

Residual-based a posteriori error estimates of mixed methods for a three-field Biot's consolidation model

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We present residual-based a posteriori error estimates of mixed finite element methods for the three-field formulation of Biot's consolidation model. The error estimator is an upper and lower bound of the space time discretization error up to data oscillation. As a by-product, we also obtain new a posteriori error estimate of mixed finite element methods for the heat equation.

Keywords: Biot's consolidation model, a posteriori error estimates, reliability, efficiency, heat equation

1. Introduction

The mathematical modeling of poro-elastic materials is aimed at describing the interactions between the deformation and fluid flow in a fluid-saturated porous medium. In this paper we provide a posteriori error estimators for the fully discrete, time dependent Biot's consolidation model for poroelastic media. A pioneering model of poroelasticity in one-dimensional setting was given in Terzaghi (1943). Nowadays, the popular formulations are in three-dimensions and they follow the model by Maurice Biot in several works, e.g., Biot (1941, 1955). The system of partial differential equations describing the Biot's consolidation model has a great deal of applications in geomechanics, petroleum engineering, and biomechanics.

The two-field formulation of Biot's consolidation model is classical and has been investigated in e.g., Ženíšek (1984); Showalter (2000); Murad *et al.* (1996); Ern & Meunier (2009). Three-field formulations, which include an unknown Darcy velocity, several conforming and non-conforming discretizations involving Stokes-stable finite-element spaces have been recently proposed as a viable approach for discretization of the Biot's model. Various three field formulations were considered in Phillips & Wheeler (2007a,b) with and a priori error estimates are presented in such a work. Recently, three-field formulation using Stokes stable elements, based on displacement, pressure, and total pressure was proposed and analysed in Oyarzúa & Ruiz-Baier (2016). A nonconforming discretization, which also provides element-wise mass conservation, is found in Hu *et al.* (2017). Parameter robust analysis using three field discontinuous Galerkin formulation is given in Hong & Kraus (2018), where a general theory for the a priori error analysis was introduced. Other stable discretizations and solvers are presented in e.g., Lee (2016); Lee *et al.* (2017); Rodrigo *et al.* (2018). Readers are referred to Lee (2016) for parameter robust error analysis for four- and five-field formulations. Finite volume and finite difference discretizations have also been used in this field and we point to Gaspar *et al.* (2003, 2006); Nordbotten (2016) for more results and references on such methods for Biot's system. We note that our further considerations are restricted to the finite element method and we will not discuss finite difference and

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finite volumen methods here.

There are a few works on a posteriori error control for the fully discretized time-dependent problem, see, e.g., Eriksson & Johnson (1991, 1995); Picasso (1998); Verfürth (2003); Makridakis & Nochetto (2003); Lakkis & Makridakis (2006); Ern & Vohralík (2015); Ern *et al.* (2019) for a posteriori error estimates of the primal formulation of the heat equation. A posteriori error estimation of the mixed formulation of the heat equation can be found in e.g., Cascón *et al.* (2006); Ern & Vohralík (2010); Larson & Målqvist (2011); Memon *et al.* (2012); Kim *et al.* (2018). For the classical two-field formulation in Biot's consolidation model, residual, equilibrated, and functional error estimators are derived in Ern & Meunier (2009); Riedlbeck *et al.* (2017); Kumar *et al.* (2018). In addition, equilibrated error estimators are developed in Ahmed *et al.* (2019, 2020) for the four- and five-field formulations and the fixed stress splitting scheme. Comparing to the equilibrated error indicators, residual error estimators are simpler to implement and do not require solving auxiliary problems on local patches. Several space-time adaptive algorithms based on residual error estimators are proven to be convergent, see, e.g., Chen & Feng (2004); Kreuzer *et al.* (2012); Gaspoz *et al.* (2019).

A main result in our paper is the construction of the reliable residual-based a posteriori error estimator for the three field Biot's system. To the best of our knowledge, there are no such error estimators for the mixed formulations of the Biot's model using more than two fields. Formulations using more than two fields have conservation properties which makes them practically interesting, however, their analysis is more challenging. In this paper, we derive residual a posteriori error estimates for the three-field formulation and prove that the estimator is reliable, that is, it provides an upper bound of the space-time error in the natural variational norm. Since the three-field formulation directly approximates the flux $\mathbf{w} \in H(\operatorname{div}, \Omega)$, special attention must be paid to energy estimates and the residual in the dual space $H(\operatorname{div}, \Omega)'$, which is a major obstacle in the construction of such error estimators. The analysis presented here with the help of regular decomposition and commuting quasi-interpolations, however, successfully tackles such problems, see Theorems 3.1 and 3.2 for details.

Another main result of this paper is the lower bound in Theorem 5.1. As far as we know, existing residual, equilibrated, and functional error estimators in Biot's consolidation model are not shown to be lower bounds of the space-time discretization error. This is partly due to the complexity of the Biot's model equations. Motivated by Verfürth's technique introduced in Verfürth (2003), we split the residual and estimator into space and time parts. The temporal estimator can be controlled by the spatial estimator and discretization error, while the spatial estimator is in turn controlled by the finite element error and a small portion of the temporal estimator, where the "smallness" is due to a weight function in time. The details are given later in Section 5.

Since the three-field formulation of Biot's consolidation model (2.2) contains the mixed formulation of the heat equation (2.3), we review existing a posteriori error estimates of mixed methods for the heat equation. Using a duality argument, Cascón *et al.* (2006) first obtained $L^2(0, T; H(\operatorname{div}, \Omega)')$ a posteriori estimates of the flux variable and $L^\infty(0, T; L^2(\Omega))$ estimates of the potential in mixed methods for the heat equation. Using the idea of elliptic reconstruction proposed by Makridakis & Nochetto (2003), the works Larson & Målqvist (2011); Memon *et al.* (2012) presented $L^2(0, T; L^2(\Omega))$ - and $L^\infty(0, T; L^2(\Omega))$ -type a posteriori error estimates of the flux variable in mixed methods for the heat equation. However, there is no proof that the estimators proposed in Cascón *et al.* (2006); Larson & Målqvist (2011); Memon *et al.* (2012) provide lower bounds of the discretization error. On the other hand, Ern & Vohralík (2010) presented an equilibrated estimator with a lower bound for the error in post-processed potential based on the $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$ -norm. Their estimator does not control the error in the flux variable. Comparing to the aforementioned error estimators, a posteriori analysis in this paper indeed yields a new estimator for the mixed discretization of the heat equation that is both an upper and lower

bound of the space time error in the natural norm, see Section 5 for details.

The rest of this paper is organized as follows. In Section 2, we present preliminaries and derive energy estimate for the three field formulation of Biot's consolidation model. Section 3 is devoted to a posteriori error estimates of a semi-discrete scheme (3.1). In Section 4, we develop a posteriori error estimator of the fully discrete scheme (4.1) and prove its reliability. In Section 5, we show that the error estimators are lower bounds of the space-time error and present a posteriori estimates of mixed methods for the heat equation. In Section 6, we present numerical experiments validating our theoretical results. Section 7 is for concluding remarks.

2. Preliminaries and Energy estimates

Given a $\mathbb{R}^d$ -valued function $\mathbf{u}$, the symmetric gradient $\boldsymbol{\varepsilon}$ and stress tensor $\boldsymbol{\sigma}$ are

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad \boldsymbol{\sigma}(\mathbf{u}) := 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\operatorname{div} \mathbf{u})\mathbf{I},$$

where $\mu > 0, \lambda > 0$ are Lamé coefficients, $\mathbf{I}$ is the $d \times d$ identity matrix. Let Ω be a Lipschitz domain in $\mathbb{R}^d$ and $T > 0$ be the final time. The three-field formulation of the Biot's consolidation model reads

$$-\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \alpha \nabla p = \mathbf{f} \text{ in } \Omega \times (0, T], \quad (2.1a)$$

$$\partial_t(\beta p + \alpha \operatorname{div} \mathbf{u}) + \operatorname{div} \mathbf{w} = g \text{ in } \Omega \times (0, T], \quad (2.1b)$$

$$\mathbf{K}^{-1} \mathbf{w} + \nabla p = \mathbf{0} \text{ in } \Omega \times (0, T], \quad (2.1c)$$

subject to the initial condition $\mathbf{u}(0) = \mathbf{u}_0$, $p(0) = p_0$ in Ω . For the simplicity of presentation, we consider homogeneous boundary conditions

$$\begin{aligned} \mathbf{u} &= \mathbf{0} \text{ on } \Gamma_1 \times (0, T], \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{0} \text{ on } \Gamma_2 \times (0, T], \\ p &= 0 \text{ on } \Gamma_2 \times (0, T], \quad (\mathbf{K} \nabla p) \cdot \mathbf{n} = 0 \text{ on } \Gamma_1 \times (0, T], \end{aligned}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. Note that the Neumann boundary condition for p on Γ_1 imposes an essential boundary condition for $\mathbf{w}$ on Γ_1 . In addition, we assume α, β are constants and $\mathbf{K} = \mathbf{K}(\mathbf{x})$ is a time-independent and uniformly elliptic matrix-valued function, i.e.,

$$C_1|\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi}^T \mathbf{K}(\mathbf{x}) \boldsymbol{\xi} \leq C_2|\boldsymbol{\xi}|^2 \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d \text{ and } \mathbf{x} \in \Omega,$$

where C_1, C_2 are positive constants. We introduce function spaces where we seek a weak solution to the system given in (2.1):

$$\begin{aligned} \mathbf{V} &= \{\mathbf{v} \in [H^1(\Omega)]^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}, \quad Q = L^2(\Omega), \\ \mathbf{W} &= \{\mathbf{w} \in [L^2(\Omega)]^d : \operatorname{div} \mathbf{w} \in L^2(\Omega), \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \Gamma_1\}. \end{aligned}$$

Let $(\cdot, \cdot)$ denote the $L^2(\Omega)$ inner product for scalar-, vector-, or matrix-valued functions. Next, we introduce several bilinear forms:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= (\boldsymbol{\sigma}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v})), \quad b(\mathbf{v}, q) := (\alpha \operatorname{div} \mathbf{v}, q), \\ c(p, q) &:= (\beta p, q), \quad d(\mathbf{z}, q) := (\operatorname{div} \mathbf{z}, q), \quad e(\mathbf{w}, \mathbf{z}) := (\mathbf{K}^{-1} \mathbf{w}, \mathbf{z}). \end{aligned}$$

The norms associated with the bilinear forms given above are

$$\begin{aligned}\|\mathbf{v}\|_a^2 &:= a(\mathbf{v}, \mathbf{v}), & \|q\|_c^2 &:= c(q, q), \\ \|\mathbf{z}\|_e^2 &:= e(\mathbf{z}, \mathbf{z}), & \|\mathbf{z}\|_{\mathbf{W}}^2 &:= \|\mathbf{z}\|_e^2 + \|\operatorname{div} \mathbf{z}\|,\end{aligned}$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. For the spaces defined earlier we have the following correspondence with the norms: $\mathbf{V}$ is equipped with the $\|\cdot\|_a$ -norm, Q is equipped with $\|\cdot\|_c$ -norm, and $\mathbf{W}$ is equipped with the $\mathbf{W}$ -norm. Because we are dealing with a time-dependent problem, we need the spaces of Hilbert-valued functions as follows: Given a Hilbert space H , we define

$$\begin{aligned}L^\infty(0, T; H) &= \{v : v(t) \in H \text{ for } t \in T, \operatorname{ess\,sup}_{0 \leq t \leq T} \|v(t)\|_H < \infty\}, \\ L^2(0, T; H) &= \{v : v(t) \in H \text{ for } t \in T, \int_0^T \|v(t)\|_H^2 dt < \infty\}, \\ H^1(0, T; H) &= \{v \in L^2(0, T; H) : \partial_t v \in L^2(0, T; H)\},\end{aligned}$$

see, e.g., Evans (2010) for more details. The variational formulation of (2.1) then is to find $\mathbf{u} \in H^1(0, T; \mathbf{V})$, $p \in H^1(0, T; Q)$, and $\mathbf{w} \in L^2(0, T; \mathbf{W})$ such that $\mathbf{u}(0) = u_0$, $p(0) = p_0$ and

$$a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \quad (2.2a)$$

$$c(\partial_t p, q) + b(\partial_t \mathbf{u}, q) + d(\mathbf{w}, q) = (g, q), \quad (2.2b)$$

$$e(\mathbf{w}, \mathbf{z}) - d(\mathbf{z}, p) = 0 \quad (2.2c)$$

for all $\mathbf{v} \in \mathbf{V}$, $q \in Q$, and $\mathbf{z} \in \mathbf{W}$ a.e. $t \in (0, T]$. It can be observed that (2.2) with $\mathbf{u} = \mathbf{v} = \mathbf{0}$ reduces to the mixed formulation of the heat equation or time-dependent Darcy flow:

$$c(\partial_t p, q) + d(\mathbf{w}, q) = (g, q), \quad q \in Q, \quad (2.3a)$$

$$e(\mathbf{w}, \mathbf{z}) - d(\mathbf{z}, p) = 0, \quad \mathbf{z} \in \mathbf{W}. \quad (2.3b)$$

In the rest of this section, we establish an energy estimate of (2.2) which is the main tool for deriving a posteriori error estimates. The well-posedness of two-field formulation can be found in e.g., Ženíšek (1984); Showalter (2000). For the three-field formulation we have the following result.

