



Curvature-Driven Wrinkling of Thin Elastic Shells

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Abstract

How much energy does it take to stamp a thin elastic shell flat? Motivated by recent experiments on the wrinkling patterns of floating shells, we develop a rigorous method via Γ -convergence for answering this question to leading order in the shell's thickness and other small parameters. The observed patterns involve “ordered” regions of well-defined wrinkles alongside “disordered” regions whose local features are less robust; as little to no tension is applied, the preference for order is not *a priori* clear. Rescaling by the energy of a typical pattern, we derive a limiting variational problem for the effective displacement of the shell. It asks, in a linearized way, to cover up a maximum area with a length-shortening map to the plane. Convex analysis yields a boundary value problem characterizing the accompanying patterns via their defect measures. Partial uniqueness and regularity theorems follow from the method of characteristics on the ordered part of the shell. In this way, we can deduce from the principle of minimum energy the leading order features of stamped elastic shells.

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1. Introduction

Thin elastic sheets subject to compressive boundary conditions or loads readily take on shapes far from their own. Sometimes such changes in shape lead to the development of fine-scale *wrinkling patterns*, indicating the presence of residual strain or excess material that is “wrinkled away”. Other times, *crumpling patterns* occur such as those seen in everyday, crumpled paper sheets [83]. An intriguing and widely open problem is to devise a method for predicting the features of the often disordered network of creases or “minimal ridges” [53] that forms. This and other simplified versions of the crumpling problem, which ask for sharp *a priori* lower bounds on the energy required to crumple [19, 80], remain far from being understood (despite some striking recent phenomenological progress identifying as a possible state variable the total length of the plastically damaged set [32]).

A cousin of the crumpling problem is the stamping one studied here, named after the manufacturing process of the same name. In stamping, a thin elastic sheet is pressed into a target shape. If the mid-surface of the sheet embeds isometrically into the target, the sheet may simply take on the imposed shape. If no such embedding exists, a pattern can instead appear [41]. The situation is reminiscent of the isometric embedding theorem of Nash and Kuiper, which guarantees the existence of a sequence of continuously differentiable isometric embeddings converging uniformly to any length-shortening map [48, 57]. However, in this paper we will not be concerned with such “pure” isometries, but rather with maps exhibiting small amounts of strain.

Our motivation to study stamping stems from our desire to understand the patterns that form when a thin elastic shell is placed onto an otherwise planar water bath [1, 2]. The water adheres to the underside of the shell, and capillary and gravitational forces act to stamp it flat. Stretching forces prefer isometric deformations, while bending forces limit the curvature that results. The authors in [1, 2] report on the striking formation of “wrinkle domains” made up of sinusoidal oscillations in a piecewise constant or otherwise slowly-varying direction. A typical floating shell divides into finitely many domains. At the interfaces are “walls”, across which the direction of wrinkling changes rapidly, or “folds”, wherein material is lost beneath the surface. The particular arrangement of wrinkles into domains is observed to depend strongly on the initial features of the shell—namely, its Gaussian curvature and boundary shape—and the authors wonder about the possibility of designing patterns at will.

The appearance of wrinkle domains in floating elastic shells is remarkable. It reminds of a key feature of other, more well-studied pattern forming systems such as shape memory alloys [12], micromagnets [27], and liquid crystals [7]. The authors in [1] highlight in particular a connection between wrinkles and the layers of a smectic liquid crystal. They describe a coarse-graining procedure in which the wrinkle direction plays the role of a director field, and the wrinkle peaks and troughs are encoded in the level sets of a phase field function φ . Setting an ansatz into the total energy E , the authors extract a coarse-grained or “effective” energy $E_{\text{eff}}(\varphi)$. Carrying over known results on liquid crystals, the authors make scaling predictions for various quantities such as the size of a typical domain and the width

of its walls. It remains unclear, however, whether the overall layout of the wrinkles, i.e., their particular arrangement into domains, can be recovered by this approach. There is reason to doubt it can be done. Careful examination of the ansatz in [1] shows it assumes the shell deforms by an oscillatory perturbation of some leading order deformation which is implicitly defined. As such, it and the corresponding phase field function φ may be prohibitively difficult to recover.

The situation becomes even more complicated as the thickness of the shell tends to zero. Forthcoming experiments on ultrathin shells [78], having thicknesses several orders of magnitude less than those in [1,2], show that sometimes no coherent wrinkling pattern occurs. In particular, spherical caps produce a “disordered”, crumpling-like response whose local features are sensitive to perturbation and vary between trials. Other less-symmetric spherical shells (e.g., triangles cut from spheres) display a mixed ordered-disordered response, in which one part remains ordered—being covered with wrinkle domains—while another part exhibits the crumpling-like response. Upon perturbation, the local features of the disordered parts tend to rearrange while their overall layout remains the same. In contrast, the ordered parts remain more or less unchanged. Notably, the opposite response occurs for saddle-shaped shells: ultrathin negatively curved shells exhibit the same ordered wrinkle domains as do their thicker counterparts.

The task of determining the features of wrinkled thin elastic sheets has been the subject of much research. When wrinkles occur in response to applied tensile forces, certain directions are stabilized and one may deduce the direction of wrinkling from tension-field theory [67, 70, 82], also known as the relaxed energy approach [62–64]. The relaxed energy density W_{rel} for a sheet with zero thickness (a “membrane”) is a function of its effective strain, which vanishes on bi-axially compressed states and is otherwise strictly positive. When applied to the *tension-driven wrinkling* of thin elastic sheets [9, 23], one finds that the extent of the un-wrinkled region is determined, as well as the direction of the wrinkles, by solving a *relaxed problem* of the form

$$\min_{\Phi_{\text{eff}}} \int_S W_{\text{rel}}(D\Phi_{\text{eff}}) dA \quad (1.1)$$

subject to boundary conditions and loads. Here, Φ_{eff} denotes the limiting or effective deformation of the mid-sheet S that arises in the vanishing thickness limit. A recent focus in tension-driven problems has been on identifying the scaling behavior(s) with respect to thickness (and other parameters) of the higher order terms in the expansion

$$\min E = C_0 + \text{higher order terms} \quad (1.2)$$

as the thickness tends to zero. The constant C_0 is given by the minimum value of (1.1) and it amounts to the work done at leading order to stretch the sheet. Evaluating the higher order terms requires identifying the lengthscale and amplitude of the wrinkles whose existence is implied. In general these quantities can vary throughout the sheet, making their analysis rather involved. Examples include the “wrinkling cascades” seen in uni-axially compressed floating sheets pulled taught by surface tension [38], as well as in hanging drapes pulled taught by gravity [10].

Wrinkling patterns also occur in situations devoid of strong tensile loads or even lacking them altogether. This is the case for the stamped and floating shells

introduced above. The hallmarks of such *curvature-driven wrinkling* are the presence of geometric incompatibilities driving the patterned response, and a lack of coherence in certain parameter regimes. The transition from ordered wrinkle domains in moderately thin floating shells [1] to an ordered wrinkling-disordered crumpling-like response in the ultrathin limit [78] is an example of this phenomenon. Other examples include the ordered “herringbone” patterns and their disordered “labyrinthine” counterparts occurring in bi-axially compressed sheets on a planar substrate [15, 17, 39, 40], as well as the hexagonal tiling and labyrinthine patterns occurring in compressed thin elastic spheres bonded to a spherical core [72, 77].

In any case where tension fails to dominate, the relaxed problem (1.1) offers little guidance as to the patterns that occur (we refer to a situation where $C_0 = 0$). Various authors working on problems for which surface tension is a small but non-negligible effect have suggested [60, 61, 84] that the shell’s response can be determined instead at leading order by solving a *limiting* or *effective area problem* of the form

$$\max_{\Phi_{\text{eff}}} \text{Area}(\Phi_{\text{eff}}(S)). \quad (1.3)$$

In analogy to (1.2), the minimum energy is expected to expand as

$$\min E = C_1 \cdot \gamma + \text{higher order terms} \quad (1.4)$$

as the surface tension coefficient γ of the exposed interface tends to zero, and the shell is predicted to maximize the area it covers at leading order. A natural question is regarding constraints: in [84] where a flat disc is confined to a liquid saddle surface, the perimeter of the sheet is taken to be fixed; in [60] where a flat disc wraps a water droplet, the effective deformation Φ_{eff} is understood to be a length-shortening map.

In this paper, we take the first step towards a mathematical analysis of curvature-driven wrinkling. We adopt the viewpoint of energy minimization (even global minimization to simplify) and set ourselves the following tasks: prove the validity of an effective area problem such as (1.3) for the leading order behavior of (almost) minimizers, and deduce from its solutions the patterns that form. We achieve these goals for a class of *weakly curved* or *shallow* shells whose intrinsic geometries are close to flat. This simplifying assumption facilitates analysis since it allows the use of a geometrically linear, von Karman-like energy. Geometrically linear models are standard in the literature on elastic pattern formation, though they have yet to enjoy the same level of rigorous derivation from fully nonlinear elasticity as have plate and shell models for finite bending deformations (for a recent review, see [56]). Motivated by a recent suggestion [24] that there exists a “bending-induced” tension proportional to the geometric mean of the shell’s bending modulus B and the substrate stiffness K , we rescale our energy functionals by

$$\gamma_{\text{eff}} = 2\sqrt{BK} + \gamma$$

and obtain their Γ -limit as $\gamma_{\text{eff}} \rightarrow 0$ in a topology well-suited to the formation of patterns. A linearized version of (1.3) and (1.4) results, in which γ is replaced by

the “effective surface tension” γ_{eff} . In proving these results, we will not assume that minimizers obey any particular ansatz, or even that they exhibit ordered wrinkle domains.

This brings us to what may be the most important contribution of this paper: via convex analysis of the limiting, linearized area problem, we derive a new and far-reaching method for proving that almost minimizers must tend towards an ordered–possibly disordered state, one whose ordered part consists of known wrinkle domains, and whose possibly disordered part is left unconstrained. Our method consists of two steps: first, we solve for a set of *stable lines* along which any oscillations (and concentrations) are ruled out; second, we recover the amplitude of the oscillations that do occur via a second order linear partial differential equation (PDE) for which the stable lines are characteristic curves. Thus, we have found a way to treat wrinkles as the characteristic curves of a family of differential operators, rather than as the level sets of some unknown phase field function as proposed in [1]. The upcoming Fig. 1 presents various arrangements of our stable lines. We were pleased to learn that the predicted ordered parts where they exist (shown as striped) compare favorably with the experiments that motivated our work. Even the leftover, possibly disordered parts (shown in blank) look to align. A separate paper is currently in preparation, where we plan to report on experimental and numerical tests of our predictions [78]. We turn to introduce the model we use and to state our main results.

1.1. Preliminaries

Section 1.1.1 introduces a geometrically linear model of elastic shells. Section 1.1.2 passes to its non-dimensional form and identifies the parameter regime of our results. Finally, in Section 1.1.3 we recall some basic facts about functions of bounded deformation and bounded Hessian. The formal statement of our results is in Section 1.2.

1.1.1. Weakly Curved Floating Shells We consider the model problem of a thin elastic shell floating on an otherwise planar liquid bath. Let the undeformed mid-shell S be the graph of a function p over some planar reference domain $\Omega \subset \mathbb{R}^2$, so that

$$S = \{(x_1, x_2, p(x)) : x \in \Omega\}.$$

Given a deformation $\Phi : S \rightarrow \mathbb{R}^3$ of the shell, introduce its in- and out-of-plane displacements $u : \Omega \rightarrow \mathbb{R}^2$ and $w : \Omega \rightarrow \mathbb{R}$ according to

$$\Phi(x_1, x_2, p(x)) = (x_1 + u_1(x), x_2 + u_2(x), w(x)), \quad x \in \Omega.$$

The plane being referenced is that of the undeformed liquid bath. So long as the shell is weakly curved, meaning that its typical “slope” $|\nabla p| \ll 1$, its deformation

can be modeled as a minimizer of the energy¹

$$E = \frac{Y}{2} \int_{\Omega} |e(u) + \frac{1}{2} \nabla w \otimes \nabla w - \frac{1}{2} \nabla p \otimes \nabla p|^2 dx + \frac{B}{2} \int_{\Omega} |\nabla \nabla w - \nabla \nabla p|^2 dx \\ + \frac{K}{2} \int_{\Omega} |w|^2 dx + \gamma_V \left(\int_{\Omega} \frac{1}{2} |\nabla p|^2 dx - \int_{\partial \Omega} u \cdot \hat{v} ds \right). \quad (1.5)$$

The notation $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ stands for the symmetrized gradient of the displacement u . We use $x \otimes y$ to denote the outer product of x and y , and \hat{v} for the outwards-pointing unit normal vector at $\partial \Omega$. Our formula for the energy is directly analogous to the one used in [74] to study the wrinkling of an internally pressurized spherical shell, as well as the one used in [11, 24, 34] to study the wrinkling of a flat disc on a spherical substrate; it is a geometrically linearized version of the energy used in [1] for general floating shells. Here, to fix ideas, we focus on the setup of a weakly curved shell on a planar liquid substrate, noting that our analysis can be adapted to the more general setup of a weakly curved shell on a weakly curved substrate. Underlying the energy E is a certain “geometric linearization” procedure we shall describe. But first, let us introduce each of the terms in (1.5).

The formula (1.5) accounts for the potential energy of the shell and liquid bath. The first two terms are the “stretching” and “bending” terms. They comprise the elastic energy of the shell. The parameters $Y = E_{st}$ and $B = \frac{1}{12}E_s t^3$ are its stretching and bending moduli, where E_s is its Young’s modulus and t is its dimensional thickness. For simplicity, and as it contains the essential mathematical details, we treat the case of an isotropic Hooke’s law with Poisson ratio $\nu = 0$. That is, we take $|\cdot|$ to denote either the standard Euclidean or Frobenius matrix norm. With this choice, the stretching energy is proportional to the sum of the squares of the components of the *geometrically linear strain*

$$\varepsilon = e(u) + \frac{1}{2} \nabla w \otimes \nabla w - \frac{1}{2} \nabla p \otimes \nabla p, \quad (1.6)$$

which it prefers to remain small. The bending energy is proportional to the sum of the squares of the components of $\nabla \nabla w - \nabla \nabla p$. It limits the curvature that develops. The remaining terms in (1.5) account for the energy of the liquid bath. The parameter $K = \rho g$ sets its “stiffness” to out-of-plane displacements (ρ is the density of the liquid and g is the gravitational acceleration), while γ_V sets the strength of the liquid–vapor surface tension pulling at the shell’s edge. Note in treating only the surface tension of the liquid–vapor interface, we assume the shell adheres completely to the surface of the bath (see [34] for more on this point).

Before non-dimensionalizing, we pause to discuss the fact that (1.5) does not report the true energy of the shell and liquid bath, but only approximates it to leading order in a “geometrically linear” setting where

¹ We picked up the term “weakly curved” from [37]. It indicates a family of shells also referred to as “shallow”, the deformations of which can be modeled using the Donnel–Mushtari–Vlasov theory [58, 81] or Marguerre’s theory of shallow shells [68]. Our stretching and bending terms become the ones from [68] under the substitution $w \rightarrow w + p$, and the ones from [58, 81] under the further substitution $u \rightarrow u - w \nabla p$.

$$|\nabla u| \sim |\nabla w|^2 \sim |\nabla p|^2 \ll 1. \quad (1.7)$$

The use of a more nonlinear model (“geometrically nonlinear” as in [1] or “fully nonlinear” as in [9]) would of course yield more accurate results, but would require several significant mathematical advances beyond the ones achieved here. As remarked above, we are not the first to make such a simplification in the study of elastic patterns: other authors including those of [11, 24, 34, 74] have used geometrically linear models as well. The picture that has emerged is that, whereas the quantitative predictions of such models can only be asymptotically correct, their qualitative predictions often reflect those of a more nonlinear model. So while we expect the analysis of (1.5) to reveal much about the experiments that motivated this work, we warn that it may fail to capture the parts of those experiments that are not weakly curved. The analysis of general floating shells is the subject of current research.

To illustrate this point further, let us briefly indicate how the geometrically linear energy (1.5) arises, informally, from a more nonlinear one. We focus on the stretching term, as the rest can be explained similarly. As in [1, 28], we note that the (geometrically) nonlinear stretching energy of the shell is given by

$$E_{\text{stretch}} = \frac{Y}{2} \int_S |\varepsilon_{\text{NL}}|_S^2 dA, \quad (1.8)$$

where ε_{NL} is the strain of Φ , dA is the area element of S , and $|\cdot|_S$ is a suitable matrix norm. Pulling back to Ω , we introduce the deformed and reference metrics $g = D\Phi^T D\Phi$ and $g_0 = D\Phi_0^T D\Phi_0$ where $\Phi_0(x) = (x_1, x_2, p(x))$, and write

$$\varepsilon_{\text{NL}} = \frac{1}{2}(g - g_0), \quad dA = \sqrt{\det g_0} dx, \quad |\cdot|_S = |g_0^{-1} \cdot|.$$

Taylor expanding about the trivial displacements $(u, w) = (0, 0)$ and the trivial shell $p = 0$ yields

$$\varepsilon_{\text{NL}} = \varepsilon + \text{h.o.t.}, \quad dA = dx + \text{h.o.t.}, \quad |\cdot|_S = |\cdot| + \text{h.o.t.},$$

where we have neglected higher order terms per (1.7). Replacing ε_{NL} , dA , and $|\cdot|_S$ in (1.8) with their leading order approximations ε , dx , and $|\cdot|$ results in the stretching energy from (1.5).

1.1.2. Bendability, Deformability, and Confinement We are interested in the minimizers of the energy (1.5), and especially in their dependence on its parameters. Here, we collapse these into three non-dimensional groups whose inverses are known as “bendability”, “deformability”, and “confinement”. Similar groups appear whenever elastic, surface tension, and substrate forces interact, as has been shown in many other works including [23, 34, 43, 74].

Let L be a representative lengthscale of the reference domain Ω , and let R be a representative radius of curvature for the shell. Consider the change of variables

$$u(x) = \epsilon^2 L \hat{u}\left(\frac{x}{L}\right), \quad w(x) = \epsilon L \hat{w}\left(\frac{x}{L}\right), \quad p(x) = \epsilon L \hat{p}\left(\frac{x}{L}\right) \quad (1.9)$$

where $\epsilon = L/R$. Hats denote dimensionless variables defined on the unit domain $\hat{\Omega} = L^{-1}\Omega$. That the shell is weakly curved corresponds to taking $\epsilon \ll 1$. Setting (1.9) into (1.5) and changing variables, we find that

$$\hat{E}(\hat{u}, \hat{w}) = \frac{1}{YL^2} \frac{1}{\epsilon^4} E(u, w)$$

satisfies

$$\begin{aligned} \hat{E} = & \frac{1}{2} \int_{\hat{\Omega}} |e(\hat{u}) + \frac{1}{2} \nabla \hat{w} \otimes \nabla \hat{w} - \frac{1}{2} \nabla \hat{p} \otimes \nabla \hat{p}|^2 + \frac{1}{2} \frac{B}{YL^2} \frac{1}{\epsilon^2} \int_{\hat{\Omega}} |\nabla \nabla \hat{w} - \nabla \nabla \hat{p}|^2 \\ & + \frac{1}{2} \frac{KL^2}{Y} \frac{1}{\epsilon^2} \int_{\hat{\Omega}} |\hat{w}|^2 + \frac{\gamma_{\text{lv}}}{Y} \frac{1}{\epsilon^2} \left(\int_{\hat{\Omega}} \frac{1}{2} |\nabla \hat{p}|^2 - \int_{\partial \hat{\Omega}} \hat{u} \cdot \hat{v} \right). \end{aligned}$$

Evidently, minimizing E is equivalent to minimizing \hat{E} , but the latter version has the benefit of collapsing the six original parameters $Y, B, K, \gamma_{\text{lv}}, L$, and R into three non-dimensional groups:

$$\begin{aligned} b &= \frac{B}{YL^2} \cdot \frac{R^2}{L^2} \quad (\text{bendability}^{-1}), \quad k = \frac{KR^2}{Y} \quad (\text{deformability}^{-1}), \\ \gamma &= \frac{\gamma_{\text{lv}}}{Y} \cdot \frac{R^2}{L^2} \quad (\text{confinement}^{-1}). \end{aligned}$$

Henceforth, we drop the hats and consider the non-dimensionalized energy

$$\begin{aligned} E_{b,k,\gamma}(u, w) = & \frac{1}{2} \int_{\Omega} |e(u) + \frac{1}{2} \nabla w \otimes \nabla w - \frac{1}{2} \nabla p \otimes \nabla p|^2 dx \\ & + \frac{b}{2} \int_{\Omega} |\nabla \nabla w - \nabla \nabla p|^2 dx + \frac{k}{2} \int_{\Omega} |w|^2 dx \\ & + \gamma \left(\int_{\Omega} \frac{1}{2} |\nabla p|^2 dx - \int_{\partial \Omega} u \cdot \hat{v} ds \right) \end{aligned} \quad (1.10)$$

with its parameters $b, k > 0$ and $\gamma \geq 0$.

