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Dynamics and stability of sessile drops with contact points

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Abstract

In an effort to study the stability of contact lines in fluids, we consider the dynamics of a drop of incompressible viscous Stokes fluid evolving above a one-dimensional flat surface under the influence of gravity. This is a free boundary problem: the interface between the fluid on the surface and the air above (modeled by a trivial fluid) is free to move and experiences capillary forces. The three-phase interface where the fluid, air, and solid vessel wall meet is known as a contact point, and the angle formed between the free interface and the flat surface is called the contact angle. We consider a model of this problem that allows for fully dynamic contact points and angles. We develop a scheme of a priori estimates for the model, which then allows us to show that for initial data sufficiently close to equilibrium, the model admits global solutions that decay to a shifted equilibrium exponentially fast.

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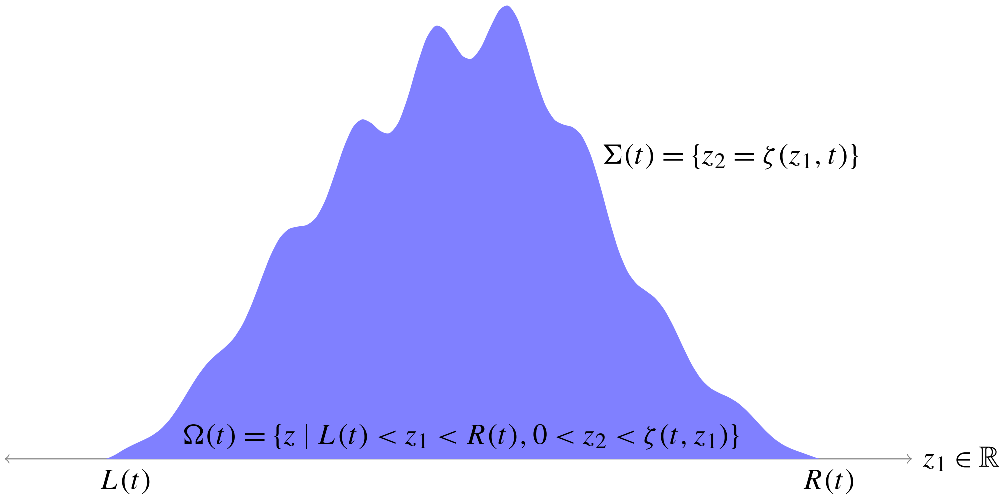

Fig. 1. An example of a droplet domain.

1. Problem formulation

Consider a two-dimensional droplet of viscous incompressible fluid evolving above a one-dimensional flat surface. Denote the spatial variable $z = (z_1, z_2) \in \mathbb{R}^2$. We then assume that at time $t \geq 0$ the fluid occupies the moving droplet domain

$$\Omega(t) := \{z \in \mathbb{R}^2 : 0 < z_2 < \zeta(t, z_1)\}, \tag{1.1}$$

where the free surface of the droplet is given by the unknown function $\zeta(t, \cdot) : [L(t), R(t)] \to \mathbb{R}^+$, which satisfies $\zeta(t, L(t)) = \zeta(t, R(t)) = 0$. Here $L(t)$ and $R(t)$ are the left and right end points of the moving droplet domain. We write the free surface at the top of the droplet as

$$\Sigma(t) := \{(z_1, z_2) : L(t) < z_1 < R(t), z_2 = \zeta(t, z_1)\}, \tag{1.2}$$

and at the bottom as

$$\Sigma_b(t) := \{(z_1, z_2) : L(t) < z_1 < R(t), z_2 = 0\}. \tag{1.3}$$

See Fig. 1 for an example of such a fluid droplet domain. For each $t \geq 0$, the fluid is described by its velocity and pressure $(u(t, \cdot), P(t, \cdot)) : \Omega(t) \to \mathbb{R}^2 \times \mathbb{R}$. The viscous stress tensor is determined in term of P and u according to $S(P, u) = PI - \mu \mathbb{D}_z u$, where I is the 2×2 identity matrix, $\mathbb{D}_z u = \nabla_z u + (\nabla_z u)^T$ is the symmetric gradient of u, and $\mu > 0$ is the viscosity of the fluid. We note that a simple computation reveals that if $\nabla_z \cdot u = 0$, then $\nabla_z \cdot S(P, u) = -\mu \Delta_z u + \nabla_z P$.

Before stating the equations of motion, we define a number of terms that will appear. We will write $g > 0$ for the strength of gravity, $\sigma > 0$ for the surface tension coefficient along the free surface, and $\beta > 0$ for the Navier slip friction coefficient on the flat surface. The coefficients $\gamma_{sv}, \gamma_{sf} \in \mathbb{R}$ are a measure of the free-energy per unit length associated to the solid-vapor and solid-fluid interaction, respectively. We set $[\gamma] = \gamma_{sv} - \gamma_{sf}$ and make the crucial assumption that

$$0 < \frac{[\gamma]}{\sigma} < 1. \tag{1.4}$$

This is equivalent to the classical Young's law, together with the extra assumption that $[\gamma] > 0$. The former is a necessary condition for the existence of any equilibrium state, while the latter is a technical condition that guarantees that any equilibrium state can be described as above by the graph of a function. Finally, we define the contact point velocity response function $\mathscr{V} : \mathbb{R} \to \mathbb{R}$ to be a C^2 increasing diffeomorphism such that $\mathscr{V}(0) = 0$. We will refer to its inverse as $\mathscr{W} = \mathscr{V}^{-1} \in C^2(\mathbb{R})$.

We require that (u, P, ζ, L, R) satisfy the gravity-driven free-boundary incompressible Stokes equations in $\Omega(t)$ for $t > 0$:

$$\begin{cases} \nabla_z \cdot S(P,u) = -\mu\Delta_z u + \nabla_z P = 0 & \text{in } \Omega(t), \\ \nabla_z \cdot u = 0 & \text{in } \Omega(t), \\ S(P,u)\nu = g\zeta\nu - \sigma\mathcal{H}(\zeta)\nu & \text{on } \Sigma(t), \\ \Big(S(P,u)\nu - \beta u\Big)\cdot\tau = 0 & \text{on } \Sigma_b(t), \\ u\cdot\nu = 0 & \text{on } \Sigma_b(t), \\ \partial_t\zeta + u_1\partial_{z_1}\zeta - u_2 = 0 & \text{on } \Sigma(t), \\ \partial_t L = \mathscr{V}\left(\sigma \left.\dfrac{1}{\sqrt{1+|\partial_{z_1}\zeta|^2}}\right|_{z_1=L} - [\gamma]\right), \\ \partial_t R = \mathscr{V}\left([\gamma] - \sigma \left.\dfrac{1}{\sqrt{1+|\partial_{z_1}\zeta|^2}}\right|_{z_1=R}\right), \end{cases} \tag{1.5}$$

for ν the outward-pointing unit normal vector, τ the associated unit tangent vector, and

$$\mathcal{H}(\zeta) := \partial_{z_1}\left(\frac{\partial_{z_1}\zeta}{\sqrt{1+|\partial_{z_1}\zeta|^2}}\right), \tag{1.6}$$

being twice the mean-curvature operator. Note that here we have already shifted the gravitational force to eliminate the atmospheric pressure P_{atm} by adjusting the actual pressure $\bar{P}$ according to $P = \bar{P} + gz_2 - P_{atm}$. Also, it is easy to see that the contact point equations are equivalent to

$$\mathscr{W}(\partial_t L) = \sigma \left.\frac{1}{\sqrt{1+|\partial_{z_1}\zeta|^2}}\right|_{z_1=L} - [\gamma] \text{ and } \mathscr{W}(\partial_t R) = [\gamma] - \sigma\left.\frac{1}{\sqrt{1+|\partial_{z_1}\zeta|^2}}\right|_{z_1=R}. \tag{1.7}$$

The mass of the fluid is conserved in time since the transport equation in (1.5) is equivalent to $\partial_t\zeta = (u\cdot\nu)\sqrt{1+|\partial_{z_1}\zeta|^2}$:

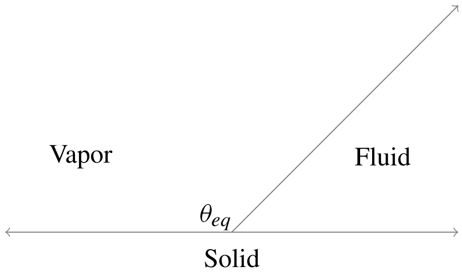

Fig. 2. Equilibrium contact angle very near the contact point.

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Omega(t)| = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{L(t)}^{R(t)} \zeta(t,z_1)\mathrm{d}z_1 = \int\limits_{L(t)}^{R(t)} \partial_t\zeta(t,z_1)\mathrm{d}z_1 + \partial_t L\zeta\Big(t, L(t)\Big) - \partial_t R\zeta\Big(t, R(t)\Big) \tag{1.8}$$

$$= \int\limits_{L(t)}^{R(t)} \partial_t\zeta(t,z_1)\mathrm{d}z_1 = \int\limits_{\Sigma(t)} u\cdot\nu = \int\limits_{\partial\Omega(t)} u\cdot\nu = \int\limits_{\Omega(t)} \nabla_z\cdot u = 0.$$

We denote this conserved mass

$$M = |\Omega(t)| = \int\limits_{L(t)}^{R(t)} \zeta(t,z_1)\mathrm{d}z_1. \tag{1.9}$$

1.1. Background and model discussion

The study of triple interfaces between fluid, solid, and vapor phases is rather old, dating to the work of Young [19], Laplace [13], and Gauss [10] in the nineteenth century. In the subsequent two centuries this problem has attracted the attention of far too many researchers for us to attempt a full survey of the literature here. Instead we refer to the exhaustive survey by de Gennes [6] for a more thorough discussion.

The initial work of Young, Laplace, and Gauss showed that equilibrium configurations not only solve a particular equation, known as the gravity-capillary equation, but also satisfy fixed contact angle conditions determined via

$$\cos(\theta_{eq}) = -\frac{[\gamma]}{\sigma}. \tag{1.10}$$

See Fig. 2 for a schematic. Note in particular that this, together with our sign assumption (1.4) allows for the possibility of describing the fluid-vapor interface by a graph.

The dynamics at a contact point are a much more complicated issue. The first issue to deal with in the context of viscous fluids is that the usual no-slip condition ($u = 0$ at the fluid-solid interface) combines with the free boundary kinematic equation (the sixth equation in (1.5)) to disallow contact point motion, which is obvious nonsense. The no-slip condition must then be replaced with the more general Navier-slip conditions, the fourth and fifth equations in (1.5), which do allow the fluid to slip along the interface at the expense of a dissipative frictional force but do not allow the fluid to penetrate the solid.

Much work has gone into the study of contact point (and line) motion: we refer to the surveys of Dussan [7] and Blake [2] for a thorough discussion of theoretical and experimental studies. What has emerged from these studies is the understanding that deviation of the dynamic contact angle θ_{dyn} from the equilibrium angle θ_{eq} (which we recall is determined via Young's relation (1.10)) causes the contact point to move in an attempt to return to the equilibrium value. More precisely, these quantities are related via

$$V_{cl} = \mathscr{V}(\sigma(\cos(\theta_{dyn}) - \cos(\theta_{eq}))), \tag{1.11}$$

where V_{cl} is the contact point normal velocity and $\mathscr{V}$ is the increasing diffeomorphism such that $\mathscr{V}(0) = 0$, mentioned above. Equations of the form (1.11) have been derived in a number of ways. Blake-Haynes [3] combined thermodynamic and molecular kinetics arguments to arrive at $\mathscr{V}(z) = A\sinh(Bz)$ for material constants $A, B > 0$. Cox [5] used matched asymptotic analysis and hydrodynamic arguments to derive (1.11) with a different $\mathscr{V}$ but of the same general form. Ren-E [14] performed molecular dynamics simulations to probe the physics near the contact point and also found an equation of the form (1.11). Ren-E [15] also derived (1.11) from constitutive equations and thermodynamic principles. The last pair of equations in (1.5) implement (1.11) in the context of the droplet problem.

In recent work, Guo-Tice [9] studied a version of (1.5), coupling the incompressible Stokes equations to the Navier-slip conditions and a contact point equation of the form (1.11), within the context of the fluid evolving inside a vessel. They proved that for data starting sufficiently close to equilibrium (measured in an appropriate Sobolev norm), solutions exist globally in time and decay to equilibrium at an exponential rate. This provides some evidence in support of the model, which was identified in total for the first time by Ren-E [14], as one expects asymptotic stability of equilibria for most physically meaningful models. The purpose of the present paper is to further press the Ren-E model by examining its behavior when used in droplet geometries. These are more complicated than the vessel geometries of [9] since there is an extra degree of freedom corresponding to the motion of the droplet endpoints.

There has also been much prior work devoted to studying contact lines and points in simplified thin-film models; we will not attempt to enumerate these results here and instead refer to the survey by Bertozzi [1]. By contrast, there are relatively few results in the literature related to models in which the full fluid mechanics are considered, and to the best of our knowledge none that allow for both dynamic contact point and dynamic contact angle. Schweizer [16] studied a 2D Navier-Stokes problem with a fixed contact angle of $\pi/2$. Bodea [4] studied a similar problem with fixed $\pi/2$ contact angle in 3D channels with periodicity in one direction. Knüpfer-Masmoudi [12] studied the dynamics of a 2D drop with fixed contact angle when the fluid is assumed to be governed by Darcy's law. Related analysis of the fully stationary Navier-Stokes system with free, but unmoving boundary, was carried out in 2D by Solonnikov [18] with contact angle fixed at π, by Jin [11] in 3D with angle $\pi/2$, and by Socolowsky [17] for 2D coating problems with fixed contact angles.

1.2. Energy-dissipation structure

The system (1.5) has a natural energy-dissipation structure, which we present in the following theorem, the proof of which we postpone to Appendix B.1.

Theorem 1.1. *We have*

$$\frac{d}{dt}\left(\int_L^R \frac{g}{2}|\zeta|^2 + \int_L^R \sigma\sqrt{1+\left|\partial_{z_1}\zeta\right|^2} - [\gamma](R-L)\right) \tag{1.12}$$

$$+\left(\int_{\Omega(t)} \frac{\mu}{2}|\mathbb{D}_z u|^2 + \int_{\Sigma_b(t)} \beta|u\cdot\tau|^2 + \Big(\mathscr{W}(\partial_t L)\partial_t L + \mathscr{W}(\partial_t R)\partial_t R\Big)\right) = 0.$$

Note that in light of the assumption (1.4) we have that $[\gamma] > 0$, and so at first glance it is possible for the energetic term (the term acted on by the time derivative) in Theorem 1.1 to be negative. However, this does not happen due to Young's condition (1.4); indeed,

$$\int_L^R \sigma\sqrt{1+\left|\partial_{z_1}\zeta\right|^2} - [\gamma](R-L) \geq \int_L^R \sigma - [\gamma](R-L) = (\sigma - [\gamma])(R-L) > 0. \tag{1.13}$$

1.3. Equilibrium

Note that in light of Theorem 1.1 any steady state equilibrium solution to (1.5) must satisfy $\zeta(t,z_1) = \zeta_0(z_1)$, $u(t,z) = 0$, $P(t,z) = P_0$, and $L(t) = L_0$, $R(t) = R_0$, with ζ_0 and P_0 satisfying a number of equilibrium conditions. Given such a solution, we define the equilibrium domain to be the set $\Omega_0 = \{(z_1,z_2) : L_0 \leq z_1 \leq R_0,\ \ 0 \leq z_2 \leq \zeta_0(z_1)\}$. Our next result provides for the existence of an equilibrium.

Theorem 1.2. *Assume* (1.4). *Then there exists a smooth equilibrium satisfying* $u = 0$ *and*

$$\begin{cases} g\zeta_0 - \sigma\partial_{z_1}\left(\dfrac{\partial_{z_1}\zeta_0}{\sqrt{1+\left|\partial_{z_1}\zeta_0\right|^2}}\right) = P_0, \\ \left|\partial_{z_1}\zeta_0(L_0)\right| = \left|\partial_{z_1}\zeta_0(R_0)\right| = \dfrac{\sqrt{\sigma^2-[\gamma]^2}}{[\gamma]}, \quad \zeta_0(L_0) = \zeta_0(R_0) = 0, \\ \displaystyle\int_{L_0}^{R_0} \zeta_0(z_1)\mathrm{d}z_1 = M,\ P_0 = C_1\Big(M,g,\sigma,[\gamma]\Big),\ \ R_0 - L_0 = C_2\Big(M,g,\sigma,[\gamma]\Big), \end{cases} \tag{1.14}$$

where $C_1\Big(M,g,\sigma,[\gamma]\Big)$ *and* $C_2\Big(M,g,\sigma,[\gamma]\Big)$ *are positive constants that depends on* M, g, σ, $[\gamma]$. *Moreover, the equilibrium is unique up to a common shift of* L_0 *and* R_0 *and a corresponding translation of* ζ_0.

The proof of this theorem may be found in Appendix B.2. Note that there is a translation invariance in the equilibrium that does not uniquely determine its location on the surface. Once one of the endpoints is selected, though, the equilibrium is unique.

1.4. Geometric reformulation

In order to analyze the PDE system (1.5) we will reformulate the equations in the equilibrium domain Ω_0. The basic idea is to define a time-dependent diffeomorphism transforming the moving domain $\Omega(t)$ into the fixed equilibrium domain Ω_0. We intend to construct a geometric mapping $\Omega_0 \to \Omega(t) : x \to z$. Without loss of generality, we assume the center of the bottom of the equilibrium domain is located at the origin. Let $\ell = \dfrac{R_0 - L_0}{2}$. Then the equilibrium domain is

$$\Omega_0 := \{x = (x_1, x_2) : -\ell < x_1 < \ell, \ \ 0 < x_2 < \zeta_0(x_1)\}. \tag{1.15}$$

Correspondingly, we denote the equilibrium top

$$\Sigma_0 := \{(x_1, x_2) : x_2 = \zeta_0(x_1), -\ell < x_1 < \ell\}, \tag{1.16}$$

the equilibrium bottom

$$\Sigma_{0b} := \{(x_1, x_2) : x_2 = 0, -\ell < x_1 < \ell\}. \tag{1.17}$$

The mapping is constructed as follows.

- Define the mapping from $\tilde{\Omega}(t) = \{y = (y_1, y_2) : -\ell < y_1 < \ell, \ \ 0 < y_2 < \zeta(t, y_1)\}$ to $\Omega(t)$ as

$$\Phi : y_1 \to z_1 = \frac{R(t) - L(t)}{2\ell} y_1 + \frac{R(t) + L(t)}{2}, \quad y_2 \to z_2 = y_2. \tag{1.18}$$

  This is a dilation in the horizontal direction.
- Let the free surface be given as a small perturbation of ζ_0, i.e. $\zeta = \zeta_0 + \eta$ for some $\eta(t, \cdot) : \mathbb{R}^+ \times [-\ell, \ell] \to \mathbb{R}$. Firstly, we extend η from $H^s(L_0, R_0)$ to $H^s(\mathbb{R})$ by means of a standard extension operator $E : H^s(L_0, R_0) \to H^s(\mathbb{R})$, which is bounded for all $0 \le s \le 3$. Then we define the upper harmonic extension of $\eta(t, x_1)$ from $x_2 = 0$ to $x_2 \ge 0$ as $\bar{\eta}(t, x_1, x_2)$ given by

$$\bar{\eta}(t, x) := \mathcal{P}\Big[E[\eta]\Big](t, x_1, \zeta_0(x_1) - x_2) \tag{1.19}$$

  for

$$\mathcal{P} f(z_1, z_2) := \int_{\mathbb{R}} \hat{f}(\xi) e^{-2\pi|\xi|z_2} e^{2\pi i z_1 \xi} d\xi. \tag{1.20}$$

  It is easy to check that $\bar{\eta}\Big(t, x_1, \zeta_0(x_1)\Big) = \mathcal{P}\Big[E[\eta]\Big](t, x_1, 0) = \eta(t, x_1)$, which means that $\eta(t, x_1) = \bar{\eta}(t, x_1, x_2)$ at Σ_0.

- Define the mapping from Ω_0 to $\tilde{\Omega}(t)$ as

$$\Psi : x_1 \to y_1, \quad x_2 \to y_2 = \left(1 + \frac{\bar{\eta}(t, x_1, x_2)}{\zeta_0(x_1)}\right) x_2. \tag{1.21}$$

  This maps $x_2 = \zeta_0(x_1)$ to $y_2 = \zeta(y_1)$ and $x_2 = 0$ to $y_2 = 0$.
- Then the composition $\Pi = \Phi \circ \Psi : \Omega_0 \to \Omega(t)$

$$\Pi : x_1 \to z_1 = \frac{R(t) - L(t)}{2\ell} x_1 + \frac{R(t) + L(t)}{2}, \quad x_2 \to z_2 = \left(1 + \frac{\bar{\eta}(t, x_1, x_2)}{\zeta_0(x_1)}\right) x_2, \tag{1.22}$$

  is the desired geometric mapping.

Write

$$J_1 = \frac{R(t) - L(t)}{2\ell}, \quad K_1 = \frac{1}{J_1}, \quad J_2 = 1 + \frac{\bar{\eta}}{\zeta_0} + \frac{x_2 \partial_2 \bar{\eta}}{\zeta_0}, \quad K_2 = \frac{1}{J_2}, \quad A = \left(\frac{\partial_1 \bar{\eta}}{\zeta_0} - \frac{\bar{\eta} \partial_1 \zeta_0}{\zeta_0^2}\right) x_2, \tag{1.23}$$

and define $K_i = \dfrac{1}{J_i}$ with $i = 1, 2$. Then the Jacobi and transform matrices are then

$$\nabla \Pi = \begin{pmatrix} J_1 & 0 \\ A & J_2 \end{pmatrix} \text{ and } \mathcal{A} = (\nabla \Pi)^{-T} = \begin{pmatrix} K_1 & -AK \\ 0 & K_2 \end{pmatrix}, \tag{1.24}$$

with the Jacobian $J = \det(\nabla \Pi) = J_1 J_2$ and $K = \det(\mathcal{A}) = \dfrac{1}{J} = K_1 K_2$. We will work within a small-energy regime that guarantees that Π is a diffeomorphism and $J, K > 0$.

In the new coordinates, $u(t, x)$, $P(t, x)$, $\zeta(t, x_1)$, $L(t)$, and $R(t)$ satisfy

$$\begin{cases} \nabla_{\mathcal{A}} \cdot S_{\mathcal{A}}(P, u) = 0 & \text{in } \Omega_0, \\ \nabla_{\mathcal{A}} \cdot u = 0 & \text{in } \Omega_0, \\ S_{\mathcal{A}}(P, u)\mathcal{N} = g\zeta\mathcal{N} - \sigma \partial_{\mathcal{A}_1}\left(\dfrac{\partial_{\mathcal{A}_1}\zeta}{\sqrt{1 + \left|\partial_{\mathcal{A}_1}\zeta\right|^2}}\right)\mathcal{N} & \text{on } \Sigma_0, \\ \Big(S_{\mathcal{A}}(P, u)\mathcal{N} - \beta u\Big) \cdot \mathcal{T} = 0 & \text{on } \Sigma_{0b}, \\ u \cdot \mathcal{N} = 0 & \text{on } \Sigma_{0b}, \\ \partial_t \zeta - K_1 \tilde{a} \partial_1 \zeta + u_1 \partial_{\mathcal{A}_1}\zeta - u_2 = 0 & \text{on } \Sigma_0, \\ \mathscr{W}(\partial_t L) = \sigma \left.\dfrac{1}{\sqrt{1 + \left|\partial_{\mathcal{A}_1}\zeta\right|^2}}\right|_{x_1 = -\ell} - [\gamma], & \\ \mathscr{W}(\partial_t R) = [\gamma] - \sigma \left.\dfrac{1}{\sqrt{1 + \left|\partial_{\mathcal{A}_1}\zeta\right|^2}}\right|_{x_1 = \ell}, & \end{cases} \tag{1.25}$$

where here

$$\tilde{a} = \frac{\partial_t R - \partial_t L}{2\ell} x_1 + \frac{\partial_t R + \partial_t L}{2} \tag{1.26}$$

and $\nabla_{\mathcal{A}}$, $\Delta_{\mathcal{A}}$ and $\mathbb{D}_{\mathcal{A}}$ are weighted operators defined as follows with Einstein summation employed

$$(\nabla_{\mathcal{A}} f)_i = \mathcal{A}_{ij}\partial_j f,\ \nabla_{\mathcal{A}} \cdot \vec{g} = \mathcal{A}_{ij}\partial_j g_i,\ \Delta_{\mathcal{A}} f = \nabla_{\mathcal{A}} \cdot \nabla_{\mathcal{A}} f,\ \text{and}\ (\mathbb{D}_{\mathcal{A}} u)_{ij} = \mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i. \tag{1.27}$$

The tensor $S_{\mathcal{A}}(P,u) = PI - \mu\mathbb{D}_{\mathcal{A}} u$ satisfies $\nabla_{\mathcal{A}} \cdot S_{\mathcal{A}}(P,u) = -\mu\Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} P$ whenever u is such that $\nabla_{\mathcal{A}} \cdot u = 0$. Also, $\partial_{\mathcal{A}_1} = K_1 \partial_1$ denotes the weighted derivative in x_1 direction. Moreover, $\mathcal{N} = J\mathcal{A}\nu_0$ and $\mathcal{T} = J\mathcal{A}\tau_0$, where ν_0 and τ_0 are the unit normal and tangential vectors on $\partial\Omega_0$. In particular, on Σ_0,

$$\nu_0 = \frac{(-\partial_1\zeta_0, 1)}{\sqrt{1 + |\partial_1\zeta_0|^2}}, \quad \tau_0 = \frac{(1, \partial_1\zeta_0)}{\sqrt{1 + |\partial_1\zeta_0|^2}}. \tag{1.28}$$

We may directly verify that

$$\mathcal{N} = \frac{(-\partial_1\zeta, J_1)}{\sqrt{1 + |\partial_1\zeta_0|^2}} \ \text{ on } \ \Sigma_0. \tag{1.29}$$

The transport equation can be rewritten as

$$\partial_t\zeta - K_1\tilde{a}\partial_1\zeta = K_1(u\cdot\mathcal{N})\sqrt{1 + |\partial_1\zeta_0|^2}, \tag{1.30}$$

or

$$J_1\partial_t\zeta - \tilde{a}\partial_1\zeta = (u\cdot\mathcal{N})\sqrt{1 + |\partial_1\zeta_0|^2}. \tag{1.31}$$

Note that the mass conservation equation in the new coordinates reads

$$M = \int_{-\ell}^{\ell} J_1\zeta(t, x_1)\mathrm{d}x_1. \tag{1.32}$$

In these new coordinates, there is a natural variant of energy-dissipation structure of Theorem 1.1. For the sake of brevity we will not prove this here but merely state it.

Theorem 1.3. *We have*

$$\partial_t\left(\int_{\Omega_0} \frac{J}{2}|u|^2 + \int_{-\ell}^{\ell} \frac{gJ_1}{2}|\zeta|^2 + \int_{-\ell}^{\ell} \sigma J_1\sqrt{1 + K_1^2|\partial_1\zeta|^2} - [\gamma](R - L)\right) \tag{1.33}$$

$$+\left(\int_{\Omega_0} \frac{\mu J}{2} |\mathbb{D}_{\mathcal{A}} u|^2 + \int_{\Sigma_{0b}} \beta |u \cdot \tau_0|^2 + \Big(\mathscr{W}(\partial_t L)\partial_t L + \mathscr{W}(\partial_t R)\partial_t R\Big)\right) = 0.$$

1.5. Perturbation form

We will consider solutions to the full problem as perturbations around the equilibrium state $(0, P_0, \zeta_0, L_0, R_0)$ given by Theorem 1.1. We will now reformulate the equations in (1.25) in terms of the perturbed unknowns.

Define the perturbation variables

$$p := P - P_0, \quad \eta := \zeta - \zeta_0, \quad l := L + \ell, \quad r := R - \ell. \tag{1.34}$$

Define $\kappa = \mathscr{W}'(0) > 0$, and let

$$\hat{\mathscr{W}}(z) = \frac{1}{\kappa}\mathscr{W}(z) - z. \tag{1.35}$$

Since

$$\partial_{\mathcal{A}_1}\left(\frac{\partial_{\mathcal{A}_1}\zeta}{\sqrt{1+|\partial_{\mathcal{A}_1}\zeta|^2}}\right) = \partial_1\left(\frac{K_1^2\Big(\partial_1\zeta_0 + \partial_1\eta\Big)}{\sqrt{1+K_1^2\Big(\partial_1\zeta_0+\partial_1\eta\Big)^2}}\right), \tag{1.36}$$

we may linearize

$$\partial_{\mathcal{A}_1}\left(\frac{\partial_{\mathcal{A}_1}\zeta}{\sqrt{1+|\partial_{\mathcal{A}_1}\zeta|^2}}\right) = \frac{\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \frac{k_1\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \frac{k_1\partial_1\zeta_0+\partial_1\eta}{\Big(\sqrt{1+|\partial_1\zeta_0|^2}\Big)^3} + \mathcal{R}, \tag{1.37}$$

where

$$k_1 = K_1 - 1 = \frac{2\ell}{R-L} - 1 = -\frac{r-l}{2\ell+r-l}, \tag{1.38}$$

and the nonlinear terms are included in $\mathcal{R}$. See Appendix C.3 for an alternate expansion of $\mathcal{R}$.

Similarly, since

$$\frac{1}{\sqrt{1+|\partial_{\mathcal{A}_1}\zeta|^2}} = \frac{1}{\sqrt{1+K_1^2\Big(\partial_1\zeta_0+\partial_1\eta\Big)^2}}, \tag{1.39}$$

we may linearize

$$\frac{1}{\sqrt{1+|\partial_{\mathcal{A}_1}\zeta|^2}} = \frac{1}{\sqrt{1+|\partial_1\zeta_0|^2}} - \frac{k_1|\partial_1\zeta_0|^2 + \partial_1\zeta_0\partial_1\eta}{\Big(\sqrt{1+|\partial_1\zeta_0|^2}\Big)^3} + \mathcal{Q}, \tag{1.40}$$

where $\mathcal{Q}$ contains the nonlinear terms. Again we refer to Appendix C.3 for an alternate expansion of $\mathcal{Q}$.

For the convenience of linearized energy-dissipation estimates, we intend to connect k_1 with $\tilde{a}$. Define a modification of $\tilde{a}$ as

$$a = \frac{2\ell(\partial_t r - \partial_t l)}{(2\ell + r - l)^2} x_1 + \frac{\partial_t r + \partial_t l}{2}, \tag{1.41}$$

which satisfies the relation $\partial_t k_1 = -\partial_1 a$, and let

$$\mathcal{O} = a - \tilde{a} = -\frac{(r-l)(4\ell + r - l)}{2(2\ell + r - l)^2}(\partial_t r - \partial_t l)x_1 \tag{1.42}$$

represent the nonlinear perturbation terms. We may linearize the transport equation in (1.25)

$$\partial_t \eta - a\partial_1 \zeta_0 = (u \cdot \mathcal{N})\sqrt{1 + |\partial_1 \zeta_0|^2} + \mathcal{S}, \tag{1.43}$$

where $\mathcal{S}$ contains the nonlinear terms. See Appendix C.3 for another expression of $\mathcal{S}$.

Considering the conservation of mass

$$M = \int_{-\ell}^{-\ell} \zeta_0(x_1)\mathrm{d}x_1 = \int_{-\ell}^{\ell} J_1 \zeta(t, x_1)\mathrm{d}x_1 = \int_{-\ell}^{\ell} J_1 \Big(\zeta_0(x_1) + \eta(t, x_1)\Big)\mathrm{d}x_1, \tag{1.44}$$

we have

$$J_1 \int_{-\ell}^{\ell} \eta(t, x_1)\mathrm{d}x_1 = \int_{-\ell}^{-\ell} (1 - J_1)\zeta_0(x_1)\mathrm{d}x_1, \text{ or } \int_{-\ell}^{\ell} \eta(t, x_1)\mathrm{d}x_1 = \int_{-\ell}^{-\ell} (K_1 - 1)\zeta_0(x_1)\mathrm{d}x_1. \tag{1.45}$$

Hence, we know the linearization of mass conservation:

$$\int_{-\ell}^{\ell} \eta = k_1 \int_{-\ell}^{\ell} \zeta_0 = k_1 M. \tag{1.46}$$

Therefore, we can rewrite the system (1.25) in the linearized variables $u(t,x)$, $p(t,x)$ and $\eta(t, x_1)$ with $l(t)$ and $r(t)$:

$$\begin{cases} \nabla_{\mathcal{A}} \cdot S_{\mathcal{A}}(p,u) = 0 & \text{in } \Omega_0, \\ \nabla_{\mathcal{A}} \cdot u = 0 & \text{in } \Omega_0, \\ S_{\mathcal{A}}(p,u)\mathcal{N} = g\eta\mathcal{N} - \sigma\partial_1 \left(\dfrac{k_1 \partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \dfrac{k_1\partial_1\zeta_0 + \partial_1\eta}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} + \mathcal{R} \right)\mathcal{N} & \text{on } \Sigma_0, \\ \Big(S_{\mathcal{A}}(p,u)\mathcal{N} - \beta u \Big) \cdot \mathcal{T} = 0 & \text{on } \Sigma_{0b}, \\ u \cdot \mathcal{N} = 0 & \text{on } \Sigma_{0b}, \\ \partial_t \eta - a\partial_1\zeta_0 = (u\cdot\mathcal{N})\sqrt{1+|\partial_1\zeta_0|^2} + \mathcal{S} & \text{on } \Sigma_0, \\ \kappa\partial_t l + \kappa\hat{\mathscr{W}}(\partial_t l) = \sigma \left(-\dfrac{k_1|\partial_1\zeta_0|^2 + \partial_1\zeta_0\partial_1\eta}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} + \mathcal{Q} \right)\Bigg|_{x_1=-\ell}, \\ \kappa\partial_t r + \kappa\hat{\mathscr{W}}(\partial_t r) = -\sigma \left(-\dfrac{k_1|\partial_1\zeta_0|^2 + \partial_1\zeta_0\partial_1\eta}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} + \mathcal{Q} \right)\Bigg|_{x_1=\ell}. \end{cases} \tag{1.47}$$

2. Main results and discussion

2.1. Energy and dissipation

In order to state our main results we must first define a number of energy and dissipation functionals. We define the basic or parallel (since temporal derivatives are the only ones parallel to the boundary) energy as

$$\mathcal{E}_{\parallel} = \sum_{j=0}^{2} \left\| \partial_t^j \eta \right\|^2_{H^1(-\ell,\ell)}, \tag{2.1}$$

and the basic dissipation as

$$\tilde{\mathcal{D}}_{\parallel} = \sum_{j=0}^{2} \left\| \partial_t^j u \right\|^2_{H^1(\Omega_0)} + \sum_{j=0}^{2} \left\| \partial_t^j u \right\|^2_{H^0(\Sigma_{0b})} + \sum_{j=0}^{2} \left(\left| \partial_t^{j+1} l \right|^2 + \left| \partial_t^{j+1} r \right|^2 \right). \tag{2.2}$$

We also define the improved basic dissipation as

$$\begin{aligned} \mathcal{D}_{\parallel} = \tilde{\mathcal{D}}_{\parallel} &+ \sum_{j=0}^{2} \left\| \partial_t^j p \right\|^2_{H^0(\Omega_0)} + \sum_{j=0}^{2} \left\| \partial_t^j \eta \right\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \\ &+ \sum_{j=0}^{2} \left(\left| \partial_t^j \partial_1 \eta(-\ell) \right|^2 + \left| \partial_t^j \partial_1 \eta(\ell) \right|^2 \right) + \sum_{j=0}^{2} \left(\left| \partial_t^j u(-\ell,0)\cdot\mathcal{N} \right|^2 + \left| \partial_t^j u(\ell,0)\cdot\mathcal{N} \right|^2 \right). \end{aligned} \tag{2.3}$$

The basic energy and dissipation arise through a version of the energy-dissipation relation (1.12). However, once we control these terms we are then able to control much more. This extra control is encoded in the full energy and dissipation, which are defined as follows:

$$\begin{aligned}\mathcal{E} = \mathcal{E}_{\|} &+ \|u\|^2_{W^2_\delta(\Omega_0)} + \|\partial_t u\|^2_{H^1(\Omega_0)} + \|p\|^2_{W^1_\delta(\Omega_0)} + \|\partial_t p\|^2_{H^0(\Omega_0)} \\ &+ \|\eta\|^2_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} + \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} + \sum_{j=0}^{1}\left(\left|\partial_t^{j+1} l(t)\right|^2 + \left|\partial_t^{j+1} r(t)\right|^2\right) \\ &+ \sum_{j=0}^{1}\left(\left|\partial_t^j \partial_1 \eta(-\ell)\right|^2 + \left|\partial_t^j \partial_1 \eta(\ell)\right|^2\right) + \sum_{j=0}^{1}\left(\left|\partial_t^j u(-\ell,0)\cdot\mathcal{N}\right|^2 + \left|\partial_t^j u(\ell,0)\cdot\mathcal{N}\right|^2\right),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\mathcal{D} =&\mathcal{D}_{\|} + \|u\|^2_{W^2_\delta(\Omega_0)} + \|\partial_t u\|^2_{W^2_\delta(\Omega_0)} + \|p\|^2_{W^1_\delta(\Omega_0)} + \|\partial_t p\|^2_{W^1_\delta(\Omega_0)} \\ &+ \|\eta\|^2_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} + \|\partial_t \eta\|^2_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} + \left\|\partial_t^2 \eta\right\|^2_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} + \left\|\partial_t^3 \eta\right\|^2_{W^{\frac{1}{2}}_\delta(-\ell,\ell)}.\end{aligned} \tag{2.5}$$

The spaces W^r_δ are weighted Sobolev spaces, as defined in Appendix A, for a fixed weight parameter $\delta \in (0,1)$.

