefore, it is feasible to present some of the cases in lecture and assign other cases as homework. These derivations have been taught to dual (undergraduate and graduate) level students in a course on computational methods for power systems for ten years. Students start by writing a short program for the power flow solution of a 2-bus network that calculates the Jacobian the traditional way in terms of trigonometric functions. After working through derivations of the concise formulation of the Jacobian, students modify the power flow solution for the 2-bus network to use the concise formulation and observe the results are identical. Finally, students write a short program for power flow solution of a 4-bus network using the concise formulation of the Jacobian.

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This paper begins with an introduction to Newton's method and its application to power flow solutions. Then we present a typical formulation of the Jacobian for power flow solution that is cumbersome, inelegant and laborious to program. Then we provide a concise formulation of the Jacobian and show how it is typically derived in research literature. We present the mathematical details that would be required for an undergraduate student to follow one of the published derivations of the concise formulation. The appendix presents alternative proofs of the concise formulations that are accessible to undergraduate students who know introductory vector calculus. The paper concludes with an outline for teaching the concise formulation using the derivations in the appendix. We have successfully taught this material in a dual (undergraduate and graduate) level course on computational methods for power systems for ten years.

II. Newton Raphson Method for power flow solution

The Newton-Raphson method for power flow solution is based on Newton's method, which can be derived from the first-order Taylor series expansion for functions of several variables. Given M functions $f_1, f_2, \ldots, f_M$ of N variables $x_1, x_2, \ldots, x_N$, define the vectors:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad \text{and} \quad \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_M(\boldsymbol{x}) \end{bmatrix} \tag{3}$$

The first-order Taylor series expansion of $\boldsymbol{f}(\boldsymbol{x})$ around the point $\boldsymbol{x}^{(k)}$ is [10]:

$$f_1(\boldsymbol{x}) \approx f_1(\boldsymbol{x}^{(k)}) + (x_1 - x_1^{(k)}) \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} + \cdots + (x_N - x_N^{(k)}) \frac{\partial f_1(\boldsymbol{x})}{\partial x_N}$$

$$\vdots \tag{4}$$

$$f_M(\boldsymbol{x}) \approx f_M(\boldsymbol{x}^{(k)}) + (x_1 - x_1^{(k)}) \frac{\partial f_M(\boldsymbol{x})}{\partial x_1} + \cdots + (x_N - x_N^{(k)}) \frac{\partial f_M(\boldsymbol{x})}{\partial x_N}$$

If we denote the Jacobian matrix compactly as:

$$\boldsymbol{J}(\boldsymbol{x}) = D_x \boldsymbol{f}(\boldsymbol{x}) = \left[\frac{\partial f_m}{\partial x_n}\right]_{M \times N} = \begin{bmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_1(\boldsymbol{x})}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_M(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_M(\boldsymbol{x})}{\partial x_N} \end{bmatrix} \tag{5}$$

then the Taylor series can be written in matrix vector form:

$$\boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}^{(k)}) + \boldsymbol{J}(\boldsymbol{x}^{(k)})(\boldsymbol{x} - \boldsymbol{x}^{(k)}) \tag{6}$$

When $M = N$, the update equation for Newton's method is derived by setting $\boldsymbol{f}(\boldsymbol{x}) = 0$ and solving the resulting equation for $\boldsymbol{x}$:

$$\boldsymbol{J}(\boldsymbol{x}^{(k)})\,\boldsymbol{x} = \boldsymbol{J}(\boldsymbol{x}^{(k)})\,\boldsymbol{x}^{(k)} - \boldsymbol{f}(\boldsymbol{x}^{(k)}) \tag{7}$$

In practice, this equation is solved by Gauss elimination and back substitution when $\boldsymbol{J}(\boldsymbol{x}^{(k)})$ is invertible but the solution is mathematically equivalent to:

$$\boldsymbol{x} = \boldsymbol{x}^{(k)} - [\boldsymbol{J}(\boldsymbol{x}^{(k)})]^{-1} \boldsymbol{f}(\boldsymbol{x}^{(k)}) \tag{8}$$

The resulting value of $\boldsymbol{x}$ is denoted $\boldsymbol{x}^{(k+1)}$ which leads to the update equation for Newton's method:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - [\boldsymbol{J}(\boldsymbol{x}^{(k)})]^{-1} \boldsymbol{f}(\boldsymbol{x}^{(k)}) \tag{9}$$

After choosing an initial value for $\boldsymbol{x}^{(0)}$, the updated equation is iterated until $\boldsymbol{f}(\boldsymbol{x}^{(k)})$ is sufficiently small.

In order to perform a power flow solution, it is necessary to identify which quantities have predetermined values and which quantities are unknown prior to the solution. There are generator buses that regulate their terminal voltage and load buses that do not regulate their terminal voltage. Turbine governors regulate the real power output of generators by adjusting their mechanical driving force.

The amount of losses are unknown prior to the power flow solution so the real power output of all the generators cannot be predetermined. For this reason, one of the generator buses is designated as the slack bus and its real power output is not one of the pre-determined quantities. Since it is possible to choose one of the phase angles as the reference, the angle of the slack bus is predetermined to be zero. Assuming there are Ng generator buses including the slack bus and the total number of buses is N, then the quantities that are known and unknown prior to the power flow solution are summarized in Table 1.

