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A duality between scattering poles and transmission eigenvalues in scattering theory

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In this paper, we develop a conceptually unified approach for characterizing and determining scattering poles and interior eigenvalues for a given scattering problem. Our approach explores a duality stemming from interchanging the roles of incident and scattered fields in our analysis. Both sets are related to the kernel of the relative scattering operator mapping incident fields to scattered fields, corresponding to the exterior scattering problem for the interior eigenvalues and the interior scattering problem for scattering poles. Our discussion includes the scattering problem for a Dirichlet obstacle where duality is between scattering poles and Dirichlet eigenvalues, and the inhomogeneous scattering problem where the duality is between scattering poles and transmission eigenvalues. Our new characterization of the scattering poles suggests a numerical method for their computation in terms of scattering data for the corresponding interior scattering problem.

1. Introduction

Spectral properties of operators associated with scattering phenomena carry essential information about the scattering media. The theory of scattering resonances is a rich and beautiful part of scattering theory, and although the notion of resonances is intrinsically dynamical, an elegant mathematical formulation comes from considering them as the poles of the meromorphic extension of the scattering operator [1,2] (we refer the reader to the comprehensive monograph [3] for an account of the vast literature on the subject). The scattering poles exist, and they are complex with negative imaginary part [4,5]. They capture physical information by identifying the rate of oscillations with the real part of a pole and the rate of decay with its imaginary part. At a scattering pole, there is a non-zero scattered field in the absence of the incident field. On the flip side of this characterization of the scattering poles, one could ask if there are frequencies for which there exists an incident field that does not scatter by the scattering object. The answer to this question leads to an interior eigenvalue problem associated with the support of the scatterer. In the case of scattering by an impenetrable obstacle with Dirichlet boundary condition, this is merely the Dirichlet eigenvalue problem for a symmetric elliptic operator; hence, all interior eigenvalues are real. A more intriguing situation arises in the scattering by an inhomogeneous medium where a new eigenvalue problem arises, referred to as the transmission eigenvalue problem [6]. At the partial differential equations level, transmission eigenvalues form the spectrum of a non-self-adjoint compact operator, which under some appropriate assumptions, is proven to have infinitely many eigenvalues in the complex plane $\mathbb{C}$, whereas at the scattering theory level, there is a profound relation between transmission eigenvalues and the kernel of the relative scattering operator [2,7]. For non-absorbing media, real transmission eigenvalues exist [8], and they can be determined from the scattering data [6,9,10], thus providing estimates of the constitutive material properties of the scattering object. The goal of this paper is to explore a duality argument between scattering poles and transmission eigenvalues, in particular to study the scattering poles in connection with the kernel of an operator that plays the same role as the relative scattering operator in relation to the transmission eigenvalues. This duality is revealed by flipping the role of interior and exterior domains. It leads to a new way of defining the scattering poles and also the possibility of a new numerical algorithm to compute them. We hope to pursue this possibility in a future publication.

To be more specific and set up the analytical framework of our paper, let us introduce transmission eigenvalues and scattering poles in connection with the relative scattering operator for an inhomogeneous medium. We assume that the medium is supported in a bounded simply connected Lipschitz region $D \subset \mathbb{R}^3$ and has the refractive index n. Let us consider the scattering of a monochromatic acoustic incident wave v that satisfies the Helmholtz equation

$$\Delta v + k^2 v = 0 \tag{1.1}$$

in $\mathbb{R}^3$ (except for possibly a subset of measure zero in the exterior of D, for example a single point for point sources or a surface for surface potentials) by this inhomogeneity. Here, $k = \omega/c_0$ is the wave number corresponding to the frequency ω, c_0 is the constant background sound speed (the refractive index of the background is normalized to one), n is a complex valued L^∞ function with $\Re(n) > 0$ and $\Im(n) \geq 0$, such that $n - 1$ is supported in $\overline{D}$. The total field u, which is decomposed as $u = u^s + v$, satisfies

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3 \tag{1.2}$$

with the scattered field u^s satisfying the outgoing Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{1.3}$$

uniformly with respect to $\hat{x} := x/|x|$, $r = |x|$ [2,7]. It is easy to see from (1.1) and (1.2) that the scattered field $u^s = u - v \in H^2_{\text{loc}}(\mathbb{R}^3)$ satisfies

$$\Delta u^s + k^2 n u^s = k^2(1 - n)v \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

The *scattering operator (matrix)* as defined by Lax & Phillips in [1] roughly maps $v \mapsto u$ (incident field to total field) and for k such that $\Im(k) \geq 0$ is an isomorphism in appropriate spaces. A heuristic argument for the latter can be given using the Lipmann–Schwinger equation for the solution of (1.4) in terms of the compact integral operator $T(k): L^2(B_R) \to L^2(B_R)$

$$(I - T(k))u = v, \quad T(k)u := k^2 \int_{\mathbb{R}^3} \Phi_k(x, y)(n(y) - 1)u(y)\,\mathrm{d}y, \tag{1.5}$$

where B_R is a large ball of radius R and $\Phi_k(x, y)$ is the radiating fundamental solution of the Helmholtz equation defined by

$$\Phi_k(x, y) = \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}. \tag{1.6}$$

A fix point theorem argument implies that for $|k|$ small enough $I - T(k)$ is invertible (see [7] for details), and hence by the analytic Fredholm theory, we have that $u := (I - T(k))^{-1}v$ is meromorphic for $k \in \mathbb{C}$. Furthermore, for k such that $\Im(k) \geq 0$, uniqueness of the scattering problem implies that u is analytic. That is its poles, which are the *scattering poles*, are located in the lower-half complex plane [11]. Later in the paper, we provide a more rigorous definition of scattering poles. To introduce the *non-scattering wave numbers* and *transmission eigenvalues,* we consider the 'incoming-to-outgoing' mapping

$$\mathcal{S}(k) : v \mapsto u^s,$$

referred to as the relative scattering operator in [2] and look for its kernel. In other words, we seek non-scattering wave numbers k for which there exists an incident field v that does not scatter, i.e. the corresponding scattered field $u^s = 0$. One can easily see that for such k, the non-trivial fields $u|_D$ and $v|_D$ satisfy the *transmission eigenvalue problem*

$$\left.\begin{aligned} &\Delta u + k^2 n(x) u = 0 \quad \text{in } D \\ &\Delta v + k^2 v = 0 \quad \text{in } D \\ &u = v \quad \text{on } \partial D \\ \text{and} \quad &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D. \end{aligned}\right\} \tag{1.7}$$

A value of $k \in \mathbb{C}$ is said to be a *transmission eigenvalue* if (1.7) has non-trivial solutions $u \in L^2(D)$ $v \in L^2(D)$, such that $u - v \in H_0^2(D)$ ($u^s := u - v$ is zero outside D). We call the pair (u, v) the corresponding eigenfunction. Thus, non-scattering wave numbers are transmission eigenvalues. The converse is in general not true. At a transmission eigenvalue the v-part of the eigenfunction must be an incident wave as defined above, i.e. satisfy the Helmholtz equation in all of $\mathbb{R}^3$ except for possibly a set of zero measure. It is already known that if ∂D contains a corner then the v-part of the eigenfunction is not extendable outside D as a solution of the Helmholtz equation [12,13].

Let us look at the particular example where the above concepts become very explicit. This is the case when the inhomogeneity $D := B_1(0)$ is the ball of radius 1 centred at the origin with radially symmetric real-valued refractive index $n(r) > 0$, $r = |x|$. We consider the incident fields that are entire solutions of the Helmholtz equation given by

$$v = j_\ell(k|x|) Y_\ell(\hat{x}),$$

where j_ℓ is the spherical Bessel function and Y_ℓ is a spherical harmonic of order $\ell \in \mathbb{N}$. Note that these incident fields are examples of so-called Herglotz functions or superposition of plane waves [7]

$$v_g(x) = \int_{\mathbb{S}^2} g(\hat{y}) \mathrm{e}^{\mathrm{i}kx\cdot\hat{y}}\,\mathrm{d}s(\hat{y}), \quad \text{with some } g \in L^2(\mathbb{S}^2),$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. Straightforward calculations by separation of variables [7] lead to the following expression for the scattered field

$$u^s(x) := \frac{C_\ell(k; n)}{W_\ell(k; n)} h_\ell^{(1)}(k|x|) Y_\ell(\hat{x}), \tag{1.8}$$

where $h_\ell^{(1)}(r)$ is the Hankel function of the first kind of order ℓ and

$$C_\ell(k;n) = \mathrm{Det}\begin{pmatrix} y_\ell(1) & -j_\ell(k) \\ y'_\ell(1) & -kj'_\ell(k) \end{pmatrix}, \quad W_\ell(k;n) = \mathrm{Det}\begin{pmatrix} y_\ell(1) & -h_\ell^{(1)}(k) \\ y'_\ell(1) & -kh_\ell^{(1)'}(k) \end{pmatrix} \tag{1.9}$$

with y_ℓ (depending on k and n) being the solution to

$$y''_\ell + \frac{2}{r}y'_\ell + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) y_\ell = 0$$

that behaves like $j_\ell(kr)$ as $r \to 0$. Thus, non-scattering wave numbers correspond to those values of $k \in \mathbb{C}$ for which $C_\ell(k;n) = 0$ whereas the scattering poles are $k \in \mathbb{C}$, for which $W_\ell(k;n) = 0$. In this case, every transmission eigenvalue is a non-scattering wave number, since by construction at a transmission eigenvalue, the eigenfunctions of (1.7) with $D := B_1(0)$ and $n := n(r)$ are linear combinations of $v = j_\ell(k|x|)Y_\ell(\hat{x})$ and $u := v + u^s$ with u^s given by (1.8). Note that by separating variables in (1.7), we can see that all transmission eigenvalues for the spherically symmetric media are obtained from $C_\ell(k;n) = 0$ for $\ell \in \mathbb{N}$. The transmission eigenvalues for spherically symmetric cases are extensively studied in [14–17]. In particular, it is shown that (except for some exceptional cases) the entire functions $C_\ell(k;n)$ have infinitely many real zeros and infinitely many complex zeros. Thus, in this spherically symmetric case, the set of transmission eigenvalues, non-scattering frequencies and the zeros of the relative scattering operator, here characterized as

