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An extended Galerkin analysis for elliptic problems

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Abstract A general analysis framework is presented in this paper for many different types of finite element methods (including various discontinuous Galerkin methods). For the second-order elliptic equation $-\mathrm{div}(\alpha\nabla u) = f$, this framework employs four different discretization variables, $u_h, \boldsymbol{p}_h, \check{u}_h$ and $\check{\boldsymbol{p}}_h$, where u_h and $\boldsymbol{p}_h$ are for approximation of u and $\boldsymbol{p} = -\alpha\nabla u$ inside each element, and $\check{u}_h$ and $\check{p}_h$ are for approximation of residual of u and $\boldsymbol{p}\cdot\boldsymbol{n}$ on the boundary of each element. The resulting 4-field discretization is proved to satisfy two types of inf-sup conditions that are uniform with respect to all discretization and penalization parameters. As a result, many existing finite element and discontinuous Galerkin methods can be analyzed using this general framework by making appropriate choices of discretization spaces and penalization parameters.

Keywords finite element method, extended Galerkin analysis, unified study

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1 Introduction

In this paper, we propose an *extended Galerkin* analysis framework for most of the existing finite element methods (FEMs). We will illustrate the main idea by using the following elliptic boundary value problem:

$$\begin{cases} -\operatorname{div}(\alpha\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ -(\alpha\nabla u)\cdot\boldsymbol{n} = 0 & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ ($d \geqslant 1$) is a bounded domain and its boundary, $\partial\Omega$, is split into Dirichlet and Neumann parts, namely $\partial\Omega = \Gamma_D \cup \Gamma_N$. For simplicity, we assume that the $(d-1)$-dimensional measure of Γ_D is nonzero. Here, $\boldsymbol{n}$ is the outward unit normal direction of Γ_N, and $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded and symmetric positive definite matrix, with its inverse denoted by $c = \alpha^{-1}$. By setting $\boldsymbol{p} = -\alpha\nabla u$, the above problem can be written as

$$\begin{cases} c\boldsymbol{p} + \nabla u = 0 & \text{in } \Omega, \\ -\operatorname{div}\boldsymbol{p} = -f & \text{in } \Omega \end{cases} \tag{1.2}$$

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with the boundary condition $u=0$ on Γ_D and $\boldsymbol{p}\cdot\boldsymbol{n}=0$ on Γ_N.

There are two major variational formulations for (1.1). The first is to find

$$u \in H_D^1(\Omega) := \{v \in H^1(\Omega) : v\,|_{\Gamma_D} = 0\}$$

such that for any $v \in H_D^1(\Omega)$,

$$\int_\Omega (\alpha \nabla u) \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x. \tag{1.3}$$

The second one is to find

$$\boldsymbol{p} \in \boldsymbol{H}_N(\mathrm{div};\Omega) := \{\boldsymbol{q} \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{q}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_N\}, \quad u \in L^2(\Omega)$$

such that for any $\boldsymbol{q} \in \boldsymbol{H}_N(\mathrm{div};\Omega)$ and $v \in L^2(\Omega)$,

$$\begin{cases} \displaystyle\int_\Omega c\boldsymbol{p}\cdot\boldsymbol{q}\,\mathrm{d}x - \int_\Omega u\,\mathrm{div}\boldsymbol{q}\,\mathrm{d}x = 0, \\ \displaystyle -\int_\Omega v\,\mathrm{div}\boldsymbol{p}\,\mathrm{d}x = -\int_\Omega f v\,\mathrm{d}x. \end{cases} \tag{1.4}$$

In correspondence to the two variational formulations, two different conforming finite element methods have been developed. The first one, which approximates $u \in H_D^1(\Omega)$, can be traced back to the 1940s [31] and the Courant element [25]. After a decade, many works, such as [17, 28, 29, 32, 37, 40, 50, 51], proposed more conforming elements and presented serious mathematical proofs concerning error analysis and, hence, established the basic theory of FEMs. These *primal FEMs* contain one unknown, namely u, to solve. The second one, which approximates $\boldsymbol{p} \in \boldsymbol{H}_N(\mathrm{div};\Omega)$ and $u \in L^2(\Omega)$ based on a mixed variational principal, is called the *mixed FEMs* [3, 7, 9, 12, 36, 43]. These mixed methods solve two variables, namely flux variable $\boldsymbol{p}$ and u, and the condition for the well-posedness of mixed formulations is known as inf-sup or the Ladyzhenskaya-Babuška-Breezi (LBB) condition [9].

Contrary to the continuous Galerkin methods, the discontinuous Galerkin (DG) methods, which can be traced back to the late 1960s [5, 34], aim to relax the conforming constraint on u or $\boldsymbol{p}\cdot\boldsymbol{n}$. To maintain consistency of the DG discretization, additional finite element spaces need to be introduced on the element boundaries. In essence, the numerical fluxes on the element boundaries were introduced explicitly and therefore eliminated. In most existing DG methods, only one such boundary space is introduced as, for example, Lagrangian multiplier space, either for u as the primal DG methods [14, 24, 27] or for $\boldsymbol{p}\cdot\boldsymbol{n}$ as the mixed DG methods [30]. Primal DG methods have been applied to purely elliptic problems; examples include the interior penalty methods studied in [2, 6, 48] and the local DG method for the elliptic problem in [24]. Primal DG methods for diffusion and elliptic problems were considered in [13]. A review of the development of DG methods up to 1999 can be found in [23].

Given $\Omega \subset \mathbb{R}^d$, for any $D \subseteq \Omega$, and any positive integer m, let $H^m(D)$ be the Sobolev space with the corresponding usual norm and semi-norm, denoted by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$, respectively. The L^2-inner product on D and ∂D are denoted by $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$, respectively. $\|\cdot\|_{0,D}$ and $\|\cdot\|_{0,\partial D}$ are the norms of Lebesgue spaces $L^2(D)$ and $L^2(\partial D)$, respectively. We abbreviate $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ by $\|\cdot\|_m$ and $|\cdot|_m$, respectively, when $D=\Omega$, and $\|\cdot\|_0 = \|\cdot\|_{0,\Omega}$.

We denote by $\{\mathcal{T}_h\}_h$ a family of shape-regular triangulations of $\overline{\Omega}$. Let $h_K = \mathrm{diam}(K)$ and $h = \max\{h_K : K \in \mathcal{T}_h\}$. We also denote by $H^m(\mathcal{T}_h)$ the space of functions on Ω whose restriction to each element K belongs to the space $H^m(K)$ for any $m \geqslant 0$. For any $K \in \mathcal{T}_h$, denote $\boldsymbol{n}_K$ as the outward unit normal of K. Denote by $\mathcal{E}_h$ the union of the boundaries of the elements K of $\mathcal{T}_h$.

Associated with the triangulation $\mathcal{T}_h$, denote V_h and $\boldsymbol{Q}_h$ to be the generic piecewise smooth scalar and vector-valued discrete spaces on the triangulation $\mathcal{T}_h$, respectively. In addition, $\check{V}_h$ and $\check{\boldsymbol{Q}}_h$ are the generic piecewise smooth discrete spaces on $\mathcal{E}_h$, respectively.

In this paper, we consider the following problem: Find $(\boldsymbol{p}_h, u_h, \check{\boldsymbol{p}}_h, \check{u}_h) \in \boldsymbol{Q}_h \times V_h \times \check{\boldsymbol{Q}}_h \times \check{V}_h$ such that

$$\begin{cases} (c\boldsymbol{p}_h, \boldsymbol{q}_h)_K - (u_h, \mathrm{div}\boldsymbol{q}_h)_K + \langle \hat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n}\rangle_{\partial K} = 0, \quad \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h, \\ (\boldsymbol{p}_h, \nabla v_h)_K - \langle \hat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, v_h\rangle_{\partial K} = -(f, v_h)_K, \quad \forall\, v_h \in V_h, \\ \langle [\![u_h]\!] - \tau^{-1}\check{\boldsymbol{p}}_h, \check{\boldsymbol{q}}_h\rangle = 0, \quad \forall\, \check{\boldsymbol{q}}_h \in \check{\boldsymbol{Q}}_h, \\ \langle [\boldsymbol{p}_h] - \eta^{-1}\check{u}_h, \check{v}_h\rangle = 0, \quad \forall\, \check{v}_h \in \check{V}_h, \end{cases} \tag{1.5}$$

where $\hat{u}_h := \bar{u}_h + \check{u}_h$, $\hat{\boldsymbol{p}}_h := \bar{\boldsymbol{p}}_h + \check{\boldsymbol{p}}_h$ and $\langle \cdot, \cdot\rangle := \sum_{e \in \mathcal{E}_h} \langle \cdot, \cdot \rangle_e$.

We will explain the relevant technical details for (1.5) in the following sections. Here, we make the following general comments:

1. Under proper choices of the discrete spaces, the formulation (1.5) recovers the analysis of H^1 conforming finite element if we eliminate all the discretization variables except u_h. By eliminating $\check{\boldsymbol{p}}_h$, the formulation (1.5) recovers some special cases of the hybrid methods [20]. If we further eliminate $\check{u}_h$, the resulting system solves two variables $\boldsymbol{p}_h$ and u_h, which recovers the $H(\mathrm{div})$ mixed finite element method.

2. The relationship between the formulation (1.5) and DG methods is twofold. First, by simply taking the trivial spaces for $\check{u}_h$ and $\check{\boldsymbol{p}}_h$, the formulation (1.5) recovers most of DG methods shown in [4]. Second, if we confine to a special choice

$$\bar{u}_h = \{u_h\} \quad \text{and} \quad \bar{\boldsymbol{p}}_h = \{\!\{\boldsymbol{p}_h\}\!\},$$

by virtue of the characterization of the hybridization and the DG method [20], the formulation (1.5) can be related to some DG methods if we eliminate both $\check{\boldsymbol{p}}_h$ and $\check{u}_h$ (see Section 4 for details).

3. In Subsection 4.1, the formulation (1.5) can be compared with most hybridized discontinuous Galerkin (HDG) methods if we eliminate $\check{\boldsymbol{p}}_h$. In 2009, a unified formulation of the hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second-order elliptic problems was presented in [21]. The resulting system needs to solve three variables, one approximating u, one approximating $\boldsymbol{p}$, and the third one approximating the trace of u on the element boundary. A projection-based error analysis of HDG methods was presented in [22], in which a projection operator was tailored to obtain the L^2 error estimates for both potential and flux. More references to the recent developments of HDG methods can be found in [18].

4. In Subsection 4.2, the formulation (1.5) can be compared with most weak Galerkin (WG) methods if we eliminate $\check{u}_h$. With the introduction of weak gradient and weak divergence, a WG method for a second-order elliptic equation formulated as a system of two first-order linear equations was proposed and analyzed in [46, 47]. In fact, the weak Galerkin methods in [46] solve two variables, one approximating u and the other one approximating the flux $\boldsymbol{p} \cdot \boldsymbol{n}$ on the element boundary, which differs from the method proposed in this paper when α is not piecewise constant. While the weak Galerkin methods in [47] solve three variables, one approximating u, one approximating $\boldsymbol{p}$, and the third one approximating the flux $\boldsymbol{p} \cdot \boldsymbol{n}$ on the element boundary. A summary of the idea and applications of WG methods for various problems can be found in [45].

In addition, we study two types of uniform inf-sup conditions for the proposed formulation in Section 3, by which the well-posedness of the formulation (1.5) follows naturally. With these uniform inf-sup conditions, we obtain some limiting of the formulation (1.5) in Subsection 4.4.

1. If the parameters in the Nitsche's trick are set to be $\tau = (\rho h_e)^{-1}$, $\eta \cong \tau^{-1}$, the formulation (1.5) is shown to converge to the H^1 conforming method as $\rho \to 0$ under certain conditions pertaining to the discrete spaces.

2. If the parameters in the Nitsche's trick are set to be $\eta = (\rho h_e)^{-1}$, $\tau \cong \eta^{-1}$, the formulation (1.5) is shown to converge to the $H(\mathrm{div})$ conforming method as $\rho \to 0$ under certain conditions pertaining to the discrete spaces.

Throughout this paper, we shall use the letter C, which is independent of mesh-size and stabilization parameters, to denote a generic positive constant which may stand for different values at different occurrences. The notations $x \lesssim y$ and $x \gtrsim y$ mean $x \leqslant Cy$ and $x \geqslant Cy$, respectively.