2.2. Main theorems

Our main result establishes the existence of global-in-time solutions that decay to equilibrium at an exponential rate.

Theorem 2.1. *Fix $0 < \delta < 1$. There exists a universal constant $\vartheta > 0$ such that if $\mathcal{E}(0) \le \vartheta$, then there exists a unique global solution (u, p, η) to (1.47) such that*

$$\begin{aligned}&\sup_{t\ge 0}\Bigg(\mathcal{E}(t) + e^{\lambda t}\Big(\mathcal{E}_{\|}(t) + \|u(t)\|^2_{H^1(\Omega_0)} + \|u(t)\cdot\tau_0\|^2_{H^0(\Sigma_{0b})} + \|p(t)\|^2_{H^0(\Omega_0)} + |\partial_t l(t)|^2 \\ &\quad + \left|\partial_t r(t)^2\right| + |\partial_1\eta(t,-\ell)|^2 + |\partial_1\eta(t,\ell)|^2 + |u(t,-\ell,0)\cdot\mathcal{N}|^2 + |u(t,\ell,0)\cdot\mathcal{N}|^2\Big)\Bigg) \\ &\quad + \int_0^\infty \mathcal{D}(t)\mathrm{d}t \lesssim \mathcal{E}(0).\end{aligned} \tag{2.6}$$

Some remarks are in order. First, the result can be interpreted as saying that the equilibrium constructed in Theorem 1.2 is asymptotically stable in the dynamics (1.47). Second, although the equilibrium droplet is asymptotically stable in the sense described in the theorem (namely, in the fixed coordinate system), its final location in Eulerian coordinates does not have to coincide with where it began. Indeed, let $X = (X_1, X_2)$ be the mass center of the fluid. Then we know

$$\partial_t X = \frac{1}{M} \int_{\Omega_0} J(t) u(t, x) \mathrm{d}x. \tag{2.7}$$

Based on the exponential decay of u, we know there exists $\lambda > 0$ such that

$$\mathrm{e}^{\lambda t} |\partial_t X| \lesssim \mathcal{E}(0). \tag{2.8}$$

Therefore, we know

$$|X(\infty) - X(0)| \lesssim \frac{\mathcal{E}(0)}{\lambda} < \infty. \tag{2.9}$$

In other words, the mass center is shifted for a finite distance. This is consistent with the fact that the equilibrium is only unique once one of its endpoints is fixed. The translation invariance gives a one-parameter family of equilibria, and our result then says that this family is stable.

Third, the theorem is valid for any weight parameter $0 < \delta < 1$. This is in contrast with the result in [9] for the vessel problem, which required δ to be tuned to the equilibrium contact angle. The reason for this is that in the present paper, in order to employ the graph formulation for the droplet we have had to enforce (1.4), which restricts to $\theta_{eq} \in (\pi/2, \pi)$. In this range there are no restrictions placed on $\delta \in (0, 1)$ in the weighted elliptic theory. It is only for acute θ_{eq} that restrictions are needed, as in [9].

Our strategy for proving Theorem 2.1 is in broad strokes the same as that employed for the vessel problem in [9]: we use a nonlinear energy method based on the physical energy-dissipation structure of 1.3, coupled to weighted elliptic estimates in the equilibrium domain. This results in a closed scheme of a priori estimates (Theorems 7.3 and 7.4) that couples to a local existence theory (Theorem 7.5) via a continuation argument to give global decaying solutions.

All of the difficulties (other than the δ restriction issue described above) from the vessel problem carry over to the droplet problem, so we refer to the introduction of [9] for a summary of these and the strategy for dealing with them. There are two principal new difficulties caused by the droplet geometry. The first is due to the extra degree of freedom present in the horizontal motion of the droplet endpoints. In our geometric coordinate system this motion is manifest in the first part of the Jacobian, J_1, due to dilation and contraction of the interval $(L(t), R(t))$. While the dissipation provides control of the time derivatives of L, R, it is not clear from Theorem 1.3 that dissipative control of J_1 is possible. This is analogous to the problem of controlling η with the dissipation when it naturally only controls $\partial_t \eta$. Acquiring dissipative control of J_1 and η, which couple together in a nontrivial way, is one of the key steps in our a priori estimates. The second new difficulty is manifest in the fact that the free surface graph intersects the flat line on which the droplet resides. This causes technical problems with the geometric coefficients near the contact points, and we must resort to delicate analysis to handle these.

The rest of the paper is organized as follows. In Section 3 we develop essential linearized analysis tools. In Section 4 we record various auxiliary estimates that couple directly to the energy-dissipation structure to give enhanced control. In Section 5 we estimate the various nonlinearities appearing in the energy-dissipation structure. For the sake of brevity we have attempted to borrow as many of these as possible from [9], but many terms in the droplet problem do not appear in the vessel problem and require delicate analysis. In Section 6 we estimate the various nonlinear terms appearing in the elliptic estimates. Again, we have borrowed what we could from [9], but much new work was needed. Finally, in Section 7 we synthesize our a priori estimates and complete the proof of Theorem 2.1.

3. Linear analysis

In this section we analyze the problem (1.47) and its time derivatives in linear form. To begin we record the form of (1.47) when time derivatives are applied. Upon doing this we find that $v(t,x) := \partial_t^m u(t,x)$, $q(t,x) := \partial_t^m p(t,x)$, $\varrho(t,x_1) := \partial_t^m \eta(t,x_1)$, $\mathscr{L}(t) := \partial_t^m l(t)$ and $\mathscr{R}(t) := \partial_t^m r(t)$ satisfy

$$\begin{cases} \nabla_{\mathcal{A}} \cdot S_{\mathcal{A}}(q,v) = \mathcal{F}_1 & \text{in } \Omega_0, \\ \nabla_{\mathcal{A}} \cdot v = \mathcal{F}_2 & \text{in } \Omega_0, \\ S_{\mathcal{A}}(q,v)\mathcal{N} = g\varrho\mathcal{N} - \sigma\partial_1\left(\dfrac{\mathscr{K}_1\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \dfrac{\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} + \partial_t^m\mathcal{R}\right)\mathcal{N} + \mathcal{F}_3 & \text{on } \Sigma_0, \\ \left(S_{\mathcal{A}}(q,v)\mathcal{N} - \beta v\right)\cdot\mathcal{T} = \mathcal{F}_4 & \text{on } \Sigma_{0b}, \\ v\cdot\mathcal{N} = 0 & \text{on } \Sigma_{0b}, \\ \partial_t\varrho - \mathfrak{a}\partial_1\zeta_0 = (v\cdot\mathcal{N})\sqrt{1+|\partial_1\zeta_0|^2} + \partial_t^m\mathcal{S} + \mathcal{F}_5 & \text{on } \Sigma_0, \\ \kappa\partial_t\mathscr{L} = \sigma\left(-\dfrac{\mathscr{K}_1|\partial_1\zeta_0|^2 + \partial_1\zeta_0\partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} + \partial_t^m\mathcal{Q}\right)\Bigg|_{-\ell} + \mathcal{F}_6, \\ \kappa\partial_t\mathscr{R} = -\sigma\left(-\dfrac{\mathscr{K}_1|\partial_1\zeta_0|^2 + \partial_1\zeta_0\partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} + \partial_t^m\mathcal{Q}\right)\Bigg|_{\ell} + \mathcal{F}_7, \end{cases} \tag{3.1}$$

where $\mathscr{K}_1 := \partial_t^m k_1$ and $\mathfrak{a} := \partial_t^m a$ satisfying $\partial_1\mathfrak{a} = -\partial_t\mathscr{K}_1$, with the mass conservation

$$\int_{-\ell}^{\ell} \varrho = \mathscr{K}_1 \int_{-\ell}^{\ell} \zeta_0. \tag{3.2}$$

Here, the nonlinear terms $\mathcal{F}_1 - \mathcal{F}_7$ are defined in Appendix C.2.

In what follows we will need the following spaces. We define the time-dependent spaces

$${}_0\mathcal{H}^1(\Omega_0) = \left\{ u : \Omega_0 \to \mathbb{R}^2 \;\middle|\; \int_{\Omega_0} \frac{\mu J}{2} |\mathbb{D}_{\mathcal{A}} u|^2 + \int_{\Sigma_{0b}} \beta |u\cdot\tau_0|^2 < \infty, \;\; u\cdot\mathcal{N} = 0 \text{ on } \Sigma_{0b} \right\}, \tag{3.3}$$

and

$$\mathcal{W}(t) = \left\{ v \in {}_0\mathcal{H}^1(\Omega_0) \mid v\cdot\mathcal{N} \in H^1(-\ell,\ell) \right\}, \tag{3.4}$$

$$\mathcal{V}(t) = \{ v \in \mathcal{W} \mid \nabla_{\mathcal{A}}\cdot v = 0 \}. \tag{3.5}$$

3.1. Weak formulation

We now aim to justify a weak formulation of (3.1).

Lemma 3.1. *Suppose that η is given and $\mathcal{A}$ and $\mathcal{N}$ are determined in terms of η. If $(v, q, \varrho, \mathscr{L}, \mathscr{R})$ are sufficiently regular and satisfy (3.1), then for any $w \in \mathcal{W}(t)$,*

$$\begin{aligned}
&\int_{\Omega_0} \frac{\mu J}{2}\mathbb{D}_{\mathcal{A}} v : \mathbb{D}_{\mathcal{A}} w - \int_{\Omega_0} Jq(\nabla_{\mathcal{A}} \cdot w) + \int_{\Sigma_{0b}} \beta(v\cdot\tau_0)(w\cdot\tau_0) + \int_{\Sigma_0} g\varrho(w\cdot\mathcal{N}) \qquad (3.6)\\
&+ \int_{\Sigma_0} \mathscr{K}_1(P_0 - g\zeta_0)(w\cdot\mathcal{N}) + \int_{\Sigma_0} \sigma\left(\frac{\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3}\right)\partial_1(w\cdot\mathcal{N})\\
&+ \left(-\frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{R}(w\cdot\mathcal{N})\Big|_{\ell} - \frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{L}(w\cdot\mathcal{N})\Big|_{-\ell}\right)\\
&= \int_{\Omega_0} Jw\cdot\mathcal{F}_1 - \int_{\Sigma_0} w\cdot\mathcal{F}_3 - \int_{\Sigma_{0b}} \mathcal{F}_4\left(w\cdot\frac{\mathcal{T}}{|\mathcal{T}|^2}\right) - \int_{\Sigma_0} \sigma\partial_1\partial_t^m\mathcal{R}(w\cdot\mathcal{N})\\
&+ \frac{\sigma}{\partial_1\zeta_0}\partial_t^m\mathcal{Q}(w\cdot\mathcal{N})\Big|_{-\ell}^{\ell} - \frac{1}{\partial_1\zeta_0}\mathcal{F}_7(w\cdot\mathcal{N})\Big|_{\ell} - \frac{1}{\partial_1\zeta_0}\mathcal{F}_6(w\cdot\mathcal{N})\Big|_{-\ell}.
\end{aligned}$$

Proof. Multiplying Jw on both sides of the Stokes equation and integrating over Ω_0 imply

$$\int_{\Omega_0} J\Big(\nabla_{\mathcal{A}}\cdot S_{\mathcal{A}}(q,v)\Big)\cdot w = \int_{\Omega_0} Jw\cdot\mathcal{F}_1.$$

Using the simple identity

$$J\mathcal{A}\nu_0 = \begin{cases} J\nu_0 & \text{on } \Sigma_{0b},\\ \mathcal{N} & \text{on } \Sigma_0,\end{cases} \tag{3.7}$$

and integrating by parts yields

$$\begin{aligned}
\int_{\Omega_0} J\Big(\nabla_{\mathcal{A}}\cdot S_{\mathcal{A}}(q,v)\Big)\cdot w &= \left(\int_{\Omega_0}\frac{\mu J}{2}\mathbb{D}_{\mathcal{A}}v : \mathbb{D}_{\mathcal{A}}w - \int_{\Omega_0} Jq(\nabla_{\mathcal{A}}\cdot w)\right) \qquad (3.8)\\
&+ \int_{\Sigma_{0b}}\Big(S_{\mathcal{A}}(q,v)\mathcal{N}\Big)\cdot w + \int_{\Sigma_0}\Big(S_{\mathcal{A}}(q,v)\mathcal{N}\Big)\cdot w = I_1 + I_2 + I_3.
\end{aligned}$$

We may then simplify

$$
\begin{aligned}
I_2 &= \int_{\Sigma_{0b}} \Big(S_{\mathcal{A}}(q,v)\mathcal{N}\Big)\cdot w \qquad (3.9)\\
&= \int_{\Sigma_{0b}} \Big(\Big(S_{\mathcal{A}}(q,v)\mathcal{N}\Big)\cdot\mathcal{N}\Big)\Big(w\cdot\frac{\mathcal{N}}{|\mathcal{N}|^2}\Big) + \int_{\Sigma_{0b}} \Big(\Big(S_{\mathcal{A}}(q,v)\mathcal{N}\Big)\cdot\mathcal{T}\Big)\Big(w\cdot\frac{\mathcal{T}}{|\mathcal{T}|^2}\Big)\\
&= \int_{\Sigma_{0b}} \Big(\Big(S_{\mathcal{A}}(q,v)\mathcal{N}\Big)\cdot\mathcal{T}\Big)\Big(w\cdot\frac{\mathcal{T}}{|\mathcal{T}|^2}\Big) = \int_{\Sigma_{0b}} \beta\Big(v\cdot\frac{\mathcal{T}}{|\mathcal{T}|}\Big)\Big(w\cdot\frac{\mathcal{T}}{|\mathcal{T}|}\Big) + \int_{\Sigma_{0b}} \mathcal{F}_4\Big(w\cdot\frac{\mathcal{T}}{|\mathcal{T}|^2}\Big)\\
&= \int_{\Sigma_{0b}} \beta(v\cdot\tau_0)(w\cdot\tau_0) + \int_{\Sigma_{0b}} \mathcal{F}_4\Big(w\cdot\frac{\mathcal{T}}{|\mathcal{T}|^2}\Big).
\end{aligned}
$$

On the other hand, we may decompose

$$
I_3 = \int_{\Sigma_0} \Big(S_{\mathcal{A}}(q,v)\mathcal{N}\Big)\cdot w = \int_{\Sigma_0} g\varrho(w\cdot\mathcal{N}) + H + \int_{\Sigma_0} w\cdot\mathcal{F}_3, \qquad (3.10)
$$

where the equilibrium equation (1.14) and integrating by parts yield

$$
\begin{aligned}
H &= -\int_{\Sigma_0} \sigma\,\partial_1\left(\frac{\mathscr{K}_1\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \frac{\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho}{\Big(\sqrt{1+|\partial_1\zeta_0|^2}\Big)^3} + \partial_t^m\mathcal{R}\right)(w\cdot\mathcal{N}) \qquad (3.11)\\
&= \int_{\Sigma_0} \mathscr{K}_1(P_0-g\zeta_0)(w\cdot\mathcal{N}) + \int_{\Sigma_0}\sigma\left(\frac{\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho}{\Big(\sqrt{1+|\partial_1\zeta_0|^2}\Big)^3}\right)\partial_1(w\cdot\mathcal{N}) - \int_{\Sigma_0}\sigma\,\partial_1\partial_t^m\mathcal{R}(w\cdot\mathcal{N})\\
&\quad -\sigma\left(\frac{\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho}{\Big(\sqrt{1+|\partial_1\zeta_0|^2}\Big)^3}\right)(w\cdot\mathcal{N})\Big|_{-\ell}^{\ell}.
\end{aligned}
$$

In particular, using the contact point equation, we have

$$
\begin{aligned}
&-\sigma\left(\frac{\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho}{\Big(\sqrt{1+|\partial_1\zeta_0|^2}\Big)^3}\right)(w\cdot\mathcal{N})\Big|_{-\ell}^{\ell} \qquad (3.12)\\
&= \frac{1}{\partial_1\zeta_0}\Big(-\kappa\,\partial_t\mathscr{R} - \sigma\,\partial_t^m\mathcal{Q} + \mathcal{F}_7\Big)(w\cdot\mathcal{N})\Big|_{\ell} + \frac{1}{\partial_1\zeta_0}\Big(-\kappa\,\partial_t\mathscr{L} + \sigma\,\partial_t^m\mathcal{Q} + \mathcal{F}_6\Big)(w\cdot\mathcal{N})\Big|_{-\ell}
\end{aligned}
$$

$$
\begin{aligned}
&= \bigg(- \frac{\kappa}{\partial_1 \zeta_0} \partial_t \mathscr{R}(w \cdot \mathcal{N}) \Big|_{\ell} - \frac{\kappa}{\partial_1 \zeta_0} \partial_t \mathscr{L}(w \cdot \mathcal{N}) \Big|_{-\ell} \bigg) \\
&\quad + \frac{1}{\partial_1 \zeta_0} \Big(- \sigma \partial_t^m \mathcal{Q} + \mathcal{F}_7 \Big) (w \cdot \mathcal{N}) \Big|_{\ell} + \frac{1}{\partial_1 \zeta_0} \Big(\sigma \partial_t^m \mathcal{Q} + \mathcal{F}_6 \Big) (w \cdot \mathcal{N}) \Big|_{-\ell}.
\end{aligned}
$$

Collecting all of the above terms, we have the weak formulation (3.6). □

3.2. Linearized energy-dissipation structure

We have the following equation for the evolution of the energy of $(v, q, \varrho, \mathscr{L}, \mathscr{R})$:

Theorem 3.2. *Suppose that η is given and $\mathcal{A}$ and $\mathcal{N}$ are determined in terms of η. If $(v, q, \varrho, \mathscr{L}, \mathscr{R})$ satisfy (3.1), we have*

$$
\begin{aligned}
&\partial_t \left(\int_{-\ell}^{\ell} \frac{\sigma}{2} \frac{\Big(\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho \Big)^2}{\Big(\sqrt{1 + |\partial_1 \zeta_0|^2} \Big)^3} + \frac{\mathscr{K}_1^2}{2} \left(P_0 M + \sigma \int_{-\ell}^{\ell} \frac{|\partial_1 \zeta_0|^2}{\sqrt{1 + |\partial_1 \zeta_0|^2}} \right) + \frac{g}{2} \int_{-\ell}^{\ell} (\mathscr{K}_1 \zeta_0 - \varrho)^2 \right) \\
&\quad + \left(\int_{\Omega_0} \frac{J \mu}{2} |\mathbb{D}_{\mathcal{A}} v|^2 + \int_{\Sigma_{0b}} \beta (v \cdot \tau_0)^2 + \kappa \Big((\partial_t \mathscr{L})^2 + (\partial_t \mathscr{R})^2 \Big) \right) \qquad (3.13) \\
&= \int_{\Omega_0} J v \cdot \mathcal{F}_1 + \int_{\Omega_0} J q \mathcal{F}_2 - \int_{\Sigma_0} v \cdot \mathcal{F}_3 - \int_{\Sigma_{0b}} \mathcal{F}_4 \Big(v \cdot \frac{\mathcal{T}}{|\mathcal{T}|^2} \Big) \\
&\quad - \sigma \int_{\Sigma_0} \partial_1 \partial_t^m \mathcal{R} (v \cdot \mathcal{N}) + \bigg(- \sigma \partial_t^m \mathcal{Q} \Big(\partial_t \mathscr{L} + \partial_t^m \mathcal{O} \Big) \Big|_{\ell} + \sigma \partial_t^m \mathcal{Q} \Big(\partial_t \mathscr{R} + \partial_t^m \mathcal{O} \Big) \Big|_{-\ell} \bigg) \\
&\quad - \kappa \bigg(\partial_t \mathscr{R} \partial_t^m \mathcal{O} \Big|_{\ell} + \partial_t \mathscr{L} \partial_t^m \mathcal{O} \Big|_{-\ell} \bigg) - \mathcal{F}_7 \Big(\partial_t \mathscr{R} + \partial_t^m \mathcal{O} \Big) \Big|_{\ell} - \mathcal{F}_6 \Big(\partial_t \mathscr{L} + \partial_t^m \mathcal{O} \Big) \Big|_{-\ell} \\
&\quad + \int_{-\ell}^{\ell} \left(g \varrho - \partial_1 \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0}{\sqrt{1 + |\partial_1 \zeta_0|^2}} + \frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\Big(\sqrt{1 + |\partial_1 \zeta_0|^2} \Big)^3} \right) \right) (\partial_t^m \mathcal{S} + \mathcal{F}_5).
\end{aligned}
$$

Proof. Letting $w = v$ in the weak formulation (3.6), we know

$$
\begin{aligned}
&\int_{\Omega_0} \frac{\mu J}{2} |\mathbb{D}_{\mathcal{A}} v|^2 + \int_{\Sigma_{0b}} \beta |v \cdot \tau_0|^2) + \int_{\Sigma_0} g \varrho (v \cdot \mathcal{N}) \qquad (3.14) \\
&\quad + \int_{\Sigma_0} \mathscr{K}_1 (p_0 - g \zeta_0)(v \cdot \mathcal{N}) + \int_{\Sigma_0} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\Big(\sqrt{1 + |\partial_1 \zeta_0|^2} \Big)^3} \right) \partial_1 (v \cdot \mathcal{N}) \\
&\quad + \bigg(- \frac{\kappa}{\partial_1 \zeta_0} \partial_t \mathscr{R}(v \cdot \mathcal{N}) \Big|_{\ell} - \frac{\kappa}{\partial_1 \zeta_0} \partial_t \mathscr{L}(v \cdot \mathcal{N}) \Big|_{-\ell} \bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Omega_0} J v \cdot \mathcal{F}_1 + \int_{\Omega_0} J q \mathcal{F}_2 - \int_{\Sigma_0} v \cdot \mathcal{F}_3 - \int_{\Sigma_{0b}} \mathcal{F}_4 \Big(v \cdot \frac{\mathcal{T}}{|\mathcal{T}|^2} \Big) - \int_{\Sigma_0} \sigma \partial_1 \partial_t^m \mathcal{R}(v \cdot \mathcal{N}) \\
&\quad + \frac{\sigma}{\partial_1 \zeta_0} \partial_t^m \mathcal{Q}(v \cdot \mathcal{N}) \Big|_{-\ell}^{\ell} - \frac{1}{\partial_1 \zeta_0} \mathcal{F}_7 (v \cdot \mathcal{N}) \Big|_{\ell} - \frac{1}{\partial_1 \zeta_0} \mathcal{F}_6 (v \cdot \mathcal{N}) \Big|_{-\ell}.
\end{aligned}
$$

We will focus on the simplification of gravitational, surface tension, and contact point terms. We may directly verify the relation

$$
\partial_1 \mathfrak{a} = -\partial_t \mathscr{K}_1, \tag{3.15}
$$

which will be used frequently in the following.

Step 1: Gravitational term: We may decompose

$$
\begin{aligned}
G &:= \int_{\Sigma_0} g \varrho (v \cdot \mathcal{N}) = \int_{-\ell}^{\ell} g \varrho (v \cdot \mathcal{N}) \sqrt{1 + |\partial_1 \zeta_0|^2} = \int_{-\ell}^{\ell} g \varrho \Big(\partial_t \varrho - \mathfrak{a} \partial_1 \zeta_0 - \partial_t^m \mathcal{S} - \mathcal{F}_5 \Big) \qquad (3.16) \\
&= G_1 + G_2 - \int_{-\ell}^{\ell} g \varrho (\partial_t^m \mathcal{S} + \mathcal{F}_5).
\end{aligned}
$$

Integration by parts and using (3.15), we obtain

$$
G_1 = \int_{-\ell}^{\ell} g \varrho \partial_t \varrho = \partial_t \left(\int_{-\ell}^{\ell} \frac{g}{2} |\varrho|^2 \right), \tag{3.17}
$$

$$
G_2 = -\int_{-\ell}^{\ell} g \varrho \mathfrak{a} \partial_1 \zeta_0 = \int_{-\ell}^{\ell} g \mathfrak{a} \zeta_0 \partial_1 \varrho + \int_{-\ell}^{\ell} g \zeta_0 \varrho \partial_1 \mathfrak{a} = \int_{-\ell}^{\ell} g \mathfrak{a} \zeta_0 \partial_1 \varrho - \int_{-\ell}^{\ell} g \partial_t \mathscr{K}_1 \zeta_0 \varrho = A_1 + A_2. \tag{3.18}
$$

We cannot simplify A_1 and A_2 at this stage and have to wait until the surface tension terms.

Step 2: Surface tension term – First stage: The transport equation in (3.1) and integrating by parts yield

$$
\begin{aligned}
H_1 &:= \int_{\Sigma_0} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1 + |\partial_1 \zeta_0|^2} \right)^3} \right) \partial_1 (v \cdot \mathcal{N}) \qquad (3.19) \\
&= \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1 + |\partial_1 \zeta_0|^2} \right)^3} \right) \partial_1 \Big(\partial_t \varrho - \mathfrak{a} \partial_1 \zeta_0 - \partial_t^m \mathcal{S} - \mathcal{F}_5 \Big)
\end{aligned}
$$

$$= \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \partial_1 \left(\partial_t \varrho - \mathfrak{a} \partial_1 \zeta_0 \right) - \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \partial_1 (\partial_t^m \mathcal{S} + \mathcal{F}_5).$$

We may use (3.15) and (1.14) to simplify

$$\begin{aligned}
&\int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \partial_1 \left(\partial_t \varrho - \mathfrak{a} \partial_1 \zeta_0 \right) \\
&= \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \left(\partial_t \partial_1 \varrho - \partial_1 \mathfrak{a} \partial_1 \zeta_0 - \mathfrak{a} \partial_{11} \zeta_0 \right) \\
&= \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \left((\partial_t \partial_1 \varrho + \partial_t \mathscr{K}_1 \partial_1 \zeta_0) - \mathfrak{a} \partial_{11} \zeta_0 \right) = B_1 + B_2,
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \left(\partial_t \partial_1 \varrho + \partial_t \mathscr{K}_1 \partial_1 \zeta_0 \right) \\
&= \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \partial_t \left(\partial_1 \varrho + \mathscr{K}_1 \partial_1 \zeta_0 \right) = \partial_t \left(\int_{-\ell}^{\ell} \frac{\sigma}{2} \frac{\left(\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho \right)^2}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right),
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
B_2 &= -\int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \mathfrak{a} \partial_{11} \zeta_0 \\
&= -\int_{-\ell}^{\ell} \sigma \mathfrak{a} (\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho) \partial_1 \left(\frac{\partial_1 \zeta_0}{\sqrt{1+|\partial_1 \zeta_0|^2}} \right) \\
&= \int_{-\ell}^{\ell} \mathfrak{a} (\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho)(P_0 - g \zeta_0) = C_1 + C_2 + C_3 + C_4.
\end{aligned} \tag{3.21}$$

We may use (3.15) to directly compute

$$C_1 = \mathscr{K}_1 P_0 \int_{-\ell}^{\ell} \mathfrak{a}\partial_1\zeta_0 = -\mathscr{K}_1 P_0 \int_{-\ell}^{\ell} \partial_1\mathfrak{a}\zeta_0 = \mathscr{K}_1 P_0 \int_{-\ell}^{\ell} \partial_t\mathscr{K}_1\zeta_0 = \partial_t\left(\frac{P_0 M}{2}\mathscr{K}_1^2\right), \tag{3.22}$$

$$C_2 = -\mathscr{K}_1 g \int_{-\ell}^{\ell} \mathfrak{a}\zeta_0\partial_1\zeta_0 = -\frac{\mathscr{K}_1 g}{2}\int_{-\ell}^{\ell} \mathfrak{a}\partial_1|\zeta_0|^2 = \frac{\mathscr{K}_1 g}{2}\int_{-\ell}^{\ell} \partial_1\mathfrak{a}\zeta_0^2 \tag{3.23}$$
$$= -\frac{\mathscr{K}_1 g}{2}\int_{-\ell}^{\ell} \partial_t\mathscr{K}_1\zeta_0^2 = -\partial_t\left(\frac{\mathscr{K}_1^2 g}{4}\int_{-\ell}^{\ell}\zeta_0^2\right),$$

and

$$C_3 = P_0\int_{-\ell}^{\ell}\mathfrak{a}\partial_1\varrho = -P_0\int_{-\ell}^{\ell}\partial_1\mathfrak{a}\varrho = P_0\int_{-\ell}^{\ell}\partial_t\mathscr{K}_1\varrho = P_0\mathscr{K}_1\partial_t\mathscr{K}_1\int_{-\ell}^{\ell}\zeta_0 = \partial_t\left(\frac{P_0 M}{2}\mathscr{K}_1^2\right), \tag{3.24}$$

which means

$$C_1 + C_3 = \partial_t\Big(P_0 M\mathscr{K}_1^2\Big) \text{ and } C_4 = -\int_{-\ell}^{\ell} g\mathfrak{a}\zeta_0\partial_1\varrho, \tag{3.25}$$

and so

$$A_1 + C_4 = \int_{-\ell}^{\ell} g\mathfrak{a}\zeta_0\partial_1\varrho - \int_{-\ell}^{\ell} g\mathfrak{a}\zeta_0\partial_1\varrho = 0. \tag{3.26}$$

Step 3: Surface tension term – Second stage: The transport equation in (3.1) and integrating by parts imply

$$H_2 := \int_{\Sigma_0}\mathscr{K}_1(P_0 - g\zeta_0)(v\cdot\mathcal{N}) = \int_{-\ell}^{\ell}\mathscr{K}_1(P_0 - g\zeta_0)\Big(\partial_t\varrho - \mathfrak{a}\partial_1\zeta_0 - \partial_t^m\mathcal{S} - \mathcal{F}_5\Big) \tag{3.27}$$
$$= D_1 + D_2 + D_3 + D_4 - \int_{-\ell}^{\ell}\mathscr{K}_1(P_0 - g\zeta_0)\Big(\partial_t^m\mathcal{S} + \mathcal{F}_5\Big),$$

where by (3.15),

$$D_1 = \mathscr{K}_1 P_0\int_{-\ell}^{\ell}\partial_t\varrho = \mathscr{K}_1 P_0\left(\partial_t\mathscr{K}_1\int_{-\ell}^{\ell}\zeta_0\right) = \partial_t\left(\frac{P_0 M}{2}\mathscr{K}_1^2\right), \tag{3.28}$$

$$D_2 = -g\mathscr{K}_1\int_{-\ell}^{\ell}\zeta_0\partial_t\varrho = -g\mathscr{K}_1\partial_t\left(\int_{-\ell}^{\ell}\zeta_0\varrho\right), \tag{3.29}$$

which means

$$A_2 + D_2 = -g\partial_t\mathscr{K}_1\left(\int_{-\ell}^{\ell}\zeta_0\varrho\right) - g\mathscr{K}_1\partial_t\left(\int_{-\ell}^{\ell}\zeta_0\varrho\right) = -\partial_t\left(g\mathscr{K}_1\int_{-\ell}^{\ell}\zeta_0\varrho\right), \tag{3.30}$$

$$D_3 = -\mathscr{K}_1P_0\int_{-\ell}^{\ell}\mathfrak{a}\partial_1\zeta_0 = \mathscr{K}_1P_0\int_{-\ell}^{\ell}\partial_1\mathfrak{a}\zeta_0 = -P_0\mathscr{K}_1\int_{-\ell}^{\ell}\partial_t\mathscr{K}_1\zeta_0 = -\partial_t\left(\frac{P_0M}{2}\mathscr{K}_1^2\right), \tag{3.31}$$

from which we see that

$$D_1 + D_3 = 0, \tag{3.32}$$

and

$$\begin{aligned} D_4 =& g\mathscr{K}_1\int_{-\ell}^{\ell}\zeta_0\mathfrak{a}\partial_1\zeta_0 = \frac{g\mathscr{K}_1}{2}\int_{-\ell}^{\ell}\mathfrak{a}\partial_1|\zeta_0|^2 = -\frac{g\mathscr{K}_1}{2}\int_{-\ell}^{\ell}\partial_1\mathfrak{a}\zeta_0^2 \\ =& \frac{g\mathscr{K}_1}{2}\int_{-\ell}^{\ell}\partial_t\mathscr{K}_1\zeta_0^2 = \partial_t\left(\frac{\mathscr{K}_1^2 g}{4}\int_{-\ell}^{\ell}\zeta_0^2\right), \end{aligned} \tag{3.33}$$

which in turn implies that

$$C_2 + D_4 = 0. \tag{3.34}$$

In summary, we have

$$\begin{aligned} G + H_1 + H_2 =& \partial_t\left(\int_{-\ell}^{\ell}\frac{g}{2}|\varrho|^2 + P_0M\mathscr{K}_1^2 - g\mathscr{K}_1\int_{-\ell}^{\ell}\zeta_0\varrho\right) - \int_{-\ell}^{\ell}g\varrho(\partial_t^m\mathcal{S} + \mathcal{F}_5) \\ &- \int_{-\ell}^{\ell}\sigma\left(\frac{\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3}\right)\partial_1(\partial_t^m\mathcal{S} + \mathcal{F}_5) - \int_{-\ell}^{\ell}\mathscr{K}_1(P_0 - g\zeta_0)\left(\partial_t^m\mathcal{S} + \mathcal{F}_5\right). \end{aligned} \tag{3.35}$$

Step 4: Contact point term – First stage: Note that $\partial_t\varrho(-\ell) = \partial_t\varrho(\ell) = 0$, and $\mathfrak{a}(-\ell) = \partial_t\mathscr{L} + \partial_t^m\mathcal{O}(-\ell)$, $\mathfrak{a}(\ell) = \partial_t\mathscr{R} + \partial_t^m\mathcal{O}(\ell)$. Using these, we have

$$
\begin{aligned}
&-\frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{R}(v\cdot\mathcal{N})\Big|_{\ell}-\frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{L}(v\cdot\mathcal{N})\Big|_{-\ell} \\
&=-\frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{R}\Big(\partial_t\varrho-\mathfrak{a}\partial_1\zeta_0-\partial_t^m\mathcal{S}-\mathcal{F}_5\Big)\Big|_{\ell}-\frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{L}\Big(\partial_t\varrho-\mathfrak{a}\partial_1\zeta_0-\partial_t^m\mathcal{S}-\mathcal{F}_5\Big)\Big|_{-\ell} \\
&=\kappa\Big((\partial_t\mathscr{L})^2+(\partial_t\mathscr{R})^2\Big)+\kappa\Big(\partial_t\mathscr{R}\partial_t^m\mathcal{O}\Big|_{\ell}+\partial_t\mathscr{L}\partial_t^m\mathcal{O}\Big|_{-\ell}\Big) \\
&\quad+\Big(\frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{R}\Big(\partial_t^m\mathcal{S}+\mathcal{F}_5\Big)\Big|_{\ell}+\frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{L}\Big(\partial_t^m\mathcal{S}+\mathcal{F}_5\Big)\Big|_{-\ell}\Big).
\end{aligned}
\tag{3.36}
$$

Step 5: Contact point term – Second stage: Using the transport equation in (3.1), we have

$$
\begin{aligned}
&\frac{\sigma}{\partial_1\zeta_0}\partial_t^m\mathcal{Q}(v\cdot\mathcal{N})\Big|_{-\ell}^{\ell}=\frac{\sigma}{\partial_1\zeta_0}\partial_t^m\mathcal{Q}\Big(\partial_t\varrho-\mathfrak{a}\partial_1\zeta_0-\partial_t^m\mathcal{S}-\mathcal{F}_5\Big)\Big|_{-\ell}^{\ell} \\
&=\Big(-\sigma\partial_t^m\mathcal{Q}\Big(\partial_t\mathscr{L}+\partial_t^m\mathcal{O}\Big)\Big|_{\ell}+\sigma\partial_t^m\mathcal{Q}\Big(\partial_t\mathscr{R}+\partial_t^m\mathcal{O}\Big)\Big|_{-\ell}\Big)-\frac{\sigma}{\partial_1\zeta_0}\partial_t^m\mathcal{Q}\Big(\partial_t^m\mathcal{S}+\mathcal{F}_5\Big)\Big|_{-\ell}^{\ell},
\end{aligned}
\tag{3.37}
$$

and

$$
\begin{aligned}
&-\frac{1}{\partial_1\zeta_0}\mathcal{F}_7(v\cdot\mathcal{N})\Big|_{\ell}-\frac{1}{\partial_1\zeta_0}\mathcal{F}_6(v\cdot\mathcal{N})\Big|_{-\ell} \\
&=-\frac{1}{\partial_1\zeta_0}\mathcal{F}_7\Big(\partial_t\varrho-\mathfrak{a}\partial_1\zeta_0-\partial_t^m\mathcal{S}-\mathcal{F}_5\Big)\Big|_{\ell}-\frac{1}{\partial_1\zeta_0}\mathcal{F}_6\Big(\partial_t\varrho-\mathfrak{a}\partial_1\zeta_0-\partial_t^m\mathcal{S}-\mathcal{F}_5\Big)\Big|_{-\ell} \\
&=\mathcal{F}_7\Big(\partial_t\mathscr{R}+\partial_t^m\mathcal{O}\Big)\Big|_{\ell}+\mathcal{F}_6\Big(\partial_t\mathscr{L}+\partial_t^m\mathcal{O}\Big)\Big|_{-\ell}+\frac{1}{\partial_1\zeta_0}\mathcal{F}_7\Big(\partial_t^m\mathcal{S}+\mathcal{F}_5\Big)\Big|_{\ell} \\
&\quad+\frac{1}{\partial_1\zeta_0}\mathcal{F}_6\Big(\partial_t^m\mathcal{S}+\mathcal{F}_5\Big)\Big|_{-\ell}.
\end{aligned}
\tag{3.38}
$$

Combining the terms related to $\partial_t^m\mathcal{S}+\mathcal{F}_5$, and using the equilibrium equation (1.14), we can get (3.13). □

4. Basic estimates

In this section we record a number of essential estimates needed for the nonlinear analysis of (1.47). These include auxiliary estimates for the pressure, free surface function, and contact points, as well as elliptic estimates for the Stokes problem.

