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ANALYSIS OF THE LINEAR SAMPLING METHOD FOR IMAGING PENETRABLE OBSTACLES IN THE TIME DOMAIN

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We consider the problem of locating and reconstructing the geometry of a penetrable obstacle from time-domain measurements of causal waves. More precisely, we assume that we are given the scattered field due to point sources placed on a surface enclosing the obstacle and that the scattered field is measured on the same surface. From these multistatic scattering data we wish to determine the position and shape of the target.

To deal with this inverse problem, we propose and analyze the time-domain linear sampling method (TDLSM) by means of localizing the interior transmission eigenvalues in the Fourier–Laplace domain. We also prove new time-domain estimates for the forward problem and the interior transmission problem, as well as analyze several time-domain operators arising in the inversion scheme.

1. Introduction

The inverse scattering problem we shall consider is the determination of the shape of an unknown penetrable acoustic scatterer from measurements of the scattered field away from the object. This is a classical model problem, and the linear sampling method (LSM) for determining the support of the unknown scatterer in the frequency domain dates back to [Colton and Kirsch 1996]. Developments and improved theory for this method are discussed in [Colton and Kress 2013]. The method has the advantages of needing very little a priori information about the scatterer, and it does not involve complex optimization calculations. The main drawback is that it requires multistatic data over a large aperture.

We shall postpone a comparison of the frequency-domain and time-domain LSM until the appropriate problems have been defined in Section 4. However we note that multifrequency versions of the frequency-domain LSM show some advantages [Guzina et al. 2010]. Since data is often gathered in the time domain, it is natural to propose a time-domain linear sampling method (TDLSM) for the inverse scattering problem. This was first done for impenetrable scatterers in [Chen et al. 2010], with an improved analysis and better numerical tests appearing in [Haddar et al. 2014]. The latter paper concluded that the TDLSM “is relatively robust under noise and that it is to some extent possible to reconstruct obstacles from limited aperture data.” More recently the TDLSM has been applied in a waveguide [Monk and Selgas 2016] and also, more pertinently to this paper, to penetrable media in [Guo et al. 2013]. The latter paper established some theoretical aspects of the TDLSM for a penetrable medium, but it was not possible to prove the standard theorems related to linear-sampling-type methods. This was because the time-domain interior

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transmission problem could not be analyzed due to the fact that the distribution of the frequency-domain interior transmission eigenvalues in the complex plane was poorly characterized at the time.

Recently, the theoretical understanding of the frequency-domain transmission eigenvalues has improved markedly. In particular, in [Vodev 2015] it is shown that a quadrant in the complex plane (sufficiently far above the real axis, we shall give precise details later) is free of transmission eigenvalues. In this case, Vodev also provides important estimates of the Dirichlet-to-Neumann operator for the transmission problem as a function of the complex frequency. Furthermore, [Vodev 2018a] provides a theoretical framework that allows us to extend the domain in which there are no transmission eigenvalues to an entire half-space of the complex plane and obtain estimates for the Dirichlet-to-Neumann operator in this case. It is then possible, using the estimates for the Fourier–Laplace transform of the time-domain solution of the interior transmission problem, to prove the existence of a unique solution to the time-domain interior transmission problem and derive suitable a priori bounds. The analysis of this solution involves the use of very weak (L^2) solutions of the Helmholtz equation, and we provide an appropriate analysis for this case based on [Cakoni et al. 2016, Remark 3.6; Cakoni and Kress 2017]. In particular, we obtain new frequency-dependent bounds for the solution.

Once the interior transmission problem is well understood, we can then use the approximation properties of time-domain single-layer potentials, and the well-posedness of the forward scattering problem, to conclude time-domain analogues of the usual theorems proved to justify the LSM. This justifies the use of the TDLSM, under rather restrictive assumptions, for penetrable time-domain scattering.

We do not provide any numerical results here, and instead direct interested readers to [Guo et al. 2013].

The layout of the paper is as follows. We start, in Section 2, by defining some notation. Then, in Section 3, we consider the forward problem in the time domain and state existence and uniqueness of the solution in certain weighted Sobolev spaces (more details are provided in the Appendix); this is necessary for our analysis and allows us to give details of the TDLSM. We describe the inverse problem under consideration and the near field operator on which the TDLSM is based in Section 4. The upcoming analysis of the interior transmission problem requires the use of L^2 solutions of the Helmholtz equation having Dirichlet boundary conditions, and we prove well-posedness of this problem (with frequency-dependent estimates) in Section 5. The main contribution of our paper is in Section 6, where we analyze the time-domain interior transmission problem. In particular, this uses heavily the results from [Vodev 2015; 2018a]; this is the key to our analysis of the TDLSM. Finally, in Section 7 we prove that the TDLSM has similar blow-up properties to its usual frequency-domain version.

2. Notation: spaces and the Fourier–Laplace transform

Following [Bamberger and Duong 1986; Ha-Duong 2003; Haddar et al. 2014], our analysis of both the forward and inverse time-domain problem is based on the Fourier–Laplace transform and related function spaces. We now give a brief summary of the relevant tools we will use, mainly from [Ha-Duong 2003]. For each Hilbert space X , we consider $\mathcal{D}'(\mathbb{R}, X)$ the space of X -valued distributions, and we denote by $\mathcal{D}'(\mathbb{R}_+, X)$ its subset of causal distributions, that is, those distributions which vanish on $(-\infty, 0)$. Similarly, $\mathcal{S}'(\mathbb{R}, X)$ and $\mathcal{S}'(\mathbb{R}_+, X)$ stand for tempered and causal tempered distributions, respectively.

Moreover, for all $\sigma > 0$ we define

$$\mathcal{L}'_{\sigma}(\mathbb{R}, X) = \{f \in \mathcal{D}'(\mathbb{R}, X) : e^{-\sigma t} f(t) \in \mathcal{S}'(\mathbb{R}, X)\}.$$

In particular, given $f \in \mathcal{L}'_{\sigma}(\mathbb{R}, X)$ with $e^{-\sigma t} f(t) \in L^1(\mathbb{R}, X)$, we take its Fourier–Laplace transform to be

$$\mathcal{L}[f](s) = \int_{-\infty}^{\infty} e^{ist} f(t) dt \quad \text{for } s \in \mathbb{C}_{\sigma}, \quad (1)$$

where $\mathbb{C}_{\sigma} = \{s \in \mathbb{C} : \Im(s) > \sigma\}$. We also consider, for each $p \in \mathbb{R}$, the Hilbert space

$$H_{\sigma}^p(\mathbb{R}, X) = \left\{ f \in \mathcal{L}'_{\sigma}(\mathbb{R}, X) : \int_{-\infty+i\sigma}^{\infty+i\sigma} |s|^{2p} \|\mathcal{L}[f](s)\|_X^2 ds < \infty \right\}$$

endowed with the norm

$$\|f\|_{H_{\sigma}^p(\mathbb{R}, X)} = \left(\int_{-\infty+i\sigma}^{\infty+i\sigma} |s|^{2p} \|\mathcal{L}[f](s)\|_X^2 ds \right)^{1/2};$$

its subspace of causal fields is denoted by $H_{\sigma}^p(\mathbb{R}_+, X)$.

In addition, for any two Hilbert spaces X, Y , we denote by $\mathcal{B}(X, Y)$ the space of linear continuous operators defined from X on Y .

Following [Ha-Duong 2003], for each Lipschitz domain $\mathcal{O} \subseteq \mathbb{R}^d$ ($d = 2$ or 3) and $s \in \mathbb{C}$ we define the frequency-dependent norms

$$\|v\|_{1,s,\mathcal{O}} = \left(\int_{\mathcal{O}} (|s v(\mathbf{x})|^2 + |\nabla v(\mathbf{x})|^2) d\mathbf{x} \right)^{1/2} \quad \text{for all } v \in H^1(\mathcal{O});$$

in general, by recurrence, we consider

$$\|v\|_{m+1,s,\mathcal{O}} = (|s|^2 \|v\|_{m,s,\mathcal{O}}^2 + |v|_{m+1,\mathcal{O}}^2)^{1/2} \quad \text{for all } v \in H^{m+1}(\mathcal{O}), \quad (2)$$

for $m = 1, 2, \dots$, where $|v|_{m+1,\mathcal{O}}$ is the $H^{m+1}(\mathcal{O})$ seminorm. Notice that these weighted norms are equivalent to the usual unweighted norms for all $s \neq 0$.

When $\partial\mathcal{O}$ is regular, the spaces $H^m(\partial\mathcal{O})$ can be defined equivalently by means of eigenfunctions of the Laplace–Beltrami operator $-\Delta_{\partial\mathcal{O}}$, which also allows us to generalize their definition to arbitrary order $m \in \mathbb{R}$. More precisely, we can follow [Terrasse 1993] and consider an orthonormal basis of $L^2(\partial\mathcal{O})$ which consists of the eigenfunctions Y_i (with $i = 0, \dots, \infty$) of $-\Delta_{\partial\mathcal{O}}$ and sorted so that the associated eigenvalues $\lambda_i \geq 0$ are such that $\lambda_0 = 0$ and $\lambda_i > 0$ for $i = 1, 2, \dots$. We then have that each distribution on $\partial\mathcal{O}$, $g \in \mathcal{D}'(\partial\mathcal{O})$, can be represented as

$$g = \sum_{i=0}^{\infty} g_i Y_i.$$

For all $m \geq 0$, we define

$$H^m(\partial\mathcal{O}) = \left\{ g \in \mathcal{D}'(\partial\mathcal{O}) : \sum_{i=0}^{\infty} (1 + \lambda_i)^m |g_i|^2 < \infty \right\} \quad (3)$$

and endow it with the weighted norm

$$\|g\|_{m,s,\partial\mathcal{O}} = \left(\sum_{i=0}^{\infty} (|s|^2 + \lambda_i)^m |g_i|^2 \right)^{1/2}. \quad (4)$$

Also notice that the corresponding dual spaces (with pivot space $L^2(\partial\mathcal{O})$) can be introduced and they are exactly the same spaces as those we would define by (3) for negative indices $m < 0$; in particular we can endow them by taking here again the norm (4), but now for $m < 0$.

The key role of the above weighted spaces in deducing information about a time-dependent operator by analyzing its Fourier–Laplace counterpart is made clear by the following result; see [Lubich 1994].