2 Derivation of the method

Let $\mathcal{E}_h^i = \mathcal{E}_h \setminus \partial\Omega$ be the set of interior edges and $\mathcal{E}_h^\partial = \mathcal{E}_h \setminus \mathcal{E}_h^i$ be the set of boundary edges. Furthermore, for any $e \in \mathcal{E}_h$, let $h_e = \operatorname{diam}(e)$. Let e be the common edge of two elements K^+ and K^-, and let $\boldsymbol{n}^i = \boldsymbol{n}\,|_{\partial K^i}$ be the unit outward normal vector on ∂K^i with $i = +, -$. For any scalar-valued function $v \in H^1(\mathcal{T}_h)$ and vector-valued function $\boldsymbol{q} \in \boldsymbol{H}^1(\mathcal{T}_h)$, let $v^\pm = v\,|_{\partial K^\pm}$ and $\boldsymbol{q}^\pm = \boldsymbol{q}\,|_{\partial K^\pm}$. Then, we define averages $\{\cdot\}$, $\{\!\{\cdot\}\!\}$ and jumps $[\![\cdot]\!]$, $[\cdot]$ as follows:

$$
\begin{aligned}
&\{v\} = \frac{1}{2}(v^+ + v^-), \quad \{\!\{\boldsymbol{q}\}\!\} = \frac{1}{2}(\boldsymbol{q}^+ + \boldsymbol{q}^-) \quad \text{on } e \in \mathcal{E}_h^i,\\
&[\![v]\!] = v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-, \quad [\boldsymbol{q}] = \boldsymbol{q}^+\cdot\boldsymbol{n}^+ + \boldsymbol{q}^-\cdot\boldsymbol{n}^- \quad \text{on } e \in \mathcal{E}_h^i,\\
&[\![v]\!] = v\boldsymbol{n}, \quad \{v\} = v, \quad \{\!\{\boldsymbol{q}\}\!\} = \boldsymbol{q}, \quad [\boldsymbol{q}] = 0 \quad \text{on } e \in \Gamma_D,\\
&[\![v]\!] = \boldsymbol{0}, \quad \{v\} = v, \quad \{\!\{\boldsymbol{q}\}\!\} = \boldsymbol{q}, \quad [\boldsymbol{q}] = \boldsymbol{q}\cdot\boldsymbol{n} \quad \text{on } e \in \Gamma_N.
\end{aligned}
\tag{2.1}
$$

The notation follows the rules: (i) $\{\!\{\cdot\}\!\}$ and $[\![\cdot]\!]$ are vector-valued operators; (ii) $\{\cdot\}$ and $[\cdot]$ are scalar-valued operators.

For simplicity of the exposition, we use the following convention:

$$
(\cdot,\cdot) := \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_K, \quad \langle\cdot,\cdot\rangle := \sum_{e\in\mathcal{E}_h} \langle\cdot,\cdot\rangle_e, \quad \langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \langle\cdot,\cdot\rangle_{\partial K}. \tag{2.2}
$$

For any scalar-valued function v and vector-valued function $\boldsymbol{q}$, we denote

$$
\langle v, \boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \langle v, \boldsymbol{q}\cdot\boldsymbol{n}_K\rangle_{\partial K}.
$$

Here, we specify the outward unit normal direction $\boldsymbol{n}$ corresponding to the element K, namely $\boldsymbol{n}_K$. In addition, let ∇_h and div_h be defined as

$$
\nabla_h v\,|_K := \nabla v\,|_K, \quad \operatorname{div}_h\boldsymbol{q}\,|_K := \operatorname{div}\boldsymbol{q}|_K, \quad \forall\, K \in \mathcal{T}_h.
$$

Lemma 2.1. *With the averages and jumps defined in* (2.1)*, we have the following identities* [4]:

$$
(v, \operatorname{div}_h\boldsymbol{q}) + (\nabla_h v, \boldsymbol{q}) = \langle v, \boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \langle \{\!\{\boldsymbol{q}\}\!\}, [\![v]\!]\rangle + \langle [\boldsymbol{q}], \{v\}\rangle, \quad \forall\, \boldsymbol{q} \in \boldsymbol{H}^1(\mathcal{T}_h), \quad v \in H^1(\mathcal{T}_h). \tag{2.3}
$$

For any $\boldsymbol{q}_h \in \boldsymbol{Q}_h$ and $v_h \in V_h$, integration by parts gives

$$
\begin{cases}
(c\boldsymbol{p}, \boldsymbol{q}_h)_K - (u, \operatorname{div}_h\boldsymbol{q}_h)_K + \langle \boldsymbol{q}_h\cdot\boldsymbol{n}, u\rangle_{\partial K} = 0, \quad \forall\, \boldsymbol{q}_h \in \boldsymbol{p}_h,\\
(\boldsymbol{p}, \nabla_h v_h)_K - \langle \boldsymbol{p}\cdot\boldsymbol{n}, v_h\rangle_{\partial K} = -(f, v_h)_K, \quad \forall\, v_h \in V_h.
\end{cases}
\tag{2.4}
$$

We introduce the 4-field discretization variables as

$$
\begin{aligned}
&u \approx u_h, \quad \boldsymbol{p} \approx \boldsymbol{p}_h \quad \text{in } K,\\
&u \approx \hat{u}_h, \quad \boldsymbol{p} \approx \hat{\boldsymbol{p}}_h \quad \text{on } \partial K.
\end{aligned}
\tag{2.5}
$$

Here, we point out that $\hat{u}_h$ and $\hat{\boldsymbol{p}}_h$ are single-valued on $\mathcal{E}_h$. Then we obtain the major but natural part of the DG formulation:

$$
\begin{cases}
(c\boldsymbol{p}_h, \boldsymbol{q}_h)_K - (u_h, \operatorname{div}_h\boldsymbol{q}_h)_K + \langle \boldsymbol{q}_h\cdot\boldsymbol{n}, \hat{u}_h\rangle_{\partial K} = 0, \quad \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h,\\
(\boldsymbol{p}_h, \nabla_h v_h)_K - \langle \hat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, v_h\rangle_{\partial K} = -(f, v_h)_K, \quad \forall\, v_h \in V_h.
\end{cases}
\tag{2.6}
$$

Next, define $\hat{u}_h$ and $\hat{\boldsymbol{p}}_h$ on the boundary

$$
\hat{u}_h := \bar{u}_h + \check{u}_h, \quad \hat{\boldsymbol{p}}_h := \bar{\boldsymbol{p}}_h + \check{\boldsymbol{p}}_h \quad \text{on } \partial K.
$$

Here, $\bar{\boldsymbol{p}}_h$ and $\bar{u}_h$ are some approximations of $\boldsymbol{p}$ and u on the element boundary in terms of u_h and $\boldsymbol{p}_h$ from inside of elements. In the simple case,

$$
\bar{\boldsymbol{p}}_h := \{\!\{\boldsymbol{p}_h\}\!\}, \quad \bar{u}_h := \{u_h\}.
$$

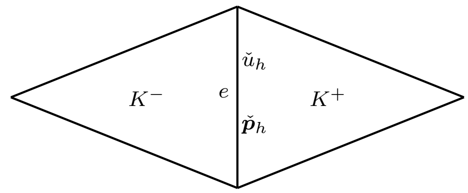

Figure 1 DG notation

We apply the Nitsche's trick to $\check{u}_h \in \check{V}_h$ and $\check{\boldsymbol{p}}_h \in \check{\boldsymbol{Q}}_h$ to impose weak continuity (see Figure 1):

$$\check{\boldsymbol{p}}_h \approx \tau[\![u_h]\!], \quad \check{u}_h \approx \eta[\boldsymbol{p}_h].$$

More precisely,

$$\begin{cases} \langle [\![u_h]\!] - \tau^{-1}\check{\boldsymbol{p}}_h, \check{\boldsymbol{q}}_h\rangle = 0, \quad \forall\, \check{\boldsymbol{q}}_h \in \check{\boldsymbol{Q}}_h, \\ \langle [\boldsymbol{p}_h] - \eta^{-1}\check{u}_h, \check{v}_h\rangle = 0, \quad \forall\, \check{v}_h \in \check{V}_h. \end{cases} \tag{2.7}$$

Collecting (2.6) and (2.7), we obtain a common case of (1.5) as

$$\begin{cases} (c\boldsymbol{p}_h, \boldsymbol{q}_h)_K - (u_h, \mathrm{div}_h \boldsymbol{q}_h)_K + \langle \hat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n}\rangle_{\partial K} = 0, \\ (\boldsymbol{p}_h, \nabla_h v_h)_K - \langle \hat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, v_h\rangle_{\partial K} = -(f, v_h)_{0,K}, \\ \langle [\![u_h]\!] - \tau^{-1}\check{\boldsymbol{p}}_h, \check{\boldsymbol{q}}_h\rangle = 0, \\ \langle [\boldsymbol{p}_h] - \eta^{-1}\check{u}_h, \check{v}_h\rangle = 0, \end{cases} \tag{2.8}$$

where $\hat{u}_h := \{u_h\} + \check{u}_h$ and $\hat{\boldsymbol{p}}_h := \{\!\{\boldsymbol{p}_h\}\!\} + \check{\boldsymbol{p}}_h$. In the context of this paper, we will mainly focus on the analysis of the 4-field formulation (2.8).

3 Unified analysis of the 4-field formulation

In this section, we shall present two types of the inf-sup condition for the formulation (2.8). In both cases, the parameter ρ is assumed to be a positive constant. For the sake of simplicity of the exposition, we also abbreviate the dependence of both ρ and mesh size h to $\rho_h := \rho h$.

3.1 Some equivalent formulations

Let $\check{\boldsymbol{\mathcal{Q}}}_h^p : \boldsymbol{L}^2(\mathcal{E}_h) \to \check{\boldsymbol{Q}}_h$ and $\check{\mathcal{Q}}_h^u : L^2(\mathcal{E}_h) \to \check{V}_h$ be the L^2 projections. We first give some equivalent formulations which will be useful in the analysis.

3-field formulation I. By (2.7), we have the explicit expression of $\check{\boldsymbol{p}}_h$ as

$$\check{\boldsymbol{p}}_h = \tau\check{\boldsymbol{\mathcal{Q}}}_h^p[\![u_h]\!] \quad \text{on } \mathcal{E}_h. \tag{3.1}$$

Denote

$$\tilde{u}_h = (u_h, \check{u}_h) \quad \text{and} \quad \tilde{v}_h = (v_h, \check{v}_h).$$

Then the formulation (2.8) can be recast as

$$\begin{cases} a_h(\boldsymbol{p}_h, \boldsymbol{q}_h) + b_h(\boldsymbol{q}_h, \tilde{u}_h) = 0, \quad \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h, \\ b_h(\boldsymbol{p}_h, \tilde{v}_h) - c_h(\tilde{u}_h, \tilde{v}_h) = -(f, v_h), \quad \forall\, \tilde{v}_h \in \tilde{V}_h, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} a_h(\boldsymbol{p}_h, \boldsymbol{q}_h) &:= (c\boldsymbol{p}_h, \boldsymbol{q}_h), \\ b_h(\boldsymbol{q}_h, \tilde{v}_h) &:= -(\mathrm{div}_h \boldsymbol{q}_h, v_h) + \langle [\boldsymbol{q}_h], \check{v}_h + \{v_h\}\rangle, \\ c_h(\tilde{u}_h, \tilde{v}_h) &:= \langle \tau\check{\boldsymbol{\mathcal{Q}}}_h^p[\![u_h]\!], \check{\boldsymbol{\mathcal{Q}}}_h^p[\![v_h]\!]\rangle + \langle \eta^{-1}\check{u}_h, \check{v}_h\rangle. \end{aligned}$$

We note that (3.2) is equivalent to the formulation (2.8). Firstly, the solution $(\boldsymbol{p}_h, u_h, \check{u}_h)$ obtained from (2.8) coincides the solution of (3.2). On the other hand, having the solution $(\boldsymbol{p}_h, u_h, \check{u}_h)$ of (3.2), we can construct $\check{\boldsymbol{p}}_h$ via (3.1). It is straightforward to show that $(\boldsymbol{p}_h, u_h, \check{\boldsymbol{p}}_h, \check{u}_h)$ is the solution of (2.8).