THEOREM 2.1 Let $u_0 \in \mathbf{V}$, $p_0 \in Q$, $\mathbf{f} \in H^1(0, T; \mathbf{V}')$, and $g \in L^2(0, T; Q)$. Then the variational formulation (2.2) admits a unique weak solution

$$(\mathbf{u}, p, \mathbf{w}) \in H^1(0, T; \mathbf{V}) \times H^1(0, T; Q) \times L^2(0, T; \mathbf{W}).$$

We skip the proof of Theorem 2.1 as it directly follows from the energy estimates in Lemma 2.1 and a standard argument using a Galerkin method in space, in the same fashion as for the linear parabolic equation (see, e.g., Evans (2010)). For the purpose of a posteriori error estimation, we consider a more general variational problem: Find $\tilde{\mathbf{u}} \in H^1(0, T; \mathbf{V})$, $\tilde{p} \in H^1(0, T; Q)$, $\tilde{\mathbf{w}} \in L^2(0, T; \mathbf{W})$, such that

$$a(\tilde{\mathbf{u}}, \mathbf{v}) - b(\mathbf{v}, \tilde{p}) = \langle \mathbf{F}_1, \mathbf{v} \rangle, \quad \mathbf{v} \in \mathbf{V}, \quad (2.4a)$$

$$c(\partial_t \tilde{p}, q) + b(\partial_t \tilde{\mathbf{u}}, q) + d(\tilde{\mathbf{w}}, q) = \langle \mathbf{F}_2, q \rangle, \quad q \in Q, \quad (2.4b)$$

$$e(\tilde{\mathbf{w}}, \mathbf{z}) - d(\mathbf{z}, \tilde{p}) = \langle \mathbf{F}_3, \mathbf{z} \rangle, \quad \mathbf{z} \in \mathbf{W}, \quad (2.4c)$$

where $\mathbf{F}_1 \in H^1(0, T; \mathbf{V}')$, $F_2 \in L^2(0, T; Q')$, $\mathbf{F}_3 \in H^1(0, T; \mathbf{W}')$ are time-dependent bounded linear functionals living in dual spaces. At each time $t \in [0, T]$, the dual norms are given by

$$\begin{aligned}\|\mathbf{F}_1\|_t &= \|\mathbf{F}_1\|_{\mathbf{V}'} := \sup_{\mathbf{v} \in \mathbf{V}, \|\mathbf{v}\|_a=1} \langle \mathbf{F}_1, \mathbf{v} \rangle, \\ \|F_2\|_t &= \|F_2\|_{Q'} := \sup_{q \in Q, \|q\|_c=1} \langle F_2, q \rangle, \\ \|\mathbf{F}_3\|_t &= \|\mathbf{F}_3\|_{\mathbf{W}'} := \sup_{\mathbf{z} \in \mathbf{W}, \|\mathbf{z}\|_w=1} \langle \mathbf{F}_3, \mathbf{z} \rangle.\end{aligned}$$

Norms of $\partial_t \mathbf{F}_1 \in \mathbf{V}'$ and $\partial_t \mathbf{F}_3 \in \mathbf{W}'$ are defined in a similar fashion. Given $t \in [0, T]$ and an interval $I \subseteq [0, T]$, we make use of the norms

$$\begin{aligned}\|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})(t)\|^2 &:= \|\tilde{\mathbf{u}}(t)\|_a^2 + \|\tilde{p}(t)\|_c^2 + \|\tilde{\mathbf{w}}(t)\|_e^2, \\ \|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})\|_{L^2(I; X)}^2 &:= \int_I (\|\tilde{\mathbf{u}}\|_a^2 + \|\partial_t \tilde{\mathbf{u}}\|_a^2 + \|\tilde{p}\|_c^2 + \|\partial_t \tilde{p}\|_c^2 + \|\tilde{\mathbf{w}}\|_w^2 + \|\partial_t \tilde{\mathbf{w}}\|_w^2) ds.\end{aligned}$$

The following energy estimate is crucial to a posteriori error estimation of numerical methods for (2.2).

LEMMA 2.1 There exists a constant C_{stab} dependent only on $\mu, \alpha, \beta, \mathbf{K}, \Omega$ such that for all $t \in (0, T]$,

$$\begin{aligned}\|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})(t)\|^2 + \|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})\|_{L^2(0, t; X)}^2 &\leq C_{\text{stab}} \{ \|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})(0)\|^2 \\ &+ \left(\int_0^t \|F_2\|_{Q'} ds \right)^2 + \int_0^t (\|\mathbf{F}_1\|_t^2 + \|\partial_t \mathbf{F}_1\|_t^2 + \|F_2\|_t^2 + \|\mathbf{F}_3\|_t^2 + \|\partial_t \mathbf{F}_3\|_t^2) ds \}.\end{aligned}$$

Proof. Setting $\mathbf{v} = \partial_t \tilde{\mathbf{u}}, \mathbf{z} = \tilde{\mathbf{w}}, q = \tilde{p}$ in (2.4) yields

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{u}}\|_a^2 + \frac{1}{2} \frac{d}{dt} \|\tilde{p}\|_c^2 + \|\tilde{\mathbf{w}}\|_e^2 = \langle \mathbf{F}_1, \partial_t \tilde{\mathbf{u}} \rangle + \langle F_2, \tilde{p} \rangle + \langle \mathbf{F}_3, \tilde{\mathbf{w}} \rangle. \quad (2.5)$$

On the other hand, differentiating (2.4a) and (2.4c) with respect to time t gives

$$\begin{aligned}a(\partial_t \tilde{\mathbf{u}}, \mathbf{v}) - b(\mathbf{v}, \partial_t \tilde{p}) &= \langle \partial_t \mathbf{F}_1, \mathbf{v} \rangle, \\ e(\partial_t \tilde{\mathbf{w}}, \mathbf{z}) - d(\mathbf{z}, \partial_t \tilde{p}) &= \langle \partial_t \mathbf{F}_3, \mathbf{z} \rangle.\end{aligned}$$

Taking as test functions $\mathbf{v} = \partial_t \tilde{\mathbf{u}}$ and $\mathbf{z} = \tilde{\mathbf{w}}$ in the equations above and using (2.4b) with $q = \partial_t \tilde{p}$ then leads to

$$\|\partial_t \tilde{\mathbf{u}}\|_a^2 + \|\partial_t \tilde{p}\|_c^2 + \frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{w}}\|_e^2 = \langle \partial_t \mathbf{F}_1, \partial_t \tilde{\mathbf{u}} \rangle + \langle F_2, \partial_t \tilde{p} \rangle + \langle \partial_t \mathbf{F}_3, \tilde{\mathbf{w}} \rangle. \quad (2.6)$$

Using (2.5), (2.6), the Cauchy–Schwarz and Young’s inequalities, we obtain

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{u}}\|_a^2 + \frac{1}{2} \frac{d}{dt} \|\tilde{p}\|_c^2 + (1 - \delta) \|\tilde{\mathbf{w}}\|_e^2 + \frac{1}{2} \|\partial_t \tilde{\mathbf{u}}\|_a^2 \\ + \frac{1}{2} \|\partial_t \tilde{p}\|_c^2 + \frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{w}}\|_e^2 \leq G + \|\mathbf{F}_2\|_{Q'} \|\tilde{p}\|_c + \delta \|\operatorname{div} \tilde{\mathbf{w}}\|^2,\end{aligned}$$

where $\delta > 0$ and

$$G = \|\mathbf{F}_1\|_t^2 + \|\partial_t \mathbf{F}_1\|_t^2 + \frac{1}{2} \|F_2\|_t^2 + \frac{\delta^{-1}}{2} \|\mathbf{F}_3\|_t^2 + \frac{\delta^{-1}}{2} \|\partial_t \mathbf{F}_3\|_t^2.$$

Integrating the previous inequality yields

$$\begin{aligned} & \frac{1}{2} \|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})(t)\|^2 + \int_0^t \left(\frac{1}{2} \|\partial_t \tilde{\mathbf{u}}\|_a^2 + \frac{1}{2} \|\partial_t \tilde{p}\|_c^2 + (1-\delta) \|\tilde{\mathbf{w}}\|_e^2 \right) ds \\ & \leq \frac{1}{2} \|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})(0)\|^2 + \int_0^t (G + \|F_2\|_r \|\tilde{p}\|_c + \delta \|\operatorname{div} \tilde{\mathbf{w}}\|^2) ds. \end{aligned} \quad (2.7)$$

Recall that $\|\tilde{p}\|_{L^\infty(0,t;Q)} := \max_{0 \leq s \leq t} \|\tilde{p}(s)\|_c$. In particular, (2.7) implies that

$$\begin{aligned} & \frac{1}{2} \|\tilde{p}(s)\|_c^2 \leq \frac{1}{2} \|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})(0)\|^2 \\ & + \int_0^t (G + \delta \|\operatorname{div} \tilde{\mathbf{w}}\|^2) ds + \|\tilde{p}\|_{L^\infty(0,t;Q)} \int_0^t \|F_2\|_r ds \end{aligned}$$

for all $0 \leq s \leq t$. Hence a combination of the previous estimate with

$$\|\tilde{p}\|_{L^\infty(0,t;Q)} \int_0^t \|F_2\|_r ds \leq \frac{1}{4} \|\tilde{p}\|_{L^\infty(0,t;Q)}^2 + \left(\int_0^t \|F_2\|_r ds \right)^2$$

shows that

$$\begin{aligned} & \frac{1}{4} \|\tilde{p}\|_{L^\infty(0,t;Q)}^2 \leq \frac{1}{2} \|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})(0)\|^2 \\ & + \int_0^t (G + \delta \|\operatorname{div} \tilde{\mathbf{w}}\|^2) ds + \left(\int_0^t \|F_2\|_r ds \right)^2. \end{aligned} \quad (2.8)$$

Using (2.7) and (2.8) and a Young's inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})(t)\|^2 + \int_0^t \left(\frac{1}{2} \|\partial_t \tilde{\mathbf{u}}\|_a^2 + \frac{1}{2} \|\partial_t \tilde{p}\|_c^2 + (1-\delta) \|\tilde{\mathbf{w}}\|_e^2 \right) ds \\ & \leq \|(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{w}})(0)\|^2 + 2 \int_0^t (G + \delta \|\operatorname{div} \tilde{\mathbf{w}}\|^2) ds + 2 \left(\int_0^t \|F_2\|_r ds \right)^2, \end{aligned} \quad (2.9)$$

Let C be a generic constant dependent only on $\alpha, \beta, \mu, \Omega$. It follows from (2.4b) with $q = \operatorname{div} \tilde{\mathbf{w}}$ that

$$\|\operatorname{div} \tilde{\mathbf{w}}\|^2 \leq C (\|F_2\|_r^2 + \|\partial_t \tilde{p}\|_c^2 + \|\partial_t \tilde{\mathbf{u}}\|_a^2). \quad (2.10)$$

Taking the derivative with respect to time on both sides of (2.4c) shows that

$$\|\partial_t \tilde{\mathbf{w}}\|_{\mathbf{W}'} \leq C (\|\partial_t \mathbf{F}_3\|_r + \|\partial_t \tilde{p}\|_c). \quad (2.11)$$

The inf-sup condition for $d(\cdot, \cdot)$ together with (2.4c) and (2.4a) then imply the following inequality:

$$\|\tilde{p}\|_c + \|\tilde{\mathbf{u}}\|_a \leq C (\|\mathbf{F}_1\|_r + \|\mathbf{F}_3\|_r + \|\tilde{\mathbf{w}}\|_e). \quad (2.12)$$

Choosing a sufficiently small $\delta > 0$ and combining (2.9) and (2.10)–(2.12) completes the proof of the lemma. $\square$

3. Error estimator for the semi-discrete problem

Let $\mathcal{T}_h$ be a conforming simplicial triangulation of Ω that is aligned with Γ_1 and Γ_2 . The mesh $\mathcal{T}_h$ is shape-regular in the sense that

$$\max_{K \in \mathcal{T}_h} \frac{r_K}{\rho_K} := \tilde{C}_{\text{shape}} < \infty,$$

where r_K, ρ_K are radii of circumscribed and inscribed spheres of K . Let $\mathbf{V}_h \subset \mathbf{V}$, $\mathbf{W}_h \subset \mathbf{W}$, $Q_h \subset Q$ be suitable finite element spaces based on $\mathcal{T}_h$. In particular, we choose $\mathbf{V}_h \times Q_h$ to be a stable mixed element pair for the Stokes equation, and $\mathbf{W}_h \times Q_h$ to be a stable mixed element pair for the mixed formulation of Poisson's equation. It has been shown in e.g., Hong & Kraus (2018); Rodrigo *et al.* (2018) that this choice leads to stable space discretization. For example, $\mathbf{V}_h \times Q_h$ can be chosen to be the $(P_1 + \text{face bubble functions}) \times P_0$ element (see Girault & Raviart (1986)) and $\mathbf{W}_h \times Q_h$ can be the lowest order Raviart–Thomas (see Raviart & Thomas (1977)) or Brezzi–Douglas–Marini element (see Brezzi *et al.* (1985)). Let $\mathcal{F}(\mathcal{T}_h)$ denote the collection of faces in $\mathcal{T}_h$ and $\mathbf{n}_F$ be a unit normal to F for any face $F \in \mathcal{F}(\mathcal{T}_h)$. Let $\mathcal{P}_k(K)$ denote the space of polynomials of degree no greater than k on K , and

$$\begin{aligned} \mathbf{V}_{h,l} &= \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_K \in [\mathcal{P}_1(K)]^d \text{ for all } K \in \mathcal{T}_h\}, \\ \mathbf{B}_h &= \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_K \in \text{span}\{\phi_F \mathbf{n}_F\}_{F \subset \partial K, F \in \mathcal{F}(\mathcal{T}_h)} \text{ for all } K \in \mathcal{T}_h\}. \end{aligned}$$

Here, ϕ_F is the face bubble function supported on union of elements having $F \in \mathcal{F}(\mathcal{T}_h)$ as a face, i.e., $\phi_F = \prod_{z_j \in F} \lambda_j$ where λ_j is the barycentric coordinate corresponding to the vertex z_j in the face F . The triple $\mathbf{V}_h \times Q_h \times \mathbf{W}_h$ can be chosen as $\mathbf{V}_h^0 \times Q_h^0 \times \mathbf{W}_h^0$, where

$$\begin{aligned} \mathbf{V}_h^0 &= \mathbf{V}_{h,l} \oplus \mathbf{B}_h, \\ Q_h^0 &= \{q \in L^2(\Omega) : q|_K \in \mathcal{P}_0(K) \text{ for all } K \in \mathcal{T}_h\}, \\ \mathbf{W}_h^0 &= \{\mathbf{z} \in \mathbf{W} : \mathbf{z}|_K \in [\mathcal{P}_0(K)]^d + \mathcal{P}_0(K)\mathbf{x} \text{ for all } K \in \mathcal{T}_h\}. \end{aligned}$$

Here $\mathbf{x} = (x_1, x_2, \dots, x_d)^T$ is the linear position vector. In general, we assume the inclusion $\mathbf{W}_h^0 \subseteq \mathbf{W}_h$.