$$\mathcal{S}(k) : j_\ell(k|x|) \mapsto \frac{C_\ell(k;n)}{W_\ell(k;n)} h_\ell^{(1)}(k|x|)$$

coincide. The scattering poles, on the other hand, are the poles of the relative scattering operator $\mathcal{S}(k)$. The existence of the scattering poles for this spherically symmetric case can be obtained from more general results contained in theorems 2.10 and 2.16 of [3]. The simple argument below shows the existence of an infinite set of scattering poles that are the zeros of $W_0(k;n) = 0$, which can be rewritten as the zeros of

$$\mathrm{Det}\begin{pmatrix} y(1) & \dfrac{\mathrm{e}^{ik}}{k} \\ y'(1) & i\mathrm{e}^{ik} \end{pmatrix} = 0,$$

where we denote $y_0(r) := y(r)/r$, i.e. $y(r)$ satisfies $y'' + k^2 n(r) y = 0$. Hence for our purpose, it suffices to analyse only $W_0(k;n)$, which corresponds to the scattering poles with spherically symmetric eigenfunctions. For this case, we first show that there exist infinitely many scattering poles. From (5.17) and (5.21) in [6], one can see that for k in a neighbourhood of zero, $y_0(1)$ behaves like $j_0(k)$. Hence, $kW_0(k;n)$ is an entire function. Furthermore, $kW_0(k;n)|_{k=0} = 1$. Thus, by Hadamard's factorizations theorem

$$kW_0(k;n) = \mathrm{e}^{\alpha k} \prod_{j=1}^{\infty} \left(1 - \frac{k}{k_j}\right) \mathrm{e}^{k/k_j}, \tag{1.10}$$

where α is a complex constant and k_j are the zeros of $kW_0(k;n)$, i.e. scattering poles, which we know are complex with negative imaginary part. Now, taking large $k > 0$ from [6, Section 4.2], we have that y behaves as

$$y(r) = \frac{1}{k\left[n(0)\,n(r)\right]^{1/4}} \sin\left(k \int_0^r \left[n(\rho)\right]^{1/2} \mathrm{d}\rho\right) + \mathcal{O}\left(\frac{1}{|k|^2}\right) \tag{1.11}$$

and

$$y'(r) = \left[\frac{n(r)}{n(0)}\right]^{1/4} \cos\left(k \int_0^r [n(\rho)]^{1/2}\, \mathrm{d}\rho\right) + \mathcal{O}\left(\frac{1}{|k|}\right). \tag{1.12}$$

In particular, we have that $kW_0(k;n)$ remains bounded oscillating as $k \to +\infty$. Obviously, if there were no zeros of $kW_0(k;n)$, then from (1.10) $kW_0(k;n) = e^{\alpha k}$, which does not match this asymptotic behaviour. On the other hand, if the product in (1.10) is finite, i.e. there are only finitely many zeros of $kW_0(k;n)$, then $kW_0(k;n)$ would either go to zero or become unbounded as $k \to +\infty$. This proves the existence of an infinite number of zeros of $W_0(k;n)$, i.e. scattering poles. It is interesting to note that for obstacle scattering, which will be discussed in the next section, for each fixed ℓ, there are only a finite number of scattering poles, i.e. zeros of $h_\ell^{(1)}(k)=0$, and in particular there are no scattering poles for obstacle scattering corresponding to $\ell=0$.

For general media, we restrict ourselves to incident waves $v:=v_g$ being a superposition of point sources located at $y\in\partial B$ (otherwise referred to as a surface potential) given by

$$v_g(x)=\int_{\partial B} g(y)\Phi_k(x,y)\,\mathrm{d}s(y), \tag{1.13}$$

where the region $B\subset\mathbb{R}^3$ is such that $D\subset B$ has Lipschitz boundary ∂B, and $\Phi_k(x,y)$ is given by (1.6). By linearity, the corresponding scattered field u_g^s is given by

$$u_g^s(x)=\int_{\partial B} g(y)u^s(x,y)\,\mathrm{d}s(y), \tag{1.14}$$

where $u^s(x,y)$ is a solution of

$$\Delta u^s(\cdot,y)+k^2nu^s(\cdot,y)=k^2(1-n)\Phi_k(\cdot,y)\quad \text{in } \mathbb{R}^3,\quad \text{for } y\in\partial B. \tag{1.15}$$

We can now explicitly characterize the relative scattering operator in terms of the compact linear operator $\mathcal{S}(k):L^2(\partial B)\to L^2(\partial B)$ defined by

$$\mathcal{S}(k):g\mapsto u_g^s|_{\partial B}. \tag{1.16}$$

The case of a spherically symmetric media discussed above will correspond to this configuration if the point sources are located at infinity. A *non-scattering wave number*, i.e. $k\in\mathbb{C}$ such that Kern $\mathcal{S}(k)\neq\emptyset$, is a transmission eigenvalue for which the v-part of the eigenfunction (u,v) in (1.7) is a surface potential $v:=v_g$ given by (1.14). In general, this is not the case. However, (1.4) defines the outgoing scattered field $u^s\in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ corresponding to a (generalized) incident field

$$v\in H_{inc}(D):=\{v\in L^2(D):\ \Delta v+k^2v=0,\ \text{in the distributional sense}\}.$$

$H_{inc}(D)$ is a Hilbert space that densely contains the superposition of point sources (surface dipoles) v_g given by (1.14) (see e.g. [6]). Thus, $\mathcal{G}(k):H_{inc}(D)\to H^{3/2}(\partial B)$ mapping $v\mapsto u^s|_{\partial B}$ is a compact linear operator, and k is a transmission eigenvalue if and only if the Kern $\mathcal{G}(k)$ is non-trivial (in fact the part v of the corresponding eigenfunction belongs to Kern($\mathcal{G}$)). Evidently, the following relation holds

$$\mathcal{S}(k)g=\mathcal{G}(k)\mathcal{H}g,\quad \text{where } \mathcal{H}:g\mapsto v_g|_D,\ \overline{\mathcal{H}(L^2(\partial B))}=H_{\mathrm{inc}}(D). \tag{1.17}$$

Hence at a transmission eigenvalue, one can construct a v_g of unit norm that produces an arbitrary small scattered field u_g^s. The above analysis leads to the following characterization of *transmission eigenvalues*.

Definition 1.1 (Equivalent definition of transmission eigenvalues). A wave number $k\in\mathbb{C}$ is a transmission eigenvalue if there there exists a sequence $g_j\in L^2(\partial B)$ such that the sequence $\{v_{g_j}\}_{j\in\mathbb{N}}$ of v_{g_j} given by (1.14) converges to a non-zero $v\in\mathrm{Ker}\,\mathcal{G}(k)$ in the $L^2(D)$ norm.

This definition together with relation (1.17) is used to determine the (interior) transmission eigenvalues from a knowledge of the exterior scattered field, i.e. the exterior relative scattering operator.

It is important to notice that the above discussion on transmission eigenvalues originated from the question of finding a non-zero incident field that gives rise to a zero scattered field. Interchanging the role of incident and scattered fields, that is, considering non-zero scattered fields with zero incident fields, gives rise to the scattering poles. Hence, we arrive at a dual relationship between the transmission eigenvalues and the scattering poles by considering an appropriate interior scattering problem, i.e. probing from inside of the scatterer, which yields an interior scattering operator whose injectivity is connected to the scattering poles. The main goal of this paper is to explore this duality and obtain a new characterization of the scattering poles that also suggests a computational method.

Our paper is organized as follows. In the next section, we consider scattering by an impenetrable obstacle with Dirichlet boundary conditions. This is a standard problem in scattering theory important in its own right but it also provides a simpler framework to introduce our new characterization of the corresponding scattering poles. For the Dirichlet scattering problem, the duality is between the (interior) Dirichlet eigenvalues for the negative Laplace operator in the support of the scatterer and scattering poles. Section 3 is dedicated to the discussion of scattering by an inhomogeneous media introduced in the Introduction, where we prove a similar result as in definition 1.1 for the scattering poles. For both problems, our new characterization provides a possible computational approach of the scattering poles from the interior scattering data in the spirit of the generalized linear sampling method, as discussed in [6] for transmission eigenvalues.

2. Scattering poles for a Dirichlet obstacle

Let D again be a bounded simply connected region in $\mathbb{R}^3$ with Lipschitz smooth boundary ∂D. The scattering problem for a Dirichlet obstacle reads: for a given incident field v which is a solution of the Helmholtz equation $\Delta v + k^2 v = 0$ in $\mathbb{R}^3$ (except for possibly a subset of measure zero in the exterior of D), find the scattered field $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{D})$ such that

$$\left.\begin{aligned} &\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D} \\ &u^s = -v \quad \text{on } \partial D \\ \text{and} \quad &\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \end{aligned}\right\} \tag{2.1}$$

Note that similarly to the scattering by an inhomogeneous media, the relative scattering operator is defined by

$$\mathcal{S}(k) : g \mapsto u^s_g|_{\partial B}, \tag{2.2}$$

where u^s_g solves (2.1) with $v := v_g$ given by (1.14) and the region $B \subset \mathbb{R}^3$ is such that $D \subset B$. One can easily check that $\mathcal{S}(k)g = 0$ if k is a Dirichlet eigenvalue of the negative Laplacian in D and with eigenfunction of the form given by (1.14). Here again, we have the relation

$$\mathcal{S}(k)g = -\mathcal{G}(k)\mathcal{H}g, \quad \text{where } \mathcal{H} : g \mapsto v_g|_{\partial D}, \ \overline{\mathcal{H}(L^2(\partial B))} = H^{1/2}(\partial D),$$

where

$$\mathcal{G}(k) : f \in H^{1/2}(\partial D) \mapsto w_f|_{\partial B_R} \in H^{1/2}(\partial B)$$

with $w := w_f$ satisfying

$$\begin{aligned} &\Delta w + k^2 w = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D} \\ &w = f \quad \text{on } \partial D \\ &\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \end{aligned}$$

Thus, we can conclude that $k \in \mathbb{R}$ is a Dirichlet eigenvalue if and only if there exists a sequence $g_j \in L^2(\partial B)$ such that the sequence $f_j := -v_{g_j}|_{\partial D}$ with v_{g_j} given by (1.14) converges to a non-zero $f \in \operatorname{Ker}\mathcal{G}(k)$ in the $H^{1/2}(\partial D)$ norm. The above characterization of Dirichlet eigenvalues can be used to compute them merely from a knowledge of the relative scattering operator $\mathcal{S}(k)$ (without knowing D) using linear sampling methods [6,9].