3-field formulation II. By (2.7), we have the explicit expression of $\check{u}_h$ as

$$\check{u}_h = \eta \check{\mathcal{Q}}_h^u[\boldsymbol{p}_h] \quad \text{on } \mathcal{E}_h. \tag{3.3}$$

Denote $\tilde{\boldsymbol{p}}_h = (\boldsymbol{p}_h, \check{\boldsymbol{p}}_h)$ and $\tilde{\boldsymbol{q}}_h = (\boldsymbol{q}_h, \check{\boldsymbol{q}}_h)$. Then the formulation (2.8) can be recast as

$$\begin{cases} a_w(\tilde{\boldsymbol{p}}_h, \tilde{\boldsymbol{q}}_h) + b_w(\tilde{\boldsymbol{q}}_h, u_h) = 0, & \forall\, \tilde{\boldsymbol{q}}_h \in \tilde{\boldsymbol{Q}}_h, \\ b_w(\tilde{\boldsymbol{p}}_h, v_h) = -(f, v_h), & \forall\, v_h \in V_h, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} a_w(\tilde{\boldsymbol{p}}_h, \tilde{\boldsymbol{q}}_h) &:= (c\boldsymbol{p}_h, \boldsymbol{q}_h) + \langle \eta \check{\mathcal{Q}}_h^u[\boldsymbol{p}_h], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle + \langle \tau^{-1}\check{\boldsymbol{p}}_h, \check{\boldsymbol{q}}_h\rangle, \\ b_w(\tilde{\boldsymbol{q}}_h, v_h) &:= (\nabla_h v_h, \boldsymbol{q}_h) - \langle [\![v_h]\!], \check{\boldsymbol{q}}_h + \{\!\{\boldsymbol{q}_h\}\!\}\rangle. \end{aligned}$$

By using a similar argument, we know that (3.4) is equivalent to the formulation (2.8).

2-field formulation. By plugging in (3.1) and (3.3) into (2.6) and the DG identity (2.3), the 4-field formulation with $(\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h)$ is equivalent to the following 2-field formulation, which seeks $(\boldsymbol{p}_h, u_h) \in \boldsymbol{Q}_h \times V_h$ such that

$$\begin{cases} a(\boldsymbol{p}_h, \boldsymbol{q}_h) + b(\boldsymbol{q}_h, u_h) = 0, & \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h, \\ b(\boldsymbol{p}_h, v_h) - c(u_h, v_h) = -(f, v_h), & \forall\, v_h \in V_h \end{cases} \tag{3.5}$$

with

$$\begin{cases} a(\boldsymbol{p}_h, \boldsymbol{q}_h) = (c\boldsymbol{p}_h, \boldsymbol{q}_h) + \langle \eta \check{\mathcal{Q}}_h^u[\boldsymbol{p}_h], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle, \\ b(\boldsymbol{p}_h, v_h) = (\boldsymbol{p}_h, \nabla_h v_h) - \langle \{\!\{\boldsymbol{p}_h\}\!\}, [\![v_h]\!]\rangle \\ \qquad\qquad\quad = -(\mathrm{div}_h \boldsymbol{p}_h, v_h) + \langle [\boldsymbol{p}_h], \{v_h\}\rangle, \\ c(u_h, v_h) = \langle \tau \check{\boldsymbol{\mathcal{Q}}}_h^p [\![u_h]\!], \check{\boldsymbol{\mathcal{Q}}}_h^p [\![v_h]\!]\rangle. \end{cases} \tag{3.6}$$

We note that (3.5) is equivalent to the formulation (2.8). Firstly, the solution $(\boldsymbol{p}_h, u_h)$ obtained from (2.8) coincides the solution of (3.5). On the other hand, having the solution $(\boldsymbol{p}_h, u_h)$ of (3.5), by using (3.1) and (3.3), we can construct $\check{\boldsymbol{p}}_h$ and $\check{u}_h$, respectively. It is straightforward to show that $(\boldsymbol{p}_h, u_h, \check{\boldsymbol{p}}_h, \check{u}_h)$ is the solution of (2.8). Hence, the 4-field formulation (2.8), the 3-field formulations (3.2) and (3.4), and the 2-field formulation (3.5) are mutually equivalent.

Furthermore, if the choice of the spaces $\boldsymbol{Q}_h, \check{V}_h, V_h$ and $\check{\boldsymbol{Q}}_h$ satisfying $[\boldsymbol{Q}_h] \subset \check{V}_h$ and $[\![V_h]\!] \subset \check{\boldsymbol{Q}}_h$, then the projections $\check{\mathcal{Q}}_h^u$ and $\check{\boldsymbol{\mathcal{Q}}}_h^p$ reduce to identities. In this case, (3.5) reduces to the local discontinuous Galerkin (LDG) method proposed in [15].

Remark 3.1 (Consistency)**.** Let $(\boldsymbol{p}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times H^1(\Omega)$ be the exact solution of (1.2). Recalling the DG notation (2.1) and the formulation (3.5), we have

$$\begin{cases} a(\boldsymbol{p}, \boldsymbol{q}_h) + b(\boldsymbol{q}_h, u) = 0, & \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h, \\ b(\boldsymbol{p}, v_h) - c(u, v_h) = -(f, v_h), & \forall\, v_h \in V_h, \end{cases} \tag{3.7}$$

which shows the consistency of the 2-field formulation (3.5). The equivalence of (3.5) and (2.8) implies the consistency of the latter. In fact, the formulation (2.8) seeks

$$(\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h) \in \boldsymbol{Q}_h \times V_h \times \check{\boldsymbol{Q}}_h \times \check{V}_h$$

is consistent since $(\boldsymbol{p}, \boldsymbol{0}, u, 0)$ satisfies the formulation (2.8).

For $k \geqslant 0$, we specify several spaces as follows:

$$\begin{aligned} V_h^k &:= \{v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}_k(K), \forall\, K \in \mathcal{T}_h\}, \\ \boldsymbol{Q}_h^k &:= \{\boldsymbol{p}_h \in \boldsymbol{L}^2(\Omega) : \boldsymbol{p}_h|_K \in \boldsymbol{\mathcal{P}}_k(K), \forall\, K \in \mathcal{T}_h\}, \\ \boldsymbol{Q}_h^{k,RT} &:= \{\boldsymbol{p}_h \in \boldsymbol{L}^2(\Omega) : \boldsymbol{p}_h|_K \in \boldsymbol{\mathcal{P}}_k(K) + \boldsymbol{x}\mathcal{P}_k(K), \forall\, K \in \mathcal{T}_h\}, \end{aligned} \tag{3.8}$$

where $\mathcal{P}_k(K)$ is the space of polynomial functions of degree at most k on K, and

$$\boldsymbol{\mathcal{P}}_k(K) := [\mathcal{P}_k(K)]^d.$$

Let $\check{\boldsymbol{Q}}(e)$ and $\check{V}(e)$ denote some local spaces on e which will be specified at their occurrences. For $k \geqslant 0$, we specify several spaces as follows:

$$\begin{aligned} \check{\boldsymbol{Q}}_h^k &:= \{\check{\boldsymbol{p}}_h \in \boldsymbol{L}^2(\mathcal{E}_h) : \check{\boldsymbol{p}}_h|_e \in \boldsymbol{\mathcal{P}}_k(e), \forall\, e \in \mathcal{E}_h^i, \check{\boldsymbol{p}}_h|_{\Gamma_N} = \boldsymbol{0}\}, \\ \check{V}_h^k &:= \{\check{v}_h \in L^2(\mathcal{E}_h) : \check{v}_h|_e \in \mathcal{P}_k(e), \forall\, e \in \mathcal{E}_h^i, \check{v}_h|_{\Gamma_D} = 0\}, \end{aligned} \tag{3.9}$$

where $\mathcal{P}_k(e)$ is the space of polynomial functions of degree at most k on e.

3.2 Grad-based uniform inf-sup condition

We shall consider the well-posedness of the formulation (2.8) in the grad-based case when $\tau = (\rho h_e)^{-1}, \eta \cong \tau^{-1} = \rho h_e$. For any $\boldsymbol{p}_h \in \boldsymbol{Q}_h, \check{\boldsymbol{p}}_h \in \check{\boldsymbol{Q}}_h, u_h \in V_h, \check{u}_h \in \check{V}_h$, define the norms

$$\begin{aligned} \|\tilde{\boldsymbol{p}}_h\|_{0,\rho_h}^2 &:= \underbrace{(c\boldsymbol{p}_h, \boldsymbol{p}_h)}_{\|\boldsymbol{p}_h\|_{0,c}^2} + \underbrace{\langle \rho h_e \check{\boldsymbol{p}}_h, \check{\boldsymbol{p}}_h\rangle}_{\|\check{\boldsymbol{p}}_h\|_{0,\rho_h}^2}, \\ \|\tilde{u}_h\|_{1,\rho_h}^2 &:= \underbrace{(\nabla_h u_h, \nabla_h u_h) + \langle \rho^{-1} h_e^{-1} \check{\boldsymbol{\mathcal{Q}}}_h^p [\![u_h]\!], \check{\boldsymbol{\mathcal{Q}}}_h^p [\![u_h]\!]\rangle}_{\|u_h\|_{1,\rho_h}^2} + \underbrace{\langle \rho^{-1} h_e^{-1} \check{u}_h, \check{u}_h\rangle}_{\|\check{u}_h\|_{0,\rho_h^{-1}}^2}. \end{aligned} \tag{3.10}$$

We note that the norms in (3.10) depend on both the parameter ρ and mesh size h, and the dependence can be abbreviated to ρ_h. Here, we assume that $\check{\boldsymbol{Q}}_h$ contains piecewise constant function space to guarantee that $\|\cdot\|_{1,\rho_h}$ is indeed a norm. We further note that the norm $\|\cdot\|_{1,\rho_h}$ can be extended to $(V_h + H^1(\Omega)) \times \check{V}_h$. The following norms are induced from (3.10):

$$\|f\|_{-1,\rho_h} := \sup_{v_h \in V_h \setminus \{0\}} \frac{(f, v_h)}{\|v_h\|_{1,\rho_h}}.$$

Using the 3-field formulation II (3.4), we will show the grad-based inf-sup condition as well as the quasi-optimal error estimate under the following assumption.

Assumption 3.2. *The spaces* $\boldsymbol{Q}_h, \check{\boldsymbol{Q}}_h$ *and* V_h *satisfy the following conditions*:

(a) $\check{\boldsymbol{Q}}_h$ *contains piecewise constant function space*;

(b) $\nabla_h V_h \subset \boldsymbol{Q}_h$;

(c) $\{\!\{\boldsymbol{Q}_h\}\!\} \subset \check{\boldsymbol{Q}}_h$.

Lemma 3.3. *Under the Assumption* 3.2*, for any* $\rho_0 > 0$ *and* $0 < \rho \leqslant \rho_0$*, we have*

$$\inf_{v_h \in V_h \setminus \{0\}} \sup_{\tilde{\boldsymbol{p}}_h \in \tilde{\boldsymbol{Q}}_h \setminus \{\boldsymbol{0}\}} \frac{b_w(\tilde{\boldsymbol{p}}_h, v_h)}{\|v_h\|_{1,\rho_h} \|\tilde{\boldsymbol{p}}_h\|_{0,\rho_h}} \geqslant \beta_w, \tag{3.11}$$

where $\beta_w > 0$ *is independent of both mesh size* h *and* ρ.