Lemma 1. *Let $\sigma > 0$ and assume that $s \in \mathbb{C}_\sigma \mapsto \hat{A}_s \in \mathcal{B}(X, Y)$ is an analytic function such that, for some $r \in \mathbb{R}$, it holds that*

$$\|\hat{A}_s\|_{\mathcal{B}(X,Y)} \leq C|s|^r \quad \text{for a.e. } s \in \mathbb{C}_\sigma.$$

Set

$$a(t) = \frac{1}{2\pi} \int_{-\infty+i\sigma}^{\infty+i\sigma} e^{-ist} \hat{A}_s ds,$$

and let

$$Ag = \int_{-\infty}^{\infty} a(t)g(\cdot - t) dt$$

be the associated convolution operator. Then, for all $p \in \mathbb{R}$, A extends to a bounded operator from $H_\sigma^{p+r}(\mathbb{R}, X)$ to $H_\sigma^p(\mathbb{R}, Y)$.

3. Forward problem

Before we can discuss the inverse problem, we need to define the corresponding forward problem. We consider an inhomogeneous penetrable obstacle with sound speed $c(x)$ occupying a Lipschitz bounded domain $\Omega \subset \mathbb{R}^d$; we denote its complement by $\Omega^+ = \mathbb{R}^d \setminus \bar{\Omega}$ and its boundary by $\Gamma = \partial\Omega$. We take $\mathbf{v}$ to be the unit vector field normal to Γ and directed outward of Ω . We assume Ω^+ is connected and Γ is C^∞ .

Setting $n(x) = c_0^2/c^2(x)$, where c_0 is the speed of sound in the background, we assume $n|_{\bar{\Omega}}$ is in $C^\infty(\bar{\Omega})$ and satisfies

$$n(\mathbf{x}) \neq 1 \quad \text{for all } \mathbf{x} \in \Gamma,$$

where $n|_\Gamma$ is taken as the limiting value from Ω . In addition, we assume that there is a positive constant n_0 such that $n(\mathbf{x}) \geq n_0$ at a.e. $\mathbf{x} \in \mathbb{R}^d$ and note that $n = 1$ in Ω^+ .

The assumptions on the smoothness of the boundary Γ and the function n are necessary to establish the solvability of the interior transmission problem (see Section 6.2) using Vodev's results. If this fact could be established without the use of pseudodifferential calculus, it might be possible to reduce these requirements.

In our inversion scheme the unknown scatterer Ω is probed by incident fields due to point sources. In particular let χ denote a modulation function taken to be a smooth function of compact support on $(0, \infty)$.

Then the incident field $u^{\text{inc}} = u^{\text{inc}}(t, \mathbf{x}; \mathbf{y})$ due to a point source at $\mathbf{y} \in \mathbb{R}^d$ satisfies the following equations in distributional sense:

$$\begin{aligned} \partial_t^2 u^{\text{inc}} - \Delta u^{\text{inc}} &= \chi(t) \delta_{\mathbf{y}} & \text{in } \mathbb{R}^d, \text{ for } t > 0, \\ u^{\text{inc}} &= \partial_t u^{\text{inc}} = 0 & \text{in } \mathbb{R}^d, \text{ for } t \leq 0, \end{aligned} \quad (5)$$

where $\delta_{\mathbf{y}}$ is the Dirac delta centered at $\mathbf{y}$.

We denote by $u = u(t, \mathbf{x}; \mathbf{y})$ the scattered field due to the incident field $u^{\text{inc}}(t, \mathbf{x}; \mathbf{y})$ emitted at $\mathbf{y} \in \Omega^+$. Then, the scattered field solves the problem

$$n \partial_t^2 u - \Delta u = f \quad \text{in } \mathbb{R}^d, \text{ for } t \in \mathbb{R}. \quad (6)$$

Above, we have $f = (1 - n) \partial_t^2 u^{\text{inc}}$ in $\mathbb{R} \times \mathbb{R}^d$, but in the following we allow this function to have a more general expression for the theoretical analysis. Notice that for f causal, the scattered field is also causal and satisfies

$$u = \partial_t u = 0 \quad \text{in } \mathbb{R}^d, \text{ for } t \leq 0.$$

In addition, in the time domain, there is no need for a radiation condition for the scattered field when $|\mathbf{x}| \rightarrow \infty$, thanks to causality and finite wave-speed propagation (so that the field starts at rest and propagates at a finite wave-speed outward from Ω).

The time-dependent problem (6) is already known to be well-posed in suitable function spaces (see [Guo et al. 2013, Proposition 3.2]); in the [Appendix](#) we sketch the analysis because of its importance for our paper. Both for this proof, and in the remainder of the paper, we use the Fourier–Laplace transform (1) for $s \in \mathbb{C}_\sigma$ with appropriate choices of $\sigma > 0$.

On one hand, formally taking the Fourier–Laplace transform in time of (5) we see that the function $\hat{u}_s^{\text{inc}} = \mathcal{L}[u^{\text{inc}}](s)$ is given by $\hat{u}_s^{\text{inc}} = \mathcal{L}[\chi](s) \Phi_s$, where the fundamental solution in the Fourier–Laplace domain, denoted by $\Phi_s = \Phi_s(\mathbf{x}, \mathbf{y})$, satisfies

$$\Delta \Phi_s + s^2 \Phi_s = -\delta_{\mathbf{y}} \quad \text{in } \mathbb{R}^d, \quad (7)$$

together with a suitable radiation condition at infinity. Indeed, since we will only be interested in the case $\Im(s) > \sigma_0 > 0$ for some fixed σ_0 , it is sufficient to require a decaying solution as $|\mathbf{x}| \rightarrow \infty$. The fundamental solution can be written explicitly as

$$\Phi_s(\mathbf{x}, \mathbf{y}) = \begin{cases} i H_0^{(1)}(s|\mathbf{x} - \mathbf{y}|)/4 & \text{if } d = 2, \\ e^{is|\mathbf{x} - \mathbf{y}|}/(4\pi|\mathbf{x} - \mathbf{y}|) & \text{if } d = 3 \end{cases} \quad (8)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ with $\mathbf{x} \neq \mathbf{y}$, and where $H_0^{(1)}$ stands for the Hankel function of order 0 and first kind.

On the other hand, from (6) we deduce that the Fourier–Laplace transform of the scattered field $\hat{u}_s(\mathbf{x}) = \mathcal{L}[u](s, \mathbf{x}; \mathbf{y})$ solves the problem

$$\Delta \hat{u}_s + n s^2 \hat{u}_s = -\hat{f}_s \quad \text{in } \mathbb{R}^d, \quad (9)$$

where we have $\hat{f}_s = \mathcal{L}[f](s, \mathbf{x})$.

As we will do later for other time-domain solutions, we study the Helmholtz-like problem (9) as an auxiliary tool to analyze the time-dependent problem (6). More precisely, in the [Appendix](#) we provide

some explicit bounds on the norm of the solution of (9) which make clear its dependence on $s \in \mathbb{C}_\sigma$; by doing so, we can later make use of Lemma 1 to deduce information about the time-dependent problem (6). In particular we have the following result:

Proposition 2. *Let us fix $\sigma_0 > 0$. Then, for each $f \in H_{\sigma_0}^{p+1}(\mathbb{R}, H^{-1}(\mathbb{R}^d))$ with $p \in \mathbb{R}$, the time-domain problem (6) is well-posed, and its unique solution $u \in H_\sigma^p(\mathbb{R}, H^1(\mathbb{R}^d))$ satisfies*

$$\|u\|_{H_\sigma^p(\mathbb{R}, H^1(\mathbb{R}^d))} \leq C \|f\|_{H_{\sigma_0}^{p+1}(\mathbb{R}, H^{-1}(\mathbb{R}^d))}$$

for all $\sigma \in \mathbb{R}$ with $\sigma \geq \sigma_0$, where C is some constant independent of p , σ and f .

We now denote by

$$S : H_{\sigma_0}^{p+1}(\mathbb{R}, H^{-1}(\mathbb{R}^d)) \rightarrow H_\sigma^p(\mathbb{R}, H^1(\mathbb{R}^d))$$

the solution operator of the forward problem (6); that is, $Sf \in H_\sigma^p(\mathbb{R}, H^1(\mathbb{R}^d))$ represents the unique solution of the time-dependent problem (6) for $f \in H_{\sigma_0}^{p+1}(\mathbb{R}, H^{-1}(\mathbb{R}^d))$. By Proposition 2, we already know that the solution operator is well-defined, linear and uniformly bounded for $p \in \mathbb{R}$ and $\sigma > \sigma_0$; moreover, we know that it preserves causality. Note that from the estimate in Proposition 11 and following Remark 12, we can also see the solution operator S as a bounded operator

$$S : H_{\sigma_0}^{p+1}(\mathbb{R}_+, L^2(\mathbb{R}^d)) \rightarrow H_\sigma^p(\mathbb{R}_+, H^1(\mathbb{R}^d)). \quad (10)$$

3.1. Analysis of the data provided by incident fields in the time domain. Let us recall that, for a given incident field u^{inc} , the datum we will consider for problem (6) will be of the form

$$f = (1 - n) \partial_{tt}^2 u^{\text{inc}} \quad \text{in } \mathbb{R} \times \mathbb{R}^d.$$

Thus, we have to see how this datum depends on the incident field itself; in particular, we have to study if $f \in H_{\sigma_0}^p(\mathbb{R}_+, H^{-1}(\mathbb{R}^d))$ for some $p \in \mathbb{R}$ and, therefore, Proposition 2 applies.

The usual data in the time domain are defined through boundary conditions written in terms of traces and normal derivatives on surfaces on each time slice. For the sake of clarity, let us first introduce these operators in the Fourier–Laplace domain. We first introduce the trace operators $\gamma^\pm$, where the superindices $-$ and $+$ identify whether the trace is taken from the interior $\Omega^- = \Omega$ or the exterior $\Omega^+ = \mathbb{R}^d \setminus \bar{\Omega}$, respectively. These traces are defined for smooth functions $v : \bar{\Omega}^\pm \rightarrow \mathbb{C}$ by $\gamma^\pm v = v|_\Gamma$ and can be extended to well-defined and bounded operators from $H^1(\Omega^\pm)$ into $H^{1/2}(\Gamma)$; in fact, we have

$$\|\gamma^\pm v\|_{1/2, s, \Gamma} \leq C_\gamma \|v\|_{1, s, \Omega^\pm} \quad \text{for all } v \in H^1(\Omega^\pm),$$

where C_γ is a constant independent of $s \in \mathbb{C}_\sigma$; see, e.g., [Ha-Duong 2003, Lemma 1; Haddar et al. 2014, Lemma 1]. We also introduce the normal derivative operators $\partial_\nu^\pm$, where we use the superindices in a similar way. The normal derivatives are defined for smooth functions $v : \bar{\Omega}^\pm \rightarrow \mathbb{C}$ by $\partial_\nu^\pm v = (\nabla v)|_\Gamma \cdot \nu$ and can be extended to a well-defined and bounded operator from $X_s(\Omega^\pm)$ into $H^{-1/2}(\Gamma)$. Here we set

$$X_s(\Omega^\pm) = \{v \in H^1(\Omega^\pm) : \Delta v + s^2 v = 0 \text{ in } \Omega^\pm\},$$

endowed with the $H^1(\Omega^\pm)$ norm. In fact, we have

$$\|\partial_\nu^\pm v\|_{-1/2, s, \Gamma} \leq C_{\partial_\nu} \|v\|_{1, s, \Omega^\pm} \quad \text{for all } v \in X_s(\Omega^\pm), \quad (11)$$

where C_{∂_v} is a constant independent of $s \in \mathbb{C}_\sigma$; see, e.g., [Ha-Duong 2003, Lemma 2] or the proof of [Haddar et al. 2014, Proposition 9].