4.1. Pressure estimates

The linearized energy-dissipation structure cannot control the pressure directly, so we need a separate argument. It is well-known that the pressure can be regarded as the Lagrangian multiplier in the fluid equation, and that this gives rise to a pressure estimate. Here we will employ estimates proved in [9], which we restate without proof here.

Theorem 4.1. *If (v, ϱ) satisfies*

$$\begin{aligned}
&\int_{\Omega_0} \frac{\mu J}{2} \mathbb{D}_{\mathcal{A}} v : \mathbb{D}_{\mathcal{A}} w + \int_{\Sigma_{0b}} \beta (v \cdot \tau_0)(w \cdot \tau_0) + \int_{\Sigma_0} g \varrho (w \cdot \mathcal{N}) \\
&+ \int_{\Sigma_0} \mathscr{K}_1 (p_0 - g\zeta_0)(w \cdot \mathcal{N}) + \int_{\Sigma_0} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1 + |\partial_1 \zeta_0|^2}\right)^3} \right) \partial_1 (w \cdot \mathcal{N}) \\
&+ \left(- \frac{\kappa}{\partial_1 \zeta_0} \partial_t \mathscr{R} (w \cdot \mathcal{N}) \Big|_{\ell} - \frac{\kappa}{\partial_1 \zeta_0} \partial_t \mathscr{L} (w \cdot \mathcal{N}) \Big|_{-\ell} \right) \\
= &\int_{\Omega_0} J w \cdot \mathcal{F}_1 - \int_{\Sigma_0} w \cdot \mathcal{F}_3 - \int_{\Sigma_{0b}} \mathcal{F}_4 \left(w \cdot \frac{\mathcal{T}}{|\mathcal{T}|^2} \right) - \int_{\Sigma_0} \sigma \partial_1 \partial_t^m \mathcal{R} (w \cdot \mathcal{N}) \\
&+ \frac{\sigma}{\partial_1 \zeta_0} \partial_t^m \mathcal{Q} (w \cdot \mathcal{N}) \Big|_{-\ell}^{\ell} - \frac{1}{\partial_1 \zeta_0} \mathcal{F}_7 (w \cdot \mathcal{N}) \Big|_{\ell} - \frac{1}{\partial_1 \zeta_0} \mathcal{F}_6 (w \cdot \mathcal{N}) \Big|_{-\ell},
\end{aligned} \tag{4.1}$$

for all $w \in \mathcal{V}(t)$, then there exists a unique $q \in \mathring{H}^0(\Omega_0)$ such that

$$\begin{aligned}
&\int_{\Omega_0} \frac{\mu J}{2} \mathbb{D}_{\mathcal{A}} v : \mathbb{D}_{\mathcal{A}} w - \int_{\Omega_0} J q (\nabla_{\mathcal{A}} \cdot w) + \int_{\Sigma_{0b}} \beta (v \cdot \tau_0)(w \cdot \tau_0) + \int_{\Sigma_0} g \varrho (w \cdot \mathcal{N}) \\
&+ \int_{\Sigma_0} \mathscr{K}_1 (p_0 - g\zeta_0)(w \cdot \mathcal{N}) + \int_{\Sigma_0} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1 + |\partial_1 \zeta_0|^2}\right)^3} \right) \partial_1 (w \cdot \mathcal{N}) \\
&+ \left(- \frac{\kappa}{\partial_1 \zeta_0} \partial_t \mathscr{R} (w \cdot \mathcal{N}) \Big|_{\ell} - \frac{\kappa}{\partial_1 \zeta_0} \partial_t \mathscr{L} (w \cdot \mathcal{N}) \Big|_{-\ell} \right) \\
= &\int_{\Omega_0} J w \cdot \mathcal{F}_1 - \int_{\Sigma_0} w \cdot \mathcal{F}_3 - \int_{\Sigma_{0b}} \mathcal{F}_4 \left(w \cdot \frac{\mathcal{T}}{|\mathcal{T}|^2} \right) - \int_{\Sigma_0} \sigma \partial_1 \partial_t^m \mathcal{R} (w \cdot \mathcal{N}) \\
&+ \frac{\sigma}{\partial_1 \zeta_0} \partial_t^m \mathcal{Q} (w \cdot \mathcal{N}) \Big|_{-\ell}^{\ell} - \frac{1}{\partial_1 \zeta_0} \mathcal{F}_7 (w \cdot \mathcal{N}) \Big|_{\ell} - \frac{1}{\partial_1 \zeta_0} \mathcal{F}_6 (w \cdot \mathcal{N}) \Big|_{-\ell},
\end{aligned} \tag{4.2}$$

for all $w \in \mathcal{W}(t)$. Moreover,

$$\|q\|_{H^0(\Omega_0)}^2 \lesssim \|v\|_{H^1(\Omega_0)}^2 + \|\mathcal{F}\|_{(H^1)^*}^2, \tag{4.3}$$

where $\mathcal{F} \in (H^1)^*$ is given by

$$\langle \mathcal{F}, w \rangle = \int_{\Omega_0} J w \cdot \mathcal{F}_1 - \int_{\Sigma_0} w \cdot \mathcal{F}_3 - \int_{\Sigma_{0b}} \mathcal{F}_4 \left(w \cdot \frac{\mathcal{T}}{|\mathcal{T}|^2} \right). \tag{4.4}$$

Proof. See the proof of [9, Theorem 4.6]. □

4.2. Free surface estimates

In this section, we prove various estimates regarding the free surface function. We begin with a useful inequality.

Lemma 4.2. *Suppose that $\varrho \in H_0^1(-\ell,\ell)$ and $\mathscr{K}_1 \in \mathbb{R}$ satisfy (3.2). Then we have the norm equivalence*

$$\|\partial_1\varrho\|_{H^0(-\ell,\ell)} \lesssim \|\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho\|_{H^0(-\ell,\ell)} \lesssim \|\partial_1\varrho\|_{H^0(-\ell,\ell)}. \tag{4.5}$$

Proof. Assume that the first inequality is not true. Then we can find a sequence $\{(\mathscr{K}_1^n, \varrho^n)\}_{n=1}^\infty \subset \mathbb{R}\times H_0^1(-\ell,\ell)$ such that

$$\mathscr{K}_1^n \int_{-\ell}^{\ell} \zeta_0(x_1)\mathrm{d}x_1 = \int_{-\ell}^{\ell} \varrho^n(x_1)\mathrm{d}x_1, \tag{4.6}$$

with

$$\left\|\partial_1\varrho^n\right\|_{H^0(-\ell,\ell)} = 1 \text{ and } \left\|\mathscr{K}_1^n\partial_1\zeta_0 + \partial_1\varrho^n\right\|_{H^0(-\ell,\ell)} \le \frac{1}{n}. \tag{4.7}$$

Then by Poincaré's inequality and the relation (3.2), we have that

$$\left|\mathscr{K}_1^n\right| = \left|\int_{-\ell}^{\ell} \varrho^n(x_1)\mathrm{d}x_1\right|\left|\int_{-\ell}^{\ell} \zeta_0(x_1)\mathrm{d}x_1\right|^{-1} \lesssim \left\|\varrho^n\right\|_{H^0(-\ell,\ell)} \lesssim \left\|\partial_1\varrho^n\right\|_{H^0(-\ell,\ell)} \lesssim 1. \tag{4.8}$$

Hence, we know $\mathscr{K}_1^n$ is bounded and up to the extraction a subsequence, we may assume $\mathscr{K}_1^n \to \mathscr{K}_1$. On the other hand, by Poincaré's inequality, we may estimate

$$\left\|\mathscr{K}_1^n\zeta_0 + \varrho^n\right\|_{H^1(-\ell,\ell)} \lesssim \left\|\partial_1\left(\mathscr{K}_1^n\zeta_0 + \varrho^n\right)\right\|_{H^0(-\ell,\ell)} \lesssim \frac{1}{n}. \tag{4.9}$$

Hence, $\mathscr{K}_1^n\zeta_0 + \varrho^n \to 0$ in H^1, but since $\mathscr{K}_1^n \to \mathscr{K}_1$, we conclude that $\varrho^n \to \varrho$ in H^1. Sending $n\to\infty$ in (4.6) and using (3.2), we conclude that

$$\mathscr{K}_1 \int_{-\ell}^{\ell} \zeta_0(x_1)\mathrm{d}x_1 = \int_{-\ell}^{\ell} \varrho(x_1)\mathrm{d}x_1 = -\mathscr{K}_1 \int_{-\ell}^{\ell} \zeta_0(x_1)\mathrm{d}x_1. \tag{4.10}$$

Hence, $\mathscr{K}_1=0$ and $\varrho=0$. This contradicts the fact that $\|\partial_1\varrho\|_{H^0(-\ell,\ell)} = \lim_{n\to\infty}\|\partial_1\varrho^n\|_{H^0(-\ell,\ell)} = 1$, which proves the first inequality. On the other hand, the second inequality follows trivially from (3.2). □

Next, we prove an elliptic estimate for the free surface function.

Lemma 4.3. *Suppose that $\varrho \in H_0^1(-\ell,\ell)$ and $\mathscr{K}_1 \in \mathbb{R}$ satisfy (3.2), and*

$$\int_{-\ell}^{\ell} g\varrho\theta + \int_{-\ell}^{\ell} \mathscr{K}_1(P_0 - g\zeta_0)\theta + \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right) \partial_1\theta = \langle \mathcal{F}, \theta \rangle , \tag{4.11}$$

for any $\theta \in H_0^1(-\ell,\ell)$. If $\mathcal{F} \in H^{-s}(-\ell,\ell)$ for $s \in [0,1]$, then $\varrho \in H_0^{2-s}(-\ell,\ell)$ and

$$\|\varrho\|_{H^{2-s}(-\ell,\ell)} + |\mathscr{K}_1| \lesssim \|\mathcal{F}\|_{H^{-s}(-\ell,\ell)} . \tag{4.12}$$

Proof. We begin with an H^1 estimate for ϱ. We rearrange the first two terms of (4.11) to see the structure

$$\int_{-\ell}^{\ell} g\Big(\varrho - \mathscr{K}_1\zeta_0\Big)\theta + P_0\mathscr{K}_1 \int_{-\ell}^{\ell} \theta + \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right) \partial_1\theta = \langle \mathcal{F}, \theta \rangle . \tag{4.13}$$

Then we take the test function $\theta = \mathscr{K}_1\zeta_0 + \varrho$ and use Lemma 4.2 to obtain

$$\begin{aligned} \int_{-\ell}^{\ell} \sigma \left(\frac{\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right) \partial_1\theta &= \int_{-\ell}^{\ell} \sigma \frac{|\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho|^2}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \\ &\gtrsim \|\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho\|_{H^0(-\ell,\ell)}^2 \gtrsim \|\varrho\|_{H^1(-\ell,\ell)}^2 . \end{aligned} \tag{4.14}$$

On the other hand,

$$\begin{aligned} &\int_{-\ell}^{\ell} g\Big(\varrho - \mathscr{K}_1\zeta_0\Big)\theta + P_0\mathscr{K}_1 \int_{-\ell}^{\ell} \theta = g\int_{-\ell}^{\ell} \Big(\varrho - \mathscr{K}_1\zeta_0\Big)\Big(\varrho + \mathscr{K}_1\zeta_0\Big) + P_0\mathscr{K}_1 \int_{-\ell}^{\ell} \Big(\varrho + \mathscr{K}_1\zeta_0\Big) \\ &= g\int_{-\ell}^{\ell} \Big(\varrho^2 - \mathscr{K}_1^2\zeta_0^2\Big) + P_0\mathscr{K}_1 \int_{-\ell}^{\ell} \varrho + P_0\mathscr{K}_1^2 \int_{-\ell}^{\ell} \zeta_0 = g\int_{-\ell}^{\ell} |\varrho|^2 - g\mathscr{K}_1^2 \int_{-\ell}^{\ell} \zeta_0^2 + 2P_0\mathscr{K}_1^2 M \\ &= g\int_{-\ell}^{\ell} |\varrho|^2 + \mathscr{K}_1^2\left(2P_0M - g\int_{-\ell}^{\ell} \zeta_0^2\right). \end{aligned} \tag{4.15}$$

Consider the second equation in (1.14). Multiplying by ζ_0 and integrating, we obtain the identity

$$P_0M - g\int_{-\ell}^{\ell} \zeta_0^2 = \sigma \int_{-\ell}^{\ell} \frac{|\partial_1\zeta_0|^2}{\sqrt{1+|\partial_1\zeta_0|^2}}. \tag{4.16}$$

Therefore, since $P_0M > 0$, we may use Lemma 4.2 to derive

$$g\int_{-\ell}^{\ell}\varrho\theta+\int_{-\ell}^{\ell}\mathscr{K}_1\Big(P_0-g\zeta_0\Big)\theta+\sigma\int_{-\ell}^{\ell}\left(\frac{\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3}\right)\partial_1\theta \tag{4.17}$$
$$=\sigma\int_{-\ell}^{\ell}\frac{|\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho|^2}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3}+g\int_{-\ell}^{\ell}|\varrho|^2+\mathscr{K}_1^2\left(P_0M+\sigma\int_{-\ell}^{\ell}\frac{|\partial_1\zeta_0|^2}{\sqrt{1+|\partial_1\zeta_0|^2}}\right)$$
$$\gtrsim\|\varrho\|_{H^1(-\ell,\ell)}^2+\mathscr{K}_1^2.$$

Also, we use Lemma 4.2 to directly estimate

$$\begin{aligned}|\langle\mathcal{F},\theta\rangle|&\lesssim\|\mathcal{F}\|_{H^{-1}(-\ell,\ell)}\|\theta\|_{H^1(-\ell,\ell)}=\|\mathcal{F}\|_{H^{-1}(-\ell,\ell)}\|\mathscr{K}_1\zeta_0+\varrho\|_{H^1(-\ell,\ell)}\\&\lesssim\|\mathcal{F}\|_{H^{-1}(-\ell,\ell)}\|\varrho\|_{H^1(-\ell,\ell)}.\end{aligned} \tag{4.18}$$

Hence,

$$\|\varrho\|_{H^1(-\ell,\ell)}+|\mathscr{K}_1|\lesssim\|\mathcal{F}\|_{H^{-1}(-\ell,\ell)}. \tag{4.19}$$

We now obtain an H^2 estimate for our solution, supposing that $\mathcal{F}\in H^0$. Let $b(x_1)=\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3$ and take an arbitrary smooth function $\psi\in C_c^\infty(-\ell,\ell)$. Plugging in the test function $\theta=\psi b$ and rearranging (4.11), we find that

$$\begin{aligned}\sigma\int_{-\ell}^{\ell}(\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho)\,\partial_1\psi&=\langle\mathcal{F},\psi b\rangle-\int_{-\ell}^{\ell}g\varrho b\psi-\int_{-\ell}^{\ell}\mathscr{K}_1 b\Big(P_0-g\zeta_0\Big)\psi\\&\quad-\sigma\int_{-\ell}^{\ell}\partial_1 b\left(\frac{\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho}{b}\right)\psi.\end{aligned} \tag{4.20}$$

Hence, we know that $\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho$ is weakly differentiable and

$$\mathscr{K}_1\partial_{11}\zeta_0+\partial_{11}\varrho=-\frac{1}{\sigma}\left(\mathcal{F}b-g\varrho b-\mathscr{K}_1 b\Big(P_0-g\zeta_0\Big)\right)+\partial_1 b\left(\frac{\mathscr{K}_1\partial_1\zeta_0+\partial_1\varrho}{b}\right). \tag{4.21}$$

Then this and the estimate (4.19) imply that

$$\begin{aligned}\|\varrho\|_{H^2(-\ell,\ell)}&\lesssim\|\varrho\|_{H^1(-\ell,\ell)}+|\mathscr{K}_1|+\|\mathscr{K}_1\partial_{11}\zeta_0+\partial_{11}\varrho\|_{H^0(-\ell,\ell)}\\&\lesssim\|\mathcal{F}\|_{H^0(-\ell,\ell)}+\|\varrho\|_{H^1(-\ell,\ell)}+|\mathscr{K}_1|\lesssim\|\mathcal{F}\|_{H^0(-\ell,\ell)}.\end{aligned} \tag{4.22}$$

With the bounds (4.19) and (4.22) in hand, we may apply a standard interpolation argument to get the desired estimate. □

The next lemma is a variant of the previous elliptic regularity result.

Lemma 4.4. *Suppose that $\varrho \in H_0^1(-\ell,\ell)$ and $\mathscr{K}_1 \in \mathbb{R}$ satisfy (3.2), and*

$$g\int_{\Sigma_0} \varrho\theta + \mathscr{K}_1 \int_{\Sigma_0} (P_0 - g\zeta_0)\theta + \sigma \int_{\Sigma_0} \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right) \partial_1\theta = \langle f, \partial_1\theta\rangle, \tag{4.23}$$

for any $\theta \in H_0^1(-\ell,\ell)$. If $f \in H^{\frac12}(-\ell,\ell)$, then $\varrho \in H^{\frac32}(-\ell,\ell)$ and

$$\|\varrho\|_{H^{\frac32}(-\ell,\ell)} \lesssim \|f\|_{H^{\frac12}(-\ell,\ell)}. \tag{4.24}$$

Proof. Arguing as in Lemma 4.3, we may deduce the bound $\|\varrho\|_{H^1(-\ell,\ell)} + |\mathscr{K}_1| \lesssim \|f\|_{H^0(-\ell,\ell)}$. Plugging in the test function $\theta = \psi \in C_c^\infty(-\ell,\ell)$ implies that

$$\sigma \int_{-\ell}^{\ell} \left(\frac{\mathscr{K}_1 \partial_1\zeta_0 + \partial_1\varrho}{b} - f \right) \partial_1\psi = -\int_{-\ell}^{\ell} g\varrho\psi - \int_{-\ell}^{\ell} \mathscr{K}_1 \Big(P_0 - g\zeta_0\Big)\psi. \tag{4.25}$$

Hence, we know that $\chi = \dfrac{\mathscr{K}_1 \partial_1\zeta_0 + \partial_1\varrho}{b} - f$ is weakly differentiable and

$$\partial_1\chi = \frac{1}{\sigma}\Big(g\varrho + \mathscr{K}_1\Big(P_0 - g\zeta_0\Big)\Big). \tag{4.26}$$

Then this implies that

$$\begin{aligned} \|\chi\|_{H^1(-\ell,\ell)} &\lesssim \|\chi\|_{H^0(-\ell,\ell)} + \|\partial_1\chi\|_{H^0(-\ell,\ell)} \\ &\lesssim |\mathscr{K}_1| + \|\varrho\|_{H^1(-\ell,\ell)} + \|f\|_{H^0(-\ell,\ell)} \lesssim \|f\|_{H^0(-\ell,\ell)}. \end{aligned} \tag{4.27}$$

Therefore, we have

$$\begin{aligned} \|\partial_1\varrho\|_{H^{\frac12}(-\ell,\ell)} &\lesssim |\mathscr{K}_1| + \|fb\|_{H^{\frac12}(-\ell,\ell)} + \|\chi b\|_{H^{\frac12}(-\ell,\ell)} \\ &\lesssim \|f\|_{H^{\frac12}(-\ell,\ell)} + \|\chi\|_{H^{\frac12}(-\ell,\ell)} \lesssim \|f\|_{H^{\frac12}(-\ell,\ell)}. \end{aligned} \tag{4.28}$$

The desired estimate follows. □

We have now developed all of the tools needed to state a central estimate of the free surface function.

Theorem 4.5. *If (v, q, ϱ) satisfies*

$$\int_{\Omega_0} \frac{\mu J}{2} \mathbb{D}_{\mathcal{A}} v : \mathbb{D}_{\mathcal{A}} w - \int_{\Omega_0} Jq(\nabla_{\mathcal{A}} \cdot w) + \int_{\Sigma_{0b}} \beta(v\cdot\tau_0)(w\cdot\tau_0) + \int_{\Sigma_0} g\varrho(w\cdot\mathcal{N}) \tag{4.29}$$

$$
\begin{aligned}
&+\int_{\Sigma_0} \mathscr{K}_1(p_0 - g\zeta_0)(w\cdot\mathcal{N}) + \int_{\Sigma_0} \sigma \left(\frac{\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right) \partial_1(w\cdot\mathcal{N}) \\
&+\left(-\frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{R}(w\cdot\mathcal{N})\Big|_{\ell} - \frac{\kappa}{\partial_1\zeta_0}\partial_t\mathscr{L}(w\cdot\mathcal{N})\Big|_{-\ell} \right) \\
=&\int_{\Omega_0} J w\cdot\mathcal{F}_1 - \int_{\Sigma_0} w\cdot\mathcal{F}_3 - \int_{\Sigma_{0b}} \mathcal{F}_4\left(w\cdot\frac{\mathcal{T}}{|\mathcal{T}|^2}\right) - \int_{\Sigma_0}\sigma\partial_1\partial_t^m\mathcal{R}(w\cdot\mathcal{N}) \\
&+\frac{\sigma}{\partial_1\zeta_0}\partial_t^m\mathcal{Q}(w\cdot\mathcal{N})\Big|_{-\ell}^{\ell} - \frac{1}{\partial_1\zeta_0}\mathcal{F}_7(w\cdot\mathcal{N})\Big|_{\ell} - \frac{1}{\partial_1\zeta_0}\mathcal{F}_6(w\cdot\mathcal{N})\Big|_{-\ell},
\end{aligned}
$$

for all $w \in \mathcal{W}(t)$, then for each $\theta \in H_0^1(-\ell,\ell)$, there exists $w[\theta] \in \mathcal{W}(t)$ such that the following hold:

(1) *$w[\theta]$ depends linearly on θ.*
(2) *$w[\theta]\cdot\mathcal{N} = \theta$ on Σ_0.*
(3) *We have the estimate*

$$
\|w[\theta]\|^2_{H^1(\Omega_0)} \lesssim \|\theta\|^2_{H^{\frac{1}{2}}(-\ell,\ell)} \quad \text{and} \quad \|w[\theta]\|^2_{\mathcal{W}(t)} \lesssim \|\theta\|^2_{H^1(-\ell,\ell)}. \tag{4.30}
$$

(4) *We have the identity*

$$
\begin{aligned}
&\int_{\Sigma_0} g\varrho\theta + \mathscr{K}_1\int_{\Sigma_0}(P_0 - g\zeta_0)\theta + \sigma\int_{\Sigma_0}\left(\frac{\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right)\partial_1\theta \qquad (4.31)\\
&= \langle \mathcal{G}, w[\theta]\rangle + \langle \mathcal{F}, w[\theta]\rangle + \int_{\Sigma_0}\sigma\partial_t^m\mathcal{R}\partial_1\theta,
\end{aligned}
$$

where $\mathcal{G}$ is given by

$$
\langle \mathcal{G}, w[\theta]\rangle = -\int_{\Omega_0}\frac{\mu J}{2}\mathbb{D}_{\mathcal{A}}v : \mathbb{D}_{\mathcal{A}}w[\theta] + \int_{\Omega_0} Jq(\nabla_{\mathcal{A}}\cdot w[\theta]) - \int_{\Sigma_{0b}}\beta(v\cdot\tau_0)(w[\theta]\cdot\tau_0), \tag{4.32}
$$

and $\mathcal{F} \in (H^1)^*$ is given by

$$
\langle \mathcal{F}, w[\theta]\rangle = \int_{\Omega_0} Jw[\theta]\cdot\mathcal{F}_1 - \int_{\Sigma_0} w[\theta]\cdot\mathcal{F}_3 - \int_{\Sigma_{0b}}\mathcal{F}_4\left(w[\theta]\cdot\frac{\mathcal{T}}{|\mathcal{T}|^2}\right). \tag{4.33}
$$

(5) *We have the estimate*

$$
\|\varrho\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \lesssim \|v\|^2_{H^1(\Omega_0)} + \|q\|^2_{H^0(\Omega_0)} + \|\mathcal{F}\|^2_{(H^1)^*} + \left\|\partial_t^m\mathcal{R}\right\|^2_{H^{\frac{1}{2}}(-\ell,\ell)}. \tag{4.34}
$$

Proof. Let $\theta \in H_0^1(-\ell, \ell)$. Then we may use standard elliptic arguments (see Theorem 4.11 in [9]) to find $w[\theta] \in \mathcal{W}(t)$ satisfying

$$\begin{cases} \nabla_{\mathcal{A}} \cdot w = C \displaystyle\int_{\ell}^{\ell} \theta(x_1)\mathrm{d}x_1 & \text{in } \Omega_0, \\ w \cdot \mathcal{N} = \theta & \text{on } \Sigma_0, \\ w \cdot \nu_0 = 0 & \text{on } \Sigma_{0b}, \end{cases} \tag{4.35}$$

where C is chosen depending on Ω_0 in order to enforce the compatibility condition for this equation. The resulting $w[\theta]$ satisfies the requirements of the statements. An integration by parts yields

$$-\int_{\Sigma_0} \sigma \partial_1 \partial_t^m \mathcal{R}(w \cdot \mathcal{N}) = \int_{\Sigma_0} \sigma \partial_t^m \mathcal{R} \partial_1 (w \cdot \mathcal{N}) - \sigma \partial_t^m \mathcal{R}(w \cdot \mathcal{N})\Big|_{-\ell}^{\ell}. \tag{4.36}$$

Also, since $\theta \in H_0^1(-\ell, \ell)$, in the weak formulation, all the contributions at contact points vanish. Hence, we have

$$\begin{aligned} &\int_{\Sigma_0} g\varrho\theta + \int_{\Sigma_0} \mathscr{K}_1(P_0 - g\zeta_0)\theta + \int_{\Sigma_0} \sigma \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1 + |\partial_1 \zeta_0|^2}\right)^3} \right) \partial_1 \theta \\ &= \langle \mathcal{G}, w[\theta] \rangle + \langle \mathcal{F}, w[\theta] \rangle + \int_{\Sigma_0} \sigma \partial_t^m \mathcal{R} \partial_1 \theta. \end{aligned} \tag{4.37}$$

Then the estimate (4.34) follows in light of Lemmas 4.3 and 4.4. □

4.3. Contact point estimates

In this section, we will prove the estimates of derivatives of η and $u \cdot \mathcal{N}$ at the contact points, which are much stronger than the results obtained through the usual trace theorem.

We start with the $\partial_1 \eta$ estimates.

Theorem 4.6. *There exists a universal $\vartheta > 0$ such that if $\|\eta\|_{H^0(-\ell,\ell)} + |\partial_t L| + |\partial_t R| < \vartheta$, then*

$$\sum_{j=0}^{2} \left(\left|\partial_t^j \partial_1 \eta(-\ell)\right|^2 + \left|\partial_t^j \partial_1 \eta(\ell)\right|^2 \right) \lesssim \sum_{j=0}^{2} \left\|\partial_t^j \eta\right\|_{H^0(-\ell,\ell)}^2 + \sum_{j=0}^{2} \left(\left|\partial_t^{j+1} l\right|^2 + \left|\partial_t^{j+1} r\right|^2 \right). \tag{4.38}$$

Proof. First note the simple fact that $\partial_t L = \partial_t l$ and $\partial_t R = \partial_t r$. Consider the equations for $\partial_t L$ and $\partial_t R$ in (1.25). Solving for $\partial_1 \eta(-\ell)$ in the $\partial_t L$ equation, we obtain

$$\partial_1\eta(-\ell) = J_1\sqrt{\frac{\sigma^2}{\left(\mathscr{W}(\partial_t L)+[\gamma]\right)^2}-1}-\partial_1\zeta_0(-\ell), \tag{4.39}$$

where here we assume ϑ is sufficiently small for the term in the square root to be nonnegative. From the equilibrium equations (1.14), we may compute

$$\partial_1\zeta_0(-\ell)=\sqrt{\frac{\sigma^2}{[\gamma]^2}-1}. \tag{4.40}$$

Further restricting the value of ϑ if necessary, employing a Taylor expansion, and using Lemma C.2, we conclude from these that

$$|\partial_1\eta(-\ell)| \lesssim |J_1-1|+|\partial_t L| \lesssim \|\eta\|_{H^0(-\ell,\ell)}+|\partial_t l|. \tag{4.41}$$

When ∂_t is applied in (4.39), we know

$$\begin{aligned}&\partial_t\partial_1\eta(-\ell)\left(\partial_1\zeta_0(-\ell)+\partial_1\eta(-\ell)\right)\\ &= J_1\partial_t J_1\left(\frac{\sigma^2}{\left(\mathscr{W}(\partial_t L)+[\gamma]\right)^2}-1\right)-J_1^2\frac{\sigma^2\mathscr{W}'(\partial_t L)\partial_t^2 L}{\left(\mathscr{W}(\partial_t L)+[\gamma]\right)^3}.\end{aligned} \tag{4.42}$$

Again restricting the value of ϑ if necessary, and considering $\partial_1\zeta_0(-\ell)\gtrsim 1$, we know

$$|\partial_1\zeta_0(-\ell)+\partial_1\eta(-\ell)| \gtrsim 1. \tag{4.43}$$

Then we may use (4.42) and (4.43) together with (1.45) to estimate

$$|\partial_t\partial_1\eta(-\ell)| \lesssim |\partial_t J_1|+\left|\partial_t^2 L\right| \lesssim \|\partial_t\eta\|_{H^0(-\ell,\ell)}+\left|\partial_t^2 l\right|. \tag{4.44}$$

When ∂_t^2 is applied in (4.39), we know

$$\begin{aligned}&\partial_t^2\partial_1\eta(-\ell)\left(\partial_1\zeta_0(-\ell)+\partial_1\eta(-\ell)\right)+\left(\partial_t\partial_1\eta(-\ell)\right)^2\\ &=\left(J_1\partial_t^2 J_1+(\partial_t J_1)^2\right)\left(\frac{\sigma^2}{\left(\mathscr{W}(\partial_t L)+[\gamma]\right)^2}-1\right)-J_1\partial_t J_1\frac{\sigma^2\mathscr{W}'(\partial_t L)\partial_t^2 L}{\left(\mathscr{W}(\partial_t L)+[\gamma]\right)^3}\\ &\quad+J_1^2\frac{3\sigma^2\left(\mathcal{W}'(\partial_t L)\right)^2\left(\partial_t^2 L\right)^2}{\left(\mathscr{W}(\partial_t L)+[\gamma]\right)^4}-J_1^2\frac{\sigma^2\left(\mathscr{W}''(\partial_t L)\left(\partial_t^2 L\right)^2+\mathscr{W}'(\partial_t L)\partial_t^3 L\right)}{\left(\mathscr{W}(\partial_t L)+[\gamma]\right)^3}.\end{aligned} \tag{4.45}$$

Hence, as above, we may estimate

$$\begin{aligned}\left|\partial_t^2\partial_1\eta(-\ell)\right| &\lesssim |\partial_t\partial_1\eta(-\ell)|^2+\left|\partial_t^2 J_1\right|+|\partial_t J_1|^2+|\partial_t J_1|\left|\partial_t^2 L\right|+\left|\partial_t^2 L\right|^2+\left|\partial_t^3 L\right| \\ &\lesssim \left\|\partial_t^2\eta\right\|_{H^0(-\ell,\ell)}+\left|\partial_t^2 l\right|+\left|\partial_t^3 l\right|.\end{aligned} \tag{4.46}$$

In summary, we have proved that if ϑ is sufficiently small, then

$$\sum_{j=0}^{2}\left|\partial_t^j\partial_1\eta(-\ell)\right| \lesssim \sum_{j=0}^{2}\left\|\partial_t^j\eta\right\|_{H^0(-\ell,\ell)}+\sum_{j=0}^{2}\left|\partial_t^{j+1} l\right|. \tag{4.47}$$

A similar argument with the $\partial_t R$ equation provides the bound

$$\sum_{j=0}^{2}\left|\partial_t^j\partial_1\eta(\ell)\right| \lesssim \sum_{j=0}^{2}\left\|\partial_t^j\eta\right\|_{H^0(-\ell,\ell)}+\sum_{j=0}^{2}\left|\partial_t^{j+1} r\right|. \quad \square \tag{4.48}$$

Next we consider the $u\cdot\mathcal{N}$ estimates.

Theorem 4.7. *There exists a universal $\vartheta>0$ such that if $\|\eta\|_{H^0(-\ell,\ell)}+|\partial_t l|+|\partial_t r|+\left|\partial_t^2 l\right|+\left|\partial_t^2 r\right|<\vartheta$, then*

$$\begin{aligned}&\sum_{j=0}^{2}\left(\left|\partial_t^j u(-\ell,0)\cdot\mathcal{N}\right|^2+\left|\partial_t^j u(\ell,0)\cdot\mathcal{N}\right|^2\right) \\ &\lesssim \sum_{j=0}^{2}\left\|\partial_t^j\eta\right\|^2_{H^0(-\ell,\ell)}+\sum_{j=0}^{2}\left(\left|\partial_t^{j+1} l\right|^2+\left|\partial_t^{j+1} r\right|^2\right).\end{aligned} \tag{4.49}$$

Proof. Throughout the proof we will abuse notation by suppressing the time dependence of the unknowns; for example, we will write $\partial_t\eta(\pm\ell)$ in place of $\partial_t\eta(\pm\ell,t)$. The transport equation reads

$$(u\cdot\mathcal{N})b=J_1\partial_t\eta-\tilde{a}(\partial_1\zeta_0+\partial_1\eta), \tag{4.50}$$

for $b=\sqrt{1+|\partial_1\zeta_0|^2}$. Hence, noting that $\partial_t\eta(-\ell)=0$, using Theorem 4.6, and taking ϑ to be sufficiently small, we know

$$|u(-\ell,0)\cdot\mathcal{N}|=|\tilde{a}(\partial_1\zeta_0+\partial_1\eta(-\ell))|\lesssim|\tilde{a}|\left(1+|\partial_1\eta(-\ell)|\right)\lesssim|\tilde{a}|\lesssim|\partial_t l|+|\partial_t r|. \tag{4.51}$$

When ∂_t is applied in (4.50), we have

$$(\partial_t u\cdot\mathcal{N})b=\partial_t J_1\partial_t\eta+J_1\partial_t^2\eta-\partial_t\tilde{a}(\partial_1\zeta_0+\partial_1\eta)-\tilde{a}\partial_t\partial_1\eta-(u\cdot\partial_t\mathcal{N})b. \tag{4.52}$$

It is easy to check that $u(-\ell,0)=(\partial_t l,0)$ and $u(\ell,0)=(\partial_t r,0)$. Hence, noting that $\partial_t^2\eta(-\ell)=0$, using Theorem 4.6, and taking ϑ to be sufficiently small, we know

$$\begin{aligned}|\partial_t u(-\ell,0)\cdot\mathcal{N}| &\lesssim |\partial_t\tilde{a}(\partial_1\zeta_0+\partial_1\eta(-\ell))|+|\tilde{a}\partial_t\partial_1\eta(-\ell)|+|(u(-\ell,0)\cdot\partial_t\mathcal{N})b| \\ &\lesssim |\partial_t\tilde{a}|\Big(1+|\partial_1\eta(-\ell)|\Big)+|\tilde{a}|\,|\partial_t\partial_1\eta(-\ell)|+|u(-\ell,0)|\,|\partial_t\partial_1\eta(-\ell)| \\ &\lesssim |\partial_t\tilde{a}|+|\partial_t\partial_1\eta(-\ell)|\lesssim \sum_{j=0}^{1}\left\|\partial_t^j\eta\right\|_{H^0(-\ell,\ell)}+\sum_{j=0}^{1}\left|\partial_t^{j+1}l\right|+\sum_{j=0}^{1}\left|\partial_t^{j+1}r\right|.\end{aligned} \tag{4.53}$$

When ∂_t^2 is applied in (4.50), we have

$$\begin{aligned}(\partial_t^2 u\cdot\mathcal{N})b &= \partial_t^2 J_1\partial_t\eta+\partial_t J_1\partial_t^2\eta+J_1\partial_t^3\zeta-\partial_t^2\tilde{a}(\partial_1\zeta_0+\partial_1\eta)-\partial_t\tilde{a}\partial_t\partial_1\eta \\ &\quad-\tilde{a}\partial_t^2\partial_1\eta-(u\cdot\partial_t^2\mathcal{N})b-(\partial_t u\cdot\partial_t\mathcal{N})b.\end{aligned} \tag{4.54}$$

Hence, noting that $\partial_t^3\eta(-\ell)=0$, using Theorem 4.6 and taking ϑ to be sufficiently small, we know

$$\begin{aligned}&|\partial_t u(-\ell,0)\cdot\mathcal{N}| \\ &\lesssim \left|\partial_t^2\tilde{a}\right||(\partial_1\zeta_0+\partial_1\eta(-\ell))|+|\partial_t\tilde{a}|\,|\partial_t\partial_1\eta(-\ell)|+|\tilde{a}|\left|\partial_t^2\partial_1\eta(-\ell)\right| \\ &+|u(-\ell,0)|\left|\partial_t^2\mathcal{N}\right|+|\partial_t u(-\ell,0)|\,|\partial_t\mathcal{N}| \\ &\lesssim \left|\partial_t^2\tilde{a}\right|\Big(1+|\partial_1\eta(-\ell)|\Big)+|\partial_t\tilde{a}|\,|\partial_t\partial_1\eta(-\ell)|+|\tilde{a}|\left|\partial_t^2\partial_1\eta(-\ell)\right| \\ &+|u(-\ell,0)|\left|\partial_t^2\partial_1\eta\right|+|\partial_t u(-\ell,0)|\,|\partial_t\partial_1\eta(-\ell)| \\ &\lesssim \left|\partial_t^2\tilde{a}\right|+|\partial_t\partial_1\eta(-\ell)|+\left|\partial_t^2\partial_1\eta(-\ell)\right|\lesssim\sum_{j=0}^{2}\left\|\partial_t^j\eta\right\|_{H^0(-\ell,\ell)}+\sum_{j=0}^{2}\left|\partial_t^{j+1}l\right|+\sum_{j=0}^{2}\left|\partial_t^{j+1}r\right|.\end{aligned} \tag{4.55}$$

In summary, we have now shown

$$\sum_{j=0}^{2}\left|\partial_t^j u(-\ell,0)\cdot\mathcal{N}\right|^2\lesssim\sum_{j=0}^{2}\left\|\partial_t^j\eta\right\|_{H^0(-\ell,\ell)}^2+\sum_{j=0}^{2}\left(\left|\partial_t^{j+1}l\right|^2+\left|\partial_t^{j+1}r\right|^2\right). \tag{4.56}$$

A similar argument with the $\partial_t R$ equation provides the bound

$$\sum_{j=0}^{2}\left|\partial_t^j u(\ell,0)\cdot\mathcal{N}\right|^2\lesssim\sum_{j=0}^{2}\left\|\partial_t^j\eta\right\|_{H^0(-\ell,\ell)}^2+\sum_{j=0}^{2}\left(\left|\partial_t^{j+1}l\right|^2+\left|\partial_t^{j+1}r\right|^2\right). \quad\square \tag{4.57}$$

4.4. *Weighted elliptic estimates for the Stokes problem*

We now turn our attention to some weighted elliptic estimates for solutions to the Stokes problem.