The semi-discrete version of (2.2) is to find $\mathbf{u}_h \in H^1(0, T; \mathbf{V}_h)$, $p_h \in H^1(0, T; Q_h)$, and $\mathbf{w}_h \in L^2(0, T; \mathbf{W}_h)$ such that $\mathbf{u}_h(0) = \mathbf{u}_h^0$, $p_h(0) = p_h^0$ and

$$a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h, \quad (3.1a)$$

$$c(\partial_t p_h, q) + b(\partial_t \mathbf{u}_h, q) + d(\mathbf{w}_h, q) = (g, q), \quad q \in Q_h, \quad (3.1b)$$

$$e(\mathbf{w}_h, \mathbf{z}) - d(\mathbf{z}, p_h) = 0, \quad \mathbf{z} \in \mathbf{W}_h. \quad (3.1c)$$

Here $\mathbf{u}_h^0 \in \mathbf{V}_h$, $p_h^0 \in Q_h$ are some finite element approximation to $\mathbf{u}_0$ and p_0 . In this section, we derive a posteriori error estimation for the semi-discrete method (3.1). To this end, let

$$e_u = \mathbf{u} - \mathbf{u}_h, \quad e_p = p - p_h, \quad e_w = \mathbf{w} - \mathbf{w}_h. \quad (3.2)$$

It follows from (2.2) that the errors satisfy

$$a(e_u, \mathbf{v}) - b(\mathbf{v}, e_p) = \langle \mathbf{r}_1, \mathbf{v} \rangle, \quad \mathbf{v} \in \mathbf{V}, \quad (3.3a)$$

$$c(\partial_t e_p, q) + b(\partial_t e_u, q) + d(e_w, q) = \langle \mathbf{r}_2, q \rangle, \quad q \in Q, \quad (3.3b)$$

$$e(e_w, \mathbf{z}) - d(\mathbf{z}, e_p) = \langle \mathbf{r}_3, \mathbf{z} \rangle, \quad \mathbf{z} \in \mathbf{W}, \quad (3.3c)$$

where the residuals $\mathbf{r}_1(t) \in \mathbf{V}'$, $r_2(t) \in Q'$, $\mathbf{r}_3(t) \in \mathbf{W}'$ are defined by

$$\begin{aligned}\langle \mathbf{r}_1, \mathbf{v} \rangle &:= (\mathbf{f}, \mathbf{v}) - a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h), \\ \langle r_2, q \rangle &:= (g, q) - c(\partial_t p_h, q) - b(\partial_t \mathbf{u}_h, q) - d(\mathbf{w}_h, q), \\ \langle \mathbf{r}_3, \mathbf{z} \rangle &:= -e(\mathbf{w}_h, \mathbf{z}) + d(\mathbf{z}, p_h).\end{aligned}$$

With the help of Lemma 2.1, we immediately obtain the following corollary.

COROLLARY 3.1 For the errors defined in (3.2) and $t \in (0, T]$, we have

$$\begin{aligned}& \left\| (e_u, e_p, e_w)(t) \right\|^2 + \|(e_u, e_p, e_w)\|_{L^2(0,t;X)}^2 \leq C_{\text{stab}} \left\{ \left\| (e_u, e_p, e_w)(0) \right\|^2 \right. \\ & \left. + \left(\int_0^t \|r_2\|_t ds \right)^2 + \int_0^t (\|\mathbf{r}_1\|_t^2 + \|\partial_t \mathbf{r}_1\|_t^2 + \|r_2\|_t^2 + \|\mathbf{r}_3\|_t^2 + \|\partial_t \mathbf{r}_3\|_t^2) ds \right\}.\end{aligned}$$

For a $\mathbb{R}^d$ -valued function $\mathbf{v} = (v_i)_{1 \leq i \leq d}$ and a scalar-valued function v , let

$$\begin{aligned}\text{curl } \mathbf{v} &= (\partial_{x_2} v_3 - \partial_{x_3} v_2, \partial_{x_3} v_1 - \partial_{x_1} v_3, \partial_{x_1} v_2 - \partial_{x_2} v_1)^T \text{ when } d = 3, \\ \text{curl } v &= (\partial_{x_2} v, -\partial_{x_1} v)^T, \quad \text{rot } \mathbf{v} = \partial_{x_1} v_2 - \partial_{x_2} v_1 \text{ when } d = 2.\end{aligned}$$

Let $\mathbf{N}_h^0$ denote the lowest order Nédélec edge element space (see Nédélec (1980))

$$\begin{aligned}\mathbf{N}_h^0 &= \{ \mathbf{v} \in [L^2(\Omega)]^3 : \text{curl } \mathbf{v} \in [L^2(\Omega)]^3, \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_1, \\ & \quad \mathbf{v}|_K \in [\mathcal{P}_0(K)]^3 + [\mathcal{P}_0(K)]^3 \times \mathbf{x} \text{ for all } K \in \mathcal{T}_h \},\end{aligned}$$

and V_h denote the *scalar* linear element space

$$V_h = \{ v \in H^1(\Omega) : v|_K \in \mathcal{P}_1(K) \text{ for all } K \in \mathcal{T}_h, v = 0 \text{ on } \Gamma_1 \}.$$

For each $K \in \mathcal{T}_h$, let $h_K = |K|^{\frac{1}{d}}$ denote the size of K , $\|\cdot\|_K$ denote the L^2 -norm on K , and $\|\cdot\|_{\partial K}$ denote the L^2 -norm on ∂K . To estimate the norms of $\mathbf{r}_3, \partial_t \mathbf{r}_3 \in \mathbf{W}'$, we need the following theorem, which is a combination of the H^1 -regular decomposition (see e.g., Hiptmair (2002); Pasciak & Zhao (2002); Demlow & Hirani (2014)) and bounded quasi-interpolation operators which commute with the exterior differentiation (see Schöberl (2008); Demlow & Hirani (2014)).

THEOREM 3.1 Let $\Omega \subset \mathbb{R}^d$ with $d = 3$ (resp. $d = 2$). There exist quasi-interpolations $\Pi_h : [L^2(\Omega)]^3 \rightarrow \mathbf{W}_h^0$ (resp. $\Pi_h : [L^2(\Omega)]^2 \rightarrow \mathbf{W}_h^0$), and $J_h : [L^2(\Omega)]^3 \rightarrow \mathbf{N}_h^0$ (resp. $J_h : L^2(\Omega) \rightarrow V_h$) such that $\Pi_h \text{curl} = \text{curl } J_h$. In addition, for any $\mathbf{z} \in \mathbf{W}$, there exist $\boldsymbol{\varphi} \in \mathbf{V}$ (resp. $\boldsymbol{\varphi} \in H^1(\Omega)$, $\boldsymbol{\varphi}|_{\Gamma_1} = 0$) and $\boldsymbol{\phi} \in \mathbf{V}$, such that

$$\mathbf{z} = \text{curl } \boldsymbol{\varphi} + \boldsymbol{\phi}, \quad \Pi_h \mathbf{z} = \text{curl } J_h \boldsymbol{\varphi} + \Pi_h \boldsymbol{\phi},$$

and

$$\begin{aligned}& \sum_{K \in \mathcal{T}_h} (h_K^{-2} \|\boldsymbol{\varphi} - J_h \boldsymbol{\varphi}\|_K^2 + h_K^{-2} \|\boldsymbol{\phi} - \Pi_h \boldsymbol{\phi}\|_K^2 \\ & \quad + h_K^{-1} \|\boldsymbol{\varphi} - J_h \boldsymbol{\varphi}\|_{\partial K}^2 + h_K^{-1} \|\boldsymbol{\phi} - \Pi_h \boldsymbol{\phi}\|_{\partial K}^2) \leq C_{\text{reg}} \|\mathbf{z}\|_{\mathbf{W}}^2,\end{aligned}\tag{3.4}$$

where C_{reg} depends only on $\mathbf{K}$, Ω , Γ_1 , $\tilde{C}_{\text{shape}}$.

Theorem 3.1 or its variants are widely used in the a posteriori error estimation of stationary problems based on $H(\text{div})$ or $H(\text{curl})$, see, e.g., Cascon *et al.* (2007); Schöberl (2008); Huang & Xu (2012); Demlow & Hirani (2014); Chen & Wu (2017); Li (2019,b); Holst *et al.* (2020).

Now we are in a position to derive a posteriori error estimator of the system given in (3.1). For each face F , let $\mathbf{n}_F$ be a unit normal to F where $\mathbf{n}_F$ is chosen to be outward pointing when F is a boundary face. For each interior $F \in \mathcal{F}(\mathcal{T}_h)$ shared by $K_1, K_2 \in \mathcal{T}_h$ and a piecewise H^1 -function χ , let $[\chi]_F := (\chi|_{K_1} - \chi|_{K_2})|_F$ denote the jump across F , where $\mathbf{n}_F$ is pointing from K_1 to K_2 . For any $F \subset \partial\Omega$ that is a boundary face in $\mathcal{T}_h$, we set $[\chi]_F := 0$ if $F \subset \Gamma_1$, and $[\chi]_F := \chi|_F$ if $F \subset \Gamma_2$. Regarding the mesh $\mathcal{T}_h$, we use the following error indicators

$$\begin{aligned} \mathcal{E}_{\mathcal{T}_h}^1(\mathbf{u}_h, p_h, \mathbf{f}) &:= \sum_{K \in \mathcal{T}_h} \{h_K^2 \|\mathbf{f} + \text{div } \boldsymbol{\sigma}(\mathbf{u}_h) - \alpha \nabla p_h\|_K^2 \\ &\quad + \sum_{F \in \mathcal{F}(\mathcal{T}_h), F \subset \partial K} h_K \|[\boldsymbol{\sigma}(\mathbf{u}_h) - \alpha p_h \mathbf{I}] \mathbf{n}_F\|_F^2\}, \\ \mathcal{E}_{\mathcal{T}_h}^2(\partial_t \mathbf{u}_h, \partial_t p_h, \mathbf{w}_h, g) &:= \sum_{K \in \mathcal{T}_h} \|g - \beta \partial_t p_h - \text{div } \partial_t \mathbf{u}_h - \text{div } \mathbf{w}_h\|_K^2. \end{aligned}$$

Another error estimator is

$$\begin{aligned} \mathcal{E}_{\mathcal{T}_h}^3(p_h, \mathbf{w}_h) &:= \sum_{K \in \mathcal{T}_h} \{h_K^2 \|\mathbf{K}^{-1} \mathbf{w}_h + \nabla p_h\|_K^2 + h_K^2 \|\text{curl}(\mathbf{K}^{-1} \mathbf{w}_h)\|_K^2 \\ &\quad + \sum_{F \in \mathcal{F}(\mathcal{T}_h), F \subset \partial K} h_K \|[(\mathbf{K}^{-1} \mathbf{w}_h) \times \mathbf{n}_F]\|_F^2 + h_K \|[p_h]\|_F^2\} \text{ when } d = 3, \end{aligned}$$

and when $d = 2$ (for a two dimensional problem)

$$\begin{aligned} \mathcal{E}_{\mathcal{T}_h}^3(p_h, \mathbf{w}_h) &:= \sum_{K \in \mathcal{T}_h} \{h_K^2 \|\mathbf{K}^{-1} \mathbf{w}_h + \nabla p_h\|_K^2 + h_K^2 \|\text{rot}(\mathbf{K}^{-1} \mathbf{w}_h)\|_K^2 \\ &\quad + \sum_{F \in \mathcal{F}(\mathcal{T}_h), F \subset \partial K} h_K \|[(\mathbf{K}^{-1} \mathbf{w}_h) \cdot \mathbf{t}_F]\|_F^2 + h_K \|[p_h]\|_F^2\}, \end{aligned}$$

where $\mathbf{t}_F$ is a unit tangent vector to F . The next theorem presents a posteriori error estimates of the semi-discrete method (3.1).