Table 1. Known and unknown quantities for different buses

Bus number	Type	Predetermined	Unknown
1	Slack	$\lvert V_1\rvert$, δ_1	P_1, Q_1
2	Generator	P_2, $\lvert V_2\rvert$	Q_2, δ_2
⋮	⋮	⋮	⋮
Ng	Generator	P_{Ng}, $\lvert V_{Ng}\rvert$	Q_{Ng}, δ_{Ng}
Ng+1	Load	P_{Ng+1}, Q_{Ng+1}	$\lvert V_{Ng+1}\rvert$, δ_{Ng+1}
⋮	⋮	⋮	⋮
N	Load	P_N, Q_N	$\lvert V_N\rvert$, δ_N

The predetermined values for P_i and Q_i are called $P_{i,sched}$ and $Q_{i,sched}$, where "sched" means scheduled. The equations and variables for Newton's method are:

$$g(x) = \begin{bmatrix} P_2 - P_{2,sched} \\ \vdots \\ P_N - P_{N,sched} \\ Q_{Ng+1} - Q_{Ng+1,sched} \\ \vdots \\ Q_N - Q_{N,sched} \end{bmatrix} = 0, \quad x = \begin{bmatrix} \delta_2 \\ \vdots \\ \delta_N \\ |V_{Ng+1}| \\ \vdots \\ |V_N| \end{bmatrix} \tag{10}$$

III. Standard formulation of the Jacobian matrix

The typical formulation of the Jacobian matrix used in the Newton-Raphson method is cumbersome, inelegant, and laborious to program. This is especially true because the expressions for partial derivatives are different when the derivative is taken with respect to a variable at the same bus as the quantity being differentiated versus a variable at a different bus than the quantity being differentiated.

Let P_i, Q_i, $|V_i|$, δ_i represent the real and reactive power injections and the voltage magnitude and phase angle at bus i. If elements of the bus admittance matrix are $Y_{ij} = |Y_{ij}| \angle \theta_{ij}$, then the equations for elements of the Jacobian matrix are given as follows when $i \neq j$ [1]:

$$J1_{ij} = \partial P_i/\partial \delta_j = |V_i Y_{ij} V_j| \sin(\delta_i\text{-}\delta_j\text{-}\theta_{ij}) \text{ , } i,j = 2,3,\ldots,\text{N} \quad (11)$$

$$J2_{ij} = \partial P_i/\partial |V_j| = |V_i Y_{ij}| \cos(\delta_i\text{-}\delta_j\text{-}\theta_{ij}) \text{ , } i,j = 2,3,\ldots,\text{N} \quad (12)$$

$$J3_{ij} = \partial Q_i/\partial \delta_j = \text{-}|V_i Y_{ij} V_j| \cos(\delta_i\text{-}\delta_j\text{-}\theta_{ij}) \text{ , } i,j = 2,3,\ldots,\text{N} \quad (13)$$

$$J4_{ij} = \partial Q_i/\partial |V_j| = |V_i Y_{ij}| \sin(\delta_i\text{-}\delta_j\text{-}\theta_{ij}) \text{ , } i,j = 2,3,\ldots,\text{N} \quad (14)$$

Equations for the Jacobian matrix elements when $i = j$ are calculated using different expressions [1]:

$$J1_{ii} = \partial P_i/\partial \delta_i = \text{-}|V_i| \sum_{\substack{n=1 \\ n \neq i}}^{N} Y_{in} V_n \sin(\delta_i\text{-}\delta_n\text{-}\theta_{in}) \quad (15)$$

$$J2_{ii} = \partial P_i/\partial |V_i|$$
$$= |V_i Y_{ii}| \cos(\theta_{ii}) + \sum_{n=1}^{N} Y_{in} V_n \cos(\delta_i\text{-}\delta_n\text{-}\theta_{in}) \quad (16)$$

$$J3_{ii} = \partial Q_i/\partial \delta_i = |V_i| \sum_{\substack{n=1 \\ n \neq i}}^{N} Y_{in} V_n \cos(\delta_i\text{-}\delta_n\text{-}\theta_{in}) \quad (17)$$

$$J4_{ii} = \partial Q_i/\partial |V_i|$$
$$= \text{-}|V_i Y_{ii}| \sin(\theta_{ii}) + \sum_{n=1}^{N} Y_{in} V_n \sin(\delta_i\text{-}\delta_n\text{-}\theta_{in}) \quad (18)$$

The expressions in [2] for a slight variant of the Jacobian matrix are similarly cumbersome, inelegant, and laborious to program. The equations in [2] are given as follows when $i \neq j$:

$$J1_{ij} = \partial P_i/\partial \delta_j = |V_i V_j Y_{ij}| \sin(\theta_{ij}+\delta_j\text{-}\delta_i) \text{ , } i,j = 2,3,\ldots,\text{N} \quad (19)$$

$$J2_{ij} = |V_j| \partial P_i/\partial V_j = |V_j| |V_i Y_{ij}| \cos(\theta_{ij}+\delta_j\text{-}\delta_i) \text{ , } i,j = 2,3,\ldots,\text{N} \quad (20)$$

$$J3_{ij} = \partial Q_i/\partial \delta_j = \text{-}|V_i V_j Y_{ij}| \cos(\theta_{ij}+\delta_j\text{-}\delta_i) \text{ , } i,j = 2,3,\ldots,\text{N} \quad (21)$$

$$J4_{ij} = |V_j| \partial Q_i/\partial \delta_j = \text{-}|V_j| |V_i Y_{ij}| \sin(\theta_{ij}+\delta_j\text{-}\delta_i), \; i,j = 2,3,\ldots,\text{N} \quad (22)$$

Equations for the Jacobian matrix elements when $i = j$ are calculated using different expressions in [2] as well:

$$J1_{ii} = \partial P_i/\partial \delta_i = \sum_{\substack{n=1 \\ n \neq i}}^{N} |Y_{in} V_i V_n| \sin(\theta_{in}+\delta_n\text{-}\delta_i) \quad (23)$$

$$J2_{ii} = |V_i| \partial P_i/\partial |V_i| = \text{P}_i + |V_i|^2 \text{ Real } \{Y_{ii}\} \quad (24)$$

$$J3_{ii} = \partial Q_i/\partial \delta_i = \sum_{\substack{n=1 \\ n \neq i}}^{N} |Y_{in} V_i V_n| \cos(\theta_{in}+\delta_n\text{-}\delta_i) \quad (25)$$

$$J4_{ii} = |V_i| \partial Q_i/\partial |V_i| = \text{Q}_i - |V_i|^2 \text{ Imag } \{Y_{ii}\} \quad (26)$$