Theorem 4.8. *There exists a $\vartheta > 0$ such that if $\eta \in W_\delta^{\frac{5}{2}}(\Sigma_0)$ and $\|\eta\|_{W_\delta^{\frac{5}{2}}(\Sigma_0)} < \vartheta$, then there is a unique solution $(v, q, \varrho) \in W_\delta^2(\Omega_0) \times \mathring{W}_\delta^1(\Omega_0) \times W_\delta^{\frac{5}{2}}(\Sigma_0)$ to the equation*

$$\begin{cases} \nabla_{\mathcal{A}} \cdot S_{\mathcal{A}}(q, v)\mathcal{N} = G_1 & \text{in } \Omega_0, \\ \nabla_{\mathcal{A}} \cdot v = G_2 & \text{in } \Omega_0, \\ v \cdot \mathcal{N} = G_3^+ & \text{on } \Sigma_0, \\ S_{\mathcal{A}}(q, v)\mathcal{N} = g\varrho\mathcal{N} - \sigma\partial_1\left(\dfrac{\mathscr{K}_1\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \dfrac{\mathscr{K}_1\partial_1\zeta_0 + \partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} + \partial_t^m\mathcal{R}\right)\mathcal{N} & \\ \qquad\qquad + G_4^+\dfrac{\mathcal{T}}{|\mathcal{T}|} + G_5\dfrac{\mathcal{N}}{|\mathcal{N}|} & \text{on } \Sigma_0, \\ v \cdot \mathcal{N} = G_3^- & \text{on } \Sigma_{0b}, \\ \Big(S_{\mathcal{A}}(q, v)\mathcal{N} - \beta v\Big) \cdot \mathcal{T} = G_4^- & \text{on } \Sigma_{0b}, \\ \varrho\Big|_{-\ell} = \varrho\Big|_{\ell} = 0, & \end{cases} \tag{4.58}$$

such that for any $\delta \in (0, 1)$,

$$\begin{aligned} \|v\|^2_{W_\delta^2(\Omega_0)} + \|q\|^2_{W_\delta^1(\Omega_0)} + \|\varrho\|^2_{W_\delta^{\frac{5}{2}}(\Sigma_0)} &\lesssim \|\varrho\|^2_{H^0(-\ell,\ell)} + \|G_1\|^2_{W_\delta^0(\Omega_0)} + \|G_2\|^2_{W_\delta^1(\Omega_0)} \\ &+ \|G_3\|^2_{W_\delta^{\frac{3}{2}}(\partial\Omega_0)} + \|G_4\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)} + \|G_5\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)} + \left\|\partial_1\partial_t^m\mathcal{R}\right\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)}. \end{aligned} \tag{4.59}$$

Proof. Using Theorem 5.10 in [9], we may find $(v, q) \in W_\delta^2(\Omega_0) \times \mathring{W}_\delta^1(\Omega_0)$ satisfying

$$\|v\|^2_{W_\delta^2(\Omega_0)} + \|q\|^2_{W_\delta^1(\Omega_0)} \lesssim \|G_1\|^2_{W_\delta^0(\Omega_0)} + \|G_2\|^2_{W_\delta^1(\Omega_0)} + \|G_3\|^2_{W_\delta^{\frac{3}{2}}(\partial\Omega_0)} + \|G_4\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)}. \tag{4.60}$$

The only remaining estimate is for ϱ, which satisfies an elliptic equation

$$\begin{aligned} g\varrho - \sigma\partial_1\left(\frac{\partial_1\varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3}\right) &= (qI - \mu\mathbb{D}_{\mathcal{A}}v)\mathcal{N} \cdot \frac{\mathcal{N}}{|\mathcal{N}|^2} \\ &+ \sigma\partial_1\left(\frac{\mathscr{K}_1\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \frac{\mathscr{K}_1\partial_1\zeta_0}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} + \partial_t^m\mathcal{R}\right) - \frac{G_5}{|\mathcal{N}|}, \end{aligned} \tag{4.61}$$

with Dirichlet boundary conditions $\varrho(-\ell) = \varrho(\ell) = 0$. We may use (4.59) to verify that

$$\left\|(qI - \mu\mathbb{D}_{\mathcal{A}}v)\mathcal{N} \cdot \frac{\mathcal{N}}{|\mathcal{N}|^2}\right\|^2_{W_\delta^{\frac{1}{2}}(\Sigma_0)} \lesssim \|v\|^2_{W_\delta^2(\Omega_0)} + \|q\|^2_{W_\delta^1(\Omega_0)}. \tag{4.62}$$

Using (3.2), we may directly estimate

$$\left\| \sigma \partial_1 \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0}{\sqrt{1+|\partial_1 \zeta_0|^2}} + \frac{\mathscr{K}_1 \partial_1 \zeta_0}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \right\|^2_{W_\delta^{\frac{1}{2}}(\Sigma_0)} \lesssim |\mathscr{K}_1| \lesssim \|\varrho\|_{H^0(-\ell,\ell)} . \tag{4.63}$$

We then combine (4.61), (4.62), and (4.63) to deduce that

$$\begin{aligned} \|\varrho\|^2_{W_\delta^{\frac{5}{2}}(\Sigma_0)} &\lesssim \left\| (qI - \mu \mathbb{D}_{\mathcal{A}} v)\mathcal{N} \cdot \frac{\mathcal{N}}{|\mathcal{N}|^2} \right\|^2_{W_\delta^{\frac{1}{2}}(\Sigma_0)} \\ &\quad + \left\| \sigma \partial_1 \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0}{\sqrt{1+|\partial_1 \zeta_0|^2}} + \frac{\mathscr{K}_1 \partial_1 \zeta_0}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \right\|^2_{W_\delta^{\frac{1}{2}}(\Sigma_0)} \\ &\quad + \left\| \partial_1 \partial_t^m \mathcal{R} \right\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)} + \left\| \frac{G_5}{|\mathcal{N}|} \right\|^2_{W_\delta^{\frac{1}{2}}(\Sigma_0)} \\ &\lesssim \|v\|^2_{W_\delta^2(\Omega_0)} + \|q\|^2_{W_\delta^1(\Omega_0)} + \|\varrho\|^2_{H^0(-\ell,\ell)} + \left\| \partial_1 \partial_t^m \mathcal{R} \right\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)} + \|G_5\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)} . \end{aligned} \tag{4.64}$$

Combining (4.60) and (4.64), we get the desired result. □

5. Nonlinear estimates in the energy-dissipation structure

We will employ the basic energy estimate of Theorem 3.2 as the starting point for our a priori estimates. In order for this to be effective, we must be able to estimate the interaction terms appearing on the right side of (3.13) when the $\mathcal{F}_i$ terms are given as in Appendix C.2. For the sake of brevity we will only present these estimates when the $\mathcal{F}_i$ terms are given for the twice temporally differentiated problem. The corresponding estimates for the once temporally differentiated problem follow from similar, though often simpler, arguments. When possible, we will present our estimates in the most general form, as estimates for general functionals generated by the $\mathcal{F}_i$ terms. It is only for a few essential terms that we must resort to employing the special structure of the interaction terms in order to close our estimates.

Throughout this section, we always assume that η is given and satisfies

$$\sup_{0\le t\le T} \left(\mathcal{E}_{\parallel}(t) + \|\eta(t)\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} + \|\partial_t \eta(t)\|_{W_\delta^{\frac{3}{2}}(-\ell,\ell)} \right) \le \vartheta < 1, \tag{5.1}$$

for some $\vartheta > 0$ sufficiently small. Also, throughout this section, we will repeatedly use without explicit statement the following techniques in our estimates:

- Sobolev embedding theorems and trace theorems for both the usual Sobolev spaces and weighted Sobolev spaces;

- the pointwise bound $\left|\partial_t^m k_1\right| \lesssim \left|\partial_t^m l\right| + \left|\partial_t^m r\right|$, as well as the bounds, for any $1 \le r < \infty$, $\left|\partial_t^m k_1\right| \lesssim \left\|\partial_t^m \eta\right\|_{L^r(-\ell,\ell)} \lesssim \left\|\partial_t^m \partial_1 \eta\right\|_{L^r(-\ell,\ell)}$, which follow due to (3.2) and Poincaré's inequality;
- for $s > \frac{1}{2}$, the estimate $\|\bar{\eta}\|_{H^s(\Omega_0)} \lesssim \|\eta\|_{H^{s-\frac{1}{2}}(-\ell,\ell)}$, which is due to the definition of harmonic extension;
- the bound $|\nabla \mathcal{A}| \lesssim \left|\nabla^2 \bar{\eta}\right| + |\nabla \bar{\eta}| \left|\frac{1}{\zeta_0}\right|$;
- Theorems 4.6 and 4.7.

Moreover, we will repeatedly use the following two lemmas:

Lemma 5.1. *Let $d = dist(\cdot, M)$, where $M = \Big\{(-\ell, 0), (\ell, 0)\Big\}$ is the set of the corner points. Suppose that $0 < \delta < 1$. Then $d^{-\delta} \in L^r(\Omega_0)$ for $1 \le r < \frac{2}{\delta}$.*

Proof. See Lemma 6.1 in [9]. $\square$

Lemma 5.2. *Let $1 < p < 2$. Then $\frac{1}{\zeta_0} \in L^p(\Omega_0)$.*

Proof. The difficulty concentrates on the neighborhood of the contact points. Near $(-\ell, 0)$, using the mean value theorem, we have for $c \in (-\ell, x_1)$,

$$\zeta_0(x_1) = \zeta_0(-\ell) + \partial_1 \zeta_0(c)(x_1 + \ell) = \partial_1 \zeta_0(c)(x_1 + \ell). \tag{5.2}$$

Then using polar coordinates, we compute

$$\int_{x \in \Omega_0, d \le R} \frac{1}{\zeta_0^p(x_1)} \mathrm{d}x_1 \mathrm{d}x_2 \le \int_0^\Theta \int_0^R \frac{r}{r^p \cos^p \theta} \mathrm{d}r \mathrm{d}\theta = \int_0^\Theta \int_0^R \frac{1}{r^{p-1} \cos^p \theta} \mathrm{d}r \mathrm{d}\theta < \infty, \tag{5.3}$$

where $\tan \Theta = \partial_1 \zeta_0(-\ell)$. The integral with respect to r is finite since $0 < p - 1 < 1$ and the integral with respect to θ is finite since $0 < \theta < \Theta < \frac{\pi}{2}$. $\square$

5.1. Estimate of the $\mathcal{F}_1$ term

The following estimate is the same as that of Proposition 6.2 of [9], but the argument needed to arrive at this estimate is slightly different due to the different structure of $\mathcal{A}$. In particular, the appearance of the term $1/\zeta_0$ in terms involving $\nabla \mathcal{A}$ is novel to the droplet problem.

Lemma 5.3. *Let $\mathcal{F}_1$ be given by (C.2.1) or (C.2.8). We have the estimate*

$$\left| \int_{\Omega_0} J v \cdot \mathcal{F}_1 \right| \lesssim \|v\|_{H^1(\Omega_0)} \left(\mathcal{E} + \sqrt{\mathcal{E}}\right) \sqrt{\mathcal{D}}. \tag{5.4}$$

Proof. We will only present the estimate of the term $-\nabla_{\partial_t^2 \mathcal{A}} \cdot \left(pI - \mu \mathbb{D}_{\mathcal{A}} u\right)$ in order to highlight how to deal with the appearance of $1/\zeta_0$. The remaining terms can be handled similarly, following the general blueprint of Proposition 6.2 of [9] with appropriate modifications to handle $1/\zeta_0$ as indicated in what follows.

To estimate $-\nabla_{\partial_t^2 \mathcal{A}} \cdot \left(pI - \mu \mathbb{D}_{\mathcal{A}} u\right)$, we begin by splitting

$$
\begin{aligned}
&\left| \int_{\Omega_0} J v \Big(-\nabla_{\partial_t^2 \mathcal{A}} \cdot \left(pI - \mu \mathbb{D}_{\mathcal{A}} u\right) \Big) \right| \qquad (5.5) \\
&\lesssim \int_{\Omega_0} |v| \left|\partial_t^2 \mathcal{A}\right| \left(\left|\nabla^2 u\right| + |\nabla p| \right) + \int_{\Omega_0} |v| \left|\partial_t^2 \mathcal{A}\right| |\nabla \mathcal{A}| \left(|\nabla u| + |p| \right) \\
&\lesssim \int_{\Omega_0} |v| \left|\partial_t^2 \nabla \bar{\eta}\right| \left(\left|\nabla^2 u\right| + |\nabla p| \right) + \int_{\Omega_0} |v| \left|\partial_t^2 \nabla \bar{\eta}\right| \left|\nabla^2 \bar{\eta}\right| \left(|\nabla u| + |p| \right) \\
&\quad + \int_{\Omega_0} |v| \left|\partial_t^2 \nabla \bar{\eta}\right| |\nabla \bar{\eta}| \left| \frac{1}{\zeta_0} \right| \left(|\nabla u| + |p| \right) =: I + II + III.
\end{aligned}
$$

For I, we choose $q \in [1, \infty)$ and $2 < r < \frac{2}{\delta}$ such that $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. Then we have

$$
\begin{aligned}
I &= \int_{\Omega_0} |v| \left|\partial_t^2 \nabla \bar{\eta}\right| \left(\left|\nabla^2 u\right| + |\nabla p| \right) \qquad (5.6) \\
&\lesssim \|v\|_{L^q(\Omega_0)} \left\|\partial_t^2 \nabla \bar{\eta}\right\|_{L^q(\Omega_0)} \left\|d^{-\delta}\right\|_{L^r(\Omega_0)} \left(\left\|d^\delta \nabla^2 u\right\|_{L^2(\Omega_0)} + \left\|d^\delta \nabla p\right\|_{L^2(\Omega_0)} \right) \\
&\lesssim \|v\|_{H^1(\Omega_0)} \left\|\partial_t^2 \nabla \bar{\eta}\right\|_{H^1(\Omega_0)} \left(\|u\|_{W^2_\delta(\Omega_0)} + \|p\|_{W^1_\delta(\Omega_0)} \right) \\
&\lesssim \|v\|_{H^1(\Omega_0)} \left\|\partial_t^2 \eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \left(\|u\|_{W^2_\delta(\Omega_0)} + \|p\|_{W^1_\delta(\Omega_0)} \right) \lesssim \|v\|_{H^1(\Omega_0)} \sqrt{\mathcal{D}} \sqrt{\mathcal{E}}.
\end{aligned}
$$

For II, we choose $m = \frac{2}{2-s}$ and $2 < r < \frac{2}{\delta}$ such that $\frac{1}{m} + \frac{1}{r} < 1$, which is possible since $\delta < 1 < s$. Then choosing $q \in [1, \infty)$ such that $\frac{3}{q} + \frac{1}{m} + \frac{1}{r} = 1$, we have

$$
\begin{aligned}
II &= \int_{\Omega_0} |v| \left|\partial_t^2 \nabla \bar{\eta}\right| \left|\nabla^2 \bar{\eta}\right| \left(|\nabla u| + |p| \right) \qquad (5.7) \\
&\lesssim \|v\|_{L^q(\Omega_0)} \left\|\partial_t^2 \nabla \bar{\eta}\right\|_{L^q(\Omega_0)} \left\|\nabla^2 \bar{\eta}\right\|_{L^m(\Omega_0)} \left\|d^{-\delta}\right\|_{L^r(\Omega_0)} \left(\left\|d^\delta \nabla u\right\|_{L^q(\Omega_0)} + \left\|d^\delta p\right\|_{L^q(\Omega_0)} \right) \\
&\lesssim \|v\|_{H^1(\Omega_0)} \left\|\partial_t^2 \nabla \bar{\eta}\right\|_{H^1(\Omega_0)} \left\|\nabla^2 \bar{\eta}\right\|_{H^{s-1}(\Omega_0)} \left(\|u\|_{W^2_\delta(\Omega_0)} + \|p\|_{W^1_\delta(\Omega_0)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|v\|_{H^1(\Omega_0)} \left\|\partial_t^2 \eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|\eta\|_{H^{s+\frac{1}{2}}(-\ell,\ell)} \left(\|u\|_{W^2_\delta(\Omega_0)} + \|p\|_{W^1_\delta(\Omega_0)}\right) \\
&\lesssim \|v\|_{H^1(\Omega_0)} \sqrt{\mathcal{D}}\sqrt{\mathcal{E}}\sqrt{\mathcal{E}} = \|v\|_{H^1(\Omega_0)} \mathcal{E}\sqrt{\mathcal{D}}.
\end{aligned}
$$

For III, we choose $2 < r < \dfrac{2}{\delta}$ and $1 < p < 2$ such that $\dfrac{1}{p} + \dfrac{1}{r} < 1$, which is possible since $r > 2$. Then choosing $q \in [1, \infty)$ such that $\dfrac{4}{q} + \dfrac{1}{p} + \dfrac{1}{r} = 1$, we have

$$
\begin{aligned}
III &= \int_{\Omega_0} |v| \left|\partial_t^2 \nabla\bar{\eta}\right| |\nabla\bar{\eta}| \left|\frac{1}{\zeta_0}\right| \Big(|\nabla u| + |p|\Big) \qquad (5.8) \\
&\lesssim \|v\|_{L^q(\Omega_0)} \left\|\partial_t^2 \nabla\bar{\eta}\right\|_{L^q(\Omega_0)} \|\nabla\bar{\eta}\|_{L^q(\Omega_0)} \left\|\frac{1}{\zeta_0}\right\|_{L^p(\Omega_0)} \left\|d^{-\delta}\right\|_{L^r(\Omega_0)} \\
&\quad \times \Big(\left\|d^\delta \nabla u\right\|_{L^q(\Omega_0)} + \left\|d^\delta p\right\|_{L^q(\Omega_0)} \Big) \\
&\lesssim \|v\|_{H^1(\Omega_0)} \left\|\partial_t^2 \nabla\bar{\eta}\right\|_{H^1(\Omega_0)} \|\nabla\bar{\eta}\|_{H^1(\Omega_0)} \left(\|u\|_{W^2_\delta(\Omega_0)} + \|p\|_{W^1_\delta(\Omega_0)}\right) \\
&\lesssim \|v\|_{H^1(\Omega_0)} \left\|\partial_t^2 \eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)} \left(\|u\|_{W^2_\delta(\Omega_0)} + \|p\|_{W^1_\delta(\Omega_0)}\right) \\
&\lesssim \|v\|_{H^1(\Omega_0)} \sqrt{\mathcal{D}}\sqrt{\mathcal{E}}\sqrt{\mathcal{E}} = \|v\|_{H^1(\Omega_0)} \mathcal{E}\sqrt{\mathcal{D}}. \quad \square
\end{aligned}
$$

5.2. *Estimate of the $\mathcal{F}_2$ term*

The estimate of the $\mathcal{F}_2$ term is available from [9]. We record it now.

Lemma 5.4. *Let $\mathcal{F}_2$ be given by* (C.2.2) *or* (C.2.9). *We have the estimate*

$$
\left|\int_{\Omega_0} J\psi\mathcal{F}_2\right| \lesssim \|\psi\|_{H^0(\Omega_0)} \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{5.9}
$$

Proof. The estimate is proved in Theorem 6.8 of [9]. $\square$

5.3. *Estimate of the $\mathcal{F}_3$ term*

We now estimate the $\mathcal{F}_3$ term.

Lemma 5.5. *Let $\mathcal{F}_3$ be given by* (C.2.3) *or* (C.2.10) *and $\mathcal{R}$ be given by* (C.3.1). *We have the estimate*

$$
\left|\int_{\Sigma_0} v\cdot\mathcal{F}_3\right| \lesssim \|v\|_{H^1(\Omega_0)} \left(\mathcal{E} + \sqrt{\mathcal{E}}\right)\sqrt{\mathcal{D}}. \tag{5.10}
$$

Proof. In the following, we choose $p = \dfrac{3+\delta}{2+2\delta}$, $q = \dfrac{6+2\delta}{1-\delta}$ and $r = \dfrac{9+3\delta}{1-\delta}$ such that $\dfrac{1}{p} + \dfrac{2}{q} = 1$ and $\dfrac{1}{p} + \dfrac{3}{r} = 1$.

Estimates of the integral involving the terms $\mu \mathbb{D}_{\partial_t^2 \mathcal{A}} u \mathcal{N}$, $\mu \mathbb{D}_{\partial_t \mathcal{A}} u \partial_t \mathcal{N}$ $2\mu \mathbb{D}_{\partial_t \mathcal{A}} \partial_t u \mathcal{N}$, $-(pI - \mu \mathbb{D}_{\mathcal{A}} u)\partial_t^2 \mathcal{N}$, $-2(\partial_t p I - \mu \mathbb{D}_{\mathcal{A}} \partial_t u)\partial_t \mathcal{N}$, $g\eta \partial_t^2 \mathcal{N}$, and $2g\partial_t \eta \partial_t \mathcal{N}$ may be found in the proof of Proposition 6.4 of [9]. The remaining three terms are novel, and we will present the estimates here.

Term: $-\sigma \partial_1 \left(\dfrac{k_1 \partial_1 \zeta_0}{\sqrt{1+|\partial_1 \zeta_0|^2}} + \dfrac{k_1 \partial_1 \zeta_0 + \partial_1 \eta}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \partial_t^2 \mathcal{N}$

We estimate

$$
\begin{aligned}
&\left| \int_{\Sigma_0} v \left(-\sigma \partial_1 \left(\frac{k_1 \partial_1 \zeta_0}{\sqrt{1+|\partial_1 \zeta_0|^2}} + \frac{k_1 \partial_1 \zeta_0 + \partial_1 \eta}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) \partial_t^2 \mathcal{N} \right) \right| \lesssim \int_{\Sigma_0} |v| \left(|k_1| + \left| \partial_1^2 \eta \right| \right) \left| \partial_t^2 \mathcal{N} \right| \\
&\lesssim \int_{\Sigma_0} |v| \left(|k_1| + \left| \partial_1^2 \eta \right| \right) \left| \partial_t^2 \partial_1 \eta \right| \lesssim \|v\|_{L^q(\Sigma_0)} \left(\|\eta\|_{L^p(-\ell,\ell)} + \left\| \partial_1^2 \eta \right\|_{L^p(-\ell,\ell)} \right) \left\| \partial_t^2 \partial_1 \eta \right\|_{L^q(-\ell,\ell)} \\
&\lesssim \|v\|_{H^{\frac{1}{2}}(\Sigma_0)} \|\eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \lesssim \|v\|_{H^1(\Omega_0)} \|\eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \\
&\lesssim \|v\|_{H^1(\Omega_0)} \sqrt{\mathcal{E}} \sqrt{\mathcal{D}}.
\end{aligned}
\tag{5.11}
$$

Term: $-\sigma(\partial_1 \mathcal{R})\partial_t^2 \mathcal{N}$

We may directly bound

$$
|\partial_1 \mathcal{R}| \lesssim |\partial_1 \eta| \left| \partial_1^2 \eta \right| + |k_1| \left| \partial_1^2 \eta \right|. \tag{5.12}
$$

Then we estimate

$$
\begin{aligned}
\left| \int_{\Sigma_0} v \Big(-\sigma(\partial_1 \mathcal{R}) \partial_t^2 \mathcal{N} \Big) \right| &\lesssim \int_{\Sigma_0} |v| \, |\partial_1 \mathcal{R}| \left| \partial_t^2 \mathcal{N} \right| \\
&\lesssim \int_{\Sigma_0} |v| \, |\partial_1 \eta| \left| \partial_1^2 \eta \right| \left| \partial_t^2 \partial_1 \eta \right| + \int_{\Sigma_0} |v| \, |k_1| \left| \partial_1^2 \eta \right| \left| \partial_t^2 \partial_1 \eta \right| \\
&\lesssim \|v\|_{L^r(\Sigma_0)} \|\partial_1 \eta\|_{L^r(-\ell,\ell)} \left\| \partial_1^2 \eta \right\|_{L^p(-\ell,\ell)} \left\| \partial_t^2 \partial_1 \eta \right\|_{L^r(-\ell,\ell)} \\
&\lesssim \|v\|_{H^{\frac{1}{2}}(\Sigma_0)} \|\partial_1 \eta\|_{H^{\frac{1}{2}}(-\ell,\ell)} \|\eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)}
\end{aligned}
\tag{5.13}
$$

$$\lesssim \|v\|_{H^1(\Omega_0)} \|\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|\eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)}$$
$$\lesssim \|v\|_{H^1(\Omega_0)} \sqrt{\mathcal{E}}\sqrt{\mathcal{E}}\sqrt{\mathcal{D}} = \|v\|_{H^1(\Omega_0)} \mathcal{E}\sqrt{\mathcal{D}}.$$

Term: $-2\sigma\partial_1\left(\frac{\partial_t k_1\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}}+\frac{\partial_t k_1\partial_1\zeta_0+\partial_t\partial_1\eta}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3}\right)\partial_t\mathcal{N}$

We estimate

$$\left|\int_{\Sigma_0} v\left(-2\sigma\partial_1\left(\frac{\partial_t k_1\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}}+\frac{\partial_t k_1\partial_1\zeta_0+\partial_t\partial_1\eta}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3}\right)\partial_t\mathcal{N}\right)\right| \tag{5.14}$$
$$\lesssim \int_{\Sigma_0} |v|\left(|\partial_t k_1| + \left|\partial_t\partial_1^2\eta\right|\right)|\partial_t\mathcal{N}| \lesssim \int_{\Sigma_0} |v|\left(|\partial_t k_1| + \left|\partial_t\partial_1^2\eta\right|\right)|\partial_t\partial_1\eta|$$
$$\lesssim \|v\|_{L^q(\Sigma_0)}\left(\|\partial_t\eta\|_{L^p(-\ell,\ell)} + \left\|\partial_t\partial_1^2\eta\right\|_{L^p(-\ell,\ell)}\right)\|\partial_t\partial_1\eta\|_{L^q(-\ell,\ell)}$$
$$\lesssim \|v\|_{H^{\frac{1}{2}}(\Sigma_0)} \|\partial_t\eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \|\partial_t\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)} \lesssim \|v\|_{H^1(\Omega_0)} \|\partial_t\eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \|\partial_t\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}$$
$$\lesssim \|v\|_{H^1(\Omega_0)}\sqrt{\mathcal{D}}\sqrt{\mathcal{E}}.$$

Term: $-2\sigma(\partial_t\partial_1\mathcal{R})\partial_t\mathcal{N}$

We may directly obtain

$$|\partial_t\partial_1\mathcal{R}| \lesssim 6\,|\partial_t k_1|\left|\partial_1^2\eta\right| + |k_1|\left|\partial_t\partial_1^2\eta\right| + \left|\partial_1^2\eta\right||\partial_t\partial_1\eta| + |\partial_1\eta|\left|\partial_t\partial_1^2\eta\right|. \tag{5.15}$$

Then we estimate

$$\left|\int_{\Sigma_0} v\Big(-2\sigma(\partial_t\partial_1\mathcal{R})\partial_t\mathcal{N}\Big)\right| \lesssim \int_{\Sigma_0} |v|\,|\partial_t\partial_1\mathcal{R}|\,|\partial_t\mathcal{N}| \tag{5.16}$$
$$\lesssim \int_{\Sigma_0} |v|\left(\left|\partial_1^2\eta\right||\partial_t\partial_1\eta| + |\partial_1\eta|\left|\partial_t\partial_1^2\eta\right|\right)|\partial_t\partial_1\eta|$$
$$\lesssim \|v\|_{L^r(\Sigma_0)}\left(\left\|\partial_1^2\eta\right\|_{L^p(-\ell,\ell)}\|\partial_t\partial_1\eta\|_{L^r(-\ell,\ell)} + \|\partial_1\eta\|_{L^r(-\ell,\ell)}\left\|\partial_t\partial_1^2\eta\right\|_{L^p(-\ell,\ell)}\right)\|\partial_t\partial_1\eta\|_{L^r(-\ell,\ell)}$$
$$\lesssim \|v\|_{H^{\frac{1}{2}}(\Sigma_0)} \|\eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \|\partial_t\eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \|\partial_t\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}$$
$$\lesssim \|v\|_{H^1(\Omega_0)}\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{E}} = \|v\|_{H^1(\Omega_0)}\mathcal{E}\sqrt{\mathcal{D}}. \quad \square$$

5.4. Estimate of the $\mathcal{F}_4$ term

The estimate for $\mathcal{F}_4$ is again available from [9].

Lemma 5.6. *Let $\mathcal{F}_4$ be given by (C.2.4) or (C.2.11). We have the estimate*

$$\left| \int_{\Sigma_{0b}} \mathcal{F}_4 \left(v \cdot \frac{\mathcal{T}}{|\mathcal{T}|^2} \right) \right| \lesssim \|v\|_{H^1(\Omega_0)} \sqrt{\mathcal{E}} \sqrt{\mathcal{D}}. \tag{5.17}$$

Proof. The estimate is proved in Proposition 6.4 of [9], though there the nonlinearity is named F_5 instead of $\mathcal{F}_4$. $\square$

5.5. Estimate of the $\mathcal{F}_6$ and $\mathcal{F}_7$ terms

We now turn to the terms $\mathcal{F}_6$ and $\mathcal{F}_7$.

Lemma 5.7. *Let $\mathcal{F}_6$ be given by (C.2.6) or (C.2.13), and $\mathcal{F}_7$ be given by (C.2.7) or (C.2.14). We have*

$$\left| -\mathcal{F}_7 \left(\partial_t \mathscr{R} + \partial_t^2 \mathcal{O} \right) \Big|_{\ell} - \mathcal{F}_6 \left(\partial_t \mathscr{L} + \partial_t^2 \mathcal{O} \right) \Big|_{-\ell} \right| \lesssim \sqrt{\mathcal{E}} \mathcal{D}. \tag{5.18}$$

Proof. We know that

$$|\partial_t \mathscr{L}| = \left| \partial_t^3 l \right| \lesssim \sqrt{\mathcal{D}} \text{ and } |\partial_t \mathscr{R}| = \left| \partial_t^3 r \right| \lesssim \sqrt{\mathcal{D}}, \tag{5.19}$$

and that

$$\left| \partial_t^2 \mathcal{O} \right| \lesssim \Big(|\partial_t l| + |\partial_t r| \Big) \Big(\left| \partial_t^2 l \right| + \left| \partial_t^2 r \right| \Big) + |k_1| \Big(\left| \partial_t^3 l \right| + \left| \partial_t^3 r \right| \Big) \lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}}. \tag{5.20}$$

Hence, we have the estimates

$$\left| \partial_t \mathscr{L} + \partial_t^2 \mathcal{O} \Big|_{-\ell} \right| \lesssim \sqrt{\mathcal{D}} \text{ and } \left| \partial_t \mathscr{R} + \partial_t^2 \mathcal{O} \Big|_{\ell} \right| \lesssim \sqrt{\mathcal{D}}. \tag{5.21}$$

We may easily check that $\left| \tilde{\mathscr{W}}'(z) \right| + \left| \tilde{\mathscr{W}}''(z) \right| \lesssim z$ for z small, and so we know

$$|\mathcal{F}_6| \lesssim \left| \tilde{\mathscr{W}}'(\partial_t l) \right| \left| \partial_t^3 l \right| + \left| \tilde{\mathscr{W}}''(\partial_t l) \right| \left| \partial_t^2 l \right|^2 \lesssim |\partial_t l| \left| \partial_t^3 l \right| + |\partial_t l| \left| \partial_t^2 l \right|^2 \lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}}.$$

Next we bound

$$|\mathcal{F}_7| \lesssim \left| \tilde{\mathscr{W}}'(\partial_t r) \right| \left| \partial_t^3 r \right| + \left| \tilde{\mathscr{W}}''(\partial_t r) \right| \left| \partial_t^2 r \right|^2 \lesssim |\partial_t r| \left| \partial_t^3 r \right| + |\partial_t r| \left| \partial_t^2 r \right|^2 \lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}}, \tag{5.22}$$

from which we deduce that

$$\left| -\mathcal{F}_7\Big(\partial_t \mathscr{R} + \partial_t^2 \mathcal{O}\Big)\Big|_{\ell} - \mathcal{F}_6\Big(\partial_t \mathscr{L} + \partial_t^2 \mathcal{O}\Big)\Big|_{-\ell} \right| \lesssim \sqrt{\mathcal{E}}\mathcal{D}. \quad \square \tag{5.23}$$

5.6. *Estimate of the $\mathcal{F}_5$ term*

Next we handle $\mathcal{F}_5$.

Lemma 5.8. *Let $\mathcal{F}_5$ be given by (C.2.5) or (C.2.12) and $\mathcal{S}$ be given by (C.3.14). We have*

$$\left| \int_{-\ell}^{\ell} \left(g\varrho - \partial_1 \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right) \right) (\partial_t^2 \mathcal{S} + \mathcal{F}_5) - \frac{\mathrm{d}\mathfrak{F}_1}{\mathrm{d}t} \right| \lesssim \left(\mathcal{E} + \sqrt{\mathcal{E}}\right)\mathcal{D}, \tag{5.24}$$

where

$$\mathfrak{F}_1 = -\int_{-\ell}^{\ell} \frac{J_1 - 1}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \left|\partial_t^2 \partial_1 \eta\right|^2. \tag{5.25}$$

Proof. It is easy to check that

$$\begin{aligned} & g\varrho - \partial_1 \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \frac{\mathscr{K}_1 \partial_1 \zeta_0 + \partial_1 \varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right) \\ &= \left(g\varrho - \partial_1 \left(\frac{\mathscr{K}_1 \partial_1 \zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \frac{\mathscr{K}_1 \partial_1 \zeta_0}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right) \right) + \partial_1 \left(\frac{\partial_1 \varrho}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} \right) =: I + II. \end{aligned} \tag{5.26}$$

We may directly verify that

$$\begin{aligned} \|I\|_{L^\infty(-\ell,\ell)} &\lesssim \|\varrho\|_{L^\infty(-\ell,\ell)} + \|\mathscr{K}_1\|_{L^\infty(-\ell,\ell)} \lesssim \left\|\partial_t^2 \eta\right\|_{L^\infty(-\ell,\ell)} + \left\|\partial_t^2 k_1\right\|_{L^\infty(-\ell,\ell)} \\ &\lesssim \left\|\partial_t^2 \eta\right\|_{L^\infty(-\ell,\ell)} \lesssim \left\|\partial_t^2 \eta\right\|_{H^1(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}}. \end{aligned} \tag{5.27}$$

Then we have

$$\left| \int_{-\ell}^{\ell} I\,(\partial_t^2 \mathcal{S} + \mathcal{F}_5) \right| \lesssim \|I\|_{L^\infty(-\ell,\ell)} \left(\int_{-\ell}^{\ell} \left|\partial_t^2 \mathcal{S}\right| + \int_{-\ell}^{\ell} |\mathcal{F}_5| \right). \tag{5.28}$$

We may directly estimate

$$\int_{-\ell}^{\ell} \left|\partial_t^2 \mathcal{S}\right| \lesssim \left(|k_1| + \sum_{j=0}^{2} \left(\left|\partial_t^{j+1} l\right| + \left|\partial_t^{j+1} r\right| \right) \right) \tag{5.29}$$

$$\times \left(\left\|\partial_t^3 \eta\right\|_{H^0(-\ell,\ell)} + \left\|\partial_t^2 \eta\right\|_{H^1(-\ell,\ell)} + \|\eta\|_{H^1(-\ell,\ell)} + \sum_{j=0}^{2} \left(\left|\partial_t^{j+1} l\right| + \left|\partial_t^{j+1} r\right| \right) \right)$$

$$\lesssim \sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \mathcal{D},$$

and

$$\int_{-\ell}^{\ell} |\mathcal{F}_5| \lesssim \left(\|u\|_{H^0(-\ell,\ell)} + \|\partial_t u\|_{H^0(-\ell,\ell)} \right) \left(\|\partial_t \eta\|_{H^1(-\ell,\ell)} + \left\|\partial_t^2 \eta\right\|_{H^1(-\ell,\ell)} \right) \tag{5.30}$$

$$\lesssim \left(\|u\|_{H^1(\Omega_0)} + \|\partial_t u\|_{H^1(\Omega_0)} \right) \left(\|\partial_t \eta\|_{H^1(-\ell,\ell)} + \left\|\partial_t^2 \eta\right\|_{H^1(-\ell,\ell)} \right) \lesssim \sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \mathcal{D}.$$

Therefore, we know that

$$\left| \int_{-\ell}^{\ell} I(\partial_t^2 \mathcal{S} + \mathcal{F}_5) \right| \lesssim \sqrt{\mathcal{E}}\mathcal{D}. \tag{5.31}$$

Then we turn to the estimate of II. We integrate by parts to obtain

$$\int_{-\ell}^{\ell} II(\partial_t^2 \mathcal{S} + \mathcal{F}_5) = \int_{-\ell}^{\ell} \partial_1 \left(\frac{\partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) (\partial_t^2 \mathcal{S} + \mathcal{F}_5) \tag{5.32}$$

$$= -\int_{-\ell}^{\ell} \left(\frac{\partial_1 \varrho}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \right) (\partial_t^2 \partial_1 \mathcal{S} + \partial_1 \mathcal{F}_5) = \int_{-\ell}^{\ell} b \partial_t^2 \partial_1 \eta \left(\partial_t^2 \partial_1 \mathcal{S} + \partial_1 \mathcal{F}_5 \right),$$

for $b(x_1) = -\left(1 + |\partial_1 \zeta_0|^2\right)^{-3/2}$.