In the time domain, we use the same notation for the traces and normal derivatives as in the Fourier–Laplace domain. More precisely, we consider the bounded operators

$$\begin{aligned}\gamma^\pm &: H_\sigma^p(\mathbb{R}, H^1(\Omega^\pm)) \rightarrow H_\sigma^p(\mathbb{R}, H^{1/2}(\Gamma)), \\ \partial_\nu^\pm &: X_\sigma^p(\mathbb{R}, H^1(\Omega^\pm)) \rightarrow H_\sigma^p(\mathbb{R}, H^{-1/2}(\Gamma)),\end{aligned}\tag{12}$$

where

$$X_\sigma^p(\mathbb{R}, H^1(\Omega^\pm)) = \{v \in H_\sigma^p(\mathbb{R}, H^1(\Omega^\pm)) : \partial_{tt}^2 v - \Delta v = 0 \text{ in } \Omega^\pm \text{ for } t \in \mathbb{R}\}$$

is endowed with the norm of subspace of $H_\sigma^p(\mathbb{R}, H^1(\Omega^\pm))$.

The datum we are dealing with in (6) is of the form $f = (1 - n)\partial_{tt}^2 u^{\text{inc}}$ and we have to clarify its dependence on the incident field u^{inc} . To this end, we introduce the operator

$$v \in X_{\sigma_0}^p(\mathbb{R}, H^1(\Omega^\pm)) \mapsto (1 - n)\partial_{tt}^2 v \in H_{\sigma_0}^{p-1}(\mathbb{R}, L^2(\Omega^\pm)),$$

which is well-defined, linear and continuous: Indeed, simply notice that we can apply Lemma 1 because, for all $v \in H_{\sigma_0}^p(\mathbb{R}, H^1(\Omega^\pm))$ and a.e. $s \in \mathbb{C}_{\sigma_0}$, the Fourier–Laplace transform of $(1 - n)\partial_{tt}^2 v$ is $(1 - n)s^2 \hat{v}_s$, which satisfies

$$\begin{aligned}\|(1 - n)s^2 \hat{v}_s\|_{0,s,\Omega^\pm} &\leq \|1 - n\|_{\infty,\Omega^\pm} |s|^2 \|\hat{v}_s\|_{0,\Omega^\pm} \\ &\leq \|1 - n\|_{\infty,\Omega^\pm} |s| \|\hat{v}_s\|_{1,s,\Omega^\pm}.\end{aligned}$$

Here and in the sequel, we use the common notation $\|\cdot\|_{\infty,\mathcal{O}}$ to denote the natural norm in $L^\infty(\mathcal{O})$.

4. The near field operator and the inversion scheme

In this section, we first introduce the near field operator in the time domain for a surface Σ assumed to be a closed smooth surface containing Ω in its interior.

We define the *near field operator* in the time domain in integral form as

$$\mathcal{N}\phi(t, \mathbf{x}_m) = \int_\Sigma \int_{-\infty}^t u(\tau, \mathbf{x}_m; \mathbf{x}_s) \phi(t - \tau, \mathbf{x}_s) d\tau dS_{\mathbf{x}_s}\tag{13}$$

for $t \in \mathbb{R}$ and $\mathbf{x}_m \in \Sigma$, where

$$u(\cdot, \cdot; \mathbf{x}_s) = S((1 - n)\partial_{tt}^2 u^{\text{inc}}(\cdot, \cdot; \mathbf{x}_s));$$

that is, $u(\cdot, \cdot; \mathbf{x}_s)$ denotes the scattered field due to the incident field $u^{\text{inc}}(\cdot, \cdot; \mathbf{x}_s)$ emitted at $\mathbf{x}_s \in \Sigma$. Recall that the incident field due to a point source was explicitly defined in Section 3, see (5), and that we have justified the regularity of $(1 - n)\partial_{tt}^2 u^{\text{inc}}$ in Section 3.1.

Now suppose we know the scattered field $u(t, \mathbf{x}; \mathbf{x}_s)$ for all time instances $t > 0$, for all possible receivers $\mathbf{x} \in \Sigma$ and for all sources $\mathbf{x}_s \in \Sigma$; obviously in practice we would have a discrete and noisy data set but we will not consider this further here; see [Guo et al. 2013] for examples using discrete noisy data. The scatterer Ω is unknown and we wish to determine its shape. To this end, we will use the TDLSM

based on the near field operator $\mathcal{N}$; see [Guo et al. 2013]. In particular, given a sampling point $\mathbf{z} \in \mathbb{R}^d$, we use Tikhonov regularization to approximately solve the *near field equation*:

$$\text{find } \varphi_{\mathbf{z}} \in H_{\sigma}^p(\mathbb{R}_+, L^2(\Sigma)) \text{ such that } \mathcal{N}\varphi_{\mathbf{z}} = u_{\mathbf{z}}^{\text{inc}} \text{ in } H_{\sigma}^p(\mathbb{R}_+, H^{1/2}(\Sigma)), \quad (14)$$

where we let $u_{\mathbf{z}}^{\text{inc}} = u^{\text{inc}}(\cdot, \cdot; \mathbf{z})|_{\mathbb{R} \times \Sigma}$. Then we can compute $\|\varphi_{\mathbf{z}}\|_{H_{\sigma}^p(\mathbb{R}_+, L^2(\Sigma))}$. We vary the sampling point (usually on a uniform grid) in the region where we suspect Ω to lie, and then use $1/\|\varphi_{\mathbf{z}}\|_{H_{\sigma}^p(\mathbb{R}_+, L^2(\Sigma))}$ as an indicator function of Ω (this should be close to zero outside the scatterer and nonzero inside). For more details of the LSM philosophy see [Colton and Kress 2013].

In the remainder of this section we recall from [Guo et al. 2013] the formal definition of the near field operator that will be used in the sequel. The relevant theorems about the inversion scheme are given in Section 7.

As we did in Section 3.1 (see (12)) and the comments therein, we introduce the usual trace operator on Σ in the time domain. Doing so, we have the linear and bounded space-time trace operator

$$\gamma_{\Sigma} : H_{\sigma}^p(\mathbb{R}, H^1(\mathbb{R}^d)) \rightarrow H_{\sigma}^p(\mathbb{R}, H^{1/2}(\Sigma))$$

such that $\gamma_{\Sigma} v = v|_{\mathbb{R} \times \Sigma}$ for all $v : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ smooth enough. We then understand in (13) that

$$u(\tau, \mathbf{x}_m; \mathbf{x}_s) = \gamma_{\Sigma} S((1-n) \partial_{tt}^2 u^{\text{inc}}(\cdot, \cdot; \mathbf{x}_s)) \quad \text{for } \tau \in \mathbb{R}, \mathbf{x}_m \in \Sigma.$$

Therefore, we have the following factorization of the near field operator:

$$\mathcal{N} = \mathcal{G} \circ \text{SL}_{\Sigma}^{\chi}, \quad (15)$$

where $\mathcal{G} = \gamma_{\Sigma} \circ S \circ ((1-n) \partial_{tt}^2)$, and the retarded single-layer potential is given by

$$\text{SL}_{\Sigma}^{\chi} \phi(t, \mathbf{x}) = \int_{\Sigma} \int_{-\infty}^t u^{\text{inc}}(\tau, \mathbf{x}; \mathbf{x}_s) \phi(t - \tau, \mathbf{x}_s) d\tau dS_{\mathbf{x}_s}$$

for $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^d$, and $\phi \in H_{\sigma}^p(\mathbb{R}, L^2(\Sigma))$.

The following result states some basic properties of the (regularized) retarded single-layer potential. In particular, the first statement is shown in [Haddar et al. 2014, Propositions 6 and 19], whereas the last statement is analogous to [Haddar et al. 2014, Proposition 19]; see also [Guo et al. 2013, Proposition 3.1].

Proposition 3. *Let $p \in \mathbb{R}$ and $\sigma > 0$. Then, the retarded single-layer potential $\text{SL}_{\Sigma}^{\chi}$ defines a bounded and injective operator*

$$\text{SL}_{\Sigma}^{\chi} : H_{\sigma}^p(\mathbb{R}, L^2(\Sigma)) \rightarrow X_{\sigma}^{p+2}(\mathbb{R}, H^1(\mathbb{R}^d \setminus \Sigma)) \cap H_{\sigma}^{p+2}(\mathbb{R}, H^1(\mathbb{R}^d)),$$

whose trace on $\mathbb{R} \times \Gamma$ is bounded and injective with dense range:

$$\gamma^{+} \circ \text{SL}_{\Sigma}^{\chi} = \gamma^{-} \circ \text{SL}_{\Sigma}^{\chi} : H_{\sigma}^p(\mathbb{R}, L^2(\Sigma)) \rightarrow H_{\sigma}^{p+2}(\mathbb{R}, H^{1/2}(\Gamma));$$

also its restriction to $\mathbb{R} \times \Omega^{\pm}$ is bounded and injective with dense range:

$$\text{SL}_{\Sigma}^{\chi}|_{\mathbb{R} \times \Omega^{\pm}} : H_{\sigma}^p(\mathbb{R}, L^2(\Sigma)) \rightarrow X_{\sigma}^{p+2}(\mathbb{R}, H^1(\Omega^{\pm})).$$

Moreover, it defines a bounded operator when understood as

$$\text{SL}_\Sigma^\chi : H_\sigma^p(\mathbb{R}, H^{-1/2}(\Sigma)) \rightarrow X_\sigma^{p-1}(\mathbb{R}, H^1(\mathbb{R}^d \setminus \Sigma)) \cap H_\sigma^{p-1}(\mathbb{R}, H^1(\mathbb{R}^d)).$$

As a direct consequence of this result, together with the factorization of the near field operator (15), the behavior of the solution and trace operators, and the properties of the data (see Proposition 2 and Section 3.1), we deduce that both

$$\mathcal{N} : H_\sigma^p(\mathbb{R}, L^2(\Sigma)) \rightarrow H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma)) \quad (16)$$

and

$$\mathcal{N} : H_\sigma^p(\mathbb{R}, H^{-1/2}(\Sigma)) \rightarrow H_\sigma^{p-3}(\mathbb{R}, H^{1/2}(\Sigma))$$

are bounded. In the sequel, we will focus on the modified near field operator acting as in (16) and consider just causal fields.

