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A posteriori error estimates of finite element methods by preconditioning☆

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ABSTRACT

We present a framework that relates preconditioning with a posteriori error estimates in finite element methods. In particular, we use standard tools in subspace correction methods to obtain reliable and efficient error estimators. As a simple example, we recover the classical residual error estimators for the second order elliptic equations.

1. Introduction

Adaptive finite element methods (AFEMs) have been an active research area since the pioneering work [1]. In contrast to finite elements based on quasi-uniform meshes, AFEMs produce a sequence of locally refined grids that is able to resolve the singularity arising from irregular data in the underlying boundary value problems. Readers are referred to e.g., [2–4] for a thorough introduction. Among the key concepts in AFEMs, a posteriori error estimates are the building block for comparing errors on different elements and marking elements with large errors for refinement. For details on various AFEM error estimation techniques we refer to works on: explicit residual estimators [4]; implicit estimators based on local problems [1,5–7]; recovery-based estimators; [8–12]; hierarchical basis estimators [13–16]; functional estimators [17]; and equilibrated estimators [18–21].

On the other hand, parallel with the development of AFEMs, there are also substantial research efforts in studying efficient preconditioning, which is a technique for approximating the inverse of a differential operator. Usually, such approximations are aimed at accelerating Krylov subspace iterative methods for solving linear systems resulting from discretized partial differential equations. Popular techniques used for preconditioning include e.g., multigrid [22–27] and domain decomposition/subspace correction methods [28–30]. In practice, subspace correction methods provide an efficient way of reducing the condition number of a large-scale but finite-dimensional linear system. However, the analysis of uniform convergence rate of those methods often benefits from the general setting of infinite-dimensional Hilbert spaces (see, for example, [31–33]).

In this paper we present a general framework relating abstract operator preconditioning [30,32–35] to a posteriori error estimates. In particular, we shall show that such standard techniques for developing preconditioners also yield reliable and efficient error estimators. Here, for clarity of presentation, we focus on the symmetric and positive-definite problems although extensions to more general cases are definitely within reach. As a simple example, with this framework, we are able to recover the classical residual error estimators for elliptic equations in primal form.

The rest of this paper is organized as follows. In Section 2, we set up the model variational problem and define the operator notation which is convenient when constructing preconditioners. In Section 3, we develop the main theory on posteriori error estimates via preconditioning. Section 4 is devoted to the example of second order elliptic equation that illustrates the aforementioned abstract theory. Concluding remarks are found in Section 5.

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2. Preliminaries

Let V be a Hilbert space and V' denote the dual space of V. Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear form and $f \in V'$. We consider the following variational problem: Find $u \in V$ such that for all $v \in V$

$$a(u, v) = \langle f, v\rangle. \tag{2.1}$$

Here $\langle\cdot,\cdot\rangle$ is the duality pairing between V' and V. Let $\|\cdot\|_V$ denote the norm on V and $\|\cdot\|_{V'}$ the dual norm of V'. For simplicity, we assume that the bilinear form $a(\cdot,\cdot)$ is symmetric and positive-definite (SPD). The continuity and positive-definiteness of $a(\cdot,\cdot)$ imply

$$a(v, w) \le \overline{\alpha}\|v\|_V\|w\|_V, \tag{2.2a}$$

$$a(v, v) \ge \underline{\alpha}\|v\|_V^2, \tag{2.2b}$$

for all $v, w \in V$, where $\overline{\alpha}, \underline{\alpha} > 0$ are absolute constants. Such a bilinear form naturally defines a bounded isomorphism $A : V \to V'$ for which we have

$$\langle Av, w\rangle := a(v, w), \quad \forall v, w \in V.$$

Hence, (2.1) is equivalent to the operator equation

$$Au = f. \tag{2.3}$$

(2.2a) and (2.2b) imply that A induces the inner product $\langle A\cdot,\cdot\rangle$ on V. For all $v \in V$, the A-norm on V is defined as $\|v\|_A := \langle Av, v\rangle^{\frac{1}{2}}$, which is equivalent to the V-norm.

2.1. Approximation from a subspace

Let us consider a general case where we approximate the solution to (2.1) by restricting it to a subspace $V_h \subset V$, namely: Find $u_h \in V_h$ such that

$$a(u_h, v) = \langle f, v\rangle \text{ for all } v \in V_h. \tag{2.4}$$

Note that the subspace V_h does not even have to be finite dimensional, although it usually is in applications. It follows from (2.2a), (2.2b) and the well-known Lax–Milgram theorem that (2.4) admits a unique solution.

For such a subspace $V_h \subset V$, we consider the natural inclusion $I_h : V_h \hookrightarrow V$ and its adjoint $Q_h := I_h' : V' \to V_h'$ defined as

$$\langle Q_h g, v_h\rangle = \langle g, I_h v_h\rangle \quad \text{for all } g \in V' \text{ and } v_h \in V_h.$$

We introduce the operator $A_h := Q_h A I_h : V_h \to V_h'$ which approximates A on V_h. In this way, the discrete problem (2.4) reads

$$A_h u_h = Q_h f.$$

3. A posteriori error estimates by preconditioning

A posteriori error estimates are of the form

$$C_1\eta_h \le \|u - u_h\|_V \le C_2\eta_h,$$

where C_1, C_2 are absolute positive constants and η_h is computed from u_h. In AFEMs, η_h is the sum of error indicators on all elements. The local error indicators can be used to compare errors on different elements and those elements with large errors will be refined. In this way, the errors estimated by η_h are equidistributed over all elements in the mesh. The optimal computational complexity of AFEMs is often attributed to the aforementioned equidistribution of errors. Rigorous analysis of convergence and optimality of AFEMs can be found in e.g., [36–40].