Proof. Since $\nabla_h V_h \subset \boldsymbol{Q}_h$ and $\{\!\{\boldsymbol{Q}_h\}\!\} \subset \check{\boldsymbol{Q}}_h$, taking

$$\boldsymbol{p}_h = \nabla_h v_h, \quad \check{\boldsymbol{p}}_h = -\rho^{-1} h_e^{-1} \check{\boldsymbol{\mathcal{Q}}}_h^p [\![v_h]\!] - \{\!\{\nabla_h v_h\}\!\},$$

we have

$$b_w(\tilde{\boldsymbol{p}}_h, v_h) = (\nabla_h v_h, \nabla_h v_h) + \langle \rho^{-1} h_e^{-1} \check{\boldsymbol{\mathcal{Q}}}_h^p [\![v_h]\!], \check{\boldsymbol{\mathcal{Q}}}_h^p [\![v_h]\!]\rangle = \|v_h\|_{1,\rho_h}^2.$$

Then for any $\rho_0 > 0$ and $\rho \leqslant \rho_0$, we obtain

$$\begin{aligned} \|\tilde{\boldsymbol{p}}_h\|_{0,\rho_h}^2 &= (c\nabla_h v_h, \nabla_h v_h) + \|\rho^{1/2} h_e^{1/2} \{\!\{\nabla_h v_h\}\!\} + \rho^{-1/2} h_e^{-1/2} \check{\boldsymbol{\mathcal{Q}}}_h^p [\![v_h]\!]\|_{0,\mathcal{E}_h}^2 \\ &\leqslant C(\rho_0)\{(\nabla_h v_h, \nabla_h v_h) + \|\rho^{-1/2} h_e^{-1/2} \check{\boldsymbol{\mathcal{Q}}}_h^p [\![v_h]\!]\|_{0,\mathcal{E}_h}^2\} \\ &\leqslant \beta_w \|v_h\|_{1,\rho_h}^2. \end{aligned} \tag{3.12}$$

Then, we obtain the desired result. □

The boundedness of $a_w(\cdot,\cdot)$ on the discrete spaces follows from the standard trace inequality. For the boundedness of $b_w(\cdot,\cdot)$, we use the Cauchy-Schwarz inequality, the trace inequality, and the following inequality:

$$h_e^{-1}\|[\![v_h]\!]\|_{0,e}^2 \lesssim |\nabla_h v_h|_{0,\omega_e}^2 + h_e^{-1}\|\check{\mathcal{Q}}_h^p[\![v_h]\!]\|_{0,e}^2 \tag{3.13}$$

provided that $\check{\boldsymbol{Q}}_h$ contains piecewise constant function space. Here,

$$\omega_e := \bigcup_{e\subset\partial K} K.$$

The coercivity of $a_w(\cdot,\cdot)$ is obvious, and hence (3.4) is well-posed on the discrete spaces. Due to the equivalence between (2.8) and (3.4), we have the first main result.

Theorem 3.4. *Under Assumption* 3.2*, if we choose* $\tau=(\rho h_e)^{-1}$*,* $\eta\cong\tau^{-1}=\rho h_e$ *in the formulation* (2.8)*, then we have the following*:

(1) *There exists* $\rho_0>0$ *such that the extended Galerkin* (*XG*) *method is uniformly well-posed with respect to the norms* $\|\cdot\|_{0,\rho_h}$ *and* $\|\cdot\|_{1,\rho_h}$ *when* $\rho\in(0,\rho_0]$ *and the following estimate holds*:

$$\|\boldsymbol{p}_h\|_{0,c}+\|\check{\boldsymbol{p}}_h\|_{0,\rho_h}+\|u_h\|_{1,\rho_h}+\|\check{u}_h\|_{0,\rho_h^{-1}}\lesssim\|f\|_{-1,\rho_h}. \tag{3.14}$$

(2) *Let* $(\boldsymbol{p},u)\in\boldsymbol{H}^{1/2+\epsilon}(\Omega)\times H^1(\Omega)$ *be the solution of* (1.2) *and* $(\tilde{\boldsymbol{p}}_h,\tilde{u}_h)\in\tilde{\boldsymbol{Q}}_h\times\tilde{V}_h$ *be the solution of* (2.8)*. We have the quasi-optimal approximation as follows*:

$$\begin{aligned}&\|\boldsymbol{p}-\boldsymbol{p}_h\|_{h,c}+\|\check{\boldsymbol{p}}_h\|_{0,\rho_h}+\|u-u_h\|_{1,\rho_h}+\|\check{u}_h\|_{0,\rho_h^{-1}}\\&\quad\lesssim\inf_{\boldsymbol{r}_h\in\boldsymbol{Q}_h,w_h\in V_h}(\|\boldsymbol{p}-\boldsymbol{r}_h\|_{h,c}+\|u-w_h\|_{1,\rho_h}),\end{aligned} \tag{3.15}$$

where

$$\|\boldsymbol{p}\|_{h,c}^2:=(c\boldsymbol{p},\boldsymbol{p})+\langle h_e\{\!\{\boldsymbol{p}\}\!\},\{\!\{\boldsymbol{p}\}\!\}\rangle+\langle h_e[\boldsymbol{p}],[\boldsymbol{p}]\rangle,\quad\forall\,\boldsymbol{p}\in\boldsymbol{Q}_h+\boldsymbol{H}^{1/2+\epsilon}(\Omega). \tag{3.16}$$

(3) *If* $\boldsymbol{p}\in\boldsymbol{H}^{k+1}(\Omega),u\in H^{k+2}(\Omega)$ $(k\geqslant0)$ *and we choose the spaces*

$$\boldsymbol{Q}_h\times\check{\boldsymbol{Q}}_h\times V_h\times\check{V}_h=\boldsymbol{Q}_h^k\times\check{\boldsymbol{Q}}_h^k\times V_h^{k+1}\times\check{V}_h$$

for any $\check{V}_h$*, then we have the error estimate*

$$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,c}+\|\check{\boldsymbol{p}}_h\|_{0,\rho_h}+\|u-u_h\|_{1,\rho_h}+\|\check{u}_h\|_{0,\rho_h^{-1}}\lesssim h^{k+1}(|\boldsymbol{p}|_{k+1}+|u|_{k+2}). \tag{3.17}$$

Proof. **Step 1.** The uniform inf-sup condition on discrete spaces. From the Brezzi theory [9, 12], the discrete well-posedness of (3.4) implies

$$\|\boldsymbol{p}_h\|_{0,c}+\|\check{\boldsymbol{p}}_h\|_{0,\rho_h}+\|u_h\|_{1,\rho_h}\lesssim\|f\|_{-1,\rho_h}.$$

Note that $\check{u}_h=\eta\check{\mathcal{Q}}_h^u[\boldsymbol{p}_h]$, and using the trace inequality, we have

$$\|\check{u}_h\|_{0,\rho_h^{-1}}^2\cong\langle\rho h_e\check{\mathcal{Q}}_h^u[\boldsymbol{p}_h],\check{\mathcal{Q}}_h^u[\boldsymbol{p}_h]\rangle\lesssim\|\boldsymbol{p}_h\|_{0,c}^2, \tag{3.18}$$

which gives (3.14).

Step 2. Quasi-optimal error estimates. Using the trace inequality and the inverse inequality, the norm $\|\cdot\|_{h,c}$ defined in (3.16) is equivalent to $\|\cdot\|_{0,c}$ on the discrete space $\boldsymbol{Q}_h$. Therefore, we have the following inf-sup condition:

$$\sup_{\tilde{\boldsymbol{q}}_h\in\tilde{\boldsymbol{Q}}_h\backslash\{0\},v_h\in V_h\backslash\{0\}}\frac{\tilde{a}_w((\tilde{\boldsymbol{r}}_h,w_h),(\tilde{\boldsymbol{q}}_h,v_h))}{\|\tilde{\boldsymbol{q}}_h\|_{h,\rho_h}+\|v_h\|_{1,\rho_h}}\gtrsim\|\tilde{\boldsymbol{r}}_h\|_{h,\rho_h}+\|w_h\|_{1,\rho_h},\quad\forall\,\tilde{\boldsymbol{r}}_h\in\tilde{\boldsymbol{Q}}_h,\quad w_h\in V_h, \tag{3.19}$$

where

$$\begin{aligned}&\tilde{a}_w((\tilde{\boldsymbol{r}}_h,w_h),(\tilde{\boldsymbol{q}}_h,v_h)):=a_w(\tilde{\boldsymbol{r}}_h,\tilde{\boldsymbol{q}}_h)+b_w(\tilde{\boldsymbol{q}}_h,w_h)+b_w(\tilde{\boldsymbol{r}}_h,v_h),\\&\|\tilde{\boldsymbol{r}}_h\|_{h,\rho_h}^2:=\|\boldsymbol{r}_h\|_{h,c}^2+\|\check{\boldsymbol{r}}_h\|_{0,\rho_h}^2=(c\boldsymbol{r}_h,\boldsymbol{r}_h)+\langle h_e\{\!\{\boldsymbol{r}_h\}\!\},\{\!\{\boldsymbol{r}_h\}\!\}\rangle+\langle h_e[\boldsymbol{r}_h],[\boldsymbol{r}_h]\rangle+\langle\rho h_e\check{\boldsymbol{r}}_h,\check{\boldsymbol{r}}_h\rangle.\end{aligned}$$

Note that the norm $\|\cdot\|_{h,\rho_h}$ can be extended to $(\boldsymbol{Q}_h + \boldsymbol{H}^{1/2+\epsilon}(\Omega)) \times \check{\boldsymbol{Q}}_h$. Recalling the consistency of the 3-field formulation, we have for any $\tilde{\boldsymbol{r}}_h \in \tilde{\boldsymbol{Q}}_h, w_h \in \tilde{V}_h$,

$$
\begin{aligned}
\|\tilde{\boldsymbol{r}}_h - \tilde{\boldsymbol{p}}_h\|_{h,\rho_h} + \|w_h - u_h\|_{1,\rho_h} &\lesssim \sup_{\tilde{\boldsymbol{q}}_h \in \tilde{\boldsymbol{Q}}_h \setminus \{0\}, v_h \in V_h \setminus \{0\}} \frac{\tilde{a}_w((\tilde{\boldsymbol{r}}_h - \tilde{\boldsymbol{p}}_h, w_h - u_h), (\tilde{\boldsymbol{q}}_h, v_h))}{\|\tilde{\boldsymbol{q}}_h\|_{h,\rho_h} + \|v_h\|_{1,\rho_h}} \\
&= \sup_{\tilde{\boldsymbol{q}}_h \in \tilde{\boldsymbol{Q}}_h \setminus \{0\}, v_h \in V_h \setminus \{0\}} \frac{\tilde{a}_w((\tilde{\boldsymbol{r}}_h - (\boldsymbol{p}, \boldsymbol{0}), w_h - u), (\tilde{\boldsymbol{q}}_h, v_h))}{\|\tilde{\boldsymbol{q}}_h\|_{h,\rho_h} + \|v_h\|_{1,\rho_h}} \\
&\lesssim \|\tilde{\boldsymbol{r}}_h - (\boldsymbol{p}, \boldsymbol{0})\|_{h,\rho_h} + \|w_h - u\|_{1,\rho_h}.
\end{aligned} \tag{3.20}
$$

Here, in the last step, the boundedness of $a_w(\cdot,\cdot)$ and $b_w(\cdot,\cdot)$ defined in (3.4) is applied by using (3.13) and the Cauchy-Schwarz inequality, namely,

$$
\begin{aligned}
a_w(\tilde{\boldsymbol{r}}_h - (\boldsymbol{p}, \boldsymbol{0}), \tilde{\boldsymbol{q}}_h) &= (c(\boldsymbol{r}_h - \boldsymbol{p}), \boldsymbol{q}_h) + \langle \eta \check{\mathcal{Q}}_h^u[\boldsymbol{r}_h - \boldsymbol{p}], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle + \langle \tau^{-1}\check{\boldsymbol{r}}_h, \check{\boldsymbol{q}}_h\rangle \\
&\lesssim \|(\tilde{\boldsymbol{r}}_h - (\boldsymbol{p}, \boldsymbol{0}))\|_{h,\rho_h} \|\tilde{\boldsymbol{q}}_h\|_{h,\rho_h}, \\
b_w(\tilde{\boldsymbol{r}}_h - (\boldsymbol{p}, \boldsymbol{0}), v_h) &= (\nabla_h v_h, \boldsymbol{r}_h - \boldsymbol{p}) - \langle [\![v_h]\!], \check{\boldsymbol{r}}_h\rangle - \langle [\![v_h]\!], \{\!\{\boldsymbol{r}_h - \boldsymbol{p}\}\!\}\rangle \\
&\lesssim \|(\tilde{\boldsymbol{r}}_h - (\boldsymbol{p}, \boldsymbol{0}))\|_{h,\rho_h} \|v_h\|_{1,\rho_h} \quad \text{(by (3.13))}, \\
b_w(\tilde{\boldsymbol{q}}_h, w_h - u) &= (\nabla(w_h - u), \boldsymbol{q}_h) - \langle [\![w_h - u]\!], \check{\boldsymbol{q}}_h + \{\!\{\boldsymbol{q}_h\}\!\}\rangle \\
&\lesssim \|\tilde{\boldsymbol{q}}_h\|_{h,\rho_h} \|w_h - u\|_{1,\rho_h} \quad (\text{by } \{\!\{\boldsymbol{Q}_h\}\!\} \subset \check{\boldsymbol{Q}}_h).
\end{aligned}
$$

The quasi-optimal approximation (3.15) then follows from (3.18), (3.20) and the triangle inequality. Consequently, the error estimate (3.17) follows directly from (3.15) and interpolation theory. □

Remark 3.5. In Step 2, with the help of norm $\|\cdot\|_{h,c}$ in (3.16), we avoid using the trace inequality in the proof of boundedness of $a_w(\cdot,\cdot)$ and $b_w(\cdot,\cdot)$ on the continuous spaces. Whence, the quasi-optimal approximation holds with the minimal regularity requirement.