Having non-dimensionalized, we can now introduce the asymptotic regime of our results. This paper studies the asymptotics of $E_{b,k,\gamma}$ and its minimizers in any limit $b \rightarrow 0, k \rightarrow \infty, \gamma \rightarrow 0$ such that

$$\frac{b}{k}, \frac{\gamma}{k}, 2\sqrt{bk} + \gamma \ll 1 \quad \text{and} \quad \left(\frac{b}{k} \right)^{1/10} \ll 2\sqrt{bk} + \gamma. \quad (1.11)$$

These conditions arise from the search for a parameter regime where minimizers satisfy

$$\varepsilon \approx 0 \quad \text{and} \quad w \approx 0 \quad (1.12)$$

so that they are nearly strain-free and nearly flat. One expects this to hold if stretching and substrate forces dominate the response. As with isometric embeddings, there exist infinitely many nearly strain-free displacements to any neighborhood of the plane. (We will construct such displacements later on. See also [49] for the case $\varepsilon = 0$.) With so much freedom, it is reasonable to think of minimizing the

bending, substrate, and surface energies while treating (1.12) as a “constraint”; this is an instance of the “Gauss–Euler elastica” variational principle proposed recently in [24]. Following the line of reasoning there, one predicts the minimum energy to scale $\sim 2\sqrt{bk}$ or γ , whichever is the larger. The typical values of the stretching and substrate forces are ~ 1 and k , so we are lead to take $2\sqrt{bk} + \gamma \ll 1$ and k as in the first part of (1.11).

The last part of (1.11) is harder to explain. It comes from the fact that in order to justify the claim that minimizers satisfy (1.12), we must be able to prove the existence of in- and out-of-plane displacements satisfying

$$\int_{\Omega} |\varepsilon|^2 \ll \min E_{b,k,\gamma} \quad \text{and} \quad k \int_{\Omega} |w|^2 \ll \min E_{b,k,\gamma}$$

in a regime where the minimum energy is expected to scale $\sim 2\sqrt{bk} + \gamma$. Furthermore, as we intend to prove a Γ -convergence result, we must accomplish this for any possible limiting in-plane displacement u_{eff} . This is not a straightforward task, and it becomes all the more difficult (perhaps eventually impossible) as $2\sqrt{bk} + \gamma \rightarrow 0$. Our choice to impose the second part of (1.11) arises from the details of our herringbone-based recovery sequences. See the discussion following Theorem 1.1 for more on this and Section 3 for the details.

1.1.3. Functions of Bounded Deformation and Bounded Hessian The goals of this paper are to obtain and analyze the Γ -limit of $\frac{1}{2\sqrt{bk} + \gamma} E_{b,k,\gamma}$ in the parameter regime (1.11). To this end, we make use of the spaces of *bounded deformation* and *bounded Hessian* functions

$$\begin{aligned} BD(\Omega) &= \left\{ u \in L^1(\Omega; \mathbb{R}^2) : e(u) \in \mathcal{M}(\Omega; \text{Sym}_2) \right\}, \\ HB(\Omega) &= \left\{ \varphi \in L^1(\Omega) : \nabla \nabla \varphi \in \mathcal{M}(\Omega; \text{Sym}_2) \right\} \end{aligned}$$

where $\mathcal{M}(\Omega; \text{Sym}_2)$ is the space of finite, symmetric 2-by-2 matrix-valued Radon measures on the reference domain $\Omega \subset \mathbb{R}^2$. As these spaces may not be immediately familiar to all, we recall their basic properties and refer to [26, 75, 76] for more details. The reader wishing to skip forward to our results should go to Section 1.2.

The spaces $BD(\Omega)$ and $HB(\Omega)$ are Banach spaces under the norms

$$\begin{aligned} \|u\|_{BD(\Omega)} &= \|u\|_{L^1(\Omega)} + \int_{\Omega} |e(u)|_1, \\ \|\varphi\|_{HB(\Omega)} &= \|\varphi\|_{L^1(\Omega)} + \int_{\Omega} |\nabla \nabla \varphi|_1 \end{aligned}$$

respectively, where we define

$$\int_{\Omega} |\mu|_1 = \sup_{\substack{\sigma \in C_c(\Omega; \text{Sym}_2) \\ |\sigma(x)|_{\infty} \leq 1 \ \forall x \in \Omega}} \int_{\Omega} \langle \sigma, \mu \rangle$$

for $\mu \in \mathcal{M}(\Omega; \text{Sym}_2)$. Although the norm $\int_{\Omega} |\mu|_1$ is equivalent to the more commonly used total variation one $\sum_{ij} |\mu_{ij}|(\Omega)$, we use the former as it appears in our results. Note $\int_{\Omega} |\mu|_1 = \text{tr } \mu(\Omega)$ if $\mu \geq 0$. The natural injections

$BD(\Omega) \hookrightarrow L^2(\Omega; \mathbb{R}^2)$ and $HB(\Omega) \hookrightarrow C(\overline{\Omega})$ are continuous in the strong topologies induced by the given norms. As it turns out, $BD(\Omega)$ and $HB(\Omega)$ are dual spaces and possess weak-* topologies. Between these and the strong topologies lie the so-called *intermediate* topologies metrized respectively by the distances

$$\begin{aligned} \|u - v\|_{L^1(\Omega)} &+ \left| \int_{\Omega} |e(u)|_1 - \int_{\Omega} |e(v)|_1 \right|, \\ \|\varphi - \psi\|_{L^1(\Omega)} &+ \left| \int_{\Omega} |\nabla \nabla \varphi|_1 - \int_{\Omega} |\nabla \nabla \psi|_1 \right|. \end{aligned}$$

The trace maps $BD(\Omega) \rightarrow L^1(\partial\Omega, \mathcal{H}^1)$, $u \mapsto u|_{\partial\Omega}$ and $HB(\Omega) \rightarrow (C(\partial\Omega), L^1(\partial\Omega, \mathcal{H}^1))$, $\varphi \mapsto (\varphi|_{\partial\Omega}, \nabla \varphi|_{\partial\Omega})$ are intermediately continuous, and are defined by continuous extension from the intermediately dense set $C^\infty(\overline{\Omega}; \mathbb{R}^2)$. We often drop the notation $\cdot|_{\partial\Omega}$ when the meaning is clear, as in the integration-by-parts identities

$$\begin{aligned} \int_{\Omega} \langle \sigma, e(u) \rangle &= - \int_{\Omega} \operatorname{div} \sigma \cdot u \, dx + \int_{\partial\Omega} \langle \sigma, u \otimes \hat{v} \rangle \, ds, \\ \int_{\Omega} \langle \sigma, \nabla \nabla \varphi \rangle &= - \int_{\Omega} \operatorname{div} \sigma \cdot \nabla \varphi \, dx + \int_{\partial\Omega} \langle \sigma, \nabla \varphi \otimes \hat{v} \rangle \, ds \end{aligned} \quad (1.13)$$

for all $\sigma \in C^1(\overline{\Omega}; \operatorname{Sym}_2)$.

Finally, we introduce the quotient space $BD(\Omega)/\mathcal{R}$. Setting $\sigma = Id$ into the first identity in (1.13) shows that $\int_{\partial\Omega} u \cdot \hat{v}$ is unchanged under the replacement $u \rightarrow u + r$ if $e(r) = 0$. By definition,

$$\mathcal{R} = \{r \in BD(\Omega) : e(r) = 0\}$$

is the space of *linearly strain-free* maps. It consists of all maps $x \mapsto Rx + b$ where R is anti-symmetric and $b \in \mathbb{R}^2$. Although Korn's inequality fails on $BD(\Omega)$, the Poincaré-type inequality

$$\min_{r \in \mathcal{R}} \|u - r\|_{L^1(\Omega)} \lesssim_{\Omega} \int_{\Omega} |e(u)|_1 \quad \forall u \in BD(\Omega)$$

holds. Thus, $\int_{\Omega} |e(u)|_1$ defines a norm on the quotient space

$$BD(\Omega)/\mathcal{R} = \{u + r : u \in BD(\Omega), r \in \mathcal{R}\}$$

under which it becomes a Banach space. By Banach-Alaoglu, norm-bounded subsets of $BD(\Omega)/\mathcal{R}$ are weakly-* precompact. Note $u_n \xrightarrow{*} u$ weakly-* in $BD(\Omega)/\mathcal{R}$ if and only if $e(u_n) \xrightarrow{*} e(u)$ weakly-* in $\mathcal{M}(\Omega; \operatorname{Sym}_2)$. In such a case, there exists $\{r_n\}_{n \in \mathbb{N}} \subset \mathcal{R}$ so that $u_n + r_n \rightarrow u$ strongly in $L^1(\Omega; \mathbb{R}^2)$.

1.2. Statement and Discussion of Results

Having introduced the (non-dimensionalized) energies $E_{b,k,\gamma}$ in (1.10), we proceed to state our results. We start in Section 1.2.1 by deriving the sought after effective energy E_{eff} of the floating shell as the Γ -limit of the rescaled energies $\frac{1}{2\sqrt{bk} + \gamma} E_{b,k,\gamma}$. There we produce a first statement of the limiting problem in terms

of the effective in-plane displacement u_{eff} . A second statement appears in Section 1.2.2 in terms of a new variable called the “defect measure” μ . We think of it as encoding the patterns. Section 1.2.3 obtains a dual description via an “Airy potential” function φ , and produces a boundary value problem for optimal μ whose coefficients depend on an optimal choice of φ . Finally, Section 1.2.4 presents our method of stable lines. For a short list of open questions, see Section 1.2.5.

Assumptions. Here we collect for the reader’s convenience a list of assumptions that will reappear throughout. The following assumptions are basic to what we do:

$$\Omega \subset \mathbb{R}^2 \text{ is a bounded, Lipschitz domain, } p \in W^{2,2}(\Omega), \quad (1.14a)$$

$$\{(b_n, k_n, \gamma_n)\}_{n \in \mathbb{N}} \subset (0, 1] \times (0, \infty) \times [0, 1] \text{ satisfies}$$

$$\frac{b_n}{k_n}, \frac{\gamma_n}{k_n}, 2\sqrt{b_n k_n} + \gamma_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (1.14b)$$

Certain of our results require the following additional assumptions to hold:

$$\Omega \text{ is strictly star-shaped, } p \in W^{2,\infty}(\Omega), \quad (1.15a)$$

$$\frac{(b_n/k_n)^{1/10}}{2\sqrt{b_n k_n} + \gamma_n} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (1.15b)$$

Unless otherwise stated, any asymptotic statement involving b , k , or γ is understood to hold on a sequence satisfying (1.14b) and (1.15b). We often mute the subscript n . Recall Ω is said to be *strictly star-shaped* if there exists $x \in \Omega$ so that for all $y \in \partial\Omega$ the open line segment from x to y belongs to Ω . Sometimes, we make use of the hypothesis that Ω is simply connected to simplify the statements of certain results.

These and other assumptions enter at various steps in our analysis. Briefly, the situation is as follows: while for our complete Γ -convergence result we must impose all of the assumptions in (1.14) and (1.15), each of its components hold in greater generality; so do our results regarding the analysis of the limiting problems. To help the reader navigate, we have included a statement at the top of each of Sections 2–6 clarifying which assumptions are needed there.

1.2.1. The Limiting Area Problem Our first result is a formula for the effective energy E_{eff} of a weakly curved, floating shell, along with the limiting (linearized) area problem it implies. Anticipating the minimum energy to scale $\sim 2\sqrt{bk} + \gamma$, we divide by this and pass to the limit in the sense of Γ -convergence. As usual, we fix the admissible set and extend the energies $E_{b,k,\gamma} : BD(\Omega) \times W^{1,2}(\Omega) \rightarrow (-\infty, \infty]$ by taking

$$E_{b,k,\gamma}(u, w) = \begin{cases} (1.10) & (u, w) \in W^{1,2}(\Omega) \times W^{2,2}(\Omega) \\ \infty & \text{otherwise} \end{cases}.$$

Define $E_{\text{eff}} : BD(\Omega) \times W^{1,2}(\Omega) \rightarrow (-\infty, \infty]$ by

$$E_{\text{eff}}(u, w) = \begin{cases} \int_{\Omega} \frac{1}{2} |\nabla p|^2 dx - \int_{\partial\Omega} u \cdot \hat{v} ds & e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p dx, w = 0 \\ \infty & \text{otherwise} \end{cases}$$

where $\mu \leq \tilde{\mu}$ if $\tilde{\mu} - \mu \in \mathcal{M}_+(\Omega; \text{Sym}_2)$, the space of finite, non-negative, Sym_2 -valued Radon measures on Ω .

Theorem 1.1. *Let Ω , p , and $\{(b, k, \gamma)\}$ satisfy the assumptions (1.14) and (1.15). The Γ -convergence*

$$\frac{1}{2\sqrt{bk} + \gamma} E_{b,k,\gamma} \xrightarrow{\Gamma} E_{\text{eff}}$$

holds with respect to the weak- $BD(\Omega)/\mathcal{R} \times W^{1,2}(\Omega)$ topology, and the rescaled energies are equi-coercive on that space. More precisely, we have the following results:*

1. (Γ -liminf inequality) *Given any weakly-* converging sequence*

$$(u_{b,k,\gamma}, w_{b,k,\gamma}) \xrightarrow{*} (u, w) \quad \text{weakly-* in } BD(\Omega)/\mathcal{R} \times W^{1,2}(\Omega),$$

there holds

$$\liminf \frac{E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma})}{2\sqrt{bk} + \gamma} \geq E_{\text{eff}}(u, w);$$

2. (recovery sequences) *Given any $(u, w) \in BD(\Omega) \times W^{1,2}(\Omega)$, there exists a sequence*

$$(u_{b,k,\gamma}, w_{b,k,\gamma}) \xrightarrow{*} (u, w) \quad \text{weakly-* in } BD(\Omega)/\mathcal{R} \times W^{1,2}(\Omega)$$

such that

$$\lim \frac{E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma})}{2\sqrt{bk} + \gamma} = E_{\text{eff}}(u, w);$$

3. (equi-coercivity) *Any sequence $\{(u_{b,k,\gamma}, w_{b,k,\gamma})\}$ that satisfies*

$$\limsup \frac{E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma})}{2\sqrt{bk} + \gamma} < \infty$$

admits a sub-sequence that converges weakly- in $BD(\Omega)/\mathcal{R} \times W^{1,2}(\Omega)$.*

Remark 1.1. At first glance, it may seem surprising that the space BD , which was originally introduced in connection with plasticity (see, e.g., [75]), should arise in a problem devoid of plastic effects. It can, however, be anticipated on the grounds that our energies are geometrically linear. For reasons that will become clear, ∇w will be bounded *a priori* in L^2 . The scaling $|e(u)| \sim |\nabla w|^2$ then indicates a bound on $e(u)$ in L^1 , implying weak-* pre-compactness in BD up to a linearly strain-free map. The equi-coercivity result justifies these claims. A similar observation was made in [20] where BD appeared in a Γ -limit analysis of clamped elastic membranes.

It is well-known that Γ -convergence combined with equi-coercivity implies the convergence of minimum values along with minimizers [22, 25]. Here, we deduce that

$$\lim \frac{\min E_{b,k,\gamma}}{2\sqrt{bk} + \gamma} = \min_{(u_{\text{eff}}, w_{\text{eff}}) \in BD(\Omega) \times W^{1,2}(\Omega)} E_{\text{eff}}(u_{\text{eff}}, w_{\text{eff}}).$$

Furthermore, the displacements $(u_{\text{eff}}, w_{\text{eff}})$ on the righthand side are optimal if and only if they are the weak-* limit of a sequence of *almost minimizers* $\{(u_{b,k,\gamma}, w_{b,k,\gamma})\}$ of $E_{b,k,\gamma}$. Such sequences satisfy

$$E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma}) = \min E_{b,k,\gamma} + o(2\sqrt{bk} + \gamma)$$

by definition. Reducing to the finite part of E_{eff} yields the following result:

Corollary 1.1. *Given the assumptions (1.14) and (1.15), the rescaled minimum energies satisfy*

$$\lim \frac{\min E_{b,k,\gamma}}{2\sqrt{bk} + \gamma} = \min_{\substack{u_{\text{eff}} \in BD(\Omega) \\ e(u_{\text{eff}}) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \, ds. \quad (1.16)$$

Furthermore, $(u_{\text{eff}}, w_{\text{eff}})$ arises as the weak-* $BD(\Omega)/\mathcal{R} \times W^{1,2}(\Omega)$ limit of almost minimizers of $E_{b,k,\gamma}$ if and only if u_{eff} solves the limiting problem on the righthand side, and $w_{\text{eff}} = 0$.

We wish to make two remarks, on the geometric meaning of the limiting problem, and on the proof of the Γ -convergence result. First, we demonstrate how the limiting problem in Corollary 1.1 can be recovered by linearizing the proposed area problem (1.3). Just as we may associate to a deformation $\Phi : S \rightarrow \mathbb{R}^3$ the in- and out-of-plane displacements u and w , we may associate to the limiting or *effective* displacements u_{eff} and $w_{\text{eff}} = 0$ the effective deformation $\Phi_{\text{eff}} : S \rightarrow \mathbb{R}^3$ given by

$$\Phi_{\text{eff}}(x, p(x)) = (x + u_{\text{eff}}(x), 0), \quad x \in \Omega.$$

Whereas the area of the undeformed mid-shell S satisfies

$$A(S) = \int_{\Omega} \sqrt{1 + |\nabla p|^2} \, dx = |\Omega| + \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx + \text{h.o.t.},$$

the area of its image under Φ_{eff} satisfies

$$A(\Phi_{\text{eff}}(S)) = \int_S \sqrt{\det D\Phi_{\text{eff}}^T D\Phi_{\text{eff}}} \, dA = |\Omega| + \int_{\Omega} \operatorname{div} u_{\text{eff}} \, dx + \text{h.o.t.}$$

to leading order in ∇p and ∇u_{eff} . (This ignores the possibility that Φ_{eff} may not be one-to-one.) Subtracting and applying the divergence theorem yields the expansion

$$A(S) - A(\Phi_{\text{eff}}(S)) = \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \, ds + \text{h.o.t.} \quad (1.17)$$

Similarly, the one-sided constraint

$$e(u_{\text{eff}}) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx$$

from (1.16)—which we refer to henceforth as the statement that u_{eff} is (*linearly*) *tension-free*—can be recovered by linearizing the statement that Φ_{eff} is *short*, i.e.,

$$d_{\mathbb{R}^2}(\Phi_{\text{eff}}(x), \Phi_{\text{eff}}(y)) \leq d_S(x, y) \quad \forall x, y \in S.$$

In this way, the limiting problem from Corollary 1.1 manifests as the leading order part of the geometric variational problem

$$\min_{\substack{\Phi_{\text{eff}}: S \rightarrow \mathbb{R}^2 \\ \text{that are short}}} A(S) - A(\Phi_{\text{eff}}(S))$$

which asks to cover up as much area as possible with a length-shortening map of S to the plane.

We turn to discuss the key ingredients in the proof of Theorem 1.1. It requires establishing *a priori* lower bounds on $E_{b,k,\gamma}$, and verifying that they are asymptotically sharp. Behind the Γ -liminf part is a sort of “geometric interpolation inequality” that quantifies the fact that two regular enough embedded surfaces cannot be both extrinsically close and intrinsically far. Here, the surfaces in question are those of the nearly isometrically deformed mid-shell $\Phi(S)$ and of its projection to the plane. In terms of the displacements u and w , the inequality states that

$$\left(\int_{\Omega} \left| \frac{1}{2} \Delta w \right|^2 \right)^{1/2} \left(\int_{\Omega} |w|^2 \right)^{1/2} \geq \int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{v} + \text{h.o.t.} \quad (1.18)$$

whenever $\varepsilon \approx 0$ and $w \approx 0$. On the lefthand side we see a trade-off between the linearized mean curvature $H \approx \frac{1}{2} \Delta w$ and the out-of-plane displacement w . On the righthand side we recognize from (1.17) the difference between the intrinsic and planar projected areas of the shell. Taking the trace of the statement that $\varepsilon \approx 0$ we see from (1.6) that

$$\operatorname{div} u + \frac{1}{2} |\nabla w|^2 \approx \frac{1}{2} |\nabla p|^2.$$

Thus, (1.18) reminds of the classic Gagliardo–Nirenberg interpolation inequality

$$C \left(\int_{\Omega} |\nabla \nabla w|^2 \right)^{1/2} \left(\int_{\Omega} |w|^2 \right)^{1/2} \geq \int_{\Omega} |\nabla w|^2 + \text{h.o.t.} \quad (1.19)$$

which holds for $w \approx 0$ and independently of the strain (see, e.g., [31]). While (1.19) implies the equi-coercivity part of Theorem 1.1, it is not strong enough to establish its Γ -liminf part. Thinking of replacing the full Hessian $\nabla \nabla w$ with Δw , which is justified when $\varepsilon \approx 0$, we were led to its sharpened form (1.18). Though we are certainly not the first to apply a Gagliardo–Nirenberg interpolation inequality to the study of elastic patterns—such inequalities play an organizing role throughout the subject of energy-driven pattern formation [46]—we know of only one other

analysis of wrinkling in which an optimal prefactor is known [8]. The suggestion that the geometric interpolation inequality (1.18) should be used in place of (1.19) appears to be new; see Section 2 for more details.