3.1. Links with operator preconditioning

Let

$$e := u - u_h,$$

$$r := f - Au_h \in V'.$$

Clearly, from our discussion above, it follows that constructing a posteriori error estimators is equivalent to estimating a norm of the error $e = A^{-1}r$ by computable bounds. We note, however, that a direct computation of the norm of $A^{-1}r$ will be, in general, impossible or too expensive, since one needs to compute the action of A^{-1} on r. As we pointed out in the introduction, approximating such action has been also studied for several decades and is known as *preconditioning*.

Following this simple observation we now borrow some simple ideas from this field and apply them in constructing a posteriori error estimators.

First, we need a bounded isomorphism (the *preconditioner*) $B : V' \to V$, whose particular form will be given later. For the time being we only assume that B is bounded and SPD, i.e., $\langle\cdot, B\cdot\rangle$ is an inner product on V'. Let $S : V' \to V$ be a SPD operator, which we will refer to as "the *smoother*" and is such that its range approximates well the high frequency part of the range of A^{-1}, i.e., the result of the action Sr provides a good approximation to the high frequency components of the error. Now, a simple choice for B is

$$B := S + I_h A_h^{-1} Q_h,$$

which is known as *additive Schwarz preconditioner*. Just to simplify the presentation, we will not consider the multiplicative preconditioner in this paper although following the abstract framework developed in [32,33] similar results can also be obtained in the multiplicative case as well. Let $\underline{\beta}, \overline{\beta}$ be two positive absolute constants. We say that B is a preconditioner for A provided there exist constants $\underline{\beta} > 0$ and $\overline{\beta} < \infty$, such that

$$\underline{\beta}\langle B^{-1}v, v\rangle \le \langle Av, v\rangle \le \overline{\beta}\langle B^{-1}v, v\rangle, \quad \forall v \in V. \tag{3.1}$$

The inequality (3.1) is known as *spectral equivalence*, or *norm equivalence*, and is a common ingredient in the analysis of convergence of iterative methods for large-scale linear systems.

3.2. Estimating the residual

We now show that the norm (spectral) equivalence (3.1) naturally yields a two-sided estimate on $\|e\|_A$. This is the central result in this paper.

Theorem 3.1. *Let (3.1) hold. Then we have the following two sided bound*

$$\overline{\beta}^{-1}\langle r, Sr\rangle \le \|e\|_A^2 \le \underline{\beta}^{-1}\langle r, Sr\rangle.$$

Proof. Since A is SPD, we use the Cauchy–Schwarz inequality to obtain

$$\langle Ae, BAe\rangle^2 \le \langle Ae, e\rangle\langle ABAe, BAe\rangle. \tag{3.2}$$

The inequality (3.1) implies

$$\langle ABAe, BAe\rangle \le \overline{\beta}\langle B^{-1}BAe, BAe\rangle = \overline{\beta}\langle Ae, BAe\rangle. \tag{3.3}$$

Combining (3.2) and (3.3) yields

$$\langle r, Br\rangle = \langle Ae, BAe\rangle \le \overline{\beta}\langle Ae, e\rangle,$$

where we used $r = Ae$ in the first equality. The upper bound

$$\langle Ae, e\rangle \le \underline{\beta}^{-1}\langle r, Br\rangle$$

can be shown in a similar fashion. In summary, we have

$$\overline{\beta}^{-1}\langle r, Br\rangle \le \langle Ae, e\rangle \le \underline{\beta}^{-1}\langle r, Br\rangle. \tag{3.4}$$

On the other hand, for any $v_h \in V_h$, (2.4) implies

$$\langle Q_h r, v_h\rangle = \langle r, v_h\rangle = \langle f, v_h\rangle - \langle A_h u_h, v_h\rangle = 0,$$

i.e., $Q_h r = 0$. Hence,

$$Br = Sr + I_h A_h^{-1}(Q_h r) = Sr. \tag{3.5}$$

Combining (3.4) and (3.5) completes the proof. □

Throughout the rest of this paper, $\langle r, Sr\rangle$ will serve as a (nearly) computable a posteriori error estimator that is proved to be both an upper and lower bound of the error $\|e\|_A$. In order to derive an error estimator within our framework, the key step is to suitably select the smoother S such that the spectral equivalence (3.1) holds.

3.3. Additive Schwarz smoother

In this subsection, we construct a particular S using the additive Schwarz method. For such a smoother, we present a lemma that serves as a criterion for verifying (3.1).