IV. Concise Derivation of the Jacobian Matrix

Recent papers in the research literature have presented a method for calculating the Jacobian matrix that is concise, elegant, and simple to program [7], [8], [9]. Those papers use [$\boldsymbol{V}$] to denote a square matrix with vector $\boldsymbol{V}$ along the diagonal and zeros elsewhere. This paper uses diag ($\boldsymbol{V}$) in place of [$\boldsymbol{V}$] . If $\boldsymbol{V}$ is the vector of bus voltages, $\boldsymbol{I}$ is the vector of bus current injections and $\boldsymbol{Y}$ is the bus admittance matrix then the vector of bus power injections is:

$$\boldsymbol{S} = \text{diag}(\boldsymbol{V})\, \boldsymbol{I}^* = \text{diag}(\boldsymbol{V})\, (\boldsymbol{Y}\boldsymbol{V})^* \quad (27)$$

and the partial derivatives required for the Jacobian are

$$D_\delta \boldsymbol{S} = \text{diag}(\boldsymbol{I}^*)\, D_\delta \boldsymbol{V} + \text{diag}(\boldsymbol{V})\, D_\delta (\boldsymbol{Y}^* \boldsymbol{V}^*) \quad (28)$$

$$= \text{diag}(\boldsymbol{I}^*)\, [\, j\, \text{diag}(\boldsymbol{V})\,]$$
$$+ \text{diag}(\boldsymbol{V})\, [\, \boldsymbol{Y}^* \, [\, j\, \text{diag}(\boldsymbol{V})\,]^* \,] \quad (29)$$

$$= j\, \text{diag}(\boldsymbol{V})\, [\, \text{diag}(\boldsymbol{I}^*) - \boldsymbol{Y}^* \text{diag}(\boldsymbol{V}^*)\,] \quad (30)$$

$$D_{|V|} \boldsymbol{S}$$
$$= \text{diag}(\boldsymbol{I}^*)\, D_{|V|} \boldsymbol{V} + \text{diag}(\boldsymbol{V})\, D_{|V|} (\boldsymbol{Y}^* \boldsymbol{V}^*) \quad (31)$$

$$= \text{diag}(\boldsymbol{I}^*)\, \text{diag}(\boldsymbol{V})\, [\, \text{diag}(|\boldsymbol{V}|)\,]^{-1}$$
$$+ \text{diag}(\boldsymbol{V})\, \boldsymbol{Y}^* \text{diag}(\boldsymbol{V}^*)\, [\, \text{diag}(|\boldsymbol{V}^*|)\,]^{-1} \quad (32)$$

$$= \text{diag}(\boldsymbol{V})\, [\text{diag}(\boldsymbol{I}^*) + \boldsymbol{Y}^* \text{diag}(\boldsymbol{V}^*)]\, [\text{diag}(|\boldsymbol{V}|)]^{-1} \quad (33)$$

V. Explaining the Concise Derivation to Students

One approach to teaching the concise formulation to students is to explain the calculations involved in the concise derivation with undergraduate level matrix math. To do this, several derivations must be introduced starting with the derivative of a Hadamard product.

A. Preliminary Mathematical Results

The Hadamard product of two matrices is the entry-wise product matrix defined by [11]. If

$$\boldsymbol{F} = [f_{mn}]_{M \times N} \quad \text{and} \quad \boldsymbol{G} = [g_{mn}]_{M \times N} \quad (34)$$

The Hadamard product of A and B is:

$$\boldsymbol{F} \circ \boldsymbol{G} = [f_{mn} g_{mn}]_{M \times N} \quad (35)$$

Result 1: Derivative of a Hadamard product of vectors.

If $\boldsymbol{F} = [f_m]_{M \times 1}$ and $\boldsymbol{G} = [g_m]_{M \times 1}$ are functions of $\boldsymbol{X} = [x_n]_{N \times 1}$ then

$$D_x (\boldsymbol{F} \circ \boldsymbol{G}) = D_x \begin{bmatrix} f_1 g_1 \\ \vdots \\ f_M g_M \end{bmatrix} \quad (36)$$

$$= \left[\frac{\partial (f_m\, g_m)}{\partial x_n} \right]_{MxN} \quad (37)$$

$$= \left[\frac{\partial f_m}{\partial x_n} g_m + f_m \frac{\partial g_m}{\partial x_n} \right]_{MxN} \quad (38)$$

$$= \left[\frac{\partial f_m}{\partial x_n} g_m\right]_{MxN} + \left[f_m \frac{\partial g_m}{\partial x_n}\right]_{MxN} \quad (39)$$

$$= \text{diag}\,(\, \boldsymbol{G}\,)\ D_X \boldsymbol{F} + \text{diag}\,(\, \boldsymbol{F}\,)\, D_X \boldsymbol{G} \quad (40)$$

Result 2: Derivative of the product of a constant matrix times a variable vector. Let

$\boldsymbol{A} = [a_{km}]_{K \times M}$ be a matrix of constants and

$\boldsymbol{F}(\boldsymbol{X}) = [f_m]_{M \times 1}$ be a function of $\boldsymbol{X} = [x_n]_{N \times 1}$

$$\boldsymbol{A}\,\boldsymbol{F} = \begin{pmatrix} a_{11}f_1 + a_{12}f_2 + \cdots + a_{1M}f_M \\ \vdots \\ a_{K1}f_1 + a_{K2}f_2 + \cdots + a_{KM}f_M \end{pmatrix} \triangleq \begin{bmatrix} g_1 \\ \vdots \\ g_K \end{bmatrix} \quad (41)$$

Then

$$D_X(\boldsymbol{A}\,\boldsymbol{F}) = D_X \begin{bmatrix} g_1 \\ \vdots \\ g_K \end{bmatrix} = \left[\frac{\partial g_k}{\partial x_n}\right]_{KxN} \quad (42)$$

$$= \left[\frac{\partial}{\partial x_n}\left(a_{k1}f_1 + a_{k2}f_2 + \cdots + a_{kM}f_M\right)\right]_{KxN} \quad (43)$$

$$= \left[a_{k1}\frac{\partial f_1}{\partial x_n} + a_{k2}\frac{\partial f_2}{\partial x_n} + \cdots + a_{kM}\frac{\partial f_M}{\partial x_n}\right]_{KxN} \quad (44)$$

$$= \boldsymbol{A}\, D_X(\boldsymbol{F}) \quad (45)$$

Result 3: Derivative of a complex matrix with respect to phase angle and magnitude. Let