3.3 Div-based uniform inf-sup condition

In light of the formulation of $b(\cdot,\cdot)$ in (3.6), we now establish the div-based inf-sup condition. For any $\boldsymbol{p}_h \in \boldsymbol{Q}_h, \check{\boldsymbol{p}}_h \in \check{\boldsymbol{Q}}_h, u_h \in V_h, \check{u}_h \in \check{V}_h$, the norms are defined by

$$
\begin{aligned}
\|\tilde{\boldsymbol{p}}_h\|^2_{\mathrm{div},\rho_h} &:= \underbrace{(c\boldsymbol{p}_h, \boldsymbol{p}_h) + (\mathrm{div}_h \boldsymbol{p}_h, \mathrm{div}_h \boldsymbol{p}_h) + \langle \rho^{-1} h_e^{-1} \check{\mathcal{Q}}_h^u[\boldsymbol{p}_h], \check{\mathcal{Q}}_h^u[\boldsymbol{p}_h]\rangle}_{\|\boldsymbol{p}_h\|^2_{\mathrm{div},\rho_h}} + \underbrace{\langle \rho^{-1} h_e^{-1} \check{\boldsymbol{p}}_h, \check{\boldsymbol{p}}_h\rangle}_{\|\check{\boldsymbol{p}}_h\|^2_{0,\rho_h^{-1}}}, \\
\|\tilde{u}_h\|^2_{0,\rho_h} &:= \underbrace{(u_h, u_h)}_{\|u_h\|_0^2} + \underbrace{\langle \rho h_e \check{u}_h, \check{u}_h\rangle}_{\|\check{u}_h\|^2_{0,\rho_h}}.
\end{aligned} \tag{3.21}
$$

Note that the norm $\|\cdot\|_{\mathrm{div},\rho_h}$ can be extended to $(\boldsymbol{Q}_h + \boldsymbol{H}(\mathrm{div}, \Omega)) \times \check{\boldsymbol{Q}}_h$.

Assumption 3.6. *The spaces $\boldsymbol{Q}_h, V_h$ and $\check{V}_h$ satisfy the following conditions*:

(a) *let $\boldsymbol{R}_h := \boldsymbol{Q}_h \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ and $\boldsymbol{R}_h \times V_h$ be a stable pair for the mixed method*;

(b) $\mathrm{div}_h \boldsymbol{Q}_h = V_h$;

(c) $\{V_h\} \subset \check{V}_h$.

In light of (3.6), the boundedness of $a(\cdot,\cdot)$ is obvious. The boundedness of $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ on discrete spaces can be derived from the Cauchy-Schwarz inequality, the trace inequality and the assumption $\{V_h\} \subset \check{V}_h$. We are now in the position to state the second main result.

Theorem 3.7. *Under Assumption* 3.6, *if we choose $\eta = (\rho h_e)^{-1}, \tau \cong \eta^{-1} = \rho h_e$ in the formulation* (2.8), *then we have the following*:

(1) *There exists $\rho_0 > 0$, such that the XG method is uniformly well-posed with respect to the norms $\|\cdot\|_{\mathrm{div},\rho_h}$ and $\|\cdot\|_{0,\rho_h}$ when $\rho \in (0, \rho_0]$ and the following estimate holds*:

$$
\|\boldsymbol{p}_h\|_{\mathrm{div},\rho_h} + \|\check{\boldsymbol{p}}_h\|_{0,\rho_h^{-1}} + \|u_h\|_0 + \|\check{u}_h\|_{0,\rho_h} \lesssim \|f\|_0. \tag{3.22}
$$

(2) *Assume further that*

$$\text{(d)} \quad [\boldsymbol{Q}_h] \subset \check{V}_h. \tag{3.23}$$

Let $(\boldsymbol{p}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times H^{1/2+\epsilon}(\Omega)$ *be the solution of* (1.2) *and* $(\tilde{\boldsymbol{p}}_h, \tilde{u}_h) \in \tilde{\boldsymbol{Q}}_h \times \tilde{V}_h$ *be the solution of* (2.8)*. We have the following quasi-optimal approximation*:

$$\begin{aligned} &\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{div},\rho_h} + \|\check{\boldsymbol{p}}_h\|_{0,\rho_h^{-1}} + \|u - u_h\|_{h,0} + \|\check{u}_h\|_{0,\rho_h} \\ &\lesssim \inf_{\boldsymbol{r}_h \in \boldsymbol{Q}_h, w_h \in V_h} (\|\boldsymbol{p} - \boldsymbol{r}_h\|_{\mathrm{div},\rho_h} + \|u - w_h\|_{h,0}), \end{aligned} \tag{3.24}$$

where

$$\|u\|_{h,0}^2 := \|u\|_0^2 + \langle h_e\{u\}, \{u\}\rangle + \langle h_e[\![u]\!], [\![u]\!]\rangle, \quad \forall\, u \in V_h + H^{1/2+\epsilon}(\Omega). \tag{3.25}$$

(3) *If* $\boldsymbol{p} \in \boldsymbol{H}^{k+2}(\Omega), u \in H^{k+1}(\Omega)$ $(k \geqslant 0)$*, and we choose the spaces*

$$\boldsymbol{Q}_h \times \check{\boldsymbol{Q}}_h \times V_h \times \check{V}_h = \boldsymbol{Q}_h^{k,RT} \times \check{\boldsymbol{Q}}_h \times V_h^k \times \check{V}_h^k$$

or $\boldsymbol{Q}_h^{k+1} \times \check{\boldsymbol{Q}}_h \times V_h^k \times \check{V}_h^{k+1}$ *for any* $\check{\boldsymbol{Q}}_h$*, then the following estimate holds*:

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{div},\rho_h} + \|\check{\boldsymbol{p}}_h\|_{0,\rho_h^{-1}} + \|u - u_h\|_0 + \|\check{u}_h\|_{0,\rho_h} \lesssim h^{k+1}(|\boldsymbol{p}|_{k+2} + |u|_{k+1}). \tag{3.26}$$

Proof. **Step 1.** The uniform inf-sup condition on discrete spaces. We first consider the inf-sup condition for $b(\cdot,\cdot)$. For any $v_h \in V_h$, there exists $\boldsymbol{q}_h^c \in \boldsymbol{R}_h \subset \boldsymbol{Q}_h$, such that $\mathrm{div}\boldsymbol{q}_h^c = v_h, [\boldsymbol{q}_h^c] = 0$ and

$$\|\boldsymbol{q}_h^c\|_0 + \|\mathrm{div}\boldsymbol{q}_h^c\|_0 \lesssim \|v_h\|_0 \quad \text{or} \quad \|\boldsymbol{q}_h^c\|_{\mathrm{div},\rho_h} \lesssim \|v_h\|_0. \tag{3.27}$$

Define the operator $B : \boldsymbol{Q}_h \to V_h'$ by $\langle B\boldsymbol{q}_h, v_h\rangle = b(\boldsymbol{q}_h, v_h)$ for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$ and $v_h \in V_h$. Let

$$\begin{aligned} K = \mathrm{Ker}B &:= \{\boldsymbol{q}_h \in \boldsymbol{Q}_h : -(\mathrm{div}_h\boldsymbol{q}_h, v_h) + \langle [\boldsymbol{q}_h], \{v_h\}\rangle = 0, \forall\, v_h \in V_h\}, \\ H = \mathrm{Ker}B' &:= \{v_h \in V_h : -(\mathrm{div}_h\boldsymbol{q}_h, v_h) + \langle [\boldsymbol{q}_h], \{v_h\}\rangle = 0, \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h\}. \end{aligned}$$

Clearly, we have $H = \{0\}$ from (3.27). Next, we consider the coercivity of $a(\cdot,\cdot)$ on K. Define the lifting operator $r : L^2(\mathcal{E}_h) \mapsto V_h$ by

$$\int_\Omega r(w) v_h \,\mathrm{d}x = \sum_{e\in\mathcal{E}_h} \int_e w\{v_h\} \,\mathrm{d}s, \quad \forall\, v_h \in V_h. \tag{3.28}$$

Then, a standard scaling argument gives

$$\|r(w)\|_0 \lesssim h_e^{-1/2} \|w\|_{0,\mathcal{E}_h}^2, \quad \forall\, w \in L^2(\mathcal{E}_h). \tag{3.29}$$

Using the definition of lifting operator r and the assumption that $\{V_h\} \subset \check{V}_h$, we have

$$r(\check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]) = \mathrm{div}_h \boldsymbol{q_h}$$

for any $\boldsymbol{q}_h \in K$. Then the coercivity of $a(\cdot,\cdot)$ on K is shown as follows: For any $\boldsymbol{q}_h \in K$,

$$\begin{aligned} a(\boldsymbol{q}_h, \boldsymbol{q}_h) &= (c\boldsymbol{q}_h, \boldsymbol{q}_h) + \langle \eta\check{\mathcal{Q}}_h^u[\boldsymbol{q}_h], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle \\ &\gtrsim (c\boldsymbol{q}_h, \boldsymbol{q}_h) + (r(\check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]), r(\check{\mathcal{Q}}_h^u[\boldsymbol{q}_h])) + \langle \rho^{-1} h_e^{-1} \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle \\ &= (c\boldsymbol{q}_h, \boldsymbol{q}_h) + (\mathrm{div}_h\boldsymbol{q}_h, \mathrm{div}_h\boldsymbol{q}_h) + \langle \rho^{-1} h_e^{-1} \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle = \|\boldsymbol{q}_h\|_{\mathrm{div},\rho_h}^2. \end{aligned}$$

By [7, Theorem 4.3.1], we have the discrete well-posedness of (3.5). Then, we have

$$\|\boldsymbol{p}_h\|_{\mathrm{div},\rho_h} + \|u_h\|_0 \lesssim \|f\|_0,$$

which implies (3.22) by using $\check{\boldsymbol{p}}_h = \tau\check{\boldsymbol{\mathcal{Q}}}_h^p[\![u_h]\!]$, $\check{u}_h = \eta\check{\mathcal{Q}}_h^u[\boldsymbol{p}_h]$ and the trace inequality.