For $n \in \mathbb{N}$, $1 \le k \le n$, let $V_k \subset V$ be subspaces providing a decomposition of V, namely,

$$V = \sum_{k=1}^{n} V_k. \tag{3.6}$$

Let $I_k : V_k \hookrightarrow V$ be the natural inclusion and $Q_k : V' \to V_k'$ denote its adjoint. We further set $A_k := Q_k A I_k$. Next, let $S_k : V_k' \to V_k$ be spectrally equivalent to A_k^{-1}. More precisely, for $1 \le k \le n$ and $v_k \in V_k$, we assume that

$$\underline{\gamma}\langle S_k^{-1} v_k, v_k\rangle \le \langle A_k v_k, v_k\rangle \le \overline{\gamma}\langle S_k^{-1} v_k, v_k\rangle, \tag{3.7}$$

where $\underline{\gamma}, \overline{\gamma}$ are positive absolute constants. The smoother S (additive Schwarz method) is then defined to be

$$S := \sum_{k=1}^{n} I_k S_k Q_k.$$

By the definition of B, we obtain

$$B = I_h A_h^{-1} Q_h + \sum_{k=1}^{n} I_k S_k Q_k.$$

The norm of B can be estimated using the following lemma, which can be found in e.g., [29,32,33,41].

Lemma 3.2. *We have the following identity*

$$\langle B^{-1} v, v\rangle = \inf_{v_h + \sum_{k=1}^{n} v_k = v} \langle A_h v_h, v_h\rangle + \sum_{k=1}^{n} \langle S_k^{-1} v_k, v_k\rangle,$$

where the infimum is taken over $v_h \in V_h$ and $v_k \in V_k$ for $1 \le k \le n$.

The proof that B is a good preconditioner for A is standard. We include it here for completeness and we follow the proof in [33].

Lemma 3.3. *For each k, let*

$$\mathcal{M}(k) := \{j : \sup_{v_j \in V_j, v_k \in V_k} a(v_j, v_k) \neq 0\},$$

and $M := \max_{1 \le k \le n} \#\mathcal{M}(k)$. In addition, assume that for all $v \in V$, there exist $v_h \in V_h$ and $v_k \in V_k$ with $1 \le k \le n$ satisfying

$$\|v_h\|_A^2 + \sum_{k=1}^{n} \|v_k\|_A^2 \le C_{stab} \|v\|_A^2, \quad v = v_h + \sum_{k=1}^{n} v_k. \tag{3.8}$$

Then (3.1) holds with constants $\overline{\beta} = 2\max(1, \overline{\gamma} M)$, $\underline{\beta} = \min(1, \underline{\gamma}) C_{stab}^{-1}$.

Proof. For $v \in V$, assume the decomposition $v = v_h + \sum_{k=1}^{n} v_k$ with $v_h \in V_h$, $v_k \in V_k$. Direct calculation shows that

$$\begin{aligned}\|v\|_A^2 &\le 2\|v_h\|_A^2 + 2\left\|\sum_{k=1}^{n} v_k\right\|_A^2 \\ &= 2\|v_h\|_A^2 + 2\sum_{j,k=1}^{n} \langle A v_j, v_k\rangle.\end{aligned} \tag{3.9}$$

The definition of $\mathcal{M}(k)$ and M implies

$$\begin{aligned}\sum_{j,k=1}^{n} \langle A v_j, v_k\rangle &= \sum_{k=1}^{n} \sum_{j \in \mathcal{M}(k)} a(v_j, v_k) \\ &\le \frac{1}{2} \sum_{k=1}^{n} \sum_{j \in \mathcal{M}(k)} \|v_j\|_A^2 + \|v_k\|_A^2 \le M \sum_{k=1}^{n} \|v_k\|_A^2.\end{aligned}$$

Combining the previous estimate with (3.9) and (3.7) gives

$$\begin{aligned}\|v\|_A^2 &\leq 2\langle A_h v_h, v_h\rangle + 2M\sum_{k=1}^{n}\langle A v_k, v_k\rangle \\ &\leq 2\max(1,\overline{\gamma}M)\left(\langle A_h v_h, v_h\rangle + \sum_{k=1}^{n}\langle S_k^{-1} v_k, v_k\rangle\right).\end{aligned} \tag{3.10}$$

Taking the infimum with respect to all decompositions and using Lemma 3.2, we obtain the upper bound

$$\|v\|_A^2 \leq 2\max(1,\overline{\gamma}M)\langle B^{-1}v, v\rangle.$$

For the lower bound in (3.1), let $v = v_h + \sum_{k=1}^{n} v_k$ be the decomposition that satisfies (3.8). It then follows from (3.7) and (3.8) that

$$\begin{aligned}\langle A_h v_h, v_h\rangle + \sum_{k=1}^{n}\langle S_k^{-1} v_k, v_k\rangle &\leq \|v_h\|_A^2 + \sum_{k=1}^{n}\underline{\gamma}^{-1}\langle A_k v_k, v_k\rangle \\ &\leq \max(1,\underline{\gamma}^{-1})\left(\|v_h\|_A^2 + \sum_{k=1}^{n}\|v_k\|_A^2\right) \leq \max(1,\underline{\gamma}^{-1})C_{\mathrm{stab}}\|v\|_A^2.\end{aligned}$$

Using the previous estimate and Lemma 3.2, we obtain

$$\langle B^{-1}v, v\rangle \leq \max(1,\underline{\gamma}^{-1})C_{\mathrm{stab}}\|v\|_A^2.$$

The proof is complete. □

4. Examples

In this section, we consider the typical example of a scalar elliptic equation. Let $V = H_0^1(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is a Lipschitz polytope. For a given $f \in L^2(\Omega)$ and $K \in [W_\infty^1(\Omega)]^{d\times d}$, the bilinear and linear forms in Eq. (2.1) are:

$$a(u, v) := \int_\Omega K\nabla u \cdot \nabla v dx, \quad \langle f, v\rangle := \int_\Omega f v dx.$$

In addition, we assume K is piecewise constant and uniformly elliptic, i.e.,

$$\underline{\alpha}|\xi|^2 \leq \xi^T K(x)\xi \leq \overline{\alpha}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, x \in \Omega.$$

Hence, (2.2a) and (2.2b) hold.