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} |V_1|e^{j\delta_1} \\ \vdots \\ |V_N|e^{j\delta_N} \end{bmatrix} \quad , \quad \boldsymbol{\delta} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_N \end{bmatrix} \quad , \quad |\boldsymbol{V}| = \begin{bmatrix} |V_1| \\ \vdots \\ |V_N| \end{bmatrix} \quad (46)$$

The matrix of derivatives of voltage with respect to angle can be expressed as:

$$D_{\boldsymbol{\delta}} \boldsymbol{V} = \begin{bmatrix} \frac{\partial V_1}{\partial \delta_1} & \cdots & \frac{\partial V_1}{\partial \delta_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial V_N}{\partial \delta_1} & \cdots & \frac{\partial V_N}{\partial \delta_N} \end{bmatrix} \quad (47)$$

However, since V_k only depends on δ_k, all off-diagonal entries are zero:

$$D_{\boldsymbol{\delta}} \boldsymbol{V} = \begin{bmatrix} j|V_1|e^{j\delta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & j|V_N|e^{j\delta_N} \end{bmatrix} \quad (48)$$

$$= j \begin{bmatrix} V_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V_N \end{bmatrix} = j\ \text{diag}\,(\boldsymbol{V}) \quad (49)$$

Similarly,

$$D_{|\mathbf{V}|}\, \boldsymbol{V} = \begin{bmatrix} e^{j\delta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{j\delta_N} \end{bmatrix} \quad (50)$$

$$= \begin{bmatrix} |V_1|e^{j\delta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |V_N|e^{j\delta_N} \end{bmatrix} \begin{bmatrix} |V_1|^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |V_N|^{-1} \end{bmatrix} \quad (51)$$

$$= \text{diag}(\boldsymbol{V})\ [\ \text{diag}\,(\,|\,\boldsymbol{V}\,|\,)\ ]^{-1} \quad (52)$$

Result 4: Commutativity of Hadamard products and of diagonal matrices.

If $\boldsymbol{F} = [f_m]_{M \times 1}$ and $\boldsymbol{G} = [g_m]_{M \times 1}$ then

$$\boldsymbol{F} \circ \boldsymbol{G} = [f_m g_m]_{M \times 1} = [g_m f_m]_{M \times 1} = \boldsymbol{G} \circ \boldsymbol{F} \quad (53)$$

$$\text{diag}\,(\,\boldsymbol{F}\,)\ \text{diag}\,(\,\boldsymbol{G}\,) = \text{diag}\,(\,[f_m g_m]_{M \times 1}\,) \quad (54)$$

$$= \text{diag}\,(\,[g_m f_m]_{M \times 1}\,) = \text{diag}\,(\,\boldsymbol{G}\,)\ \text{diag}\,(\,\boldsymbol{F}\,) \quad (55)$$

B. Application of Results to Explain Concise Proof to Students

Then the concise proof can be explained to dual level students by starting with the complex power equation:

$$\boldsymbol{S} = \boldsymbol{V} \circ \boldsymbol{I}^* = \boldsymbol{V} \circ (\,\boldsymbol{Y}\,\boldsymbol{V}\,)^* \quad (56)$$

The derivative of the complex power with respect to the angle can be found using Result 1:

$$D_\delta \boldsymbol{S} = \text{diag}\,(\,\boldsymbol{I}^*\,)\, D_\delta \boldsymbol{V} + \text{diag}\,(\,\boldsymbol{V}\,)\ D_\delta\ (\,\boldsymbol{Y}^*\,\boldsymbol{V}^*\,) \quad (57)$$

The equation can be further simplified by using Result 2 and Result 3:

$$D_\delta \boldsymbol{S} = \text{diag}\,(\,\boldsymbol{I}^*\,)\,[\,j\ \text{diag}\,(\,\boldsymbol{V}\,)\,] + \text{diag}\,(\,\boldsymbol{V}\,)\ [\ \boldsymbol{Y}^*\ [\,j\ \text{diag}\,(\,\boldsymbol{V}\,)\,]^*\ ] \quad (58)$$

Diag ($\boldsymbol{V}$) can be placed at the front of the equation by using Result 4:

$$D_\delta \boldsymbol{S} = j\ \text{diag}\,(\,\boldsymbol{V}\,)\,[\ \text{diag}\,(\,\boldsymbol{I}^*\,) - \boldsymbol{Y}^*\ \text{diag}\,(\,\boldsymbol{V}^*\,)\,] \quad (59)$$

Likewise, the derivative of the complex power with respect to the voltage magnitude can be found starting with Result 1:

$$D_{|V|}\, \boldsymbol{S} = \text{diag}\,(\,\boldsymbol{I}^*\,)\, D_{|V|}\, \boldsymbol{V} + \text{diag}\,(\,\boldsymbol{V}\,)\, D_{|V|}\,(\,\boldsymbol{Y}^*\,\boldsymbol{V}^*\,) \quad (60)$$

The equation can be further simplified using Result 2 and Result 3:

$$D_{|V|}\, \boldsymbol{S} = \text{diag}\,(\,\boldsymbol{I}^*\,)\ \text{diag}\,(\,\boldsymbol{V}\,)\,[\ \text{diag}\,(\,|\,\boldsymbol{V}\,|\,)\,]^{-1} + \text{diag}\,(\,\boldsymbol{V}\,)\ \ \boldsymbol{Y}^*\ \text{diag}\,(\,\boldsymbol{V}^*\,)\ \ [\ \text{diag}\,(|\,\boldsymbol{V}^*\,|)\,]^{-1} \quad (61)$$