Much of our work is devoted to the construction of recovery sequences verifying the optimality of our lower bounds. Given any candidate tension-free displacement u_{eff} , we construct in Section 3 an admissible sequence $\{(u_{b,k,\gamma}, w_{b,k,\gamma})\}$ converging weakly-* to $(u_{\text{eff}}, 0)$ and whose energy satisfies

$$E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma}) = (2\sqrt{bk} + \gamma) \left(\int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{v} \right) + O \left(\left(\frac{b}{k} \right)^{1/10} \right).$$

The out-of-plane parts of three such constructions are depicted in Fig. 1. Their essential character is given by

$$w(x) = \sqrt{2\text{tr} \langle \varepsilon_{\text{eff}} \rangle} \cdot l_{\text{wr}} \cos \left(\frac{x \cdot \hat{\eta}_{\text{herr}}(x)}{l_{\text{wr}}} \right) \quad (1.20)$$

where we denote the *effective strain* of u_{eff} by

$$\varepsilon_{\text{eff}} = e(u_{\text{eff}}) - \frac{1}{2} \nabla p \otimes \nabla p \, dx. \quad (1.21)$$

We envision a “piecewise herringbone” pattern consisting of multiple herringbones, one of which appears in each bold square in Panel (a) of Fig. 1. Herringbones are made of twinned uni-directional wrinkles superimposed on alternating bands of in-plane shear. We select them as our basic building blocks as they are highly effective at accommodating constant bi-axial compressive strains [47]. Simply put, our idea is that with enough herringbones, one should be able to accommodate any non-constant ε_{eff} —even the measure-valued ones in Theorem 1.1.

At the smallest scales, the ansatz (1.20) features uni-directional wrinkles at a lengthscale l_{wr} and in the direction of $\hat{\eta}_{\text{herr}}$. Other larger lengthscales include one associated with the size of the individual herringbones (also with the “averaging” operator $\langle \cdot \rangle$ through which ε_{eff} will be approximated as piecewise constant), and one associated with the oscillations in $\hat{\eta}_{\text{herr}}$ representing the wrinkle twins. Before moving on, we would like to motivate the locally sinusoidal character of our piecewise herringbones—which are closer to the herringbones treated in [5,47] than, say, the origami-based ones in [6]—on the grounds of energy minimization. Consider what it takes for the two sides of the geometric interpolation inequality (1.18) to be nearly the same: in its additive form, the inequality can be improved to say that

$$\begin{aligned} & \frac{b}{2} \int_{\Omega} |\Delta w|^2 + \frac{k}{2} \int_{\Omega} |w|^2 - 2\sqrt{bk} \left(\int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{v} \right) \\ & \geq \frac{1}{2} \int_{\Omega} |b^{1/2} \Delta w + k^{1/2} w|^2 + \text{h.o.t.} \end{aligned}$$

so long as $\varepsilon \approx 0$ and $w \approx 0$. Hence, minimizers must satisfy

$$-\Delta w \approx \sqrt{\frac{k}{b}} w$$

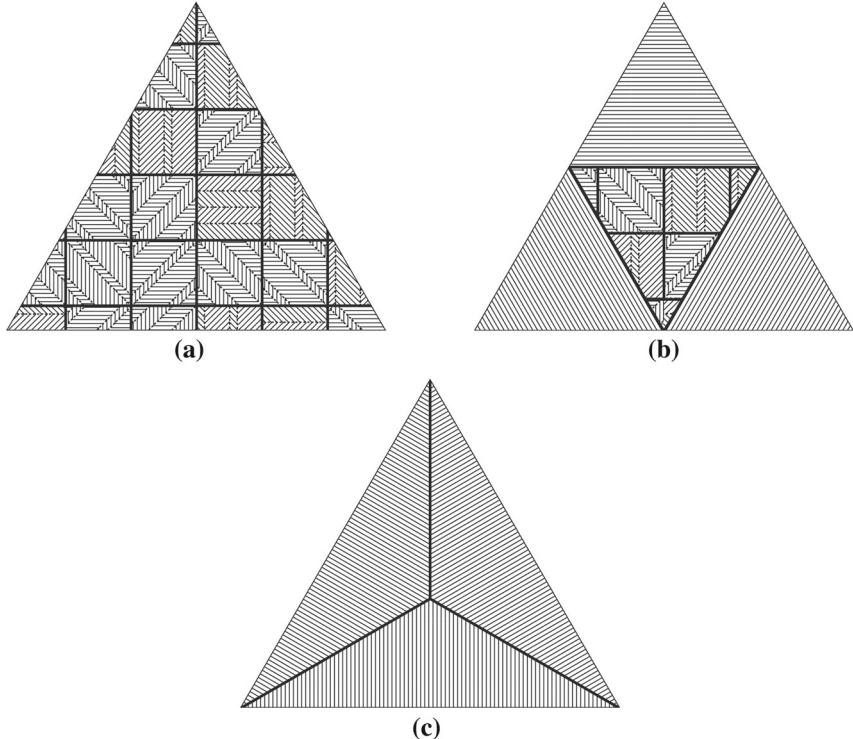


Fig. 1. Three possible patterns formed by a floating triangular shell. Panel (a) depicts the “piecewise herringbone” pattern we use to construct arbitrary recovery sequences. It consists of multiple herringbones, one per square, each of which is made up of twinned wrinkles and alternating in-plane shear. Panel (b) depicts an almost minimal pattern in the positively curved case, consisting of ordered, uni-directional wrinkles and a piecewise herringbone to model its disordered part. Panel (c) depicts an almost minimal pattern in the negatively curved case. The emergence of ordered “wrinkle domains” such as in (b) and (c) will be shown to follow from the principle of minimum energy

consistent with the locally sinusoidal character of (1.20). Note this also explains the choice $l_{\text{wr}} = (b/k)^{1/4}$ we will eventually make. It is well-appreciated in the literature on elastic pattern formation that such a lengthscale should emerge from a competition between bending and substrate effects (see, e.g., [16]). We refer to Section 2 for more on the geometric interpolation inequality and Section 3 for the details of our piecewise herringbones.

1.2.2. Defect Measures Thus far, our results have focused on the effective displacements that arise as limits of almost minimizers of $E_{b,k,\gamma}$. As explained in Corollary 1.1, these can be found by solving the (linearized) area problem

$$\min_{\substack{u_{\text{eff}} \in BD(\Omega) \\ e(u_{\text{eff}}) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \, ds \quad (1.22)$$

for the effective in-plane displacement u_{eff} , and recalling that $w_{\text{eff}} = 0$. In light of our previous discussion of the experiments on floating shells from [1,2,78], the reader may wonder whether solving (1.22) actually recovers the observed wrinkle domains and possibly disordered parts. Indeed, deducing these is the goal of the rest of our results, which culminate in Section 1.2.4 with our method of stable lines. We start by rewriting (1.22) as a minimization over the effective strain ε_{eff} from (1.21) or, as we prefer to think of it, over a quantity called the “defect measure”.

Defect measures are a basic tool for encoding the properties of high-frequency oscillations (and concentrations) governed by PDEs [51,52]. We define them in the present context as follows. Whenever a sequence $\{(u_n, w_n)\}_{n \in \mathbb{N}}$ converges weakly-* in $BD(\Omega)/\mathcal{R} \times W^{1,2}(\Omega)$ to $(u_{\text{eff}}, 0)$ and satisfies

$$e(u_n) + \frac{1}{2} \nabla w_n \otimes \nabla w_n \rightarrow \frac{1}{2} \nabla p \otimes \nabla p \quad \text{strongly in } L^2(\Omega) \quad (1.23)$$

so that it is *asymptotically strain-free*, we may associate to it a non-negative Sym_2 -valued *defect measure*

$$\mu = \text{weak-}*\lim \nabla w_n \otimes \nabla w_n dx \quad \text{in } \mathcal{M}(\Omega; \text{Sym}_2). \quad (1.24)$$

Taking limits, we deduce the important identity

$$e(u_{\text{eff}}) + \frac{1}{2} \mu = \frac{1}{2} \nabla p \otimes \nabla p dx \quad (1.25)$$

which couples μ back to u_{eff} thus guaranteeing it is well-defined. In particular, the limit in (1.24) holds *a posteriori* since any converging sub-sequence must yield the same result (for a complete proof, see Lemma 2.2). Combining this with (1.21) yields that

$$\mu = -2\varepsilon_{\text{eff}} \quad \text{where} \quad \varepsilon_{\text{eff}} = e(u_{\text{eff}}) - \frac{1}{2} \nabla p \otimes \nabla p dx.$$

Evidently, solving for the defect measure of a given sequence is tantamount to finding its effective strain.

Some examples are in order. Consider a uni-directional wrinkling pattern with lengthscale $l_{\text{wr}} \ll 1$ and constant direction $\hat{\eta}$. Thinking that the out-of-plane part should satisfy

$$w(x) = \sqrt{2}l_{\text{wr}} \cos\left(\frac{x \cdot \hat{\eta}}{l_{\text{wr}}}\right) \quad \text{yields} \quad \mu = \hat{\eta} \otimes \hat{\eta} dx$$

as $l_{\text{wr}} \rightarrow 0$. The same measure results for non-constant $\hat{\eta}$, so long as its variations are sufficiently mild. If $\hat{\eta}$ varies rapidly, as it does for the piecewise herringbone patterns in (1.20), μ can end up being rank two. Folds with various profiles can also be handled. Let $l_f \ll 1$ and fix $\hat{\eta}$. Taking

$$w(x) = \frac{l_f^{\frac{1}{2}}}{(2\pi)^{\frac{1}{4}}} e^{-\frac{1}{2}\left(\frac{x \cdot \hat{\eta}}{l_f}\right)^2} \quad \text{yields} \quad \mu = \hat{\eta} \otimes \hat{\eta} \mathcal{H}^1 \lfloor \{x : x \cdot \hat{\eta} = 0\}$$

as $l_f \rightarrow 0$. The notation on the right indicates the restriction of the one-dimensional Hausdorff measure \mathcal{H}^1 to the given line. The motifs of wrinkles and folds are ubiquitous in thin elastic sheets [13, 65]. We propose to model them using defect measures in the vanishing thickness limit.

Returning to the context of weakly curved, floating shells, we now change variables from u_{eff} to μ . We do so by identifying the set of defect measures associated to the recovery sequences from Theorem 1.1. In fact, all recovery sequences are asymptotically strain-free (see Section 2). Hence, $\mu \in \mathcal{M}(\Omega; \text{Sym}_2)$ arises as the defect measure of a recovery sequence if and only if (1.25) holds for some tension-free $u_{\text{eff}} \in BD(\Omega)$. That u_{eff} is tension-free is equivalent to the statement that $\mu \geq 0$. Recall the Saint-Venant compatibility conditions which state, for simply connected domains, that a Sym_2 -valued matrix field m is a linear strain, i.e., $m = e(u)$ for some u if and only if

$$\partial_{11}m_{22} + \partial_{22}m_{11} - 2\partial_{12}m_{12} = 0. \quad (1.26)$$

That this holds in the smooth setting appears in standard references on elasticity (e.g., [54]). By a straightforward approximation argument, it also holds when $m \in \mathcal{M}$ and $u \in BD$. Denote

$$\text{curlcurl } m = \partial_{11}m_{22} + \partial_{22}m_{11} - 2\partial_{12}m_{12} \quad (1.27)$$

and observe the “very weak Hessian” identity

$$-\frac{1}{2} \text{curlcurl } \nabla w \otimes \nabla w = \det \nabla \nabla w, \quad (1.28)$$

so named as it allows to define $\det \nabla \nabla w$ for $w \in W^{1,2}$ [42, 49]. Combining (1.26) and (1.28) yields the following fact: provided Ω is simply connected, there exists $u \in BD(\Omega)$ satisfying (1.25) if and only if $\mu \in \mathcal{M}(\Omega; \text{Sym}_2)$ satisfies

$$-\frac{1}{2} \text{curlcurl } \mu = \det \nabla \nabla p \quad (1.29)$$

in the sense of distributions. Therefore, we may exchange the set of admissible u_{eff} from (1.22) with the new set of admissible defect measures μ characterized by their non-negativity and the PDE (1.29). To finish the exchange, note the identity

$$\frac{1}{2} \int_{\Omega} |\mu|_1 = \int_{\Omega} \frac{1}{2} |\nabla p|^2 dx - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} ds,$$

which follows from (1.25) upon integrating its trace. The following result is proved:

Corollary 1.2. *Given the assumptions (1.14) and (1.15), the rescaled minimum energies satisfy*

$$\lim_{\gamma \rightarrow 0} \frac{\min E_{b,k,\gamma}}{2\sqrt{bk} + \gamma} = \min_{\substack{\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2) \\ -\frac{1}{2} \text{curlcurl } \mu = \det \nabla \nabla p}} \frac{1}{2} \int_{\Omega} |\mu|_1. \quad (1.30)$$

Furthermore, μ arises as the defect measure of almost minimizers of $E_{b,k,\gamma}$ if and only if it solves the limiting problem on the righthand side.

The limiting problems in Corollaries 1.1 and 1.2 are two sides of the same coin: Corollary 1.1 determines the limiting displacement of the shell via optimal u_{eff} , and Corollary 1.2 determines the limiting features of its patterns via optimal μ . Defect measures play a similar role for curvature-driven wrinkling to that of the “wrinkling strain” identified in [64] for tension-driven wrinkling. Both quantities specify how much material must be “gotten rid of” in an appropriate limit. However, μ does not derive from the relaxation of a fixed energy density, but rather from the Γ -limit of a sequence of rescaled energies.

Before proceeding to discuss the optimizers of the limiting problems in detail, we pause to answer the question of whether the minimum energy actually scales $\sim 2\sqrt{bk} + \gamma$, under the assumptions given at the start of Section 1.2. Rearranging (1.16) or (1.30) yields the expansion

$$\min E_{b,k,\gamma} = C_1 \cdot (2\sqrt{bk} + \gamma) + o(2\sqrt{bk} + \gamma)$$

where C_1 is the minimum value of the limiting problems. As (1.30) makes clear,

$$C_1 = 0 \iff \det \nabla \nabla p = 0 \text{ a.e. on } \Omega.$$

Thus, the minimum energy scales $\sim 2\sqrt{bk} + \gamma$ if and only if the initial Gaussian curvature of the shell, which is proportional to $\det \nabla \nabla p$, is other than zero.

1.2.3. Convex Analysis of the Limiting Problems The previous results, in particular Corollaries 1.1 and 1.2, established the role of the limiting problems

$$\min_{\substack{u_{\text{eff}} \in BD(\Omega) \\ e(u_{\text{eff}}) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial \Omega} u_{\text{eff}} \cdot \hat{v} \, ds \quad \text{and} \quad \min_{\substack{\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2) \\ -\frac{1}{2} \text{curl curl } \mu = \det \nabla \nabla p}} \frac{1}{2} \int_{\Omega} |\mu|_1 \quad (1.31)$$

for the leading order response of weakly curved, floating elastic shells. We turn to study their minimizers. Each of the problems in (1.31) is convex. On general grounds, such “primal” minimization problems should admit “dual” maximization problems, the solutions of which are paired via “complementary slackness” conditions. What distinguishes the present discussion of convex duality from the typical example (as in, e.g., [29]) is that, here, the natural pairing will turn out to require an “inner product” between matrix-valued measures. The situation reminds of the duality between stress and strain in Hencky plasticity, where similar issues arise [45] (see also [4]).

Let $\rho \in C_c^\infty(B_1)$ be non-negative and supported on the open unit disc $B_1 = B(0, 1) \subset \mathbb{R}^2$, and let $\int_{B_1} \rho \, dx = 1$. Given $\mu \in \mathcal{M}(\Omega; \text{Sym}_2)$, we define its *mollifications* $\{\mu_\delta\}_{\delta > 0} \subset C^\infty(\overline{\Omega}; \text{Sym}_2)$ by

$$\mu_\delta(x) = \int_{\Omega} \frac{1}{\delta^2} \rho\left(\frac{x-y}{\delta}\right) d\mu(y), \quad x \in \overline{\Omega}. \quad (1.32)$$

Denote

$$\nabla^\perp \nabla^\perp = \text{cof } \nabla \nabla = \begin{pmatrix} \partial_{22} & -\partial_{12} \\ -\partial_{12} & \partial_{11} \end{pmatrix}. \quad (1.33)$$

Theorem 1.2. *Let Ω and p satisfy the assumption (1.14a) and suppose that Ω is simply connected. The minimization problems in (1.31) are each dual to the maximization problem*

$$\max_{\substack{\varphi: \mathbb{R}^2 \rightarrow \mathbb{R} \\ \varphi \text{ is convex} \\ \varphi = \frac{1}{2}|x|^2 \text{ on } \mathbb{R}^2 \setminus \Omega}} \int_{\Omega} \left(\varphi - \frac{1}{2}|x|^2 \right) \det \nabla \nabla p \, dx. \quad (1.34)$$

In particular, the optimal values in (1.31) and (1.34) are the same, and admissible μ and φ are optimal if and only if the complementary slackness conditions

$$0 = \lim_{\delta \rightarrow 0} \int_{\Omega} \left| \langle \mu_{\delta}, \nabla^{\perp} \nabla^{\perp} \varphi \rangle \right| \quad \text{and} \quad 0 = \lim_{\delta \rightarrow 0} \int_{\partial \Omega} |\hat{v} \cdot [\nabla \varphi] (\hat{\tau} \otimes \hat{\tau}, \mu_{\delta})| \, ds \quad (1.35)$$

hold; the same is true for admissible u_{eff} with $\nabla p \otimes \nabla p \, dx - 2e(u_{\text{eff}})$ in place of μ . Here, $[\nabla \varphi]$ denotes the jump in $\nabla \varphi$ across $\partial \Omega$ in the direction of \hat{v} . It equals to $x - \nabla \varphi|_{\partial \Omega}$ where the trace is taken from Ω .

Remark 1.2. The admissible set in (1.34) consists of all convex extensions of $\frac{1}{2}|x|^2$ from $\mathbb{R}^2 \setminus \Omega$ into Ω . The use of \mathbb{R}^2 is immaterial, and it can be replaced by any convex neighborhood of Ω . In fact, φ can be shown to be admissible if and only if it restricts to Ω as an element of HB with $\nabla \nabla \varphi \geq 0$ and such that the boundary conditions

$$\varphi = \frac{1}{2}|x|^2 \quad \text{and} \quad \hat{v} \cdot \nabla \varphi \leq \hat{v} \cdot x \quad \text{at } \partial \Omega \quad (1.36)$$

hold in the sense of trace. See Lemma 4.5. Regarding traces at $\partial \Omega$, our convention will be that whenever we refer to the values of a quantity there, we mean those of its trace from Ω unless otherwise explicitly specified.

Remark 1.3. The complementary slackness conditions (1.35) hold regardless of the choice of the kernel ρ in (1.32), so long as it belongs to $C_c^{\infty}(B_1)$, is non-negative, and integrates to one. Other equivalent statements hold using the approximating sequences in Proposition 4.1. The freedom to approximate μ as needed will come in handy later on in Section 5 when it comes time to justify the upcoming assertions of our method of stable lines. Even more equivalent statements of complementary slackness can be obtained by approximating φ . We leave these to the reader.

Remark 1.4. Other, more general versions of the dual problem appear in Section 4, including ones that apply when Ω is multiply connected. See Proposition 4.1 and the discussion immediately thereafter.

While we explain in Section 4 how one can anticipate the form of the dual problem (1.34) on general grounds—it follows from a minimax procedure using the divergence-free “stress” $\sigma = \nabla^{\perp} \nabla^{\perp} \varphi$ as a Lagrange multiplier for the tension-free constraint—here we demonstrate how the complementary slackness conditions

(1.35) arise. As the primal problems (1.31) are convex, their solutions are completely characterized by first order optimality. The key is an integration by parts identity that says, roughly speaking, that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mu|_1 - \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p \\ &= \frac{1}{2} \int_{\Omega} \langle \nabla^{\perp} \nabla^{\perp} \varphi, \mu \rangle + \frac{1}{2} \int_{\partial \Omega} \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu \rangle \end{aligned} \quad (1.37)$$

whenever μ and φ are admissible for (1.31) and (1.34). The integrands on the righthand side of the equals sign are non-negative, while the difference on the lefthand side vanishes if and only if μ and φ are optimal. Hence, optimality should be equivalent to the complementary slackness conditions

$$\langle \nabla^{\perp} \nabla^{\perp} \varphi, \mu \rangle = 0 \quad \text{on } \Omega \quad \text{and} \quad \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu \rangle = 0 \quad \text{at } \partial \Omega. \quad (1.38)$$

The only problem with this is that the terms appearing in (1.38) are not obviously well-defined: evaluating the first one requires making sense of an “inner product” between the matrix-valued measures μ and $\nabla^{\perp} \nabla^{\perp} \varphi$; evaluating the second one requires assigning boundary values to the $\hat{\tau} \hat{\tau}$ -component of μ . While it may be possible to take advantage of the relationship between the formally adjoint operators $\nabla^{\perp} \nabla^{\perp}$ and curlcurl to treat (1.38) in some more intrinsic way, we choose to regularize instead. Integrating by parts with the mollifications $\{\mu_{\delta}\}$ from (1.32), we obtain (1.37) upon sending $\delta \rightarrow 0$. The asserted complementary slackness conditions follow. See Section 4 for the complete proof of Theorem 1.2, as well as for a discussion of duality for general Ω .