Let $\mathcal{T}_h$ be a conforming and shape-regular simplicial partition of Ω aligned with discontinuities of K. Let $\mathcal{P}_p(D)$ denote the set of polynomials of degree at most p on a domain D. The subspace $V_h \subset V$ is

$$V_h =: \{v_h \in V : v_h|_T \in \mathcal{P}_p(T) \text{ for all } T \in \mathcal{T}_h\},$$

where $p \geq 1$ is an integer.

Let $\{x_k\}_{k=1}^n$ denote the set of vertices in $\mathcal{T}_h$. For each x_k, let ϕ_k denote the continuous piecewise linear function that takes the value 1 at x_k and 0 at other vertices. Furthermore, we denote $\Omega_k := \operatorname{supp}\phi_k$ for $1 \leq k \leq n$. Obviously we have

$$\Omega = \bigcup_{k=1}^{n}\Omega_k, \quad \sum_{k=1}^{n}\phi_k(x) = 1, \tag{4.1}$$

$$\|\nabla\phi_k\|_{L^\infty(\Omega)} \eqsim h_k^{-1} := (\operatorname{diam}\Omega_k)^{-1}. \tag{4.2}$$

4.1. A posteriori error estimates for Lagrange elements

Now, let $V_k = H_0^1(\Omega_k)$ which is a subspace of $V = H_0^1(\Omega)$ by zero extension. The partition of unity (4.1) implies

$$V = \sum_{k=1}^{n} V_k.$$

We note that the framework also works for other local patches, as long as their union covers Ω.

For a fixed k, the set $\mathcal{M}(k)$ defined in Lemma 3.3 translates into

$$\mathcal{M}(k) = \{j : \Omega_k \cap \Omega_j \neq \emptyset\}.$$

In this case, $M = \max_{1\leq k\leq n}\#\mathcal{M}(k)$ is an absolute constant by the shape-regularity of $\mathcal{T}_h$.

Throughout the rest of this paper, we adopt the notation $C_1 \lesssim C_2$ provided $C_1 \leq C_3 C_2$ with C_3 being a generic constant dependent only on K and M. We say $C_1 \approx C_2$ provided $C_1 \lesssim C_2$ and $C_2 \lesssim C_1$. Given an element T and a face e, let h_T and h_e denote the diameter of T and e, respectively. The shape-regularity of $\mathcal{T}_h$ implies that $h_k \approx h_T \approx h_e$ if $x_k \in \overline{T} \cap \overline{e}$ and we will use these notions interchangeably.

We set $S_k := A_k^{-1}$ and thus $\overline{\gamma} = \underline{\gamma} = 1$ in (3.7). The corresponding smoother S yields an error estimator. In order to show the reliability and efficiency, we need to verify (3.8) in Lemma 3.3.

Corollary 4.1. *We have the following estimate*

$$\|e\|_A^2 \approx \sum_{k=1}^n \langle Q_k r, A_k^{-1} Q_k r\rangle.$$

Proof. To verify (3.8), we take $v_h = \Pi_h v \in V_h$, where Π_h is a H^1-stable interpolation which also enjoys standard approximation properties:

$$|\Pi_h v|_{H^1(\Omega)}^2 + \sum_{k=1}^n h_k^{-2} \|v - \Pi_h v\|_{L^2(\Omega_k)}^2 + |v - \Pi_h v|_{H^1(\Omega_k)}^2 \lesssim |v|_{H^1(\Omega)}^2. \tag{4.3}$$

A simple choice for Π_h is the Clément interpolation [42]. We now set $v_k = \phi_k(v - \Pi_h v)$. Hence, $v = v_h + \sum_{k=1}^n v_k$ is a decomposition. It follows from (4.2) and (4.3) that

$$\begin{aligned}
\|v_h\|_A^2 + \sum_{k=1}^n \|v_k\|_A^2 &\approx |\Pi_h v|_{H^1(\Omega)}^2 + \sum_{k=1}^n |\phi_k(v - \Pi_h v)|_{H^1(\Omega_k)}^2 \\
&\lesssim |\Pi_h v|_{H^1(\Omega)}^2 + \sum_{k=1}^n h_k^{-2} \|v - \Pi_h v\|_{L^2(\Omega_k)}^2 + |v - \Pi_h v|_{H^1(\Omega_k)}^2 \\
&\lesssim |v|_{H^1(\Omega)}^2 \lesssim \|v\|_A^2.
\end{aligned}$$

Hence, (3.8) are verified. Finally, we conclude Corollary 4.1 from Theorem 3.1 and Lemma 3.3. □

For $\varphi \in V_k$, we have

$$\langle Q_k r, \varphi\rangle = \int_{\Omega_k} f\varphi dx - a(u_h, \varphi).$$

Hence, computing $\eta_k := A_k^{-1} Q_k r \in V_k$ amounts to solving the variational problem:

$$a(\eta_k, \varphi) = \int_{\Omega_k} f\varphi dx - a(u_h, \varphi), \quad \forall \varphi \in V_k. \tag{4.4}$$