First Term of $\partial_t^2 \partial_1 \mathcal{S}$: $\partial_t^2 \partial_1 \left((J_1 - 1)\partial_t \eta \right)$
We can directly compute

$$\partial_t^2 \partial_1 \left((J_1 - 1)\partial_t \eta \right) = \partial_t^2 J_1 \partial_t \partial_1 \eta + 2\partial_t J_1 \partial_t^2 \partial_1 \eta + (J_1 - 1)\partial_t^3 \partial_1 \eta. \tag{5.33}$$

For the first two terms we estimate

$$\left| \int_{-\ell}^{\ell} b \left(\partial_t^2 \partial_1 \eta \right) \left(\partial_t^2 J_1 \partial_t \partial_1 \eta \right) \right| \lesssim \left\| \partial_t^2 \partial_1 \eta \right\|_{L^2(-\ell,\ell)} \left\| \partial_t^2 J_1 \right\|_{L^\infty(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{L^2(-\ell,\ell)} \tag{5.34}$$

$$\lesssim \left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)} \left(\left|\partial_t^2 l\right| + \left|\partial_t^2 r\right|\right) \|\partial_t\eta\|_{H^1(-\ell,\ell)}$$
$$\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D},$$

and

$$\left|\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(2\partial_t J_1\partial_t^2\partial_1\eta\right)\right| \lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)} \|\partial_t J_1\|_{L^\infty(-\ell,\ell)} \left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)} \tag{5.35}$$
$$\lesssim \left\|\partial_t^2\eta\right\|^2_{H^1(-\ell,\ell)} \left(|\partial_t l| + |\partial_t r|\right) \lesssim \left(\sqrt{\mathcal{D}}\right)^2\sqrt{\mathcal{E}} = \sqrt{\mathcal{E}}\mathcal{D}.$$

For the third term we then write

$$\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left((J_1-1)\partial_t^3\partial_1\eta\right) = \partial_t\left(\int_{-\ell}^{\ell} b(J_1-1)\left|\partial_t^2\partial_1\eta\right|^2\right) - \int_{-\ell}^{\ell} b\partial_t J_1\left|\partial_t^2\partial_1\eta\right|^2, \tag{5.36}$$

where we have

$$\left|\int_{-\ell}^{\ell} b\partial_t J_1\left|\partial_t^2\partial_1\eta\right|^2\right| \lesssim \|\partial_t J_1\|_{L^\infty(-\ell,\ell)} \left\|\partial_t^2\partial_1\eta\right\|^2_{L^2(-\ell,\ell)} \tag{5.37}$$
$$\lesssim \left(|\partial_t l| + |\partial_t r|\right)\left\|\partial_t^2\eta\right\|^2_{H^1(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}}\left(\sqrt{\mathcal{D}}\right)^2 = \sqrt{\mathcal{E}}\mathcal{D}.$$

Second Term of $\partial_t^2\partial_1\mathcal{S}$: $\partial_t^2\partial_1\left(\tilde{a}\partial_1\eta\right)$

We can directly compute

$$\partial_t^2\partial_1\left(\tilde{a}\partial_1\eta\right) = -\partial_t^3 k_1\partial_1\eta + \partial_t^2\tilde{a}\partial_1^2\eta - 2\partial_t^2 k_1\partial_t\partial_1\eta + 2\partial_t\tilde{a}\partial_t\partial_1^2\eta - \partial_t k_1\partial_t^2\partial_1\eta + \tilde{a}\partial_t^2\partial_1^2\eta. \tag{5.38}$$

We estimate each term as follows

$$\left|\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(\partial_t^3 k_1\partial_1\eta\right)\right| \lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)} \left\|\partial_t^3 k_1\right\|_{L^\infty(-\ell,\ell)} \|\partial_1\eta\|_{L^2(-\ell,\ell)} \tag{5.39}$$
$$\lesssim \left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)} \left(\left|\partial_t^3 l\right| + \left|\partial_t^3 r\right|\right) \|\eta\|_{H^1(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D},$$

$$\left|\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(\partial_t^2\tilde{a}\partial_1^2\eta\right)\right| \lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)} \left\|\partial_t^2\tilde{a}\right\|_{L^\infty(-\ell,\ell)} \left\|\partial_1^2\eta\right\|_{L^2(-\ell,\ell)} \tag{5.40}$$
$$\lesssim \left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)} \left(\left|\partial_t^3 l\right| + \left|\partial_t^3 r\right|\right) \|\eta\|_{H^2(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D},$$

$$\left|\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(2\partial_t^2 k_1\partial_t\partial_1\eta\right)\right| \lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)}\left\|\partial_t^2 k_1\right\|_{L^\infty(-\ell,\ell)}\|\partial_t\partial_1\eta\|_{L^2(-\ell,\ell)} \tag{5.41}$$
$$\lesssim \left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)}\left(\left|\partial_t^2 l\right|+\left|\partial_t^2 r\right|\right)\|\partial_t\eta\|_{H^1(-\ell,\ell)}$$
$$\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}}=\sqrt{\mathcal{E}}\mathcal{D},$$

$$\left|\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(2\partial_t\tilde{a}\partial_t\partial_1^2\eta\right)\right| \lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)}\|\partial_t\tilde{a}\|_{L^\infty(-\ell,\ell)}\left\|\partial_t\partial_1^2\eta\right\|_{L^2(-\ell,\ell)} \tag{5.42}$$
$$\lesssim \left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)}\left(\left|\partial_t^2 l\right|+\left|\partial_t^2 r\right|\right)\|\partial_t\eta\|_{H^2(-\ell,\ell)}$$
$$\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}}=\sqrt{\mathcal{E}}\mathcal{D}$$

$$\left|\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(\partial_t k_1\partial_t^2\partial_1\eta\right)\right| \lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)}\|\partial_t k_1\|_{L^\infty(-\ell,\ell)}\left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)} \tag{5.43}$$
$$\lesssim \left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)}\left(|\partial_t l|+|\partial_t r|\right)\left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)}$$
$$\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}}=\sqrt{\mathcal{E}}\mathcal{D}.$$

For the remaining term we write

$$\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(\tilde{a}\partial_t^2\partial_1^2\eta\right)=\int_{-\ell}^{\ell} b\tilde{a}\partial_1\left|\partial_t^2\partial_1\eta\right|^2=-\int_{-\ell}^{\ell}\partial_1\left(b\tilde{a}\right)\left|\partial_t^2\partial_1\eta\right|^2+\left(b\tilde{a}\left|\partial_t^2\partial_1\eta\right|^2\right)\Big|_{-\ell}^{\ell} \tag{5.44}$$

and then estimate

$$\left|\int_{-\ell}^{\ell}\partial_1\left(b\tilde{a}\right)\left|\partial_t^2\partial_1\eta\right|^2\right| \lesssim \left\|\partial_1\left(b\tilde{a}\right)\right\|_{L^\infty(-\ell,\ell)}\left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)}^2 \tag{5.45}$$
$$\lesssim\left(|\partial_t l|+|\partial_t r|\right)\left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)}^2\lesssim\sqrt{\mathcal{E}}\left(\sqrt{\mathcal{D}}\right)^2=\sqrt{\mathcal{E}}\mathcal{D},$$

and

$$\left|\left(b\tilde{a}\left|\partial_t^2\partial_1\eta\right|^2\right)\Big|_{-\ell}^{\ell}\right|\lesssim|b\tilde{a}|\left(\left|\partial_t^2\partial_1\eta\right|^2\Big|_{-\ell}+\left|\partial_t^2\partial_1\eta\right|^2\Big|_{\ell}\right)\lesssim\sqrt{\mathcal{E}}\left(\sqrt{\mathcal{D}}\right)^2=\sqrt{\mathcal{E}}\mathcal{D}. \tag{5.46}$$

Third Term of $\partial_t^2\partial_1\mathcal{S}$: $\partial_t^2\partial_1\left(\mathcal{O}\partial_1\zeta_0\right)$

We can directly compute

$$\partial_t^2\partial_1\left(\mathcal{O}\partial_1\zeta_0\right)=\partial_t^2\partial_1\mathcal{O}\partial_1\zeta_0+\partial_t^2\mathcal{O}\partial_1^2\zeta_0. \tag{5.47}$$

We estimate each term via:

$$\begin{aligned}\left|\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(\partial_t^2\partial_1\mathcal{O}\partial_1\zeta_0\right)\right| &\lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)}\left\|\partial_t^2\partial_1\mathcal{O}\right\|_{L^2(-\ell,\ell)} \\ &\lesssim \left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)}\left(\sum_{j=0}^{2}\left(\left|\partial_t^{j+1}l\right|+\left|\partial_t^{j+1}r\right|\right)\right)^2 \\ &\lesssim \sqrt{\mathcal{E}}\left(\sqrt{\mathcal{D}}\right)^2=\sqrt{\mathcal{E}}\mathcal{D},\end{aligned} \tag{5.48}$$

and

$$\begin{aligned}\left|\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(\partial_t^2\mathcal{O}\partial_1^2\zeta_0\right)\right| &\lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^2(-\ell,\ell)}\left\|\partial_t^2\mathcal{O}\right\|_{L^2(-\ell,\ell)} \\ &\lesssim \left\|\partial_t^2\eta\right\|_{H^1(-\ell,\ell)}\left(\sum_{j=0}^{2}\left(\left|\partial_t^{j+1}l\right|+\left|\partial_t^{j+1}r\right|\right)\right)^2 \\ &\lesssim \sqrt{\mathcal{E}}\left(\sqrt{\mathcal{D}}\right)^2=\sqrt{\mathcal{E}}\mathcal{D}.\end{aligned} \tag{5.49}$$

First Term of $\partial_1\mathcal{F}_5$**:** $\partial_1\left(u\cdot\partial_t^2\mathcal{N}\right)$

We can directly compute

$$\partial_1\left(u\cdot\partial_t^2\mathcal{N}\right)=\partial_1 u\cdot\partial_t^2\mathcal{N}+u\cdot\partial_t^2\partial_1\mathcal{N}. \tag{5.50}$$

For the first term, we choose $p=\dfrac{3+\delta}{2+2\delta}$ and $q=\dfrac{6+2\delta}{1-\delta}$ such that $\dfrac{1}{p}+\dfrac{2}{q}=1$.

$$\begin{aligned}\left|\int_{-\ell}^{\ell} b\left(\partial_t^2\partial_1\eta\right)\left(\partial_1 u\cdot\partial_t^2\mathcal{N}\right)\right| &\lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^q(-\ell,\ell)}\left\|\partial_1 u\right\|_{L^p(-\ell,\ell)}\left\|\partial_t^2\partial_1\eta\right\|_{L^q(-\ell,\ell)} \\ &\lesssim \left\|\partial_t^2\partial_1\eta\right\|_{H^{\frac{1}{2}}(-\ell,\ell)}\left\|\partial_1 u\right\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)}\left\|\partial_t^2\partial_1\eta\right\|_{H^{\frac{1}{2}}(-\ell,\ell)} \\ &\lesssim \left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)}\left\|u\right\|_{W_\delta^2(\Omega_0)}\left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \\ &\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}}=\sqrt{\mathcal{E}}\mathcal{D}.\end{aligned} \tag{5.51}$$

For the second term, we have

$$\int_{-\ell}^{\ell} b\left(\partial_t^2 \partial_1 \eta\right)\left(u \cdot \partial_t^2 \partial_1 \mathcal{N}\right) = \int_{-\ell}^{\ell} b u_1 \partial_1 \left|\partial_t^2 \partial_1 \eta\right|^2 \tag{5.52}$$
$$= -\int_{-\ell}^{\ell} \partial_1\left(b u_1\right) \left|\partial_t^2 \partial_1 \eta\right|^2 + \left(b u_1 \left|\partial_t^2 \partial_1 \eta\right|^2\right)\Big|_{-\ell}^{\ell},$$

and then bound

$$\left|\int_{-\ell}^{\ell} \partial_1(b u_1) \left|\partial_t^2 \partial_1 \eta\right|^2\right| \lesssim \|\partial_1(b u_1)\|_{L^p(-\ell,\ell)} \left\|\partial_t^2 \partial_1 \eta\right\|_{L^q(-\ell,\ell)}^2 \tag{5.53}$$
$$\lesssim \|u\|_{W_\delta^2(\Omega_0)} \left\|\partial_t^2 \eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)}^2 \lesssim \sqrt{\mathcal{E}}\left(\sqrt{\mathcal{D}}\right)^2 = \sqrt{\mathcal{E}}\mathcal{D}$$

and

$$\left|\left(b u_1 \left|\partial_t^2 \partial_1 \eta\right|^2\right)\Big|_{-\ell}^{\ell}\right| \lesssim \left(|u_1|\Big|_{-\ell} + |u_1|\Big|_{\ell}\right)\left(\left|\partial_t^2 \partial_1 \eta\right|^2\Big|_{-\ell} + \left|\partial_t^2 \partial_1 \eta\right|^2\Big|_{\ell}\right) \tag{5.54}$$
$$\lesssim \left(|\partial_t l| + |\partial_t r|\right)\left(\left|\partial_t^2 \partial_1 \eta\right|^2\Big|_{-\ell} + \left|\partial_t^2 \partial_1 \eta\right|^2\Big|_{\ell}\right) \lesssim \sqrt{\mathcal{E}}\left(\sqrt{\mathcal{D}}\right)^2 = \sqrt{\mathcal{E}}\mathcal{D}.$$

Second Term of $\partial_1 \mathcal{F}_5$: $\partial_1\left(\partial_t u \cdot \partial_t \mathcal{N}\right)$
We can directly compute

$$\partial_1\left(\partial_t u \cdot \partial_t \mathcal{N}\right) = \partial_t \partial_1 u \cdot \partial_t \mathcal{N} + \partial_t u \cdot \partial_t \partial_1 \mathcal{N}. \tag{5.55}$$

We estimate each term as follows:

$$\left|\int_{-\ell}^{\ell} b\left(\partial_t^2 \partial_1 \eta\right)\left(\partial_t \partial_1 u \cdot \partial_t \mathcal{N}\right)\right| \lesssim \left\|\partial_t^2 \partial_1 \eta\right\|_{L^q(-\ell,\ell)} \|\partial_t \partial_1 u\|_{L^p(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)} \tag{5.56}$$
$$\lesssim \left\|\partial_t^2 \partial_1 \eta\right\|_{H^{\frac{1}{2}}(-\ell,\ell)} \|\partial_t \partial_1 u\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{H^{\frac{1}{2}}(-\ell,\ell)}$$
$$\lesssim \left\|\partial_t^2 \eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|\partial_t u\|_{W_\delta^2(\Omega_0)} \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}$$
$$\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D}$$

and

$$\left|\int_{-\ell}^{\ell} b\left(\partial_t^2 \partial_1 \eta\right)\left(\partial_t u \cdot \partial_t \partial_1 \mathcal{N}\right)\right| \lesssim \left\|\partial_t^2 \partial_1 \eta\right\|_{L^q(-\ell,\ell)} \|\partial_t u\|_{L^q(-\ell,\ell)} \left\|\partial_t \partial_1^2 \eta\right\|_{L^p(-\ell,\ell)} \tag{5.57}$$

$$\lesssim \left\|\partial_t^2\partial_1\eta\right\|_{H^{\frac{1}{2}}(-\ell,\ell)} \|\partial_t u\|_{H^{\frac{1}{2}}(-\ell,\ell)} \left\|\partial_t\partial_1^2\eta\right\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)}$$
$$\lesssim \left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|\partial_t u\|_{H^1(\Omega_0)} \|\partial_t\eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)}$$
$$\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D}. \quad \square$$

5.7. Other nonlinear estimates

We now turn our attention to bounding various other nonlinear terms that appear in the analysis. We begin with $\mathcal{O}$.

Lemma 5.9. *Let $\mathcal{O}$ be given by (C.3.18). We have*

$$\left|\kappa\left(\partial_t\mathscr{R}\partial_t^2\mathcal{O}\Big|_\ell + \partial_t\mathscr{L}\partial_t^2\mathcal{O}\Big|_{-\ell}\right)\right| \lesssim \sqrt{\mathcal{E}}\mathcal{D}.$$

Proof. By definition $\partial_t\mathscr{R} = \partial_t^3 R$ and $\partial_t\mathscr{L} = \partial_t^3 L$. Then we estimate

$$\left|\kappa\left(\partial_t\mathscr{R}\partial_t^2\mathcal{O}\Big|_\ell + \partial_t\mathscr{L}\partial_t^2\mathcal{O}\Big|_{-\ell}\right)\right| \lesssim \left(\left|\partial_t^3 l\right| + \left|\partial_t^3 r\right|\right)\left(|\partial_t l| + |\partial_t r|\right)\left(\left|\partial_t^2 l\right| + \left|\partial_t^2 r\right|\right) \tag{5.58}$$
$$+\left(\left|\partial_t^3 r\right| + \left|\partial_t^3 l\right|\right)|k_1|\left(\left|\partial_t^3 l\right| + \left|\partial_t^3 r\right|\right) \lesssim \sqrt{\mathcal{D}}\sqrt{\mathcal{D}}\sqrt{\mathcal{E}} + \sqrt{\mathcal{D}}\sqrt{\mathcal{E}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D}. \quad \square$$

Next we consider $\mathcal{Q}$ and $\mathcal{O}$.

Lemma 5.10. *Let $\mathcal{Q}$ be given by (C.3.8) and $\mathcal{O}$ be given by (C.3.18). We have*

$$\left|\sigma\partial_t^2\mathcal{Q}\left(\partial_t\mathscr{L} + \partial_t^2\mathcal{O}\right)\Big|_\ell - \sigma\partial_t^2\mathcal{Q}\left(\partial_t\mathscr{R} + \partial_t^2\mathcal{O}\right)\Big|_{-\ell}\right| \lesssim \sqrt{\mathcal{E}}\mathcal{D}. \tag{5.59}$$

Proof. By Lemma C.6, we have

$$\left|\left(\partial_t\mathscr{L} + \partial_t^2\mathcal{O}\right)\Big|_\ell\right| + \left|\left(\partial_t\mathscr{L} + \partial_t^2\mathcal{O}\right)\Big|_{-\ell}\right| \lesssim \sqrt{\mathcal{D}}. \tag{5.60}$$

Also, it is easy to check

$$\left|\partial_t^2\mathcal{Q}\Big|_{\pm\ell}\right| \lesssim |k_1|\left|\partial_t^2 k_1\right| + |\partial_t k_1|^2 + \left|\partial_t^2 k_1\right||\partial_1\eta(\pm\ell)| + |k_1|\left|\partial_t^2\partial_1\eta(\pm\ell)\right| \tag{5.61}$$
$$+ |\partial_1\eta(\pm\ell)|\left|\partial_t^2\partial_1\eta(\pm\ell)\right| + |\partial_t\partial_1\eta(\pm\ell)|^2 \lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}.$$

Therefore, our result easily follows. $\square$

We may write $\mathcal{R} = \mathcal{R}(\varpi_1, \varpi_2)$ where $\varpi_1 = k_1$ and $\varpi_2 = \partial_1\eta$. Let $\partial_{\varpi_i}\mathcal{R}$ denote the derivative of $\mathcal{R}$ with respect to ϖ_i for $i = 1, 2$.

Lemma 5.11. *Let $\mathcal{R}$ be given by (C.3.1). We have the estimate*

$$\left| \sigma \int_{\Sigma_0} \partial_1 \partial_t^2 \mathcal{R}(v \cdot \mathcal{N}) - \frac{\mathrm{d}\mathfrak{F}_2}{\mathrm{d}t} \right| \lesssim \sqrt{\mathcal{E}} \mathcal{D}, \tag{5.62}$$

where

$$\mathfrak{F}_2 = \int_{-\ell}^{\ell} \frac{\partial_{\varpi_2} \mathcal{R} J_1}{\sqrt{1 + |\partial_1 \zeta_0|^2}} \left| \partial_t^2 \partial_1 \eta \right|^2. \tag{5.63}$$

Proof. Using the transport equation in (1.25) and integrating by parts, for $v = \partial_t^2 u$, we know

$$\begin{aligned} \sigma \int_{\Sigma_0} \partial_1 \partial_t^2 \mathcal{R}(v \cdot \mathcal{N}) &= \sigma \int_{\Sigma_0} \partial_1 \partial_t^2 \mathcal{R}\Big(bJ_1 \partial_t^3 \eta + \partial_t^2 (b\tilde{a} \partial_1 \eta) + \partial_t u \cdot \partial_t \mathcal{N} + u \cdot \partial_t^2 \mathcal{N} \Big) \qquad (5.64) \\ &= -\sigma \int_{\Sigma_0} \partial_t^2 \mathcal{R}\Big(\partial_1 \Big(bJ_1 \partial_t^3 \eta \Big) + \partial_1 \Big(\partial_t^2 (b\tilde{a} \partial_1 \eta) \Big) \\ &\quad + \partial_1 \Big(\partial_t u \cdot \partial_t \mathcal{N} \Big) + \partial_1 \Big(u \cdot \partial_t^2 \mathcal{N} \Big) \Big) \\ &\quad + \sigma \Big(\partial_t^2 \mathcal{R} \Big(bJ_1 \partial_t^3 \eta + \partial_t^2 (b\tilde{a} \partial_1 \eta) + \partial_t u \cdot \partial_t \mathcal{N} + u \cdot \partial_t^2 \mathcal{N} \Big) \Big) \Big|_{-\ell}^{\ell}, \end{aligned}$$

where here we have written $b(x_1) = \left(1 + |\partial_1 \zeta_0|^2\right)^{-1/2}$.

We may directly verify that

$$\partial_t \mathcal{R} = \partial_{\varpi_1} \mathcal{R} \partial_t k_1 + \partial_{\varpi_2} \mathcal{R} \partial_t \partial_1 \eta, \tag{5.65}$$

$$\partial_t^2 \mathcal{R} = \partial_{\varpi_1} \mathcal{R} \partial_t^2 k_1 + \partial_{\varpi_1}^2 \mathcal{R} (\partial_t k_1)^2 + \partial_{\varpi_1} \partial_{\varpi_2} \mathcal{R} \partial_t k_1 \partial_t \partial_1 \eta + \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 + \partial_{\varpi_2} \mathcal{R} \partial_t^2 \partial_1 \eta. \tag{5.66}$$

It is easy to check that

$$\left| \partial_{\varpi_1} \mathcal{R} \right| + \left| \partial_{\varpi_2} \mathcal{R} \right| + \left| \partial_{\varpi_1}^2 \mathcal{R} \right| + \left| \partial_{\varpi_1} \partial_{\varpi_2} \mathcal{R} \right| + \left| \partial_{\varpi_2}^2 \mathcal{R} \right| \lesssim |k_1| + |\partial_1 \eta|. \tag{5.67}$$

Since the terms related to k_1 are easier to estimate, we will focus on the terms related to $\partial_{\varpi_2}^2 (\partial_t \partial_1 \eta)^2$ and $\partial_{\varpi_2} \partial_t^2 \partial_1 \eta$. We will proceed with the estimates term by term.

First Integral Term, $\partial_1 \Big(bJ_1 \partial_t^3 \eta \Big)$: We can directly compute

$$\partial_1 \Big(bJ_1 \partial_t^3 \eta \Big) = \partial_1 bJ_1 \partial_t^3 \eta + bJ_1 \partial_t^3 \partial_1 \eta. \tag{5.68}$$

In the following, we choose $p = \dfrac{3+\delta}{2+2\delta}$ and $q = \dfrac{6+2\delta}{1-\delta}$ such that $\dfrac{1}{p} + \dfrac{2}{q} = 1$. The $\partial_1 b J_1 \partial_t^3 \eta$ terms can be directly estimated:

$$
\begin{aligned}
\left| \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \Big(\partial_1 b J_1 \partial_t^3 \eta \Big) \right| &\lesssim \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)} \left\| \partial_t^3 \eta \right\|_{L^p(-\ell,\ell)} \\
&\lesssim \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)} \left\| \partial_t^3 \eta \right\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \\
&\lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D},
\end{aligned}
\tag{5.69}
$$

and

$$
\begin{aligned}
\left| \int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} \partial_t^2 \partial_1 \eta \Big(\partial_1 b J_1 \partial_t^3 \eta \Big) \right| &\lesssim \|\partial_1 \eta\|_{L^q(-\ell,\ell)} \left\| \partial_t^2 \partial_1 \eta \right\|_{L^q(-\ell,\ell)} \left\| \partial_t^3 \eta \right\|_{L^p(-\ell,\ell)} \\
&\lesssim \|\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)} \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \left\| \partial_t^3 \eta \right\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \\
&\lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D}.
\end{aligned}
\tag{5.70}
$$

The $b J_1 \partial_t^3 \partial_1 \eta$ terms require more efforts. We integrate by parts to obtain

$$
\begin{aligned}
\int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \Big(b J_1 \partial_t^3 \partial_1 \eta \Big) &= -\int_{-\ell}^{\ell} \partial_1 \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 (b J_1) \partial_t^3 \eta \\
- \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)(\partial_t \partial_1^2 \eta)(b J_1) \partial_t^3 \eta &- \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \partial_1 (b J_1) \partial_t^3 \eta = I + II + III,
\end{aligned}
\tag{5.71}
$$

where the contact point terms vanish since $\partial_t^3 \eta(-\ell) = \partial_t^3 \eta(\ell) = 0$. Then we have

$$
\begin{aligned}
|I| = \left| \int_{-\ell}^{\ell} \partial_1 \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 (b J_1) \partial_t^3 \eta \right| &\lesssim \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)}^2 \left\| \partial_t^3 \eta \right\|_{L^p(-\ell,\ell)} \\
\lesssim \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}^2 \left\| \partial_t^3 \eta \right\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} &\lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D},
\end{aligned}
\tag{5.72}
$$

$$
\begin{aligned}
|II| = \left| \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)(\partial_t \partial_1^2 \eta)(b J_1) \partial_t^3 \eta \right| &\lesssim \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)} \left\| \partial_t \partial_1^2 \eta \right\|_{L^q(-\ell,\ell)} \left\| \partial_t^3 \eta \right\|_{L^p(-\ell,\ell)} \\
\lesssim \|\partial_t \eta\|_{H^{\frac{1}{2}}(-\ell,\ell)} \|\partial_t \eta\|_{H^{\frac{5}{2}}(-\ell,\ell)} \left\| \partial_t^3 \eta \right\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} &\lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D},
\end{aligned}
\tag{5.73}
$$

and

$$|III| = \left| \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \partial_1 (b J_1) \partial_t^3 \eta \right| \lesssim \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)}^2 \left\| \partial_t^3 \eta \right\|_{L^p(-\ell,\ell)} \tag{5.74}$$

$$\lesssim \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}^2 \left\| \partial_t^3 \eta \right\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D}.$$

For the remaining term we compute

$$\int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} \partial_t^2 \partial_1 \eta \Big(b J_1 \partial_t^3 \partial_1 \eta \Big) = \partial_t \Bigg(\int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} b J_1 \left| \partial_t^2 \partial_1 \eta \right|^2 \Bigg) - \int_{-\ell}^{\ell} b \partial_t (\partial_{\varpi_2} \mathcal{R} J_1) \left| \partial_t^2 \partial_1 \eta \right|^2, \tag{5.75}$$

and then estimate

$$\left| \int_{-\ell}^{\ell} b \partial_t (\partial_{\varpi_2} \mathcal{R} J_1) \left| \partial_t^2 \partial_1 \eta \right|^2 \right| \lesssim \Big(\|\partial_t J_1\|_{L^\infty(-\ell,\ell)} + \|\partial_t \partial_1 \eta\|_{L^\infty(-\ell,\ell)} \Big) \left\| \partial_t^2 \partial_1 \eta \right\|_{L^2(-\ell,\ell)}^2 \tag{5.76}$$

$$\lesssim \Big(|\partial_t l| + |\partial_t r| + \|\partial_t \eta\|_{H^2(-\ell,\ell)} \Big) \left\| \partial_t^2 \eta \right\|_{H^1(-\ell,\ell)}^2$$

$$\lesssim \sqrt{\mathcal{E}} \Big(\sqrt{\mathcal{D}} \Big)^2 = \sqrt{\mathcal{E}} \mathcal{D}.$$

Second Integral Term, $\partial_1 \Big(\partial_t^2 (b \tilde{a} \partial_1 \eta) \Big)$**:** We begin by computing

$$\begin{aligned} \partial_t^2 \partial_1 \Big(b \tilde{a} \partial_1 \eta \Big) = {} & \partial_1 b \partial_t^2 \tilde{a} \partial_1 \eta + \partial_1 b \partial_t \tilde{a} \partial_t \partial_1 \eta + \partial_1 b \tilde{a} \partial_t^2 \partial_1 \eta \\ & - b \partial_t^3 k_1 \partial_1 \eta + b \partial_t^2 \tilde{a} \partial_1^2 \eta - 2 b \partial_t^2 k_1 \partial_t \partial_1 \eta \\ & + 2 b \partial_t \tilde{a} \partial_t \partial_1^2 \eta - b \partial_t k_1 \partial_t^2 \partial_1 \eta + b \tilde{a} \partial_t^2 \partial_1^2 \eta. \end{aligned} \tag{5.77}$$

For the first two of these terms we may bound

$$\left| \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \Big(\partial_1 b \partial_t^2 \tilde{a} \partial_1 \eta \Big) \right| \lesssim \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)}^2 \left\| \partial_t^2 \tilde{a} \right\|_{L^\infty(-\ell,\ell)} \|\partial_1 \eta\|_{L^p(-\ell,\ell)} \tag{5.78}$$

$$\lesssim \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}^2 \Big(\left| \partial_t^3 l \right| + \left| \partial_t^3 r \right| \Big) \|\eta\|_{W_\delta^{\frac{3}{2}}(-\ell,\ell)}$$

$$\lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D}.$$

$$\left| \int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} \partial_t^2 \partial_1 \eta \Big(\partial_1 b \partial_t^2 \tilde{a} \partial_1 \eta \Big) \right| \lesssim \left\| \partial_t^2 \partial_1 \eta \right\|_{L^2(-\ell,\ell)} \left\| \partial_t^2 \tilde{a} \right\|_{L^\infty(-\ell,\ell)} \|\partial_1 \eta\|_{L^2(-\ell,\ell)} \tag{5.79}$$

$$\lesssim \left\| \partial_t^2 \eta \right\|_{H^1(-\ell,\ell)}^2 \left(\left| \partial_t^3 l \right| + \left| \partial_t^3 r \right| \right) \|\eta\|_{H^1(-\ell,\ell)}$$
$$\lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D}.$$

Arguing similarly for all but the last term, we conclude that

$$\left| \sigma \int_{\Sigma_0} \partial_t^2 \mathcal{R} \Big(\partial_1 \partial_t^2 (b\tilde{a} \partial_1 \eta) - b\tilde{a} \partial_t^2 \partial_1^2 \eta \Big) \right| \lesssim \sqrt{\mathcal{E}} \mathcal{D}. \tag{5.80}$$

The last term is much more complicated. To handle it we first integrate by parts to obtain

$$\begin{aligned}
&\int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \Big(b\tilde{a} \partial_t^2 \partial_1^2 \eta \Big) = -\int_{-\ell}^{\ell} \partial_1 \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 (b\tilde{a}) \partial_t^2 \partial_1 \eta \\
&- \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)(\partial_t \partial_1^2 \eta)(b\tilde{a}) \partial_t^2 \partial_1 \eta \\
&- \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \partial_1 (b\tilde{a}) \partial_t^2 \partial_1 \eta + \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \Big(b\tilde{a} \partial_t^2 \partial_1 \eta \Big) \Big|_{-\ell}^{\ell} \\
&= I + II + III + IV.
\end{aligned} \tag{5.81}$$

Then we have

$$\begin{aligned}
|I| &= \left| \int_{-\ell}^{\ell} \partial_1 \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 (b\tilde{a}) \partial_t^2 \partial_1 \eta \right| \lesssim \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)}^2 \left\| \partial_t^2 \partial_1 \eta \right\|_{L^p(-\ell,\ell)} \\
&\lesssim \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}^2 \left\| \partial_t^2 \eta \right\|_{W_\delta^{\frac{3}{2}}(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D},
\end{aligned} \tag{5.82}$$

$$\begin{aligned}
|II| &= \left| \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)(\partial_t \partial_1^2 \eta)(b\tilde{a}) \partial_t^2 \partial_1 \eta \right| \lesssim \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)} \left\| \partial_t \partial_1^2 \eta \right\|_{L^p(-\ell,\ell)} \left\| \partial_t^2 \partial_1 \eta \right\|_{L^q(-\ell,\ell)} \\
&\lesssim \|\partial_t \eta\|_{H^{\frac{1}{2}}(-\ell,\ell)} \|\partial_t \eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D},
\end{aligned} \tag{5.83}$$

and

$$|III| = \left| \int_{-\ell}^{\ell} \partial^2_{\varpi_2} \mathcal{R} (\partial_t \partial_1 \eta)^2 \partial_1 (b\tilde{a}) \partial_t^2 \partial_1 \eta \right| \lesssim \|\partial_t \partial_1 \eta\|^2_{L^q(-\ell,\ell)} \left\| \partial_t^2 \partial_1 \eta \right\|_{L^p(-\ell,\ell)} \tag{5.84}$$
$$\lesssim \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \left\| \partial_t^2 \eta \right\|_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D}.$$

The contact point term

$$|IV| \lesssim \left| \partial^2_{\varpi_2} \mathcal{R} (\partial_t \partial_1 \eta)^2 \Big(b\tilde{a} \partial_t^2 \partial_1 \eta \Big) \Big|_{-\ell}^{\ell} \right| \tag{5.85}$$
$$\lesssim \Big(|\partial_t \partial_1 \eta(-\ell)|^2 + |\partial_t \partial_1 \eta(\ell)|^2 \Big) \Big(\left| \partial_t^2 \partial_1 \eta(-\ell) \right| + \left| \partial_t^2 \partial_1 \eta(\ell) \right| \Big) \lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D}.$$

On the other hand,

$$\int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} \partial_t^2 \partial_1 \eta \Big(b\tilde{a} \partial_t^2 \partial_1^2 \eta \Big) = \int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} b\tilde{a} \partial_1 \left| \partial_t^2 \partial_1 \eta \right|^2 \tag{5.86}$$
$$= -\int_{-\ell}^{\ell} \partial_1 \Big(\partial_{\varpi_2} \mathcal{R} b\tilde{a} \Big) \left| \partial_t^2 \partial_1 \eta \right|^2 + \Big(\partial_{\varpi_2} \mathcal{R} b\tilde{a} \left| \partial_t^2 \partial_1 \eta \right|^2 \Big) \Big|_{-\ell}^{\ell},$$

where we have the bounds

$$\left| \int_{-\ell}^{\ell} \partial_1 \Big(\partial_{\varpi_2} \mathcal{R} b\tilde{a} \Big) \left| \partial_t^2 \partial_1 \eta \right|^2 \right| \lesssim \left\| \partial_1 \Big(\partial_{\varpi_2} \mathcal{R} b\tilde{a} \Big) \right\|_{L^p(-\ell,\ell)} \left\| \partial_t^2 \partial_1 \eta \right\|^2_{L^q(-\ell,\ell)} \tag{5.87}$$
$$\lesssim \Big(|\partial_t l| + |\partial_t r| + \|\eta\|_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} \Big) \left\| \partial_t^2 \eta \right\|^2_{H^{\frac{3}{2}}(-\ell,\ell)}$$
$$\lesssim \sqrt{\mathcal{E}} \Big(\sqrt{\mathcal{D}} \Big)^2 = \sqrt{\mathcal{E}}\mathcal{D},$$

and

$$\left| \Big(\partial_{\varpi_2} \mathcal{R} b\tilde{a} \left| \partial_t^2 \partial_1 \eta \right|^2 \Big) \Big|_{-\ell}^{\ell} \right| \lesssim |\tilde{a}| \Big(\left| \partial_t^2 \partial_1 \eta \right|^2 \Big|_{-\ell} + \left| \partial_t^2 \partial_1 \eta \right|^2 \Big|_{\ell} \Big) \lesssim \sqrt{\mathcal{E}} \Big(\sqrt{\mathcal{D}} \Big)^2 = \sqrt{\mathcal{E}}\mathcal{D}. \tag{5.88}$$