We finally notice that the problems under study conserve causality; see the incident field characterization (5) and problem (6). In this sense, we have already emphasized after Proposition 2 that (6) preserves causality (see also Remark 12 in the Appendix). Using this result together with the causality preserving properties of the retarded single-layer potential and the trace operator, the factorization (15) implies that also

$$\mathcal{N} : H_\sigma^p(\mathbb{R}_+, L^2(\Sigma)) \rightarrow H_\sigma^p(\mathbb{R}_+, H^{1/2}(\Sigma))$$

is well defined and bounded.

Having defined $\mathcal{N}$ we briefly compare the frequency and time-domain LSM in practice. A key point for either LSM is that it does not depend on a knowledge of the scatterer (i.e., it is independent of whether the scatterer is penetrable or not penetrable, connected or disconnected). Of course a theoretical justification of the method does heavily depend on the scattering mechanism.

In both methods we choose the auxiliary source point $\mathbf{z}$ to lie on a grid covering the region where we expect the unknown scatterer to lie. For each $\mathbf{z}$ we compute an indicator function that is expected to be almost zero outside the scatterer and nonzero inside. This is derived from the solution of an appropriate near field equation. The justification of this approach in the time domain is the main result of our paper stated in Theorem 10, while for the frequency domain there is a large literature summarized in [Cakoni et al. 2016].

Time-domain LSM. For each $\mathbf{z}$ we solve a regularized and discretized version of (14). The indicator function is then the reciprocal of a discrete norm of the approximation to the solution $\varphi_{\mathbf{z}}$. For details see [Guo et al. 2013; Haddar et al. 2014]. Because the resulting matrices can be very large, we use the truncated singular-value decomposition to regularize the problem and compute this iteratively. As shown in [Guo et al. 2013; Haddar et al. 2014], in comparison to the frequency-domain approach similar results can often be computed for fewer sources and in a smaller aperture.

Frequency domain LSM. The frequency-domain LSM requires solving a discretized version of the Fourier transform with respect to time of the near field equation in (14). This reduces the problem to solving an ill-posed linear integral equation on Σ frequency by frequency. Because the discrete problems are smaller, we often use Tikhonov regularization to regularize the problem together with the Morozov

discrepancy principle to choose the regularization parameter; see [Cakoni et al. 2016]. Multifrequency versions of the frequency-domain LSM show some advantages compared to single-frequency inversions [Guzina et al. 2010]. By using all available frequencies in the time-domain method we hope to reap the benefits of the multifrequency LSM as much as possible.

5. Very weak solutions of the Helmholtz problem

When we study the interior transmission problem, we shall encounter solutions of the wave equation that are only L^2 regular in space. These very weak solutions can easily be characterized variationally. Our goal in this section is to give a precise definition of such a very weak solution, and derive s -dependent bounds, and hence space-time bounds, on the time-dependent solution (these being new). Our analysis is motivated by [Cakoni et al. 2016, Remark 3.6; Cakoni and Kress 2017].

In particular, we want to define a solution of the following interior Dirichlet problem with a weak boundary datum $\hat{u}_\Gamma \in H^{-1/2}(\Gamma)$:

$$\begin{aligned} \Delta \hat{u} + s^2 \varepsilon \hat{u} &= 0 & \text{in } \Omega, \\ \hat{u} &= \hat{u}_\Gamma & \text{on } \Gamma, \end{aligned} \quad (17)$$

where either $\varepsilon = 1$ or $\varepsilon = n$ and we define $\varepsilon_0 = \min_{\mathbf{x} \in \Omega} \varepsilon(\mathbf{x}) > 0$. As before, we consider $s \in \mathbb{C}_\sigma$ for $\sigma > 0$; in particular we know that $\Im(s) > 0$.

In order to define the very weak solution of (17), we let $L_\Delta^2(\Omega) = \{\hat{v} \in L^2(\Omega) : \Delta \hat{v} \in L^2(\Omega)\}$ endowed with the weighted norm

$$\|\hat{v}\|_{\Delta, s, \Omega} = \sqrt{\|\Delta \hat{v}\|_{0, \Omega}^2 + |s|^4 \|\hat{v}\|_{0, \Omega}^2}.$$

We also consider its subspace $L_{\Delta, 0}^2(\Omega) = \{\hat{v} \in L_\Delta^2(\Omega) : \hat{v} = 0 \text{ on } \Gamma\}$, which is well-defined; see [Cakoni and Kress 2017]. Note that, under our assumptions and using elliptic regularity, the normal derivative $\partial_\nu : L_{\Delta, 0}^2(\Omega) \rightarrow H^{1/2}(\Gamma)$ is well-defined and bounded. Now we deduce a variational formulation of (17) by formally using integration by parts twice:

$$\begin{aligned} 0 &= \int_\Omega (\Delta \hat{u} + s^2 \varepsilon \hat{u}) \bar{\hat{v}} \, d\mathbf{x} = \int_\Omega (-\nabla \hat{u} \cdot \nabla \bar{\hat{v}} + s^2 \varepsilon \hat{u} \bar{\hat{v}}) \, d\mathbf{x} \\ &= \int_\Omega (\hat{u} \Delta \bar{\hat{v}} + s^2 \varepsilon \hat{u} \bar{\hat{v}}) \, d\mathbf{x} - \int_\Gamma \hat{u} \overline{\partial_\nu \hat{v}} \, dS \quad \text{for all } \hat{v} \in L_{\Delta, 0}^2(\Omega). \end{aligned}$$

Thus, we understand that a *very weak* solution of problem (17) is a function $\hat{u} \in L^2(\Omega)$ such that

$$\int_\Omega (\hat{u} \Delta \bar{\hat{v}} + s^2 \varepsilon \hat{u} \bar{\hat{v}}) \, d\mathbf{x} = \int_\Gamma \hat{u}_\Gamma \overline{\partial_\nu \hat{v}} \, dS \quad \text{for all } \hat{v} \in L_{\Delta, 0}^2(\Omega). \quad (18)$$

To analyze this problem, we define the sesquilinear form

$$B(\hat{v}_1, \hat{v}_2) = \int_\Omega (\hat{v}_1 \Delta \bar{\hat{v}}_2 + s^2 \varepsilon \hat{v}_1 \bar{\hat{v}}_2) \, d\mathbf{x} \quad \text{for all } \hat{v}_1, \hat{v}_2 \in L_\Delta^2(\Omega).$$

Let us verify the Nečas–Babuška conditions in the form given in [Braess 2001, Theorem 3.6].

(i) We first show *continuity* by noticing that

$$\begin{aligned} |B(\hat{v}_1, \hat{v}_2)| &\leq \|\hat{v}_1\|_{0,\Omega} (\|\Delta \hat{v}_2\|_{0,\Omega} + |s|^2 \|\varepsilon\|_{\infty,\Omega} \|\hat{v}_2\|_{0,\Omega}) \\ &\leq \sqrt{2} \max(1, \|\varepsilon\|_{\infty,\Omega}) \|\hat{v}_1\|_{0,\Omega} \|\hat{v}_2\|_{\Delta,s,\Omega} \quad \text{for all } \hat{v}_1, \hat{v}_2 \in L^2_{\Delta}(\Omega). \end{aligned}$$

(ii) We next prove an *inf-sup* (or Babuška–Brezzi) condition. To this end, let us consider $\hat{v}_1 \in L^2(\Omega)$ and denote by $\hat{v}_2 \in H^1(\Omega)$ the solution of

$$\begin{aligned} \Delta \hat{v}_2 + \bar{s}^2 \varepsilon \hat{v}_2 &= \hat{v}_1 \quad \text{in } \Omega, \\ \hat{v}_2 &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{19}$$

Since $\Im(s) > 0$ and the function ε is real and strictly positive, the interior Dirichlet problem (19) is solvable. Indeed, by multiplying by a test function $\hat{w}$, integrating over Ω and integrating by parts,

$$-\int_{\Omega} \nabla \hat{v}_2 \cdot \overline{\nabla \hat{w}} \, d\mathbf{x} + \bar{s}^2 \int_{\Omega} \varepsilon \hat{v}_2 \bar{\hat{w}} \, d\mathbf{x} = \int_{\Omega} \hat{v}_1 \cdot \bar{\hat{w}} \, d\mathbf{x};$$

we can apply the Lax–Milgram lemma thanks to the ellipticity of the sesquilinear form on the left-hand side of the above equation: when $\Re(s) \neq 0$, we may take its imaginary part; otherwise, if $\Re(s) = 0$, it is straightforward. We also have the estimate

$$-s \|\nabla \hat{v}_2\|_{0,\Omega}^2 + |s|^2 \bar{s} \int_{\Omega} \varepsilon |\hat{v}_2|^2 \, d\mathbf{x} = s \int_{\Omega} \hat{v}_1 \bar{\hat{v}}_2 \, d\mathbf{x};$$

therefore, by taking the imaginary part and using the fact that $\Im(s) \geq \sigma > 0$ and $\varepsilon(x) \geq \varepsilon_0$ in Ω , we have

$$\min(1, \varepsilon_0) (\|\nabla \hat{v}_2\|_{0,\Omega}^2 + |s|^2 \|\hat{v}_2\|_{0,\Omega}^2) \leq \|\hat{v}_1\|_{0,\Omega} \|\hat{v}_2\|_{0,\Omega};$$

as a consequence,

$$|s|^2 \|\hat{v}_2\|_{0,\Omega} \leq \frac{1}{\min(1, \varepsilon_0)} \|\hat{v}_1\|_{0,\Omega}.$$

Moreover, according to (19) it holds $\Delta \hat{v}_2 = -\bar{s}^2 \varepsilon \hat{v}_2 + \hat{v}_1$ in Ω , so that

$$\|\Delta \hat{v}_2\|_{0,\Omega} \leq |s|^2 \|\varepsilon\|_{\infty,\Omega} \|\hat{v}_2\|_{0,\Omega} + \|\hat{v}_1\|_{0,\Omega}.$$

Summing up, we have

$$\|\hat{v}_2\|_{\Delta,s,\Omega} \leq C \|\hat{v}_1\|_{0,\Omega}, \tag{20}$$

where $C > 0$ is independent of $s \in \mathbb{C}_{\sigma}$. Returning to the inf-sup condition, we can make the choice $\hat{w} = \hat{v}_2$ to deduce that

$$\sup_{\hat{w} \in L^2_{\Delta,0}(\Omega)} \frac{|B(\hat{v}_1, \hat{w})|}{\|\hat{w}\|_{\Delta,s,\Omega}} \geq \frac{\|\hat{v}_1\|_{0,\Omega}^2}{\|\hat{v}_2\|_{\Delta,s,\Omega}} \geq \frac{1}{C} \|\hat{v}_1\|_{0,\Omega}$$

and the inf-sup condition is satisfied.