Taking $\varphi = \eta_k$ in (4.4) implies that

$$\|\eta_k\|_A^2 = \langle Q_k r, A_k^{-1} Q_k r\rangle.$$

It then follows from the previous identity and Corollary 4.1 that

$$\|e\|_A^2 \approx \sum_{k=1}^n \|\eta_k\|_A^2. \tag{4.5}$$

4.2. Computable error estimator

Unfortunately, $\|\eta_k\|_A$ is not available in practice because (4.4) is local but still not fully computable. To implement the estimator in Corollary 4.1, we consider the approximate problem: Find $\widetilde{\eta}_k \in \widetilde{V}_k$ such that

$$a(\widetilde{\eta}_k, \varphi) = \int_{\Omega_k} f\varphi dx - a(u_h, \varphi), \quad \forall \varphi \in \widetilde{V}_k, \tag{4.6}$$

where $\widetilde{V}_k \subset V_k$ is a subspace of piecewise polynomials. Roughly speaking, the approximate estimator $\left(\sum_{k=1}^n \|\widetilde{\eta}_k\|_A^2\right)^{\frac{1}{2}}$ is expected to be an accurate upper and lower bound of $\|e\|_A$ provided the degree of piecewise polynomials in $\widetilde{V}_k$ is sufficiently high.

Taking $\varphi = \widetilde{\eta}_k$ in (4.6) and (4.4), we obtain

$$\|\widetilde{\eta}_k\|_A^2 = a(\eta_k, \widetilde{\eta}_k) \leq \|\eta_k\|_A \|\widetilde{\eta}_k\|_A.$$

Combining the previous estimate with (4.5) provides the following computable lower bound for the error

$$\sum_{k=1}^{n}\|\widetilde{\eta}_k\|_A^2 \leq \sum_{k=1}^{n}\|\eta_k\|_A^2 \lesssim \|e\|_A^2. \tag{4.7}$$

To derive a computable upper bound for $\|e\|_A$, let us first write the action of the residual on $V_k = H_0^1(\Omega_k)$. We denote the set of all $(d-1)$-dimensional faces in the triangulation $\mathcal{T}_h$ by $\mathcal{E}_h$. Clearly, $\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^\partial$ where $\mathcal{E}_h^\partial$ denotes the set of all boundary faces and $\mathcal{E}^o$ the interior faces. We further denote $\mathcal{T}_h|_{\Omega_k}$ by $\mathcal{T}_k$ and $\mathcal{E}_h|_{\mathring{\Omega}_k}$ by $\mathcal{E}_k$, respectively. Note that $\mathcal{E}_k$ does not include the faces on $\partial\Omega_k$. For each $T \in \mathcal{T}_h$, let

$$r_T := (f + \operatorname{div} K\nabla u_h)|_T.$$

Further, for each $e \in \mathcal{E}_h^o$, let $T_1, T_2 \in \mathcal{T}_h$ be the two elements sharing e, n_1 (resp. n_2) the outward unit normal to ∂K_1 (resp. ∂K_2), and

$$r_e := K\nabla u_h|_{T_1}\cdot n_1 + K\nabla u_h|_{T_2}\cdot n_2.$$

It then follows from (4.4) and integration by parts that

$$a(\eta_k, \varphi) = \sum_{T\in\mathcal{T}_k}\int_T r_T\varphi dx + \sum_{e\in\mathcal{E}_k}\int_e r_e\varphi ds, \quad \forall \varphi \in V_k. \tag{4.8}$$

Now, let us introduce the computable quantity

$$\zeta_k := \left(\sum_{T\in\mathcal{T}_k} h_T^2\|r_T\|_{L^2(T)}^2 + \sum_{e\in\mathcal{E}_k} h_e\|r_e\|_{L^2(e)}^2\right)^{\frac{1}{2}},$$

which is the standard explicit residual error estimator. We take $\varphi = \eta_k$ and use (4.8) and the Cauchy–Schwarz inequality to obtain that

$$\begin{aligned}\|\eta_k\|_A^2 = a(\eta_k,\eta_k) &= \sum_{T\in\mathcal{T}_k}\int_T r_T\eta_k dx + \sum_{e\in\mathcal{E}_k}\int_e r_e\eta_k ds\\ &\leq \sum_{T\in\mathcal{T}_k}\|\eta_k\|_{L^2(T)}\|r_T\|_{L^2(T)} + \sum_{e\in\mathcal{E}_k}\|\eta_k\|_{L^2(e)}\|r_e\|_{L^2(e)}\\ &\leq \zeta_k\left(\sum_{T\in\mathcal{T}_k} h_T^{-2}\|\eta_k\|_{L^2(T)}^2 + \sum_{e\in\mathcal{E}_k} h_e^{-1}\|\eta_k\|_{L^2(e)}^2\right)^{\frac{1}{2}}.\end{aligned}$$

Finally, combining the previous inequality with the trace inequality and the Poincaré inequality $\|\eta_k\|_{L^2(\Omega_k)} \lesssim h_k\|\nabla\eta_k\|_{L^2(\Omega_k)}$ yields

$$\|\eta_k\|_A^2 \lesssim \zeta_k\left(h_k^{-2}\|\eta_k\|_{L^2(\Omega_k)}^2 + \|\nabla\eta_k\|_{L^2(\Omega_k)}^2\right)^{\frac{1}{2}} \lesssim \zeta_k\|\eta_k\|_A.$$