Step 2. Quasi-optimal error estimates. Using the trace inequality and the inverse inequality, the norm $\|\cdot\|_{h,0}$ defined in (3.25) is equivalent to $\|\cdot\|_0$ on the discrete space V_h. Therefore, we have the following uniform inf-sup condition:

$$\sup_{\boldsymbol{q}_h\in \boldsymbol{Q}_h\setminus\{\boldsymbol{0}\}, v_h\in V_h\setminus\{0\}} \frac{\tilde{a}((\boldsymbol{r}_h, w_h),(\boldsymbol{q}_h, v_h))}{\|\boldsymbol{q}_h\|_{\mathrm{div},\rho_h}+\|v_h\|_{h,0}} \gtrsim \|\boldsymbol{r}_h\|_{\mathrm{div},\rho_h}+\|w_h\|_{h,0}, \quad \forall\, \boldsymbol{r}_h\in \boldsymbol{Q}_h, \quad w_h\in V_h, \tag{3.30}$$

where

$$\tilde{a}((\boldsymbol{r}_h, w_h),(\boldsymbol{q}_h, v_h)) := a(\boldsymbol{r}_h,\boldsymbol{q}_h)+b(\boldsymbol{q}_h,w_h)+b(\boldsymbol{r}_h,v_h)-c(w_h,v_h).$$

Recalling the consistency of the 2-field formulation, we have for any $\boldsymbol{r}_h\in \boldsymbol{Q}_h$, $w_h\in V_h$,

$$\begin{aligned}\|\boldsymbol{r}_h-\boldsymbol{p}_h\|_{\mathrm{div},\rho_h}+\|w_h-u_h\|_{h,0} &\lesssim \sup_{\boldsymbol{q}_h\in \boldsymbol{Q}_h\setminus\{\boldsymbol{0}\}, v_h\in V_h\setminus\{0\}} \frac{\tilde{a}((\boldsymbol{r}_h-\boldsymbol{p}_h, w_h-u_h),(\boldsymbol{q}_h, v_h))}{\|\boldsymbol{q}_h\|_{\mathrm{div},\rho_h}+\|v_h\|_{h,0}}\\ &= \sup_{\boldsymbol{q}_h\in \boldsymbol{Q}_h\setminus\{\boldsymbol{0}\}, v_h\in V_h\setminus\{0\}} \frac{\tilde{a}((\boldsymbol{r}_h-\boldsymbol{p}, w_h-u),(\boldsymbol{q}_h, v_h))}{\|\boldsymbol{q}_h\|_{\mathrm{div},\rho_h}+\|v_h\|_{h,0}}\\ &\lesssim \|\boldsymbol{r}_h-\boldsymbol{p}\|_{\mathrm{div},\rho_h}+\|w_h-u\|_{0,h}.\end{aligned} \tag{3.31}$$

Here, in the last step, we apply the boundedness of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ as follows:

$$\begin{aligned} a(\boldsymbol{r}_h-\boldsymbol{p},\boldsymbol{q}_h) &= (c(\boldsymbol{r}_h-\boldsymbol{p}),\boldsymbol{q}_h)+\langle \eta\check{\mathcal{Q}}_h^u[\boldsymbol{r}_h-\boldsymbol{p}], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle\\ &\lesssim \|\boldsymbol{r}_h-\boldsymbol{p}\|_{\mathrm{div},\rho_h}\|\boldsymbol{q}_h\|_{\mathrm{div},\rho_h},\\ b(\boldsymbol{r}_h-\boldsymbol{p},v_h) &= -(\mathrm{div}_h(\boldsymbol{r}_h-\boldsymbol{p}),v_h)+\langle[\boldsymbol{r}_h-\boldsymbol{p}],\{v_h\}\rangle\\ &\lesssim \|\boldsymbol{r}_h-\boldsymbol{p}\|_{\mathrm{div},\rho_h}\|v_h\|_{h,0} \quad (\text{by } \{V_h\}\subset \check{V}_h),\\ b(\boldsymbol{q}_h,w_h-u) &= -(\mathrm{div}_h\boldsymbol{q}_h,w_h-u)+\langle[\boldsymbol{q}_h],\{w_h-u\}\rangle\\ &\lesssim \|\boldsymbol{q}_h\|_{\mathrm{div},\rho_h}\|w_h-u\|_{h,0} \quad (\text{by } [\boldsymbol{Q}_h]\subset \check{V}_h),\\ c(w_h-u,v_h) &= \langle \tau\check{\boldsymbol{\mathcal{Q}}}_h^p[\![w_h-u]\!], \check{\boldsymbol{\mathcal{Q}}}_h^p[\![v_h]\!]\rangle \lesssim \|w_h-u\|_{h,0}\|v_h\|_{h,0}.\end{aligned}$$

Similar to (3.20) in Theorem 3.4, we have the the quasi-optimal approximation (3.24), which directly leads to the error estimate (3.26) from the interpolation theory. □

Remark 3.8. Similar to the grad-based analysis, a norm $\|\cdot\|_{h,0}$ is introduced in (3.25) to obtain the quasi-optimal approximation under the minimal regularity assumption. The assumption (d) in (3.23) is necessary for the quasi-optimal error estimate, though it is not required in the discrete well-posedness.

4 Relationship with existing methods and analysis

In this section, we exploit the relationship between the formulation (2.8) and several existing numerical methods, which leads to the well-posedness and error estimates of the existing numerical methods. We follow the three different variants of the 4-field system by eliminating either $\check{\boldsymbol{p}}_h$ or $\check{u}_h$, or both.

4.1 Eliminating $\check{\boldsymbol{p}}_h$

After eliminating $\check{\boldsymbol{p}}_h$ via (3.1), the resulting 3-field formulation (3.2) is a generalization of the stabilized hybrid mixed method [30], or some special cases of the HDG method [20–22, 33, 38].

Some special cases. More precisely, let us consider the case in which

$$\begin{aligned}&\eta=\frac{1}{4}\tau^{-1}, \quad \check{V}_h=\check{\boldsymbol{Q}}_h\cdot\boldsymbol{n},\\ &[\boldsymbol{Q}_h]\subset \check{V}_h \quad \text{or} \quad \{V_h\}\subset \check{V}_h.\end{aligned} \tag{4.1}$$

If we denote

$$\hat{u}_h := \check{\mathcal{Q}}_h^u\{u_h\}+\check{u}_h\in \check{V}_h \quad \text{and} \quad \hat{v}_h := \check{\mathcal{Q}}_h^u\{v_h\}+\check{v}_h\in \check{V}_h,$$

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then the 3-field formulation (3.2) is shown to be: Find $(\boldsymbol{p}_h, u_h, \hat{u}_h) \in \boldsymbol{Q}_h \times V_h \times \check{V}_h$ such that for any $(\boldsymbol{q}_h, v_h, \hat{v}_h) \in \boldsymbol{Q}_h \times V_h \times \check{V}_h$,

$$\begin{cases} (c\boldsymbol{p}_h, \boldsymbol{q}_h)_K - (u_h, \mathrm{div}_h \boldsymbol{q}_h)_K + \langle \hat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n} \rangle_{\partial K} = 0, \\ -(\mathrm{div}_h \boldsymbol{p}_h, v_h)_K + \langle 2\tau(\hat{u}_h - \check{\mathcal{Q}}_h^u u_h), \check{\mathcal{Q}}_h^u v_h \rangle_{\partial K} = -(f, v_h)_K, \\ \langle \boldsymbol{p}_h \cdot \boldsymbol{n}, \hat{v}_h \rangle_{\partial K} - \langle 2\tau(\hat{u}_h - \check{\mathcal{Q}}_h^u u_h), \hat{v}_h \rangle_{\partial K} = 0. \end{cases} \tag{4.2}$$

The above formulation shows that $\boldsymbol{p}_h$ and u_h can be represented by $\hat{u}_h$ locally from the first and the second equations. As a result, a globally coupled equation solely for $\hat{u}_h$ on $\mathcal{E}_h$ can be obtained.

It can be shown that (4.2) reduces to the standard HDG method [20–22], if

$$\check{\mathcal{Q}}_h^u u_h = u_h, \check{\mathcal{Q}}_h^u v_h = v_h \quad \text{or} \quad V_h|_{\mathcal{E}_h} \subset \check{V}_h. \tag{4.3}$$

Moreover, the formulation (4.2) coincides with the HDG with the reduced stabilization method [33, 38]. In particular, the grad-based analysis in Subsection 3.2 is fit for the HDG with the reduced stabilization method, with the choice

$$\boldsymbol{Q}_h \times V_h \times \check{V}_h = \boldsymbol{Q}_h^k \times V_h^{k+1} \times \check{V}_h^k.$$

Specific choices of the discrete space and the corresponding numerical methods are summarized in Table 1. We refer to [30] for discussion from the HDG to the hybrid mixed methods [3, 12, 20] and the mixed methods [7, 10–12, 36, 43].

Remark 4.1. We should note that the uniform inf-sup condition for the HDG method when $\eta = \frac{1}{4}\tau^{-1} = \mathcal{O}(1)$, $\boldsymbol{Q}_h = \boldsymbol{Q}_h^k$, $V_h = V_h^k$, $\hat{V}_h = \hat{V}_h^k$ is not proved in Section 3.

Minimal stabilized div-based method. In light of Theorem 3.7, the div-based inf-sup condition holds for *any* $\check{\boldsymbol{Q}}_h$. Hence, when choosing $\check{\boldsymbol{Q}}_h = \{0\}$, the 3-field formulation (3.2) reduces to a stabilized div-based method with minimal stabilization, whose well-posedness and the quasi-optimal error estimate are guaranteed under Assumptions 3.6 and (3.23).

Furthermore, with the assumption (3.23) and $\check{\boldsymbol{Q}}_h = \{0\}$, the 2-field formulation (3.5) reduces to the mixed DG method [30]. This implies that the mixed DG method proposed in [30] can be interpreted as the minimal stabilized div-based method.

Mixed method. Finally, we remark that, if we take $\tau = \rho h$, $\rho \to 0$, and choose

$$\boldsymbol{Q}_h \times V_h \times \check{V}_h = \boldsymbol{Q}_h^{k+1} \times V_h^k \times \check{V}_h^{k+1}$$

or

$$\boldsymbol{Q}_h \times V_h \times \check{V}_h = \boldsymbol{Q}_h^{k,RT} \times V_h^k \times \check{V}_h^k,$$

then the 3-field formulation (3.2) implies the mixed method by eliminating $\check{u}_h$. We make this statement rigorous in Subsection 4.4.2.

Table 1 From (2.8) to existing methods

$\boldsymbol{Q}_h$	$\check{\boldsymbol{Q}}_h$	V_h	$\check{V}_h$	Reference	inf-sup condition
$\boldsymbol{Q}_h^k$	$\check{\boldsymbol{Q}}_h^{k+1}$	V_h^{k+1}	$\check{V}_h^{k+1}$	HDG in [33]	gradient-based
$\boldsymbol{Q}_h^{k+1}$	$\check{\boldsymbol{Q}}_h^{k+1}$	V_h^k	$\check{V}_h^{k+1}$	HDG in [21]	div-based
$\boldsymbol{Q}_h^{k,\mathrm{RT}}$	$\check{\boldsymbol{Q}}_h^k$	V_h^k	$\check{V}_h^k$	HDG in [21]	div-based
$\boldsymbol{Q}_h^k$	$\check{\boldsymbol{Q}}_h^k$	V_h^{k+1}	$\check{V}_h^k$	HDG with reduced stabilization in [33, 38]	gradient-based
$\boldsymbol{Q}_h^k$	$\check{\boldsymbol{Q}}_h^k$	V_h^k	$\check{V}_h^k$	HDG in [19, 22]	not proved
$\boldsymbol{Q}_h^{k+1}$	$\{0\}$	V_h^k	$\check{V}_h^k$	Mixed DG in [30]	div-based
$\boldsymbol{Q}_h^{k,\mathrm{RT}}$	$\check{\boldsymbol{Q}}_h^k$	V_h^k	$\check{V}_h^{k+1}$	WG in [46]	div-based
$\boldsymbol{Q}_h^k$	$\check{\boldsymbol{Q}}_h^k$	V_h^{k+1}	$\check{V}_h^k$	WG-MFEM in [47]	grad-based
$\boldsymbol{Q}_h^k$	$\check{\boldsymbol{Q}}_h^k$	V_h^{k+1}	$\{0\}$	LDG in [24]	grad-based

4.2 Eliminating $\check{u}_h$

After eliminating $\check{u}_h$ via (3.3), the resulting 3-field formulation (3.4) is a generalization of the stabilized hybrid primal method [30], or some special cases of the WG-MFEM method [47]. In fact, in the case when α is not piecewise constant, the WG-MFEM method in [47] differs from the formulation (3.4).