THEOREM 3.2 When $d = 2$ or 3 , there exists a constant C_{rel} dependent only on $\mu, \alpha, \beta, \mathbf{K}, \Omega, \Gamma_1$ and the shape regularity of $\mathcal{T}_h$, such that

$$\begin{aligned} &\| (e_u, e_p, e_w)(t) \|^2 + \|(e_u, e_p, e_w)\|_{L^2(0,t;X)}^2 \leq C_{\text{rel}} \{ \| (e_u, e_p, e_w)(0) \|^2 \\ &\quad + \left(\int_0^t \mathcal{E}_{\mathcal{T}_h}^2(\partial_t \mathbf{u}_h, \partial_t p_h, \mathbf{w}_h, g) \frac{1}{2} ds \right)^2 + \int_0^t (\mathcal{E}_{\mathcal{T}_h}^1(\mathbf{u}_h, p_h, \mathbf{f}) + \mathcal{E}_{\mathcal{T}_h}^1(\partial_t \mathbf{u}_h, \partial_t p_h, \partial_t \mathbf{f}) \\ &\quad + \mathcal{E}_{\mathcal{T}_h}^2(\partial_t \mathbf{u}_h, \partial_t p_h, \mathbf{w}_h, g) + \mathcal{E}_{\mathcal{T}_h}^3(p_h, \mathbf{w}_h) + \mathcal{E}_{\mathcal{T}_h}^3(\partial_t p_h, \partial_t \mathbf{w}_h)) ds \}. \end{aligned}$$

Proof. We focus on the case $d = 3$ since the proof when $d = 2$ is similar. In the proof, we use C to denote generic constant dependent only on $\mu, \alpha, \beta, \mathbf{K}, \tilde{C}_{\text{shape}}$, and Ω, Γ_1 . In view of Corollary 3.1, it remains to estimate the norm of each residual. Let $I_h : \mathbf{V} \rightarrow \mathbf{V}_h$ denote the Clément interpolation (see Clément (1975); Verfürth (2013)). Thanks to (3.1a), it holds that for each $\mathbf{v} \in \mathbf{V}$,

$$\langle \mathbf{r}_1, \mathbf{v} \rangle = (\mathbf{f}, \mathbf{v} - I_h \mathbf{v}) - a(\mathbf{u}_h, \mathbf{v} - I_h \mathbf{v}) + b(\mathbf{v} - I_h \mathbf{v}, p_h). \quad (3.5)$$

Element-wise integration by parts leads to

$$\begin{aligned} \langle \mathbf{r}_1, \mathbf{v} \rangle &= (\mathbf{f}, \mathbf{v} - I_h \mathbf{v}) + \sum_{K \in \mathcal{T}_h} \left\{ - \int_K \boldsymbol{\sigma}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v} - I_h \mathbf{v}) + \int_K \operatorname{div}(\alpha(\mathbf{v} - I_h \mathbf{v})) p_h \right\} \\ &= \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{f} + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h) - \alpha \nabla p_h) \cdot (\mathbf{v} - I_h \mathbf{v}) \\ &\quad + \sum_{F \in \mathcal{F}(\mathcal{T}_h)} \int_F ([-\boldsymbol{\sigma}(\mathbf{u}_h) + \alpha p_h \mathbf{I}] \mathbf{n}_F) \cdot (\mathbf{v} - I_h \mathbf{v}). \end{aligned}$$

Combining the previous equation with the Cauchy–Schwarz inequality and shape-regularity of $\mathcal{T}_h$, we obtain

$$\langle \mathbf{r}_1, \mathbf{v} \rangle \leq C \mathcal{E}_{\mathcal{T}_h}^1(\mathbf{u}_h, p_h, \mathbf{f})^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}_h} h_K^{-2} \|\mathbf{v} - I_h \mathbf{v}\|_K^2 + h_K^{-1} \|\mathbf{v} - I_h \mathbf{v}\|_{\partial K}^2 \right)^{\frac{1}{2}}. \quad (3.6)$$

It then follows from (3.6), the well-known approximation result

$$\sum_{K \in \mathcal{T}_h} h_K^{-2} \|\mathbf{v} - I_h \mathbf{v}\|_K^2 + h_K^{-1} \|\mathbf{v} - I_h \mathbf{v}\|_{\partial K}^2 \leq C |\mathbf{v}|_{H^1(\Omega)}^2,$$

and the Korn's inequality (cf. Kondratiev & Oleinik (1989))

$$|\mathbf{v}|_{H^1(\Omega)} \leq C \|\mathbf{v}\|_a, \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.7)$$

that

$$\|\mathbf{r}_1\|_l \leq C \mathcal{E}_{\mathcal{T}_h}^1(\mathbf{u}_h, p_h, \mathbf{f})^{\frac{1}{2}}. \quad (3.8)$$

Similarly (3.5) implies that for each $\mathbf{v} \in \mathbf{V}$,

$$\langle \partial_t \mathbf{r}_1, \mathbf{v} \rangle = (\partial_t \mathbf{f}, \mathbf{v} - I_h \mathbf{v}) - a(\partial_t \mathbf{u}_h, \mathbf{v} - I_h \mathbf{v}) + b(\mathbf{v} - I_h \mathbf{v}, \partial_t p_h).$$

Then the next estimate

$$\|\partial_t \mathbf{r}_1\|_l^2 \leq C \mathcal{E}_{\mathcal{T}_h}^1(\partial_t \mathbf{u}_h, \partial_t p_h, \partial_t \mathbf{f}). \quad (3.9)$$

can be proved in the same way as (3.8). The norm of $\mathbf{r}_2$ is trivially estimated by

$$\|\mathbf{r}_2\|_l \leq C \|g - \beta \partial_t p_h - \operatorname{div} \partial_t \mathbf{u}_h - \operatorname{div} \mathbf{w}_h\|. \quad (3.10)$$

To estimate $\|\mathbf{r}_3\|_l$, we use (3.1c) to obtain

$$\langle \mathbf{r}_3, \mathbf{z} \rangle = -e(\mathbf{w}_h, \mathbf{z} - \Pi_h \mathbf{z}) + d(\mathbf{z} - \Pi_h \mathbf{z}, p_h). \quad (3.11)$$

Due to Theorem 3.1, there exists $\boldsymbol{\varphi} \in \mathbf{V}$ and $\boldsymbol{\phi} \in \mathbf{V}$ such that

$$\mathbf{z} - \Pi_h \mathbf{z} = \operatorname{curl}(\boldsymbol{\varphi} - J_h \boldsymbol{\varphi}) + \boldsymbol{\phi} - \Pi_h \boldsymbol{\phi}, \quad (3.12)$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\phi}$ satisfy (3.4). Using (3.11), (3.12), and element-wise integration by parts, we arrive at

$$\begin{aligned} \langle \mathbf{r}_3, \mathbf{z} \rangle &= -(\mathbf{K}^{-1} \mathbf{w}_h, \operatorname{curl}(\boldsymbol{\varphi} - J_h \boldsymbol{\varphi})) - (\mathbf{K}^{-1} \mathbf{w}_h, \boldsymbol{\phi} - \Pi_h \boldsymbol{\phi}) + (\operatorname{div}(\boldsymbol{\phi} - \Pi_h \boldsymbol{\phi}), p_h) \\ &= \sum_{K \in \mathcal{T}_h} \left\{ - \int_K \operatorname{curl}(\mathbf{K}^{-1} \mathbf{w}_h) \cdot (\boldsymbol{\varphi} - J_h \boldsymbol{\varphi}) - \int_K (\mathbf{K}^{-1} \mathbf{w}_h + \nabla p_h) \cdot (\boldsymbol{\phi} - \Pi_h \boldsymbol{\phi}) \right\} \\ &\quad + \sum_{F \in \mathcal{F}(\mathcal{T}_h)} \left\{ \int_F -[(\mathbf{K}^{-1} \mathbf{w}_h) \times \mathbf{n}_F] \cdot (\boldsymbol{\varphi} - J_h \boldsymbol{\varphi}) + \int_F [p_h](\boldsymbol{\phi} - \Pi_h \boldsymbol{\phi}) \cdot \mathbf{n}_F \right\}. \end{aligned}$$

It then follows from the previous equation, the Cauchy–Schwarz inequality, and (3.4) that

$$\langle \mathbf{r}_3, \mathbf{z} \rangle \leq C \mathcal{E}_{\mathcal{T}_h}^3(p_h, \mathbf{w}_h)^{\frac{1}{2}} \|\mathbf{z}\|_{\mathbf{W}}. \quad (3.13)$$

Similarly, it holds that

$$\|\partial_t \mathbf{r}_3\|_t \leq C \mathcal{E}_{\mathcal{T}_h}^3(\partial_t p_h, \partial_t \mathbf{w}_h)^{\frac{1}{2}}. \quad (3.14)$$

Combining (3.8)–(3.10), (3.13), (3.14) completes the proof. $\square$

4. Fully discrete method

Let $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$ and $\tau_n = t_n - t_{n-1}$ for $n = 1, 2, \dots, N$. Let $\mathcal{T}_h^n$ be a conforming simplicial triangulation of Ω aligned with Γ_1 and Γ_2 . Let $\mathbf{V}_h^n, Q_h^n, \mathbf{W}_h^n$ be finite element subspaces of $\mathbf{V}, Q, \mathbf{W}$ described in Section 3 based on grid $\mathcal{T}_h^n$, respectively. We assume that $\{\mathcal{T}_h^n\}_{n=0}^N$ is uniformly shape-regular w.r.t. n , that is,

$$\max_{0 \leq n \leq N} \max_{K \in \mathcal{T}_h^n} \frac{r_K}{\rho_K} := C_{\text{shape}} < \infty.$$

Given a sequence $\{\chi^n\}_{n=0}^N$, we define the backward difference as

$$\delta_t \chi^n = \frac{\chi^n - \chi^{n-1}}{\tau_n},$$

and the continuous linear interpolant χ^τ on $[0, T]$ as

$$\chi^\tau(t) = \frac{t - t_{n-1}}{\tau_n} \chi^n + \frac{t_n - t}{\tau_n} \chi^{n-1}, \quad t \in [t_{n-1}, t_n].$$

Notice that $\partial_t \chi^\tau = \delta_t \chi^n$ over $[t_{n-1}, t_n]$. Let $\mathbf{u}_h^0, p_h^0, \mathbf{w}_h^0$ be suitable approximation to $\mathbf{u}(0), p(0), \mathbf{w}(0)$, and $\mathbf{f}^n = \mathbf{f}(t_n), g^n = g(t_n)$. The fully discrete scheme for (2.2) is to find $\mathbf{u}_h^n \in \mathbf{V}_h^n, p_h^n \in Q_h^n, \mathbf{w}_h^n \in \mathbf{W}_h^n$ with $n = 1, 2, \dots, N$, such that

$$a(\mathbf{u}_h^n, \mathbf{v}) - b(\mathbf{v}, p_h^n) = (\mathbf{f}^n, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h^n, \quad (4.1a)$$

$$c(\delta_t p_h^n, q) + b(\delta_t \mathbf{u}_h^n, q) + d(\mathbf{w}_h^n, q) = (g^n, q), \quad q \in Q_h^n, \quad (4.1b)$$

$$e(\mathbf{w}_h^n, \mathbf{z}) - d(\mathbf{z}, p_h^n) = 0, \quad \mathbf{z} \in \mathbf{W}_h^n. \quad (4.1c)$$

Here, (4.1) is derived using the implicit Euler discretization in time. To apply Lemma 2.1, let $\mathbf{u}_h^\tau(t), p_h^\tau(t), \mathbf{w}_h^\tau(t)$ be the continuous linear interpolants of $\mathbf{u}_h^n, p_h^n, \mathbf{w}_h^n$ defined above, respectively. Let

$$E_u = \mathbf{u} - \mathbf{u}_h^\tau, \quad E_p = p - p_h^\tau, \quad E_w = \mathbf{w} - \mathbf{w}_h^\tau. \quad (4.2)$$

Rewriting (2.2) gives the following the error equation

$$\begin{aligned} a(E_u, \mathbf{v}) - b(\mathbf{v}, E_p) &= \langle \mathbf{R}_1, \mathbf{v} \rangle, \quad \mathbf{v} \in \mathbf{V}, \\ c(\partial_t E_p, q) + b(\partial_t E_u, q) + d(E_w, q) &= \langle \mathbf{R}_2, q \rangle, \quad q \in Q, \\ e(E_w, \mathbf{z}) - d(\mathbf{z}, E_p) &= \langle \mathbf{R}_3, \mathbf{z} \rangle, \quad \mathbf{z} \in \mathbf{W}, \end{aligned} \quad (4.3)$$

where residuals $\mathbf{R}_1(t) \in \mathbf{V}', \mathbf{R}_2(t) \in Q', \mathbf{R}_3(t) \in \mathbf{W}'$ for each $t \in [0, T]$ are

$$\begin{aligned} \langle \mathbf{R}_1, \mathbf{v} \rangle &:= (\mathbf{f}, \mathbf{v}) - a(\mathbf{u}_h^\tau, \mathbf{v}) + b(\mathbf{v}, p_h^\tau), \\ \langle \mathbf{R}_2, q \rangle &:= (g, q) - c(\delta_t p_h^\tau, q) - b(\delta_t \mathbf{u}_h^\tau, q) - d(\mathbf{w}_h^\tau, q), \\ \langle \mathbf{R}_3, \mathbf{z} \rangle &:= -e(\mathbf{w}_h^\tau, \mathbf{z}) + d(\mathbf{z}, p_h^\tau). \end{aligned} \quad (4.4)$$

Similarly to the error analysis of the semi-discrete problem, it suffices to analyze the dual norm of the residuals, defined in (4.4). However, the mesh used in the fully discrete scheme is allowed to change at different time levels and we introduce several useful operations in the following. Given two triangulations $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$ of Ω , let $\mathcal{T}_{h_1} \vee \mathcal{T}_{h_2}$ denote the minimal common refinement, i.e., $\mathcal{T}_{h_1} \vee \mathcal{T}_{h_2}$ is the coarsest conforming triangulation that is a refinement of both $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$. Similarly, let $\mathcal{T}_{h_1} \wedge \mathcal{T}_{h_2}$ denote the maximal common coarsening, i.e., $\mathcal{T}_{h_1} \wedge \mathcal{T}_{h_2}$ is the finest conforming triangulation that is a coarsening of both $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$. The operations $\wedge$ and $\vee$ on triangulations are widely used in adaptivity literature, see, e.g., Cascon *et al.* (2008); Diening *et al.* (2016).