Hence, using (4.5) and the previous inequality, we obtain the following computable upper bound for the error

$$\|e\|_A^2 \eqsim \sum_{k=1}^{n}\|\eta_k\|_A^2 \lesssim \sum_{k=1}^{n}\zeta_k^2. \tag{4.9}$$

So far the finite element error $\|e\|_A$ is estimated from below and above by two different estimators. To show that either $\left(\sum_{k=1}^n\|\widetilde{\eta}_k\|_A^2\right)^{\frac{1}{2}}$ or $\left(\sum_{k=1}^n\zeta_k^2\right)^{\frac{1}{2}}$ is a *two-sided* bound of $\|e\|_A$, we use the bubble function technique due Verfürth, which seems to be indispensable tool in deriving such estimates. To keep the presentation self-contained as much as possible, we give the details of deriving the two-sided estimates below and we note that such arguments are standard, see, e.g., [43]. For each $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$, the volume and face bubble functions are defined as

$$\phi_T := \prod_{x_k\in\overline{T}}\phi_k, \quad \phi_e := \prod_{x_k\in\overline{e}}\phi_k,$$

respectively. Let Ω_e denote the union of elements sharing e as a face. We note that $\|\phi_T\|_{L^\infty(T)} \eqsim 1$, $\|\phi_e\|_{L^\infty(\Omega_e)} \eqsim 1$, and $\operatorname{supp}\phi_T \subseteq T$, $\operatorname{supp}\phi_e \subseteq \Omega_e$. Given an integer $m \geq 0$, it is well-known (see [43]) that for $v \in \mathcal{P}_m(T)$ and $w \in \mathcal{P}_m(e)$, we have the following estimates:

$$\|\phi_T v\|_T \lesssim \|v\|_T \lesssim \|\phi_T^{\frac{1}{2}} v\|_T, \tag{4.10a}$$

$$\|\phi_e w\|_e \lesssim \|w\|_e \lesssim \|\phi_e^{\frac{1}{2}} w\|_e, \tag{4.10b}$$

$$\|E_e w\|_{\Omega_e} \eqsim h_e^{\frac{1}{2}} \|w\|_e, \tag{4.10c}$$

where $E_e w \in \mathcal{P}_m(\Omega_e)$ is an extension of w, such that $(E_e w)|_e = w$.

To show that $\sum_{k=1}^n \|\widetilde{\eta}_k\|_A^2$ is an upper bound for $\|e\|_A^2$, we take $\widetilde{V}_k$ in (4.6) as

$$\widetilde{V}_k := \sum_{T\in\mathcal{T}_k} \phi_T \mathcal{P}_{p-1}(\Omega_k) + \sum_{e\in\mathcal{E}_k} \phi_e \mathcal{P}_{p-1}(\Omega_k),$$

which is clearly a subspace of V_k. Similarly to (4.8), one can rewrite (4.6) as

$$a(\widetilde{\eta}_k, \varphi) = \sum_{T\in\mathcal{T}_k} \int_T r_T \varphi dx + \sum_{e\in\mathcal{E}_k} \int_e r_e \varphi ds, \quad \forall \varphi \in \widetilde{V}_k. \tag{4.11}$$

Let Q_T denote the L^2-projection onto $\mathcal{P}_{p-1}(T)$. Using (4.10a), (4.11) with $\varphi = \phi_T Q_T r_T \in \widetilde{V}_k$, the Cauchy–Schwarz and inverse inequalities, we have

$$\begin{aligned}
\|Q_T r_T\|_T^2 &\lesssim \int_T r_T \varphi dx + \int_T (Q_T r_T - r_T)\varphi dx \\
&= a(\widetilde{\eta}_k, \varphi) + \int_T (Q_T r_T - r_T)\varphi dx \\
&\lesssim h_T^{-1} \|\widetilde{\eta}_k\|_A \|\varphi\|_T + \|Q_T r_T - r_T\|_T \|\varphi\|_T \\
&\lesssim \big(h_T^{-1} \|\widetilde{\eta}_k\|_A + \|Q_T r_T - r_T\|_T\big) \|Q_T r_T\|_T.
\end{aligned} \tag{4.12}$$

It then follows from (4.12), (4.10a), and $(\mathrm{id} - Q_T)(\mathrm{div} K \nabla u_h) = 0$ that

$$\begin{aligned}
\|r_T\|_T &\le \|Q_T r_T\|_T + \|r_T - Q_T r_T\|_T \\
&\lesssim h_T^{-1} \|\widetilde{\eta}_k\|_A + \|f - Q_T f\|_T.
\end{aligned} \tag{4.13}$$

On the other hand, taking $\varphi = \phi_e E_e r_e \in \widetilde{V}_k$ in (4.11) and using (4.10b), (4.10c), we have

$$\begin{aligned}
\|r_e\|_e^2 &\lesssim \int_e r_e \varphi ds = a(\widetilde{\eta}_k, \varphi) - \sum_{T\in\mathcal{T}_k, T\subset\Omega_e} \int_T r_T \varphi dx \\
&\lesssim h_e^{-1} \|\widetilde{\eta}_k\|_A \|\varphi\|_{\Omega_e} + \sum_{T\in\mathcal{T}_k, T\subset\Omega_e} \|r_T\|_T \|\varphi\|_T \\
&\lesssim \Big(h_e^{-\frac{1}{2}} \|\widetilde{\eta}_k\|_A + \sum_{T\in\mathcal{T}_k, T\subset\Omega_e} h_T^{\frac{1}{2}} \|r_T\|_T\Big) \|r_e\|_e.
\end{aligned} \tag{4.14}$$