Third Integral Term, $\partial_1 \Big(\partial_t u \cdot \partial_t \mathcal{N} \Big)$: We can directly compute

$$\partial_1 \Big(\partial_t u \cdot \partial_t \mathcal{N} \Big) = \partial_t \partial_1 u \cdot \partial_t \mathcal{N} + \partial_t u \cdot \partial_t \partial_1 \mathcal{N}. \tag{5.89}$$

We estimate each term. In the following, we choose $p = \frac{3+\delta}{2+2\delta}$ and $q = \frac{6+2\delta}{1-\delta}$ such that $\frac{1}{p} + \frac{2}{q} = 1$:

$$\begin{aligned}\left| \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R}(\partial_t \partial_1 \eta)^2 \Big(\partial_t \partial_1 u \cdot \partial_t \mathcal{N} \Big) \right| &\lesssim \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)}^2 \|\partial_t \partial_1 u\|_{L^p(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{L^\infty(-\ell,\ell)} \\ &\lesssim \|\partial_t \partial_1 \eta\|_{H^{\frac{1}{2}}(-\ell,\ell)}^2 \|\partial_t \partial_1 u\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{H^1(-\ell,\ell)} \\ &\lesssim \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}^2 \|\partial_t u\|_{W_\delta^2(\Omega_0)} \|\partial_t \eta\|_{H^2(-\ell,\ell)} \\ &\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D},\end{aligned} \tag{5.90}$$

$$\begin{aligned}\left| \int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} \partial_t^2 \partial_1 \eta \Big(\partial_t \partial_1 u \cdot \partial_t \mathcal{N} \Big) \right| &\lesssim \left\| \partial_t^2 \partial_1 \eta \right\|_{L^q(-\ell,\ell)} \|\partial_t \partial_1 u\|_{L^p(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)} \\ &\lesssim \left\| \partial_t^2 \partial_1 \eta \right\|_{H^{\frac{1}{2}}(-\ell,\ell)} \|\partial_t \partial_1 u\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{H^{\frac{1}{2}}(-\ell,\ell)} \\ &\lesssim \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|\partial_t u\|_{W_\delta^2(\Omega_0)} \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)} \\ &\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D},\end{aligned} \tag{5.91}$$

$$\begin{aligned}\left| \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R}(\partial_t \partial_1 \eta)^2 \Big(\partial_t u \cdot \partial_t \partial_1 \mathcal{N} \Big) \right| &\lesssim \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)}^2 \|\partial_t u\|_{L^\infty(-\ell,\ell)} \left\| \partial_t \partial_1^2 \eta \right\|_{L^p(-\ell,\ell)} \\ &\lesssim \|\partial_t \partial_1 \eta\|_{H^{\frac{1}{2}}(-\ell,\ell)}^2 \|\partial_t u\|_{H^1(-\ell,\ell)} \left\| \partial_t \partial_1^2 \eta \right\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \\ &\lesssim \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}^2 \|\partial_t u\|_{W_\delta^2(\Omega_0)} \|\partial_t \eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \\ &\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D},\end{aligned} \tag{5.92}$$

$$\begin{aligned}\left| \int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} \partial_t^2 \partial_1 \eta \Big(\partial_t u \cdot \partial_t \partial_1 \mathcal{N} \Big) \right| &\lesssim \left\| \partial_t^2 \partial_1 \eta \right\|_{L^q(-\ell,\ell)} \|\partial_t u\|_{L^q(-\ell,\ell)} \left\| \partial_t \partial_1^2 \eta \right\|_{L^p(-\ell,\ell)} \\ &\lesssim \left\| \partial_t^2 \partial_1 \eta \right\|_{H^{\frac{1}{2}}(-\ell,\ell)} \|\partial_t u\|_{H^{\frac{1}{2}}(-\ell,\ell)} \left\| \partial_t \partial_1^2 \eta \right\|_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \\ &\lesssim \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|\partial_t u\|_{H^1(\Omega_0)} \|\partial_t \eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \\ &\lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}} = \sqrt{\mathcal{E}}\mathcal{D}.\end{aligned} \tag{5.93}$$

Fourth Integral Term, $\partial_1\Big(u\cdot\partial_t^2\mathcal{N}\Big)$: We can directly compute

$$\partial_1\Big(u\cdot\partial_t^2\mathcal{N}\Big)=\partial_1 u\cdot\partial_t^2\mathcal{N}+u\cdot\partial_t^2\partial_1\mathcal{N}. \tag{5.94}$$

We estimate each term. In the following, we choose $p=\dfrac{3+\delta}{2+2\delta}$ and $q=\dfrac{6+2\delta}{1-\delta}$ such that $\dfrac{1}{p}+\dfrac{2}{q}=1$:

$$\left|\int_{-\ell}^{\ell}\partial_{\varpi_2}^2\mathcal{R}(\partial_t\partial_1\eta)^2\Big(\partial_1 u\cdot\partial_t^2\mathcal{N}\Big)\right| \tag{5.95}$$

$$\begin{aligned}
&\lesssim \|\partial_t\partial_1\eta\|_{L^\infty(-\ell,\ell)}\|\partial_t\partial_1\eta\|_{L^q(-\ell,\ell)}\|\partial_1 u\|_{L^p(-\ell,\ell)}\left\|\partial_t^2\partial_1\eta\right\|_{L^q(-\ell,\ell)}\\
&\lesssim \|\partial_t\eta\|_{H^2(-\ell,\ell)}\|\partial_t\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}\|u\|_{W^2_\delta(\Omega_0)}\left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)}\lesssim\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}}=\sqrt{\mathcal{E}}\mathcal{D},
\end{aligned}$$

$$\begin{aligned}
\left|\int_{-\ell}^{\ell}\partial_{\varpi_2}\mathcal{R}\partial_t^2\partial_1\eta\Big(\partial_1 u\cdot\partial_t^2\mathcal{N}\Big)\right| &\lesssim \left\|\partial_t^2\partial_1\eta\right\|_{L^q(-\ell,\ell)}\|\partial_1 u\|_{L^p(-\ell,\ell)}\left\|\partial_t^2\partial_1\eta\right\|_{L^q(-\ell,\ell)} \qquad (5.96)\\
&\lesssim \left\|\partial_t^2\partial_1\eta\right\|_{H^{\frac{1}{2}}(-\ell,\ell)}\|\partial_1 u\|_{W^{\frac{1}{2}}_\delta(-\ell,\ell)}\left\|\partial_t^2\partial_1\eta\right\|_{H^{\frac{1}{2}}(-\ell,\ell)}\\
&\lesssim \left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)}\|u\|_{W^2_\delta(\Omega_0)}\left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)}\\
&\lesssim\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}\sqrt{\mathcal{D}}=\sqrt{\mathcal{E}}\mathcal{D}.
\end{aligned}$$

The last term is much more complicated. We integrate by parts to obtain

$$\begin{aligned}
\int_{-\ell}^{\ell}\partial_{\varpi_2}^2\mathcal{R}(\partial_t\partial_1\eta)^2\Big(u\cdot\partial_t^2\partial_1\mathcal{N}\Big) &= -\int_{-\ell}^{\ell}\partial_1\partial_{\varpi_2}^2\mathcal{R}(\partial_t\partial_1\eta)^2 u_1\partial_t^2\partial_1\eta\\
&\quad-\int_{-\ell}^{\ell}\partial_{\varpi_2}^2\mathcal{R}(\partial_t\partial_1\eta)(\partial_t\partial_1^2\eta)u_1\partial_t^2\partial_1\eta\\
&\quad-\int_{-\ell}^{\ell}\partial_{\varpi_2}^2\mathcal{R}(\partial_t\partial_1\eta)^2\partial_1 u_1\partial_t^2\partial_1\eta+\partial_{\varpi_2}^2\mathcal{R}(\partial_t\partial_1\eta)^2\Big(u_1\partial_t^2\partial_1\eta\Big)\Big|_{-\ell}^{\ell}\\
&=: I+II+III+IV.
\end{aligned}$$

Then we have

$$
\begin{aligned}
|I| &= \left| \int_{-\ell}^{\ell} \partial_1 \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 u_1 \partial_t^2 \partial_1 \eta \right| \qquad (5.97)\\
&\lesssim \|\partial_t \partial_1 \eta\|_{L^\infty(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)} \|u\|_{L^p(-\ell,\ell)} \left\| \partial_t^2 \partial_1 \eta \right\|_{L^q(-\ell,\ell)} \\
&\lesssim \|\partial_t \eta\|_{H^2(-\ell,\ell)} \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|u\|_{W_\delta^2(\Omega_0)} \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D},
\end{aligned}
$$

$$
\begin{aligned}
|II| &= \left| \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)(\partial_t \partial_1^2 \eta) u_1 \partial_t^2 \partial_1 \eta \right| \qquad (5.98)\\
&\lesssim \|\partial_t \partial_1 \eta\|_{L^\infty(-\ell,\ell)} \left\| \partial_t \partial_1^2 \eta \right\|_{L^p(-\ell,\ell)} \|u\|_{L^q(-\ell,\ell)} \left\| \partial_t^2 \partial_1 \eta \right\|_{L^q(-\ell,\ell)} \\
&\lesssim \|\partial_t \eta\|_{H^2(-\ell,\ell)} \|\partial_t \eta\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \|u\|_{H^1(\Omega_0)} \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D},
\end{aligned}
$$

and

$$
\begin{aligned}
|III| &= \left| \int_{-\ell}^{\ell} \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \partial_1 u_1 \partial_t^2 \partial_1 \eta \right| \qquad (5.99)\\
&\lesssim \|\partial_t \partial_1 \eta\|_{L^\infty(-\ell,\ell)} \|\partial_t \partial_1 \eta\|_{L^q(-\ell,\ell)} \|\partial_1 u\|_{L^p(-\ell,\ell)} \left\| \partial_t^2 \partial_1 \eta \right\|_{L^q(-\ell,\ell)} \\
&\lesssim \|\partial_t \eta\|_{H^2(-\ell,\ell)} \|\partial_t \eta\|_{H^{\frac{3}{2}}(-\ell,\ell)} \|u\|_{W_\delta^2(\Omega_0)} \left\| \partial_t^2 \eta \right\|_{H^{\frac{3}{2}}(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D}.
\end{aligned}
$$

The contact point term

$$
\begin{aligned}
|IV| &\lesssim \left| \partial_{\varpi_2}^2 \mathcal{R} (\partial_t \partial_1 \eta)^2 \Big(u_1 \partial_t^2 \partial_1 \eta \Big) \Big|_{-\ell}^{\ell} \right| \qquad (5.100)\\
&\lesssim \bigg(|\partial_t \partial_1 \eta(-\ell)|^2 + |\partial_t \partial_1 \eta(\ell)|^2 \bigg) \bigg(\left| \partial_t^2 \partial_1 \eta(-\ell) \right| + \left| \partial_t^2 \partial_1 \eta(\ell) \right| \bigg) \Big(|\partial_t L| + |\partial_t R| \Big) \\
&\lesssim \sqrt{\mathcal{E}} \sqrt{\mathcal{D}} \sqrt{\mathcal{D}} = \sqrt{\mathcal{E}} \mathcal{D}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} \partial_t^2 \partial_1 \eta \Big(u \cdot \partial_t^2 \partial_1 \mathcal{N} \Big) &= \int_{-\ell}^{\ell} \partial_{\varpi_2} \mathcal{R} u_1 \partial_1 \left| \partial_t^2 \partial_1 \eta \right|^2 \qquad (5.101)\\
&= - \int_{-\ell}^{\ell} \partial_1 \Big(\partial_{\varpi_2} \mathcal{R} u_1 \Big) \left| \partial_t^2 \partial_1 \eta \right|^2 + \left(\partial_{\varpi_2} \mathcal{R} u_1 \left| \partial_t^2 \partial_1 \eta \right|^2 \right) \Big|_{-\ell}^{\ell},
\end{aligned}
$$

where we have

$$\left|\int_{-\ell}^{\ell} \partial_1\left(\partial_{\varpi_2}\mathcal{R}u_1\right)\left|\partial_t^2\partial_1\eta\right|^2\right| \lesssim \left\|\partial_1\left(\partial_{\varpi_2}\mathcal{R}u_1\right)\right\|_{L^p(-\ell,\ell)}\left\|\partial_t^2\partial_1\eta\right\|_{L^q(-\ell,\ell)}^2 \tag{5.102}$$
$$\lesssim \|u\|_{W_\delta^2(\Omega_0)}\left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)}^2 \lesssim \sqrt{\mathcal{E}}\left(\sqrt{\mathcal{D}}\right)^2 = \sqrt{\mathcal{E}}\mathcal{D},$$

and

$$\left|\left(\partial_{\varpi_2}\mathcal{R}u_1\left|\partial_t^2\partial_1\eta\right|^2\right)\Big|_{-\ell}^{\ell}\right| \lesssim \left(|u_1|\Big|_{-\ell} + |u_1|\Big|_{\ell}\right)\left(\left|\partial_t^2\partial_1\eta\right|^2\Big|_{-\ell} + \left|\partial_t^2\partial_1\eta\right|^2\Big|_{\ell}\right) \tag{5.103}$$
$$\lesssim \left(|\partial_t L| + |\partial_t R|\right)\left(\left|\partial_t^2\partial_1\eta\right|^2\Big|_{-\ell} + \left|\partial_t^2\partial_1\eta\right|^2\Big|_{\ell}\right)$$
$$\lesssim \sqrt{\mathcal{E}}\left(\sqrt{\mathcal{D}}\right)^2 = \sqrt{\mathcal{E}}\mathcal{D}.$$

Next we turn to the contact point terms:

$$\sigma\left(\partial_t^2\mathcal{R}\left(bJ_1\partial_t^3\eta + \partial_t^2(b\tilde{a}\partial_1\eta) + \partial_t u\cdot\partial_t\mathcal{N} + u\cdot\partial_t^2\mathcal{N}\right)\right)\Big|_{-\ell}^{\ell} \tag{5.104}$$
$$= \sigma\left(\partial_t^2\mathcal{R}\left(\partial_t^2(b\tilde{a}\partial_1\eta) + \partial_t u\cdot\partial_t\mathcal{N} + u\cdot\partial_t^2\mathcal{N}\right)\right)\Big|_{-\ell}^{\ell},$$

which follows because $\partial_t^3\eta(-\ell) = \partial_t^3\eta(\ell) = 0$. Note the fact that

$$\left|\partial_t^2\mathcal{R}(\pm\ell)\right| \lesssim \left|\partial_{\varpi_2}^2\mathcal{R}(\pm\ell)\right||\partial_t\partial_1\eta(\pm\ell)|^2 + \left|\partial_{\varpi_2}\mathcal{R}(\pm\ell)\right|\left|\partial_t^2\partial_1\eta(\pm\ell)\right| \lesssim \sqrt{\mathcal{D}}. \tag{5.105}$$

First Contact Point Term, $\partial_t^2(b\tilde{a}\partial_1\eta)$: We can directly compute

$$\partial_t^2(b\tilde{a}\partial_1\eta) = b\partial_t^2\tilde{a}\partial_1\eta + 2b\partial_t\tilde{a}\partial_t\partial_1\eta + b\tilde{a}\partial_t^2\partial_1\eta. \tag{5.106}$$

Then we have

$$\left|b\partial_t^2\tilde{a}\partial_1\eta\Big|_{\pm\ell}\right| \lesssim \left(\left|\partial_t^3 l\right| + \left|\partial_t^3 r\right|\right)|\partial_1\eta(\pm\ell)| \lesssim \sqrt{\mathcal{D}}\sqrt{\mathcal{E}}, \tag{5.107}$$

$$\left|2b\partial_t\tilde{a}\partial_t\partial_1\eta\Big|_{\pm\ell}\right| \lesssim \left(\left|\partial_t^2 l\right| + \left|\partial_t^2 r\right|\right)|\partial_t\partial_1\eta(\pm\ell)| \lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}, \tag{5.108}$$

and

$$\left|b\tilde{a}\partial_t^2\partial_1\eta\Big|_{\pm\ell}\right| \lesssim \left(|\partial_t l| + |\partial_t r|\right)\left|\partial_t^2\partial_1\eta(\pm\ell)\right| \lesssim \sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{5.109}$$

Second Contact Point Term, $\partial_t u\cdot\partial_t\mathcal{N}$: Notice that $u(-\ell,0) = (\partial_t l, 0)$ and $u(\ell,0) = (\partial_t r, 0)$. We have

$$\left|\partial_t u\cdot\partial_t\mathcal{N}\Big|_{\pm\ell}\right|\lesssim\left(\left|\partial_t^2 l\right|+\left|\partial_t^2 r\right|\right)|\partial_t\partial_1\eta(\pm\ell)|\lesssim\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{5.110}$$

Third Contact Point Term, $u\cdot\partial_t^2\mathcal{N}$: We have

$$\left|u\cdot\partial_t^2\mathcal{N}\Big|_{\pm\ell}\right|\lesssim\left(|\partial_t l|+|\partial_t r|\right)\left|\partial_t^2\partial_1\eta(\pm\ell)\right|\lesssim\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}.\quad\square \tag{5.111}$$

5.8. Nonlinear estimates in the pressure estimates and free surface estimates

Lemma 5.12. *Define the functional $H^1(\Omega_0)\ni w\to\langle\mathcal{F},w\rangle\in\mathbb{R}$ via*

$$\langle\mathcal{F},w\rangle=\int_{\Omega_0}Jw\cdot\mathcal{F}_1-\int_{\Sigma_0}w\cdot\mathcal{F}_3-\int_{\Sigma_{0b}}\mathcal{F}_4\left(w\cdot\frac{\mathcal{T}}{|\mathcal{T}|^2}\right). \tag{5.112}$$

Then

$$|\langle\mathcal{F},w\rangle|\lesssim\|w\|_{H^1(\Omega_0)}\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{5.113}$$

Proof. This is a summary of previous estimates for $\mathcal{F}_1$, $\mathcal{F}_3$ and $\mathcal{F}_4$. $\square$

Lemma 5.13. *Let $\mathcal{R}$ be given by* (C.3.1)*. We have the estimate*

$$\left\|\partial_t^2\mathcal{R}\right\|_{H^{\frac{1}{2}}(-\ell,\ell)}\lesssim\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{5.114}$$

Proof. Similar to the proof of Lemma 5.11, for $s-\frac{1}{2}>\frac{1}{2}$, we estimate

$$\begin{aligned}\left\|\partial_{\varpi_2}\mathcal{R}\partial_t^2\partial_1\eta\right\|_{H^{\frac{1}{2}}(-\ell,\ell)}&\lesssim\left(|k_1|+\|\partial_1\eta\|_{H^{s-\frac{1}{2}}(-\ell,\ell)}\right)\left\|\partial_t^2\partial_1\eta\right\|_{H^{\frac{1}{2}}(-\ell,\ell)}\\&\lesssim\|\eta\|_{H^{s+\frac{1}{2}}(-\ell,\ell)}\left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)}\lesssim\sqrt{\mathcal{E}}\sqrt{\mathcal{D}},\end{aligned} \tag{5.115}$$

and

$$\begin{aligned}\left\|\partial_{\varpi_2}^2\mathcal{R}(\partial_t\partial_1\eta)^2\right\|_{H^{\frac{1}{2}}(-\ell,\ell)}&\lesssim\|\partial_t\partial_1\eta\|_{H^{s-\frac{1}{2}}(-\ell,\ell)}\|\partial_t\partial_1\eta\|_{H^{\frac{1}{2}}(-\ell,\ell)}\\&\lesssim\|\partial_t\eta\|_{H^{s+\frac{1}{2}}(-\ell,\ell)}\|\partial_t\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}\lesssim\sqrt{\mathcal{D}}\sqrt{\mathcal{E}}.\quad\square\end{aligned} \tag{5.116}$$

6. Nonlinear estimates in the Stokes problem

We will now prove the nonlinear estimate in the Stokes problem when applying ∂_t on both sides of the equation. Throughout this section, we always assume η is given and satisfies

$$\sup_{0\le t\le T}\left(\mathcal{E}_{\parallel}(t)+\|\eta(t)\|_{W_\delta^{\frac{5}{2}}(-\ell,\ell)}+\|\partial_t\eta(t)\|_{W_\delta^{\frac{3}{2}}(-\ell,\ell)}\right)\le\vartheta<1, \tag{6.1}$$

for some $\vartheta > 0$ sufficiently small. Here, we will employ the same techniques as developed in [9] and Section 5.

6.1. Estimate of the G_1 term

We now handle the term G_1.

Lemma 6.1. *Let $G_1 = \mathcal{F}_1$. We have the estimate*

$$\|G_1\|^2_{W^0_\delta(\Omega_0)} \lesssim \left(\mathcal{E}^2 + \mathcal{E}\right)\mathcal{D}. \tag{6.2}$$

Proof. We will only present the estimate for the term $-\nabla_{\partial_t\mathcal{A}} \cdot \left(pI - \mu\mathbb{D}_{\mathcal{A}}u\right)$. The term $\mu\nabla_{\mathcal{A}} \cdot \mathbb{D}_{\partial_t\mathcal{A}}u$ may be handled with a similar argument. We begin by bounding

$$\begin{aligned}
&\left\|-\nabla_{\partial_t\mathcal{A}} \cdot \left(pI - \mu\mathbb{D}_{\mathcal{A}}u\right)\right\|^2_{W^0_\delta(\Omega_0)} \qquad (6.3)\\
&\lesssim \|\partial_t\mathcal{A}\nabla p\|^2_{W^0_\delta(\Omega_0)} + \left\|\partial_t\mathcal{A}\mathcal{A}\nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} + \|\partial_t\mathcal{A}\nabla\mathcal{A}\nabla u\|^2_{W^0_\delta(\Omega_0)}\\
&=: I + II + III.
\end{aligned}$$

For I, we have

$$\begin{aligned}
I = \|\partial_t\mathcal{A}\nabla p\|^2_{W^0_\delta(\Omega_0)} &\lesssim \|\partial_t\mathcal{A}\|^2_{L^\infty(\Omega_0)}\|\nabla p\|^2_{W^0_\delta(\Omega_0)} \lesssim \|\partial_t\nabla\bar{\eta}\|^2_{L^\infty(\Omega_0)}\|\nabla p\|^2_{W^0_\delta(\Omega_0)} \qquad (6.4)\\
&\lesssim \|\partial_t\eta\|^2_{H^{s+\frac{1}{2}}(-\ell,\ell)}\|p\|^2_{W^1_\delta(\Omega_0)} \lesssim \mathcal{D}\mathcal{E}.
\end{aligned}$$

For II, we have

$$\begin{aligned}
II = \left\|\partial_t\mathcal{A}\mathcal{A}\nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} &\lesssim \|\partial_t\mathcal{A}\|^2_{L^\infty(\Omega_0)}\|\mathcal{A}\|^2_{L^\infty(\Omega_0)}\left\|\nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} \qquad (6.5)\\
&\lesssim \|\partial_t\nabla\bar{\eta}\|^2_{L^\infty(\Omega_0)}\left\|\nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} \lesssim \|\partial_t\eta\|^2_{H^{s+\frac{1}{2}}(-\ell,\ell)}\|u\|^2_{W^2_\delta(\Omega_0)} \lesssim \mathcal{D}\mathcal{E}.
\end{aligned}$$

For III, we choose $q \in [1,\infty)$ such that $\frac{3}{q} + \frac{1}{2+2\delta} = \frac{1}{2}$, and also $p \in [1,\infty)$ such that $\frac{2}{p} + \frac{2-s}{2} = \frac{1}{2}$. Then we have

$$\begin{aligned}
III = \|\partial_t\mathcal{A}\nabla\mathcal{A}\nabla u\|^2_{W^0_\delta(\Omega_0)} &\lesssim \left\|\partial_t\nabla\bar{\eta}\nabla^2\bar{\eta}\nabla u\right\|^2_{W^0_\delta(\Omega_0)} + \left\|\partial_t\nabla\bar{\eta}\nabla\bar{\eta}\frac{1}{\zeta_0}\nabla u\right\|^2_{W^0_\delta(\Omega_0)} \qquad (6.6)\\
&\lesssim \|\partial_t\nabla\bar{\eta}\|^2_{L^p(\Omega_0)}\left\|\nabla^2\bar{\eta}\right\|^2_{L^{\frac{2}{2-s}}(\Omega_0)}\left\|d^\delta\nabla u\right\|^2_{L^p(\Omega_0)}\\
&\quad + \|\partial_t\nabla\bar{\eta}\|^2_{L^q(\Omega_0)}\|\nabla\bar{\eta}\|^2_{L^q(\Omega_0)}\|\nabla u\|^2_{L^q(\Omega_0)}\left\|\frac{d^\delta}{\zeta_0}\right\|_{L^{2+2\delta}(\Omega_0)}
\end{aligned}$$

$$\lesssim \|\partial_t \bar{\eta}\|^2_{H^2(\Omega_0)} \|\bar{\eta}\|^2_{H^{s+1}(\Omega_0)} \|\nabla u\|^2_{W^1_\delta(\Omega_0)} + \|\partial_t \bar{\eta}\|^2_{H^2(\Omega_0)} \|\bar{\eta}\|^2_{H^1(\Omega_0)} \|\nabla u\|^2_{W^1_\delta(\Omega_0)}$$
$$\lesssim \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \|\eta\|^2_{H^{s+\frac{1}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} + \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \|\eta\|^2_{H^{\frac{1}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)}$$
$$\lesssim \mathcal{D}\mathcal{E}^2. \quad \square$$

6.2. *Estimate of the G_2 term*

Now we handle the term G_2.

Lemma 6.2. *Let $G_2 = \mathcal{F}_2$. We have the estimate*

$$\|G_2\|^2_{W^1_\delta(\Omega_0)} \lesssim \mathcal{E}\mathcal{D}. \tag{6.7}$$

Proof. The estimate may be proved as in Proposition 7.2 of [9] with minor modifications to accommodate the $1/\zeta_0$ term as in the proof of Lemma 6.1. $\square$

6.3. *Estimate of the G_3 term*

Next we handle G_3.

Lemma 6.3. *Let G_3 be given by $G_3^- = 0$ and*

$$G_3^+ = \frac{\partial_t\left(J_1 \partial_t \eta\right) - \partial_t\left(\tilde{a}\partial_1 \zeta\right)}{\sqrt{1+|\partial \zeta_0|^2}} - u \cdot \partial_t \mathcal{N}. \tag{6.8}$$

We have the estimate

$$\|G_3\|^2_{W^{\frac{3}{2}}_\delta(\partial\Omega_0)} \lesssim \mathcal{D}_\parallel + \mathcal{E}\mathcal{D}. \tag{6.9}$$

Proof. We estimate

$$\begin{aligned} \|G_3\|^2_{W^{\frac{3}{2}}_\delta(\partial\Omega_0)} &\lesssim \left\|G_3^+\right\|^2_{W^{\frac{3}{2}}_\delta(\Sigma_0)} \\ &\lesssim \left\|\partial_t\left(J_1 \partial_t \eta\right)\right\|^2_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} + \left\|\partial_t\left(\tilde{a}\partial_1 \zeta\right)\right\|^2_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} + \|u \cdot \partial_t \mathcal{N}\|^2_{W^{\frac{3}{2}}_\delta(\Sigma_0)} \\ &=: I + II + III. \end{aligned} \tag{6.10}$$

We then bound

$$\begin{aligned} I = \left\|\partial_t\left(J_1 \partial_t \eta\right)\right\|^2_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} &\lesssim \|\partial_t J_1 \partial_t \eta\|^2_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} + \left\|J_1 \partial_t^2 \eta\right\|^2_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} \\ &\lesssim |\partial_t J_1|^2 \|\partial_t \eta\|^2_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} + |J_1|^2 \left\|\partial_t^2 \eta\right\|^2_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} \end{aligned} \tag{6.11}$$

$$
\begin{aligned}
&\lesssim\Big(|\partial_t l|^2+|\partial_t r|^2\Big)\,\|\partial_t\eta\|_{H^{\frac{3}{2}}(-\ell,\ell)}+\left\|\partial_t^2\eta\right\|_{H^{\frac{3}{2}}(-\ell,\ell)}\lesssim\mathcal{E}\mathcal{D}+\mathcal{D}_{\|},\\
II&=\left\|\partial_t\Big(\tilde a\partial_1\zeta\Big)\right\|^2_{W_\delta^{\frac{3}{2}}(-\ell,\ell)}\lesssim\Big(\left|\partial_t^2 l\right|^2+\left|\partial_t^2 r\right|^2\Big)\,\|\zeta\|^2_{W_\delta^{\frac{3}{2}}(-\ell,\ell)}+\Big(|\partial_t l|^2+|\partial_t r|^2\Big)\,\|\partial_t\eta\|^2_{W_\delta^{\frac{3}{2}}(-\ell,\ell)}\\
&\lesssim\Big(\left|\partial_t^2 l\right|^2+\left|\partial_t^2 r\right|^2\Big)+\Big(|\partial_t l|^2+|\partial_t r|^2\Big)\,\|\partial_t\eta\|^2_{W_\delta^{\frac{3}{2}}(-\ell,\ell)}\lesssim\mathcal{D}_{\|}+\mathcal{E}\mathcal{D},
\end{aligned}
\tag{6.12}
$$

and

$$
\begin{aligned}
III&=\|u\cdot\partial_t\mathcal{N}\|^2_{W_\delta^{\frac{3}{2}}(\Sigma_0)}\lesssim\|u\cdot\partial_t\mathcal{N}\|_{W_\delta^1(\Sigma_0)}+\left\|\partial_1\Big(u\cdot\partial_t\mathcal{N}\Big)\right\|_{W_\delta^{\frac{1}{2}}(\Sigma_0)}\\
&\lesssim\|u\|^2_{W_\delta^{\frac{3}{2}}(\Sigma_0)}\|\partial_t\partial_1\eta\|^2_{W_\delta^{\frac{3}{2}}(-\ell,\ell)}+\|\partial_1 u\|^2_{W_\delta^{\frac{1}{2}}(\Sigma_0)}\|\partial_t\partial_1\eta\|^2_{W_\delta^{\frac{3}{2}}(-\ell,\ell)}\\
&\quad+\|u\|^2_{W_\delta^{\frac{3}{2}}(\Sigma_0)}\left\|\partial_t\partial_1^2\eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)}\\
&\lesssim\|u\|^2_{W_\delta^2(\Omega_0)}\|\partial_t\eta\|^2_{W_\delta^{\frac{5}{2}}(-\ell,\ell)}\lesssim\mathcal{E}\mathcal{D}.\quad\square
\end{aligned}
\tag{6.13}
$$

6.4. Estimate of the G_4 term

Next to estimate is G_4.

Lemma 6.4. *Let $G_4=\mathcal{F}_4$ be given by $G_4^-=\mathcal{F}_4$ and $G_4^+=\mathcal{F}_3\cdot\dfrac{\mathcal{T}}{|\mathcal{T}|}$. We have the estimate*

$$
\|G_4\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)}\lesssim\Big(\mathcal{E}^2+\mathcal{E}\Big)\mathcal{D}.
\tag{6.14}
$$

Proof. It is easy to check that

$$
\|G_4\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)}\lesssim\left\|G_4^-\right\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)}+\left\|G_4^+\right\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)}\lesssim\|\mathcal{F}_4\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)}+\|\mathcal{F}_3\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)}.
\tag{6.15}
$$

We will proceed by estimating these term by term.

G_4^- **Term** $\mu\mathbb{D}_{\partial_t\mathcal{A}}u\nu\cdot\tau$**:** We have

$$
\begin{aligned}
\left\|\mu\mathbb{D}_{\partial_t\mathcal{A}}u\nu\cdot\tau\right\|^2_{W_\delta^{\frac{1}{2}}(\Sigma_{0b})}&\lesssim\left\|\mathbb{D}_{\partial_t\mathcal{A}}ue_2\cdot e_1\right\|^2_{W_\delta^1(\Omega_0)}\\
&\lesssim\|\partial_t\mathcal{A}\nabla u\|^2_{W_\delta^0(\Omega_0)}+\left\|\partial_t\mathcal{A}\nabla^2 u\right\|^2_{W_\delta^0(\Omega_0)}+\|\partial_t\nabla\mathcal{A}\nabla u\|^2_{W_\delta^0(\Omega_0)}\\
&=:I+II+III.
\end{aligned}
\tag{6.16}
$$

For I, we have

$$I = \|\partial_t \mathcal{A} \nabla u\|^2_{W^0_\delta(\Omega_0)} \lesssim \|\partial_t \mathcal{A}\|^2_{L^4(\Omega_0)} \left\|d^\delta \nabla u\right\|^2_{L^4(\Omega_0)} \lesssim \|\partial_t \nabla \bar{\eta}\|^2_{L^4(\Omega_0)} \left\|d^\delta \nabla u\right\|^2_{L^4(\Omega_0)} \tag{6.17}$$
$$\lesssim \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} \lesssim \mathcal{D}\mathcal{E}.$$

For II, we have

$$II = \left\|\partial_t \mathcal{A} \nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} \lesssim \|\partial_t \mathcal{A}\|^2_{L^\infty(\Omega_0)} \left\|\nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} \lesssim \|\partial_t \nabla \bar{\eta}\|^2_{L^\infty(\Omega_0)} \left\|\nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} \tag{6.18}$$
$$\lesssim \|\partial_t \eta\|^2_{H^{s+\frac{1}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} \lesssim \mathcal{D}\mathcal{E}.$$

For III, we choose $q \in [1, \infty)$ such that $\dfrac{2}{q} + \dfrac{1}{2+2\delta} = \dfrac{1}{2}$. Then we have

$$III = \|\partial_t \nabla \mathcal{A} \nabla u\|^2_{W^0_\delta(\Omega_0)} \lesssim \left\|\partial_t \nabla^2 \bar{\eta} \nabla u\right\|^2_{W^0_\delta(\Omega_0)} + \left\|\partial_t \nabla \bar{\eta} \frac{1}{\zeta_0} \nabla u\right\|^2_{W^0_\delta(\Omega_0)} \tag{6.19}$$
$$\lesssim \left\|\partial_t \nabla^2 \bar{\eta}\right\|^2_{L^{\frac{2}{2-s}}(\Omega_0)} \left\|d^\delta \nabla u\right\|^2_{L^{\frac{2}{s-1}}(\Omega_0)} + \|\partial_t \nabla \bar{\eta}\|^2_{L^q(\Omega_0)} \left\|\frac{d^\delta}{\zeta_0}\right\|_{L^{2+2\delta}(\Omega_0)} \|\nabla u\|^2_{L^q(\Omega_0)}$$
$$\lesssim \|\partial_t \bar{\eta}\|^2_{H^{s+1}(\Omega_0)} \|\nabla u\|^2_{W^1_\delta(\Omega_0)} + \|\partial_t \bar{\eta}\|^2_{H^2(\Omega_0)} \|\nabla u\|^2_{W^1_\delta(\Omega_0)}$$
$$\lesssim \|\partial_t \eta\|^2_{H^{s+\frac{1}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} + \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} \lesssim \mathcal{D}\mathcal{E}.$$

First Term of G_4^+, $\mu \mathbb{D}_{\partial_t \mathcal{A}} u \mathcal{N}$: We have

$$\left\|\mu \mathbb{D}_{\partial_t \mathcal{A}} u \mathcal{N}\right\|^2_{W^{\frac{1}{2}}_\delta(\Sigma_0)} \lesssim \left\|\mathbb{D}_{\partial_t \mathcal{A}} u (J_1 e_2 + e_1 \partial_1 \bar{\eta})\right\|^2_{W^1_\delta(\Omega_0)} \tag{6.20}$$
$$\lesssim \|\partial_t \mathcal{A} \nabla u\|^2_{W^0_\delta(\Omega_0)} + \left\|\partial_t \mathcal{A} \nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} + \|\partial_t \nabla \mathcal{A} \nabla u\|^2_{W^0_\delta(\Omega_0)} + \left\|\partial_t \mathcal{A} \nabla u \partial_1^2 \bar{\eta}\right\|^2_{W^0_\delta(\Omega_0)}$$
$$=: I + II + III + IV.$$

For I, we have

$$I = \|\partial_t \mathcal{A} \nabla u\|^2_{W^0_\delta(\Omega_0)} \lesssim \|\partial_t \mathcal{A}\|^2_{L^4(\Omega_0)} \left\|d^\delta \nabla u\right\|^2_{L^4(\Omega_0)} \lesssim \|\partial_t \nabla \bar{\eta}\|^2_{L^4(\Omega_0)} \left\|d^\delta \nabla u\right\|^2_{L^4(\Omega_0)} \tag{6.21}$$
$$\lesssim \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} \lesssim \mathcal{D}\mathcal{E}.$$

For II, we have

$$II = \left\|\partial_t \mathcal{A} \nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} \lesssim \|\partial_t \mathcal{A}\|^2_{L^\infty(\Omega_0)} \left\|\nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} \lesssim \|\partial_t \nabla \bar{\eta}\|^2_{L^\infty(\Omega_0)} \left\|\nabla^2 u\right\|^2_{W^0_\delta(\Omega_0)} \tag{6.22}$$
$$\lesssim \|\partial_t \eta\|^2_{H^{s+\frac{1}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} \lesssim \mathcal{D}\mathcal{E}.$$