(iii) We finally verify the *uniqueness condition*. With this aim, we consider $\hat{v}_1 \in L^2_{\Delta,0}(\Omega)$ such that $\hat{v}_1 \neq 0$. Then, we may choose $\hat{v}_2 = \Delta \hat{v}_1 + s^2 \varepsilon \hat{v}_1 \in L^2(\Omega)$ and obtain

$$B(\hat{v}_2, \hat{v}_1) = \|\Delta \hat{v}_1 + s^2 \varepsilon \hat{v}_1\|_{0,\Omega}^2 > 0,$$

where we have used that $\Im(s) > 0$ and ε is real and strictly positive: Indeed, if $\hat{v}_1 \in L^2_{\Delta,0}(\Omega)$ satisfies $\Delta \hat{v}_1 + s^2 \varepsilon \hat{v}_1 = 0$ in Ω , by multiplying by $\hat{v}_1$, integrating over Ω and integrating by parts,

$$-\int_{\Omega} |\nabla \hat{v}_1|^2 d\mathbf{x} + s^2 \int_{\Omega} \varepsilon |\hat{v}_1|^2 d\mathbf{x} = 0;$$

when $\Re(s) \neq 0$, we may take the imaginary part above to deduce that $\hat{v}_1 = 0$ in Ω ; on the other hand, if $\Re(s) = 0$, we have that $s^2 = -\Im(s)^2$ and we also deduce that $\hat{v}_1 = 0$ in Ω .

Having verified the above Nečas–Babuška conditions, we know there is a unique solution $\hat{u} \in L^2_{\Delta}(\Omega)$ of (18). Furthermore using the solution $\hat{v} = \hat{v}_2 \in L^2_{\Delta,0}(\Omega)$ of (19) for $\hat{v}_1 = \hat{u}$, we have (see [Cakoni et al. 2016, Remark 3.6])

$$\|\hat{u}\|_{0,\Omega}^2 = \left| \int_{\Gamma} \hat{u}_{\Gamma} \overline{\partial_{\mathbf{v}} \hat{v}} dS \right| \leq \|\hat{u}_{\Gamma}\|_{-1/2,\Gamma} \|\partial_{\mathbf{v}} \hat{v}\|_{1/2,\Gamma} \leq C'_{\partial_{\mathbf{v}}} \|\hat{u}_{\Gamma}\|_{-1/2,\Gamma} \|\Delta \hat{v}\|_{0,\Omega},$$

where again $C'_{\partial_{\mathbf{v}}}$ is independent of s (indeed, this is an a priori estimate for the Poisson problem). But using (20) we know that $\|\hat{v}\|_{\Delta,s,\Omega} \leq C \|\hat{u}\|_{0,\Omega}$; therefore

$$\|\hat{u}\|_{0,\Omega} \leq C'_{\partial_{\mathbf{v}}} C \|\hat{u}_{\Gamma}\|_{-1/2,\Gamma},$$

and the bound does not depend on $s \in \mathbb{C}_{\sigma}$.

Summarizing, we have proved the following result.

Proposition 4. *Provided $s \in \mathbb{C}_{\sigma}$ for some $\sigma > 0$, there is a unique very weak solution $\hat{u} \in L^2(\Omega)$ to (17). Furthermore we have the a priori estimate*

$$\|\hat{u}\|_{0,\Omega} \leq C'_{\partial_{\mathbf{v}}} C \|\hat{u}_{\Gamma}\|_{-1/2,\Gamma},$$

where C is independent of $s \in \mathbb{C}_{\sigma}$.

Notice that, by Lemma 1, we also deduce the well-posedness of the counterpart of the very weak problem in the time domain, whose unique solution satisfies

$$\|u\|_{H^p_{\sigma}(\mathbb{R}, L^2(\Omega))} \leq C \|u_{\Gamma}\|_{H^p_{\sigma}(\mathbb{R}, H^{-1/2}(\Gamma))}.$$

6. The interior transmission problem

In this section, we study the *interior transmission problem* (ITP) in both in the Fourier–Laplace domain and the time domain, which provides us with a key tool for the analysis of the TDLSM in Section 7.

6.1. The ITP in the Fourier–Laplace domain. Let us consider the following interior transmission problem in the Fourier–Laplace domain (FDITP):

$$\begin{aligned} \Delta \hat{u}_1 + s^2 \hat{u}_1 &= 0 & \text{in } \Omega, \\ \Delta \hat{u}_2 + s^2 \hat{u}_2 &= 0 & \text{in } \Omega, \\ \hat{u}_1 - \hat{u}_2 &= \hat{g}, \quad \partial_{\mathbf{v}} \hat{u}_1 - \partial_{\mathbf{v}} \hat{u}_2 = \hat{h} & \text{on } \Gamma. \end{aligned} \tag{21}$$

We need to show that this problem has a unique solution for s in an appropriate half of the complex plane and establish s -dependent bounds in appropriate norms on the solution. Indeed, we shall localize the *interior transmission eigenvalues* by extending slightly the results of [Vodev 2015; 2018a]. More precisely, let us consider the homogeneous counterpart of the FDITP, which we simply name FDITP_0 . We then say that $s \in \mathbb{C}$ is a *transmission eigenvalue* if the problem FDITP_0 has a nontrivial solution $\hat{u}_1, \hat{u}_2 \in L^2(\Omega)$ with $\hat{u}_1 - \hat{u}_2 \in H_0^2(\Omega)$.

For later use, we will need the auxiliary interior Dirichlet problem

$$\begin{aligned} \Delta \hat{w} + s^2 \hat{w} &= 0 & \text{in } \Omega, \\ \hat{w} &= \hat{g} & \text{on } \Gamma. \end{aligned} \quad (22)$$

Because of the regularity of the boundary, this problem is well-posed for $\hat{g} \in H^{m-1/2}(\Gamma)$ when $m \geq 1$, in which case $\hat{w} \in H^m(\Omega)$ and $\partial_\nu \hat{w} \in H^{m-3/2}(\Gamma)$. We can then use $\hat{w}$ to rewrite the FDITP in terms of $\hat{v} = \hat{u}_1 - \hat{w}$ and $\hat{u} = \hat{u}_2$ as

$$\begin{aligned} \Delta \hat{v} + s^2 \hat{v} &= 0 & \text{in } \Omega, \\ \Delta \hat{u} + s^2 \hat{u} &= 0 & \text{in } \Omega, \\ \hat{v} - \hat{u} &= 0, \quad \partial_\nu \hat{v} - \partial_\nu \hat{u} = \hat{h} - \partial_\nu \hat{w} & \text{on } \Gamma. \end{aligned} \quad (23)$$

This problem can be equivalently written as the problem of finding $\hat{\psi} \in L^2(\Gamma)$ such that

$$(\mathcal{M}_1(s) - \mathcal{M}_n(s))\hat{\psi} = \hat{h} - \partial_\nu \hat{w},$$

where $\hat{\psi} := \hat{u}|_\Gamma = \hat{v}|_\Gamma$ and $\mathcal{M}_\varepsilon(s)$ is the Dirichlet-to-Neumann operator associated with $\Delta + s^2\varepsilon$ in Ω .

Theorem 5. Assume that $\hat{h} \in H^1(\Gamma)$ and $\partial_\nu \hat{w} \in H^1(\Gamma)$. Then there exists $\sigma_* > 0$ sufficiently large such that there exists no interior transmission eigenvalue in $\mathbb{C}_{\sigma_*}$. Furthermore, for $s \in \mathbb{C}_{\sigma_*}$ the modified FDITP problem (23) has a unique solution $\hat{u}, \hat{v} \in L^2(\Omega)$ and satisfies the following a priori estimate on its trace:

$$\|\hat{u}\|_{0,\Gamma} \leq C \|\hat{h} - \partial_\nu \hat{w}\|_{1,\Gamma}.$$

Remark 6. The constant σ_* cannot in general be arbitrarily close to zero. In [Leung and Colton 2012] they show an example of a spherically stratified medium for which infinitely many complex eigenvalues are uniformly bounded away from the real axis.

Remark 7. It would be desirable to prove the above theorem for the optimal boundary data in $\hat{h} \in H^{1/2}(\Gamma)$ and $\partial_\nu \hat{w} \in H^{-1/2}(\Gamma)$, resulting in $\hat{u} \in H^{-1/2}(\Gamma)$. This would then improve the regularity requirements in later theorems.

Proof. The proof of this theorem can be done by distinguishing two cases for $s \in \mathbb{C}_{\sigma_*}$, depending on the conditions $|\Re(s)| \geq 1$ and $|\Re(s)| < 1$, and where σ_* will be defined shortly:

Case 1: $|\Re(s)| \geq 1$. In this case, the results [Vodev 2018a, Theorems 2.1 and 6.1] guarantee that, for $\sigma_* > 0$ sufficiently large, problem (23) has a unique solution that satisfies

$$\|\hat{u}\|_{0,\Gamma} \leq C |s|^{-1} \|\hat{h} - \partial_\nu \hat{w}\|_{1,\Gamma}, \quad (24)$$

where $C > 0$ does not depend on $s \in \mathbb{C}_\sigma$ with $\sigma \geq \sigma_* > 0$.

Case 2: $|\Re(s)| < 1$. Let us set some notation following [Vodev 2015]. Consider the interior Dirichlet problem of finding $\hat{w}$ such that

$$\Delta \hat{w} + s^2 m \hat{w} = 0 \quad \text{in } \Omega \quad \text{and} \quad \hat{w}|_{\Gamma} = \hat{\psi}.$$

Define

- r_0 to be the principal symbol of the Laplace–Beltrami operator on Γ ($-\Delta_{\Gamma}$) equipped with the Riemannian metric induced by the Euclidean metric in $\mathbb{R}^d$;
- $\rho_m^2 \equiv \rho_m^2(z) = mz - r_0$ on Γ with the square root chosen so that $\Im(\rho_m) > 0$.

We make use of the following theorem which is a special case of Theorem 3.3 from [Vodev 2015].