To handle simultaneously $(\mathbf{u}_h^n, p_h^n, \mathbf{w}_h^n)$ and $(\mathbf{u}_h^{n-1}, p_h^{n-1}, \mathbf{w}_h^{n-1})$, we assume that for $1 \leq n \leq N$ and consecutive meshes $\mathcal{T}_h^n$ and $\mathcal{T}_h^{n-1}$, the maximal common coarsening $\mathcal{T}_h^n \wedge \mathcal{T}_h^{n-1}$ and the minimal common refinement $\mathcal{T}_h^n \vee \mathcal{T}_h^{n-1}$ exist. As is well known, this assumption is true when $\{\mathcal{T}_h^n\}_{n=0}^N$ are newest vertex bisection refinement of the same macrotriangulation, cf. Lakkis & Makridakis (2006); Cascon *et al.* (2008). In addition, there is a uniform bound on the ratio of the sizes of elements in $K \in \mathcal{T}_h^n \vee \mathcal{T}_h^{n-1}$ and of elements $K' \in \mathcal{T}_h^n \wedge \mathcal{T}_h^{n-1}$ contained in K , that is,

$$\sup_{1 \leq n \leq N} \sup_{K' \subset K, K' \in \mathcal{T}_h^n \vee \mathcal{T}_h^{n-1}} \sup_{K \in \mathcal{T}_h^n \wedge \mathcal{T}_h^{n-1}} \frac{h_K}{h_{K'}} := C_{\text{ratio}} < \infty. \quad (4.5)$$

Similar assumptions are made in a posteriori error estimation of the heat equation, see, e.g., Verfürth (2003). Let $\mathbf{f}_h^n \in \mathbf{V}_h^n$ and $g_h^n \in Q_h^n$ be approximations to $\mathbf{f}^n$ and g^n , respectively. Within the interval $[t_{n-1}, t_n]$, we split the residuals into

$$\mathbf{R}_1 = \mathbf{f} - \mathbf{f}_h^n + \mathbf{S}_1^n + \mathbf{T}_1^n, \quad (4.6a)$$

$$R_2 = g - g_h^n + S_2^n + T_2^n, \quad (4.6b)$$

$$\mathbf{R}_3 = \mathbf{S}_3^n + \mathbf{T}_3^n, \quad (4.6c)$$

where the spatial residuals $\mathbf{S}_1^n \in \mathbf{V}'$, $S_2^n \in Q'$, $\mathbf{S}_3^n \in \mathbf{W}'$ are defined as

$$\begin{aligned} \langle \mathbf{S}_1^n, \mathbf{v} \rangle &:= \langle \mathbf{f}_h^n, \mathbf{v} \rangle - a(\mathbf{u}_h^n, \mathbf{v}) + b(\mathbf{v}, p_h^n), \\ \langle S_2^n, q \rangle &:= \langle g_h^n, q \rangle - c(\delta_t p_h^n, q) - b(\delta_t \mathbf{u}_h^n, q) - d(\mathbf{w}_h^n, q), \\ \langle \mathbf{S}_3^n, \mathbf{z} \rangle &:= -e(\mathbf{w}_h^n, \mathbf{z}) + d(\mathbf{z}, p_h^n), \end{aligned}$$

and the temporal residuals $\mathbf{T}_1^n(t) \in \mathbf{V}'$, $T_2^n(t) \in Q'$, $\mathbf{T}_3^n(t) \in \mathbf{W}'$ for $t \in [t_{n-1}, t_n]$ are

$$\begin{aligned} \langle \mathbf{T}_1^n, \mathbf{v} \rangle &:= a(\mathbf{u}_h^n - \mathbf{u}_h^\tau, \mathbf{v}) - b(\mathbf{v}, p_h^n - p_h^\tau), \\ \langle T_2^n, q \rangle &:= d(\mathbf{w}_h^n - \mathbf{w}_h^\tau, q), \\ \langle \mathbf{T}_3^n, \mathbf{z} \rangle &:= e(\mathbf{w}_h^n - \mathbf{w}_h^\tau, \mathbf{z}) - d(\mathbf{z}, p_h^n - p_h^\tau). \end{aligned}$$

By (4.4), the temporal derivatives of $\mathbf{R}_1$ and $\mathbf{R}_3$ over $[t_{n-1}, t_n]$ are

$$\langle \partial_t \mathbf{R}_1, \mathbf{v} \rangle = \langle \partial_t \mathbf{f} - \delta_t \mathbf{f}_h^n, \mathbf{v} \rangle + \langle \delta_t \mathbf{S}_1^n, \mathbf{v} \rangle, \quad (4.7a)$$

$$\langle \partial_t \mathbf{R}_3, \mathbf{v} \rangle = \langle \delta_t \mathbf{S}_3^n, \mathbf{v} \rangle. \quad (4.7b)$$

Next, we use the following fully discrete spatial error indicators which can be viewed as fully discrete

counterparts of the error estimators introduced in Section 3. We set,

$$\begin{aligned}\mathcal{E}_1^n &:= \mathcal{E}_{\mathcal{T}_h^n}^1(\mathbf{u}_h^n, p_h^n, \mathbf{f}_h^n), & \mathcal{E}_{1,t}^n &:= \mathcal{E}_{\mathcal{T}_h^n \vee \mathcal{T}_h^{n-1}}^1(\delta_t \mathbf{u}_h^n, \delta_t p_h^n, \delta_t \mathbf{f}_h^n), \\ \mathcal{E}_2^n &:= \mathcal{E}_{\mathcal{T}_h^n}^2(\delta_t \mathbf{u}_h^n, \delta_t p_h^n, \mathbf{w}_h^n, \mathbf{g}_h^n), \\ \mathcal{E}_3^n &:= \mathcal{E}_{\mathcal{T}_h^n}^3(p_h^n, \mathbf{w}_h^n), & \mathcal{E}_{3,t}^n &:= \mathcal{E}_{\mathcal{T}_h^n \vee \mathcal{T}_h^{n-1}}^3(\delta_t p_h^n, \delta_t \mathbf{w}_h^n).\end{aligned}$$

Throughout the rest of the presentation, we shall write $A \lesssim B$ provided $A \leq CB$, where C is a constant depending only on $\mu, \alpha, \beta, \mathbf{K}, \Omega, \Gamma_1, C_{\text{shape}}, C_{\text{ratio}}$. Since the spatial residuals are time-independent, their norms can be estimated as in the proof of Theorem 3.2. We have the following result.

LEMMA 4.1 For $1 \leq n \leq N$, it holds that

$$\|\mathbf{S}_1^n\|_I^2 \lesssim \mathcal{E}_1^n, \quad (4.8a)$$

$$\|\delta_t \mathbf{S}_1^n\|_I^2 \lesssim \mathcal{E}_{1,t}^n, \quad (4.8b)$$

$$\|\mathbf{S}_2^n\|_I^2 \lesssim \mathcal{E}_2^n, \quad (4.8c)$$

$$\|\mathbf{S}_3^n\|_I^2 \lesssim \mathcal{E}_3^n, \quad (4.8d)$$

$$\|\delta_t \mathbf{S}_3^n\|_I^2 \lesssim \mathcal{E}_{3,t}^n. \quad (4.8e)$$

Proof. For $\mathbf{v} \in \mathbf{V}$, let $\mathbf{v}_h$ be the Cl  ment interpolant on $\mathcal{T}_h^n$. It follows from (4.1a) and element-wise integration by parts that

$$\begin{aligned}\langle \mathbf{S}_1^n, \mathbf{v} \rangle &= \langle \mathbf{S}_1^n, \mathbf{v} - \mathbf{v}_h \rangle = \sum_{K \in \mathcal{T}_h^n} \int_K (\mathbf{f}_h^n + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h) - \alpha \nabla p_h) \cdot (\mathbf{v} - \mathbf{v}_h) \\ &\quad - \sum_{F \in \mathcal{F}(\mathcal{T}_h^n)} \int_F ([\boldsymbol{\sigma}(\mathbf{u}_h) - \alpha p_h \mathbf{I}] \mathbf{n}_F) \cdot (\mathbf{v} - \mathbf{v}_h).\end{aligned} \quad (4.9)$$

Using (4.9) and the same analysis of estimating $\|\mathbf{r}_1\|_I$ in Theorem 3.2, we obtain

$$\|\mathbf{S}_1^n\|_I = \sup_{\mathbf{v} \in \mathbf{V}, \|\mathbf{v}\|_a=1} \langle \mathbf{S}_1^n, \mathbf{v} \rangle \lesssim (\mathcal{E}_1^n)^{\frac{1}{2}}.$$

For any $\mathbf{v} \in \mathbf{V}$, let $\tilde{\mathbf{v}}_h$ be the Cl  ment interpolant of $\mathbf{v}$ on $\mathcal{T}_h^n \wedge \mathcal{T}_h^{n-1}$. Using (4.5), (4.1a), and integrating by parts over $\mathcal{T}_h^n \vee \mathcal{T}_h^{n-1}$, and following again the same analysis of estimating $\|\mathbf{r}_1\|_I$ in Theorem 3.2, we obtain a similar estimate:

$$\begin{aligned}\|\delta_t \mathbf{S}_1^n\|_I &= \sup_{\mathbf{v} \in \mathbf{V}, \|\mathbf{v}\|_a=1} \langle \delta_t \mathbf{S}_1^n, \mathbf{v} \rangle = \sup_{\mathbf{v} \in \mathbf{V}, \|\mathbf{v}\|_a=1} \langle \delta_t \mathbf{S}_1^n, \mathbf{v} - \tilde{\mathbf{v}}_h \rangle \\ &= \sup_{\mathbf{v} \in \mathbf{V}, \|\mathbf{v}\|_a=1} (\delta_t \mathbf{f}_h^n, \mathbf{v} - \tilde{\mathbf{v}}_h) - a(\delta_t \mathbf{u}_h^n, \mathbf{v} - \tilde{\mathbf{v}}_h) + b(\mathbf{v} - \tilde{\mathbf{v}}_h, \delta_t p_h^n) \lesssim (\mathcal{E}_{1,t}^n)^{\frac{1}{2}}.\end{aligned}$$

The remaining estimates can be proved in the same way. $\square$

Let $\|\mathbf{f} - \mathbf{f}_h^i\|_I = \|\mathbf{f} - \mathbf{f}_h^i\|_{\mathbf{V}'}$ and $\|\partial_t \mathbf{f} - \delta_t \mathbf{f}_h^i\|_I = \|\partial_t \mathbf{f} - \delta_t \mathbf{f}_h^i\|_{\mathbf{V}'}$. We present the first main result of this paper in the following theorem.