Theorem 1.2 separates the problem of determining the overall layout of the patterns encoded by optimal μ from that of determining their amplitude. We envision a two-step procedure, where in the first step an optimal *Airy potential* φ is found by solving the dual problem (1.34), and in the second step the complementary slackness conditions are systematically applied. To lighten the notation, we use (1.38) from now on to refer to the complementary slackness conditions (1.35) with a remark that they hold *in the regularized sense*. Note the meaning of this is independent of the choice of the mollifying kernel ρ .

Corollary 1.3. *Let Ω , p , and $\{(b, k, \gamma)\}$ satisfy the assumptions (1.14) and (1.15), and let φ solve the dual problem (1.34). Then $\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2)$ arises as the defect measure of a sequence of almost minimizers $\{(u_{b,k,\gamma}, w_{b,k,\gamma})\}$ of $E_{b,k,\gamma}$, i.e.,*

$$\mu = \text{weak-}* \lim \nabla w_{b,k,\gamma} \otimes \nabla w_{b,k,\gamma} dx \quad \text{in } \mathcal{M}(\Omega; \text{Sym}_2) \quad (1.39)$$

if and only if

$$\begin{cases} -\frac{1}{2} \text{curlcurl } \mu = \det \nabla \nabla p & \text{on } \Omega \\ \langle \nabla^{\perp} \nabla^{\perp} \varphi, \mu \rangle = 0 & \text{on } \Omega \\ \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu \rangle = 0 & \text{at } \partial \Omega \end{cases} \quad (1.40)$$

The first equation holds in the sense of distributions, while the second and third equations hold in the regularized sense.

1.2.4. Stable Lines We come at last to our method of stable lines. This is a way to deduce from knowledge of an optimal φ solving the dual problem (1.34) that optimal μ solving the primal problem (1.31) are rank one, absolutely continuous, and uniquely determined on a subset of Ω . In other words, the method asserts the existence of an “ordered” part, where stable lines exist and the convergence of almost minimizers towards known patterns is implied. (As usual, any reference to the almost minimizers of $E_{b,k,\gamma}$ is contingent on the Γ -convergence in Theorem 1.1.) At its heart is an analysis of (1.40) as a boundary value problem for μ via the method of characteristics. As such, it is a bit difficult to describe the method in a manner that is both general and precise. The following contains only the essence of what we achieve in Section 5 and Section 6. See Section 1.2.5 for a list of open questions that remain.

The first task is to explain what we mean by the “stable lines” and the “ordered part” of the shell. The definitions we present here are only preliminary, as they require more regularity than generally holds. More general definitions appear in Section 5. Suppose, for the sake of argument, that μ and φ are not only optimal in (1.31) and (1.34) but are also smooth, at least off of some small (say, Hausdorff one-dimensional) set. Then, the first complementary slackness condition in (1.40) gives that

$$\nabla^\perp \nabla^\perp \varphi \perp \mu$$

in the pointwise sense. Since $\nabla \nabla \varphi$ and μ are non-negative, it follows that the sum of their ranks is at most two. Where $\text{rank } \nabla \nabla \varphi = 2$, it must be that $\mu = 0$. On the other hand, where $\text{rank } \mu = 2$ we see that φ is affine. The part where $\text{rank } \nabla \nabla \varphi = 1$ can be said to be *ordered*, as there

$$\mu = \lambda \hat{\eta} \otimes \hat{\eta} \quad \text{for some } \lambda \geq 0 \text{ and } \hat{\eta} \in R(\nabla \nabla \varphi). \quad (1.41)$$

Given the interpretation of μ as a defect measure of almost minimizers in (1.39), we see that

$$\hat{\eta}^\perp \cdot \nabla w_{b,k,\gamma} \rightarrow 0 \quad \text{strongly in } L^2 \text{ on the ordered part.}$$

Put another way, the peaks and troughs of any wrinkles that persist must become asymptotically perpendicular to the unit vector field $\hat{\eta}$ throughout the ordered part. To help keep track of this, we propose the following geometric construction: given an optimal φ , plot its

$$\text{stable lines}—\text{curves parallel to } N(\nabla \nabla \varphi) \text{ where } \text{rank } \nabla \nabla \varphi = 1. \quad (1.42)$$

In an asymptotic sense, these are the wrinkle peaks and troughs. They form domains, as is apparent in Fig. 2. Naturally, one wonders if their geometry can be described. First, let us give an argument for their existence.

It is a well-known fact of differential geometry that any smooth enough *developable surface*—which by definition has vanishing Gaussian curvature throughout its extent—is the disjoint union of planar regions and an otherwise *ruled* part consisting of line segments that extend between boundary points. Such segments define the *generators* or *ruling lines* of the given surface (see, e.g., [71, 73]). Now where

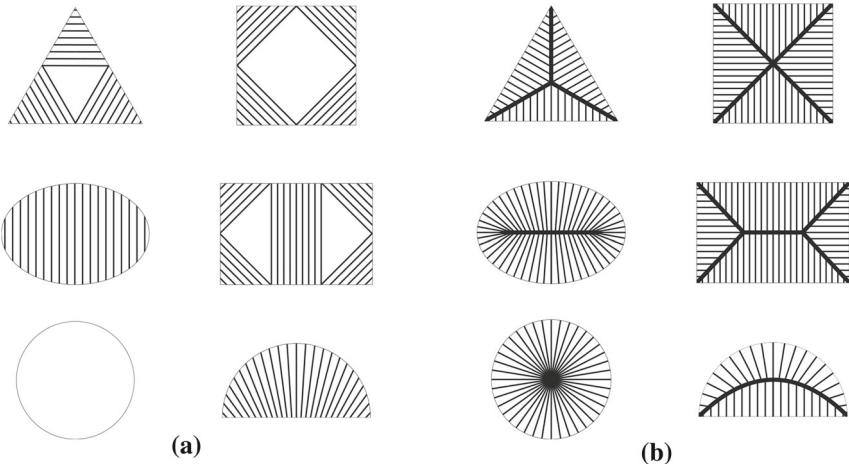


Fig. 2. Optimal wrinkle patterns arrange themselves according to the plotted “stable lines”. Panel (a) depicts the stable lines of various positively curved shells, and Panel (b) does the same for negatively curved shells. By definition, stable lines fill out the ordered part of the shell; any disordered response is confined to regions absent these lines (shown as blank). Stable lines are also the characteristic curves of a certain second order, linear PDE governing the defect measures of almost minimizers. When combined with appropriate boundary conditions, uniqueness and regularity theorems can be proved throughout the ordered part

$\det \nabla \nabla \varphi = 0$ the Gaussian curvature of the graph of φ vanishes, so that it describes a developable surface. Where $\text{rank } \nabla \nabla \varphi = 1$, it consists of ruling lines. Upon projection to the plane we recover the desired stable lines. In fact, this argument shows a little more: any smooth curve picked out by (1.42) is the planar projection of a ruling line.

Evidently the layout of the stable lines, and that of the ordered wrinkle domains they describe, is tied up with the geometry of developable surfaces. The ruled part of any (piecewise) smooth developable surface can be decomposed into developable pieces of the following three elementary types: cylinders, whose ruling lines are parallel; cones, whose ruling lines intersect at a point; and “tangential developables”, whose ruling lines are tangent to a space curve. Upon projection, we deduce the following classification scheme for stable lines: we say that

- a family of stable lines is of the *cylindrical* type if it consists of parallel segments;
- a family of stable lines is of the *conical* type if its segments, when extended, meet at a point; and
- a family of stable lines is of the *tangential* type if its segments, when extended, meet along a curve.

General arrangements of stable lines are built from these. From the twelve shells depicted in Fig. 2 we count seven consisting of only the cylindrical type; two with only the conical type (the positively curved half-disc and the negatively curved disc); one featuring both cylindrical and conical types (the negative half-disc); and one with only the tangential type (the negative ellipse).

The following result is the key to the stable lines shown in Fig. 2. It contains recipes for solving the dual problem when the initial Gaussian curvature is of one sign.

Proposition 1.1. *Let Ω and p satisfy the assumption (1.14a) and suppose that Ω is simply connected. If $\det \nabla \nabla p \geq 0$ a.e., the dual problem (1.34) is solved by the largest convex extension φ_+ of $\frac{1}{2}|x|^2$ into Ω . It satisfies*

$$\varphi_+(x) = \min_{\{y_i\} \subset \partial\Omega} \sum_{i=1}^3 \theta_i \frac{1}{2}|y_i|^2 \quad \text{for } x \in \Omega, \quad (1.43)$$

where the minimization is taken over all pairs and triples $\{y_i\} \subset \partial\Omega$ such that

$$x = \sum_i \theta_i y_i \quad \text{where } \{\theta_i\} \subset (0, 1) \text{ satisfies } \sum_i \theta_i = 1.$$

If instead $\det \nabla \nabla p \leq 0$ a.e., the dual problem is solved by the smallest convex extension φ_- of $\frac{1}{2}|x|^2$ into Ω . It satisfies

$$\varphi_-(x) = \frac{1}{2}|x|^2 - \frac{1}{2}d_{\partial\Omega}^2(x) \quad \text{where } d_{\partial\Omega}(x) = \min_{y \in \partial\Omega} |x - y| \quad \text{for } x \in \Omega. \quad (1.44)$$

Remark 1.5. Optimal φ solving (1.34) are not in general unique. However, if $\det \nabla \nabla p$ is strictly positive or strictly negative a.e., then φ_+ or φ_- is the unique solution of (1.34). See Section 6.1.

We just finished describing how stable lines dictate the geometry of optimal μ , and how we were able to solve for the ones in Fig. 2. There is a second, equally as important role played by the stable lines. We claim that they are characteristic curves along which the PDE in (1.40) becomes an ordinary differential equation (ODE) for the only possibly non-zero eigenvalue λ of μ on the ordered part. Going back to our previous assumptions of smoothness for μ and φ , and again postponing precise statements to Section 5, we note the existence of a function $\varrho > 0$ such that

$$-\frac{1}{2} \operatorname{curlcurl}(\hat{\eta} \otimes \hat{\eta} \cdot) = -\frac{1}{2\varrho} \partial_{\hat{\eta}^\perp}^2(\varrho \cdot)$$

where $\partial_{\hat{\eta}^\perp} = \hat{\eta}^\perp \cdot \nabla$ and where $\hat{\eta}$ is a unit eigenvector associated to λ . Since $\hat{\eta}$ points perpendicularly to the stable lines, we recognize this operator as a directional derivative along their extent. Setting (1.41) into the first part of (1.40), we deduce that

$$-\frac{1}{2\varrho} \partial_{\hat{\eta}^\perp}^2(\varrho \lambda) = \det \nabla \nabla p \quad \text{along the stable lines.} \quad (1.45)$$

Thus, we have arrived at a second order linear ODE for λ . Boundary data can be extracted from (1.40), after which integration yields uniqueness, regularity, and even explicit solution formulas across the ordered part.

Let us briefly comment on the sort of boundary data that can be deduced. In some cases, stable lines pass between boundary points, such as for the positively

curved shells in Panel (a) of Fig. 2. Provided φ is suitably non-degenerate, the last part of (1.40) can be shown to imply the Dirichlet-type condition

$$\varrho\lambda = 0 \quad \text{where stable lines meet } \partial\Omega. \quad (1.46)$$

Our proof of this assumes in particular that the coefficient $\hat{v} \cdot [\nabla\varphi]$ is non-zero, and also that the stable lines in question meet $\partial\Omega$ transversely. Together, (1.45) and (1.46) constitute a family of two-point boundary value problems indexed by the stable lines. See Corollary 5.1 for a precise statement of this result.

In other cases, stable lines meet in the interior. This happens for the negatively curved shells in Panel (b) of Fig. 2. Since we expect $\nabla\nabla\varphi$ to explode where stable lines meet, the second equation in (1.40) should provide Dirichlet data. The first equation should yield matching conditions. Altogether we will show that

$$\varrho\lambda = 0 \quad \text{and} \quad \partial_{\hat{\eta}^\perp}(\varrho\lambda) = 0 \quad \text{where stable lines meet,} \quad (1.47)$$

again subject to non-degeneracy conditions on φ (e.g., if the stable lines meet along a curve, we assume they meet it transversely). Combining (1.45) and (1.47) yields a family of Cauchy problems indexed by the stable lines. See Corollary 5.2 and Corollary 5.3 for precise statements of this result.

The preceding observations suggest a significantly more general result: optimal μ should be uniquely determined wherever stable lines exist. While we do not yet know how to prove this for general shells, we will show in Section 6 that it holds for each of the shells in Fig. 2, along with other related ones as well. That section also contains the proof of Proposition 1.1. We hope our general description of the method of stable lines here and in Section 5 helps the reader see the bigger picture behind what it implies at the level of the examples in Section 6.

1.2.5. Open Questions We close this introduction with a few open questions. Besides the obvious ones regarding the extension of our results beyond the assumptions given at the start of Section 1.2 and beyond the realm of weakly curved shells, there are some important issues that remain regarding the method of stable lines.

First, it is an admittedly awkward point throughout that $\nabla\nabla\varphi$ is only *a priori* a measure. For this reason, we do not yet have a generally useful definition of stable lines. What we lack is a regularity theory for optimal φ solving (1.34) or, failing that, a classification of developable surfaces of regularity HB . For now, we note that each of the examples in Section 6 enjoys the following additional regularity: optimal φ are smooth off of a singular set of finite length. This is more than enough to justify our approach. More generally, building off of the theory of $W^{2,2}$ developable surfaces in [35, 36, 59] we show how to make sense of it where φ is (locally) $W^{2,2}$. See Section 5.

Second, we wonder if optimal μ solving (1.31) are unique under the condition that there exists an optimal φ that is nowhere affine. Conjecture 6.1 at the very end gives a concrete version of this question for negatively curved shells. Our reasoning is simply that stable lines (suitably defined) should be characteristic curves for the boundary value problem (1.40), and that the given hypothesis on φ should imply

their density in Ω . This should lead to the uniqueness of μ . Though we do not have a general theorem to this effect, we have achieved it in the context of several examples, including the positively curved ellipse and half-disc in Panel (a) of Fig. 2, as well as each of the negatively curved shells in Panel (b). See Section 6.2 and Section 6.3.

Conversely, we wonder if there must exist infinitely many optimal μ provided there exists a region on which optimal φ are affine. Only when φ is affine on all of Ω have we shown this to be true—see Example 6.2 for the highly degenerate case of a positively curved disc. When combined with our Γ -convergence results, the existence of infinitely many optimal μ implies the existence of infinitely many almost minimizing sequences for $E_{b,k,\gamma}$, a situation that could perhaps explain the disorder seen in ultrathin shells [78]. Whether this disorder arises from an overall flatness of the energy landscape, or instead to a prevalence of local minimizers remains to be understood.

1.3. Outline of the Paper

The remainder of the paper establishes the results outlined above. Section 2 covers the Γ -liminf and equi-coercivity parts of Theorem 1.1, while Section 3 handles the recovery sequence part. Section 4 establishes Theorem 1.2. Section 5 discusses the method of stable lines. Finally, Section 6 proves Proposition 1.1 and presents the details behind the patterns sketched in Fig. 2. Since Corollary 1.1–Corollary 1.3 follow more or less immediately from the theorems as above, we do not repeat their proofs below.

1.4. Notation

We use big O and little o notation as well as their abbreviations \lesssim and \ll . We write $f = o(g)$ and $f \ll g$ to mean that the functions f and g satisfy $\frac{f}{g} \rightarrow 0$ in a relevant limit, and $f = O(g)$ and $f \lesssim g$ to mean that there exists a constant $C > 0$ such that $f \leq Cg$. If $C = C(\alpha)$ we indicate this using a subscript, as in $f \lesssim_\alpha g$. We write $f \sim g$ to mean that $f \lesssim g$ and $g \lesssim f$. We abbreviate $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$.

Dots and angle brackets denote the Euclidean vector and Frobenius matrix inner products $x \cdot y = \sum_i x_i y_i$ and $\langle A, B \rangle = \sum_{ij} A_{ij} B_{ij}$. Single lines without subscripts denote Euclidean and Frobenius norms. The open Euclidean ball centered at x with radius r is $B_r(x) = B(x, r) = \{y : |x - y| < r\}$. We abbreviate $B_r = B_r(0)$. The shortest Euclidean distance from x to a set S is $d_S(x) = d(x, S) = \inf_{y \in S} |x - y|$. We use the matrix norms

$$|A| = |A|_2 = \sqrt{\sum_{ij} |A_{ij}|^2}, \quad |A|_1 = \sum_{ij} |A_{ij}|, \quad |A|_\infty = \max_{ij} |A_{ij}|$$

throughout. Double lines $\|\cdot\|$ are reserved for function space norms.

Regarding function spaces, we use $C^k(X)$ and $\text{Lip}(X)$ to mean the spaces of real-valued, k -times differentiable and Lipschitz continuous functions on some

appropriate domain or metric space X . Subscripts are used as normal, with b for uniformly bounded functions and c for compactly supported ones. Semi-colons indicate ranges other than \mathbb{R} . For instance, $C_b(\Omega; \text{Sym}_d)$ indicates the space of continuous, uniformly bounded, symmetric d -by- d matrix-valued functions on Ω . The Sobolev spaces $W^{k,r}(\Omega)$ and their local versions $W_{\text{loc}}^{k,r}(\Omega)$ are defined as usual. We take the convention of referring to the (unique) continuous representative of a function that is a.e. defined, if it exists. See Section 1.1.3 for $BD(\Omega)$ and $HB(\Omega)$.

Regarding measures, we write $\mathcal{M}(X)$ to mean the space of finite, real-valued Radon measures on some locally compact Hausdorff space X . Semi-colons indicate values in vector spaces other than \mathbb{R} . The subscript $+$ indicates non-negativity. Given $\sigma \in C_b(X; \text{Sym}_d)$ and $\mu \in \mathcal{M}(X; \text{Sym}_d)$, we denote their Frobenius inner product and its integral on X by

$$\langle \sigma, \mu \rangle = \sum_{ij} \sigma_{ij} \mu_{ij} \quad \text{and} \quad \int_X \langle \sigma, \mu \rangle = \langle \sigma, \mu \rangle(X).$$

Given a Borel measurable set S , we write $\mu|_S$ to mean the restriction of μ to S defined by $\mu|_S(\cdot) = \mu(S \cap \cdot)$. The two-dimensional Lebesgue and one-dimensional Hausdorff measures \mathcal{L}^2 and \mathcal{H}^1 appear throughout. We also use the notations dx and ds . If a property is stated simply as holding “a.e.”, we mean that it holds with respect to Lebesgue unless the situation dictates otherwise. We denote $|S| = \mathcal{L}^2(S)$.

Finally, by a “curve” we mean a homeomorphic copy of an open interval $I \subset \mathbb{R}$, i.e., its image under a continuous and one-to-one map. Such a map “parameterizes” the given curve. Any further regularity will be specified, e.g., a smooth curve is one that admits a C^∞ parameterization. Given a Lipschitz curve $\Gamma \subset \mathbb{R}^2$, its tangent lines $T_s \Gamma$ are defined for \mathcal{H}^1 -a.e. $s \in \Gamma$, along with a choice of unit tangent and unit normal vector $\hat{\tau}_\Gamma$ and $\hat{\nu}_\Gamma$ satisfying

$$\hat{\tau}_\Gamma(s) \parallel T_s \Gamma \quad \text{and} \quad \hat{\nu}_\Gamma(s) \perp T_s \Gamma \quad \text{for } \mathcal{H}^1\text{-a.e. } s \in \Gamma.$$

We refer to the unit tangent and outwards-pointing unit normal at $\partial\Omega$ simply as $\hat{\tau}$ and $\hat{\nu}$. By convention, $\hat{\tau} = \hat{\nu}^\perp$ where \perp denotes counterclockwise rotation by $\pi/2$.

2. A priori Lower Bounds and Tension-Free Limits

This section establishes the equi-coercivity and Γ -liminf parts of Theorem 1.1. These results do not rely on the full set of assumptions listed at the start of Section 1.2, and make use of only the basic ones in (1.14). For the reader’s convenience, we recall these assumptions in the formal statement of what we prove.