Diag ($\boldsymbol{V}$) can be placed at the front of the equation by using Result 4:

$$D_{|V|}\boldsymbol{S} = \text{diag}(\boldsymbol{V})\ [\text{diag}(\boldsymbol{I}^*) + \boldsymbol{Y}^*\ \text{diag}(\boldsymbol{V}^*)]\ [\text{diag}(|\boldsymbol{V}|)]^{-1} \quad (62)$$

C. Expressing Concise Formulation in Terms of Proofs in Appendix

In order to express the concise formulation of the Jacobian in terms that are suitable for undergraduate students, it is necessary to define $\boldsymbol{P} = \text{Real}\{\boldsymbol{S}\}$, $\boldsymbol{Q} = \text{Imag}\{\boldsymbol{S}\}$, and it is helpful to define:

$$\boldsymbol{Sdiag} = \text{diag}(\boldsymbol{S}) \quad (63)$$

$$\boldsymbol{A} = \text{diag}(\boldsymbol{V})\ \boldsymbol{Y}^*\ \text{diag}(\boldsymbol{V}^*) = \boldsymbol{V}\,\boldsymbol{V}^{*T} \circ \boldsymbol{Y}^* \quad (64)$$

Then (59) and (62) become:

$$D_{\delta}\boldsymbol{S} = j\,(\boldsymbol{Sdiag} - \boldsymbol{A}) \tag{65}$$

$$(D_{|V|}\boldsymbol{S})\,\mathrm{diag}(|\boldsymbol{V}|) = \boldsymbol{Sdiag} + \boldsymbol{A} \tag{66}$$

which yield the equations derived in the appendix:

$$\partial P_i/\partial\delta_j = -\,\mathrm{Imag}\,\{\, Sdiag_{ij} - A_{ij} \,\} \tag{67}$$

$$\partial Q_i/\partial\delta_j = \mathrm{Real}\,\{\, Sdiag_{ij} - A_{ij} \,\} \tag{68}$$

$$|V_j|\,\partial P_i/\partial|V_j| = \mathrm{Real}\,\{\, Sdiag_{ij} + A_{ij} \,\} \tag{69}$$

$$|V_j|\,\partial Q_i/\partial|V_j| = \mathrm{Imag}\,\{\, Sdiag_{ij} + A_{ij} \,\} \tag{70}$$

D. Calculating Concise Formulation in MATLAB

Matrix A can be calculated as the element by element product of two matrices, namely the outer product $\boldsymbol{V}^* \boldsymbol{V}^T$ and the bus admittance matrix $\boldsymbol{Y}$. In MATLAB, the operator "`.*`" performs element by element multiplication of vectors or matrices. The MATLAB function "`diag(S)`" creates a matrix with the elements of vector $\boldsymbol{S}$ on the diagonal. Matrices containing all the partial derivatives required to construct the Jacobian matrix as formulated in [2] can be calculated in MATLAB with the following six lines of code:

```
Sdiag = diag ( V .* conj(Y*V) ) ;
A = ( V * conj(V.') ) .* conj(Y) ;
DPDdelta = -imag(Sdiag-A);
VDPDV = real(Sdiag+A);
DQDdelta = real(Sdiag-A);
VDQDV = imag(Sdiag+A);
```

The Jacobian matrix in [2] can then be constructed from the partial derivatives. For example, suppose a four-bus network has generator buses 1, 2 and load buses 3, 4. The slack bus has $\boldsymbol{V}_1 = 1 \angle 0°$ and the voltage magnitude at the second bus $|\boldsymbol{V}_2|$ is also specified in advance because it is a voltage-controlled bus. There are five nonlinear expressions for the mismatch values of P_2, P_3, P_4, Q_3, Q_4, and five variables that must be solved for include δ_2, δ_3, δ_4, $|\boldsymbol{V}_3|$, and $|\boldsymbol{V}_4|$. The Jacobian matrix can be constructed from the matrices in the previous section of code with the following four lines of MATLAB code:

```
J(1:3,1:3) = DPDdelta(2:4,2:4);
J(1:3,4:5) = VDPDV(2:4,3:4);
J(4:5,1:3) = DQDdelta(3:4,2:4);
J(4:5,4:5) = VDQDV(3:4,3:4);
```

VI. USE OF CONCISE FORMULATION IN TEACHING

Teaching the concise formulation of the Jacobian to lower-level students has not been documented in the literature. The derivations of the concise formulation in the appendix are suitable for undergraduate students, where some cases can be presented in lecture while other cases are assigned as exercises. These derivations have been successfully taught in a dual (undergraduate and graduate) level course on computational methods for power systems for ten years.

A. Introducing Newton's Method With Traditional Jacobian

Students first learn to apply Newton's method to a polynomial function of one variable such as:

$$g(x) = x^3 - 5x^2 + 2x + 8 = 0 \tag{70}$$

Then students apply Newton's method to the power flow equations for a 2-bus network with a generator connected to bus 1. The equations and variable are:

$$\boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} P_2 - P_{2,sched} \\ Q_2 - Q_{2,sched} \end{bmatrix} = 0 \quad , \quad \boldsymbol{x} = \begin{bmatrix} \delta_2 \\ |V_2| \end{bmatrix} \tag{71}$$

The first time students apply Newton's method to the two bus network they start with the power flow equations in terms of trigonometric functions and calculate partial derivatives:

$$g_1(\boldsymbol{x}) = |V_2|\,|V_1|\,|Y_{21}|\cos(\theta_{21} + \delta_1 - \delta_2) + |V_2|\,|V_2|\,|Y_{22}|\cos(\theta_{22}) - P_{2,\mathrm{sched}} = 0 \tag{72}$$

$$g_2(\boldsymbol{x}) = -|V_2|\,|V_1|\,|Y_{21}|\sin(\theta_{21} + \delta_1 - \delta_2) - |V_2|\,|V_2|\,|Y_{22}|\sin(\theta_{22}) - Q_{2,\mathrm{sched}} = 0 \tag{73}$$

B. Deriving Concise Formulation of the Jacobian

Then we work through the proofs of the concise formulation of the Jacobian with nearly equal participation by the students. For example, we prove the following in class:

$$\partial P_i/\partial\delta_j = -\,\mathrm{Imag}\,\{\, Sdiag_{ij} - A_{ij} \,\} \tag{74}$$

and assign the proof of the following as homework:

$$\partial Q_i/\partial\delta_j = \mathrm{Real}\,\{\, Sdiag_{ij} - A_{ij} \,\} \tag{75}$$

It helps to preface the class derivation of (74) by calculating the partial derivatives for a 3 bus network with every term written out similar to the expressions for $g_1(\boldsymbol{x})$ and $g_2(\boldsymbol{x})$ above.