If we look at the scattering of $v = j_\ell(k|x|)Y_\ell(\hat{x})$ (which is a superposition of point sources located at infinity) by a Dirichlet ball of radius one in $\mathbb{R}^3$, we have that the Dirichlet eigenvalues are the zeros of $j_\ell(k) = 0$ (for such k, $j_\ell(k|x|)Y_\ell(\hat{x})$ is the corresponding eigenfunction) whereas the scattering poles are the zeros of $h^{(1)}_\ell(k) = 0$. We notice that $h^{(1)}_\ell(k|x|)Y_\ell(\hat{x})$ are superpositions of point

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sources located at the origin [7]. This duality motivates us to consider an appropriate scattering problem defined inside D, namely the scattering problem defined by (2.3), which will be the basis of our characterization of scattering poles that is dual to the Dirichlet eigenvalues stated above. To this end, we consider a Lipschitz closed surface $\partial\mathcal{C}$ circumscribing a simply connected region $\mathcal{C}\subset D$. From now on, since we are interested in scattering poles which exclude real values of k, without loss of generality the following assumption is valid:

Assumption 2.1. k^2 is not a Dirichlet eigenvalue of the negative Laplacian in $\mathcal{C}$ and in D.

For a point $z\in D$, let $u^s(\cdot,z)\in H^1(D)$ be the unique solution of

$$\left.\begin{aligned}\Delta u^s(\cdot,z)+k^2u^s(\cdot,z)&=0\quad \text{in } D\\ u^s(\cdot,z)&=-\Phi_k(\cdot,z)\quad \text{on } \partial D,\end{aligned}\right\}\tag{2.3}$$

where $\Phi_k(\cdot,\cdot)$ is the fundamental solution of the Helmholtz equation defined by (1.6). Next, we define the interior scattering operator $\mathcal{N}_k: L^2(\partial\mathcal{C})\to L^2(\partial\mathcal{C})$

$$\mathcal{N}_k:\varphi\mapsto u^s_\varphi|_{\partial\mathcal{C}},\tag{2.4}$$

where $u^s_\varphi\in H^1(D)$ is the unique solution of

$$\left.\begin{aligned}\Delta u^s_\varphi+k^2u^s_\varphi&=0\quad \text{in } D\\ u^s_\varphi&=-\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)\quad \text{on } \partial D\end{aligned}\right\}\tag{2.5}$$

with $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)$ being the surface dipole given as the superposition of point sources

$$\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)(x)=\int_{\partial\mathcal{C}}\varphi(z)\Phi_k(x,z)\,\mathrm{d}s(z).\tag{2.6}$$

Obviously, $\mathcal{N}_k$ is a compact operator. Also by linearity, $\mathcal{N}_k$ can be written as

$$\mathcal{N}_k\varphi(x)=\int_{\partial\mathcal{C}}\varphi(z)u^s(x,z)\,\mathrm{d}s(z),\quad x\in\partial\mathcal{C}.\tag{2.7}$$

We recall the definition of the single-layer potential $\mathrm{SL}^k_{\partial D}: H^{s-1/2}(\partial D)\to H^{s+1}_{\mathrm{loc}}(\mathbb{R}^3\backslash\partial D)$ (see e.g. [18] for the mapping properties)

$$\mathrm{SL}^k_{\partial D}(\psi)(x):=\int_{\partial D}\psi(y)\Phi_k(x,y)\,ds_y,\quad x\in\mathbb{R}^3\backslash\partial D\tag{2.8}$$

and double-layer potential $\mathrm{DL}^k_{\partial D}: H^{s+1/2}(\partial D)\to H^{s+1}_{\mathrm{loc}}(\mathbb{R}^3\backslash\partial D)$

$$\mathrm{DL}^k_{\partial D}(\psi)(x):=\int_{\partial D}\psi(y)\frac{\partial\Phi_k(x,y)}{\partial\nu_y}\,\mathrm{d}s_y,\quad x\in\mathbb{R}^3\backslash\partial D,\tag{2.9}$$

where $-1\le s\le 1$. Next, let us consider the following problem: for a given $f\in H^{1/2}(\partial D)$, look for $w\in H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash D)$ such that

$$\begin{cases}\Delta w+k^2w=0 & \text{in } \mathbb{R}^3\backslash\overline{D}\\ w=f & \text{on } \partial D\\ w=\mathrm{SL}^k_{\partial D}\left(\dfrac{\partial w}{\partial\nu}\right)-\mathrm{DL}^k_{\partial D}(w) & \text{in } \mathbb{R}^3\backslash\overline{D}.\end{cases}\tag{2.10}$$

The following equivalent definition of scattering poles is proven in [19, Theorem 7.11].

Proposition 2.2. *$k\in\mathbb{C}$ is a pole of the scattering matrix (i.e. a scattering pole) of the Dirichlet scattering problem for D if and only if the homogeneous problem (2.10), i.e. with $f=0$, has a non-trivial solution.*

The next theorem connects the scattering poles to the injectivity of $\mathcal{N}_k$. We highlight here a connection between the Dirichlet (interior) eigenvalues and the (exterior) relative scattering operator $\mathcal{S}(k)$.

Theorem 2.3. *Assume that $k \in \mathbb{C}$ is not a scattering pole and satisfies assumption 2.1. Then, the operator $\mathcal{N}_k : L^2(\partial\mathcal{C}) \to L^2(\partial\mathcal{C})$ is injective.*

Proof. Let $\mathcal{N}_k\varphi = 0$. This means that $u^s_\varphi = 0$ on $\partial\mathcal{C}$. Since from (2.5), $\Delta u^s_\varphi + k^2 u^s_\varphi = 0$ in D and hence in $\mathcal{C}$, assumption 2.1 guarantees that $u_\varphi = 0$ in $\mathcal{C}$. Therefore, by a unique continuation argument, $u_\varphi = 0$ in D. Therefore, $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi) = 0$ on ∂D. This means that $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)$ satisfies (2.10) with $f = 0$, where we use lemma 2.4 below. Since k is not a scattering pole, by proposition 2.2 we can conclude that $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi) \equiv 0$ in $\mathbb{R}^3\backslash\overline{D}$ (see lemma 2.4). Finally, unique continuation, assumption 2.1 and the jump relation for the normal derivative of the single-layer potential across $\partial\mathcal{C}$ imply that $\varphi = 0$. This proves that $\mathcal{N}_k$ is injective. ■

In our discussion, we use the following technical result.

Lemma 2.4. *Let $k \in \mathbb{C}$ and $\varphi \in L^2(\partial\mathcal{C})$. The single-layer potential $w := SL^k_{\partial\mathcal{C}}(\varphi)$ is in $H^2_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{\mathcal{C}})$ and it satisfies $w = SL^k_{\partial D}(\partial w/\partial\nu) - DL^k_{\partial D}(w)$ in $\mathbb{R}^3\backslash\overline{D}$.*

Proof. The mapping property (2.8) implies that $w \in H^2_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{\mathcal{C}})$. Next, for $p \in \mathbb{R}^3\backslash\overline{D}$, we recall from the definition of $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)$ that

$$w(p) = \int_{\partial\mathcal{C}} \varphi(y)\Phi_k(p,y)\,\mathrm{d}s(y).$$

We also have that for such p, the mapping

$$y \mapsto \Phi_k(y,p) = \Phi_k(p,y), \quad y \in D$$

satisfies the Helmholtz equation in D, where we use the symmetry of the fundamental solution. Hence for fixed $p \in \mathbb{R}^3\backslash D$, the Green's representation theorem implies that

$$\Phi_k(p,y) = \Phi_k(y,p) = \int_{\partial D} \Phi_k(x,p)\frac{\partial\Phi_k(x,y)}{\partial\nu_x} - \frac{\partial\Phi_k(x,p)}{\partial\nu_x}\Phi_k(x,y)\,\mathrm{d}s_x \quad \forall\, y \in D. \tag{2.11}$$

Multiplying the above by $\varphi \in L^2(\partial\mathcal{C})$, integrating over $\partial\mathcal{C}$ and exchanging the order of integration, we obtain

$$w(p) = \mathrm{SL}^k_{\partial D}\left(\frac{\partial w}{\partial\nu}\right)(p) - \mathrm{DL}^k_{\partial D}(w)(p).$$ ■

Lemma 2.5. *The operator $\mathcal{N}_k : L^2(\partial\mathcal{C}) \to L^2(\partial\mathcal{C})$ is symmetric, i.e. $\mathcal{N}_k^\top = \mathcal{N}_k$, where $\mathcal{N}_k^\top$ denotes the transpose operator defined by*

$$\int_{\partial\mathcal{C}} \mathcal{N}_k\varphi\,\psi\,\mathrm{d}s = \int_{\partial\mathcal{C}} \varphi\,\mathcal{N}_k^\top\psi\,\mathrm{d}s \quad \forall\,\varphi,\,\psi \in L^2(\partial\mathcal{C}).$$