Proposition 2.1. (equi-coercivity and Γ -liminf inequality) *Suppose*

$$\Omega \text{ is bounded and Lipschitz, } \quad p \in W^{2,2}(\Omega), \quad \frac{b}{k}, \frac{\gamma}{k}, \frac{2\sqrt{bk}}{k} + \gamma \ll 1 \quad (2.1)$$

and let the sequence $\{(u_{b,k,\gamma}, w_{b,k,\gamma})\}$ satisfy

$$\limsup \frac{E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma})}{2\sqrt{bk} + \gamma} < \infty. \quad (2.2)$$

Then the following statements hold:

1. $\{(u_{b,k,\gamma}, w_{b,k,\gamma})\}$ is weakly-* pre-compact in $BD(\Omega)/\mathcal{R} \times W^{1,2}(\Omega)$;
2. each of its weak-* limit points are of the form $(u_{\text{eff}}, 0)$ where u_{eff} is tension-free, i.e.,

$$e(u_{\text{eff}}) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx.$$

Moreover, if

$$(u_{b,k,\gamma}, w_{b,k,\gamma}) \xrightarrow{*} (u_{\text{eff}}, 0) \text{ weakly-* in } BD(\Omega)/\mathcal{R} \times W^{1,2}(\Omega)$$

then

$$\liminf \frac{E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma})}{2\sqrt{bk} + \gamma} \geq \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \, ds.$$

Remark 2.1. In the course of proving this result, we will show that any sequence that obeys (2.2) is asymptotically strain-free in that (1.23) holds—this follows in particular from the first inequality in Lemma 2.1. Lemma 2.2 then explains how a unique defect measure μ can be associated to any asymptotically strain-free and weakly-* converging sequence. Together, these facts justify the introduction of defect measures in Section 1.2.2.

We begin with a list of *a priori* bounds. Recall the definition of the strain ε in (1.6).

Lemma 2.1. *The inequalities*

$$\begin{aligned} \int_{\Omega} |\varepsilon|^2 &\lesssim_{\Omega} E_{b,k,\gamma} + \gamma^2, \quad \int_{\Omega} |e(u)| \lesssim_{\Omega,p} 1 + \sqrt{E_{b,k,\gamma}} + \frac{E_{b,k,\gamma}}{\sqrt{bk} \vee \gamma}, \\ \int_{\Omega} |w|^2 &\lesssim_{\Omega} \frac{E_{b,k,\gamma} + \gamma^2}{k}, \quad \int_{\Omega} |\nabla w|^2 \lesssim_{\Omega,p} 1 + \frac{E_{b,k,\gamma}}{\sqrt{bk} \vee \gamma}, \\ \int_{\Omega} |\nabla \nabla w|^2 &\lesssim_{\Omega,p} 1 + \frac{E_{b,k,\gamma} + \gamma^2}{b} \end{aligned}$$

hold for all $0 < b \leq k$ and $0 \leq \gamma \leq 1$.

Proof. First, we add a suitable constant to $E_{b,k,\gamma}$ to make the result non-negative. Integrating the pointwise identity

$$\begin{aligned} \frac{1}{2} |\varepsilon - \gamma Id|^2 &= \frac{1}{2} |\varepsilon|^2 - \langle \varepsilon, \gamma Id \rangle + \frac{1}{2} |\gamma Id|^2 = \frac{1}{2} |\varepsilon|^2 - \gamma \text{tr } \varepsilon + \gamma^2 \\ &= \frac{1}{2} |\varepsilon|^2 + \gamma \left(\frac{1}{2} |\nabla p|^2 - \text{div } u \right) - \frac{\gamma}{2} |\nabla w|^2 + \gamma^2 \end{aligned}$$

and applying the divergence theorem, there results

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\varepsilon - \gamma Id|^2 + \frac{\gamma}{2} \int_{\Omega} |\nabla w|^2 &= \frac{1}{2} \int_{\Omega} |\varepsilon|^2 + \gamma \left(\int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{v} \right) \\ &\quad + \gamma^2 |\Omega|. \end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{E}_{b,k,\gamma} &= E_{b,k,\gamma} + \gamma^2 |\Omega| \\
&= \frac{1}{2} \int_{\Omega} |\varepsilon - \gamma Id|^2 + \frac{\gamma}{2} \int_{\Omega} |\nabla w|^2 \\
&\quad + \frac{b}{2} \int_{\Omega} |\nabla \nabla w - \nabla \nabla p|^2 + \frac{k}{2} \int_{\Omega} |w|^2 \geq 0.
\end{aligned} \tag{2.3}$$

Being a sum of squares, $\tilde{E}_{b,k,\gamma}$ easily admits lower bounds. Bounds on the original energy follow.

We proceed to prove the inequalities from the claim. It follows from (2.3) that

$$\int_{\Omega} |w|^2 \lesssim \frac{\tilde{E}_{b,k,\gamma}}{k}$$

and so the third inequality holds. The first and last inequalities are just as easily shown. Using the triangle inequality and (2.3) we see that

$$\int_{\Omega} |\varepsilon|^2 \lesssim \int_{\Omega} |\varepsilon - \gamma Id|^2 + \gamma^2 |\Omega| \lesssim \tilde{E}_{b,k,\gamma} + \gamma^2 |\Omega|, \tag{2.4}$$

$$\int_{\Omega} |\nabla \nabla w|^2 \lesssim \int_{\Omega} |\nabla \nabla w - \nabla \nabla p|^2 + \int_{\Omega} |\nabla \nabla p|^2 \lesssim \frac{\tilde{E}_{b,k,\gamma}}{b} + \|\nabla \nabla p\|_{L^2}^2. \tag{2.5}$$

The first and last inequalities follow. We turn now to control ∇w and $e(u)$.

Two separate arguments yield bounds on ∇w , depending on whether $\gamma \geq \sqrt{bk}$ or not. The inequality

$$\|\nabla w\|_{L^2}^2 \lesssim \frac{\tilde{E}_{b,k,\gamma}}{\gamma} \tag{2.6}$$

follows directly from (2.3) in any case. At the same time, we can interpolate between the bending and substrate terms to obtain another bound. Using the triangle inequality with (2.3) as we did in the proof of (2.5), we note that

$$\tilde{E}_{b,k,\gamma} + b \|\nabla \nabla p\|_{L^2}^2 \gtrsim b \|\nabla \nabla w\|_{L^2}^2 + k \|w\|_{L^2}^2 \gtrsim \sqrt{bk} \|\nabla \nabla w\|_{L^2} \|w\|_{L^2} + k \|w\|_{L^2}^2 \tag{2.7}$$

by an elementary Young's inequality. Recall the Gagliardo–Nirenberg interpolation inequality

$$\|\nabla w\|_{L^2(\Omega)} \lesssim \|\nabla \nabla w\|_{L^2(\Omega)}^{1/2} \|w\|_{L^2(\Omega)}^{1/2} + C(\Omega) \|w\|_{L^2(\Omega)}, \tag{2.8}$$

which holds for all $w \in W^{2,2}(\Omega)$ [31]. Since by hypothesis $b \leq k$, it follows from (2.7) and (2.8) that

$$\|\nabla w\|_{L^2}^2 \lesssim_{\Omega} \frac{\sqrt{bk} \|\nabla \nabla w\|_{L^2} \|w\|_{L^2} + k \|w\|_{L^2}^2}{\sqrt{bk}} \lesssim \frac{\tilde{E}_{b,k,\gamma}}{\sqrt{bk}} + \|\nabla \nabla p\|_{L^2}^2. \tag{2.9}$$

Combining (2.6) and (2.9) yields the fourth inequality from the claim.

Finally, we handle $e(u)$. By the definition of ε and Hölder's inequality,

$$\begin{aligned} \int_{\Omega} |e(u)| &\leq \int_{\Omega} |\varepsilon| + \int_{\Omega} \left| \frac{1}{2} \nabla w \otimes \nabla w \right| + \int_{\Omega} \left| \frac{1}{2} \nabla p \otimes \nabla p \right| \\ &\lesssim |\Omega|^{1/2} \|\varepsilon\|_{L^2} + \|\nabla w\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 \\ &\lesssim \sqrt{\tilde{E}_{b,k,\gamma} + \gamma^2} + \frac{\tilde{E}_{b,k,\gamma}}{\sqrt{bk} \vee \gamma} + \|\nabla \nabla p\|_{L^2}^2 + \|\nabla p\|_{L^2}^2, \end{aligned}$$

where in the last line we used (2.4), (2.6), and (2.9). The remaining inequality follows. \square

Next, we verify that the weak-* limits of asymptotically strain-free sequences are tension-free. At the same time, we justify the notion of defect measures introduced in Section 1.2.2.

Lemma 2.2. *Let*

$$(u_n, w_n) \xrightarrow{*} (u, 0) \text{ weakly-* in } BD(\Omega)/\mathcal{R} \times W^{1,2}(\Omega)$$

and suppose it is asymptotically strain-free in that

$$e(u_n) + \frac{1}{2} \nabla w_n \otimes \nabla w_n \rightarrow \frac{1}{2} \nabla p \otimes \nabla p \text{ stongly in } L^2(\Omega; \text{Sym}_2).$$

Then $\{\nabla w_n \otimes \nabla w_n dx\}$ converges weakly-* in $\mathcal{M}(\Omega; \text{Sym}_2)$ to a non-negative, Sym_2 -valued Radon measure μ called the defect measure of the given sequence. The defect measure satisfies

$$e(u) + \frac{1}{2} \mu = \frac{1}{2} \nabla p \otimes \nabla p dx. \quad (2.10)$$

As a result, the limiting in-plane displacement u must be tension-free.

Proof. Evidently,

$$\nabla w_n \otimes \nabla w_n dx = 2 \left(e(u_n) + \frac{1}{2} \nabla w_n \otimes \nabla w_n dx \right) - 2e(u_n) \quad \forall n.$$

On the righthand side, we see the difference between a sequence converging strongly to $\nabla p \otimes \nabla p dx$, and another converging weakly-* to $2e(u)$. Therefore, the lefthand side converges weakly-*. Passing to the limit and rearranging yields (2.10). Non-negativity is preserved by weak-* convergence. Therefore, $\mu \geq 0$. \square

At this point, we have enough to deduce the first part of Proposition 2.1 on compactness. In order to prove the second part on *a priori* lower bounds, we must identify the optimal prefactor in the bound on ∇w from Lemma 2.1. That bound was a consequence of the Gagliardo–Nirenberg interpolation inequality (2.8) so, naturally, we seek a sharpened version of it now.

Lemma 2.3. *Let $\chi \in C_c^\infty(\Omega)$ have $0 \leq \chi \leq 1$. If $b, k > 0$ and $w \in W^{2,2}(\Omega)$,*

$$\begin{aligned} b \int_{\Omega} |\nabla \nabla w|^2 + k \int_{\Omega} |w|^2 &\geq 2\sqrt{bk} \int_{\Omega} |\nabla w|^2 \chi + \int_{\Omega} |b^{1/2} \Delta w + k^{1/2} w|^2 \chi \\ &\quad - 2\sqrt{bk} \|\nabla \chi\|_{L^\infty(\Omega)} \|w\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \\ &\quad - b \|\nabla \nabla \chi\|_{L^\infty(\Omega)} \|\nabla w\|_{L^2(\Omega)}^2. \end{aligned}$$

Remark 2.2. Below, we shall apply this to sequences $\{w_n\}$ converging weakly-* to zero in $W^{1,2}(\Omega)$, under the condition that $b \ll k$. Dividing by \sqrt{bk} we see that the terms appearing on the second line above behave as errors. These arise, respectively, from estimates on $\operatorname{div}(w \nabla w)$ and $\det \nabla \nabla w$ in negative norms.

Remark 2.3. Following up on the previous remark, we note that if $\varepsilon \approx 0$ then $\det \nabla \nabla w \approx \det \nabla \nabla p$ as a result of the very weak Hessian identity (1.28), the Saint-Venant compatibility conditions (1.26), and the definition of the strain. Hence, $|\nabla \nabla w| \approx |\Delta w|$ explaining the appearance of the mean curvature $H \approx \frac{1}{2} \Delta w$ on the lefthand side of the geometric interpolation inequality (1.18).

Proof. Note the pointwise identities

$$|\nabla \nabla w|^2 = |\Delta w|^2 - 2 \det \nabla \nabla w \quad (2.11)$$

$$b|\Delta w|^2 + k|w|^2 = 2\sqrt{bk} \left(|\nabla w|^2 - \operatorname{div}(w \nabla w) \right) + |b^{1/2} \Delta w + k^{1/2} w|^2 \quad (2.12)$$

as well as the very weak Hessian identity (1.28). Now let χ be as in the statement. Testing the first identity (2.11) against χ and integrating by parts using (1.28), we obtain that

$$\begin{aligned} \int_{\Omega} |\nabla \nabla w|^2 \chi &= \int_{\Omega} |\Delta w|^2 \chi - 2 \det \nabla \nabla w \chi \\ &= \int_{\Omega} |\Delta w|^2 \chi + \left\langle \nabla w \otimes \nabla w, \nabla^\perp \nabla^\perp \chi \right\rangle \\ &\geq \int_{\Omega} |\Delta w|^2 \chi - \|\nabla \nabla \chi\|_{L^\infty} \|\nabla w\|_{L^2}^2. \end{aligned}$$

Testing the second identity (2.12) against χ and integrating by parts, there follows

$$\begin{aligned} \int_{\Omega} (b|\Delta w|^2 + k|w|^2) \chi &= 2\sqrt{bk} \int_{\Omega} |\nabla w|^2 \chi + w \nabla w \cdot \nabla \chi \\ &\quad + \int_{\Omega} |b^{1/2} \Delta w + k^{1/2} w|^2 \chi \\ &\geq 2\sqrt{bk} \int_{\Omega} |\nabla w|^2 \chi + \int_{\Omega} |b^{1/2} \Delta w + k^{1/2} w|^2 \chi \\ &\quad - 2\sqrt{bk} \|\nabla \chi\|_{L^\infty} \|w\|_{L^2} \|\nabla w\|_{L^2}. \end{aligned}$$

Combining these and using that $0 \leq \chi \leq 1$, we deduce that

$$\begin{aligned} b \int_{\Omega} |\nabla \nabla w|^2 + k \int_{\Omega} |w|^2 &\geq \int_{\Omega} (b|\nabla \nabla w|^2 + k|w|^2) \chi \\ &\geq \int_{\Omega} (b|\Delta w|^2 + k|w|^2) \chi - b \|\nabla \nabla \chi\|_{L^\infty} \|\nabla w\|_{L^2}^2 \\ &\geq 2\sqrt{bk} \int_{\Omega} |\nabla w|^2 \chi + \int_{\Omega} |b^{1/2} \Delta w - k^{1/2} w|^2 \chi \\ &\quad - 2\sqrt{bk} \|\nabla \chi\|_{L^\infty} \|w\|_{L^2} \|\nabla w\|_{L^2} \\ &\quad - b \|\nabla \nabla \chi\|_{L^\infty} \|\nabla w\|_{L^2}^2. \end{aligned}$$

This completes the proof. \square

We are ready to prove the Γ -liminf and equi-coercivity parts of Theorem 1.1.

Proof of Proposition 2.1. First, consider an admissible sequence $\{(u_{b,k,\gamma}, w_{b,k,\gamma})\}$ whose energy satisfies

$$E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma}) \lesssim \sqrt{bk} \vee \gamma \ll 1 \quad (2.13)$$

but is otherwise arbitrary. We must prove that it is weakly-* pre-compact and identify its limit points. According to Lemma 2.1 and the assumptions on the parameters in (2.1), any sequence satisfying (2.13) enjoys the estimates

$$\int_{\Omega} |e(u_{b,k,\gamma}) + \frac{1}{2} \nabla w_{b,k,\gamma} \otimes \nabla w_{b,k,\gamma} - \frac{1}{2} \nabla p \otimes \nabla p|^2 \lesssim_{\Omega} \sqrt{bk} \vee \gamma + \gamma^2 \ll 1, \quad (2.14)$$

$$\int_{\Omega} |e(u_{b,k,\gamma})| \lesssim_{\Omega,p} 1 + \sqrt{\sqrt{bk} \vee \gamma} + \frac{\sqrt{bk} \vee \gamma}{\sqrt{bk} \vee \gamma} \lesssim 1, \quad (2.15)$$

$$\int_{\Omega} |w_{b,k,\gamma}|^2 \lesssim_{\Omega} \frac{\sqrt{bk} \vee \gamma + \gamma^2}{k} \lesssim \sqrt{\frac{b}{k}} \vee \frac{\gamma}{k} \ll 1, \quad (2.16)$$

$$\int_{\Omega} |\nabla w_{b,k,\gamma}|^2 \lesssim_{\Omega,p} 1 + \frac{\sqrt{bk} \vee \gamma}{\sqrt{bk} \vee \gamma} \lesssim 1, \quad (2.17)$$

$$\int_{\Omega} |\nabla \nabla w_{b,k,\gamma}|^2 \lesssim_{\Omega,p} 1 + \frac{\sqrt{bk} \vee \gamma + \gamma^2}{b} \lesssim \sqrt{\frac{k}{b}} \vee \frac{\gamma}{b}. \quad (2.18)$$

The first estimate shows that $\{(u_{b,k,\gamma}, w_{b,k,\gamma})\}$ is asymptotically strain-free. The second proves that $u_{b,k,\gamma}$ remains uniformly bounded in BD/\mathcal{R} . The third and fourth estimates show that $w_{b,k,\gamma}$ remains uniformly bounded in $W^{1,2}$ and converges to zero strongly in L^2 . In combination with the Banach–Alaoglu theorem, these imply that $\{(u_{b,k,\gamma}, w_{b,k,\gamma})\}$ is weakly-* pre-compact in $BD/\mathcal{R} \times W^{1,2}$. Applying Lemma 2.2 we learn that its limit points are of the form $(u_{\text{eff}}, 0)$ where the in-plane part u_{eff} is tension-free. The first part of Proposition 2.1 is proved.

Now consider an admissible sequence satisfying

$$(u_{b,k,\gamma}, w_{b,k,\gamma}) \xrightarrow{*} (u_{\text{eff}}, 0) \text{ weakly-* in } BD/\mathcal{R} \times W^{1,2}$$

and for which the bound (2.13) holds. In terms of the defect measure μ from Lemma 2.2, we must prove that

$$\liminf \frac{E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma})}{2\sqrt{bk} + \gamma} \geq \frac{1}{2} \int_{\Omega} |\mu|_1. \quad (2.19)$$

Indeed, according to (2.10) and the first integration by parts identity in (1.13),

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\mu|_1 &= \frac{1}{2} \int_{\Omega} \langle Id, \mu \rangle = \int_{\Omega} \left\langle Id, \frac{1}{2} \nabla p \otimes \nabla p \, dx - e(u_{\text{eff}}) \right\rangle \\ &= \int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \, ds. \end{aligned}$$

Our plan is to pass to the limit in the bending and substrate terms from $E_{b,k,\gamma}$ using the sharp Gagliardo–Nirenberg inequality from Lemma 2.3. Passing to the limit in the surface energy presents no additional difficulties.