Theorem 8. *Let*

$$\mathcal{Z} := \{z_0 \in \mathbb{C} : \Re(z_0) = -1, |\Im(z_0)| \leq 1\}.$$

Then there exists some h_0 small enough such that for all h with $0 < h \leq h_0$ the solution $\hat{w}$ of the interior Dirichlet problem satisfies

$$|-ih\partial_{\nu}\hat{w} - \text{Op}_h(\rho_m(z) + hb)\hat{\psi}|_{H_{\text{sc}}^1(\Gamma)} \leq Ch\|\hat{\psi}\|_{0,\Gamma}, \quad (25)$$

where $b \in \mathcal{S}_{0,1}^0$ does not depend on h , $z \in \mathcal{Z}$ or the function m .

In the above theorem and in the sequel $\text{Op}_h(q)$ stands for the h -pseudodifferential operator with symbol q ; see Section 2 in [Vodev 2015]. Further, $|\cdot|_{H_{\text{sc}}^1(\Gamma)}$ is the semiclassical $H^1(\Gamma)$ norm and $\mathcal{S}_{0,1}^0$ is the class of symbols defined in [Vodev 2018b].

Assuming that $\sigma > \sigma_*$ where σ_* is sufficiently large, we shall show that (23) has a unique solution which satisfies

$$\|\hat{u}\|_{0,\Gamma} \leq C\|\hat{h} - \partial_{\nu}\hat{w}\|_{1,\Gamma},$$

with C independent of $s \in \mathbb{C}_{\sigma}$. Together with (24) this proves the theorem since $|s|^{-1} \leq 1/\sigma_*$.

It is straightforward to see that $|\Re(s^2)| > |\Im(s^2)|$ as long as $s \in \mathbb{C}_{\sigma}$ with

$$\sigma \geq \frac{1}{\sqrt{2}-1},$$

and also that $\Re(s^2) \ll -1$ if $\sigma \geq \sqrt{2}$. This places us in the case considered in Theorem 8 when we define

$$h = \frac{1}{\sqrt{-\Re(s^2)}}, \quad z = -\frac{s^2}{\Re(s^2)}.$$

Indeed, it holds that $h > 0$ is small (the bigger the σ , the smaller the h) and that $z \in \mathcal{Z}$. Then, by Theorem 8 with $m := 1$, we know that, for each small enough h (that is, when $\sigma > \sigma_*$ big enough), we have a smooth function b (independent of h small enough and $z \in \mathcal{Z}$, and therefore independent of σ) such that

$$|-ih\partial_{\nu}\hat{v} - \text{Op}_h(\rho_1 + hb)\hat{v}|_{\Gamma}|_{H_{\text{sc}}^1(\Gamma)} \leq Ch\|\hat{v}|_{\Gamma}\|_{L^2(\Gamma)}.$$

Similarly, applying Theorem 8 with $m := n$ for each small enough h ,

$$|-ih\partial_{\nu}\hat{u} - \text{Op}_h(\rho_n + hb)\hat{u}|_{\Gamma}|_{H_{\text{sc}}^1(\Gamma)} \leq Ch\|\hat{u}|_{\Gamma}\|_{L^2(\Gamma)}.$$

By using the triangle inequality and recalling that $\hat{v}|_\Gamma = \hat{u}|_\Gamma$, we deduce that

$$\left| (\text{Op}_h(\rho_1 + hb) - \text{Op}_h(\rho_n + hb))\hat{u}|_\Gamma \right|_{H_{\text{sc}}^1(\Gamma)} \leq h|\partial_\nu \hat{v} - \partial_\nu \hat{u}|_{H_{\text{sc}}^1(\Gamma)} + Ch\|\hat{u}|_\Gamma\|_{L^2(\Gamma)}.$$

Above and in the sequel, we denote by C a positive constant which does not depend on h nor z . We also know, by (5.9) and (5.11) in [Vodev 2015] applied to $\varepsilon = \frac{1}{2}$, using the notation of that paper,

$$\begin{aligned} \left\| \text{Op}_h\left(\frac{1}{\rho_1 - \rho_n}\right) \right\|_{H_{\text{sc}}^1(\Gamma) \rightarrow L^2(\Gamma)} &\leq C, \\ \left\| \text{Op}_h\left(\frac{1}{\rho_1 - \rho_n}\right) \text{Op}_h(\rho_1 - \rho_n) - \text{Id} \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} &\leq Ch, \end{aligned}$$

where Id is the identity for h -pseudodifferential operators.

Summing up, we have

$$\begin{aligned} \|\hat{u}|_\Gamma\|_{0,\Gamma} &\leq \left\| \left(\text{Op}_h\left(\frac{1}{\rho_1 - \rho_n}\right) \text{Op}_h(\rho_1 - \rho_n) - \text{Id} \right) \hat{u}|_\Gamma \right\|_{0,\Gamma} + \left\| \text{Op}_h\left(\frac{1}{\rho_1 - \rho_n}\right) \text{Op}_h(\rho_1 - \rho_n) \hat{u}|_\Gamma \right\|_{0,\Gamma} \\ &\leq Ch\|\hat{u}|_\Gamma\|_{0,\Gamma} + C\left| \text{Op}_h(\rho_1 - \rho_n) \hat{u}|_\Gamma \right|_{H_{\text{sc}}^1(\Gamma)} \\ &\leq Ch\|\hat{u}|_\Gamma\|_{0,\Gamma} + C\left| \text{Op}_h(\rho_1 + hb) - \text{Op}_h(\rho_n + hb) \hat{u}|_\Gamma \right|_{H_{\text{sc}}^1(\Gamma)} \\ &\leq Ch\|\hat{u}|_\Gamma\|_{0,\Gamma} + Ch|\partial_\nu \hat{v} - \partial_\nu \hat{u}|_{H_{\text{sc}}^1(\Gamma)}, \end{aligned}$$

and we conclude that

$$\|\hat{u}|_\Gamma\|_{0,\Gamma} \leq C\|\partial_\nu \hat{v} - \partial_\nu \hat{u}\|_{1,\Gamma}$$

for h small enough and fixed (i.e., for $\sigma \geq \sigma_*$, with σ_* large enough). $\square$

We now may use the preceding theorem and our estimates for the very weak solution of the Dirichlet problem in Proposition 4 to prove the following result.

Corollary 9. *Under the hypotheses of Theorem 5, and assuming $\hat{g} \in H^2(\Gamma)$, there exists a unique solution $\hat{u}_j \in L^2(\Omega)$, $j = 1, 2$, of (21) for each $s \in \mathbb{C}_{\sigma_*}$. Furthermore the following estimates hold:*

$$\|\hat{u}_j\|_{0,\Omega} \leq C(\|\hat{h}\|_{1,s,\Gamma} + |s|^{5/2}\|\hat{g}\|_{2,s,\Gamma}),$$

where $C > 0$ does not depend on $s \in \mathbb{C}_{\sigma_*}$ and $j = 1, 2$.

Proof. We now return to the FDITP in the form (21) and will show that Theorem 5 is sufficient to deduce existence of a unique solution to this problem. We need to bound $\partial_\nu \hat{w}$ when $\hat{w}$ is taken to be the solution of (22). Because of the smoothness of the domain, the normal derivative operator is well-defined and bounded from $H_{\Delta}^{5/2}(\Omega) = \{\hat{w} \in H^{5/2}(\Omega) : \Delta \hat{w} \in H^{3/2}(\Omega)\}$ into $H^1(\Gamma)$, so that

$$\|\partial_\nu \hat{w}\|_{1,\Gamma} \leq C(\|\hat{w}\|_{5/2,\Omega} + \|\Delta \hat{w}\|_{3/2,\Omega}). \quad (26)$$

Moreover, when $\hat{w}$ is taken to be the solution of (22), we have $\Delta \hat{w} = -s^2 \hat{w}$ in Ω and, since the domain is smooth, it satisfies the estimate

$$\|\hat{w}\|_{m,\Omega} \leq C(|s|^2\|\hat{w}\|_{m-2,\Omega} + \|\hat{g}\|_{m-1/2,\Gamma}) \quad \text{for } m \geq 1. \quad (27)$$

This estimate is well known: for $m \geq 2$, it follows by using the right inverse of the trace operator and Theorem 5 of [Evans 2010], and for $2 \geq m \geq 1$ it follows by interpolating between the estimate for $m = 2$ and the standard estimate in $H^1(\Omega)$. Thus, from (26) and (27) we have

$$\|\partial_{\mathbf{v}} \hat{w}\|_{1,\Gamma} \leq C(|s|^2 \|\hat{w}\|_{1/2,\Omega} + |s|^4 \|\hat{w}\|_{-1/2,\Omega} + \|\hat{g}\|_{2,\Gamma} + |s|^2 \|\hat{g}\|_{1,\Gamma}).$$

On the other hand, by the definition of the weighted norms (2) we know

$$\|\hat{w}\|_{1/2,\Omega} + |s| \|\hat{w}\|_{-1/2,\Omega} \leq \|\hat{w}\|_{1,\Omega} + |s| \|\hat{w}\|_{0,\Omega} \leq C \|\hat{w}\|_{1,s,\Omega},$$

which combined with a bound for the solution of the interior Dirichlet problem in the Fourier–Laplace domain (see the Appendix for similar reasoning), leads to

$$\|\hat{w}\|_{1/2,\Omega} + |s| \|\hat{w}\|_{-1/2,\Omega} \leq C |s| \|\hat{g}\|_{1/2,s,\Gamma}.$$

Therefore,

$$\|\partial_{\mathbf{v}} \hat{w}\|_{1,\Gamma} \leq C(|s|^4 \|\hat{g}\|_{1/2,s,\Gamma} + \|\hat{g}\|_{2,\Gamma} + |s|^2 \|\hat{g}\|_{1,\Gamma}),$$

but, because of the definition of the weighted norms (4),

$$|s|^{3/2} \|\hat{g}\|_{1/2,s,\Gamma} + |s| \|\hat{g}\|_{1,\Gamma} + \|\hat{g}\|_{2,\Gamma} \leq C \|\hat{g}\|_{2,s,\Gamma},$$

so that

$$\|\partial_{\mathbf{v}} \hat{w}\|_{1,\Gamma} \leq C |s|^{5/2} \|\hat{g}\|_{2,s,\Gamma}.$$

This allows us to conclude from Theorem 5 that

$$\|\hat{u}\|_{0,\Gamma} \leq C(\|\hat{h}\|_{1,\Gamma} + |s|^{5/2} \|\hat{g}\|_{2,s,\Gamma}).$$