Hence, combining (4.9), (4.13), (4.14) and using the shape regularity of $\mathcal{T}_h$, we obtain the computable upper bound based on $\widetilde{\eta}_k$:

$$\|e\|_A^2 \lesssim \sum_{k=1}^n \|\zeta_k\|_A^2 \lesssim \sum_{k=1}^n \|\widetilde{\eta}_k\|_A^2 + \mathrm{osc}_{\mathcal{T}_h}(f)^2,$$

where $\mathrm{osc}_{\mathcal{T}_h}(f) := \big(\sum_{T\in\mathcal{T}_h} h_T^2 \|f - Q_T f\|_T^2\big)^{\frac{1}{2}}$ is called the data oscillation in the literature. Compared with $\|e\|_A$, the quantity $\mathrm{osc}_{\mathcal{T}_h}(f)$ is a higher order term provided f is piecewise smooth.

Similarly, using (4.8) with $\varphi = \phi_T Q_T r_T$, $\varphi = \phi_e E_e r_e$, and (4.5) we obtain

$$\sum_{k=1}^n \|\zeta_k\|_A^2 \lesssim \sum_{k=1}^n \|\eta_k\|_A^2 + \mathrm{osc}_{\mathcal{T}_h}(f)^2 \lesssim \|e\|_A^2 + \mathrm{osc}_{\mathcal{T}_h}(f)^2,$$

which is a lower bound based on ζ_k.

5. Concluding remarks

For SPD problems, we have shown how preconditioning can be used to derive a posteriori error estimates. Extensions of this abstract theoretical framework and its application to derive estimators for indefinite, nonconforming, and discontinuous Galerkin methods are ongoing. A close inspection of the arguments shows that not only preconditioning can give a unified way to derive a posteriori error estimators. This is a two-way street: the a posteriori error estimators may provide efficient smoothers for multilevel methods. For example, the operator S we have introduced in our framework is a clear analogue of smoothing (relaxation) operator. We hope that some of the error indicators and estimators may give efficient smoothers in case of non-symmetric and or indefinite problems which are, in general, hard to precondition.