Consider the bending term. Due to (2.18) and our assumption from (2.1) that $b \ll k$, it satisfies

$$\begin{aligned} b \int_{\Omega} |\nabla \nabla w_{b,k,\gamma} - \nabla \nabla p|^2 &\geq b \int_{\Omega} |\nabla \nabla w_{b,k,\gamma}|^2 - 2b \|\nabla \nabla w_{b,k,\gamma}\|_{L^2} \|\nabla \nabla p\|_{L^2} \\ &\quad - b \|\nabla \nabla p\|_{L^2}^2 \\ &\geq b \int_{\Omega} |\nabla \nabla w_{b,k,\gamma}|^2 - o(\sqrt{bk}). \end{aligned}$$

Fix a cutoff function $\chi \in C_c^{\infty}(\Omega)$ that satisfies $0 \leq \chi \leq 1$ but is otherwise arbitrary. Using the sharp Gagliardo–Nirenberg inequality from Lemma 2.3 and the bound just obtained, we conclude that

$$\begin{aligned} b \int_{\Omega} |\nabla \nabla w_{b,k,\gamma} - \nabla \nabla p|^2 + k \int_{\Omega} |w_{b,k,\gamma}|^2 &\geq 2\sqrt{bk} \int_{\Omega} |\nabla w_{b,k,\gamma}|^2 \chi - 2\sqrt{bk} \|\nabla \chi\|_{L^{\infty}} \|w\|_{L^2} \|\nabla w\|_{L^2} \\ &\quad - b \|\nabla \nabla \chi\|_{L^{\infty}} \|w\|_{L^2}^2 - o(\sqrt{bk}) \\ &\geq 2\sqrt{bk} \int_{\Omega} |\nabla w_{b,k,\gamma}|^2 \chi - o(\sqrt{bk}) \end{aligned}$$

by (2.16) and (2.17). Combining this with the definition (1.10) of the energy and (2.14), there results

$$\begin{aligned} E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma}) &\geq \frac{b}{2} \int_{\Omega} |\nabla \nabla w_{b,k,\gamma} - \nabla \nabla p|^2 + \frac{k}{2} \int_{\Omega} |w_{b,k,\gamma}|^2 \\ &\quad + \gamma \int_{\Omega} \frac{1}{2} |\nabla p|^2 - \operatorname{div} u_{b,k,\gamma} \\ &\geq (2\sqrt{bk} + \gamma) \int_{\Omega} \left\langle \chi Id, \frac{1}{2} \nabla w_{b,k,\gamma} \otimes \nabla w_{b,k,\gamma} \right\rangle \\ &\quad - o(\sqrt{bk} \vee \gamma). \end{aligned}$$

Dividing by $2\sqrt{bk} + \gamma$ and passing to the limit via the defect measure μ from Lemma 2.2 yields

$$\begin{aligned} \liminf \frac{E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma})}{2\sqrt{bk} + \gamma} &\geq \lim \int_{\Omega} \left\langle \chi Id, \frac{1}{2} \nabla w_{b,k,\gamma} \otimes \nabla w_{b,k,\gamma} \right\rangle \\ &= \frac{1}{2} \int_{\Omega} \langle \chi Id, \mu \rangle. \end{aligned}$$

Letting $\chi \uparrow 1$ and noting that $\mu \geq 0$ we obtain (2.19). \square

3. The Piecewise Herringbone Construction

The previous section established the Γ -liminf and equi-coercivity parts of Theorem 1.1. Here, we complete its proof by producing the required recovery sequences. To do so we will make use of all but one of the assumptions present in Section 1.2. The main result of this section is as follows:

Proposition 3.1. (recovery sequences) *Suppose*

$$\begin{aligned} \Omega &\text{ is bounded, Lipschitz, and strictly star-shaped,} \\ p &\in W^{2,\infty}(\Omega), \quad \text{and} \quad \left(\frac{b}{k}\right)^{1/10} \ll 2\sqrt{bk} + \gamma \ll 1 \end{aligned} \quad (3.1)$$

and let $u_{\text{eff}} \in BD(\Omega)$ be tension-free, meaning that

$$e(u_{\text{eff}}) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx.$$

Then there exists a sequence

$$(u_{b,k,\gamma}, w_{b,k,\gamma}) \xrightarrow{*} (u_{\text{eff}}, 0) \quad \text{weakly-}* \text{ in } BD(\Omega) \times W^{1,2}(\Omega)$$

such that

$$\lim \frac{E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma})}{2\sqrt{bk} + \gamma} = \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \, ds. \quad (3.2)$$

Remark 3.1. The missing hypothesis is that $\gamma \ll k$ (see (1.14b)). While the form of the Γ -limit depends on this hypothesis—as can be anticipated from the discussion surrounding (1.12)—it is not necessary here as the recovery sequences we describe do not depend on γ . Nevertheless, we include γ as a subscript to keep the notation consistent, and also to remind that other recovery sequences may very well depend on all three parameters.

Our proof of Proposition 3.1 centers around the notion of the *target defect measure* μ specified by u_{eff} . Following the discussion in Section 1.2.2 in the introduction, we note that any recovery sequence must satisfy

$$\nabla w_{b,k,\gamma} \otimes \nabla w_{b,k,\gamma} \, dx \xrightarrow{*} \mu \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega; \text{Sym}_2)$$

$$\text{where } \mu = -2\varepsilon_{\text{eff}} \text{ and } \varepsilon_{\text{eff}} = e(u_{\text{eff}}) - \frac{1}{2} \nabla p \otimes \nabla p \, dx.$$

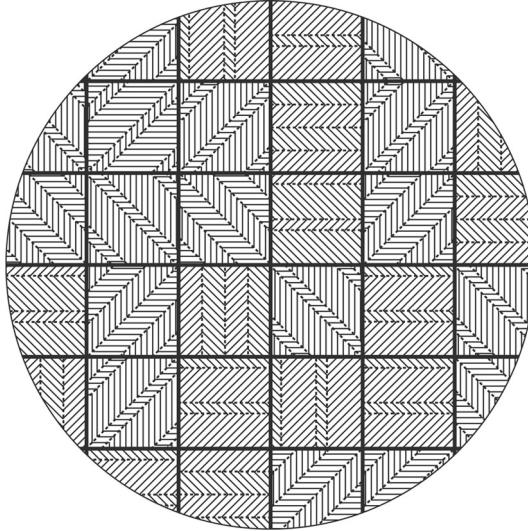


Fig. 3. A “piecewise herringbone” pattern representative of the ones we use to construct recovery sequences. Herringbones adapted to constant target defects occupy individual squares. Each herringbone consists of twinned uni-directional wrinkles and bands of in-plane shear. Wrinkles are indicated by thin lines, and dashed lines indicate “internal walls” across which their direction rapidly varies. Bold lines indicate “external walls” separating the herringbones. The number of squares, the number of twins, and the width of the walls will be optimized

We think of $\frac{1}{2}\mu$ as a “misfit” to be alleviated by some well-chosen pattern. Note $\mu \geq 0$ as u_{eff} is tension-free.

The proof proceeds in three steps. First, we reduce to the case where μ is Lipschitz and strictly positive. The key lemma is in Section 3.1, where we show how to approximate tension-free $u \in BD$ with $u \in C^\infty$ that are *uniformly tension-free*, meaning that

$$e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p - \lambda Id \quad \text{for some } \lambda > 0. \quad (3.3)$$

Though our proof of this relies crucially on the supposed strict star-shapedness of Ω , we wonder whether it holds in greater generality. It is not difficult to understand why we would like μ to be Lipschitz, as then it can be approximated by a piecewise constant target defect $\langle \mu \rangle$ obtained from averaging μ on a lattice of squares (shown in Fig. 3 in bold). On the other hand, we pass from $\mu \geq 0$ to $\mu > 0$ simply because it shortens the proof. Of course this does not preclude the possibility that optimal μ —i.e., those minimizing the righthand side of (3.2), see also (1.30)—may turn out to be rank one or even to vanish somewhere.

The second step zooms into the squares where $\mu \approx \langle \mu \rangle$. Section 3.2 produces a two-scale wrinkling pattern known alternatively as the “herringbone”, “chevron”, or “zigzag” one, and which can be adapted to any constant target defect (e.g., the ones prescribed by $\langle \mu \rangle$). Such patterns occur naturally in bi-axially compressed sheets whose displacements are suppressed [15, 17, 39, 40]. We were inspired by

their analyses in [5, 6, 47], the last of which comes the closest to what we do here. That reference identifies the scaling law of the minimum energy in a setup favoring herringbones. In Section 3.2, we sharpen this result with a version of the herringbone ansatz whose energy is optimal at leading order.

Finally, Section 3.3 assembles the individually herringboned squares into the “piecewise herringbone” pattern depicted in Fig. 3, and estimates its energy. We discuss “walls” of two types where the direction of wrinkling can rapidly change: “internal walls” that reside within the squares, and “external walls” at the interfaces between neighboring squares. The total cost of the walls increases in proportion to their area. It scales with the total number of squares. On the other hand, the cost associated with the approximation $\mu \approx \langle \mu \rangle$ decays with an increasing number of squares. Balancing these, we eventually deduce that the excess energy implicit in (3.2) can be made $\lesssim (b/k)^{1/10}$ by using $\sim (k/b)^{1/10}$ squares—see Corollary 3.1 for the details. Section 3.4 concludes with the formal proof of Proposition 3.1.

3.1. Smooth Approximation of Tension-Free Displacements

We start by showing how to approximate tension-free displacements by smooth and uniformly tension-free ones. It will be important later on that we work in a topology for which the functional on the righthand side of (3.2) is continuous. Although the trace map $u \mapsto u|_{\partial\Omega}$ fails to be continuous in the weak-* topology on BD , it is continuous in the intermediate topology. See Section 1.1.3 for the relevant definitions.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be bounded, Lipschitz, and strictly star-shaped, and let $p \in W^{2,\infty}(\Omega)$. The set of smooth and uniformly tension-free displacements is intermediately dense in the tension-free ones. That is, given any tension-free $u \in BD(\Omega)$, there exists $\{u_n\}_{n \in \mathbb{N}} \subset C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying (3.3) such that*

$$u_n \rightarrow u \text{ strongly in } L^1(\Omega) \text{ and } \int_{\Omega} |e(u_n)|_1 \rightarrow \int_{\Omega} |e(u)|_1 \text{ as } n \rightarrow \infty.$$

Proof. After a translation we can take Ω to be strictly star-shaped with respect to the origin. Also, since u is the intermediate limit of $u_\lambda = u - \lambda x$ as $\lambda \rightarrow 0$, it suffices to prove the result for displacements that are uniformly tension-free. So let $u \in BD(\Omega)$ be uniformly tension-free and let $\lambda > 0$ be as in (3.3). We construct the desired approximations $\{u_n\}$ in a two step process of dilation and mollification.

First, we dilate: given $\tau \in (0, 1)$, set $\Omega_\tau = \frac{1}{\tau} \Omega$ and let $u_\tau : \Omega_\tau \rightarrow \mathbb{R}^2$ and $p_\tau : \Omega_\tau \rightarrow \mathbb{R}$ be given by

$$u_\tau(x) = \frac{1}{\tau} u(\tau x) \quad \text{and} \quad p_\tau(x) = \frac{1}{\tau} p(\tau x) \quad \text{for } x \in \Omega_\tau. \quad (3.4)$$

Since $e(u_\tau) = e(u)(\tau \cdot)$ it follows from (3.3) that

$$e(u_\tau) \leq \left(\frac{1}{2} \nabla p_\tau \otimes \nabla p_\tau - \lambda Id \right) dx \quad \text{on } \Omega_\tau. \quad (3.5)$$

Next, we mollify: fix $\rho \in C_c^\infty(B_1)$ with $\rho \geq 0$ and $\int_{B_1} \rho \, dx = 1$, and denote by $(\cdot)_\delta$ the standard mollification

$$(f)_\delta(x) = \int_{\mathbb{R}^2} \frac{1}{\delta^2} \rho\left(\frac{x-y}{\delta}\right) f(y) \, dy \quad \text{for } \delta > 0.$$

Recall Ω was taken to be strictly star-shaped with respect to 0. Thus, there exists $c_0(\Omega) > 0$ such that

$$0 < \delta < c_0(\Omega)(1 - \tau) \implies \bar{\Omega} + B_\delta \subset \Omega_\tau. \quad (3.6)$$

So long as τ and δ satisfy (3.6), we may define $u_{\tau,\delta} : \bar{\Omega} \rightarrow \mathbb{R}^2$ by writing

$$u_{\tau,\delta}(x) = (u_\tau)_\delta(x), \quad x \in \bar{\Omega}.$$

Evidently, these are smooth. We proceed to take $\delta \rightarrow 0$ and $\tau \rightarrow 1$.

We claim that $u_{\tau,\delta}$ is uniformly tension-free so long as τ is close enough to one and δ is sufficiently small. To see this, note it follows from (3.5) and our choice to take $\rho \geq 0$ that

$$e(u_{\tau,\delta}) = (e(u_\tau))_\delta \leq \left(\frac{1}{2} \nabla p_\tau \otimes \nabla p_\tau - \lambda Id \right)_\delta = \frac{1}{2} (\nabla p_\tau \otimes \nabla p_\tau)_\delta - \lambda Id \quad \text{on } \bar{\Omega}.$$

Recalling the definition of p_τ from (3.4), we see that

$$\begin{aligned} & |(\nabla p_\tau \otimes \nabla p_\tau)_\delta(x) - \nabla p \otimes \nabla p(x)| \\ & \leq \int \rho(y) |\nabla p \otimes \nabla p(\tau(x - \delta y)) - \nabla p \otimes \nabla p(x)| \, dy \\ & \leq \|\nabla p\|_{L^\infty(\Omega)} \|\nabla \nabla p\|_{L^\infty(\Omega)} (|\tau - 1| |x| + |\tau| |\delta|) \lesssim_{\Omega, p} |\tau - 1| + |\delta| \end{aligned}$$

for $x \in \bar{\Omega}$. Taking τ close enough to one and δ sufficiently small ensures that

$$e(u_{\tau,\delta}) \leq \frac{1}{2} \nabla p \otimes \nabla p - \frac{\lambda}{2} Id \quad \text{on } \bar{\Omega},$$

as desired.

It remains to choose sequences $\tau_n \rightarrow 1$ and $\delta_n \rightarrow 0$ such that

$$u_{\tau_n, \delta_n} \rightarrow u \text{ strongly in } L^1(\Omega) \quad \text{and} \quad \int_{\Omega} |e(u_{\tau_n, \delta_n})|_1 \rightarrow \int_{\Omega} |e(u)|_1 \quad \text{as } n \rightarrow \infty.$$

The L^1 -convergence holds in any case. For the second convergence, note that

$$e(u_{\tau,\delta}) \xrightarrow{*} e(u_\tau) \quad \text{weakly-} * \text{ in } \mathcal{M}(\Omega_\tau; \text{Sym}_2) \quad \text{as } \delta \rightarrow 0$$

for each fixed $\tau \in (0, 1)$. As $|e(u)|(\Omega) < \infty$, there exist at most countably many τ for which $|e(u_\tau)|(\partial\Omega) \neq 0$. Therefore, we can find $\tau_n \rightarrow 1$ and $\delta_n \rightarrow 0$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |e(u_{\tau_n, \delta_n})|_1 = \lim_{n \rightarrow \infty} \lim_{\delta \rightarrow 0} \int_{\Omega} |e(u_{\tau_n, \delta})|_1 = \lim_{n \rightarrow \infty} \int_{\Omega} |e(u_{\tau_n})|_1 = \int_{\Omega} |e(u)|_1.$$

Taking n large enough and setting $u_n = u_{\tau_n, \delta_n}$ proves the result. \square

Lemma 3.1 allows us to restrict the proof of Proposition 3.1 to u_{eff} that are smooth and uniformly tension-free, in which case the target defect can be given the pointwise definition

$$\mu(x) = \nabla p \otimes \nabla p(x) - 2e(u_{\text{eff}})(x) \quad \forall x \in \Omega. \quad (3.7)$$

Note $\mu > 0$ uniformly on Ω . Thanks to our assumption that $p \in W^{2,\infty}$ so that $\nabla p \in \text{Lip}$, we see that $\mu \in \text{Lip}$.

We now begin the process of constructing admissible displacements satisfying

$$u \approx u_{\text{eff}}, \quad w \approx 0, \quad e(u) + \frac{1}{2} \nabla w \otimes \nabla w \approx \frac{1}{2} \nabla p \otimes \nabla p$$

with nearly minimal energy. Consider the change of variables $u \rightarrow u_{\text{eff}} + v$. Since $e(\cdot)$ is linear, the energy depends on (v, w) as

$$\begin{aligned} E_{b,k,\gamma}(u_{\text{eff}} + v, w) &= \frac{1}{2} \int_{\Omega} |e(v) + \frac{1}{2} \nabla w \otimes \nabla w - \frac{1}{2} \mu|^2 \\ &\quad + \frac{b}{2} \int_{\Omega} |\nabla \nabla w - \nabla \nabla p|^2 + \frac{k}{2} \int_{\Omega} |w|^2 \\ &\quad + \gamma \left(\int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \right) - \gamma \int_{\partial\Omega} v \cdot \hat{v} \end{aligned}$$

where we have introduced μ from (3.7) into the stretching term. We treat the simplest case where μ is constant in Section 3.2, and then proceed to discuss more general μ in Section 3.3.

3.2. Herringbone Patterns Adapted to Constant Defect

Let $Q \subset \mathbb{R}^2$ be a square and consider a constant target defect $\mu \in \text{Sym}_2$ where $\mu > 0$. Here, we describe a family of displacements

$$\{(v_{\text{herr}}, w_{\text{herr}})\} \subset W^{1,\infty}(Q; \mathbb{R}^2) \times W^{2,\infty}(Q)$$

adapted to μ in that

$$v_{\text{herr}} \approx 0, \quad w_{\text{herr}} \approx 0, \quad e(v_{\text{herr}}) + \frac{1}{2} \nabla w_{\text{herr}} \otimes \nabla w_{\text{herr}} \approx \frac{1}{2} \mu$$

and that have nearly minimal energy. Fig. 4 depicts the herringbone patterns we intend to construct. Solid lines indicate wrinkle peaks and troughs. Their direction alternates in twin pairs, in tandem with bands of alternating in-plane shear. The “area fraction” referred to there is set by the parameter

$$\theta = \frac{\lambda_1}{\lambda_1 + \lambda_2} \in (0, \frac{1}{2}] \quad \text{where } 0 < \lambda_1 \leq \lambda_2 \text{ are the eigenvalues of } \mu. \quad (3.8)$$

Panel (a) depicts the isotropic case $\theta = \frac{1}{2}$ in which μ is a multiple of the identity. Panel (b) shows an anisotropic case where $\theta \in (0, \frac{1}{2})$. Sending $\theta \rightarrow 0$ recovers uni-directional wrinkles as in Panel (c).

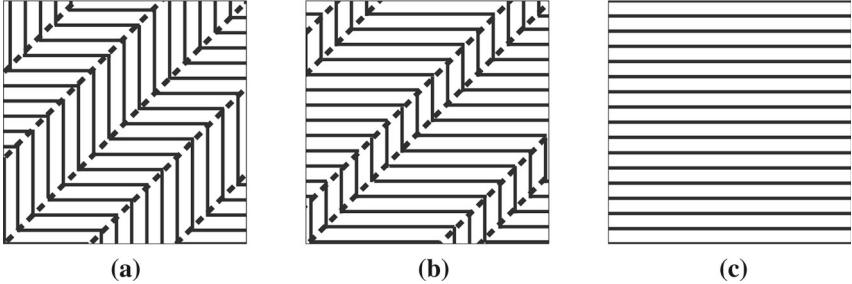


Fig. 4. Herringbone patterns with variable area fractions adapted to constant defects. Solid lines depict wrinkle peaks and troughs, while dashed lines indicate the presence of “internal walls” across which the wrinkling direction changes. Panel (a) depicts an “isotropic” herringbone for isotropic defect. Panel (b) shows an “anisotropic” herringbone for anisotropic defect. Panel (c) depicts uni-directional wrinkles arising for defect of rank one

Our herringbones will be parameterized by

$$l_{\text{wr}} \in (0, \infty), \quad l_{\text{sh}} \in (0, \infty), \quad \delta_{\text{int}} \in \left(0, \frac{1}{2}\theta l_{\text{sh}}\right). \quad (3.9)$$

The first parameter l_{wr} sets the lengthscale of the wrinkles. The second parameter l_{sh} sets the magnitude of the in-plane shear. There is an energetic cost associated to changing the direction of wrinkling, and δ_{int} sets the thickness of the associated “internal walls”. (Such walls are internal in the sense that they lie within the herringbone, as opposed to the “external walls” introduced in Section 3.3.) The energy estimates obtained in this section apply so long as (3.9) holds. However, it will be convenient going forward to keep in mind the special case where

$$\left(\frac{b}{k}\right)^{1/4} = l_{\text{wr}} \ll l_{\text{sh}} \ll \text{diam } Q \quad \text{and} \quad l_{\text{wr}} \lesssim \delta_{\text{int}} \lesssim_{\mu} l_{\text{sh}}$$

in which case there holds

$$E_{b,k,\gamma}(v_{\text{herr}}, w_{\text{herr}}) = (2\sqrt{bk} + \gamma) \cdot \frac{1}{2} \text{tr } \mu |Q| + O\left(\frac{\delta_{\text{int}}}{l_{\text{sh}}}\right).$$

The error term is due to the internal walls. Combining this estimate with the *a priori* lower bounds from Section 2, we see that herringbones for which $\frac{\delta_{\text{int}}}{l_{\text{sh}}} \ll 2\sqrt{bk} + \gamma$ are optimal at leading order. We turn to construct a general herringbone now, and to estimate its energy.

3.2.1. Constructing the Herringbone

Decompose the target defect as

$$\mu = \lambda_1 \hat{\eta}_1 \otimes \hat{\eta}_1 + \lambda_2 \hat{\eta}_2 \otimes \hat{\eta}_2$$

where $\hat{\eta}_1$ and $\hat{\eta}_2$ are orthonormal eigenvectors corresponding to the eigenvalues λ_1 and λ_2 .