THEOREM 4.1 For $1 \leq i \leq n$, let $\mathbf{f}_h^i$ be the L^2 -projection of $\mathbf{f}^i$ onto $\mathbf{V}_h^i$, g_h^i the L^2 -projection of g^i onto Q_h^i . There exists a constant C_{drel} dependent only on $\mu, \alpha, \beta, \mathbf{K}, \Omega, C_{\text{shape}}, C_{\text{ratio}}$ such that for $n = 1, 2, \dots, N$, the error defined in (4.2) satisfies

$$\begin{aligned} & \left| (E_u, E_p, E_w)(t_n) \right|^2 + \|(E_u, E_p, E_w)\|_{L^2(0, t_n; X)}^2 \\ & \leq C_{\text{drel}} \left\{ \left| (E_u, E_p, E_w)(0) \right|^2 + \left(\sum_{i=1}^n \tau_i \tilde{\mathcal{E}}_{\text{time}}^i + \tau_i (\mathcal{E}_2^i)^{\frac{1}{2}} + \tilde{\mathcal{E}}_{\text{data}}^i \right)^2 \right. \\ & \quad \left. + \sum_{i=1}^n \tau_i \mathcal{E}_{\text{time}}^i + \tau_i \mathcal{E}_{\text{space}}^i + \mathcal{E}_{\text{data}}^i \right\}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathcal{E}}_{\text{time}}^i &= \|\operatorname{div}(\mathbf{w}_h^i - \mathbf{w}_h^{i-1})\|, \quad \tilde{\mathcal{E}}_{\text{data}}^i = \int_{t_{i-1}}^{t_i} \|g - g_h^i\| dt, \\ \mathcal{E}_{\text{time}}^i &= \|\mathbf{u}_h^i - \mathbf{u}_h^{i-1}\|_a^2 + \|p_h^i - p_h^{i-1}\|_c^2 + \|\mathbf{w}_h^i - \mathbf{w}_h^{i-1}\|_{\mathbf{w}}^2, \\ \mathcal{E}_{\text{space}}^i &= \mathcal{E}_1^i + \mathcal{E}_{1,t}^i + \mathcal{E}_2^i + \mathcal{E}_3^i + \mathcal{E}_{3,t}^i, \\ \mathcal{E}_{\text{data}}^i &= \int_{t_{i-1}}^{t_i} (\|\mathbf{f} - \mathbf{f}_h^i\|_r^2 + \|\partial_t \mathbf{f} - \delta_t \mathbf{f}_h^i\|_r^2 + \|g - g_h^i\|^2) dt. \end{aligned}$$

Proof. Applying Lemma 2.1 to (4.3) yields

$$\begin{aligned} & \left| (E_u, E_p, E_w)(t_n) \right|^2 + \|(E_u, E_p, E_w)\|_{L^2(0, t_n; X)}^2 \\ & \leq C_{\text{stab}} \left(\left| (E_u, E_p, E_w)(0) \right|^2 + \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|R_2\|_r dt \right)^2 \right. \\ & \quad \left. + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\|\mathbf{R}_1\|_r^2 + \|\partial_t \mathbf{R}_1\|_r^2 + \|R_2\|_r^2 + \|\mathbf{R}_3\|_r^2 + \|\partial_t \mathbf{R}_3\|_r^2) dt \right). \end{aligned}$$

For any $\mathbf{v} \in \mathbf{V}$, the continuity of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ implies that

$$\|\mathbf{T}_1^i\|_r = \sup_{\mathbf{v} \in \mathbf{V}, \|\mathbf{v}\|_a=1} \langle \mathbf{T}_1^i, \mathbf{v} \rangle \lesssim \|\mathbf{u}_h^i - \mathbf{u}_h^\tau\|_a + \|p_h^i - p_h^\tau\|_c. \quad (4.10)$$

A combination of (4.6a), (4.8a) and (4.10) then shows that when $t \in [t_{i-1}, t_i]$,

$$\|\mathbf{R}_1\|_r \lesssim \|\mathbf{f} - \mathbf{f}_h^i\|_r + \|\mathbf{u}_h^i - \mathbf{u}_h^\tau\|_a + \|p_h^i - p_h^\tau\|_c + (\mathcal{E}_1^i)^{\frac{1}{2}}. \quad (4.11)$$

For $t \in [t_{i-1}, t_i]$, it is readily checked that

$$\begin{aligned} \|\mathbf{u}_h^i - \mathbf{u}_h^\tau\|_a &= \frac{t_i - t}{\tau_i} \|\mathbf{u}_h^i - \mathbf{u}_h^{i-1}\|_a, \\ \|p_h^i - p_h^\tau\|_c &= \frac{t_i - t}{\tau_i} \|p_h^i - p_h^{i-1}\|_c. \end{aligned} \quad (4.12)$$

Integrating (4.11) over $[t_{i-1}, t_i]$ and using (4.12), we obtain

$$\int_{t_{i-1}}^{t_i} \|\mathbf{R}_1\|_r^2 dt \lesssim \int_{t_{i-1}}^{t_i} \|\mathbf{f} - \mathbf{f}_h^i\|_r^2 dt + \tau_i \|\mathbf{u}_h^i - \mathbf{u}_h^{i-1}\|_a^2 + \tau_i \|p_h^i - p_h^{i-1}\|_c^2 + \tau_i \mathcal{E}_1^i. \quad (4.13)$$

On the other hand, using (4.7a) and (4.8b), we obtain for $t \in [t_{i-1}, t_i]$,

$$\|\partial_t \mathbf{R}_1\|_r \lesssim \|\partial_t \mathbf{f} - \delta_t \mathbf{f}_h^i\|_r + (\mathcal{E}_{1,t}^i)^{\frac{1}{2}}. \quad (4.14)$$

Similarly, using (4.6), (4.7b), Lemma 4.1, and (4.12), one can estimate $\mathbf{R}_3$, $\partial_t \mathbf{R}_3$, $\mathbf{R}_2$ and obtain the following bounds

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \|\mathbf{R}_3\|_r^2 dt &\lesssim \tau_i \|\mathbf{w}_h^i - \mathbf{w}_h^{i-1}\|_{\mathbf{w}}^2 + \tau_i \|p_h^i - p_h^{i-1}\|_c^2 + \tau_i \mathcal{E}_{3,t}^i, \\ \int_{t_{i-1}}^{t_i} \|\partial_t \mathbf{R}_3\|_r^2 dt &\lesssim \tau_i \mathcal{E}_{3,t}^i, \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \|\mathbf{R}_2\|_r dt &\lesssim \int_{t_{i-1}}^{t_i} \|g - g_h^i\| dt + \tau_i \|\operatorname{div}(\mathbf{w}_h^{i-1} - \mathbf{w}_h^i)\| + \tau_i (\mathcal{E}_2^i)^{\frac{1}{2}}, \\ \int_{t_{i-1}}^{t_i} \|\mathbf{R}_2\|_r^2 dt &\lesssim \int_{t_{i-1}}^{t_i} \|g - g_h^i\|^2 dt + \tau_i \|\operatorname{div}(\mathbf{w}_h^{i-1} - \mathbf{w}_h^i)\|^2 + \tau_i \mathcal{E}_2^i. \end{aligned} \quad (4.16)$$

Combining (4.13)–(4.16) completes the proof. $\square$

REMARK 4.1 The first two terms in $\mathcal{E}_{\text{data}}^i$ can be further estimated by

$$\begin{aligned} \|\mathbf{f} - \mathbf{f}_h^i\|_r &\lesssim \sum_{K \in \mathcal{T}_h^i} h_K^2 \|\mathbf{f} - \mathbf{f}_h^i\|_K^2, \\ \|\partial_t \mathbf{f} - \delta_t \mathbf{f}_h^i\|_r &\lesssim \sum_{K \in \mathcal{T}_h^i \wedge \mathcal{T}_h^{i-1}} h_K^2 \|\partial_t \mathbf{f} - \delta_t \mathbf{f}_h^i\|_K^2. \end{aligned}$$

5. Lower bound

In this section, we show that $\tau_n \mathcal{E}_{\text{time}}^n$ and $\tau_n \mathcal{E}_{\text{space}}^n$ are lower bounds of the space-time discretization error of the fully discrete scheme (4.1). First we present a lemma comparing the spatial residual with the spatial error indicators. Since the spatial error estimators and residuals are time-independent, the proof follows from the well-known Verfürth bubble function technique for a posteriori error estimates for stationary Stokes and Poisson's equations, see, e.g., Verfürth (1991); Alonso (1996); Demlow & Hirani (2014). Throughout the rest, $\underline{C}$ is a generic constant that depends only on $\lambda, \mu, \alpha, \beta, \mathbf{K}, \Omega, \Gamma_1, C_{\text{shape}}, C_{\text{ratio}}$. Hence the constant in some lower bounds may not be locking free.

LEMMA 5.1 Let $\lambda, \mu, \mathbf{K}$ be piecewise constants over $\mathcal{T}_h^n$. For $1 \leq n \leq N$, it holds that

$$\mathcal{E}_1^n \leq \underline{C} \|\mathbf{S}_1^n\|_r^2, \quad (5.1a)$$

$$\mathcal{E}_{1,t}^n \leq \underline{C} \|\delta_t \mathbf{S}_1^n\|_r^2, \quad (5.1b)$$

$$\mathcal{E}_2^n \lesssim \|\mathbf{S}_2^n\|_r^2, \quad (5.1c)$$

$$\mathcal{E}_3^n \lesssim \|\mathbf{S}_3^n\|_r^2, \quad (5.1d)$$

$$\mathcal{E}_{3,t}^n \lesssim \|\delta_t \mathbf{S}_3^n\|_r^2. \quad (5.1e)$$

Proof. To prove (5.1a), it suffices to find $\mathbf{v} \in \mathbf{V}$, such that

$$\mathcal{E}_1^n \leq \underline{C} \langle \mathbf{S}_1^n, \mathbf{v} \rangle, \quad \|\mathbf{v}\|_a^2 \leq \underline{C} \mathcal{E}_1^n.$$

For each $K \in \mathcal{T}_h$, let $R_K = (\mathbf{f}_h^n + \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_h) - \alpha \nabla p_h)|_K$ and let ϕ_K denote the volume bubble function supported on K so that the maximum is 1. For each $F \in \mathcal{F}(\mathcal{T}_h)$, let $J_F = -[\boldsymbol{\sigma}(\mathbf{u}_h) - \alpha p_h \mathbf{I}]|_F \mathbf{n}_F$, and recall that ϕ_F is the face bubble function supported on union of neighboring elements of F . The desired $\mathbf{v}$ is then defined as

$$\mathbf{v} = \gamma_1 \sum_{K \in \mathcal{T}_h} h_K^2 R_K \phi_K + \gamma_2 \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F J_F \phi_F,$$

where h_F is the diameter of F , and γ_1, γ_2 are undetermined constants. Using the Cauchy–Schwarz inequality and finite overlapping of supports of $\{\phi_K\}$ and $\{\phi_F\}$, one can easily show that $\|\mathbf{v}\|_a^2 \leq \underline{C} \mathcal{E}_1^n$. On the other hand, (4.9) implies

$$\langle \mathbf{S}_1^n, \mathbf{v} \rangle = \sum_{K \in \mathcal{T}_h} \int_K R_K \cdot \mathbf{v} + \sum_{F \in \mathcal{F}(\mathcal{T}_h)} \int_F J_F \cdot \mathbf{v}.$$

$\langle \mathbf{S}_1^n, \mathbf{v} \rangle \geq \underline{C} \mathcal{E}_1^n$ then follows from Young's inequality and suitably chosen γ_1, γ_2 , see, e.g., Lemma 5.1 in Verfürth (2003) for details. Other lower bounds can be shown in an analogous fashion. $\square$

REMARK 5.1 Based on the $\|\cdot\|_a$ -norm, it seems that the dependence on λ in lower bounds (5.1a) and (5.1b) cannot be avoided. To obtain an error estimator that is a *robust* lower bound, one can apply the analysis here to the four- or five-field formulation Lee (2016); Ahmed *et al.* (2019) in Biot's consolidation model.

We present the second main result in the following theorem. Similar technique in the proof was used in Verfürth (2003) for proving the lower bound in a posteriori error estimation for the primal formulation of the heat equation.

THEOREM 5.1 Let $\lambda, \mu, \mathbf{K}$ be piecewise constants on $\mathcal{T}_h^n$. For $n = 1, 2, \dots, N$,

$$\begin{aligned} \tau_n \mathcal{E}_{\text{time}}^n + \tau_n \mathcal{E}_{\text{space}}^n &\leq \underline{C} \left\{ \|(E_u, E_p, E_w)\|_{L^2(t_{n-1}, t_n; X)}^2 \right. \\ &\quad \left. + \int_{t_{n-1}}^{t_n} (\|\mathbf{f} - \mathbf{f}_h^n\|_r^2 + \|\partial_t \mathbf{f} - \delta_t \mathbf{f}_h^n\|_r^2 + \|g - g_h^n\|^2) dt \right\}. \end{aligned}$$

Proof. First by (2.2) and definitions of residuals in (4.4), we have

$$\|\mathbf{R}_1\|_r \lesssim \|E_u\|_a + \|E_p\|_c, \quad (5.2a)$$

$$\|R_2\|_r \lesssim \|\partial_t E_p\|_c + \|\partial_t \operatorname{div} E_u\| + \|\operatorname{div} E_w\|, \quad (5.2b)$$

$$\|\mathbf{R}_3\|_r \lesssim \|E_w\|_e + \|E_p\|_c, \quad (5.2c)$$

$$\|\partial_t \mathbf{R}_1\|_r \lesssim \|\partial_t E_u\|_a + \|\partial_t E_p\|_c, \quad (5.2d)$$

$$\|\partial_t \mathbf{R}_3\|_r \lesssim \|\partial_t E_w\|_{\mathbf{W}'} + \|\partial_t E_p\|_c. \quad (5.2e)$$

Consider the bilinear form

$$B(\mathbf{w}, p; \mathbf{z}, q) = e(\mathbf{w}, \mathbf{z}) - d(\mathbf{z}, p) + d(\mathbf{w}, q)$$

of the mixed formulation of the elliptic equation. Due to the inf-sup condition of B , there exist $\mathbf{z} \in \mathbf{W}$ and $q \in Q$ with $\|\mathbf{z}\|_{\mathbf{W}} = 1, \|q\|_c = 1$ such that

$$\begin{aligned} \|\mathbf{w}_h^n - \mathbf{w}_h^\tau\|_{\mathbf{W}} + \|p_h^n - p_h^\tau\|_c &\lesssim B(\mathbf{w}_h^n - \mathbf{w}_h^\tau, p_h^n - p_h^\tau; \mathbf{z}, q) \\ &= \langle T_2^n, q \rangle + \langle \mathbf{T}_3^n, \mathbf{z} \rangle, \quad t \in [t_{n-1}, t_n]. \end{aligned}$$