After students complete the proof for (75) we derive:

$$|V_j|\,\partial P_i/\partial\delta_j = \mathrm{Real}\,\{\, Sdiag_{ij} + A_{ij} \,\} \tag{76}$$

in class and assign the proof of the following as homework:

$$|V_j|\,\partial Q_i/\partial\delta_j = \mathrm{Imag}\,\{\, Sdiag_{ij} + A_{ij} \,\} \tag{77}$$

By this point it is not necessary to preface the class derivation of (76) by calculating the partial derivatives for a 3 bus network with every term written out.

C. Updating Power Flow Solution to use Concise Jacobian

We show students how to calculate matrices **`Sdiag, A, DPDdelta, VDPDV, DQDdelta, VDQDV`** using the six lines of code in the previous section of this paper. We point out that the first row of the Jacobian can be calculated as:

```
J(1,1) = DPDdelta(2,2) ;
J(1,2) = VDPDV(2,2)/V(2) ;
```

and let students figure out how to calculate the second row of the Jacobian. Then students use Newton's method to solve the 2 bus network a second time using the new (concise) formulation of the Jacobian and see that they get the same answer as before. Then we make a slight variation in the power flow calculation to use the update equations in [2] which eliminates the need to divide by `V(2)` in the expression for `J(1,2)` above.

Then we explain which equations and variables are used in power flow solutions for larger networks and use the concise Jacobian to calculate the power flow solution for a network with 4 buses and 2 generators such as shown in Fig. 1. There are five nonlinear expressions for the mismatch values of P_2 , P_3 , P_4 , Q_3 , Q_4 , and five unknown variables δ_2, δ_3, δ_4, $|V_3|$, and $|V_4|$. We show students how to assign values to a submatrix of the Jacobian with the following line of code:

```
J(1:3,1:3) = DPDdelta(2:4,2:4);
```

and require students to figure out how to complete the Jacobian with the remaining three lines of code:

```
J(1:3,4:5) = VDPDV(2:4,3:4);
J(4:5,1:3) = DQDdelta(3:4,2:4);
J(4:5,4:5) = VDQDV(3:4,3:4);
```

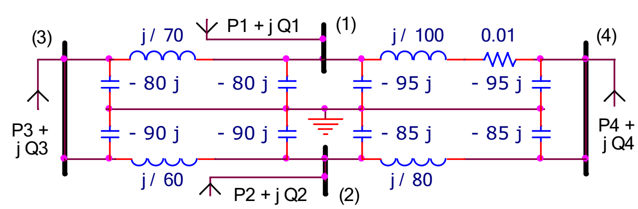

Fig 1: 4-bus network for power flow solution.

We provide values for all the predetermined quantities and an initial starting point such as the following:

$$(P_{2Sch} , P_{3Sch} , P_{4Sch} , Q_{3Sch} , Q_{4Sch}) = (8 , -8 , -8 , -4 , -4) \quad (78)$$

$$\delta_1 = 0 \text{ and } |V_1| = |V_2| = 1 \quad (79)$$

$$x^{(0)} = [0 , 0 , 0 , 1 , 1]^T . \quad (80)$$

and students adapt the power flow program they wrote for the 2-bus network to calculate the power flow for the 4-bus network in Fig. 1 using the concise formulation of the Jacobian. Finally, we ask students to consider how the concise formulation of the Jacobian matrix facilitated writing a power flow solution for the 4 bus network.

VII. Conclusions

The power flow program is an essential part of power systems analysis and design. An iterative solution method such as Newton-Raphson is necessary because the power flow equations are nonlinear. Common formulations of the Jacobian matrix in the NR calculation are cumbersome, inelegant, and laborious to program. Recent literature provides a concise formulation of the Jacobian matrix but its derivation requires mathematical manipulations that are more advanced than what most undergraduate students know. While the proofs in the research literature are difficult for students, the concise formulation is elegant and easy to program which facilitates teaching the NR power flow solution method to undergraduates.

This paper presents two methods for teaching the derivation of the concise formulation of the Jacobian to dual level (undergraduate and graduate) students. The first method involves explaining Hadamard products and higher order derivatives at an undergraduate level in order to fill in the details of derivations in the research literature. The second method only requires undergraduate vector calculus and focuses more on keeping track of which terms in the power flow expressions contain which variables. While the first method has advantages such as introducing higher dimensional derivatives, the proofs of most of the details are going to require an instructor to present details that the students are unlikely to figure out on their own. The second method, in contrast, involves several different cases where the instructor can prove some cases in class and assign others as exercises. Using the second method, students learn to work through all the details of all the cases from start to finish with an amount of repetition that is effective for teaching dual level students.

This paper describes how we have successfully taught NR power flow solution to dual level students for 10 years at Indiana University-Purdue University Indianapolis. We start using Newton's method to find the roots of a polynomial function of a single variable. Then we apply Newton's method to a two-dimensional problem arising from a 2-bus network. The first time we solve the 2-bus power flow, students calculate partial derivatives of the power flow equations in terms of trigonometric functions with all of the terms written out explicitly, that is, without summation notation. Then we derive the concise formulation of the Jacobian for an arbitrary number of variables by dividing the proofs into several cases and assigning proofs of some of the cases as exercises. Then students modify the 2-bus power flow solution to use the concise formulation of the Jacobian and see that the code is much simpler than using trigonometric functions while the numerical results are the same. Finally, students use the concise formulation of the Jacobian to calculate the power flow solution for a 4-bus network with two generators. We have successfully used this method to teach a total of about 130 dual level students, mostly undergraduates, to program the Newton-Raphson power flow solution for small networks.