Some special cases. Denote

$$\hat{\boldsymbol{p}}_h := \check{\boldsymbol{\mathcal{Q}}}_h^p\{\!\{\boldsymbol{p}_h\}\!\} + \check{\boldsymbol{p}}_h \in \check{\boldsymbol{Q}}_h, \quad \hat{\boldsymbol{q}}_h := \check{\boldsymbol{\mathcal{Q}}}_h^p\{\!\{\boldsymbol{q}_h\}\!\} + \check{\boldsymbol{q}}_h \in \check{\boldsymbol{Q}}_h.$$

Again, under the conditions that

$$\begin{aligned} &\tau = \frac{1}{4}\eta^{-1}, \quad \check{\boldsymbol{Q}}_h \cdot \boldsymbol{n} = \check{V}_h, \\ &[\![V_h]\!] \subset \check{\boldsymbol{Q}}_h \quad \text{or} \quad \{\!\{\boldsymbol{Q}_h\}\!\} \subset \check{\boldsymbol{Q}}_h, \end{aligned} \tag{4.4}$$

(3.4) can be recast as: Find $(\boldsymbol{p}_h, u_h, \check{\boldsymbol{p}}_h) \in \boldsymbol{Q}_h \times V_h \times \check{\boldsymbol{Q}}_h$ such that for any $(\boldsymbol{q}_h, v_h, \check{\boldsymbol{q}}_h) \in \boldsymbol{Q}_h \times V_h \times \check{\boldsymbol{Q}}_h$,

$$\begin{cases} (c\boldsymbol{p}_h, \boldsymbol{q}_h)_K + (\nabla_h u_h, \boldsymbol{q}_h)_K - \langle 2\eta(\hat{\boldsymbol{p}}_h - \check{\boldsymbol{\mathcal{Q}}}_h^p \boldsymbol{p}_h), \check{\boldsymbol{\mathcal{Q}}}_h^p \boldsymbol{q}_h\rangle_{\partial K} = 0, \\ (\boldsymbol{p}_h, \nabla_h v_h)_K - \langle \hat{\boldsymbol{p}}_h, v_h \boldsymbol{n}\rangle_{\partial K} = -(f, v_h)_K, \\ -\langle \nabla_h u_h, \hat{\boldsymbol{q}}_h\rangle_{\partial K} + \langle 2\eta(\hat{\boldsymbol{p}}_h - \check{\boldsymbol{\mathcal{Q}}}_h^p \boldsymbol{p}_h), \hat{\boldsymbol{q}}_h\rangle_{\partial K} = 0. \end{cases} \tag{4.5}$$

Similarly, the above formulation shows that $\boldsymbol{p}_h$ and u_h can be represented by $\hat{\boldsymbol{p}}_h$, and hence, a globally coupled equation solely for $\hat{\boldsymbol{p}}_h$ on $\mathcal{E}_h$ can be obtained. Moreover, this formulation is the weak Galerkin mixed finite element method (WG-MFEM) [47], if

$$\check{\boldsymbol{\mathcal{Q}}}_h^p \boldsymbol{p}_h = \boldsymbol{p}_h, \quad \check{\boldsymbol{\mathcal{Q}}}_h^p \boldsymbol{q}_h = \boldsymbol{q}_h \quad \text{or} \quad \boldsymbol{Q}_h|_{\mathcal{E}_h} \subset \check{\boldsymbol{Q}}_h. \tag{4.6}$$

Several possible discrete spaces and the corresponding analysis tools are listed as follows:

1. $\boldsymbol{Q}_h = \boldsymbol{Q}_h^{k,RT}$, $V_h = V_h^k$, $\check{\boldsymbol{Q}}_h = \check{\boldsymbol{Q}}_h^k$, $\tau = \mathcal{O}(h)$: div-based analysis;
2. $\boldsymbol{Q}_h = \boldsymbol{Q}_h^{k+1}$, $V_h = V_h^k$, $\check{\boldsymbol{Q}}_h = \check{\boldsymbol{Q}}_h^{k+1}$, $\tau = \mathcal{O}(h)$: div-based analysis;
3. $\boldsymbol{Q}_h = \boldsymbol{Q}_h^k$, $V_h = V_h^{k+1}$, $\check{\boldsymbol{Q}}_h = \check{\boldsymbol{Q}}_h^k$ (or $\check{\boldsymbol{Q}}_h^{k+1}$), $\tau = \mathcal{O}(h^{-1})$: grad-based analysis.

We refer to [30] for discussion from the WG to the hybrid primal methods [41, 42, 49] and the primal methods [1, 16, 26, 28, 35, 39, 44].

Minimal stabilized grad-based method. In light of Theorem 3.4, the grad-based inf-sup condition holds for *any* $\check{V}_h$. Hence, when choosing $\check{V}_h = \{0\}$, the 3-field formulation (3.4) reduces to a stabilized grad-based method with minimal stabilization, whose well-posedness and the quasi-optimal error estimate are guaranteed under Assumption 3.2.

Furthermore, with the assumption (3.23) and $\check{V}_h = \{0\}$, the 2-field formulation (3.5) reduces to an LDG method [24] in the mixed form.

Primal method. We remark that, if we take $\eta \to 0$ and choose

$$\boldsymbol{Q}_h \times V_h \times \check{\boldsymbol{Q}}_h = \boldsymbol{Q}_h^0 \times V_h^1 \times \check{\boldsymbol{Q}}_h^0,$$

the WG method is equivalent to the nonconforming finite element method discretized by the Crouzeix-Raviart element. When choosing

$$\boldsymbol{Q}_h \times V_h \times \check{\boldsymbol{Q}}_h = \boldsymbol{Q}_h^1 \times V_h^2 \times \check{\boldsymbol{Q}}_h^1$$

and taking $\eta \to 0$, the formulation (4.4) is getting unstable. In this case, the stabilization is needed for the hybrid primal method which induces to the WG method.

4.3 Eliminating both $\check{\boldsymbol{p}}_h$ and $\check{u}_h$

By plugging in (3.1) and (3.3) into (2.6) and the DG identity (2.3), the 4-field formulation with $(\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h)$ turns out to the 2-field formulation (3.5) with $(\boldsymbol{p}_h, u_h)$, which can be viewed as a special DG scheme in the mixed form. Taking the advantage of unified analysis in Section 3, the grad-based or div-based analysis can be applied to the DG scheme (3.5) with various choices of the discrete spaces.

Remark 4.2. There are four fields in the original formulation (2.8): $\boldsymbol{p}_h, \check{\boldsymbol{p}}_h, u_h, \check{u}_h$. Theoretically, by eliminating any m fields for $m \leqslant 3$, the number of possible formulations can be

$$C_4^1 + C_4^2 + C_4^3 = 4 + 6 + 4 = 14.$$

Some of them may be hybridizable. Under the special assumption, these algorithms lead to some special interesting cases, e.g., the primal method or the mixed method.

Remark 4.3. In this paper, the analysis framework is established with a special choice

$$\bar{\boldsymbol{p}}_h = \{\!\{\boldsymbol{p}_h\}\!\}, \quad \bar{u}_h = \{u_h\},$$

which includes the mixed DG [30] and mixed LDG [15]. Furthermore, the 4-field formulation (1.5) has the capacity to recover more involved DG schemes by choosing different $\bar{\boldsymbol{p}}_h$ and $\bar{u}_h$ [30].

4.4 Some limiting case of 2-field formulation as $\rho \to 0$

With the uniform inf-sup conditions, we revisit some limiting of the formulation in the case of $\rho \to 0$ [30].

4.4.1 *A limiting case based on the grad-based analysis*

First, having the grad-based inf-sup condition, we discuss the limiting of the formulation (3.5) in the case of $\tau = (\rho h_e)^{-1}$, $\eta \cong \tau^{-1} = \rho h_e$ as $\rho \to 0$. Given the H^1 conforming subspace $V_h^c := V_h \cap H_D^1(\Omega) \subset V_h$, we consider the following scheme for solving the Poisson equation (1.1): Find $(u_h^c, \boldsymbol{p}_h^c) \in V_h^c \times \boldsymbol{Q}_h$ such that

$$\begin{cases}(c\boldsymbol{p}_h^c, \boldsymbol{q}_h) + (\nabla u_h^c, \boldsymbol{q}_h) = 0, & \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h,\\ (\boldsymbol{p}_h^c, \nabla v_h^c) = -(f, v_h^c), & \forall\, v_h^c \in V_h^c.\end{cases} \tag{4.7}$$

Then, under the condition that $\nabla V_h^c \subset \nabla_h V_h \subset \boldsymbol{Q}_h$, the well-posedness of (4.7) implies that (see [8])

$$\|\boldsymbol{p}_h^c\|_{0,c} + \|u_h^c\|_1 \lesssim \sup_{v_h^c \in V_h^c \setminus \{0\}} \frac{-(f, v_h^c)}{\|v_h^c\|_1} \lesssim \|f\|_0. \tag{4.8}$$

We further note that, when c is piecewise constant, the scheme (4.7) is equivalent to the primal finite element by eliminating $\boldsymbol{p}_h^c$ through $\boldsymbol{p}_h^c = \alpha \nabla u_h^c$. Thanks to the Theorem 3.4, we have the following theorem.

Theorem 4.4. *Assume that the spaces $\boldsymbol{Q}_h, V_h$ and $\check{\boldsymbol{Q}}_h$ satisfy*

(a) $\nabla_h V_h \subset \boldsymbol{Q}_h$;

(b) $\{\!\{\boldsymbol{Q}_h\}\!\} \subset \check{\boldsymbol{Q}}_h$, $[\![V_h]\!] \subset \check{\boldsymbol{Q}}_h$;

(c) $V_h = V_h^k$ $(k \geqslant 1)$.

Then the formulation (3.5) *with* $\tau = (\rho h_e)^{-1}, \eta \cong \tau^{-1} = \rho h_e$ *converges to the primal method* (4.7) *as* $\rho \to 0$. *Furthermore, let* $(\boldsymbol{p}_h^\tau, u_h^\tau)$ *be the solution of* (3.5) *and* $(\boldsymbol{p}_h^c, u_h^c)$ *be the solution of* (4.7). *We have*

$$\|\boldsymbol{p}_h^\tau - \boldsymbol{p}_h^c\|_{0,c} + (\|\nabla_h(u_h^\tau - u_h^c)\|_0^2 + \|h_e^{-1/2}[\![u_h^\tau - u_h^c]\!]\|_{0,\mathcal{E}_h}^2)^{\frac{1}{2}} \lesssim \rho^{\frac{1}{2}}\|f\|_0. \tag{4.9}$$

Proof. Recall the 2-field formulation (3.5),

$$\begin{cases}(c\boldsymbol{p}_h^\tau, \boldsymbol{q}_h) + \langle \eta \check{\mathcal{Q}}_h^u[\boldsymbol{p}_h^\tau], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle + (\nabla_h u_h^\tau, \boldsymbol{q}_h) - \langle [\![u_h^\tau]\!], \{\!\{\boldsymbol{q}_h\}\!\}\rangle = 0, & \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h,\\ (\boldsymbol{p}_h^\tau, \nabla_h v_h) - \langle \{\!\{\boldsymbol{p}_h^\tau\}\!\}, [\![v_h]\!]\rangle - \langle \tau [\![u_h^\tau]\!], [\![v_h]\!]\rangle = -(f, v_h), & \forall\, v_h \in V_h,\end{cases} \tag{4.10}$$

where the condition $[\![V_h]\!] \subset \check{\boldsymbol{Q}}_h$ is used.

Note that for any given $v_h \in V_h$, there exists a $v_h^I \in V_h^c$, such that

$$\|v_h - v_h^I\|_0 + \|\nabla_h(v_h - v_h^I)\|_0 \lesssim \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![v_h]\!]\|_{0,e}^2\right)^{\frac{1}{2}} \leqslant \rho^{\frac{1}{2}}\|v_h\|_{1,\rho_h}. \tag{4.11}$$

Define

$$\boldsymbol{\delta}_h^p := \boldsymbol{p}_h^\tau - \boldsymbol{p}_h^c, \quad \delta_h^u := u_h^\tau - u_h^c.$$

Taking $v_h^c = v_h^I$ in (4.7), and subtracting (4.7) from the equation (4.10), we have

$$\begin{cases}(c\boldsymbol{\delta}_h^p, \boldsymbol{q}_h) + \langle \eta \check{\mathcal{Q}}_h^u[\boldsymbol{\delta}_h^p], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle + (\nabla_h \delta_h^u, \boldsymbol{q}_h) - \langle [\![\delta_h^u]\!], \{\!\{\boldsymbol{q}_h\}\!\}\rangle \\ \quad = -\langle \eta \check{\mathcal{Q}}_h^u[\boldsymbol{p}_h^c], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle, \quad \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h, \\ (\boldsymbol{\delta}_h^p, \nabla_h v_h) - \langle \{\!\{\boldsymbol{\delta}_h^p\}\!\}, [\![v_h]\!]\rangle - \langle \tau [\![\delta_h^u]\!], [\![v_h]\!]\rangle \\ \quad = -(f, v_h - v_h^I) - (\boldsymbol{p}_h^c, \nabla_h(v_h - v_h^I)) + \langle \{\!\{\boldsymbol{p}_h^c\}\!\}, [\![v_h]\!]\rangle, \quad \forall\, v_h \in V_h.\end{cases}$$