Further, we can recover both $\hat{u}_1 = \hat{v} + \hat{w}$ and $\hat{u} = \hat{u}_2$ on the boundary Γ by using that $\hat{u}_1 - \hat{u}_2 = \hat{g}$ on Γ . Indeed, we already have a bound for $\gamma \hat{u}_2 = \gamma \hat{u}$ in $L^2(\Gamma)$ and we also have a similar one for $\gamma \hat{u}_1 = \gamma \hat{u}_2 + \hat{g}$:

$$\|\gamma \hat{u}_1\|_{0,\Gamma} \leq \|\gamma \hat{u}\|_{0,\Gamma} + \|\hat{g}\|_{0,\Gamma} \leq C(\|\hat{h}\|_{1,\Gamma} + |s|^{5/2} \|\hat{g}\|_{2,s,\Gamma}).$$

However, we still have to recover the two functions $\hat{u}_1$ and $\hat{u}_2$ inside Ω by solving a Dirichlet problem when their traces are just in the weak space $L^2(\Gamma)$, which is exactly what we have studied in detail in Section 5. This strategy allows us to bound each function inside Ω by using a Dirichlet formulation with weak data and applying Proposition 4 to deduce that

$$\|\hat{u}_j\|_{0,\Omega} \leq C(\|\hat{h}\|_{1,s,\Gamma} + |s|^{5/2} \|\hat{g}\|_{2,s,\Gamma}),$$

where, as before, $C > 0$ does not depend on $s \in \mathbb{C}_{\sigma_*}$ and $j = 1, 2$. □

6.2. The ITP in the time domain. Now we have at hand all the ingredients we need to study the ITP in the time domain (TDITP). More precisely, we consider the problem of finding u_1 and u_2 which satisfy

the following, in weak form and suitable function spaces:

$$\begin{aligned} \partial_{tt}^2 u_1 - \Delta u_1 &= 0 && \text{in } \mathbb{R} \times \Omega, \\ n \partial_{tt}^2 u_2 - \Delta u_2 &= 0 && \text{in } \mathbb{R} \times \Omega, \\ u_1 - u_2 &= g, \quad \partial_\nu u_1 - \partial_\nu u_2 = h && \text{on } \mathbb{R} \times \Gamma, \\ u_1 &= u_{1,t} = 0, \quad u_2 = u_{2,t} = 0 && \text{in } \Omega \text{ for } t = 0. \end{aligned}$$

The initial conditions in the last equation imply that $u_1 = u_2 = 0$ for $t \leq 0$; i.e., they are causal fields. Notice the Fourier–Laplace counterpart of this problem is just the problem studied in the previous section. Therefore, by [Lemma 1](#) and [Corollary 9](#) we know that the TDITP is well-posed for smooth enough data, and its solution satisfies the bound

$$\|u_j\|_{H_\sigma^p(\mathbb{R}, L^2(\Omega))} \leq C(\|h\|_{H_\sigma^p(\mathbb{R}, H^1(\Gamma))} + \|g\|_{H_\sigma^{p+5/2}(\mathbb{R}, H^2(\Gamma))})$$

for each $p \in \mathbb{R}$ and $C > 0$ independent of σ for $\sigma \geq \sigma_*$ with $\sigma_* > 0$ sufficiently large.

7. Justification of the linear sampling in the time domain

Recall the near field equation (14) for $\varphi_z \in H_\sigma^p(\mathbb{R}_+, L^2(\Sigma))$:

$$\mathcal{N}\varphi_z = u_z^{\text{inc}} \quad \text{in } H_\sigma^p(\mathbb{R}_+, H^{1/2}(\Sigma)),$$

where $\mathcal{N} : H_\sigma^p(\mathbb{R}_+, L^2(\Sigma)) \rightarrow H_\sigma^p(\mathbb{R}_+, H^{1/2}(\Sigma))$ is the near field operator defined in (13), and u_z^{inc} is the field due to a source at z defined by (5). Note that in Proposition 3.4 from [\[Guo et al. 2013\]](#) it is shown that $\mathcal{N}$ is injective with dense range. As discussed in [Section 4](#), $\mathcal{N} = \mathcal{G} \circ \text{SL}_\Sigma^\chi$ where, from (10), the bounded operator

$$\mathcal{G} : u \in X_\sigma^{p+2}(\mathbb{R}_+, L^2(\Omega)) \mapsto \gamma_\Sigma w \in H_\sigma^p(\mathbb{R}_+, H^{1/2}(\Sigma)),$$

where $w \in H_\sigma^p(\mathbb{R}_+, H^1(\mathbb{R}^d))$, is the unique causal solution of

$$n \partial_{tt}^2 w - \Delta w = (1 - n) \partial_{tt}^2 u \quad \text{in } \mathbb{R}^d, \text{ for } t \in \mathbb{R}. \quad (28)$$

Let us first consider a sampling point z in Ω . We want to show that there are approximate solutions to the near field equation (14) such that the corresponding single-layer potentials remain bounded as the accuracy of the approximation improves. To this end, we first consider the incident field associated to a point source located at z with enough time regularity, that is, $u_z^{\text{inc}} := u^{\text{inc}}(\cdot, \cdot, z) \in X_\sigma^{p+9/2}(\mathbb{R}_+, H^{5/2}(\mathbb{R}^d \setminus \{z\}))$ for some $p \in \mathbb{R}$ and fixed $\sigma > 0$ large enough. We take both the negative of its trace and its normal derivative on Γ , which we denote by

$$g_z := -\gamma u_z^{\text{inc}} \in H_\sigma^{p+9/2}(\mathbb{R}_+, H^2(\Gamma))$$

and

$$h_z := -\partial_\nu u_z^{\text{inc}} \in H_\sigma^{p+9/2}(\mathbb{R}_+, H^1(\Gamma)) \subset H_\sigma^{p+2}(\mathbb{R}_+, H^1(\Gamma)).$$

Let us consider $u_1 := u_z^1 \in H_\sigma^{p+2}(\mathbb{R}_+, L^2(\Omega))$ and $u_2 := u_z^2 \in H_\sigma^{p+2}(\mathbb{R}_+, L^2(\Omega))$ the corresponding solution of the TDITP in Ω with these transmission boundary data:

$$\begin{aligned} \partial_{tt}^2 u_1 - \Delta u_1 &= 0 & \text{in } \mathbb{R} \times \Omega, \\ n \partial_{tt}^2 u_2 - \Delta u_2 &= 0 & \text{in } \mathbb{R} \times \Omega, \\ u_2 - u_1 &= g_z, \quad \partial_\nu u_2 - \partial_\nu u_1 = h_z & \text{on } \mathbb{R} \times \Gamma, \\ u_1 &= \partial_t u_1 = 0, \quad u_2 = \partial_t u_2 = 0 & \text{in } \Omega, \text{ for } t \leq 0. \end{aligned} \quad (29)$$

We have already shown that such solution exists and it is unique, and it satisfies the bound

$$\|u_j\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))} \leq C(\|h_z\|_{H_\sigma^{p+2}(\mathbb{R}, H^1(\Gamma))} + \|g_z\|_{H_\sigma^{p+9/2}(\mathbb{R}, H^2(\Gamma))}) \quad (30)$$

for $j = 1, 2$ and $p \in \mathbb{R}$, and uniformly in $\sigma > \sigma_*$. Thus w defined as $w := u_z^2 - u_z^1$ in Ω and $w := u_z^{\text{inc}}$ in Ω^c is in $H_\sigma^p(\mathbb{R}_+, H^1(\mathbb{R}^d))$ and solves (28) with $u := u_z^1$; i.e.,

$$\mathcal{G}u_z^1 = u_z^{\text{inc}} \quad \text{in } H_\sigma^p(\mathbb{R}_+, H^{1/2}(\Sigma)). \quad (31)$$

Now, let us recall that the regularized retarded single-layer potentials of fields in $H_\sigma^p(\mathbb{R}_+, L^2(\Sigma))$ define a dense subspace in $X_\sigma^{p+2}(\mathbb{R}_+, H^1(\mathbb{R}^d \setminus \Sigma))$ (see Proposition 3), which in turn is dense in $X_\sigma^{p+2}(\mathbb{R}_+, L^2(\mathbb{R}^d \setminus \Sigma))$ and whose restriction to $\mathbb{R} \times \Omega$ is dense in $X_\sigma^{p+2}(\mathbb{R}_+, L^2(\Omega))$. Thus, we know that for all $\varepsilon > 0$ there exists $\varphi_z^\varepsilon \in H_\sigma^{p+2}(\mathbb{R}_+, L^2(\Sigma))$ such that

$$\|u_z^1 - \text{SL}_\Sigma^\chi \varphi_z^\varepsilon\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))} < \varepsilon,$$

which by definition and (31) together with the boundedness of $\mathcal{G}$ implies

$$\begin{aligned} \|\mathcal{N}\varphi_z^\varepsilon - u_z^{\text{inc}}\|_{H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))} &= \|\mathcal{G}u_z^1 - \mathcal{G}(\text{SL}_\Sigma^\chi \varphi_z^\varepsilon)\|_{H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))} \\ &\leq C\|u_z^1 - \text{SL}_\Sigma^\chi \varphi_z^\varepsilon\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))} < C\varepsilon. \end{aligned} \quad (32)$$

Thus for $z \in \Omega$ the near field equation has an approximate solution whose single-layer potential converges to u_z^1 in the $H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))$ norm. Furthermore for fixed $\varepsilon > 0$, from the a priori estimate (30) and the fact that due to the singularity of the point source g_z and g_z blow-up in their respective norms as $z \rightarrow \Gamma$ from inside Ω , we have that $\|\text{SL}_\Sigma^\chi \varphi_z^\varepsilon\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))} \rightarrow \infty$ and $\|\varphi_z^\varepsilon\|_{H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))} \rightarrow \infty$ as $z \rightarrow \Gamma$.