VIII. References

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IX. Appendix

Equations (67) through (70) are derived starting from the expressions for P_i and Q_i.

$$P_i = \sum_n |Y_{in}V_iV_n| \cos(\theta_{in} + \delta_n - \delta_i) \quad (81)$$

$$Q_i = -\sum_n |Y_{in}V_iV_n| \sin(\theta_{in} + \delta_n - \delta_i) \quad (82)$$

The proof of each equation involves separate cases for $i = j$ and $i \neq j$ and using $\boldsymbol{A}$ defined in equation (64). The results show that each equation is valid for both cases.

A. Partial Derivatives of P_i With Respect to δ_j

For the case where $i \neq j$, only the j'th term contains a δ_j so:

$$\partial P_i/\partial\delta_j = -|Y_{ij}V_iV_j| \sin(\theta_{ij} + \delta_j - \delta_i) \quad (83)$$

For $i \neq j$, $Sdiag_{ij} = 0$ and

$$\begin{aligned}\text{Imag}\{-Sdiag_{ij} + A_{ij}\} &= \text{Imag}\{A_{ij}\} && (84)\\ &= \text{Imag}\{|V_iV_jY_{ij}|\angle(\theta_{ij}+\delta_j-\delta_i)\} && (85)\\ &= |Y_{ij}V_iV_j| \sin(\theta_{ij} + \delta_j - \delta_i) && (86)\\ &= \partial P_i/\partial\delta_j && (87)\end{aligned}$$

For the case where $i = j$, start by pulling out the i'th term of equation (81):

$$P_i = |Y_{ii}V_i^2| \cos(\theta_{ii} + \delta_i - \delta_i) + \sum_{n\neq i} |Y_{in}V_iV_n| \cos(\theta_{in} + \delta_n - \delta_i) \quad (88)$$

The partial with respect to the i'th term is zero. After differentiating the summation, it is convenient to include two additional terms that add up to zero so that they do not change the partial but do allow for further manipulation of the summation.

$$\partial P_i/\partial\delta_i = |Y_{ii}V_i^2| \sin(\theta_{ii} + \delta_i - \delta_i) - |Y_{ii}V_i^2| \sin(\theta_{ii} + \delta_i - \delta_i) + \sum_{n\neq i} |Y_{in}V_iV_n| \sin(\theta_{in} + \delta_n - \delta_i) \quad (89)$$

$$\partial P_i/\partial\delta_i = \sum_n |Y_{in}V_iV_n| \sin(\theta_{in} + \delta_n - \delta_i) - |Y_{ii}V_i^2| \sin(\theta_{ii} + \delta_i - \delta_i) \quad (90)$$

Notice that the first half of $\partial P_i/\partial\delta_i$ is equal to the negative reactive power:

$$\partial P_i/\partial\delta_i = -Q_i + \text{Imag}\{A_{ii}\} \quad (91)$$
$$\partial P_i/\partial\delta_i = \text{Imag}\{-S_i + A_{ii}\} \quad (92)$$
$$\partial P_i/\partial\delta_i = \text{Imag}\{-Sdiag_{ii} + A_{ii}\} \quad (93)$$

In summary, the following is valid for both $i \neq j$, and for $i = j$:

$$\partial P_i/\partial\delta_j = -\text{Imag}\{Sdiag_{ij} - A_{ij}\} \quad (94)$$

END OF PROOF

B. Partial Derivatives of P_i With Respect to $|V_j|$

For the case where $i \neq j$, only the j'th term of equation (81) contains a $|V_j|$ so:

$$\partial P_i/\partial|V_j| = |Y_{ij}V_i| \cos(\theta_{ij} + \delta_j - \delta_i) \quad (95)$$

Equation (95) is then multiplied by $|V_j|$ to obtain:

$$|V_j|\,\partial P_i/\partial|V_j| = |Y_{ij}V_jV_i| \cos(\theta_{ij} + \delta_j - \delta_i) \quad (96)$$

For $i \neq j$, $Sdiag_{ij} = 0$ and

$$\begin{aligned}\text{Real}\{Sdiag_{ij} + A_{ij}\} &= \text{Real}\{A_{ij}\} && (97)\\ &= \text{Real}\{|V_iV_jY_{ij}|\angle(\theta_{ij}+\delta_j - \delta_i)\} && (98)\\ &= |Y_{ij}V_iV_j| \cos(\theta_{ij} + \delta_j - \delta_i) && (99)\\ &= |V_j|\,\partial P_i/\partial|V_j| && (100)\end{aligned}$$

For the case where $i = j$, again start by pulling out the i'th term of equation (81) to get equation (88) and take the partial with respect to $|V_i|$:

$$\partial P_i/\partial|V_i| = 2|Y_{ii}V_i| \cos(\theta_{ii} + \delta_i - \delta_i) + \sum_{n\neq i} |Y_{in}V_n| \cos(\theta_{in} + \delta_n - \delta_i) \quad (101)$$

Multiply by $|V_i|$ and put half of the first term back into the summation:

$$|V_i|\,\partial P_i/\partial|V_i| = |Y_{ii}V_iV_i| \cos(\theta_{ii} + \delta_i - \delta_i) + \sum_n |Y_{in}V_nV_i| \cos(\theta_{in} + \delta_n - \delta_i) \quad (102)$$

Notice that the second half of equation (102) is equal to the real power:

$$|V_i|\,\partial P_i/\partial|V_i| = P_i + \text{Real}\{A_{ii}\} \quad (103)$$
$$|V_i|\,\partial P_i/\partial|V_i| = \text{Real}\{S_i + A_{ii}\} \quad (104)$$

$$|V_i|\,\partial P_i/\partial|V_i| = \text{Real}\{\,Sdiag_{ii} + A_{ii}\,\} \quad (105)$$