Proof. Using (2.12), a simple exchange of integration yields that the transpose operator $\mathcal{N}_k^\top : L^2(\partial\mathcal{C}) \to L^2(\partial\mathcal{C})$ is given by

$$\left(\mathcal{N}_k^\top\varphi\right)(x) = \int_{\partial\mathcal{C}} \varphi(z)u^s(z,x)\,\mathrm{d}s(z), \quad x \in \partial\mathcal{C}. \tag{2.12}$$

Next, we show that $u^s(x,z) = u^s(z,x)$ for all $x, z \in D$. Indeed, viewing $u^s(x,z)$ as a function $x \mapsto u^s(x,z)$, which solves the Helmholtz equation in D, we have

$$u^s(x,z) = -\int_{\partial D} \Phi_k(y,x)\frac{\partial u^s(y,z)}{\partial\nu_y} - \frac{\partial\Phi_k(y,x)}{\partial\nu_y}u^s(y,z)\,\mathrm{d}s_y$$

and viewing $u^s(z,x)$ as a function $z \mapsto u^s(z,x)$, which solves the Helmholtz equation in D, we have

$$u^s(z,x) = -\int_{\partial D} \Phi_k(y,z)\frac{\partial u^s(y,x)}{\partial\nu_y} - \frac{\partial\Phi_k(y,z)}{\partial\nu_y}u^s(y,x)\,\mathrm{d}s_y.$$

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Therefore,

$$u^s(x,z) - u^s(z,x) = \int_{\partial D} \Phi_k(y,z)\frac{\partial u^s(y,x)}{\partial \nu_y} - \Phi_k(y,x)\frac{\partial u^s(y,z)}{\partial \nu_y}\,\mathrm{d}s_y + \int_{\partial D} \frac{\partial \Phi_k(y,x)}{\partial \nu_y}u^s(y,z) - \mathrm{d}s_y\frac{\partial \Phi_k(y,z)}{\partial \nu_y}u^s(y,x)\,\mathrm{d}s_y. \tag{2.13}$$

From the boundary conditions and Green's second identity applied to the two solutions of the Helmholtz equation in D, namely, $x \mapsto u^s(x,z)$ and $z \mapsto u^s(z,x)$, we have

$$\int_{\partial D} \Phi_k(y,z)\frac{\partial u^s(y,x)}{\partial \nu_y} - \Phi_k(y,x)\frac{\partial u^s(y,z)}{\partial \nu_y}\,\mathrm{d}s_y = \int_{\partial D} u^s(y,z)\frac{\partial u^s(y,x)}{\partial \nu_y} - u^s(y,x)\frac{\partial u^s(y,z)}{\partial \nu_y}\,\mathrm{d}s_y = 0.$$

Using the boundary conditions and Green's representation theorem for $x, z \in D$, we have that

$$\int_{\partial D} \frac{\partial \Phi_k(y,x)}{\partial \nu_y}u^s(y,z) - \frac{\partial \Phi_k(y,z)}{\partial \nu_y}u^s(y,x)\,\mathrm{d}s_y = \int_{\partial D} \frac{\partial \Phi_k(y,x)}{\partial \nu_y}\Phi_k(y,z) - \frac{\partial \Phi_k(y,z)}{\partial \nu_y}\Phi_k(y,x)\,\mathrm{d}s_y = \Phi(z,x) - \Phi(x,z) = 0.$$

The last two identities and (2.13) imply that $u^s(x,z) = u^s(z,x)$ for all $x, z \in D$, which concludes the proof. ∎

Combining lemma 2.5 with theorem 2.3, we have the following result:

Theorem 2.6. *Assume that $k \in \mathbb{C}$ is not a scattering pole. Then the operator $\mathcal{N}_k : L^2(\partial\mathcal{C}) \to L^2(\partial\mathcal{C})$ has dense range.*

In addition, we can also prove the following result:

Lemma 2.7. *Assume that $k \in \mathbb{C}$ is not a scattering pole of the Dirichlet scattering problem for D. Then the operator $S : L^2(\partial\mathcal{C}) \to H^{1/2}(\partial D)$ defined by*

$$\varphi \mapsto SL^k_{\partial\mathcal{C}}(\varphi)|_{\partial D}$$

is injective and has dense range.

Proof. The injectivity is seen from the last part of the proof of theorem 2.3. Next, we have that the transpose operator $S^\top : H^{-1/2}(\partial D) \to L^2(\partial\mathcal{C})$ is given by

$$(S^\top\psi)(x) := \int_{\partial D} \psi(y)\Phi_k(x,y)\,\mathrm{d}s(y), \quad x \in \partial\mathcal{C}. \tag{2.14}$$

To show that S has dense range it suffices to show that $S^\top$ is injective. To this end, let $S^\top\psi = 0$ on $\partial\mathcal{C}$. Observing that $S^\top\psi := \mathrm{SL}^k_{\partial D}\psi|_{\partial\mathcal{C}}$ and by the uniqueness of the Dirichlet problem in $\mathcal{C}$, we have that $\mathrm{SL}^k_{\partial D}\psi \equiv 0$ in $\mathcal{C}$, whence by unique continuation this is true in all of D since it satisfies the Helmholtz equation in D. Thus, the trace of $\mathrm{SL}^k_{\partial D}\psi$ on ∂D vanishes. Thanks to lemma 2.4, $\mathrm{SL}^k_{\partial D}\psi$ satisfies the homogeneous case of (2.10), i.e. with $f = 0$. Since k is not a scattering pole, proposition 2.2 implies that $\mathrm{SL}^k_{\partial D}\psi \equiv 0$ in $\mathbb{R}^3\backslash\overline{D}$. The jump relation for the normal derivative of a single-layer potential implies that $\psi = 0$. ∎

For any $k \in \mathbb{C}$, a function $w \in H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{D})$ that satisfies

$$\left.\begin{aligned} &\Delta w + k^2 w = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D} \\ \text{and} \quad &w = \mathrm{SL}^k_{\partial D}\left(\frac{\partial w}{\partial \nu}\right) - \mathrm{DL}^k_{\partial D}(w) \text{ in } \mathbb{R}^3\backslash\overline{D} \end{aligned}\right\} \tag{2.15}$$

is referred to as a radiating solution to the Helmholtz equation in $\mathbb{R}^3\backslash\overline{D}$. We then denote the space of radiating solutions by

$$H^e_{\text{inc}}(D) = \{w \in H^1_{\text{loc}}(\mathbb{R}^3\backslash D),\ w \text{ satisfies (2.15)}\}.$$

Functions in this space will serve as generalized incident fields for the interior problem (2.5). For k^2 not a Dirichlet eigenvalue of the negative Laplacian in D (see assumption 2.1), we can define $\mathcal{G}_k : H^e_{\text{inc}}(D) \to L^2(\partial\mathcal{C})$ as the mapping $w \mapsto u_w|_{\partial\mathcal{C}}$ with u_w the unique solution of

$$\Delta u_w + k^2 u_w = 0 \text{ in } D \quad \text{and} \quad u_w = w \text{ on } \partial D. \tag{2.16}$$

The arguments in the proof of theorem 2.3 show that $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi) \in H^e_{\text{inc}}(D)$, and we obviously have that $\mathcal{N}_k$ assumes the factorization

$$\mathcal{N}_k\varphi = \mathcal{G}_k\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi). \tag{2.17}$$

Based on the above discussion, we can provide the following equivalent definition of scattering poles for a Dirichlet obstacle. We remark that this definition uses the operator $\mathcal{G}_k$; hence, it still involves the solution of the exterior scattering problem. To this end, let $\mathbb{C}_-$ denote the complex half plane of complex numbers with negative imaginary parts.

Definition 2.8. $k \in \mathbb{C}_-$ is a scattering pole for a Dirichlet obstacle if and only if $\mathcal{G}_k$ is not injective.

To see that this is equivalent to the definition provided in proposition 2.2, we note that if $k \in \mathbb{C}_-$ is a scattering pole with w the associated eigenfunction, then $w \in H^e_{\text{inc}}(D)$ and $w = 0$ on ∂D. Therefore, $u_w = 0$ and $\mathcal{G}_k w = 0$. Conversely, if $\mathcal{G}_k w = 0$ then following the same unique continuation arguments as in the proof of theorem 2.3 yields $u_w = 0$, and therefore, $w = 0$ on ∂D. Since $w \in H^e_{\text{inc}}(D)$, we obtain a non-trivial solution to the homogeneous version of (2.10), which proves that k is a scattering pole.

Next, as a consequence of lemma 2.7 and the fact that the exterior Dirichlet problem (2.10) is well posed if k is not a scattering pole, we have the following lemma.

Lemma 2.9. *Assume that $k \in \mathbb{C}$ is not a scattering pole. Then the operator $SL^k_{\partial\mathcal{C}} : L^2(\partial\mathcal{C}) \to H^e_{\text{inc}}(D)$ is injective and has dense range.*

This now allows us to obtain another characterization of the scattering poles that uses only the operator $\mathcal{N}_k$. Notice that this operator does not require any solutions to the exterior scattering problem. Therefore, this characterization can also be seen as a method for computing the scattering poles without solving the exterior scattering problem. The method is inspired by a similar technique developed in [9] to compute the Dirichlet eigenvalues from a knowledge of the exterior relative scattering operator $\mathcal{S}(k)$. To state our main result, we need to prove the following important ingredient.