Step 1: produce bands of alternating in-plane shear at scale l_{sh} . We start by introducing an in-plane displacement to transform the target defect from rank two to rank one. Define $v_{\text{sh}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$v_{\text{sh}}(x) = \sqrt{2}l_{\text{sh}}A \left(\frac{x \cdot (\hat{\eta}_2 - \hat{\eta}_1)}{\sqrt{2}l_{\text{sh}}} \right) (\hat{\eta}_2 + \hat{\eta}_1), \quad x \in \mathbb{R}^2 \quad (3.10)$$

where $A : \mathbb{R} \rightarrow \mathbb{R}$ is the one-periodic extension of

$$A(t) = \begin{cases} \frac{\lambda_2}{2}t & 0 \leq t < \theta \\ \frac{\lambda_2}{2}\theta - \frac{\lambda_1}{2}(t - \theta) & \theta \leq t \leq 1 \end{cases}, \quad t \in [0, 1].$$

It follows from the definition of θ in (3.8) that A is Lipschitz. Indeed, $A'(t)$ equals to $\frac{\lambda_2}{2}$ for $t \in (0, \theta)$ and $-\frac{\lambda_1}{2}$ for $t \in (\theta, 1)$, so that it integrates to zero.

Now, as

$$\nabla v_{\text{sh}} = A' \left(\frac{x \cdot (\hat{\eta}_2 - \hat{\eta}_1)}{\sqrt{2}l_{\text{sh}}} \right) (\hat{\eta}_2 + \hat{\eta}_1) \otimes (\hat{\eta}_2 - \hat{\eta}_1), \quad (3.11)$$

we see that

$$\begin{aligned} e(v_{\text{sh}}) - \frac{1}{2}\mu &= \left[A' \left(\frac{x \cdot (\hat{\eta}_2 - \hat{\eta}_1)}{\sqrt{2}l_{\text{sh}}} \right) - \frac{\lambda_2}{2} \right] \hat{\eta}_2 \otimes \hat{\eta}_2 \\ &\quad - \left[A' \left(\frac{x \cdot (\hat{\eta}_2 - \hat{\eta}_1)}{\sqrt{2}l_{\text{sh}}} \right) + \frac{\lambda_1}{2} \right] \hat{\eta}_1 \otimes \hat{\eta}_1. \end{aligned}$$

Recalling that A' is alternatively equal to $\frac{\lambda_2}{2}$ or $-\frac{\lambda_1}{2}$, we deduce that

$$e(v_{\text{sh}}) - \frac{1}{2}\mu = -\frac{1}{2}\text{tr } \mu \cdot \hat{\eta}_{\text{herr}} \otimes \hat{\eta}_{\text{herr}} \quad \text{on } \mathbb{R}^2, \quad (3.12)$$

where the unit vector field $\hat{\eta}_{\text{herr}} : \mathbb{R}^2 \rightarrow S^1$ satisfies

$$\hat{\eta}_{\text{herr}}(x) = \begin{cases} \hat{\eta}_1 & 0 \leq x \cdot \frac{\hat{\eta}_2 - \hat{\eta}_1}{\sqrt{2}} < \theta l_{\text{sh}} \\ \hat{\eta}_2 & \theta l_{\text{sh}} \leq x \cdot \frac{\hat{\eta}_2 - \hat{\eta}_1}{\sqrt{2}} \leq 1 \end{cases} \quad \text{when } x \cdot \frac{\hat{\eta}_2 - \hat{\eta}_1}{\sqrt{2}} \in [0, 1] \quad (3.13)$$

and is otherwise periodic. Thus v_{sh} transforms μ into a defect which is piecewise constant and rank one.

In the next step, we introduce uni-directional wrinkles in the direction of $\hat{\eta}_{\text{herr}}$. Note the jump set of $\hat{\eta}_{\text{herr}}$ is

$$J_{\hat{\eta}_{\text{herr}}} = \left\{ x \in \mathbb{R}^2 : x \cdot \frac{\hat{\eta}_2 - \hat{\eta}_1}{\sqrt{2}} \in l_{\text{sh}}\mathbb{Z} \right\} \cup \left\{ x \in \mathbb{R}^2 : x \cdot \frac{\hat{\eta}_2 - \hat{\eta}_1}{\sqrt{2}} \in \theta l_{\text{sh}} + l_{\text{sh}}\mathbb{Z} \right\}. \quad (3.14)$$

It consists of (countably many) parallel lines at distances θl_{sh} and $(1 - \theta)l_{\text{sh}}$ apart. The pointwise estimates

$$\|v_{\text{sh}}\|_{L^\infty} \lesssim \text{tr } \mu \cdot l_{\text{sh}} \quad \text{and} \quad \|\nabla v_{\text{sh}}\|_{L^\infty} \lesssim \text{tr } \mu \quad (3.15)$$

follow from (3.10) and (3.11).

Step 2: superimpose twin bands of wrinkles at scale l_{wr} . We now construct uni-directional wrinkles to alleviate the strain left over from Step 1. Define $v_{\text{wr}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $w_{\text{wr}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\begin{aligned} v_{\text{wr}}(x) &= \frac{1}{2} \text{tr } \mu \cdot l_{\text{wr}} V \left(\frac{x \cdot \hat{\eta}_{\text{herr}}(x)}{l_{\text{wr}}} \right) \hat{\eta}_{\text{herr}}(x), \quad x \in \mathbb{R}^2, \\ w_{\text{wr}}(x) &= \sqrt{\text{tr } \mu} \cdot l_{\text{wr}} W \left(\frac{x \cdot \hat{\eta}_{\text{herr}}(x)}{l_{\text{wr}}} \right), \quad x \in \mathbb{R}^2. \end{aligned} \quad (3.16)$$

Here, $W : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$W(t) = \sqrt{2} \cos(t), \quad t \in \mathbb{R}$$

and $V : \mathbb{R} \rightarrow \mathbb{R}$ is the unique 2π -periodic solution of

$$V'(t) + |W'(t)|^2 = 1 \quad \forall t \in \mathbb{R}, \quad V(0) = 0.$$

Such a solution exists as $\int_0^{2\pi} |W'|^2 = 1$.

Evidently,

$$\begin{aligned} \nabla v_{\text{wr}} &= \frac{1}{2} \text{tr } \mu \cdot V' \left(\frac{x \cdot \hat{\eta}_{\text{herr}}}{l_{\text{wr}}} \right) \hat{\eta}_{\text{herr}} \otimes \hat{\eta}_{\text{herr}} \quad \text{on } \mathbb{R}^2 \setminus J_{\hat{\eta}_{\text{herr}}}, \\ \nabla w_{\text{wr}} &= \sqrt{\text{tr } \mu} \cdot W' \left(\frac{x \cdot \hat{\eta}_{\text{herr}}}{l_{\text{wr}}} \right) \hat{\eta}_{\text{herr}} \quad \text{on } \mathbb{R}^2 \setminus J_{\hat{\eta}_{\text{herr}}} \end{aligned} \quad (3.17)$$

and so

$$e(v_{\text{wr}}) + \frac{1}{2} \nabla w_{\text{wr}} \otimes \nabla w_{\text{wr}} = \frac{1}{2} \text{tr } \mu \cdot \hat{\eta}_{\text{herr}} \otimes \hat{\eta}_{\text{herr}} \quad \text{on } \mathbb{R}^2 \setminus J_{\hat{\eta}_{\text{herr}}}. \quad (3.18)$$

Adding up (3.12) and (3.18), we see that

$$e(v_{\text{sh}} + v_{\text{wr}}) + \frac{1}{2} \nabla w_{\text{wr}} \otimes \nabla w_{\text{wr}} = \frac{1}{2} \mu \quad \text{on } \mathbb{R}^2 \setminus J_{\hat{\eta}_{\text{herr}}}. \quad (3.19)$$

As v_{wr} and w_{wr} may jump across $J_{\hat{\eta}_{\text{herr}}}$, this identity may fail to hold there. The pointwise estimates

$$\|v_{\text{wr}}\|_{L^\infty} \lesssim \text{tr } \mu \cdot l_{\text{wr}}, \quad \|\nabla v_{\text{wr}}\|_{L^\infty(\mathbb{R}^2 \setminus J_{\hat{\eta}_{\text{herr}}})} \lesssim \text{tr } \mu, \quad (3.20)$$

$$\|w_{\text{wr}}\|_{L^\infty} \lesssim \sqrt{\text{tr } \mu} \cdot l_{\text{wr}}, \quad \|\nabla w_{\text{wr}}\|_{L^\infty(\mathbb{R}^2 \setminus J_{\hat{\eta}_{\text{herr}}})} \lesssim \sqrt{\text{tr } \mu},$$

$$\|\nabla \nabla w_{\text{wr}}\|_{L^\infty(\mathbb{R}^2 \setminus J_{\hat{\eta}_{\text{herr}}})} \lesssim \frac{\sqrt{\text{tr } \mu}}{l_{\text{wr}}} \quad (3.21)$$

follow from (3.16) and (3.17).

Step 3: join the wrinkles across internal walls at scale δ_{int} . Finally, in order to ensure their bending energy is finite, we must smooth the wrinkles across the jump set $J_{\hat{\eta}_{\text{herr}}}$. For simplicity, and because it will not affect the estimates at leading order, we use a cutoff function χ_{int} supported away from $J_{\hat{\eta}_{\text{herr}}}$ to define the internal walls. (Our choice to use a simple cutoff here manifests in the assumption (1.15b).)

The shortest distance between two lines belonging to $J_{\hat{\eta}_{\text{herr}}}$ is θl_{sh} . Since by hypothesis $\delta_{\text{int}} < \theta l_{\text{sh}}$, there exists a smooth cutoff function $\chi_{\text{int}} \in C^\infty(\mathbb{R}^2)$ with

- $0 \leq \chi_{\text{int}} \leq 1$
- $\chi_{\text{int}}(x) = 0$ if $d(x, J_{\hat{\eta}_{\text{herr}}}) \leq \frac{1}{2}\delta_{\text{int}}$ and $\chi_{\text{int}}(x) = 1$ if $d(x, J_{\hat{\eta}_{\text{herr}}}) \geq \delta_{\text{int}}$,
- $\|\nabla \chi_{\text{int}}\|_{L^\infty} \lesssim \frac{1}{\delta_{\text{int}}}$ and $\|\nabla \nabla \chi_{\text{int}}\|_{L^\infty} \lesssim \frac{1}{\delta_{\text{int}}^2}$

where the constants implicit in the above are independent of all parameters. Let $v_{\text{herr}} : Q \rightarrow \mathbb{R}^2$ and $w_{\text{herr}} : Q \rightarrow \mathbb{R}$ be given by

$$v_{\text{herr}} = v_{\text{sh}} + v_{\text{wr}} \chi_{\text{int}} \quad \text{and} \quad w_{\text{herr}} = w_{\text{wr}} \chi_{\text{int}}.$$

This completes our construction of the herringbone. Note it follows from (3.19) and the definition of χ_{int} that

$$e(v_{\text{herr}}) + \frac{1}{2} \nabla w_{\text{herr}} \otimes \nabla w_{\text{herr}} = \frac{1}{2} \mu \quad \text{on } d(\cdot, J_{\hat{\eta}_{\text{herr}}}) \geq \delta_{\text{int}}. \quad (3.22)$$

The pointwise estimates

$$\|v_{\text{herr}}\|_{L^\infty} \lesssim \text{tr } \mu \cdot l_{\text{sh}} \left(1 \vee \frac{l_{\text{wr}}}{l_{\text{sh}}} \right), \quad \|\nabla v_{\text{herr}}\|_{L^\infty} \lesssim \text{tr } \mu \cdot \left(1 \vee \frac{l_{\text{wr}}}{\delta_{\text{int}}} \right) \quad (3.23)$$

$$\begin{aligned} \|w_{\text{herr}}\|_{L^\infty} &\lesssim \sqrt{\text{tr } \mu} \cdot l_{\text{wr}}, \quad \|\nabla w_{\text{herr}}\|_{L^\infty} \lesssim \sqrt{\text{tr } \mu} \cdot \left(1 \vee \frac{l_{\text{wr}}}{\delta_{\text{int}}} \right) \\ \|\nabla \nabla w_{\text{herr}}\|_{L^\infty} &\lesssim \frac{\sqrt{\text{tr } \mu}}{l_{\text{wr}}} \cdot \left(1 \vee \frac{l_{\text{wr}}^2}{\delta_{\text{int}}^2} \right) \end{aligned} \quad (3.24)$$

carry over from (3.15), (3.20), (3.21), and the properties of χ_{int} listed above.

3.2.2. Energy Estimates for the Herringbone We turn to estimate the energy of the herringbones defined above. It will be convenient to decompose Q into its “wall” and “bulk” regions given by

$$Q_{\text{wall}} = \{x \in Q : d(x, J_{\hat{\eta}_{\text{herr}}}) < \delta_{\text{int}}\} \quad \text{and} \quad Q_{\text{bulk}} = Q \setminus Q_{\text{wall}}. \quad (3.25)$$

Define

$$a_0(l_{\text{sh}}, l_{\text{wr}}; \mu, Q) = \text{tr } \mu \cdot \int_Q \left| W \left(\frac{x \cdot \hat{\eta}_{\text{herr}}(x)}{l_{\text{wr}}} \right) \right|^2 dx, \quad (3.26)$$

$$a_1(l_{\text{sh}}, l_{\text{wr}}, \delta_{\text{int}}; \mu, Q) = \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) |Q_{\text{wall}}|. \quad (3.27)$$

Lemma 3.2. *Let Q be a square, let $\mu \in \text{Sym}_2$ have $\mu > 0$, and let l_{wr} , l_{sh} , and δ_{int} satisfy (3.9). The herringbones constructed in Section 3.2.1 obey the following estimates:*

- the stretching energy satisfies

$$\int_Q \left| e(v_{\text{herr}}) + \frac{1}{2} \nabla w_{\text{herr}} \otimes \nabla w_{\text{herr}} - \frac{1}{2} \mu \right|^2 \lesssim |\mu|^2 a_1(\mu, Q);$$

- the bending energy satisfies

$$\int_Q |\nabla \nabla w_{\text{herr}}|^2 \leq \frac{a_0(\mu, Q) + C|\mu|a_1(\mu, Q)}{l_{\text{wr}}^2}$$

where the constant C is independent of all parameters;

- the substrate energy satisfies

$$\int_Q |w_{\text{herr}}|^2 \leq a_0(\mu, Q)l_{\text{wr}}^2.$$

Proof. We start by estimating the stretching energy, which requires estimating the strain ε . Recall from (3.22) that

$$\varepsilon = e(v_{\text{herr}}) + \frac{1}{2} \nabla w_{\text{herr}} \otimes \nabla w_{\text{herr}} - \frac{1}{2} \mu = 0 \quad \text{on } Q_{\text{bulk}}. \quad (3.28)$$

To handle the wall region, we apply the pointwise estimates from (3.23) and (3.24) to get that

$$|\varepsilon| \leq |\nabla v_{\text{herr}}| + \frac{1}{2} |\nabla w_{\text{herr}}|^2 + \frac{1}{2} |\mu| \lesssim |\mu| \cdot \left(1 \vee \frac{l_{\text{wr}}^2}{\delta_{\text{int}}^2} \right) \quad \text{on } Q_{\text{wall}}. \quad (3.29)$$

Combining (3.28) and (3.29) yields that

$$\int_Q |\varepsilon|^2 = \left(\int_{Q_{\text{wall}}} + \int_{Q_{\text{bulk}}} \right) |\varepsilon|^2 \lesssim |\mu|^2 \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) |Q_{\text{wall}}| = |\mu|^2 a_1$$

according to (3.27). This proves the desired estimate on the stretching energy.

Next we estimate the bending energy, being careful to keep track of the important prefactors. Since $w_{\text{herr}} = w_{\text{wr}}$ in the bulk region and $\hat{\eta}_{\text{herr}}$ is locally constant there, we see from (3.16) that

$$\nabla \nabla w_{\text{herr}} = \frac{\sqrt{\text{tr } \mu}}{l_{\text{wr}}} W'' \left(\frac{x \cdot \hat{\eta}_{\text{herr}}}{l_{\text{wr}}} \right) \hat{\eta}_{\text{herr}} \otimes \hat{\eta}_{\text{herr}} \quad \text{on } Q_{\text{bulk}}. \quad (3.30)$$

On the other hand, it follows from the last estimate in (3.24) that

$$|\nabla \nabla w_{\text{herr}}| \lesssim \frac{\sqrt{|\mu|}}{l_{\text{wr}}} \left(1 \vee \frac{l_{\text{wr}}^2}{\delta_{\text{int}}^2} \right) \quad \text{on } Q_{\text{wall}}. \quad (3.31)$$

Using (3.30) and (3.31) and the fact that $|W| = |W''|$ we deduce that

$$\begin{aligned} \int_Q |\nabla \nabla w_{\text{herr}}|^2 &= \left(\int_{Q_{\text{bulk}}} + \int_{Q_{\text{wall}}} \right) |\nabla \nabla w_{\text{herr}}|^2 \\ &\leq \frac{\text{tr } \mu}{l_{\text{wr}}^2} \int_Q \left| W \left(\frac{x \cdot \hat{\eta}_{\text{herr}}}{l_{\text{wr}}} \right) \right|^2 + C \frac{|\mu|}{l_{\text{wr}}^2} \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) |Q_{\text{wall}}| \\ &= \frac{a_0 + C|\mu|a_1}{l_{\text{wr}}^2} \end{aligned}$$

by the definitions of a_0 and a_1 in (3.26) and (3.27).

We finish with the substrate energy. Evidently, there holds

$$|w_{\text{herr}}| \leq |w_{\text{wr}}| = \sqrt{\text{tr } \mu} \cdot l_{\text{wr}} \left| W \left(\frac{x \cdot \hat{\eta}_{\text{herr}}}{l_{\text{wr}}} \right) \right| \quad \text{on } Q$$

given that $\chi_{\text{int}} \leq 1$ and due to the out-of-plane part of (3.16). It follows that

$$\int_Q |w_{\text{herr}}|^2 \leq \int_Q |w_{\text{wr}}|^2 = \text{tr } \mu \cdot l_{\text{wr}}^2 \int_Q \left| W \left(\frac{x \cdot \hat{\eta}_{\text{herr}}}{l_{\text{wr}}} \right) \right|^2 = l_{\text{wr}}^2 a_0,$$

as desired. \square

Next, we estimate the quantities a_0 and a_1 defined in (3.26) and (3.27).

Lemma 3.3. *We have the estimates*

$$\begin{aligned} |a_0(\mu, Q) - \text{tr } \mu \cdot |Q|| &\lesssim \text{tr } \mu \cdot \frac{l_{\text{wr}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{sh}}}{\text{diam } Q} \right) |Q|, \\ a_1(\mu, Q) &\lesssim \frac{\delta_{\text{int}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) \left(1 \vee \frac{l_{\text{sh}}}{\text{diam } Q} \right) |Q|. \end{aligned}$$

Proof. We start with a_1 . Recall the definitions of $J_{\hat{\eta}_{\text{herr}}}$ and Q_{wall} from (3.14) and (3.25). The former consists of parallel lines at distances θl_{sh} and $(1 - \theta)l_{\text{sh}}$ apart, the total number of which intersecting Q is $\lesssim \frac{\text{diam } Q}{l_{\text{sh}}} \vee 1$. Each such line contributes area $\lesssim \delta_{\text{int}} \cdot \text{diam } Q$ to Q_{wall} . Hence,

$$|Q_{\text{wall}}| \lesssim \left(\frac{\text{diam } Q}{l_{\text{sh}}} \vee 1 \right) \cdot (\delta_{\text{int}} \cdot \text{diam } Q) = \frac{\delta_{\text{int}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{sh}}}{\text{diam } Q} \right) |Q|. \quad (3.32)$$

It follows from (3.27) that

$$a_1 \lesssim \frac{\delta_{\text{int}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) \left(1 \vee \frac{l_{\text{sh}}}{\text{diam } Q} \right) |Q|.$$

We turn to estimate a_0 . First, we claim that

$$\left| \int_S W^2 \left(\frac{x \cdot \hat{\eta}}{l} \right) dx - |S| \right| \lesssim \mathcal{H}^1(\partial S)l \quad \forall l \in (0, \infty) \quad (3.33)$$

if $\hat{\eta} \in S^1$ and if $S \subset \mathbb{R}^2$ is a bounded, measurable set. The constant implicit in (3.33) is independent of l , $\hat{\eta}$, and S . To prove it, note that $W^2(\cdot) + W^2(\cdot + \frac{\pi}{2}) = 2$, from which it follows that

$$\int_S W^2 \left(\frac{x \cdot \hat{\eta}}{l} \right) dx + \int_S W^2 \left(\frac{(x + \frac{\pi}{2}l\hat{\eta}) \cdot \hat{\eta}}{l} \right) dx = 2|S|. \quad (3.34)$$

On the other hand, a change of variables shows that

$$\begin{aligned} \left| \int_S W^2 \left(\frac{(x + \frac{\pi}{2}l\hat{\eta}) \cdot \hat{\eta}}{l} \right) dx - \int_S W^2 \left(\frac{x \cdot \hat{\eta}}{l} \right) dx \right| &= \left| \left(\int_{S + \frac{\pi}{2}l\hat{\eta}} - \int_S \right) W^2 \left(\frac{x \cdot \hat{\eta}}{l} \right) \right| \\ &\lesssim \left| S \Delta \left(S + \frac{\pi}{2}l\hat{\eta} \right) \right|. \end{aligned} \quad (3.35)$$

To control the righthand side we use that

$$|S\Delta(S + l\hat{\eta})| \leq \mathcal{H}^1(\partial S)l \quad \forall l \in (0, \infty), \quad (3.36)$$

a direct consequence of [69, Theorem 3]. Applying (3.34)–(3.36) proves (3.33).