In summary, the following is valid for both $i \neq j$, and for $i = j$:

$$|V_j|\,\partial P_i/\partial|V_j| = \text{Real}\{\,Sdiag_{ij} + A_{ij}\,\} \quad (106)$$

END OF PROOF

C. Partial Derivatives of Q_i With Respect to δ_j

The proof of the reactive power partials will be formulated similarly to the real power partials. For the case where $i \neq j$, only the j'th term contains a δ_j so:

$$\partial Q_i/\partial\delta_j = -\,|V_iV_jY_{ij}|\cos(\theta_{ij} + \delta_j - \delta_i) \quad (107)$$

For $i \neq j$, $Sdiag_{ij} = 0$ and

$$\text{Real}\{\,Sdiag_{ij} - A_{ij}\,\} = \text{Real}\{\,-A_{ij}\,\} \quad (108)$$
$$= \text{Real}\{\,-|V_iV_jY_{ij}|\angle(\theta_{ij}+\delta_j - \delta_i)\,\} \quad (109)$$
$$= -|Y_{ij}V_iV_j|\cos(\theta_{ij} + \delta_j - \delta_i) \quad (110)$$
$$= \partial Q_i/\partial\delta_j \quad (111)$$

For the case where $i = j$, start by pulling out the i'th term of equation (82):

$$Q_i = -\,|Y_{ii}V_i^2|\sin(\theta_{ii} + \delta_i - \delta_i) - \textstyle\sum_{n\neq i}|Y_{in}V_iV_n|\sin(\theta_{in} + \delta_n - \delta_i) \quad (112)$$

The partial with respect to the i'th term is zero. After differentiating the summation, it is convenient to include two additional terms that add up to zero so that they do not change the partial but do allow for further manipulation of the summation.

$$\partial Q_i/\partial\delta_i = |Y_{ii}V_i^2|\cos(\theta_{ii} + \delta_i - \delta_i) - |Y_{ii}V_i^2|\cos(\theta_{ii} + \delta_i - \delta_i) + \textstyle\sum_{n\neq i}|Y_{in}V_iV_n|\cos(\theta_{in} + \delta_n - \delta_i) \quad (113)$$

$$\partial Q_i/\partial\delta_i = \textstyle\sum_{n}|Y_{in}V_iV_n|\cos(\theta_{in} + \delta_n - \delta_i) - |Y_{ii}V_i^2|\cos(\theta_{ii} + \delta_i - \delta_i) \quad (114)$$

Notice that the first half of $\partial Q_i/\partial\delta_i$ is equal to the real power:

$$\partial Q_i/\partial\delta_i = P_i + \text{Real}\{\,-A_{ii}\,\} \quad (115)$$
$$\partial Q_i/\partial\delta_i = \text{Real}\{\,S_i - A_{ii}\,\} \quad (116)$$
$$\partial Q_i/\partial\delta_i = \text{Real}\{\,Sdiag_{ii} - A_{ii}\,\} \quad (117)$$

In summary, the following is valid for both $i \neq j$, and for $i = j$:

$$\partial Q_i/\partial\delta_j = \text{Real}\{\,Sdiag_{ij} - A_{ij}\,\} \quad (118)$$

END OF PROOF

D. Partial Derivatives of Q_i With Respect to $|V_j|$

For the case where $i \neq j$, only the j'th term contains a $|V_j|$ so:

$$\partial Q_i/\partial|V_j| = -\,|Y_{ij}V_i|\sin(\theta_{ij} + \delta_j - \delta_i) \quad (119)$$

Equation (119) is then multiplied by $|V_j|$ to obtain:

$$|V_j|\,\partial Q_i/\partial|V_j| = -\,|Y_{ij}V_jV_i|\sin(\theta_{ij} + \delta_j - \delta_i) \quad (120)$$

For $i \neq j$, $Sdiag_{ij} = 0$ and

$$\text{Imag}\{\,Sdiag_{ij} + A_{ij}\,\} = \text{Imag}\{\,A_{ij}\,\} \quad (121)$$
$$= \text{Imag}\{\,|V_iV_jY_{ij}|\angle(\theta_{ij}+\delta_j - \delta_i)\,\} \quad (122)$$
$$= |Y_{ij}V_iV_j|\sin(\theta_{ij} + \delta_j - \delta_i) \quad (123)$$
$$= |V_j|\,\partial Q_i/\partial|V_j| \quad (124)$$

For the case where $i = j$, again start by pulling out the i'th term of equation (82) to get equation (112) and take the partial with respect to $|V_i|$:

$$\partial Q_i/\partial|V_i| = -\,2\,|Y_{ii}V_i|\sin(\theta_{ii} + \delta_i - \delta_i) - \textstyle\sum_{n\neq i}|Y_{in}V_n|\sin(\theta_{in} + \delta_n - \delta_i) \quad (125)$$

Multiply by $|V_i|$ and put half of the first term back into the summation:

$$|V_i|\,\partial Q_i/\partial|V_i| = -\,|Y_{ii}V_iV_i|\sin(\theta_{ii} + \delta_i - \delta_i) - \textstyle\sum_{n}|Y_{in}V_nV_i|\sin(\theta_{in} + \delta_n - \delta_i) \quad (126)$$

Notice that the second half of equation (126) is equal to the reactive power:

$$|V_i|\,\partial Q_i/\partial|V_i| = Q_i + \text{Imag}\{\,A_{ii}\,\} \quad (127)$$
$$|V_i|\,\partial Q_i/\partial|V_i| = \text{Imag}\{\,S_i + A_{ii}\,\} \quad (128)$$
$$|V_i|\,\partial Q_i/\partial|V_i| = \text{Imag}\{\,Sdiag_{ii} + A_{ii}\,\} \quad (129)$$

In summary, the following is valid for both $i \neq j$, and for $i = j$:

$$|V_j|\,\partial Q_i/\partial|V_j| = \text{Imag}\{\,Sdiag_{ij} + A_{ij}\,\} \quad (130)$$

END OF PROOF

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