Lemma 2.10. *Assume that $k \in \mathbb{C}$ is not a scattering pole and satisfies assumption 2.1. Let $z \in \mathbb{R}^3\backslash\overline{\mathcal{C}}$. Then $\Phi_k(\cdot, z)$ is in the range of $\mathcal{G}_k$ if and only if $z \in \mathbb{R}^3\backslash\overline{D}$.*

Proof. If $z \in \mathbb{R}^3\backslash\overline{D}$ then evidently $\mathcal{G}_k w = \Phi_k(\cdot, z)|_{\partial\mathcal{C}}$, where $w \in H^e_{\text{inc}}(D)$ is the solution of (2.10) with $f = \Phi_k(\cdot, z)|_{\partial D}$ (since in this case $u_w = \Phi_k(\cdot, z)$ in D). Conversely, assume that for $z \in D\backslash\overline{\mathcal{C}}$ there exists $w \in H^e_{\text{inc}}(D)$ such that the solution u_w of (2.16) satisfies $u_w = \Phi_k(\cdot, z)$ on $\partial\mathcal{C}$ and hence thanks to assumption 2.1 and unique continuation $u_w = \Phi_k(\cdot, z)$ in D. This is a contradiction since $u_w \in H^1(D\backslash\overline{\mathcal{C}})$ while $\Phi_k(\cdot, z) \notin H^1(D\backslash\overline{\mathcal{C}})$ due to its singularity at z. ∎

The following theorem is a simple consequence of lemma 2.9 and lemma 2.10.

Theorem 2.11. *Assume that $z \in \mathbb{R}^3\backslash\overline{D}$ and $k \in \mathbb{C}$ is not a Dirichlet scattering pole and satisfies assumption 2.1. Then for every $\epsilon > 0$, there exists $\varphi^z_\epsilon \in L^2(\partial\mathcal{C})$ such that*

$$\lim_{\epsilon\to 0} \|\mathcal{N}_k\varphi^z_\epsilon - \Phi_k(\cdot, z)\|_{L^2(\partial\mathcal{C})} = 0 \quad \textit{and} \quad \|SL^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{H^{1/2}(\partial D)} < C.$$

In fact, $SL^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)$ converges to $\Phi_k(\cdot, z)$ in $H^{1/2}(\partial D)$.

Note that in the above theorem, since k is not a Dirichlet scattering pole, $\|\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{H^{1/2}(\partial D)} < C$ is equivalent to $\|\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{H^1(K)} < C(K)$ for any compact set K in $\mathbb{R}^3\backslash D$. Since $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)$ satisfies the Helmholtz equation in $\mathbb{R}^3\backslash D$, we deduce from trace theorems that $\|\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{\mathcal{H}^{1/2}(\partial D)} < C$, where we have set for $w \in H^e_{\mathrm{inc}}(D)$

$$\|w\|_{\mathcal{H}^{1/2}(\partial D)} := \|w\|_{H^{1/2}(\partial D)} + \|\partial w/\partial \nu\|_{H^{-1/2}(\partial D)}.$$

Theorem 2.12. *Assume that $k \in \mathbb{C}$ is a scattering pole. Let $\varphi^z_\epsilon \in L^2(\partial\mathcal{C})$ be a sequence such that*

$$\lim_{\epsilon\to 0} \|\mathcal{N}_k\varphi^z_\epsilon - \Phi_k(\cdot, z)\|_{L^2(\partial\mathcal{C})} = 0.$$

Then $\|SL^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{\mathcal{H}^{1/2}(\partial D)}$ cannot be bounded for all z in a ball $B \subset \mathbb{R}^3\backslash\overline{D}$.

Proof. Corresponding to the scattering pole k there is a non-zero (the corresponding eigenfunction) $w_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{D})$ that satisfies (2.10) with $f = 0$. Assume to the contrary that there exists a sequence $\{\varphi^z_\epsilon\}$ in $L^2(\partial\mathcal{C})$ and a small ball $B \subset \mathbb{R}^3\backslash\overline{D}$ such that $\|\mathcal{N}_k\varphi^z_\epsilon - \Phi_k(\cdot, z)\|_{L^2(\partial\mathcal{C})} \to 0$ and $\|\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{\mathcal{H}^{1/2}(\partial D)} < C$ for all $z \in B$. From the latter and lemma 2.4, we can assume that without loss of generality, $SL^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)$ converges weakly to $w_z \in H^e_{\mathrm{inc}}(D)$ as $\epsilon \to 0$. Let $u_z = \mathcal{G}_k w_z$. Evidently from (2.17) and the convergence assumption, we have that $u_z = \Phi_k(\cdot, z)$ on $\partial\mathcal{C}$, and hence $u_z = \Phi_k(\cdot, z)$ in $\mathcal{C}$ by the uniqueness of the Dirichlet problem in $\mathcal{C}$ and consequently in D by analyticity. Thus, $u_z = \Phi_k(\cdot, z)$ on ∂D and since by definition $u_z = w_z$ on ∂D, we have that $w_z \in H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{D})$ satisfies (2.15) and $w_z := \Phi_k(\cdot, z)$ on ∂D. This means that w_z is a solution of (2.10) with $f := \Phi_k(\cdot, z)$.

Now let us consider (see [18] for details on these boundary integral operators)

$$S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D), \quad S\psi(x) = \int_{\partial D} \psi(y)\Phi_k(x, y)\,\mathrm{d}s_y$$

$$K' : H^{-1/2}(\partial D) \to H^{-1/2}(\partial D), \quad K'\psi(x) = \frac{\partial}{\partial \nu_x}\int_{\partial D} \psi(y)\Phi_k(x, y)\,\mathrm{d}s_y$$

$$K : H^{1/2}(\partial D) \to H^{1/2}(\partial D), \quad K\psi(x) = \int_{\partial D} \psi(y)\frac{\partial \Phi_k(x, y)}{\partial \nu_y}\,\mathrm{d}s_y.$$

Since w_0 satisfies (2.10) with $f := w_0|_{\partial D} = 0$, taking the trace of the third equation in (2.10) and its normal derivative on ∂D, and using the jump relations of the single- and double-layer potential, we have that

$$S\frac{\partial w_0}{\partial \nu} = 0 \quad \text{and} \quad \frac{1}{2}\frac{\partial w_0}{\partial \nu} - K'\frac{\partial w_0}{\partial \nu} = 0 \quad \text{on } \partial D. \tag{2.18}$$

Now taking the trace of the third equation in (2.10) satisfied by w_z with $f := w_0|_{\partial D} = \Phi_k(\cdot, z)$, yields

$$w_z = S\frac{\partial w_z}{\partial \nu} + \frac{1}{2}w_z - Kw_z \quad \text{on } \partial D. \tag{2.19}$$

Multiplying (2.19) by $\partial w_0/\partial \nu$ and integrating over ∂D, from the facts that K' is the transpose of K and S is self-adjoint in the duality pairing between $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$ with $L^2(\partial D)$ pivot space (without conjugation) and (2.18), we obtain

$$\begin{aligned}\int_{\partial D} \Phi(\cdot, z)\frac{\partial w_0}{\partial \nu}\,\mathrm{d}s &= \int_{\partial D} w_z\frac{\partial w_0}{\partial \nu}\,\mathrm{d}s = \int_{\partial D} S\frac{\partial w_z}{\partial \nu}\frac{\partial w_0}{\partial \nu}\,\mathrm{d}s + \int_{\partial D}\left(\frac{1}{2}w_z - Kw_z\right)\frac{\partial w_0}{\partial \nu}\,\mathrm{d}s \\ &= \int_{\partial D} \frac{\partial w_z}{\partial \nu}S\frac{\partial w_0}{\partial \nu}\,\mathrm{d}s + \int_{\partial D} w_z\left(\frac{1}{2}\frac{\partial w_0}{\partial \nu} - K'\frac{\partial w_0}{\partial \nu}\right)\mathrm{d}s = 0 \quad \text{for } z \in B.\end{aligned}$$

Unique continuation now implies that

$$\mathrm{SL}^k_{\partial D}\left(\frac{\partial w_0}{\partial \nu}\right)(z) := \int_{\partial D} \Phi(\cdot, z)\frac{\partial w_0}{\partial \nu}\,\mathrm{d}s = 0 \quad \text{for } z \in \mathbb{R}^3\backslash\overline{D}.$$

By taking the trace on ∂D and using the uniqueness of the Dirichlet problem in D we can conclude that

$$\mathrm{SL}^k_{\partial D}\left(\frac{\partial w_0}{\partial \nu}\right)(z) = 0 \quad \text{also for } z \in \overline{D}.$$

From the jump relation for the derivative of the single-layer potential, we can conclude that $\frac{\partial w_0}{\partial \nu} = 0$ on ∂D. Since the Cauchy data of w_0 are zero, Holmgren's theorem implies that $w_0 = 0$, which is a contradiction. This proves the theorem. ■

We can combine theorem 2.11 and theorem 2.12 to formulate the following criteria for the determination of the scattering poles.

Corollary 2.13. *Let $k \in \mathbb{C}$ satisfy assumption 2.1 and $\mathcal{C} \subset D$. For $z \in \mathbb{R}^3 \backslash \overline{D}$ and every sequence $\{\varphi_\epsilon^z\}$ in $L^2(\partial\mathcal{C})$ such that*

$$\lim_{\epsilon \to 0} \|\mathcal{N}_k \varphi_\epsilon^z - \Phi_k(\cdot, z)\|_{L^2(\partial\mathcal{C})} = 0, \tag{2.20}$$

we have that

$$\|SL^k_{\partial\mathcal{C}}(\varphi_\epsilon^z)\|_{\mathcal{H}^{1/2}(\partial D)} < C \quad \text{for all } z \text{ in a ball } B \subset \mathbb{R}^3 \backslash \overline{D}$$

if and only if k is not a scattering pole for the Dirichlet obstacle.

Remark 2.14. For the existence of a sequence φ_ϵ^z that satisfies (2.20), we need that $\Phi_k(\cdot, z)$ is in the closure of the range of $\mathcal{N}_k$. This is the case when k is not a scattering pole by Theorem (2.6). However, if k is a scattering pole, this information is not available and may depend on D. Thus, if k is a scattering pole, either such φ_ϵ^z exists and $\|\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)_\epsilon^z\|_{\mathcal{H}^{1/2}(\partial D)}$ becomes unbounded, or no φ_ϵ^z satisfying (2.20) exists, as opposed to if k is not a scattering pole. This fact can potentially be used in the detection of scattering poles.