Next, we consider a point z in Ω^+ and study the blow-up of the single-layer potential when the density is taken to be any sequence of approximate solutions to the near field equation (14) as the approximation parameter ε approaches to 0. In order to reason by contradiction, we assume that there exists some sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that $\varepsilon_n > 0$ for all $n \in \mathbb{N}$ and $\varepsilon_n \rightarrow 0$ when $n \rightarrow \infty$, and for which we can build $\{\varphi_z^{\varepsilon_n}\}_{n \in \mathbb{N}} \subset H_\sigma^p(\mathbb{R}_+, L^2(\Sigma))$ satisfying

$$\|\mathcal{N}\varphi_z^{\varepsilon_n} - u_z^{\text{inc}}\|_{H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))} < \varepsilon_n \quad (33)$$

and such that

$$\|\text{SL}_\Sigma^\chi \varphi_z^{\varepsilon_n}\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))} \leq C$$

for all $n \in \mathbb{N}$ and some generic constant $C > 0$. Then, we can take a weakly convergent subsequence of $\text{SL}_\Sigma^\chi \varphi_z^{\varepsilon_n} \subset X_\sigma^{p+2}(\mathbb{R}_+, L^2(\Omega))$, which for the sake of simplicity we identify with the original sequence, and we represent its weak limit by u_z ; i.e.,

$$\text{SL}_\Sigma^\chi \varphi_z^{\varepsilon_n} \rightharpoonup u_z \quad \text{in } X_\sigma^{p+2}(\mathbb{R}_+, L^2(\Omega)).$$

In particular $u_z \in X_\sigma^{p+2}(\mathbb{R}_+, L^2(\Omega))$; i.e., u_z is a solution of the wave equation. Let us set $w_z := \mathcal{G}u_z \in H_\sigma^p(\mathbb{R}_+, H^{1/2}(\Sigma))$. The factorization of $\mathcal{N}$ and continuity of $\mathcal{G}$ implies

$$\mathcal{N}\varphi_z^{\varepsilon_n} = \mathcal{G}(\text{SL}_\Sigma^\chi \varphi_z^{\varepsilon_n}) \rightharpoonup w_z \quad \text{in } H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))$$

as $n \rightarrow \infty$. The latter implies that $w_z = u_z^{\text{inc}}$ in $H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))$.

We move on to the Fourier–Laplace domain and denote by $\hat{w}_s$ and $\hat{u}_s^{\text{inc}}$ the transforms of w_z and $u^{\text{inc}}(\cdot, \cdot; z)$ for $s \in \mathbb{C}_\sigma$, respectively. Notice that, in the unbounded domain $\Omega^+ \setminus \{z\}$, we have that both $\hat{w}_s$ and $\hat{u}_s^{\text{inc}}$ satisfy the homogeneous Helmholtz equation. In particular, $\hat{w}_s - \hat{u}_s^{\text{inc}}$ solves the homogeneous exterior Dirichlet problem in Ω^+ . The uniqueness of the exterior Dirichlet problem for the Helmholtz equation implies

$$\hat{w}_s = \hat{u}_s^{\text{inc}}(\cdot, \cdot; z) \quad \text{in } \Omega^+ \setminus \{z\}.$$

Since we have assumed that $z \in \Omega^+$, we arrive to a contradiction by noticing that $\hat{w}_s \in H^1(\Omega^+)$, whereas $\hat{u}_s^{\text{inc}}(\cdot, \cdot; z) \notin H^1(\Omega^+)$. Thus, we conclude that for any sequence $\{\varphi_z^{\varepsilon_n}\} \subset H_\sigma^p(\mathbb{R}, L^2(\Sigma))$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, satisfying $\mathcal{N}\varphi_z^{\varepsilon_n} \rightarrow u_z^{\text{inc}}$, we have $\|\text{SL}_\Sigma^\chi \varphi_z^{\varepsilon_n}\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))} \rightarrow \infty$ and consequently $\|\varphi_z^{\varepsilon_n}\|_{H_\sigma^p(\mathbb{R}, L^2(\Sigma))} \rightarrow \infty$ as $n \rightarrow \infty$.

Summarizing the above discussion, we have shown the following justification of the linear sampling method for the inverse problem for the inhomogeneous media in the time domain.

Theorem 10. *For any $\sigma > \sigma_*$, with fixed $\sigma_* > 0$ depending on the scattering problem, and any $p \in \mathbb{R}$ the following hold:*

- (1) *If z in Ω , then for every $\varepsilon > 0$ there exists some $\varphi_z^\varepsilon \in H_\sigma^p(\mathbb{R}, L^2(\Sigma))$ such that*

$$\|\mathcal{N}\varphi_z^\varepsilon - u_z^{\text{inc}}\|_{H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))} < \varepsilon$$

satisfying

$$\|\text{SL}_\Sigma^\chi \varphi_z^\varepsilon\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))} < C \quad \text{as } \varepsilon \rightarrow 0$$

(in fact $\text{SL}_\Sigma^\chi \varphi_z^\varepsilon$ converges in $H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))$ to u_1 that solves the TDITP (29)).

- (2) *If z in Ω^+ , every sequence $\{\varphi_z^\varepsilon\}_{\varepsilon>0} \subset H_\sigma^p(\mathbb{R}, L^2(\Sigma))$ satisfying*

$$\|\mathcal{N}\varphi_z^\varepsilon - u_z^{\text{inc}}\|_{H_\sigma^p(\mathbb{R}, H^{1/2}(\Sigma))} < \varepsilon$$

is such that

$$\|\text{SL}_\Sigma^\chi \varphi_z^\varepsilon\|_{H_\sigma^{p+2}(\mathbb{R}, L^2(\Omega))} \rightarrow \infty \quad \text{and} \quad \|\varphi_z^\varepsilon\|_{H_\sigma^p(\mathbb{R}, L^2(\Sigma))} \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0.$$

Of course as is always the case with the linear sampling method, the above justification is in terms of a norm of single-layer potentials that depends on the unknown domain Ω . However numerical examples indicate that the above distinct behavior for z inside or outside Ω is inherited by the solution of the near field equation and is independent of Ω . For numerical examples and implementation details see [Guo et al. 2013].

8. Conclusion

We have provided a justification of the TDLSM under stringent hypotheses. It would be interesting to reduce the regularity needed for both the coefficients and the boundary of Ω . It would be also be desirable to derive a generalized time-domain linear sampling method that could provide a mathematical justification also for noisy data; for the time harmonic case, see [Audibert and Haddar 2014].

Appendix: Analysis of the time-domain forward scattering problem

In this appendix we sketch a proof of the well-posedness of the time-domain problem (6). To this end, we start by analyzing the forward problem in the Fourier–Laplace domain (9) and then move to the time domain.

The Fourier–Laplace domain problem (9) can be studied reasoning as for the Dirichlet problem (19) in Section 5. More precisely, we use the following standard weak formulation of (9): Find $\hat{u}_s \in H^1(\mathbb{R}^d)$ such that, for all $v \in H^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (\nabla \hat{u}_s \cdot \overline{\nabla v} - s^2 \mathbf{n} \hat{u}_s \bar{v}) d\mathbf{x} = \langle \hat{f}_s, v \rangle_{\mathbb{R}^d}. \quad (34)$$

Here $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ represents the duality product $H^{-1}(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ with pivot space $L^2(\mathbb{R}^d)$; in particular,

$$\langle g, v \rangle_{\mathbb{R}^d} = \int_{\mathbb{R}^d} g \bar{v} d\mathbf{x} \quad \text{for all } g \in L^2(\mathbb{R}^d), v \in H^1(\mathbb{R}^d).$$

As for the Dirichlet problem (19), the weak formulation (34) motivates the definition of the sesquilinear form $A^s: H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \rightarrow \mathbb{C}$ such that

$$A^s(v_1, v_2) = \int_{\mathbb{R}^d} (\nabla v_1 \cdot \overline{\nabla v_2} - s^2 \mathbf{n} v_1 \bar{v}_2) d\mathbf{x} \quad \text{for all } v_1, v_2 \in H^1(\mathbb{R}^d).$$

Then, problem (34) consists of finding $\hat{u}_s \in H^1(\mathbb{R}^d)$ such that

$$A^s(\hat{u}_s, v) = \langle \hat{f}_s, v \rangle_{\mathbb{R}^d} \quad \text{for all } v \in H^1(\mathbb{R}^d).$$

Thanks to the Lax–Milgram lemma, it is enough to study the continuity and the ellipticity of the sesquilinear form $A^s: H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \rightarrow \mathbb{C}$ (making clear the dependence on $s \in \mathbb{C}_\sigma$).

- It is straightforward to bound it in weighted norms uniformly in $s \in \mathbb{C}_\sigma$ as follows:

$$|A^s(v_1, v_2)| \leq \|\mathbf{n}\|_{\infty, \mathbb{R}^d} \|v_1\|_{1,s, \mathbb{R}^d} \|v_2\|_{1,s, \mathbb{R}^d} \quad \text{for all } v_1, v_2 \in H^1(\mathbb{R}^d). \quad (35)$$

- Concerning ellipticity, we can proceed in the standard way [Bamberger and Duong 1986] by noticing that

$$i\bar{s}A^s(v, v) = \int_{\mathbb{R}^d} (i\bar{s}|\nabla v|^2 - is|s|^2 n|v|^2) d\mathbf{x} \quad \text{for all } v \in H^1(\mathbb{R}^d);$$

then,

$$\Re(i\bar{s}A^s(v, v)) = \Im(s) \int_{\mathbb{R}^d} (|\nabla v|^2 + |s|^2 n|v|^2) d\mathbf{x}$$

and we deduce the following coercivity bound in weighted norms, which holds uniformly in $s \in \mathbb{C}_\sigma$:

$$\Re(i\bar{s}A^s(v, v)) \geq \Im(s)n_0\|v\|_{1,s,\mathbb{R}^d}^2 \quad \text{for all } v \in H^1(\mathbb{R}^d). \quad (36)$$

The following result is then a straightforward consequence of (35) and (36) by the Lax–Milgram lemma.

Proposition 11. *For all $s \in \mathbb{C}_\sigma$ and $\hat{f}_s \in H^{-1}(\mathbb{R}^d)$, the Fourier–Laplace domain problem (34) is well-posed, and its unique solution $\hat{u}_s \in H^1(\mathbb{R}^d)$ satisfies*

$$\|\hat{u}_s\|_{1,s,\mathbb{R}^d} \leq |s| \frac{1}{\sigma n_0} \|\hat{f}_s\|_{-1,s,\mathbb{R}^d}.$$

We can then analyze the forward problem in the time domain (6), taking advantage of what we already know about the behavior of the solution of the corresponding problem in the Fourier–Laplace domain (9). In particular, we can use Proposition 11 together with Lemma 1 to deduce Proposition 2.

Remark 12. Let us notice that we can apply Paley–Wiener theorem (see the second statement in [Ha-Duong 2003, Theorem 1]) to guarantee that problem (6) preserves causality. Indeed, thanks to the linearity of problem (6) and the stability bound provided in Proposition 11 (which is uniform with respect to $s \in \mathbb{C}_\sigma$), we know that $s \in \mathbb{C}_{\sigma_0} \mapsto \hat{u}_s \in L^2(\mathbb{R}^d)$ is holomorphic in $\mathbb{C}_{\sigma_0}$ and of temperate growth in $\mathbb{C}_\sigma$ as long as $s \in \mathbb{C}_{\sigma_0} \mapsto \hat{f}_s \in H^{-1}(\mathbb{R}^d)$ is.

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