By the grad-based stability result for the 2-field formulation (3.5), we obtain

$$\begin{aligned}&\|\boldsymbol{\delta}_h^p\|_{0,c} + \|\delta_h^u\|_{1,\rho_h} \\ &\quad\lesssim \sup_{\boldsymbol{q}_h\in\boldsymbol{Q}_h\setminus\{\boldsymbol{0}\}} \frac{-\langle \eta \check{\mathcal{Q}}_h^u[\boldsymbol{p}_h^c], \check{\mathcal{Q}}_h^u[\boldsymbol{q}_h]\rangle}{\|\boldsymbol{q}_h\|_{0,c}} + \sup_{v_h\in V_h\setminus\{0\}} \frac{-(f, v_h - v_h^I) - (\boldsymbol{p}_h^c, \nabla_h(v_h - v_h^I)) + \langle \{\!\{\boldsymbol{p}_h^c\}\!\}, [\![v_h]\!]\rangle}{\|v_h\|_{1,\rho_h}} \\ &\quad\lesssim \rho\|\boldsymbol{p}_h^c\|_{0,c} + \sup_{v_h\in V_h\setminus\{0\}} \frac{\|f\|_0\|v_h - v_h^I\|_0 + \|\boldsymbol{p}_h^c\|_{0,c}\|\nabla_h(v_h - v_h^I)\|_0}{\|v_h\|_{1,\rho_h}} + \rho^{\frac{1}{2}}\|\boldsymbol{p}_h^c\|_{0,c} \\ &\quad\lesssim \rho\|\boldsymbol{p}_h^c\|_{0,c} + \rho^{\frac{1}{2}}(\|f\|_0 + \|\boldsymbol{p}_h^c\|_{0,c}) \quad \text{(by (4.11))}.\end{aligned} \tag{4.12}$$

Finally, by the stability estimate (4.8), we have

$$\|\boldsymbol{\delta}_h^p\|_{0,c} + \|\delta_h^u\|_{1,\rho_h} \lesssim \rho\|f\|_0 + \rho^{\frac{1}{2}}\|f\|_0 \lesssim \rho^{\frac{1}{2}}\|f\|_0. \tag{4.13}$$

This completes the proof. □

Remark 4.5. The equivalence between $\|v_h\|_{1,\rho_h}^2$ and $(\nabla_h u_h, \nabla_h u_h) + \langle \rho^{-1} h_e^{-1}[\![u_h]\!], [\![u_h]\!]\rangle$ is not uniform as $\rho \to 0$. Therefore, the assumption $[\![V_h]\!] \subset \check{\boldsymbol{Q}}_h$ is necessary in (4.11).

Remark 4.6. A typical example that satisfies the assumption in Theorem 4.4 is $\boldsymbol{Q}_h = \boldsymbol{Q}_h^{k-1}$, $V_h = V_h^k$, and $\check{\boldsymbol{Q}}_h = \check{\boldsymbol{Q}}_h^k$ for $k \geqslant 1$.

4.4.2 *A limiting case based on the div-based analysis*

Next, having the div-based inf-sup condition, we discuss the limiting of the formulation (2.8) in the case of $\eta = (\rho h_e)^{-1}$, $\tau \cong \eta^{-1} = \rho h_e$ as $\rho \to 0$. Consider the $\boldsymbol{H}(\mathrm{div})$ conforming subspace

$$\boldsymbol{Q}_h^c := \boldsymbol{Q}_h \cap \boldsymbol{H}_N(\mathrm{div}, \Omega) \subset \boldsymbol{Q}_h,$$

the mixed method when applying to the Poisson equation (1.1) can be written as: Find $(\boldsymbol{p}_h^c, u_h^c) \in \boldsymbol{Q}_h^c \times V_h$ such that

$$\begin{cases}(c\boldsymbol{p}_h^c, \boldsymbol{q}_h^c) - (u_h^c, \mathrm{div}\boldsymbol{q}_h^c) = 0, \quad \forall\, \boldsymbol{q}_h^c \in \boldsymbol{Q}_h^c, \\ -(\mathrm{div}\boldsymbol{p}_h^c, v_h) = -(f, v_h), \quad \forall\, v_h \in V_h.\end{cases} \tag{4.14}$$

Then, under the condition that $\mathrm{div}\boldsymbol{Q}_h^c = \mathrm{div}_h\boldsymbol{Q}_h = V_h$, the well-posedness of the mixed method (see [7, 12]) implies that

$$\|\boldsymbol{p}_h^c\|_{\boldsymbol{H}(\mathrm{div})} + \|v_h^c\|_0 \lesssim \sup_{v_h\in V_h\setminus\{0\}} \frac{-(f, v_h)}{\|v_h\|_0} \lesssim \|f\|_0. \tag{4.15}$$

Thanks to Theorem 3.7, we have the following theorem.

Theorem 4.7. *Assume that the spaces $\boldsymbol{Q}_h$, $\check{V}_h$ and V_h satisfy*

(a) $\mathrm{div}_h\boldsymbol{Q}_h = V_h$;

(b) $\{V_h\} \subset \check{V}_h$, $[\boldsymbol{Q}_h] \subset \check{V}_h$;

(c) $\boldsymbol{Q}_h = \boldsymbol{Q}_h^{k,RT}$ *or* $\boldsymbol{Q}_h^{k+1}$, $k \geqslant 0$.

Then the formulation (3.5) *with* $\eta = (\rho h_e)^{-1}$, $\tau \cong \eta^{-1} = \rho h_e$ *converges to the mixed method* (4.14) *as* $\rho \to 0$. *Furthermore, let* $(\boldsymbol{p}_h^\eta, u_h^\eta)$ *be the solution of* (3.5) *and* $(\boldsymbol{p}_h^c, u_h^c)$ *be the solution of* (4.14). *We have*

$$\|\boldsymbol{p}_h^\eta - \boldsymbol{p}_h^c\|_{0,c} + \|\mathrm{div}_h(\boldsymbol{p}_h^\eta - \boldsymbol{p}_h^c)\|_0 + \|u_h^\eta - u_h^c\|_0 \lesssim \rho^{\frac{1}{2}}\|f\|_0. \tag{4.16}$$

Proof. Recall the two-field formulation (3.5),

$$\begin{cases}(c\boldsymbol{p}_h^\tau, \boldsymbol{q}_h) + \langle \eta[\boldsymbol{p}_h^\tau], [\boldsymbol{q}_h]\rangle - (u_h^\tau, \mathrm{div}_h\boldsymbol{q}_h) + \langle \{u_h^\tau\}, [\boldsymbol{q}_h]\rangle = 0, \quad \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h,\\ -(\mathrm{div}_h\boldsymbol{p}_h^\tau, v_h) + \langle [\boldsymbol{p}_h^\tau], \{v_h\}\rangle - \langle \tau\check{\boldsymbol{\mathcal{Q}}}_h^p[\![u_h^\tau]\!], \check{\boldsymbol{\mathcal{Q}}}_h^p[\![v_h]\!]\rangle = -(f, v_h), \quad \forall\, v_h \in V_h,\end{cases} \tag{4.17}$$

where the condition $[\boldsymbol{Q}_h] \subset \check{V}_h$ is used.

For any given $\boldsymbol{q}_h \in \boldsymbol{Q}_h$, there exists $\boldsymbol{q}_h^I \in \boldsymbol{Q}_h^c$, such that

$$\|\mathrm{div}_h(\boldsymbol{q}_h - \boldsymbol{q}_h^I)\|_0 + \|\boldsymbol{q}_h - \boldsymbol{q}_h^I\|_0 \lesssim \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{q}_h]\|_{0,e}^2\right)^{\frac{1}{2}} \leqslant \rho^{\frac{1}{2}}\|\boldsymbol{q}_h\|_{\mathrm{div},\rho_h}. \tag{4.18}$$

Define

$$\boldsymbol{\delta}_h^p = \boldsymbol{p}_h^\tau - \boldsymbol{p}_h^c, \quad \delta_h^u = u_h^\tau - u_h^c.$$

Taking $\boldsymbol{q}_h^c = \boldsymbol{q}_h^I$ in (4.14), and subtracting (4.14) from (4.17), we have

$$\begin{cases}(c\boldsymbol{\delta}_h^p, \boldsymbol{q}_h) + \langle \eta[\boldsymbol{\delta}_h^p], [\boldsymbol{q}_h]\rangle - (\delta_h^u, \mathrm{div}_h\boldsymbol{q}_h) + \langle \{\delta_h^u\}, [\boldsymbol{q}_h]\rangle\\ \quad = (u_h^c, \mathrm{div}_h(\boldsymbol{q}_h - \boldsymbol{q}_h^I)) - (c\boldsymbol{p}_h^c, \boldsymbol{q}_h - \boldsymbol{q}_h^I) - \langle \{u_h^c\}, [\boldsymbol{q}_h]\rangle, \quad \forall\, \boldsymbol{q}_h \in \boldsymbol{Q}_h,\\ -(\mathrm{div}_h\boldsymbol{\delta}_h^p, v_h) + \langle [\boldsymbol{\delta}_h^p], \{v_h\}\rangle - \langle \tau\check{\boldsymbol{\mathcal{Q}}}_h^p[\![\delta_h^u]\!], \check{\boldsymbol{\mathcal{Q}}}_h^p[\![v_h]\!]\rangle = \langle \tau\check{\boldsymbol{\mathcal{Q}}}_h^p[\![u_h^c]\!], \check{\boldsymbol{\mathcal{Q}}}_h^p[\![v_h]\!]\rangle, \quad \forall\, v_h \in V_h.\end{cases}$$

By the div-based stability result for the 2-field formulation (3.5), we obtain

$$\begin{aligned}&\|\boldsymbol{\delta}_h^p\|_{\mathrm{div},\rho_h} + \|\delta_h^u\|_0\\ &\quad\lesssim \sup_{\boldsymbol{q}_h\in\boldsymbol{Q}_h\setminus\{\boldsymbol{0}\}} \frac{(u_h^c, \mathrm{div}_h(\boldsymbol{q}_h - \boldsymbol{q}_h^I)) - (c\boldsymbol{p}_h^c, \boldsymbol{q}_h - \boldsymbol{q}_h^I) - \langle \{u_h^c\}, [\boldsymbol{q}_h]\rangle}{\|\boldsymbol{q}_h\|_{\mathrm{div},\rho_h}} + \sup_{v_h\in V_h\setminus\{0\}} \frac{\langle \tau\check{\boldsymbol{\mathcal{Q}}}_h^p[\![u_h^c]\!], \check{\boldsymbol{\mathcal{Q}}}_h^p[\![v_h]\!]\rangle}{\|v_h\|_0}\\ &\quad\lesssim \sup_{\boldsymbol{q}_h\in\boldsymbol{Q}_h\setminus\{\boldsymbol{0}\}} \frac{\|u_h^c\|_0\|\mathrm{div}_h(\boldsymbol{q}_h - \boldsymbol{q}_h^I)\|_0 + \|\boldsymbol{p}_h^c\|_0\|\boldsymbol{q}_h - \boldsymbol{q}_h^I\|_0}{\|\boldsymbol{q}_h\|_{\mathrm{div},\rho_h}} + \rho^{\frac{1}{2}}\|u_h^c\|_0 + \rho\|u_h^c\|_0\\ &\quad\lesssim \rho^{\frac{1}{2}}(\|u_h^c\|_0 + \|\boldsymbol{p}_h^c\|_0) + \rho^{\frac{1}{2}}\|u_h^c\|_0 \lesssim \rho^{\frac{1}{2}}(\|u_h^c\|_0 + \|\boldsymbol{p}_h^c\|_0) \quad (\text{by } (4.18)).\end{aligned} \tag{4.19}$$

By the stability estimate (4.15), we have

$$\|\boldsymbol{\delta}_h^p\|_{\mathrm{div},\rho_h} + \|\delta_h^u\|_0 \lesssim \rho^{\frac{1}{2}}(\|u_h^c\|_0 + \|\boldsymbol{p}_h^c\|_0) \lesssim \rho^{\frac{1}{2}}\|f\|_0. \tag{4.20}$$

This completes the proof. □

5 Conclusion

The extended Galerkin analysis, presented in this paper, is based on a unified 4-field formulation for a large class of numerical methods for solving second-order partial differential equations. Furthermore, we establish two types of uniform inf-sup conditions for the formulation, which naturally lead to uniform optimal error estimates for almost all the major finite element or Galerkin methods, such as HDG, WG and DG.

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