Using the previous estimate, the triangle and Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned}
 & \| \mathbf{w}_h^n - \mathbf{w}_h^\tau \|_{\mathbf{W}} + \| p_h^n - p_h^\tau \|_c \\
 & \lesssim \langle \mathbf{R}_2, q \rangle + \langle \mathbf{R}_3, \mathbf{z} \rangle - \langle g - g_h^n, q \rangle - \langle \mathbf{S}_2^n, q \rangle - \langle \mathbf{S}_3^n, \mathbf{z} \rangle \\
 & \lesssim \| \mathbf{R}_2 \|_{\mathcal{I}} + \| \mathbf{R}_3 \|_{\mathcal{I}} + \| g - g_h^n \| + \| \mathbf{S}_2^n \|_{\mathcal{I}} + \| \mathbf{S}_3^n \|_{\mathcal{I}}.
 \end{aligned} \tag{5.3}$$

Setting $\mathbf{v} = (\mathbf{u}_h^n - \mathbf{u}_h^\tau) / \| \mathbf{u}_h^n - \mathbf{u}_h^\tau \|_a$ in the definition of $\mathbf{T}_1^n$, we have for $t \in [t_{n-1}, t_n]$,

$$\begin{aligned}
 \| \mathbf{u}_h^n - \mathbf{u}_h^\tau \|_a &= \langle \mathbf{T}_1^n, \mathbf{v} \rangle + b(\mathbf{v}, p_h^n - p_h^\tau) \\
 &= \langle \mathbf{R}_1, \mathbf{v} \rangle - \langle \mathbf{f} - \mathbf{f}_h^n, \mathbf{v} \rangle - \langle \mathbf{S}_1^n, \mathbf{v} \rangle + b(\mathbf{v}, p_h^n - p_h^\tau) \\
 &\leq \| \mathbf{R}_1 \|_{\mathcal{I}} + \| \mathbf{f} - \mathbf{f}_h^n \|_{\mathcal{I}} + \| \mathbf{S}_1^n \|_{\mathcal{I}} + \| p_h^n - p_h^\tau \|_c.
 \end{aligned} \tag{5.4}$$

A combination of (5.3), (5.4), (4.12) and Lemma 4.1 shows that

$$\begin{aligned}
 \tau_n \mathcal{E}_{\text{time}}^n &\lesssim \int_{t_{n-1}}^{t_n} (\| \mathbf{f} - \mathbf{f}_h^n \|_{\mathcal{I}}^2 + \| g - g_h^n \|^2 + \| \mathbf{R}_1 \|_{\mathcal{I}}^2 \\
 &\quad + \| \mathbf{R}_2 \|_{\mathcal{I}}^2 + \| \mathbf{R}_3 \|_{\mathcal{I}}^2) dt + \tau_n (\mathcal{E}_1^n + \mathcal{E}_2^n + \mathcal{E}_3^n).
 \end{aligned} \tag{5.5}$$

It remains to estimate $\mathcal{E}_1^n$, $\mathcal{E}_2^n$, and $\mathcal{E}_3^n$. Let

$$\phi(t) = (\alpha + 1) \left(\frac{t - t_{n-1}}{\tau_n} \right)^\alpha,$$

where $\alpha > 0$ is a constant that will be specified later. It follows from Lemma 5.1 and the triangle inequality that

$$\begin{aligned}
 \tau_n (\mathcal{E}_1^n + \mathcal{E}_2^n + \mathcal{E}_3^n) &= \int_{t_{n-1}}^{t_n} \phi(t) (\mathcal{E}_1^n + \mathcal{E}_2^n + \mathcal{E}_3^n) dt \\
 &\leq \underline{C} \int_{t_{n-1}}^{t_n} \phi(t) (\| \mathbf{S}_1^n \|_{\mathcal{I}}^2 + \| \mathbf{S}_2^n \|_{\mathcal{I}}^2 + \| \mathbf{S}_3^n \|_{\mathcal{I}}^2) dt \\
 &\leq \underline{C} (\alpha + 1) \int_{t_{n-1}}^{t_n} (\| \mathbf{R}_1 \|_{\mathcal{I}}^2 + \| \mathbf{f} - \mathbf{f}_h^n \|_{\mathcal{I}}^2 + \| \mathbf{R}_2 \|_{\mathcal{I}}^2 \\
 &\quad + \| g - g_h^n \|_{\mathcal{I}}^2 + \| \mathbf{R}_3 \|_{\mathcal{I}}^2) dt + \underline{C} \int_{t_{n-1}}^{t_n} \phi(t) (\| \mathbf{T}_1^n \|_{\mathcal{I}}^2 + \| \mathbf{T}_2^n \|_{\mathcal{I}}^2 + \| \mathbf{T}_3^n \|_{\mathcal{I}}^2) dt,
 \end{aligned} \tag{5.6}$$

where the generic constant $\underline{C}$ is independent of α . For any $\mathbf{v} \in \mathbf{V}$ with $\| \mathbf{v} \|_a = 1$, direct calculation shows that

$$\begin{aligned}
 \int_{t_{n-1}}^{t_n} \phi(t) \langle \mathbf{T}_1^n, \mathbf{v} \rangle^2 dt &= \int_{t_{n-1}}^{t_n} \phi(t) \{ a(\mathbf{u}_h^n - \mathbf{u}_h^\tau, \mathbf{v}) - b(\mathbf{v}, p_h^n - p_h^\tau) \}^2 dt \\
 &= \{ a(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}) - b(\mathbf{v}, p_h^n - p_h^{n-1}) \}^2 \int_{t_{n-1}}^{t_n} \phi(t) \left(\frac{t_n - t}{\tau_n} \right)^2 dt \\
 &\leq \underline{C} (\| \mathbf{u}_h^n - \mathbf{u}_h^{n-1} \|_a^2 + \| p_h^n - p_h^{n-1} \|_c^2) \tau_n \int_0^1 s^\alpha (1 - s)^2 ds \\
 &\leq \underline{C} \tau_n \mathcal{E}_{\text{time}}^n F(\alpha),
 \end{aligned}$$

where $F(\alpha) = 1 - \frac{2(\alpha+1)}{\alpha+2} + \frac{\alpha+1}{\alpha+3}$. Hence

$$\int_{t_{n-1}}^{t_n} \phi(t) \|\mathbf{T}_1^n\|_r^2 dt = \sup_{\mathbf{v} \in \mathbf{V}, \|\mathbf{v}\|_a=1} \int_{t_{n-1}}^{t_n} \phi(t) \langle \mathbf{T}_1^n, \mathbf{v} \rangle^2 dt \leq \underline{C} \tau_n \mathcal{E}_{\text{time}}^n F(\alpha). \quad (5.7)$$

Similarly, we have

$$\int_{t_{n-1}}^{t_n} \phi(t) (\|T_2^n\|_r^2 + \|\mathbf{T}_3^n\|_r^2) dt \leq \underline{C} \tau_n \mathcal{E}_{\text{time}}^n F(\alpha). \quad (5.8)$$

Combining (5.6), (5.7) and (5.8), we obtain

$$\begin{aligned} \tau_n (\mathcal{E}_1^n + \mathcal{E}_2^n + \mathcal{E}_3^n) &\leq \underline{C}(\alpha+1) \int_{t_{n-1}}^{t_n} (\|\mathbf{R}_1\|_r^2 + \|\mathbf{f} - \mathbf{f}_h^n\|_r^2 + \|\mathbf{R}_2\|_r^2 \\ &\quad + \|g - g_h^n\|_r^2 + \|\mathbf{R}_3\|_r^2) dt + \underline{C} \tau_n \mathcal{E}_{\text{time}}^n F(\alpha). \end{aligned} \quad (5.9)$$

Note that $F(\alpha) \rightarrow 0$ as $\alpha \rightarrow \infty$. It then follows from (5.5) and (5.9) with sufficiently large α that

$$\tau_n \mathcal{E}_{\text{time}}^n \leq \underline{C} \int_{t_{n-1}}^{t_n} (\|\mathbf{f} - \mathbf{f}_h^n\|_r^2 + \|g - g_h^n\|_r^2 + \|\mathbf{R}_1\|_r^2 + \|\mathbf{R}_2\|_r^2 + \|\mathbf{R}_3\|_r^2) dt. \quad (5.10)$$

Clearly, the bounds (5.9) and (5.10) imply that

$$\begin{aligned} \tau_n (\mathcal{E}_1^n + \mathcal{E}_2^n + \mathcal{E}_3^n) \\ \leq \underline{C} \int_{t_{n-1}}^{t_n} (\|\mathbf{f} - \mathbf{f}_h^n\|_r^2 + \|g - g_h^n\|_r^2 + \|\mathbf{R}_1\|_r^2 + \|\mathbf{R}_2\|_r^2 + \|\mathbf{R}_3\|_r^2) dt. \end{aligned} \quad (5.11)$$

Finally, using Lemma 5.1 and (4.7a), (4.7b), we have

$$\begin{aligned} \mathcal{E}_{1,t}^n + \mathcal{E}_{3,t}^n &\leq \underline{C} (\|\delta_t \mathbf{S}_1^n\|_r^2 + \|\delta_t \mathbf{S}_3^n\|_r^2) \\ &\leq \underline{C} (\|\partial_t \mathbf{f} - \delta_t \mathbf{f}_h^n\|_r^2 + \|\partial_t \mathbf{R}_1\|_r^2 + \|\partial_t \mathbf{R}_3\|_r^2) \end{aligned} \quad (5.12)$$

on $[t_{n-1}, t_n]$. Combining (5.10), (5.11), (5.12), and (5.2) then completes the proof. $\square$

In practice, one can use $\mathcal{E}_{\text{time}}^n$ and $\mathcal{E}_{\text{data}}^n$ to adjust the time step size and $\mathcal{E}_{\text{space}}^n$ to refine and coarsen the spatial mesh. Due to the complexity of space-time adaptivity, we shall not present a concrete adaptive algorithm for the Biot's system. Such algorithms can be found in e.g., Ern & Meunier (2009); Riedlbeck *et al.* (2017); Ahmed *et al.* (2019). For the heat equation, readers are referred to e.g., Chen & Feng (2004); Kreuzer *et al.* (2012); Verfürth (2013); Gaspoz *et al.* (2019) for space-time adaptive algorithms as well as convergence analysis of adaptive methods. We point out that, once a space-time error indicator is available, a corresponding adaptive strategy follows and is largely independent of the equations.

We now also present a new error estimator for mixed methods for time dependent Darcy flow described by (2.3). The fully discrete scheme (4.1) with $\mathbf{u}_h = \mathbf{v} = \mathbf{0}$ reduces to

$$c(\delta_t p_h^n, q) + d(\mathbf{w}_h^n, q) = (g^n, q), \quad q \in Q_h^n, \quad (5.13a)$$

$$e(\mathbf{w}_h^n, \mathbf{z}) - d(\mathbf{z}, p_h^n) = 0, \quad \mathbf{z} \in \mathbf{W}_h^n, \quad (5.13b)$$

which obviously is a discretization of the heat equation or time-dependent Darcy flow (2.3). Therefore, the a posteriori analysis for (4.1) directly applies to (5.13). Here $\mathbf{W}_h^n \times Q_h^n$ is the Raviart–Thomas and Brezzi–Douglas–Marini mixed element space. Given an interval $I \subseteq [0, T]$, we define the norm

$$\|(q, \mathbf{z})\|_{L^2(I; Y)}^2 := \int_I (\|q\|_c^2 + \|\partial_t q\|_c^2 + \|\mathbf{z}\|_{\mathbf{W}}^2 + \|\partial_t \mathbf{z}\|_{\mathbf{W}'}^2) ds.$$

Going through the proof of Theorems 4.1 and 5.1, $\mathbf{R}_1$ disappears when deriving the upper and lower bounds for the error of (5.13). Therefore we obtain the following a posteriori error estimates.

COROLLARY 5.1 For $n = 1, 2, \dots, N$, the error of (5.13) satisfies

$$\begin{aligned} & \|E_p(t_n)\|_c^2 + \|E_w(t_n)\|_e^2 + \|(E_p, E_w)\|_{L^2(0, t_n; Y)}^2 \\ & \lesssim \eta_{\text{init}} + \left(\sum_{i=1}^n \tau_i \tilde{\mathcal{E}}_{\text{time}}^i + \tau_i (\tilde{\eta}_{\text{space}}^i)^{\frac{1}{2}} + \tilde{\mathcal{E}}_{\text{data}}^i \right)^2 \\ & + \sum_{i=1}^n \left(\tau_i \eta_{\text{time}}^i + \tau_i \eta_{\text{space}}^i + \int_{t_{i-1}}^{t_i} \|g - g_h^i\|^2 dt \right), \end{aligned}$$

where

$$\begin{aligned} \eta_{\text{init}} &= \|p(0) - p_h^0\|_c^2 + \|\mathbf{w}(0) - \mathbf{w}_h^0\|_e^2, \\ \eta_{\text{time}}^i &= \|p_h^i - p_h^{i-1}\|_c^2 + \|\mathbf{w}_h^i - \mathbf{w}_h^{i-1}\|_{\mathbf{w}}^2, \\ \tilde{\eta}_{\text{space}}^i &= \|g_h^i - \beta \delta_t p_h^i - \text{div } \mathbf{w}_h^i\|^2, \\ \eta_{\text{space}}^i &= \tilde{\eta}_{\text{space}}^i + \mathcal{E}_{\mathcal{T}_h^i}^3(p_h^i, \mathbf{w}_h^i) + \mathcal{E}_{\mathcal{T}_h^i \vee \mathcal{T}_h^{i-1}}^3(\delta_t p_h^i, \delta_t \mathbf{w}_h^i). \end{aligned}$$

In addition, when $\mathbf{K}$ is a piecewise constant on $\mathcal{T}_h^n$ it holds that,

$$\tau_n \eta_{\text{time}}^n + \tau_n \eta_{\text{space}}^n \lesssim \|(E_p, E_w)\|_{L^2(t_{n-1}, t_n; Y)}^2 + \int_{t_{i-1}}^{t_i} \|g - g_h^i\|^2 dt.$$

6. Numerical examples

To support the theoretical results and show the behavior of the fully discrete error indicators, we present a two-dimensional numerical example. The domain $\Omega = (0, 1) \times (0, 1)$ is the unit square in $\mathbb{R}^2$, and we consider the three field Biot's problem (2.1) given in § 2. The rest of the setup is as follows.