For III, we choose $q \in [1, \infty)$ such that $\dfrac{2}{q} + \dfrac{1}{2+2\delta} = \dfrac{1}{2}$. Then we have

$$
\begin{aligned}
III = \|\partial_t \nabla \mathcal{A} \nabla u\|^2_{W^0_\delta(\Omega_0)} &\lesssim \left\| \partial_t \nabla^2 \bar{\eta} \nabla u \right\|^2_{W^0_\delta(\Omega_0)} + \left\| \partial_t \nabla \bar{\eta} \frac{1}{\zeta_0} \nabla u \right\|^2_{W^0_\delta(\Omega_0)} \\
&\lesssim \left\| \partial_t \nabla^2 \bar{\eta} \right\|^2_{L^{\frac{2}{2-s}}(\Omega_0)} \left\| d^\delta \nabla u \right\|^2_{L^{\frac{2}{s-1}}(\Omega_0)} + \|\partial_t \nabla \bar{\eta}\|^2_{L^q(\Omega_0)} \left\| \frac{d^\delta}{\zeta_0} \right\|_{L^{2+2\delta}(\Omega_0)} \|\nabla u\|^2_{L^q(\Omega_0)} \\
&\lesssim \|\partial_t \bar{\eta}\|^2_{H^{s+1}(\Omega_0)} \|\nabla u\|^2_{W^1_\delta(\Omega_0)} + \|\partial_t \bar{\eta}\|^2_{H^2(\Omega_0)} \|\nabla u\|^2_{W^1_\delta(\Omega_0)} \\
&\lesssim \|\partial_t \eta\|^2_{H^{s+\frac{1}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} + \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} \lesssim \mathcal{D}\mathcal{E}.
\end{aligned}
\tag{6.23}
$$

For IV, we have

$$
\begin{aligned}
IV = \left\| \partial_t \mathcal{A} \nabla u \partial_1^2 \bar{\eta} \right\|^2_{W^0_\delta(\Omega_0)} &\lesssim \|\partial_t \mathcal{A}\|^2_{L^\infty(\Omega_0)} \|\nabla u\|^2_{L^4(\Omega_0)} \left\| \partial_1^2 \bar{\eta} \right\|^2_{L^4(\Omega_0)} \\
&\lesssim \|\partial_t \nabla \bar{\eta}\|^2_{L^\infty(\Omega_0)} \|\nabla u\|^2_{W^1_\delta(\Omega_0)} \left\| \partial_1^2 \bar{\eta} \right\|_{W^1_\delta(\Omega_0)} \\
&\lesssim \|\partial_t \eta\|^2_{H^{s+\frac{1}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} \|\eta\|_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} \lesssim \mathcal{D}\mathcal{E}^2.
\end{aligned}
$$

Second Term of G_4^+, $-(pI - \mu \mathbb{D}_{\mathcal{A}} u)\partial_t \mathcal{N}$: We have

$$
\begin{aligned}
\|-(pI - \mu \mathbb{D}_{\mathcal{A}} u)\partial_t \mathcal{N}\|^2_{W^{\frac{1}{2}}_\delta(\Sigma_0)} &\lesssim \|(pI - \mu \mathbb{D}_{\mathcal{A}} u)\partial_t (J_1 e_2 + e_1 \partial_1 \bar{\eta})\|^2_{W^1_\delta(\Omega_0)} \\
&\lesssim \|(p + \nabla u)\partial_t \partial_1 \bar{\eta}\|^2_{W^0_\delta(\Omega_0)} + \left\| \nabla^2 u \partial_t \partial_1 \bar{\eta} \right\|^2_{W^0_\delta(\Omega_0)} \\
&\quad + \|\nabla \mathcal{A} \nabla u \partial_t \partial_1 \bar{\eta}\|^2_{W^0_\delta(\Omega_0)} + \left\| (p + \nabla u)\partial_t \partial_1^2 \bar{\eta} \right\|^2_{W^0_\delta(\Omega_0)} \\
&=: I + II + III + IV.
\end{aligned}
\tag{6.24}
$$

For I, we have

$$
\begin{aligned}
I = \|(p + \nabla u)\partial_t \partial_1 \bar{\eta}\|^2_{W^0_\delta(\Omega_0)} &\lesssim \left\| d^\delta (p + \nabla u) \right\|^2_{L^4(\Omega_0)} \|\partial_t \nabla \bar{\eta}\|^2_{L^4(\Omega_0)} \\
&\lesssim \left\| d^\delta (p + \nabla u) \right\|^2_{L^4(\Omega_0)} \|\partial_t \nabla \bar{\eta}\|^2_{L^4(\Omega_0)} \lesssim \left(\|u\|^2_{W^2_\delta(\Omega_0)} + \|p\|^2_{W^1_\delta(\Omega_0)} \right) \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}.
\end{aligned}
\tag{6.25}
$$

For II, we have

$$
\begin{aligned}
II = \left\| \nabla^2 u \partial_t \partial_1 \bar{\eta} \right\|^2_{W^0_\delta(\Omega_0)} &\lesssim \left\| \nabla^2 u \right\|^2_{W^0_\delta(\Omega_0)} \|\partial_t \nabla \bar{\eta}\|^2_{L^\infty(\Omega_0)} \lesssim \left\| \nabla^2 u \right\|^2_{W^0_\delta(\Omega_0)} \|\partial_t \nabla \bar{\eta}\|^2_{L^\infty(\Omega_0)} \\
&\lesssim \|u\|^2_{W^2_\delta(\Omega_0)} \|\partial_t \eta\|^2_{H^{s+\frac{1}{2}}(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}.
\end{aligned}
\tag{6.26}
$$

For III, we choose $q \in [1, \infty)$ such that $\dfrac{3}{q} + \dfrac{1}{2+2\delta} = \dfrac{1}{2}$. Then we have

$$
\begin{aligned}
III &= \|\nabla \mathcal{A} \nabla u \partial_t \partial_1 \bar{\eta}\|^2_{W^0_\delta(\Omega_0)} \lesssim \left\| \nabla^2 \bar{\eta} \nabla u \partial_t \partial_1 \bar{\eta} \right\|^2_{W^0_\delta(\Omega_0)} + \left\| \nabla \bar{\eta} \frac{1}{\zeta_0} \nabla u \partial_t \partial_1 \bar{\eta} \right\|^2_{W^0_\delta(\Omega_0)} \\
&\lesssim \left\| \nabla^2 \bar{\eta} \right\|^2_{L^3(\Omega_0)} \left\| d^\delta \nabla u \right\|^2_{L^3(\Omega_0)} \|\partial_t \partial_1 \bar{\eta}\|^2_{L^3(\Omega_0)} \\
&\quad + \|\nabla \bar{\eta}\|^2_{L^q(\Omega_0)} \left\| \frac{d^\delta}{\zeta_0} \right\|_{L^{2+2\delta}(\Omega_0)} \|\nabla u\|^2_{L^q(\Omega_0)} \|\partial_t \partial_1 \bar{\eta}\|^2_{L^q(\Omega_0)} \\
&\lesssim \|\bar{\eta}\|^2_{H^3(\Omega_0)} \|\nabla u\|^2_{W^1_\delta(\Omega_0)} \|\partial_t \partial_1 \bar{\eta}\|^2_{H^1(\Omega_0)} + \|\bar{\eta}\|^2_{H^2(\Omega_0)} \|\nabla u\|^2_{W^1_\delta(\Omega_0)} \|\partial_t \partial_1 \bar{\eta}\|^2_{H^1(\Omega_0)} \\
&\lesssim \|\eta\|^2_{H^{\frac{5}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} + \|\eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \|u\|^2_{W^2_\delta(\Omega_0)} \|\partial_t \eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \\
&\lesssim \mathcal{D}\mathcal{E}^2.
\end{aligned}
\tag{6.27}
$$

For IV, we have

$$
\begin{aligned}
IV &= \left\| (p + \nabla u) \partial_t \partial_1^2 \bar{\eta} \right\|^2_{W^0_\delta(\Omega_0)} \lesssim \|p + \nabla u\|^2_{L^4(\Omega_0)} \left\| \partial_t \partial_1^2 \bar{\eta} \right\|^2_{L^4(\Omega_0)} \\
&\lesssim \left(\|\nabla u\|^2_{W^1_\delta(\Omega_0)} + \|p\|^2_{W^1_\delta(\Omega_0)} \right) \left\| \partial_t \partial_1^2 \bar{\eta} \right\|_{W^1_\delta(\Omega_0)} \\
&\lesssim \left(\|u\|^2_{W^2_\delta(\Omega_0)} + \|p\|^2_{W^1_\delta(\Omega_0)} \right) \|\partial_t \eta\|_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}.
\end{aligned}
\tag{6.28}
$$

Third Term of G_4^+, $g\eta \partial_t \mathcal{N}$: We estimate

$$
\|g\eta \partial_t \mathcal{N}\|^2_{W^{\frac{1}{2}}_\delta(\Sigma_0)} \lesssim \|\eta\|^2_{W^{\frac{1}{2}}_\delta(-\ell,\ell)} \|\partial_t \partial_1 \bar{\eta}\|^2_{W^{s-\frac{1}{2}}_\delta(\Sigma_0)} \lesssim \|\eta\|^2_{W^{\frac{1}{2}}_\delta(-\ell,\ell)} \|\partial_t \eta\|^2_{W^s_\delta(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}.
$$

Fourth Term of G_4^+, $\sigma \partial_1 \left(\dfrac{k_1 \partial_1 \zeta_0}{\sqrt{1 + |\partial_1 \zeta_0|^2}} + \dfrac{k_1 \partial_1 \zeta_0 + \partial_1 \eta}{\left(\sqrt{1 + |\partial_1 \zeta_0|^2} \right)^3} \right) \partial_t \mathcal{N}$: We estimate

$$
\begin{aligned}
&\left\| \sigma \partial_1 \left(\frac{k_1 \partial_1 \zeta_0}{\sqrt{1 + |\partial_1 \zeta_0|^2}} + \frac{k_1 \partial_1 \zeta_0 + \partial_1 \eta}{\left(\sqrt{1 + |\partial_1 \zeta_0|^2} \right)^3} \right) \partial_t \mathcal{N} \right\|^2_{W^{\frac{1}{2}}_\delta(\Sigma_0)} \\
&\lesssim \left\| \partial_1^2 \eta \right\|^2_{W^{\frac{1}{2}}_\delta(-\ell,\ell)} \|\partial_t \partial_1 \bar{\eta}\|^2_{W^{s-\frac{1}{2}}_\delta(\Sigma_0)} \lesssim \|\eta\|^2_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} \|\partial_t \eta\|^2_{W^s_\delta(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}.
\end{aligned}
\tag{6.29}
$$

Fifth Term of G_4^+, $\sigma\partial_1\mathcal{R}\partial_t\mathcal{N}$: We estimate

$$\|\sigma\partial_1\mathcal{R}\partial_t\mathcal{N}\|^2_{W_\delta^{\frac{1}{2}}(\Sigma_0)} \lesssim \|\partial_1\eta\|^2_{W_\delta^1(-\ell,\ell)}\left\|\partial_1^2\eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)}\|\partial_t\partial_1\bar{\eta}\|^2_{W_\delta^{s-\frac{1}{2}}(\Sigma_0)} \tag{6.30}$$
$$\lesssim \|\eta\|^2_{W_\delta^{\frac{5}{2}}(-\ell,\ell)}\|\partial_t\eta\|^2_{W_\delta^s(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}. \quad \square$$

6.5. *Estimate of the G_5 term*

Finally, we handle G_5.

Lemma 6.5. *Let $G_5 = \mathcal{F}_3 \cdot \dfrac{\mathcal{N}}{|\mathcal{N}|}$. We have the estimate*

$$\|G_5\|^2_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)} \lesssim \left(\mathcal{E}^2+\mathcal{E}\right)\mathcal{D}. \tag{6.31}$$

Proof. This estimate follows from an argument similar to that of Lemma 6.2. $\square$

6.6. *Other nonlinear estimates*

Here we record some other nonlinear estimates.

Lemma 6.6. *Let $\mathcal{R}$ be given by* (C.3.1). *We have the estimate*

$$\|\partial_1\partial_t\mathcal{R}\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}. \tag{6.32}$$

Proof. We have

$$\|\partial_1\partial_t\mathcal{R}\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \lesssim \left\|\partial_t k_1\partial_1^2\eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} + \left\|k_1\partial_t\partial_1^2\eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \tag{6.33}$$
$$+\left\|\partial_1^2\eta\partial_t\partial_1\eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} + \left\|\partial_1\eta\partial_t\partial_1^2\eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} =: I+II+III+IV.$$

We estimate each term as follows:

$$I \lesssim |\partial_t k_1|^2\left\|\partial_1^2\eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \lesssim \left(|\partial_t l|^2+|\partial_t r|^2\right)\|\eta\|^2_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}, \tag{6.34}$$

$$II \lesssim |k_1|^2\left\|\partial_t\partial_1^2\eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \lesssim \|\eta\|^2_{H^1(-\ell,\ell)}\|\partial_t\eta\|^2_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}, \tag{6.35}$$

$$III \lesssim \left\|\partial_1^2\eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)}\|\partial_t\partial_1\eta\|^2_{W_\delta^{s-\frac{1}{2}}(-\ell,\ell)} \lesssim \|\eta\|^2_{W_\delta^{\frac{5}{2}}(-\ell,\ell)}\|\partial_t\eta\|^2_{W_\delta^{s+\frac{1}{2}}(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}, \tag{6.36}$$

and

$$IV \lesssim \|\partial_1 \eta\|^2_{W_\delta^{s-\frac{1}{2}}(-\ell,\ell)} \left\|\partial_t \partial_1^2 \eta\right\|^2_{W_\delta^{\frac{1}{2}}(-\ell,\ell)} \lesssim \|\eta\|^2_{W_\delta^{s+\frac{1}{2}}(-\ell,\ell)} \|\partial_t \eta\|^2_{W_\delta^{\frac{5}{2}}(-\ell,\ell)} \lesssim \mathcal{E}\mathcal{D}. \quad \square \tag{6.37}$$

7. Well-posedness and decay

Define the functionals

$$\mathfrak{E} = \sum_{j=0}^{2} \int_{-\ell}^{\ell} \frac{\sigma}{2} \frac{\left(\partial_t^j k_1 \partial_1 \zeta_0 + \partial_t^j \partial_1 \eta\right)^2}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3}, \tag{7.1}$$

$$\mathfrak{D} = \sum_{j=0}^{2} \left(\int_{\Omega_0} \frac{J\mu}{2} \left|\mathbb{D}_{\mathcal{A}} \partial_t^j u\right|^2 + \int_{\Sigma_{0b}} \beta(\partial_t^j u \cdot \tau_0)^2 + \kappa\left((\partial_t^{j+1}\mathscr{L})^2 + (\partial_t^{j+1}\mathscr{R})^2\right) \right), \tag{7.2}$$

and

$$\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2 = -\int_{-\ell}^{\ell} \frac{J_1 - 1}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \left|\partial_t^2 \partial_1 \eta\right|^2 + \int_{-\ell}^{\ell} \frac{\partial_{\varpi_2} \mathcal{R} J_1}{\sqrt{1+|\partial_1 \zeta_0|^2}} \left|\partial_t^2 \partial_1 \eta\right|^2. \tag{7.3}$$

Our next result shows various comparability results for these functionals.

Lemma 7.1. *There exists a universal constant $\vartheta > 0$ such that if*

$$\sup_{0 \le t \le T} \mathcal{E}(t) \le \vartheta, \tag{7.4}$$

then

$$\mathfrak{E} \lesssim \mathcal{E}_{\parallel} \lesssim \mathfrak{E}, \ \mathfrak{D} \lesssim \tilde{\mathcal{D}}_{\parallel} \lesssim \mathfrak{D}, \ \textit{and} \ |\mathfrak{F}| \le \frac{1}{2}\mathfrak{E}. \tag{7.5}$$

Proof. The first two inequalities in (7.5) follow directly from Lemma C.2. It remains to prove the estimate of $\mathfrak{F}$. Lemma C.3 implies that

$$|\mathfrak{F}_1| = \left| \int_{-\ell}^{\ell} \frac{J_1 - 1}{\left(\sqrt{1+|\partial_1 \zeta_0|^2}\right)^3} \left|\partial_t^2 \partial_1 \eta\right|^2 \right| \lesssim \|\eta\|_{H^0(-\ell,\ell)} \left\|\partial_t^2 \eta\right\|^2_{H^1(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}}\mathfrak{E}, \tag{7.6}$$

$$\begin{aligned} |\mathfrak{F}_2| = \left| \int_{-\ell}^{\ell} \frac{\partial_{\varpi_2} \mathcal{R} J_1}{\sqrt{1+|\partial_1 \zeta_0|^2}} \left|\partial_t^2 \partial_1 \eta\right|^2 \right| &\lesssim \left|\partial_{\varpi_2} \mathcal{R}\right| \left\|\partial_t^2 \partial_1 \eta\right\|^2_{H^0(-\ell,\ell)} \\ &\lesssim \|\eta\|_{H^2(-\ell,\ell)} \left\|\partial_t^2 \eta\right\|^2_{H^1(-\ell,\ell)} \lesssim \sqrt{\mathcal{E}}\mathfrak{E}. \end{aligned} \tag{7.7}$$

Hence, for ϑ small, the desired estimate follows directly. $\square$

Next we state a synthesized version of the energy-dissipation equation.

Lemma 7.2. *There exists a universal constant $\vartheta > 0$ such that if*

$$\sup_{0\le t\le T} \mathcal{E}(t) + \int_0^T \mathcal{D}(t)\mathrm{d}t \le \vartheta, \tag{7.8}$$

then there exists a universal constant $C > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{E} - \mathfrak{F}) + C\mathcal{D} \le 0. \tag{7.9}$$

Proof. Let ϑ be as in Lemma 7.1. The linearized energy-dissipation structure (3.13) for ∂_t^2, ∂_t and no time derivatives, combined with the estimates in Lemmas 5.3–5.11, imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{E} - \mathfrak{F}) + C\mathfrak{D} \le \sqrt{\mathcal{E}}\mathcal{D}. \tag{7.10}$$

In Lemma 7.1, we have shown

$$\mathfrak{D} \lesssim \tilde{\mathcal{D}}_{\parallel} \lesssim \mathfrak{D}, \tag{7.11}$$

Theorem 4.1 and Lemma 5.12 imply the pressure estimate

$$\|p\|^2_{H^0(\Omega_0)} + \|\partial_t p\|^2_{H^0(\Omega_0)} + \left\|\partial_t^2 p\right\|^2_{H^0(\Omega_0)} \lesssim \tilde{\mathcal{D}}_{\parallel} + \sqrt{\mathcal{E}}\mathcal{D}. \tag{7.12}$$

Theorem 4.4 and Lemmas 5.12 and 5.13, imply the free surface estimate

$$\begin{aligned} &\|\eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} + \|\partial_t\eta\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} + \left\|\partial_t^2\eta\right\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \\ &\quad\lesssim \|p\|^2_{H^0(\Omega_0)} + \|\partial_t p\|^2_{H^0(\Omega_0)} + \left\|\partial_t^2 p\right\|^2_{H^0(\Omega_0)} + \tilde{\mathcal{D}}_{\parallel} + \sqrt{\mathcal{E}}\mathcal{D} \lesssim \tilde{\mathcal{D}}_{\parallel} + \sqrt{\mathcal{E}}\mathcal{D}. \end{aligned} \tag{7.13}$$

Theorems 4.6 and 4.7 imply the contact point estimate

$$\begin{aligned} &\sum_{j=0}^{2}\left(\left|\partial_t^j\partial_1\eta(-\ell)\right|^2 + \left|\partial_t^j\partial_1\eta(\ell)\right|^2\right) + \sum_{j=0}^{2}\left(\left|\partial_t^j u(-\ell,0)\cdot\mathcal{N}\right|^2 + \left|\partial_t^j u(\ell,0)\cdot\mathcal{N}\right|^2\right) \\ &\lesssim \tilde{\mathcal{D}}_{\parallel} + \sqrt{\mathcal{E}}\mathcal{D}. \end{aligned} \tag{7.14}$$

Combining these, we deduce that

$$\mathcal{D}_{\parallel} \lesssim \tilde{\mathcal{D}}_{\parallel} + \sqrt{\mathcal{E}}\mathcal{D} \lesssim \mathfrak{D} + \sqrt{\mathcal{E}}\mathcal{D}. \tag{7.15}$$

The Stokes problem estimate in Theorem 4.8 for at most ∂_t, combined with the estimates in Lemma 6.1, 6.2, 6.3, 6.4, 6.5, and 6.6, imply

$$\|u\|^2_{W^2_\delta(\Omega_0)} + \|p\|^2_{W^1_\delta(\Omega_0)} + \|\eta\|^2_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} + \|\partial_t u\|^2_{W^2_\delta(\Omega_0)} + \|\partial_t p\|^2_{W^1_\delta(\Omega_0)} + \|\partial_t \eta\|^2_{W^{\frac{5}{2}}_\delta(-\ell,\ell)}$$
$$\lesssim \mathcal{D}_\| + (\mathcal{E}^2 + \mathcal{E})\mathcal{D}. \tag{7.16}$$

Using the transport equation in (3.1), we may further obtain

$$\left\|\partial_t^2 \eta\right\|^2_{W^{\frac{3}{2}}_\delta(-\ell,\ell)} + \left\|\partial_t^3 \eta\right\|^2_{W^{\frac{1}{2}}_\delta(-\ell,\ell)} \lesssim \mathcal{D}_\| + (\mathcal{E}^2 + \mathcal{E})\mathcal{D}. \tag{7.17}$$

Collecting all above, we have

$$\mathcal{D} \lesssim \mathcal{D}_\| + \sqrt{\mathcal{E}}\mathcal{D} \lesssim \tilde{\mathcal{D}}_\| + \sqrt{\mathcal{E}}\mathcal{D} \lesssim \mathfrak{D} + \sqrt{\mathcal{E}}\mathcal{D}. \tag{7.18}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{E} - \mathfrak{F}) + C\mathcal{D} \le \sqrt{\mathcal{E}}\mathcal{D}. \tag{7.19}$$

Thus, if ϑ is sufficiently small, then we may conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{E} - \mathfrak{F}) + C\mathcal{D} \le 0. \quad \square \tag{7.20}$$

We now present the main a priori decay estimates.

Theorem 7.3. *There exists a universal constant $\vartheta > 0$ such that if*

$$\sup_{0\le t\le T} \mathcal{E}(t) + \int_0^T \mathcal{D}(t)\mathrm{d}t \le \vartheta, \tag{7.21}$$

then there exists a universal constant $\lambda > 0$ such that

$$\sup_{0\le t\le T} \mathrm{e}^{\lambda t}\Big(\mathcal{E}_\|(t) + \|u(t)\|^2_{H^1(\Omega_0)} + \|u(t)\cdot\tau_0\|^2_{H^0(\Sigma_{0b})} + \|p(t)\|^2_{H^0(\Omega_0)} + |\partial_t l(t)|^2 \tag{7.22}$$
$$+ \left|\partial_t r(t)^2\right| + |\partial_1\eta(t,-\ell)|^2 + |\partial_1\eta(t,\ell)|^2 + |u(t,-\ell,0)\cdot\mathcal{N}|^2 + |u(t,\ell,0)\cdot\mathcal{N}|^2\Big) \lesssim \mathcal{E}_\|(0).$$

Proof. In Lemma 7.2, we have shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{E} - \mathfrak{F}) + C\mathcal{D} \le 0. \tag{7.23}$$

Also, in Lemma 7.1, we have proved that

$$\mathfrak{E} \lesssim \mathcal{E}_\| \lesssim \mathfrak{E} \text{ and } 0 < \frac{1}{2}\mathfrak{E} \le \mathfrak{E} - \mathfrak{F} \le \frac{3}{2}\mathfrak{E}. \tag{7.24}$$

On the other hand, it is clear that $\mathfrak{E} \lesssim \mathcal{D}$, and so we deduce the bound

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{E}-\mathfrak{F})+\lambda(\mathfrak{E}-\mathfrak{F}) \leq 0. \tag{7.25}$$

Upon integrating this differential inequality, we find that

$$\mathfrak{E}(t) \lesssim \mathfrak{E}(t)-\mathfrak{F}(t) \lesssim \mathrm{e}^{-\lambda t}\Big(\mathfrak{E}(0)-\mathfrak{F}(0)\Big) \lesssim \mathrm{e}^{-\lambda t}\mathfrak{E}(0), \tag{7.26}$$

which, in light of (7.24), then implies that

$$\mathcal{E}_{\|}(t) \lesssim \mathrm{e}^{-\lambda t}\mathcal{E}_{\|}(0). \tag{7.27}$$

Next we consider the linearized energy-dissipation structure given in Theorem 3.2 for the problem with no derivatives applied. Using the transport equation in (1.25), and following similar arguments as Lemma 5.11, 5.7, 5.8, 5.9, 5.10, noting $\eta(\pm\ell)=0$, we obtain the bounds

$$\begin{aligned}\left|\sigma\int_{\Sigma_0}\partial_1\mathcal{R}(u\cdot\mathcal{N})\right| &\lesssim \sqrt{\mathcal{E}}\Big(\|\eta\|^2_{H^1(-\ell,\ell)}+\|\partial_t\eta\|^2_{H^1(-\ell,\ell)}+|\partial_t l|^2+|\partial_t r|^2\Big) \\ &\lesssim \sqrt{\mathcal{E}}\mathcal{E}_{\|}+\sqrt{\mathcal{E}}\Big(|\partial_t l|^2+|\partial_t r|^2\Big),\end{aligned} \tag{7.28}$$

$$\begin{aligned}&\left|\int_{-\ell}^{\ell}\left(g\eta-\partial_1\left(\frac{k_1\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}}+\frac{k_1\partial_1\zeta_0+\partial_1\eta}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3}\right)\right)\mathcal{S}\right| \\ &\lesssim \sqrt{\mathcal{E}}\Big(\|\eta\|^2_{H^1(-\ell,\ell)}+\|\partial_t\eta\|^2_{H^1(-\ell,\ell)}+|\partial_t l|^2+|\partial_t r|^2\Big) \lesssim \sqrt{\mathcal{E}}\mathcal{E}_{\|}+\sqrt{\mathcal{E}}\Big(|\partial_t l|^2+|\partial_t r|^2\Big),\end{aligned} \tag{7.29}$$

and

$$\left|-\sigma\mathcal{Q}\Big(\partial_t l+\mathcal{O}\Big)\Big|_{\ell}+\sigma\mathcal{Q}\Big(\partial_t r+\mathcal{O}\Big)\Big|_{-\ell}\right| \lesssim \sqrt{\mathcal{E}}\Big(|\partial_t l|^2+|\partial_t r|^2\Big), \tag{7.30}$$

$$\left|\kappa\Big(\partial_t r\mathcal{O}\Big|_{\ell}+\partial_t l\mathcal{O}\Big|_{-\ell}\Big)\right| \lesssim \sqrt{\mathcal{E}}\Big(|\partial_t l|^2+|\partial_t r|^2\Big), \tag{7.31}$$

$$\left|-\mathcal{F}_7\Big(\partial_t r+\mathcal{O}\Big)\Big|_{\ell}+\mathcal{F}_6\Big(\partial_t l+\mathcal{O}\Big)\Big|_{-\ell}\right| \lesssim \sqrt{\mathcal{E}}\Big(|\partial_t l|^2+|\partial_t r|^2\Big). \tag{7.32}$$

Therefore,

$$\begin{aligned}&\partial_t\left(\int_{-\ell}^{\ell}\frac{\sigma}{2}\frac{\Big(k_1\partial_1\zeta_0+\partial_1\eta\Big)^2}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3}+\frac{k_1^2}{2}\left(P_0M+\sigma\int_{-\ell}^{\ell}\frac{|\partial_1\zeta_0|^2}{\sqrt{1+|\partial_1\zeta_0|^2}}\right)+\frac{g}{2}\int_{-\ell}^{\ell}(k_1\zeta_0-\eta)^2\right) \\ &\lesssim \|\eta\|^2_{H^1(-\ell,\ell)}+\|\partial_t\eta\|^2_{H^1(-\ell,\ell)} \lesssim \mathcal{E}_{\|},\end{aligned} \tag{7.33}$$

and hence Theorem 3.2 allows us to bound

$$\int_{\Omega_0} \frac{J\mu}{2} |\mathbb{D}_{\mathcal{A}} u|^2 + \int_{\Sigma_{0b}} \beta (u\cdot\tau_0)^2 + \kappa\Big((\partial_t l)^2 + (\partial_t r)^2\Big) \lesssim \mathcal{E}_\parallel + \sqrt{\mathcal{E}}\Big(|\partial_t l|^2 + |\partial_t r|^2 \Big), \tag{7.34}$$

which in turn means that for $\mathcal{E}$ small,

$$\int_{\Omega_0} \frac{J\mu}{2} |\mathbb{D}_{\mathcal{A}} u|^2 + \int_{\Sigma_{0b}} \beta (u\cdot\tau_0)^2 + \kappa\Big((\partial_t l)^2 + (\partial_t r)^2\Big) \lesssim \mathcal{E}_\parallel. \tag{7.35}$$

Using the pressure estimate in Theorem 4.1, we know

$$\|p(t)\|^2_{H^0(\Omega_0)} \lesssim \|u(t)\|^2_{H^1(\Omega_0)} \lesssim \mathcal{E}_\parallel. \tag{7.36}$$

Using the contact point estimates in Theorem 4.6 and 4.7, we know

$$|\partial_1\eta(t,-\ell)|^2 + |\partial_1\eta(t,\ell)|^2 \lesssim \|\eta\|^2_{H^0(-\ell,\ell)} + |\partial_t l|^2 + |\partial_t r|^2 \lesssim \mathcal{E}_\parallel, \tag{7.37}$$

$$|u(t,-\ell,0)\cdot\mathcal{N}|^2 + |u(t,\ell,0)\cdot\mathcal{N}|^2 \lesssim \|\eta\|^2_{H^0(-\ell,\ell)} + |\partial_t l|^2 + |\partial_t r|^2 \lesssim \mathcal{E}_\parallel. \tag{7.38}$$

Combining the above, we conclude that the stated estimates hold. □

Next we present the a priori bounds at the higher level of regularity.

Theorem 7.4. *There exists a universal constant $\vartheta > 0$ such that if*

$$\sup_{0\le t\le T} \mathcal{E}(t) + \int_0^T \mathcal{D}(t)\mathrm{d}t \le \vartheta, \tag{7.39}$$

then

$$\sup_{0\le t\le T} \mathcal{E}(t) + \int_0^T \mathcal{D}(t)\mathrm{d}t \lesssim \mathcal{E}(0). \tag{7.40}$$

Proof. Again, we know from Lemmas 7.1 and 7.2 that, provided ϑ is sufficiently small, we have the bounds

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{E} - \mathfrak{F}) + C\mathcal{D} \le 0,\ \ \mathfrak{E} \lesssim \mathcal{E}_\parallel \lesssim \mathfrak{E},\ \text{ and } 0 < \frac{1}{2}\mathfrak{E} \le \mathfrak{E} - \mathfrak{F} \le \frac{3}{2}\mathfrak{E}. \tag{7.41}$$

This allows us to integrate in time to deduce that

$$\frac{1}{2}\mathfrak{E}(t) + C\int_0^t \mathcal{D}(s)\mathrm{d}s \le \frac{3}{2}\mathfrak{E}(0), \tag{7.42}$$

which implies for any $t \in [0, T]$,

$$\mathcal{E}_{\parallel}(t) + \int_0^t \mathcal{D}(s)\mathrm{d}s \lesssim \mathcal{E}_{\parallel}(0). \tag{7.43}$$

For a Hilbert space X and $f \in H^1([0, T]; X)$, we know that

$$\|f(t)\|_X^2 \lesssim \|f(0)\|_X^2 + \int_0^t \Big(\|f(s)\|_X^2 + \|\partial_t f(s)\|_X^2 \Big)\mathrm{d}s. \tag{7.44}$$

Hence, we have

$$\begin{aligned}
&\|u(t)\|^2_{W^2_\delta(\Omega_0)} + \|\partial_t u(t)\|^2_{W^2_\delta(\Omega_0)} + \|p(t)\|^2_{W^1_\delta(\Omega_0)} \\
&+ \|\partial_t p(t)\|^2_{W^1_\delta(\Omega_0)} + \|\eta(t)\|^2_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} + \|\partial_t \eta(t)\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \\
&+ \sum_{j=0}^1 \bigg(\Big|\partial_t^j \partial_1 \eta(t)\Big|_{-\ell}\Big|^2 + \Big|\partial_t^j \partial_1 \eta(t)\Big|_{\ell}\Big|^2 \bigg) + \sum_{j=0}^1 \bigg(\Big|\partial_t^j u(t)\cdot \mathcal{N}\Big|_{-\ell}\Big|^2 + \Big|\partial_t^j u(t)\cdot \mathcal{N}\Big|_{\ell}\Big|^2 \bigg) \\
\lesssim &\int_0^t \mathcal{D}(s)\mathrm{d}s + \|u(0)\|^2_{W^2_\delta(\Omega_0)} + \|\partial_t u(0)\|^2_{W^2_\delta(\Omega_0)} \\
&+ \|p(0)\|^2_{W^1_\delta(\Omega_0)} + \|\partial_t p(0)\|^2_{W^1_\delta(\Omega_0)} + \|\eta(0)\|^2_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} + \|\partial_t \eta(0)\|^2_{H^{\frac{3}{2}}(-\ell,\ell)} \\
&+ \sum_{j=0}^1 \bigg(\Big|\partial_t^j \partial_1 \eta(0)\Big|_{-\ell}\Big|^2 + \Big|\partial_t^j \partial_1 \eta(0)\Big|_{\ell}\Big|^2 \bigg) + \sum_{j=0}^1 \bigg(\Big|\partial_t^j u(0)\cdot \mathcal{N}\Big|_{-\ell}\Big|^2 + \Big|\partial_t^j u(0)\cdot \mathcal{N}\Big|_{\ell}\Big|^2 \bigg).
\end{aligned} \tag{7.45}$$

Then (7.43) and (7.45) may be combined to conclude that (7.40) holds. □

Now we record the local well-posedness result without giving detailed proof. It can be done using a variant of the argument developed in [20].

Theorem 7.5. *There exists a universal constant $\vartheta > 0$ and $T_0 > 0$ such that if $0 < T < T_0$ and $\mathcal{E}(0) \leq \vartheta$, then there exists a unique solution (u, p, η) on the interval $t \in [0, T]$ such that*

$$\sup_{t\in[0,T]} \mathcal{E}(t) + \int_0^T \mathcal{D}(t)\mathrm{d}t \lesssim \mathcal{E}(0). \tag{7.46}$$

Now we provide the global well-posedness result.

Proof of Theorem 2.1. This follows from the local well-posedness, Theorem 7.5, and a standard continuation argument using the a priori estimates of Theorem 7.3 and 7.4. For details we refer to [9]. □

Appendix A. Analysis tools

The proofs in this section can be found in Appendix C and Appendix D in [9].

A.1. Weighted Sobolev spaces

Let $M = dist(\cdot, M)$ where $M = \Big\{(-\ell, 0), (\ell, 0)\Big\}$ is the set of the corner points. Define the weighted Sobolev norm

$$\|f\|^2_{W^k_\delta(\Omega_0)} = \sum_{|\alpha|\le k} \int_{\Omega_0} \Big(dist(x, M)\Big)^{2\delta} \left|\partial^\alpha f\right|^2 \mathrm{d}x. \tag{A.1.1}$$

Then we say $f \in W^k_\delta(\Omega_0)$ if and only if $\|f\|_{W^<_\delta(\Omega_0)} \infty$. We will define the trace space $\|f\|_{W_\delta^{k-\frac{1}{2}}(\partial\Omega_0)}$ in the obvious way and it can be shown that

$$\int_{\partial\Omega_0} f(v\cdot\tau_0) \lesssim \|f\|_{W_\delta^{\frac{1}{2}}(\partial\Omega_0)} \|v\|_{H^1(\Omega_0)}. \tag{A.1.2}$$

Finally, define the zero-average space

$$\mathring{W}^k_\delta(\Omega_0) = \left\{ f \in W^k_\delta(\Omega_0) : \int_{\Omega_0} f(x)\mathrm{d}x = 0 \right\}. \tag{A.1.3}$$

Lemma A.1. *We have the continuous embedding*

$$W^1_\delta(\Omega_0) \hookrightarrow H^0(\Omega_0), \quad W^2_\delta(\Omega_0) \hookrightarrow H^1(\Omega_0), \quad H^{-1}(\Omega_0) \hookrightarrow W^0_{-\delta}(\Omega_0). \tag{A.1.4}$$

Lemma A.2. *Let $k \in \mathbb{N}$ and $\delta_1, \delta_2 \in \mathbb{R}$ with $\delta_1 < \delta_2$. Then we have that*

$$W^k_{\delta_1}(\Omega_0) \hookrightarrow W^k_{\delta_2}(\Omega_0). \tag{A.1.5}$$

Lemma A.3. *Let $k \in \mathbb{N}$ and $0 < \delta < 1$. Then for $1 \le q < \frac{2}{1+\delta}$ we have that*

$$W^k_\delta(\Omega_0) \hookrightarrow W^{k,q}(\Omega_0). \tag{A.1.6}$$

In particular, for $1 \le q < \frac{2}{\delta}$ we have

$$W^1_\delta(\Omega_0) \hookrightarrow L^q(\Omega_0). \tag{A.1.7}$$

Lemma A.4. *Suppose that* $0 < \delta < 1$ *and* $1 \le q < \frac{2}{1+\delta}$. *Then we have that*

$$W_\delta^{\frac{1}{2}}(\partial\Omega_0) \hookrightarrow L^q(\partial\Omega_0). \tag{A.1.8}$$

Lemma A.5. *Suppose that* $0 < \delta < 1$. *Then we have that*

$$W_\delta^1(\Omega_0) \hookrightarrow W_{\delta-1}^0(\Omega_0). \tag{A.1.9}$$

Lemma A.6. *Suppose that* $0 < \delta < 1$. *Then for each* $q \in [1, \infty)$, *we have that*

$$\left\| \Big(dist(\cdot, M) \Big)^\delta f \right\|_{L^q(\Omega_0)} \lesssim \|f\|_{W_\delta^1(\Omega_0)}. \tag{A.1.10}$$

Corollary A.7. *Assume* $1 < s < \min\left\{\frac{\pi}{\omega}, 2\right\}$. *Then we have*

$$W_\delta^2(\Omega_0) \hookrightarrow H^s(\Omega_0), \quad W_\delta^1(\Omega_0) \hookrightarrow H^{s-1}(\Omega_0), \quad W_\delta^{\frac{5}{2}}(\Sigma_0) \hookrightarrow H^{s+\frac{1}{2}}(\Sigma_0). \tag{A.1.11}$$

A.2. Product estimates

Lemma A.8. *Let* $\Omega_0 \in \mathbb{R}^2$. *Suppose that* $f \in H^r(\Omega_0)$ *for* $r \in (0, 1)$ *and* $g \in H^1(\Omega_0)$. *Then* $fg \in H^\sigma(\Omega_0)$ *for every* $\sigma \in (0, r)$, *and*

$$\|fg\|_{H^\sigma(\Omega_0)} \le C(r, \sigma) \|f\|_{H^r(\Omega_0)} \|g\|_{H^1(\Omega_0)}. \tag{A.2.1}$$

Lemma A.9. *Suppose that* $f \in W_\delta^1(\Omega_0)$ *for* $0 < \delta < 1$ *and that* $g \in H^{1+\kappa}(\Omega_0)$ *for* $0 < \kappa < 1$. *Then* $fg \in W_\delta^1(\Omega_0)$ *and*

$$\|fg\|_{W_\delta^1(\Omega_0)} \lesssim \|f\|_{W_\delta^1(\Omega_0)} \|g\|_{H^{1+\kappa}(\Omega_0)}. \tag{A.2.2}$$

Lemma A.10. *Suppose that* $f \in W_\delta^{\frac{1}{2}}(\Sigma_0)$ *for* $0 < \delta < 1$ *and that* $g \in H^{\frac{1}{2}+\kappa}(\Sigma_0)$ *for* $0 < \kappa < 1$. *Then* $fg \in W_\delta^{\frac{1}{2}}(\Sigma_0)$ *and*

$$\|fg\|_{W_\delta^{\frac{1}{2}}(\Sigma_0)} \lesssim \|f\|_{W_\delta^{\frac{1}{2}}(\Sigma_0)} \|g\|_{H^{\frac{1}{2}+\kappa}(\Sigma_0)}. \tag{A.2.3}$$

Appendix B. Energy-dissipation structure and equilibrium

B.1. Proof of Theorem 1.1

Proof of Theorem 1.1. We multiply u on both sides of the Stokes equation and integrate it over $\Omega(t)$ to obtain

$$I = \int\limits_{\Omega(t)} \Big(\nabla_z \cdot S(P,u)\Big) \cdot u = 0. \tag{B.1.1}$$

We divide it into several steps.