Note that the statement of corollary 2.13 and remark 2.14 suggest a computational method for the scattering poles from a knowledge of the operator $\mathcal{N}_k$, or in other words the interior scattering data. The algorithm would be to first numerically build an approximation of the operator $\mathcal{N}_k$ by approximating $u^s(x, y)$, $x \in C$, $y \in C$. Then, for a collection of points $z \in B_0 \subset \mathbb{R}^3 \backslash \overline{D}$ compute a nearby solution φ_k^z

$$\mathcal{N}_k \varphi_k^z \simeq \Phi_k(\cdot, z)$$

by means of some regularization strategy. Corollary 2.13 suggests for instance a Tikhonov regularization with a penalization term proportional to $\|\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi_k^z)\|^2_{\mathcal{H}^{1/2}(\partial D)}$. One then evaluates

$$k \mapsto \int_{B_0} \|\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi_k^z)\|_{\mathcal{H}^{1/2}(\partial D)} dz,$$

which would have peaks at scattering poles. Drawing a parallel with numerical experimentation related to identifying transmission eigenvalues [20], one may expect the algorithm to also work if we replace $\|\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi_k^z)\|_{\mathcal{H}^{1/2}}$ with $\|\varphi_k^z\|_{L^2(C)}$, which is numerically simpler to implement. We refer the reader to [21] for the application of the scattering poles to inverse scattering for a Dirichlet obstacle.

3. Scattering poles for inhomogeneous media

We now turn our attention to the scattering problem for an inhomogeneous medium (n, D) governed by (1.4), which we have already described in the Introduction. We assume that the refractive index n is a complex valued L^∞ function with $\Re(n) > 0$ and $\Im(n) \geq 0$, such that $n - 1$ is supported in $\overline{D}$. Unless otherwise indicated, we assume that the boundary ∂D is Lipschitz smooth. We start by providing an equivalent definition of the scattering poles for this problem. Given $v \in L^2(D)$, we define the scattering problem associated with an incident field v as determining the scattered field $w \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\left.\begin{aligned} &\Delta w + k^2 n w = k^2(1-n)v \quad \text{in } \mathbb{R}^3 \\ \text{and}\quad &w = \mathrm{SL}^k_{\partial D}\left(\frac{\partial w}{\partial \nu}\right) - \mathrm{DL}^k_{\partial D}(w) \quad \text{in } \mathbb{R}^3 \backslash \overline{D}. \end{aligned}\right\} \tag{3.1}$$

Proposition 3.1. *$k \in \mathbb{C}$ is a pole of the scattering matrix (i.e. a scattering pole) of the medium scattering problem (n, D), if and only if the homogeneous problem (3.1), i.e. with $v = 0$, has a non-trivial solution $w \in H^2_{\text{loc}}(\mathbb{R}^3)$.*

Proof. Poles of the scattering matrix of the medium scattering problem with support D and refractive index n can be characterized as the values of k for which the operator $I - T(k) : L^2(D) \to L^2(D)$ has a non-trivial kernel, where the compact operator $T(k) : L^2(D) \to L^2(D)$ is given by (1.5) (which is the same as saying the k- meromorphic function $(I - T(k))^{-1}$ has a pole). Assume that

$$w = T(k)(w) \text{ in } D.$$

We extend w to all of $\mathbb{R}^3$ using the representation

$$w(p) = \int_D k^2(n(y) - 1)w(y)\Phi_k(y, p)\,\mathrm{d}y \quad p \in \mathbb{R}^3.$$

Properties of volume potentials ensure that $w \in H^2_{\text{loc}}(\mathbb{R}^3)$ and satisfies $\Delta w + k^2 n w = 0$ in $\mathbb{R}^3$ [7]. The integral representation of w given in the second equation of (3.1) is obtained by multiplying (2.11) by $k^2(n(y) - 1)w(y)$ and then integrating over D.

Conversely, consider $w \in H^2_{\text{loc}}(\mathbb{R}^3)$ satisfying (3.1) with $v = 0$. Let B be a bounded domain with Lipschitz boundary containing $\overline{D}$ in its interior. Green's representation theorem in B implies that

$$w(p) = \int_D k^2(n(y) - 1)w(y)\Phi_k(y, p)\,\mathrm{d}y - \left(\mathrm{SL}^k_{\partial B}\left(\frac{\partial w}{\partial \nu}\right) - \mathrm{DL}^k_{\partial B}(w)\right)(p) \quad p \in B.$$

On the other hand, for $p \in \mathbb{R}^3 \backslash B$ and using the fact that w and $\Phi(\cdot, p)$ satisfy the Helmholtz equation in the domain between B and D, Green's second identity yields

$$\left(\mathrm{SL}^k_{\partial B}\left(\frac{\partial w}{\partial \nu}\right) - \mathrm{DL}^k_{\partial B}(w)\right)(p) = \left(\mathrm{SL}^k_{\partial D}\left(\frac{\partial w}{\partial \nu}\right) - \mathrm{DL}^k_{\partial D}(w)\right)(p) \quad p \in \mathbb{R}^3 \backslash B.$$

Therefore,

$$w(p) = \left(\mathrm{SL}^k_{\partial B}\left(\frac{\partial w}{\partial \nu}\right) - \mathrm{DL}^k_{\partial B}(w)\right)(p) \quad p \in \mathbb{R}^3 \backslash B.$$

We then infer from the continuity of w across ∂B and jump relations for single- and double-layer potentials across ∂B that

$$w(p) = \int_D k^2(n(y) - 1)w(y)\Phi_k(y, p)\,\mathrm{d}y \quad p \in \partial B. \tag{3.2}$$

Since ∂B is an arbitrarily chosen boundary enclosing D, the latter identity holds for all $p \in \mathbb{R}^3 \backslash \overline{D}$. Both sides of the equality satisfy $\Delta u + k^2 u = -k^2(n - 1)w$ in $\mathbb{R}^3$; hence, unique continuation arguments imply that (3.2) holds for all $p \in \mathbb{R}^3$ and in particular $w = T(k)(w)$ in D, which concludes the proof. ∎

Similarly to §2, we again consider a Lipschitz closed surface $\partial\mathcal{C}$ circumscribing a simply connected region $\mathcal{C} \subset D$.

Assumption 3.2. k^2 is not a Dirichlet eigenvalue of the negative Laplacian in $\mathcal{C}$, and k is not a transmission eigenvalue of (1.7).

Assume that assumption 3.2 holds. For a point $z \in D$, let $u(\cdot, z), v(\cdot, z) \in L^2(D) \times L^2(D)$ be such that $u(\cdot, z) - v(\cdot, z) \in H^2(D)$ and satisfy

$$\left.\begin{aligned}
&\Delta u(\cdot, z) + k^2 n(x) u(\cdot, z) = 0 \quad \text{in } D \\
&\Delta v(\cdot, z) + k^2 v(\cdot, z) = 0 \quad \text{in } D \\
&u(\cdot, z) - v(\cdot, z) = \Phi_k(\cdot, z) \quad \text{on } \partial D \\
\text{and}\quad &\frac{\partial u(\cdot, z)}{\partial \nu} - \frac{\partial v(\cdot, z)}{\partial \nu} = \frac{\partial \Phi_k(\cdot, z)}{\partial \nu} \quad \text{on } \partial D.
\end{aligned}\right\} \tag{3.3}$$

This problem, the so-called interior transmission problem, will play the role of a forward (interior) scattering problem that provides a new equivalent definition of scattering poles. Here, we have assumed that the refractive index $n \in L^\infty(D)$, with $\Re(n) > 0$ and $\Im(n) \geq 0$, is such that the resolvent of (1.7) is Fredholm, i.e. (3.3) has a unique solution if k is not a transmission eigenvalue. This is for example true if $\Re(n) - 1 \geq n_0 > 0$ or $1 - \Re(n) \geq n_0 > 0$ in a neighbourhood of ∂D ([6], Section 3.1).

Remark 3.3. Complex transmission eigenvalues in the lower half plane may exist in general and in fact for spherically symmetric media it is proven that they do exist [7,15]. It is not clear how to fully understand the intersection of the set of transmission eigenvalues and the scattering poles. However, in general, there are infinitely many scattering poles that are not transmission eigenvalues. Indeed in [4,5] it is proven that for inhomogeneous media there exist infinitely many scattering poles lying along the complex axis without a finite accumulation point. On the other hand, for media (n, D) satisfying $\Re(n) - 1 \geq n_0 > 0$ or $1 - \Re(n) \geq n_0 > 0$ in a neighbourhood of ∂D, it is known that $k := i\kappa$ for $|\kappa|$ large enough are not transmission eigenvalues [6,22].

Accordingly, we now redefine the space of exterior incident waves as

$$H^e_{\text{inc}}(D) = \{w \in H^2_{\text{loc}}(\mathbb{R}^3 \backslash D),\ w \text{ satisfies (2.15)}\}. \tag{3.4}$$

(Note this space is similar to $H^e_{\text{inc}}(D)$ used in §2 where we have changed the space to $H^2_{\text{loc}}(\mathbb{R}^3 \backslash D)$ taking into account the H^2-regularity of the scattered field for the transmission problem.) Then, the interior scattering operator $\mathcal{N}_k : L^2(\partial\mathcal{C}) \to L^2(\partial\mathcal{C})$ is now defined as

$$\mathcal{N}_k\varphi(x) = \int_{\partial\mathcal{C}} \varphi(z) v(x,z)\, \mathrm{d}s(z), \quad x \in \partial\mathcal{C}. \tag{3.5}$$

Obviously,

$$\mathcal{N}_k : \varphi \mapsto \tilde{v}_\varphi|_{\partial\mathcal{C}}, \tag{3.6}$$

where $(\tilde{u}_\varphi, \tilde{v}_\varphi) \in L^2(D) \times L^2(D)$ is the solution to (3.3) with $\Phi_k(\cdot, z)$ replaced by $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)$. Hence, similar to the Dirichlet case,

$$\mathcal{N}_k\varphi = \mathcal{G}_k \mathrm{SL}^k_{\partial\mathcal{C}}(\varphi), \tag{3.7}$$

where $\mathcal{G}_k : H^e_{\text{inc}}(D) \to L^2(\partial\mathcal{C})$ is now defined as the mapping

$$w \mapsto v_w|_{\partial\mathcal{C}} \tag{3.8}$$

with $(u_w, v_w) \in L^2(D) \times L^2(D)$ being the solution to (3.3), where $\Phi_k(\cdot, z)$ is replaced by w.