- We set the Lamé parameters as $\lambda = \mu = 0.4$.
- $\beta = 1$, $\alpha = 1$, and $K = I \in \mathbb{R}^{2 \times 2}$, the 2×2 identity matrix.
- The analytic solution to the problem is

$$\begin{aligned} \mathbf{u}(t, \mathbf{x}) &= \cos t \begin{pmatrix} \sin(\pi x_1) \sin(\pi x_2) \\ \sin(\pi x_1) \sin(\pi x_2) \end{pmatrix}, \quad \mathbf{w} = -\nabla p = \pi \sin t \begin{pmatrix} \sin(\pi x_1) \cos(\pi x_2) \\ \cos(\pi x_1) \sin(\pi x_2) \end{pmatrix}, \\ p(t, \mathbf{x}) &= \sin t \cos(\pi x_1) \cos(\pi x_2) \end{aligned}$$

- As boundary conditions we take homogeneous Dirichlet condition for $\mathbf{u}$ and homogeneous Neumann condition for p . Correspondingly, $\Gamma_2 = \emptyset$ in Section 2.
- We use a sequence of uniform triangular grids $\mathcal{T}_{h_k}$ with mesh sizes $h_k = 2^{-k}$, $k = 1 : J$, $J = 6$ or $J = 7$. In Figure 1 we have shown the coarsest mesh and a finer mesh.
- The spatial discretization for (2.1) is based on the lowest order finite space triple $\mathbf{V}_{h_k}^0 \times \mathcal{Q}_{h_k}^0 \times \mathbf{W}_{h_k}^0$ on grid $\mathcal{T}_{h_k}$ in Section 3.

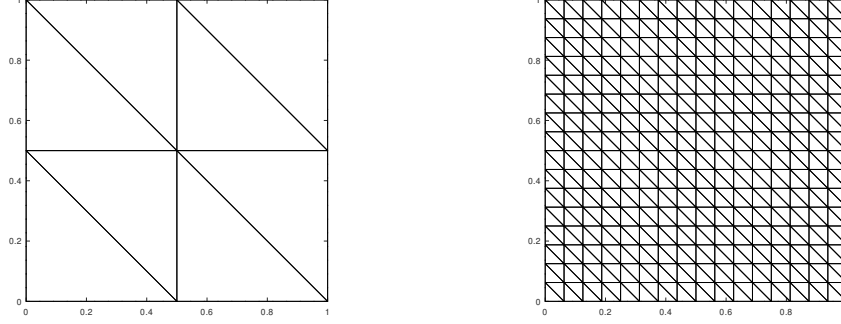


FIG. 1. Coarsest mesh used in the experiments is with mesh size $h = 2^{-1}$ (left). On the right we have shown a finer mesh with meshsize $h = 2^{-4}$.

- We start the simulations at $t = 0$ and reach the final time $T = 1$ after $N = T/\tau$ time steps with uniform time step size τ . We present two sets of tests: (1) a test where $h = h_k = 2^{-k}$, $k = 1 : 7$ and $\tau = \tau_k = 0.4h_k$ change simultaneously; and (2) a test with fixed $\tau = 5 \times 10^{-5}$ and varying $h = h_k = 2^{-k}$, $k = 1 : 6$.
- We set $p_h^0 = 0$, $\mathbf{w}_h^0 = \mathbf{0}$, and choose $\mathbf{u}_h^0$ such that (4.1a) is satisfied at $t = 0$, i.e. the equation (4.1a) holds for $n = 0$.
- In computing the a priori error, we have omitted the term containing the error term $\|\partial_t(\mathbf{w} - \mathbf{w}_h^\tau)\|_{\mathbf{w}'}^2$ from the space-time norm $\|(\cdot, \cdot, \cdot)\|_{L^2(I; X)}$ and the modified norm is still denoted as $\|(\cdot, \cdot, \cdot)\|_{L^2(I; X)}$ by abuse of notation. This is motivated by the fact that such term only enters the computation of the a priori error and its computation is rather involved. However, the ratio between the true error and our indicator might be exaggerated by dropping that term.

We first present a simple test illustrating the efficiency of the indicator numerically by decreasing h and the time step τ simultaneously. Let $(\mathbf{u}_h^\tau, p_{h_k}^\tau, \mathbf{w}_{h_k}^\tau)$ be the continuous temporal linear interpolant of the solution from (4.1) with $\mathcal{T}_h = \mathcal{T}_{h_k}$ and time step size $\tau = \tau_k = 0.4h_k$. In the first test, we compute the a priori error $E_k = \|(\mathbf{u} - \mathbf{u}_{h_k}^\tau, p - p_{h_k}^\tau, \mathbf{w} - \mathbf{w}_{h_k}^\tau)\|_{L^2(0, T; X)}$ in Theorem 4.1 with 5-point Gaussian quadrature on each time interval $[t_{n-1}, t_n]$ and 25-point Gaussian quadrature on every element in $\mathcal{T}_{h_k}$. The global error E_k is compared with following a posteriori error indicator

$$\mathcal{E}_k = \left(\sum_{n=1}^N \tau_k \mathcal{E}_{\text{time}}^n + \sum_{n=1}^N \tau_k \mathcal{E}_{\text{space}}^n \right)^{1/2}$$

on the stationary mesh $\mathcal{T}_{h_k}$. Here we do not include the data oscillation $\mathcal{E}_{\text{data}}^n$ in $\mathcal{E}_k$ because it is generally small and dominated by other terms. The results are shown in Table 1. As is seen from these results the convergence of the method is of order $(h + \tau)$. Moreover, the ratio between the a priori error and the value of the a posteriori error indicator approaches 2.5.

The next set of tests is for a fixed relatively small time step τ and aimed at comparing various characteristics of the indicators for different mesh sizes. For a fixed time t_n , $n = 1 : N$ on a grid of size

$h = 2^{-k}$	E_k	$\mathcal{E}_k$	$\mathcal{E}_k/E_k$	E_{k-1}/E_k	$\mathcal{E}_{k-1}/\mathcal{E}_k$
$k = 1$	0.947	2.375	2.51	N/A	N/A
$k = 2$	0.499	1.250	2.51	1.906	1.900
$k = 3$	0.253	0.633	2.50	1.974	1.976
$k = 4$	0.127	0.317	2.50	1.992	1.994
$k = 5$	0.064	0.159	2.50	1.998	1.998
$k = 6$	0.032	0.079	2.50	2.000	2.000
$k = 7$	0.016	0.039	2.50	2.000	2.000

Table 1. A priori and a posteriori errors for simultaneously decreasing h and τ with $\tau = 0.4h$.

$h_k = 2^{-k}$ we denote the “true” error at $t = t_n$ as

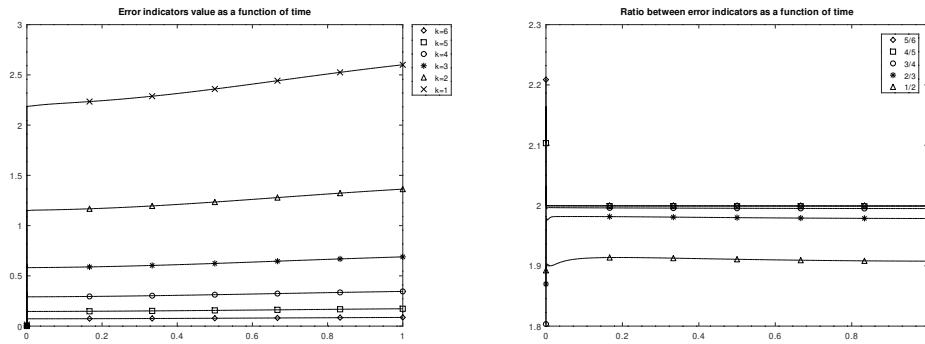
$$e_k^n = (\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_a^2 + \|\partial_t \mathbf{u}(t_n) - \delta_t \mathbf{u}_h^n\|_a^2 + \|p(t_n) - p_h^n\|_c^2 + \|\partial_t p(t_n) - \delta_t p_h^n\|_c^2 + \|\mathbf{w}(t_n) - \mathbf{w}_h^n\|_{\mathbf{w}}^2)^{\frac{1}{2}},$$

which is again computed using 25-point Gaussian quadrature element-wise. We compare e_k^n with the following instantaneous error estimator at t_n

$$\mathcal{E}_k^n = (\mathcal{E}_{\text{time}}^n + \mathcal{E}_{\text{space}}^n)^{1/2}.$$

Note that $E_k \approx (\sum_{n=1}^N \tau [e_k^n]^2)^{\frac{1}{2}}$ and $\mathcal{E}_k = (\sum_{n=1}^N \tau [\mathcal{E}_k^n]^2)^{\frac{1}{2}}$.

In Figure 2 we show the plot of $\mathcal{E}_k^n$ for $n = 1 : N$ with $N = 20,000$ for different mesh size h_k as well as the ratios between the indicators on two consecutive meshes, namely, $(\mathcal{E}_k^n / \mathcal{E}_{k-1}^n)$. Similar behavior is observed in Figure 3, where we have plotted the error reduction as predicted by the fully discrete error indicators.

FIG. 2. Plot of $\{\mathcal{E}_k^n\}_{n=1}^N$, $k = 1 : 6$ (left) and $(\mathcal{E}_{k-1}^n / \mathcal{E}_k^n)$ for $k = 2 : 6$ (right)

In Figure 4 we plotted the ratio between the error indicators and the norm of the error as a function of time for varying mesh sizes. As is seen from this plot, this ratio remains bounded. Notice that the theory developed earlier shows reliability and efficiency when we integrate the indicators and the error

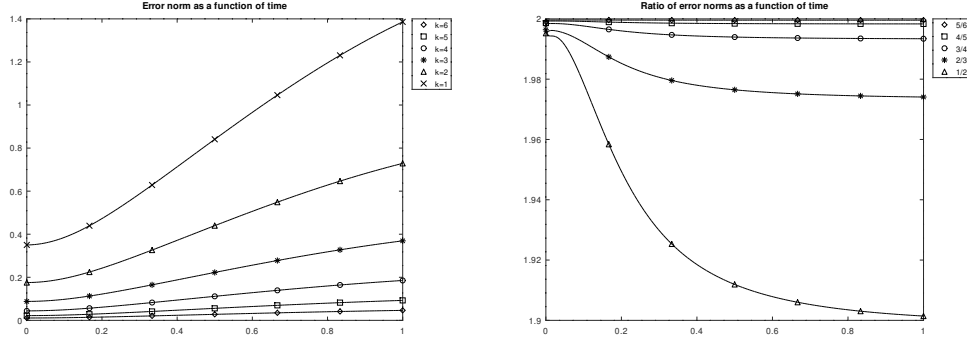


FIG. 3. Plot of $\{e_k^n\}_{n=1}^N$, $k = 1 : 6$ (left) and (e_{k-1}^n/e_k^n) for $k = 2 : 6$ (right)

norm over the interval $[0, T]$. Such results (after integrations) are found in Table 6 where we illustrate the conclusions of Theorem 4.1. As seen from this table, and expected from the theoretical results presented

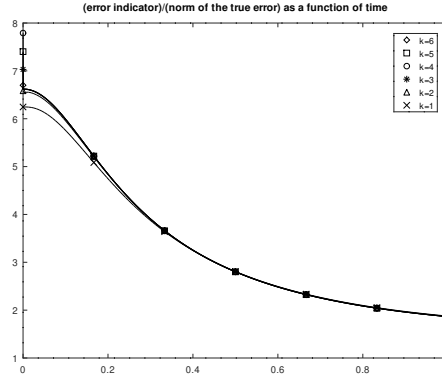


FIG. 4. Ratio between the values of the error indicators and the norm of the error as functions of time.

earlier, the ratio between the indicators shows reduction by 2. Moreover we see that the proposed fully discrete error indicators provide two sided bounds for the error up to reasonable multiplicative constant.

7. Conclusion

In this paper, we obtain a two-sided residual a posteriori error estimator for the three-field mixed method in Biot's consolidation model. It is expected that our a posteriori error analysis generalizes to mixed methods for the five-field formulation based on weakly symmetric stress tensor (see Lee (2016)), although to present such a generalization might require an elaborate analysis. Combining our analysis with a posteriori error estimation of mixed methods for elasticity using strong symmetric stress (see e.g., Carstensen *et al.* (2019); Chen *et al.* (2018); Li (2019a)), we hope to obtain a two-sided residual estimator for the four-field formulation.

$(h = 2^{-k})$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$\mathcal{E}_{k-1}/\mathcal{E}_k$	N/A	1.910	1.976	1.994	1.997	2.003
E_{k-1}/E_k	N/A	1.910	1.980	1.995	1.999	2.000
$\mathcal{E}_k/E_k$	2.63	2.63	2.63	2.62	2.62	2.62

Table 2. Robustness and efficiency of the error indicators for $\tau = 5 \times 10^{-5}$ and varying mesh sizes.

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