References

[1] I. Babuška, W.C. Rheinboldt, Error estimates for adaptive finite element computations, SIAM J. Numer. Anal. 15 (4) (1978) 736–754.
[2] Ivo Babuška, Theofanis Strouboulis, The Finite Element Method and Its Reliability, in: Numerical Mathematics and Scientific Computation, The Clarendon Press, Oxford University Press, New York, 2001.
[3] Ricardo H. Nochetto, Kunibert G. Siebert, Andreas Veeser, Theory of Adaptive Finite Element Methods: An Introduction, Springer, Berlin, 2009, pp. 409–542.
[4] Rüdiger Verfürth, A Posteriori Error Estimation Techniques for Finite Element Methods, in: Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2013.
[5] R.E. Bank, A. Weiser, Some a posteriori error estimators for elliptic partial differential equations, Math. Comp. 44 (170) (1985) 283–301.
[6] Carsten Carstensen, Stefan A. Funken, Fully reliable localized error control in the FEM, SIAM J. Sci. Comput. 21 (4) (1999/00) 1465–1484.
[7] Pedro Morin, Ricardo H. Nochetto, Kunibert G. Siebert, Local problems on stars: a posteriori error estimators, convergence, and performance, Math. Comp. 72 (243) (2003) 1067–1097.
[8] O.C. Zienkiewicz, J. Zhu, The superconvergent patch recovery and a posteriori error estimates. II. error estimates and adaptivity, Internat. J. Numer. Methods Engrg. 33 (7) (1992) 1365–1382.
[9] Carsten Carstensen, Sören Bartels, Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. I. Low order conforming, nonconforming, and mixed FEM, Math. Comp. 71 (239) (2002) 945–969.
[10] Randolph E. Bank, Jinchao Xu, Asymptotically exact a posteriori error estimators. I. Grids with superconvergence, SIAM J. Numer. Anal. 41 (6) (2003) 2294–2312.
[11] Randolph E. Bank, Jinchao Xu, Asymptotically exact a posteriori error estimators. II. General unstructured grids, SIAM J. Numer. Anal. 41 (6) (2003) 2313–2332.
[12] Zhimin Zhang, Ahmed Naga, A new finite element gradient recovery method: superconvergence property, SIAM J. Sci. Comput. 26 (4) (2005) 1192–1213.
[13] Randolph E. Bank, Hierarchical bases and the finite element method, in: Acta Numerica, 1996, in: Acta Numer., vol. 5, Cambridge Univ. Press, Cambridge, 1996, pp. 1–43.
[14] Randolph E. Bank, R. Kent Smith, A posteriori error estimates based on hierarchical bases, SIAM J. Numer. Anal. 30 (4) (1993) 921–935.
[15] Hengguang Li, Jeffrey S. Ovall, A posteriori error estimation of hierarchical type for the Schrödinger operator with inverse square potential, Numer. Math. 128 (4) (2014) 707–740.
[16] Harri Hakula, Michael Neilan, Jeffrey S. Ovall, A posteriori estimates using auxiliary subspace techniques, J. Sci. Comput. 72 (1) (2017) 97–127.
[17] Sergey Repin, A Posteriori Estimates for Partial Differential Equations, in: Radon Series on Computational and Applied Mathematics, vol. 4, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.
[18] Mark Ainsworth, J. Tinsley Oden, A Posteriori Error Estimation in Finite Element Analysis, in: Pure and Applied Mathematics (New York), Wiley-Interscience [John Wiley & Sons], New York, 2000.
[19] R. Luce, B.I. Wohlmuth, A local a posteriori error estimator based on equilibrated fluxes, SIAM J. Numer. Anal. 42 (4) (2004) 1394–1414.
[20] Dietrich Braess, Joachim Schöberl, Equilibrated residual error estimator for edge elements, Math. Comp. 77 (262) (2008) 651–672.
[21] Alexandre Ern, Martin Vohralík, Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous galerkin, and mixed discretizations, SIAM J. Numer. Anal. 53 (2) (2015) 1058–1081.
[22] U. Trottenberg, C.W. Oosterlee, A. Schüller, Multigrid, Academic Press, Inc., San Diego, CA, 2001, With contributions by A. Brandt, P. Oswald and K. Stüben.
[23] Achi Brandt, Multi-level adaptive technique (MLAT) for fast numerical solution to boundary value problems, in: Proc. 3rd Internat. Conf. on Numerical Methods in Fluid Mechanics, Paris, 1972, in: Lecture Notes in Physics, vol. 18, Springer-Verlag, Berlin, New York, 1973, pp. 82–89.
[24] W. Hackbusch, A fast iterative method for solving Poisson's equation in a general region, in: Numerical Treatment of Differential Equations (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1976), in: Lecture Notes in Math., vol. 631, Springer, Berlin, 1978, pp. 51–62, Longer version: Ein Iteratives Verfahren zur Schnellen Auflösung Elliptischer Randwertprobleme, Math. Inst., Universität zu Köln, Report 76-12 (November 1976).
[25] Wolfgang Hackbusch, Multigrid Methods and Applications, in: Springer Series in Computational Mathematics, vol. 4, Springer-Verlag, Berlin, 1985.
[26] Harry Yserentant, Old and new convergence proofs for multigrid methods, in: Acta Numerica, 1993, in: Acta Numer., Cambridge Univ. Press, Cambridge, 1993, pp. 285–326.
[27] Jinchao Xu, Ludmil Zikatanov, Algebraic multigrid methods, Acta Numer. 26 (2017) 591–721.
[28] Maksymilian Dryja, Olof B. Widlund, An Additive Variant of the Schwarz Alternating Method for the Case of Many Subregions, Technical Report 339, Department of Computer Science, Courant Institute, 1987, also Ultracomputer Note 131.
[29] Andrea Toselli, Olof Widlund, Domain Decomposition Methods—Algorithms and Theory, in: Springer Series in Computational Mathematics, vol. 34, Springer-Verlag, Berlin, 2005.
[30] Jinchao Xu, Iterative methods by space decomposition and subspace correction, SIAM Rev. 34 (4) (1992) 581–613.
[31] S.V. Nepomnyaschikh, Decomposition and fictitious domains methods for elliptic boundary value problems, in: Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, Norfolk, VA, 1991, SIAM, Philadelphia, PA, 1992, pp. 62–72.
[32] M. Griebel, P. Oswald, On the abstract theory of additive and multiplicative Schwarz algorithms, Numer. Math. 70 (2) (1995) 163–180.
[33] Jinchao Xu, Ludmil Zikatanov, The method of alternating projections and the method of subspace corrections in Hilbert space, J. Amer. Math. Soc. 15 (3) (2002) 573–597.
[34] D. Loghin, A.J. Wathen, Analysis of preconditioners for saddle-point problems, SIAM J. Sci. Comput. 25 (6) (2004) 2029–2049 (electronic).
[35] Kent-Andre Mardal, Ragnar Winther, Preconditioning discretizations of systems of partial differential equations, Numer. Linear Algebra Appl. 18 (1) (2011) 1–40.
[36] Willy Dörfler, A convergent adaptive algorithm for Poisson's equation, SIAM J. Numer. Anal. 33 (3) (1996) 1106–1124.
[37] Pedro Morin, Ricardo H. Nochetto, Kunibert G. Siebert, Data oscillation and convergence of adaptive FEM, SIAM J. Numer. Anal. 38 (2) (2000) 466–488.
[38] Peter Binev, Wolfgang Dahmen, Ron DeVore, Adaptive finite element methods with convergence rates, Numer. Math. 97 (2) (2004) 219–268.
[39] Rob Stevenson, Optimality of a standard adaptive finite element method, Found. Comput. Math. 7 (2) (2007) 245–269.

[40] J. Manuel Cascon, Christian Kreuzer, Ricardo H. Nochetto, Kunibert G. Siebert, Quasi-optimal convergence rate for an adaptive finite element method, SIAM J. Numer. Anal. 46 (5) (2008) 2524–2550.
[41] Susanne C. Brenner, L. Ridgway Scott, The Mathematical Theory of Finite Element Methods, Vol. 15, third ed., in: Texts in Applied Mathematics, vol. 35, Springer, New York, 2008.
[42] Ph. Clément, Approximation by finite element functions using local regularization, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. 9 (R-2) (1975) 77–84.
[43] Rüdiger Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, in: Advances in Numerical Mathematics, Wiley-Teubner, 1996.