In what follows, we shall keep using the notation $(\tilde{u}_\varphi, \tilde{v}_\varphi)$ and (u_w, v_w) to refer to solutions of (3.3) with boundary, data respectively, $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)$ and w, as in the above discussion.

Theorem 3.4. *Assume that $k \in \mathbb{C}$ is not a scattering pole of the medium scattering problem (n, D) and satisfies assumption 3.2. Then the operator $\mathcal{N}_k : L^2(\partial\mathcal{C}) \to L^2(\partial\mathcal{C})$ is symmetric and injective with dense range.*

Proof. The proof of symmetry follows the same lines as for the Dirichlet case in lemma 2.5 and is left to the reader. Since the symmetry and injectivity imply the denseness of the range, we only need to prove the injectivity. To this end, let $\mathcal{N}_k\varphi = 0$. This means that $\tilde{v}_\varphi = 0$ on $\partial\mathcal{C}$. Since $\Delta\tilde{v}_\varphi + k^2\tilde{v}_\varphi = 0$ in D and hence in $\mathcal{C}$, assumption 3.2 guarantees that $\tilde{v}_\varphi = 0$ in $\mathcal{C}$. Therefore, by a unique continuation argument, $\tilde{v}_\varphi = 0$ in D. Consequently, the function w defined as

$$w = \tilde{u}_\varphi \ \text{ in } D \text{ and } w = \mathrm{SL}^k_{\partial\mathcal{C}}(\varphi) \ \text{ in } \mathbb{R}^3 \backslash D$$

is in $H^2_{\text{loc}}(\mathbb{R}^3)$ and satisfies (3.1). The fact that w satisfies the integral representation in (3.1) follows from the same arguments as in the proof of lemma 2.4. Since k is not a scattering pole, from proposition 3.1, we conclude that $\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi) \equiv 0$ in $\mathbb{R}^3 \backslash \overline{D}$. Finally, the unique continuation principle, assumption 3.2 and the jump relation for the normal derivative of the single-layer potential across $\partial\mathcal{C}$ imply that $\varphi = 0$. This proves that $\mathcal{N}_k$ is injective and finishes the proof. ■

Lemma 3.5. *Assume that $k \in \mathbb{C}$ is not a scattering pole and satisfies assumption 3.2. Let $z \in \mathbb{R}^3 \backslash \overline{\mathcal{C}}$. Then $\Phi_k(\cdot, z)$ is in the range of $\mathcal{G}_k$ if and only if $z \in \mathbb{R}^3 \backslash \overline{D}$.*

Proof. For $z \in \mathbb{R}^3\backslash\overline{D}$, we define $w \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ to be the solution of (3.1) with $v = \Phi_k(\cdot, z)|_D$. Since $\Phi_k(\cdot, z)$ satisfies the Helmholtz equation in D, we have $v_w = \Phi_k(\cdot, z)$ and therefore $\mathcal{G}_k w = \Phi_k(\cdot, z)|_{\partial\mathcal{C}}$.

Conversely, assume that for $z \in D\backslash\overline{\mathcal{C}}$, there exists $w \in H^e_{\mathrm{inc}}(D)$ such that the solution v_w satisfies $v_w = \Phi_k(\cdot, z)$ on $\partial\mathcal{C}$ and hence, thanks to assumption 3.2 and unique continuation, $v_w = \Phi_k(\cdot, z)$ in D. This is a contradiction since $\Delta v_w \in L^2(D)$ while $\Delta\Phi_k(\cdot, z)$ is not. ■

We now prove a denseness lemma similar to lemma 2.9. For this, one needs to exclude exceptional values of k that correspond to being both Dirichlet and Neumann scattering poles, simultaneously, i.e. the values of $k \in \mathbb{C}$ for which exists a non-zero $w_d \in H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash D)$ solving

$$\left.\begin{aligned}
&\Delta w_d + k^2 w_d = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D}\\
&w_d = 0 \quad \text{on } \partial D\\
&w = \mathrm{SL}^k_{\partial D}\left(\frac{\partial w_d}{\partial \nu}\right) - \mathrm{DL}^k_{\partial D}(w_d) \quad \text{in } \mathbb{R}^3\backslash\overline{D}
\end{aligned}\right\} \tag{3.9}$$

and non-zero $w_n \in H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash D)$ solving

$$\left.\begin{aligned}
&\Delta w_n + k^2 w_n = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D}\\
&\frac{\partial w_n}{\partial \nu} = 0 \quad \text{on } \partial D\\
&w = \mathrm{SL}^k_{\partial D}\left(\frac{\partial w_n}{\partial \nu}\right) - \mathrm{DL}^k_{\partial D}(w_n) \quad \text{in } \mathbb{R}^3\backslash\overline{D}.
\end{aligned}\right\} \tag{3.10}$$

Note that for a unit ball, they correspond to common zeros of $h^{(1)}_{\ell_1}$ and $h^{(1)'}_{\ell_2}$ for some $\ell_1, \ell_2 \in \mathbb{N}$.

Lemma 3.6. *Let the boundary ∂D be of class $C^{1,1}$. Assume that k^2 is not an eigenvalue of the negative Laplacian in $\mathcal{C}$ and assume that k is not simultaneously both a scattering pole for the Dirichlet scattering problem and the Neumann scattering problem for D. Then the operator $SL^k_{\partial\mathcal{C}} : L^2(\partial\mathcal{C}) \to H^e_{\mathrm{inc}}(D)$ is injective with dense range.*

Proof. The injectivity of this operator has already been proved in lemma 2.7 if k is not a Dirichlet scattering pole. In case that k is a Dirichlet scattering pole, then from our assumption, it is not a Neumann scattering pole and the proof of injectivity in lemma 2.7 can be accordingly modified by considering $(\partial/\partial\nu)\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi) = 0$ on ∂D and concluded in the same way based on (3.10).

As for the denseness of the range, according to lemma 2.4, it is sufficient to prove that either the operator $S : L^2(\partial\mathcal{C}) \to H^{3/2}(\partial D)$ or the operator $K : L^2(\partial\mathcal{C}) \to H^{1/2}(\partial D)$ defined by

$$S(\varphi) := \mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)_{|\partial D} \quad \text{and} \quad K(\varphi) := \frac{\partial \mathrm{SL}^k_{\partial\mathcal{C}}(\varphi)}{\partial \nu}|_{\partial D},$$

respectively, has a dense range when k is not a scattering pole for the Dirichlet (respectively, Neumann) scattering problem for D. We will follow the same lines as in the proof of the denseness of the range of S in $H^{1/2}(\partial D)$ in lemma 2.7.

Assume first that k is not a scattering pole for the Dirichlet scattering problem for D. Let $\psi \in H^{-3/2}(\partial D)$ be such that $S^\top\psi = 0$ on $\partial\mathcal{C}$ where the transpose operator $S^\top : H^{-3/2}(\partial D) \to L^2(\partial\mathcal{C})$ is defined by (2.14) in the proof of lemma 2.7. We observe that $S^\top\psi := \mathrm{SL}^k_{\partial D}\psi|_{\partial\mathcal{C}}$ and $\mathrm{SL}_{\partial D}\psi$ defines a $L^2(D)$ solution of the Helmholtz equation (see [23]). By the uniqueness of the Dirichlet problem in $\mathcal{C}$, we have that $\mathrm{SL}^k_{\partial D}\psi \equiv 0$ in $\mathcal{C}$ and by unique continuation in all of D. Thus, the trace of $\mathrm{SL}^k_{\partial D}\psi$ on ∂D defined as an element in $H^{-1/2}(\partial D)$ [18], vanishes. Let us now define $w =: \mathrm{SL}^k_{\partial D}\psi$ in $\mathbb{R}^3\backslash D$. Again from [23], we obtain that w is an L^2 solution of the Helmholtz equation in $\mathbb{R}^3\backslash D$ with homogeneous Dirichlet boundary conditions on ∂D. Elliptic regularity implies that this solution is in $H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash D)$. Let B be a bounded domain with $C^{1,1}$ boundary such that $\overline{D} \subset B$. Lemma 2.4 (where ∂D plays the role $\partial\mathcal{C}$ and B plays the role of D) implies that

$$w(p) = \left(\mathrm{SL}^k_{\partial B}\left(\frac{\partial w}{\partial \nu}\right) - \mathrm{DL}^k_{\partial B}(w)\right)(p) \quad p \in \mathbb{R}^3\backslash B.$$

(The application of lemma 2.4 can be easily extended to densities that are only in $H^{-3/2}$ using a density argument). Then, applying the second Green formula in the domain between B and D yields

$$w(p) = \left(\mathrm{SL}^k_{\partial D}\left(\frac{\partial w}{\partial \nu}\right) - \mathrm{DL}^k_{\partial D}(w)\right)(p) \quad p \in \mathbb{R}^3 \setminus B.$$

Since B is arbitrary, we have that w satisfies the integral representation in (2.10) and therefore $w = 0$ by our assumption on k. The jump relations for normal derivatives of single-layer potentials with $H^{-3/2}(\partial D)$ densities [23] implies that $\psi = 0$ and this finishes the proof for the first case.