Step 1: Diffusion and fixed boundary terms: Integrating by parts implies

$$\begin{aligned} I &= \Bigg(\int\limits_{\Omega(t)} \frac{\mu}{2} |\mathbb{D}_z u|^2 - \int\limits_{\Omega(t)} P(\nabla_z \cdot u) \Bigg) + \int\limits_{\Sigma_b(t)} \Big(S(P,u)\nu \Big) \cdot u + \int\limits_{\Sigma(t)} \Big(S(P,u)\nu \Big) \cdot u \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{B.1.2}$$

where we may use the divergence-free condition and the boundary condition to simplify

$$I_1 = \int\limits_{\Omega(t)} \frac{\mu}{2} |\mathbb{D}_z u|^2 - \int\limits_{\Omega(t)} P(\nabla_z \cdot u) = \int\limits_{\Omega(t)} \frac{\mu}{2} |\mathbb{D}_z u|^2, \tag{B.1.3}$$

$$\begin{aligned} I_2 &= \int\limits_{\Sigma_b(t)} \Big(S(P,u)\nu \Big) \cdot u \\ &= \int\limits_{\Sigma_b(t)} \Bigg(\Big(S(P,u)\nu \Big) \cdot \nu \Bigg) (u \cdot \nu) + \int\limits_{\Sigma_b(t)} \Bigg(\Big(S(P,u)\nu \Big) \cdot \tau \Bigg) (u \cdot \tau) \\ &= \int\limits_{\Sigma_b(t)} \Bigg(\Big(S(P,u)\nu \Big) \cdot \tau \Bigg) (u \cdot \tau) = \int\limits_{\Sigma_b(t)} \beta |u \cdot \tau|^2 . \end{aligned} \tag{B.1.4}$$

Step 2: Free surface terms: We directly simplify to obtain

$$\begin{aligned} I_3 &= \int\limits_{\Sigma(t)} \Big(S(P,u)\nu \Big) \cdot u = \int\limits_{\Sigma(t)} \left(g\zeta - \sigma \partial_{z_1} \left(\frac{\partial_{z_1}\zeta}{\sqrt{1 + \left|\partial_{z_1}\zeta\right|^2}} \right) \right) (u \cdot \nu) \\ &= \int\limits_{\Sigma(t)} \left(g\zeta - \sigma \partial_{z_1} \left(\frac{\partial_{z_1}\zeta}{\sqrt{1 + \left|\partial_{z_1}\zeta\right|^2}} \right) \right) \frac{\partial_t \zeta}{\sqrt{1 + \left|\partial_{z_1}\zeta\right|^2}} \\ &= \int_{L(t)}^{R(t)} \left(g\zeta - \sigma \partial_{z_1} \left(\frac{\partial_{z_1}\zeta}{\sqrt{1 + \left|\partial_{z_1}\zeta\right|^2}} \right) \right) \partial_t \zeta = I_{3,1} + I_{3,2}. \end{aligned} \tag{B.1.5}$$

Step 3: Gravitational term: We have

$$I_{3,1} = \int_{L(t)}^{R(t)} g\zeta\,\partial_t\zeta = \int_{L(t)}^{R(t)} \frac{g}{2}\partial_t\,|\zeta|^2 = \partial_t \int_{L(t)}^{R(t)} \frac{g}{2}\,|\zeta|^2 - \frac{g}{2}\,|\zeta(R)|^2\,\partial_t R + \frac{g}{2}\,|\zeta(L)|^2\,\partial_t L \tag{B.1.6}$$
$$= \partial_t \int_{L(t)}^{R(t)} \frac{g}{2}\,|\zeta|^2,$$

since $\zeta(L) = \zeta(R) = 0$.

Step 4: Surface tension terms: Integrating by parts and using Reynold's transport equation imply

$$I_{3,2} = -\int_{L(t)}^{R(t)} \sigma\,\partial_{z_1}\left(\frac{\partial_{z_1}\zeta}{\sqrt{1+\left|\partial_{z_1}\zeta\right|^2}}\right)\partial_t\zeta \tag{B.1.7}$$
$$= \int_{L(t)}^{R(t)} \sigma\frac{\partial_{z_1}\zeta}{\sqrt{1+\left|\partial_{z_1}\zeta\right|^2}}\partial_t\partial_{z_1}\zeta - \sigma\frac{\partial_{z_1}\zeta(R)}{\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2}}\partial_t\zeta(R) + \sigma\frac{\partial_{z_1}\zeta(L)}{\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2}}\partial_t\zeta(L)$$
$$= A + B + C,$$

where we may simplify

$$A = \int_{L(t)}^{R(t)} \sigma\frac{\partial_{z_1}\zeta}{\sqrt{1+\left|\partial_{z_1}\zeta\right|^2}}\partial_t\partial_{z_1}\zeta = \int_{L(t)}^{R(t)} \sigma\,\partial_t\sqrt{1+\left|\partial_{z_1}\zeta\right|^2} \tag{B.1.8}$$
$$= \partial_t \int_{L(t)}^{R(t)} \sigma\sqrt{1+\left|\partial_{z_1}\zeta\right|^2} - \sigma\,\partial_t R\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2} + \sigma\,\partial_t L\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2}$$
$$= A_1 + A_2 + A_3,$$

and using the transport equation with $u_2(L) = u_2(R) = 0$, we have

$$B = -\sigma\frac{\partial_{z_1}\zeta(R)}{\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2}}\partial_t\zeta(R) = -\sigma\frac{\partial_{z_1}\zeta(R)}{\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2}}\Big(-u_1(R)\partial_{z_1}\zeta(R) + u_2(R)\Big) \tag{B.1.9}$$
$$= \sigma u_1(R)\frac{\left|\partial_{z_1}\zeta(R)\right|^2}{\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2}},$$

and

$$C = \sigma \frac{\partial_{z_1}\zeta(L)}{\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2}}\partial_t\zeta(L) = \sigma \frac{\partial_{z_1}\zeta(L)}{\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2}}\Big(-u_1(L)\partial_{z_1}\zeta(L)+u_2(L)\Big) \tag{B.1.10}$$
$$= -\sigma u_1(L)\frac{\left|\partial_{z_1}\zeta(L)\right|^2}{\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2}}.$$

Step 5: Contact point terms: Note that fact that $\partial_t R = u_1(R)$ and $\partial_t L = u_1(L)$, we have

$$A_2 + B = -\sigma\partial_t R\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2} + \sigma u_1(R)\frac{\left|\partial_{z_1}\zeta(R)\right|^2}{\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2}} \tag{B.1.11}$$
$$= -\sigma\partial_t R\left(\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2} - \frac{\left|\partial_{z_1}\zeta(R)\right|^2}{\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2}}\right) = -\sigma\partial_t R\frac{1}{\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2}},$$
$$A_3 + C = \sigma\partial_t L\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2} - \sigma u_1(L)\frac{\left|\partial_{z_1}\zeta(L)\right|^2}{\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2}} \tag{B.1.12}$$
$$= \sigma\partial_t L\left(\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2} - \frac{\left|\partial_{z_1}\zeta(L)\right|^2}{\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2}}\right) = \sigma\partial_t L\frac{1}{\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2}}. \tag{B.1.13}$$

Using the contact point condition, we have

$$A_2 + B = -\sigma\partial_t R\frac{1}{\sqrt{1+\left|\partial_{z_1}\zeta(R)\right|^2}} = -\partial_t R\Big([\gamma] - \mathcal{W}(\partial_t R)\Big), \tag{B.1.14}$$
$$A_3 + C = \sigma\partial_t L\frac{1}{\sqrt{1+\left|\partial_{z_1}\zeta(L)\right|^2}} = \partial_t L\Big(\mathcal{W}(\partial_t L) + [\gamma]\Big). \tag{B.1.15}$$

Hence, we have

$$A_2 + A_3 + B + C = -\partial_t\Big([\gamma](R-L)\Big) + \Big(\mathcal{W}(\partial_t L)\partial_t L + \mathcal{W}(\partial_t R)\partial_t R\Big). \tag{B.1.16}$$

Step 6: Synthesis: Collecting all terms, we arrive at (1.12). □

B.2. Proof of Theorem 1.2

Proof of Theorem 1.2. It is well known (see for instance [8]) that, given the width of the droplet domain, there exists a unique solution to the second and third equations in (1.14) that is smooth, even, and monotonically decreasing from the center. In the present context, we must choose the equilibrium width in order to satisfy the fourth and fifth equations. Here for the sake of completeness, we will give a quick sketch of the construction of solutions and details of how to determine P_0 and $R_0 - L_0$ given that we specify the mass M.

Due to horizontal translational invariance, we may assume without loss of generality that $(R_0 + L_0)/2 = 0$, i.e. the droplet is centered at the origin. Then consider the new unknown $\ell = (R_0 - L_0)/2$, in which case $L_0 = -\ell$ and $R_0 = \ell$. After integrating on both sides of the equilibrium equation, we find that P_0 is determined by M and ℓ via

$$P_0 = \frac{Mg + 2\sqrt{\sigma^2 - [\gamma]^2}}{2\ell} > 0. \tag{B.2.1}$$

It remains to construct the solution with given mass M, and ℓ chosen so that the equilibrium equations are satisfied for P_0 determined by (B.2.1).

Due to reflectional symmetry, it suffices to construct the solution for $z_1 \in [-\ell, 0]$. Let $r = -z_1$ and $\tan\psi = -\partial_r \zeta_0 = \partial_{z_1}\zeta_0$. The variable ψ is the angle formed between the tangent line of ζ_0 and a line parallel to the z_1 axis through $(z_1, \zeta_0(z_1))$, which ranges from $\psi = 0$ at the maximum of ζ_0 in the center to $\psi = \psi_0$ for

$$\psi_0 := \arctan\left(\frac{\sqrt{\sigma^2 - [\gamma]^2}}{[\gamma]}\right) \in (0, \pi/2) \tag{B.2.2}$$

at the contact point. Then in these coordinates the equilibrium equation is

$$-\sigma\partial_r(\sin\psi) = g\zeta_0 - P_0, \tag{B.2.3}$$

which, considering $\dfrac{\mathrm{d}r}{\mathrm{d}\psi} = \left(\dfrac{\mathrm{d}\psi}{\mathrm{d}r}\right)^{-1}$ and $\dfrac{\mathrm{d}\zeta_0}{\mathrm{d}\psi} = \dfrac{\mathrm{d}\zeta_0}{\mathrm{d}r}\dfrac{\mathrm{d}r}{\mathrm{d}\psi}$, is equivalent to

$$\frac{\mathrm{d}r}{\mathrm{d}\psi} = -\frac{\sigma\cos\psi}{g\zeta_0 - P_0}, \quad \frac{\mathrm{d}\zeta_0}{\mathrm{d}\psi} = \frac{\sigma\sin\psi}{g\zeta_0 - P_0}. \tag{B.2.4}$$

Setting $\zeta_0 = 0$ at ψ_0, we may solve $\dfrac{\mathrm{d}\zeta_0}{\mathrm{d}\psi}$ equation to get

$$\frac{g}{2}\zeta_0^2 - P_0\zeta_0 - [\gamma] + \sigma\cos\psi = 0 \text{ and hence } \zeta_0(\psi) = \frac{P_0 - \sqrt{P_0^2 - 2g(\sigma\cos\psi - [\gamma])}}{g}. \tag{B.2.5}$$

Then plugging this into the $\mathrm{d}r/\mathrm{d}\psi$ equation and setting $r = 0$ at $\psi = 0$, we obtain

$$r(\psi) = \int_0^{\psi} \frac{\sigma\cos\psi}{\sqrt{P_0^2 - 2g(\sigma\cos\psi - [\gamma])}}\mathrm{d}\psi. \tag{B.2.6}$$

It remains only to enforce the condition $r(\psi_0) = \ell$, which, in light of (B.2.1), is equivalent to

$$1 = \int_0^{\psi_0} \frac{2\sigma \cos\psi}{\sqrt{\left(Mg + 2\sqrt{\sigma^2 - [\gamma]^2}\right)^2 - 8g\ell^2(\sigma\cos\psi - [\gamma])}} \mathrm{d}\psi. \tag{B.2.7}$$

When $\ell \to 0$, we have

$$\begin{aligned} &\int_0^{\psi_0} \frac{2\sigma \cos\psi}{\sqrt{\left(Mg + 2\sqrt{\sigma^2 - [\gamma]^2}\right)^2 - 8g\ell^2(\sigma\cos\psi - [\gamma])}} \mathrm{d}\psi \\ &\to \int_0^{\psi_0} \frac{2\sigma\cos\psi}{Mg + 2\sqrt{\sigma^2 - [\gamma]^2}} \mathrm{d}\psi = \frac{2\sqrt{\sigma^2 - [\gamma]^2}}{Mg + 2\sqrt{\sigma^2 - [\gamma]^2}} < 1. \end{aligned} \tag{B.2.8}$$

Also, as $\ell \to \dfrac{Mg + 2\sqrt{\sigma^2 - [\gamma]^2}}{\sqrt{8g(\sigma - [\gamma])}}$, considering the Taylor expansion of $\cos\psi$ around $\psi = 0$, the integral monotonically increases to ∞. Hence, there exists a unique ℓ such that the integral is exactly 1. With this choice of ℓ and P_0, the equilibrium equations are satisfied. □

Appendix C. Nonlinear quantities

In this section we record a number of results about the nonlinear terms appearing in our analysis.

C.1. Estimates of J_1, J_2, and A

Recall that are J_1, J_2, and A given by (1.23).

Lemma C.1. *Suppose that $\|\eta(t)\|_{H^{\frac{1}{2}}(-\ell,\ell)} < \vartheta$ for some $\vartheta > 0$ sufficiently small. Then*

$$|J_1 - 1| \lesssim \|\eta\|_{H^0(-\ell,\ell)} \text{ and } |J_2| + |A| \lesssim \|\eta\|_{H^{\frac{1}{2}}(-\ell,\ell)}. \tag{C.1.1}$$

Proof. Using the conservation of mass in (1.45),

$$J_1 \int_{-\ell}^{\ell} \eta(t, x_1)\mathrm{d}x_1 = \int_{-\ell}^{-\ell} (1 - J_1)\zeta_0(x_1)\mathrm{d}x_1, \tag{C.1.2}$$

we have

$$J_1 - 1 = -\frac{\int_{-\ell}^{-\ell} \eta(x_1)\mathrm{d}x_1}{\int_{-\ell}^{-\ell} \zeta_0(x_1)\mathrm{d}x_1 + \int_{-\ell}^{-\ell} \eta(x_1)\mathrm{d}x_1}. \tag{C.1.3}$$

Hence, when

$$\left|\int_{-\ell}^{-\ell} \eta(x_1)\mathrm{d}x_1\right| \lesssim \|\eta\|_{H^0(-\ell,\ell)} < \frac{1}{2}\int_{-\ell}^{-\ell} \zeta_0(x_1)\mathrm{d}x_1 = \frac{M}{2}, \tag{C.1.4}$$

we know

$$|J_1 - 1| \lesssim \frac{2}{M}\int_{-\ell}^{-\ell} \eta(x_1)\mathrm{d}x_1 \lesssim \|\eta\|_{H^0(-\ell,\ell)}. \tag{C.1.5}$$

We now turn to the proof of the J_2 estimate, noting first that

$$J_2(x) = 1 + \frac{\bar{\eta}(x)}{\zeta_0(x_1)} + \frac{x_2}{\zeta_0(x_1)}\partial_2\bar{\eta}(x). \tag{C.1.6}$$

The difficulty lies in when x is close to the contact point. In a neighborhood of the contact points, for $\Omega_0 \ni x = (x_1, x_2)$, let $s = \dfrac{x_2}{x_1 \mp \ell}$. Using Cauchy's Mean Value Theorem, we know that,

$$\frac{\bar{\eta}(x)}{\zeta_0(x_1)} = \frac{\bar{\eta}\Big(x_1, s(x_1 \mp \ell)\Big) - \bar{\eta}(\pm\ell, 0)}{\zeta_0(x_1) - \zeta_0(\pm\ell)} = \frac{\partial_1\bar{\eta}\Big(c, s(c \mp \ell)\Big) + s\partial_2\bar{\eta}\Big(c, s(c \mp \ell)\Big)}{\partial_1\zeta_0(c)}, \tag{C.1.7}$$

for some c close to $\pm\ell$. Since $x \in \Omega_0$ and Ω_0 is convex, we may directly estimate that $s \leq |\partial_1\zeta_0(\pm\ell)|$ and

$$\left|\frac{\bar{\eta}(x)}{\zeta_0(x_1)}\right| \lesssim \|\bar{\eta}\|_{H^1(\Omega_0)} \lesssim \|\eta\|_{H^{\frac{1}{2}}(-\ell,\ell)}. \tag{C.1.8}$$

Also, since for $x \in \Omega_0$, $0 \leq x_2 \leq \zeta_0(x_1)$, we have

$$\left|\frac{x_2}{\zeta_0(x_1)}\right| \leq 1. \tag{C.1.9}$$

In total, we have

$$|J_2| \lesssim \|\bar{\eta}\|_{H^1(\Omega_0)} \lesssim \|\eta\|_{H^{\frac{1}{2}}(-\ell,\ell)}. \tag{C.1.10}$$

Finally, we turn to the A estimate. We begin by decomposing

$$A(x) = \frac{x_2}{\zeta_0(x_1)}\left(\partial_1 \bar{\eta}(x) - \frac{\bar{\eta}(x)}{\zeta_0(x_1)}\partial_1\zeta_0(x_1)\right). \tag{C.1.11}$$

As in the estimate of J_2 above, we have

$$\left|\frac{x_2}{\zeta_0(x_1)}\right| \leq 1. \tag{C.1.12}$$

Hence

$$|\partial_1\bar{\eta}(x)| \lesssim \|\bar{\eta}\|_{H^1(\Omega_0)} \lesssim \|\eta\|_{H^{\frac{1}{2}}(-\ell,\ell)}. \tag{C.1.13}$$

Also, we know

$$\left|\frac{\bar{\eta}(x)}{\zeta_0(x_1)}\right| \lesssim \|\bar{\eta}\|_{H^1(\Omega_0)} \lesssim \|\eta\|_{H^{\frac{1}{2}}(-\ell,\ell)}. \tag{C.1.14}$$

Combining these then provides the desired bound. $\square$

Using a similar argument as in Lemma C.1, we obtain the following:

Lemma C.2. *Let $0 < \delta < 1$. There exists a universal $\vartheta \in (0,1)$ such that if $\|\eta\|_{W^{\frac{5}{2}}_\delta(-\ell,\ell)} \leq \vartheta$, then*

$$\|J - 1\|_{L^\infty(\Omega_0)} + \|A\|_{L^\infty(\Omega_0)} \leq \frac{1}{2}, \tag{C.1.15}$$

$$\|\mathcal{N} - 1\|_{L^\infty(\partial\Omega_0)} + \|K - 1\|_{L^\infty(\Omega_0)} \leq \frac{1}{2}, \tag{C.1.16}$$

$$\|K\|_{L^\infty(\Omega_0)} + \|\mathcal{A}\|_{L^\infty(\Omega_0)} \lesssim 1. \tag{C.1.17}$$

Also, the map Π is a diffeomorphism.

C.2. Nonlinear terms in the energy-dissipation estimates

In this subsection we record the nonlinearities that appear in (3.1). We begin with the form when ∂_t is applied. In this case we have:

$$\mathcal{F}_1 = -\nabla_{\partial_t\mathcal{A}} \cdot \left(pI - \mu\mathbb{D}_{\mathcal{A}}u\right) + \mu\nabla_{\mathcal{A}} \cdot \mathbb{D}_{\partial_t\mathcal{A}}u, \tag{C.2.1}$$

$$\mathcal{F}_2 = -\nabla_{\partial_t\mathcal{A}} \cdot u, \tag{C.2.2}$$

$$\begin{aligned}\mathcal{F}_3 = {} & \mu\mathbb{D}_{\partial_t\mathcal{A}}u\mathcal{N} - (\tilde{p}I - \mu\mathbb{D}_{\mathcal{A}}u)\partial_t\mathcal{N} + g\eta\partial_t\mathcal{N} \\ & - \sigma\partial_1\left(\frac{k_1\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}} + \frac{k_1\partial_1\zeta_0 + \partial_1\eta}{\left(\sqrt{1+|\partial_1\zeta_0|^2}\right)^3} + \mathcal{R}\right)\partial_t\mathcal{N},\end{aligned} \tag{C.2.3}$$

$$\mathcal{F}_4 = \mu \mathbb{D}_{\partial_t \mathcal{A}} u \nu \cdot \tau, \tag{C.2.4}$$

$$\mathcal{F}_5 = (u \cdot \partial_t \mathcal{N}) \sqrt{1 + |\partial_1 \zeta_0|^2}, \tag{C.2.5}$$

$$\mathcal{F}_6 = -\kappa \tilde{\mathscr{W}}'(\partial_t l) \partial_t^2 l, \tag{C.2.6}$$

and

$$\mathcal{F}_7 = -\kappa \tilde{\mathscr{W}}'(\partial_t r) \partial_t^2 r. \tag{C.2.7}$$

Next we record the form when ∂_t^2 is applied:

$$\mathcal{F}_1 = -2\nabla_{\partial_t \mathcal{A}} \cdot \left(\partial_t p I - \mu \mathbb{D}_{\mathcal{A}} \partial_t u \right) + 2\mu \nabla_{\mathcal{A}} \cdot \mathbb{D}_{\partial_t \mathcal{A}} \partial_t u - \nabla_{\partial_t^2 \mathcal{A}} \cdot \left(p I - \mu \mathbb{D}_{\mathcal{A}} u \right) \tag{C.2.8}$$
$$+ 2\mu \nabla_{\partial_t \mathcal{A}} \cdot \mathbb{D}_{\partial_t \mathcal{A}} u + \mu \nabla_{\mathcal{A}} \cdot \mathbb{D}_{\partial_t^2 \mathcal{A}} u,$$

$$\mathcal{F}_2 = -\nabla_{\partial_t^2 \mathcal{A}} u - 2\nabla_{\partial_t \mathcal{A}} \cdot \partial_t u, \tag{C.2.9}$$

$$\begin{aligned} \mathcal{F}_3 = {} & \mu \mathbb{D}_{\partial_t^2 \mathcal{A}} u \mathcal{N} + \mu \mathbb{D}_{\partial_t \mathcal{A}} u \partial_t \mathcal{N} + 2\mu \mathbb{D}_{\partial_t \mathcal{A}} \partial_t u \mathcal{N} \\ & - (p I - \mu \mathbb{D}_{\mathcal{A}} u) \partial_t^2 \mathcal{N} - 2(\partial_t p I - \mu \mathbb{D}_{\mathcal{A}} \partial_t u) \partial_t \mathcal{N} \\ & + g \eta \partial_t^2 \mathcal{N} - \sigma \partial_1 \left(\frac{k_1 \partial_1 \zeta_0}{\sqrt{1 + |\partial_1 \zeta_0|^2}} + \frac{k_1 \partial_1 \zeta_0 + \partial_1 \eta}{\left(\sqrt{1 + |\partial_1 \zeta_0|^2} \right)^3} + \mathcal{R} \right) \partial_t^2 \mathcal{N} \\ & + 2g \partial_t \eta \partial_t \mathcal{N} - 2\sigma \partial_1 \left(\frac{\partial_t k_1 \partial_1 \zeta_0}{\sqrt{1 + |\partial_1 \zeta_0|^2}} + \frac{\partial_t k_1 \partial_1 \zeta_0 + \partial_t \partial_1 \eta}{\left(\sqrt{1 + |\partial_1 \zeta_0|^2} \right)^3} + \partial_t \mathcal{R} \right) \partial_t \mathcal{N}, \end{aligned} \tag{C.2.10}$$

$$\mathcal{F}_4 = \mu \mathbb{D}_{\partial_t^2 \mathcal{A}} u \nu \cdot \tau + 2\mu \mathbb{D}_{\partial_t \mathcal{A}} \partial_t u \nu \cdot \tau, \tag{C.2.11}$$

$$\mathcal{F}_5 = (u \cdot \partial_t^2 \mathcal{N}) \sqrt{1 + |\partial_1 \zeta_0|^2} + 2(\partial_t u \cdot \partial_t \mathcal{N}) \sqrt{1 + |\partial_1 \zeta_0|^2}, \tag{C.2.12}$$

$$\mathcal{F}_6 = -\kappa \tilde{\mathscr{W}}'(\partial_t l) \partial_t^3 l - \kappa \tilde{\mathscr{W}}''(\partial_t l)(\partial_t^2 l)^2, \tag{C.2.13}$$

and

$$\mathcal{F}_7 = -\kappa \tilde{\mathscr{W}}'(\partial_t r) \partial_t^3 r - \kappa \tilde{\mathscr{W}}''(\partial_t r)(\partial_t^2 r)^2. \tag{C.2.14}$$

C.3. Estimates of $\mathcal{R}$, $\mathcal{Q}$, $\mathcal{S}$ and $\mathcal{O}$

In this section we record estimates for the terms $\mathcal{R}$, $\mathcal{Q}$, $\mathcal{S}$ and $\mathcal{O}$, which we recall are defined in (1.37), (1.40), (1.43), and (1.42), respectively. To arrive at the estimates we first expand each term using the fundamental theorem of calculus. With the expansions in hand, the estimates then follow from elementary applications of the product rule and Sobolev embedding. As such we will omit the proofs of the bounds and only expansions and the form of the estimates.

We begin with the term $\mathcal{R}$, rewriting it as

$$\mathcal{R} = \frac{K_1^2\Big(\partial_1\zeta_0 + \partial_1\eta\Big)}{\Big(1 + K_1^2\Big(\partial_1\zeta_0 + \partial_1\eta\Big)^2\Big)^{\frac{1}{2}}} - \frac{\partial_1\zeta_0}{\Big(1 + |\partial_1\zeta_0|^2\Big)^{\frac{1}{2}}} - \frac{k_1\partial_1\zeta_0}{\Big(1 + |\partial_1\zeta_0|^2\Big)^{\frac{1}{2}}} - \frac{k_1\partial_1\zeta_0 + \partial_1\eta}{\Big(1 + |\partial_1\zeta_0|^2\Big)^{\frac{3}{2}}} \tag{C.3.1}$$

$$= \int_0^1 \left(\frac{2k_1\Big(k_1(\partial_1\zeta_0 + \omega\partial_1\eta) + 2(1+\omega k_1)\partial_1\eta\Big)}{\Big(1 + (1+\omega k_1)^2\Big(\partial_1\zeta_0 + \omega\partial_1\eta\Big)^2\Big)^{\frac{3}{2}}} \right. \tag{C.3.2}$$

$$\left. - \frac{3(1+\omega k_1)^2(\partial_1\zeta_0 + \omega\partial_1\eta)\Big(k_1(\partial_1\zeta_0 + \omega\partial_1\eta) + (1+\omega k_1)\partial_1\eta\Big)^2}{\Big(1 + (1+\omega k_1)^2\Big(\partial_1\zeta_0 + \omega\partial_1\eta\Big)^2\Big)^{\frac{5}{2}}} \right)(1-\omega)\mathrm{d}\omega.$$

The estimates for $\mathcal{R}$ are recorded in the following lemma.

Lemma C.3. *We have the bounds*

$$|\mathcal{R}| \lesssim |k_1|^2 + |\partial_1\eta|^2\,, \tag{C.3.3}$$

$$|\partial_t\mathcal{R}| \lesssim |k_1|\,|\partial_t k_1| + |\partial_t k_1|\,|\partial_1\eta| + |k_1|\,|\partial_t\partial_1\eta| + |\partial_1\eta|\,|\partial_t\partial_1\eta|\,, \tag{C.3.4}$$

$$\left|\partial_t^2\mathcal{R}\right| \lesssim |k_1|\left|\partial_t^2 k_1\right| + |\partial_t k_1|^2 + \left|\partial_t^2 k_1\right||\partial_1\eta| + |k_1|\left|\partial_t^2\partial_1\eta\right| + |\partial_1\eta|\left|\partial_t^2\partial_1\eta\right| + |\partial_t\partial_1\eta|^2\,, \tag{C.3.5}$$

$$|\partial_1\mathcal{R}| \lesssim |\partial_1\eta|\left|\partial_1^2\eta\right| + |k_1|\left|\partial_1^2\eta\right|, \tag{C.3.6}$$

$$|\partial_t\partial_1\mathcal{R}| \lesssim |\partial_t k_1|\left|\partial_1^2\eta\right| + |k_1|\left|\partial_t\partial_1^2\eta\right| + \left|\partial_1^2\eta\right||\partial_t\partial_1\eta| + |\partial_1\eta|\left|\partial_t\partial_1^2\eta\right|. \tag{C.3.7}$$

We now expand the $\mathcal{Q}$ term via

$$\mathcal{Q} = \frac{1}{\Big(1 + K_1^2\Big(\partial_1\zeta_0 + \partial_1\eta\Big)^2\Big)^{\frac{1}{2}}} - \frac{1}{\Big(1 + |\partial_1\zeta_0|^2\Big)^{\frac{1}{2}}} + \frac{\partial_1\zeta_0(k_1\partial_1\zeta_0 + \partial_1\eta)}{\Big(1 + |\partial_1\zeta_0|^2\Big)^{\frac{3}{2}}} \tag{C.3.8}$$

$$= \int_0^1 \left(-\frac{\Big(k_1(\partial_1\zeta_0 + \omega\partial_1\eta) + (1+\omega k_1)\partial_1\eta\Big)^2 + 2k_1\partial_1\eta(1+\omega k_1)(\partial_1\zeta_0 + \omega\partial_1\eta)}{\Big(1 + (1+\omega k_1)^2\Big(\partial_1\zeta_0 + \omega\partial_1\eta\Big)^2\Big)^{\frac{3}{2}}} \right.$$

$$\left. + \frac{3(1+\omega k_1)^2(\partial_1\zeta_0 + \omega\partial_1\eta)^2\Big(k_1(\partial_1\zeta_0 + \omega\partial_1\eta) + (1+\omega k_1)\partial_1\eta\Big)^2}{\Big(1 + (1+\omega k_1)^2\Big(\partial_1\zeta_0 + \omega\partial_1\eta\Big)^2\Big)^{\frac{5}{2}}} \right)(1-\omega)\mathrm{d}\omega.$$

The estimates for $\mathcal{Q}$ are then recorded in the following lemma.

Lemma C.4. *We have the estimates*

$$|\mathcal{Q}| \lesssim |k_1|^2 + |\partial_1 \eta|^2 , \tag{C.3.9}$$

$$|\partial_t \mathcal{Q}| \lesssim |k_1| \, |\partial_t k_1| + |\partial_t k_1| \, |\partial_1 \eta| + |k_1| \, |\partial_t \partial_1 \eta| + |\partial_1 \eta| \, |\partial_t \partial_1 \eta| , \tag{C.3.10}$$

$$\left|\partial_t^2 \mathcal{Q}\right| \lesssim |k_1| \left|\partial_t^2 k_1\right| + |\partial_t k_1|^2 + \left|\partial_t^2 k_1\right| |\partial_1 \eta| + |k_1| \left|\partial_t^2 \partial_1 \eta\right| + |\partial_1 \eta| \left|\partial_t^2 \partial_1 \eta\right| + |\partial_t \partial_1 \eta|^2 , \tag{C.3.11}$$

$$|\partial_1 \mathcal{Q}| \lesssim |\partial_1 \eta| \left|\partial_1^2 \eta\right| + |k_1| \left|\partial_1^2 \eta\right| , \tag{C.3.12}$$

$$|\partial_t \partial_1 \mathcal{Q}| \lesssim |\partial_t k_1| \left|\partial_1^2 \eta\right| + |k_1| \left|\partial_t \partial_1^2 \eta\right| + \left|\partial_1^2 \eta\right| |\partial_t \partial_1 \eta| + |\partial_1 \eta| \left|\partial_t \partial_1^2 \eta\right| . \tag{C.3.13}$$

Next we write the $\mathcal{S}$ term as

$$\mathcal{S} = (J_1 - 1)\partial_t \eta + \left(\tilde{a}\partial_1 \zeta - a\partial_1 \zeta_0\right) = (J_1 - 1)\partial_t \eta + \tilde{a}\partial_1 \eta + (\tilde{a} - a)\partial_1 \zeta_0 . \tag{C.3.14}$$

The estimates for $\mathcal{S}$ are in the following lemma.

Lemma C.5. *We have the bounds*

$$|\mathcal{S}| \lesssim |k_1| \, |\partial_t \eta| + \Big(|\partial_t L| + |\partial_t R| \Big) |\partial_1 \eta| + |\mathcal{O}| , \tag{C.3.15}$$

$$|\partial_t \mathcal{S}| \lesssim |k_1| \left|\partial_t^2 \eta\right| + \Big(|\partial_t L| + |\partial_t R| \Big) |\partial_t \partial_1 \eta| + \Big(\left|\partial_t^2 L\right| + \left|\partial_t^2 R\right| \Big) |\partial_1 \eta| + |\partial_t \mathcal{O}| , \tag{C.3.16}$$

$$\left|\partial_t^2 \mathcal{S}\right| \lesssim |k_1| \left|\partial_t^3 \eta\right| + \Big(|\partial_t L| + |\partial_t R| \Big) \left|\partial_t^2 \partial_1 \eta\right| + \Big(\left|\partial_t^3 L\right| + \left|\partial_t^3 R\right| \Big) |\partial_1 \eta| + \left|\partial_t^2 \mathcal{O}\right| . \tag{C.3.17}$$

Finally, we write the term $\mathcal{O}$ as

$$\begin{aligned} \mathcal{O} = a - \tilde{a} &= -(\partial_t r - \partial_t l)\left(\frac{1}{2\ell} - \frac{2\ell}{(2\ell + r - l)^2}\right) x_1 \\ &= -(\partial_t r - \partial_t l)\frac{(r-l)(4\ell + r - l)}{2\ell(2\ell + r - l)^2} x_1 \\ &= k_1 (\partial_t r - \partial_t l)\frac{4\ell + r - l}{2\ell(2\ell + r - l)} x_1 = k_1 (\partial_t r - \partial_t l)\left(\frac{1}{2\ell} + \frac{1}{2\ell + r - l}\right) x_1 . \end{aligned} \tag{C.3.18}$$

The $\mathcal{O}$ estimates are recorded in the following.

Lemma C.6. *We have the bounds*

$$|\mathcal{O}| \lesssim |k_1| \Big(|\partial_t L| + |\partial_t R| \Big), \tag{C.3.19}$$

$$|\partial_t \mathcal{O}| \lesssim \Big(|\partial_t L|^2 + |\partial_t R|^2 \Big) + |k_1| \Big(\left|\partial_t^2 L\right| + \left|\partial_t^2 R\right| \Big), \tag{C.3.20}$$

$$\left|\partial_t^2 \mathcal{O}\right| \lesssim \Big(|\partial_t L| + |\partial_t R| \Big)\Big(\left|\partial_t^2 L\right| + \left|\partial_t^2 R\right| \Big) + |k_1| \Big(\left|\partial_t^3 L\right| + \left|\partial_t^3 R\right| \Big). \tag{C.3.21}$$

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