We now consider the case where k is not a scattering pole for the Neumann scattering problem for D and shall prove that $K : L^2(\partial\mathcal{C}) \to H^{1/2}(\partial D)$ has dense range. The proof follows along the same lines as in the previous case and we will only give an outline. The transpose operator $K^\top : H^{-1/2}(\partial D) \to L^2(\partial\mathcal{C})$ is defined by

$$K^\top \psi := \mathrm{DL}^k_{\partial D}(\psi)|_{\partial\mathcal{C}}.$$

Let us set $w = \mathrm{DL}^k_{\partial D}\psi$. Properties of double-layer potentials with densities in $H^{-1/2}$ can be found in [23]. Similar considerations as above show that if $K^\top \psi = 0$ then $w = 0$ in D. Since the normal derivative of $\mathrm{DL}^k_{\partial D}\psi$ is continuous across ∂D, we obtain that w is an L^2 solution of the Helmholtz equation in $\mathbb{R}^3 \setminus D$ with homogeneous Neumann boundary conditions on ∂D. The result of lemma 2.4 holds true (and can be proven exactly in the same way) if we replace the single-layer potential with the double-layer potential. Therefore, applying this lemma together with elliptic regularity for the Neumann problem and the same argument as above for justifying the integral representation of w outside D, we get that w is associated with a scattering pole for the Neumann problem. Hence, $w = 0$ and the jump relation for the trace of the double-layer potential on ∂D implies that $\psi = 0$. ■

As a consequence of lemma 3.6, combined with lemma 3.5, we can prove the following theorem. In order to simplify the notation, for $w \in H^e_{\mathrm{inc}}(D)$ we set

$$\|w\|_{\mathcal{H}^{3/2}(\partial D)} := \|w\|_{H^{3/2}(\partial D)} + \left\|\frac{\partial w}{\partial \nu}\right\|_{H^{1/2}(\partial D)},$$

which clearly defines an equivalent norm on $H^e_{\mathrm{inc}}(D)$.

Theorem 3.7. *Let $z \in \mathbb{R}^3 \setminus \overline{D}$ and ∂D is of class $C^{1,1}$. Assume that $k \in \mathbb{C}$ is not a scattering pole of the medium scattering problem (n, D), satisfies assumption 3.2, and in addition k is not simultaneously both a Dirichlet and Neumann scattering pole for D. Then for every $\epsilon > 0$, there exists $\varphi^z_\epsilon \in L^2(\partial\mathcal{C})$ such that*

$$\lim_{\epsilon \to 0} \|\mathcal{N}_k \varphi^z_\epsilon - \Phi_k(\cdot, z)\|_{L^2(\partial\mathcal{C})} = 0 \quad \textit{and} \quad \|SL^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{\mathcal{H}^{3/2}(\partial D)} < C.$$

Finally, we now state the complementary result to the above theorem at a scattering pole.

Theorem 3.8. *Assume that $k \in \mathbb{C}$ is a scattering pole of the medium scattering problem (n, D) and satisfies assumption 3.2 and ∂D is of class $C^{1,1}$. Let $\varphi^z_\epsilon \in L^2(\partial\mathcal{C})$ be a sequence such that*

$$\lim_{\epsilon \to 0} \|\mathcal{N}_k \varphi^z_\epsilon - \Phi_k(\cdot, z)\|_{L^2(\partial\mathcal{C})} = 0.$$

Then $\|SL^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{\mathcal{H}^{3/2}(\partial D)}$ cannot be bounded for all z in a ball $B \subset \mathbb{R}^3 \setminus \overline{D}$.

Proof. Corresponding to the scattering pole k there is a non-zero (the corresponding eigenfunction) $w_0 \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ that satisfies (3.1) with $v = 0$. Assume to the contrary that there exists a sequence $\{\varphi^z_\epsilon\}$ in $L^2(\partial\mathcal{C})$ and a small ball $B \subset \mathbb{R}^3 \setminus \overline{D}$ such that $\mathcal{N}_k \varphi^z_\epsilon$ converges to $\Phi_k(\cdot, z)$ in $L^2(\partial\mathcal{C})$ and $\|\mathrm{SL}^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{\mathcal{H}^{3/2}(\partial D)} < C$ for all $z \in B$. From the latter, we can assume without loss of generality that $SL^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)$ converges weakly to $w_z \in H^e_{\mathrm{inc}}(D)$ as $\epsilon \to 0$, with $H^e_{\mathrm{inc}}(D)$ given by (3.4).

Let $v_z = \mathcal{G}_k w_z$, where (v_z, u_z) solve the interior transmission problem (3.3) with $\Phi_k(\cdot, z)$ replaced by w_z, which for $\tilde{w}_z := u_z - v_z \in H^2(D)$ can be written as

$$\left.\begin{aligned} &\Delta \tilde{w}_z + k^2 n \tilde{w}_z = k^2(1-n)v_z \quad \text{in } D \\ &\tilde{w}_z = w_z \quad \text{and} \quad \frac{\partial \tilde{w}_z}{\partial \nu} = \frac{\partial w_z}{\partial \nu} \quad \text{on } \partial D. \end{aligned}\right\} \tag{3.11}$$

It is clear from (2.17) and the convergence of $\mathcal{N}_k \varphi_\epsilon^z$ to $\Phi_k(\cdot, z)$ in $L^2(\partial \mathcal{C})$ that $v_z = \Phi_k(\cdot, z)$ on $\partial \mathcal{C}$, and hence $v_z = \Phi_k(\cdot, z)$ in $\mathcal{C}$ by the uniqueness of the Dirichlet problem in $\mathcal{C}$ and consequently in D by analyticity. Considering $W_z := \tilde{w}_z$ in D and $W_z := w_z$ in $\mathbb{R}^3 \backslash D$ from (3.11) and the facts that $w_z \in H^e_{\text{inc}}(D)$ in (3.4) and $v_z = \Phi(\cdot, z)$, we have that $W_z \in H^2_{\text{loc}}(\mathbb{R}^3)$ satisfies (3.1) with $v := \Phi(\cdot, z)$. The latter means that

$$(I - T(k))(W_z + \Phi(\cdot, z)) = \Phi(\cdot, z) \quad \text{in } D.$$

Multiplying this equation by $k^2(n-1)w_0$ and then integrating over D and changing the order of integration implies that

$$\int_D k^2(n-1)(W_z + \Phi(\cdot, z))(I - T(k))w_0 \, \mathrm{d}x = \int_D \Phi(y, z) k^2 (n-1) w_0(y) \, \mathrm{d}y.$$

Therefore,

$$\int_D \Phi(y, z) k^2 (n-1) w_0(y) \, \mathrm{d}y = 0, \quad \text{for } z \in B.$$

Unique continuation for solutions of the Helmholtz equation yields

$$P(z) := \int_D \Phi(y, z) k^2 (n-1) w_0(y) \, \mathrm{d}y = 0, \quad \text{for } z \in \mathbb{R}^3 \backslash \overline{D},$$

and hence $P(z) = 0$ and $\partial P(z)/\partial \nu = 0$ on ∂D. Now inside D, we have that $P(z) \in H^2(D)$ satisfies

$$\Delta P + k^2 P = -k^2(n-1)w_0.$$

Since w_0 solves $\Delta w_0 + k^2 n w_0 = 0$ in D, we conclude that (w_0, v), with $v := w_0 - P$ satisfies the homogeneous interior transmission problem and from assumption 3.2, i.e. k is not a transmission eigenvalue, we conclude that $w_0 = 0$ in D and therefore in $\mathbb{R}^3$ (by unique continuation), which is a contradiction. This proves the theorem. ■

We can combine theorem 3.7 and theorem 3.8 to formulate the following criteria for the determination of the scattering poles.

Corollary 3.9. *Let $k \in \mathbb{C}$ satisfies assumption 3.2, ∂D is of class $C^{1,1}$ and let $\mathcal{C} \subset D$. For $z \in \mathbb{R}^3 \backslash \overline{D}$ and every sequence $\{\varphi_\epsilon^z\}$ in $L^2(\partial \mathcal{C})$ such that*

$$\lim_{\epsilon \to 0} \|\mathcal{N}_k \varphi_\epsilon^z - \Phi_k(\cdot, z)\|_{L^2(\partial \mathcal{C})} = 0, \tag{3.12}$$

we have that

$$\|SL^k_{\partial \mathcal{C}}(\varphi_\epsilon^z)\|_{\mathcal{H}^{3/2}(D)} < C \quad \textit{for all } z \textit{ in a ball } B \subset \mathbb{R}^3 \backslash \overline{D}$$

if and only if k is not a scattering pole of the inhomogeneous media (n, D).

We can make here the same comments as in remark 2.14, namely, to guarantee the existence of a sequence φ_ϵ^z that satisfies (3.12), we need that $\Phi_k(\cdot, z)$ is in the closure of the range of $\mathcal{N}_k$. By Theorem (2.6), this is the case when $k \in \mathbb{C}_-$ is not a scattering pole of the medium scattering problem (n, D) with ∂D in $C^{1,1}$, and in addition, $k \in \mathbb{C}_-$ is not a transmission eigenvalue and is not simultaneously both a Dirichlet and a Neumann scattering pole for D. However, if k is a scattering pole for the inhomogeneous media, this information is not available. Thus, if k is a scattering pole, either φ_ϵ^z satisfying (3.12) exists and $\|\mathrm{SL}^k_{\partial \mathcal{C}}(\varphi)_\epsilon^z\|_{\mathcal{H}^{3/2}(D)}$ becomes unbounded, or no φ_ϵ^z satisfying (3.12) exists, as opposed to the case if k is not a scattering pole. This fact can be used for the detection of scattering poles. We also remark that our new characterization of the scattering poles for inhomogeneous media is inconclusive if $k \in \mathbb{C}_-$ is either a transmission eigenvalue, or is simultaneously both a Dirichlet and Neumann scattering pole for D. The characterization of the intersection of the set of scattering poles for inhomogeneous media and the above anomalous sets is an open question of interest.

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