With the estimate (3.33) in hand we can easily handle a_0 . From its definition in (3.13), $\hat{\eta}_{\text{herr}}$ takes on only the values $\hat{\eta}_1$ and $\hat{\eta}_2$. Decompose Q according to

$$Q = S_1 \cup S_2 \quad \text{where} \quad S_i = \{x \in Q : \hat{\eta}_{\text{herr}} = \hat{\eta}_i\}, \quad i = 1, 2.$$

As in the proof of (3.32), we note that

$$\mathcal{H}^1(\partial S_i) \lesssim \left(\frac{\text{diam } Q}{l_{\text{sh}}} \vee 1 \right) \cdot \text{diam } Q = \frac{1}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{sh}}}{\text{diam } Q} \right) |Q| \quad (3.37)$$

for $i = 1, 2$. Hence,

$$\begin{aligned} |a_0 - \text{tr } \mu \cdot |Q|| &= \text{tr } \mu \cdot \left| \int_Q W^2 \left(\frac{x \cdot \hat{\eta}_{\text{herr}}}{l_{\text{wr}}} \right) - |Q| \right| \\ &\leq \text{tr } \mu \cdot \sum_{i=1}^2 \left| \int_{S_i} W^2 \left(\frac{x \cdot \hat{\eta}_i}{l_{\text{wr}}} \right) - |S_i| \right| \\ &\lesssim \text{tr } \mu \cdot \sum_{i=1}^2 \mathcal{H}^1(\partial S_i) l_{\text{wr}} \lesssim \text{tr } \mu \cdot \frac{l_{\text{wr}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{sh}}}{\text{diam } Q} \right) |Q|, \end{aligned}$$

where in the second line we applied (3.33) and (3.37). \square

3.3. Piecewise Herringbone Patterns Adapted to Variable Defect

We return to $\Omega \subset \mathbb{R}^2$ which for our present purposes must only be a bounded and Lipschitz domain, and consider a target defect $\mu : \Omega \rightarrow \text{Sym}_2$ that is positive definite and Lipschitz continuous. Our task is to construct a family of displacements

$$\{(v_{\text{p.h.}}, w_{\text{p.h.}})\} \subset W^{1,\infty}(\Omega; \mathbb{R}^2) \times W^{2,\infty}(\Omega)$$

adapted to μ in that

$$v_{\text{p.h.}} \approx 0, \quad w_{\text{p.h.}} \approx 0, \quad e(v_{\text{p.h.}}) + \frac{1}{2} \nabla w_{\text{p.h.}} \otimes \nabla w_{\text{p.h.}} \approx \frac{1}{2} \mu$$

and whose energy is optimal at leading order. After approximating μ by a piecewise constant target defect $\langle \mu \rangle$ defined on a lattice of squares, we piece together a well-chosen family of herringbones from Section 3.2 to form our ‘‘piecewise herringbone’’ pattern. The result is depicted in Fig. 3 (see also Panel (a) in Fig. 1).

Our piecewise herringbones will be parameterized by

$$\begin{aligned} l_{\text{wr}} &\in (0, \infty), \quad l_{\text{sh}} \in (0, \infty), \quad l_{\text{avg}} \in (0, \infty), \\ \delta_{\text{int}} &\in \left(0, \frac{1}{4} \frac{\lambda}{\Lambda} l_{\text{sh}} \right), \quad \delta_{\text{ext}} \in \left(0, \frac{1}{2} l_{\text{avg}} \right), \end{aligned} \quad (3.38)$$

where $\lambda, \Lambda \in (0, \infty)$ satisfy

$$\lambda Id \leq \mu(x) \leq \Lambda Id \quad \forall x \in \Omega. \quad (3.39)$$

The parameters l_{wr} , l_{sh} , and δ_{int} should already be familiar from Section 3.2: these set the lengthscales of the wrinkles, the in-plane shear, and the internal walls of the herringbones. The first new parameter l_{avg} gives the “averaging” lengthscale across which we treat μ as if it were constant. It will be proportional to the diameter of the herringboned squares. The parameter δ_{ext} sets the thickness of the “external walls” between neighboring squares. We shall construct a piecewise herringbone for any choice of parameters satisfying (3.38). However, in anticipation of the optimization that is to come, we note that ones for which

$$\left(\frac{b}{k}\right)^{1/4} = l_{\text{wr}} \ll l_{\text{sh}} \ll l_{\text{avg}} \ll \text{diam } \Omega, \\ l_{\text{wr}} \lesssim \delta_{\text{int}} \lesssim_{\mu} l_{\text{sh}}, \quad l_{\text{sh}} \lesssim \delta_{\text{ext}} \lesssim l_{\text{avg}}$$

satisfy

$$E_{b,k,\gamma}(v_{\text{p.h.}}, w_{\text{p.h.}}) = (2\sqrt{bk} + \gamma) \cdot \frac{1}{2} \int_{\Omega} \text{tr } \mu \, dx + O\left(l_{\text{avg}}^2 \vee \frac{\delta_{\text{int}}}{l_{\text{sh}}} \vee \frac{\delta_{\text{ext}}}{l_{\text{avg}}}\right).$$

The error term accounts for the cost of the approximation $\mu \approx \langle \mu \rangle$ as well as that of the walls. When it is negligible, our piecewise herringbones are optimal at leading order (again, the requisite lower bound is contained in the results of Section 2). Minimizing over the free parameters maximizes the range of this result—see Corollary 3.1 for the details. We turn now to construct a general piecewise herringbone, and to estimate its energy.

3.3.1. Constructing the Piecewise Herringbone

Step 1: assemble an l_{avg} -by- l_{avg} lattice of herringbones. Define the squares

$$Q_{\alpha} = \alpha + (0, l_{\text{avg}})^2 \quad \forall \alpha \in \mathbb{Z}^2$$

and let the index set \mathcal{I} be the smallest subset of \mathbb{Z}^2 with the property that

$$\Omega \subset \bigcup_{\alpha \in \mathcal{I}} \overline{Q_{\alpha}}.$$

Define the locally averaged target defect $\langle \mu \rangle$ whose value on the α th square is

$$\mu_{\alpha} = \fint_{\Omega \cap Q_{\alpha}} \mu(x) \, dx, \quad \alpha \in \mathcal{I}. \quad (3.40)$$

We produce a family of herringbone constructions $\{(v_{\text{herr}}^{\alpha}, w_{\text{herr}}^{\alpha})\}_{\alpha \in \mathcal{I}}$ using the results of Section 3.2: given $\alpha \in \mathcal{I}$, we define $v_{\text{herr}}^{\alpha} : Q_{\alpha} \rightarrow \mathbb{R}^2$ and $w_{\text{herr}}^{\alpha} : Q_{\alpha} \rightarrow \mathbb{R}$ following the procedure from Section 3.2.1 with μ_{α} as the target defect and l_{sh} , l_{wr} , and δ_{int} as above. (We take the parameters to be independent of α for ease of

exposition, and as it will not affect the estimates at leading order.) Copying over the pointwise bounds (3.23) and (3.24), we note that

$$\|v_{\text{herr}}^\alpha\|_{L^\infty} \lesssim \text{tr } \mu_\alpha \cdot l_{\text{sh}} \left(1 \vee \frac{l_{\text{wr}}}{l_{\text{sh}}} \right), \quad \|\nabla v_{\text{herr}}^\alpha\|_{L^\infty} \lesssim \text{tr } \mu_\alpha \cdot \left(1 \vee \frac{l_{\text{wr}}}{\delta_{\text{int}}} \right) \quad (3.41)$$

$$\begin{aligned} \|w_{\text{herr}}^\alpha\|_{L^\infty} &\lesssim \sqrt{\text{tr } \mu_\alpha} \cdot l_{\text{wr}}, \quad \|\nabla w_{\text{herr}}^\alpha\|_{L^\infty} \lesssim \sqrt{\text{tr } \mu_\alpha} \cdot \left(1 \vee \frac{l_{\text{wr}}}{\delta_{\text{int}}} \right), \\ \|\nabla \nabla w_{\text{herr}}^\alpha\|_{L^\infty} &\lesssim \frac{\sqrt{\text{tr } \mu_\alpha}}{l_{\text{wr}}} \cdot \left(1 \vee \frac{l_{\text{wr}}^2}{\delta_{\text{int}}^2} \right) \end{aligned} \quad (3.42)$$

with constants independent of $\alpha \in \mathcal{I}$.

Before proceeding to the next step of the construction, let us quickly verify that the parameters l_{sh} , l_{wr} , and δ_{int} are indeed admissible for use in Section 3.2. According to (3.9), we must check that

$$l_{\text{sh}} \in (0, \infty), \quad l_{\text{wr}} \in (0, \infty), \quad \text{and} \quad \delta_{\text{int}} \in \left(0, \frac{1}{2} \theta_\alpha l_{\text{sh}} \right) \quad \forall \alpha \in \mathcal{I}, \quad (3.43)$$

where

$$\theta_\alpha = \frac{\lambda_1^\alpha}{\lambda_1^\alpha + \lambda_2^\alpha} \quad \text{and} \quad 0 < \lambda_1^\alpha \leq \lambda_2^\alpha \text{ are the eigenvalues of } \mu_\alpha.$$

It follows from (3.39) and (3.40) that $\theta_\alpha \in (\frac{\lambda}{2\Lambda}, \frac{\Lambda}{2\lambda})$. As by hypothesis $\delta_{\text{int}} < \frac{1}{4} \frac{\lambda}{\Lambda} l_{\text{sh}}$, we conclude (3.43).

Step 2: join the herringbones across external walls at scale δ_{ext} . The next step is to join the herringbones obtained above into a single, globally defined piecewise herringbone pattern. We employ a family of smooth cutoff functions supported away from $\cup_{\alpha \in \mathcal{I}} \partial Q_\alpha$ for the external walls. Since by hypothesis $\delta_{\text{ext}} < \frac{1}{2} l_{\text{avg}}$, there exists a family of smooth cutoff functions $\{\chi_{\text{ext}}^\alpha\}_{\alpha \in \mathcal{I}}$ such that

- $\chi_{\text{ext}}^\alpha \in C_c^\infty(Q_\alpha)$ and $0 \leq \chi_\alpha \leq 1$,
- $\chi_{\text{ext}}^\alpha(x) = 0$ if $d(x, \partial Q_\alpha) \leq \frac{1}{2} \delta_{\text{ext}}$ and $\chi_{\text{ext}}^\alpha(x) = 1$ if $d(x, \partial Q_\alpha) \geq \delta_{\text{ext}}$,
- $\|\nabla \chi_{\text{ext}}^\alpha\|_{L^\infty} \lesssim \frac{1}{\delta_{\text{ext}}}$, $\|\nabla \nabla \chi_{\text{ext}}^\alpha\|_{L^\infty} \lesssim \frac{1}{\delta_{\text{ext}}^2}$.

The constants implicit above are independent of all parameters (including α). Finally, we define $v_{\text{p.h.}} : \Omega \rightarrow \mathbb{R}^2$ and $w_{\text{p.h.}} : \Omega \rightarrow \mathbb{R}$ by

$$v_{\text{p.h.}} = \sum_{\alpha \in \mathcal{I}} v_{\text{herr}}^\alpha \chi_{\text{ext}}^\alpha \quad \text{and} \quad w_{\text{p.h.}} = \sum_{\alpha \in \mathcal{I}} w_{\text{herr}}^\alpha \chi_{\text{ext}}^\alpha.$$

This completes our construction of the piecewise herringbone. Note the pointwise estimates

$$\begin{aligned} \|v_{\text{p.h.}}\|_{L^\infty} &\lesssim \|\mu\|_{L^\infty} \cdot l_{\text{sh}} \left(1 \vee \frac{l_{\text{wr}}}{l_{\text{sh}}} \right), \\ \|\nabla v_{\text{p.h.}}\|_{L^\infty} &\lesssim \|\mu\|_{L^\infty} \cdot \left(1 \vee \frac{l_{\text{sh}}}{\delta_{\text{ext}}} \vee \frac{l_{\text{wr}}}{\delta_{\text{ext}}} \vee \frac{l_{\text{wr}}}{\delta_{\text{int}}} \right), \end{aligned} \quad (3.44)$$

$$\begin{aligned} \|w_{\text{p.h.}}\|_{L^\infty} &\lesssim \sqrt{\|\mu\|_{L^\infty}} \cdot l_{\text{wr}}, \quad \|\nabla w_{\text{p.h.}}\|_{L^\infty} \lesssim \sqrt{\|\mu\|_{L^\infty}} \cdot \left(1 \vee \frac{l_{\text{wr}}}{\delta_{\text{ext}}} \vee \frac{l_{\text{wr}}}{\delta_{\text{int}}} \right), \\ \|\nabla \nabla w_{\text{p.h.}}\|_{L^\infty} &\lesssim \frac{\sqrt{\|\mu\|_{L^\infty}}}{l_{\text{wr}}} \cdot \left(1 \vee \frac{l_{\text{wr}}^2}{\delta_{\text{ext}}^2} \vee \frac{l_{\text{wr}}^2}{\delta_{\text{int}}^2} \right). \end{aligned} \quad (3.45)$$

These follow from (3.41), (3.42), and the properties of $\{\chi_\text{ext}^\alpha\}$ listed above.

3.3.2. Energy Estimates for the Piecewise Herringbone Here we estimate the energy of the piecewise herringbones just defined. Decompose Ω into its “wall” and “bulk” regions given by

$$\Omega_{\text{wall}} = \{x \in \Omega : d(x, \cup_\alpha \partial Q_\alpha) < \delta_{\text{ext}}\} \quad \text{and} \quad \Omega_{\text{bulk}} = \Omega \setminus \Omega_{\text{wall}} \quad (3.46)$$

and define the quantities

$$A_0(l_{\text{avg}}; \mu) = \sum_{\alpha \in \mathcal{I}} \text{tr } \mu_\alpha \cdot |Q_\alpha| \quad (3.47)$$

$$\begin{aligned} A_1(l_{\text{avg}}, l_{\text{sh}}, l_{\text{wr}}, \delta_{\text{int}}, \delta_{\text{ext}}; \mu) &= \left(1 \vee \frac{l_{\text{sh}}^2}{\delta_{\text{ext}}^2} \vee \frac{l_{\text{wr}}^4}{\delta_{\text{ext}}^4} \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) |\Omega_{\text{wall}}| \\ &\quad + \frac{\delta_{\text{int}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) \left(1 \vee \frac{l_{\text{sh}}}{l_{\text{avg}}} \right) |\cup_{\alpha \in \mathcal{I}} Q_\alpha| \end{aligned} \quad (3.48)$$

$$A_2(l_{\text{avg}}) = l_{\text{avg}}^2 |\Omega|. \quad (3.49)$$

Lemma 3.4. *Let Ω be bounded and Lipschitz, and let $\mu : \Omega \rightarrow \text{Sym}_2$ be positive definite and Lipschitz continuous. Let l_{wr} , l_{sh} , l_{avg} , δ_{int} , and δ_{ext} satisfy (3.38). The piecewise herringbones constructed in Section 3.3.1 satisfy the following estimates:*

- the stretching energy satisfies

$$\int_\Omega \left| e(v_{\text{p.h.}}) + \frac{1}{2} \nabla w_{\text{p.h.}} \otimes \nabla w_{\text{p.h.}} - \frac{1}{2} \mu \right|^2 \lesssim \|\mu\|_{L^\infty}^2 A_1 + \|\nabla \mu\|_{L^\infty}^2 A_2;$$

- the bending energy satisfies

$$\int_\Omega |\nabla \nabla w_{\text{p.h.}}|^2 \leq \frac{A_0 + C \|\mu\|_{L^\infty} A_1}{l_{\text{wr}}^2}$$

where the constant C is independent of all parameters;

- the substrate energy satisfies

$$\int_{\Omega} |w_{\text{p.h.}}|^2 \leq (A_0 + C \|\mu\|_{L^\infty} A_1) l_{\text{wr}}^2.$$

Proof. We begin with the stretching energy. Introduce the strains

$$\begin{aligned} \varepsilon &= e(v_{\text{p.h.}}) + \frac{1}{2} \nabla w_{\text{p.h.}} \otimes \nabla w_{\text{p.h.}} - \frac{1}{2} \mu, \\ \varepsilon_\alpha &= e(v_{\text{herr}}^\alpha) + \frac{1}{2} \nabla w_{\text{herr}}^\alpha \otimes \nabla w_{\text{herr}}^\alpha - \frac{1}{2} \mu_\alpha \quad \text{for } \alpha \in \mathcal{I}. \end{aligned}$$

Using the definition of the cutoff function χ_{ext}^α we find that

$$\varepsilon = \varepsilon_\alpha + \frac{1}{2}(\mu_\alpha - \mu) \quad \text{on } \Omega_{\text{bulk}} \cap Q_\alpha,$$

hence by the triangle inequality

$$|\varepsilon| \leq |\varepsilon_\alpha| + \frac{1}{2}|\mu_\alpha - \mu| \lesssim |\varepsilon_\alpha| + \|\nabla \mu\|_{L^\infty} l_{\text{avg}} \quad \text{on } \Omega_{\text{bulk}} \cap Q_\alpha. \quad (3.50)$$

On the other hand, the pointwise estimates from (3.44) and (3.45) imply that

$$|\varepsilon| \leq |\nabla v_{\text{p.h.}}| + \frac{1}{2} |\nabla w_{\text{p.h.}}|^2 + \frac{1}{2} |\mu| \lesssim \|\mu\|_{L^\infty} \left(1 \vee \frac{l_{\text{sh}}}{\delta_{\text{ext}}} \vee \frac{l_{\text{wr}}^2}{\delta_{\text{ext}}^2} \vee \frac{l_{\text{wr}}^2}{\delta_{\text{int}}^2} \right) \quad \text{on } \Omega_{\text{wall}}. \quad (3.51)$$

Applying (3.50), (3.51), and the stretching part of Lemma 3.2 we deduce that

$$\begin{aligned} \int_{\Omega} |\varepsilon|^2 &= \left(\int_{\Omega_{\text{wall}}} + \int_{\Omega_{\text{bulk}}} \right) |\varepsilon|^2 = \int_{\Omega_{\text{wall}}} |\varepsilon|^2 + \sum_{\alpha \in \mathcal{I}} \int_{\Omega_{\text{bulk}} \cap Q_\alpha} |\varepsilon|^2 \\ &\lesssim \|\mu\|_{L^\infty}^2 \left(1 \vee \frac{l_{\text{sh}}^2}{\delta_{\text{ext}}^2} \vee \frac{l_{\text{wr}}^4}{\delta_{\text{ext}}^4} \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) |\Omega_{\text{wall}}| \\ &\quad + \sum_{\alpha \in \mathcal{I}} \left[\|\mu\|_{L^\infty}^2 a_1(\mu_\alpha, Q_\alpha) + \|\nabla \mu\|_{L^\infty}^2 l_{\text{avg}}^2 |\Omega_{\text{bulk}} \cap Q_\alpha| \right] \\ &\lesssim \|\mu\|_{L^\infty}^2 \left[\left(1 \vee \frac{l_{\text{sh}}^2}{\delta_{\text{ext}}^2} \vee \frac{l_{\text{wr}}^4}{\delta_{\text{ext}}^4} \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) |\Omega_{\text{wall}}| \right. \\ &\quad \left. + \frac{\delta_{\text{int}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) \left(1 \vee \frac{l_{\text{sh}}}{l_{\text{avg}}} \right) |\cup_{\alpha \in \mathcal{I}} Q_\alpha| \right] + \|\nabla \mu\|_{L^\infty}^2 l_{\text{avg}}^2 |\Omega| \\ &\lesssim \|\mu\|_{L^\infty}^2 A_1 + \|\nabla \mu\|_{L^\infty}^2 A_2 \end{aligned}$$

according to the definitions of A_1 and A_2 in (3.48) and (3.49). Note we used the estimate

$$\sum_{\alpha \in \mathcal{I}} a_1(\mu_\alpha, Q_\alpha) \lesssim \frac{\delta_{\text{int}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) \left(1 \vee \frac{l_{\text{sh}}}{l_{\text{avg}}} \right) |\cup_{\alpha \in \mathcal{I}} Q_\alpha| \quad (3.52)$$

in the third line, which follows from Lemma 3.3 and the fact that $\text{diam } Q_\alpha \sim l_{\text{avg}}$ uniformly in α . The desired estimate on the stretching energy is proved.

We turn to estimate the bending energy. Note that

$$w_{\text{p.h.}} = w_{\text{herr}}^\alpha \quad \text{on } \Omega_{\text{bulk}} \cap Q_\alpha \quad (3.53)$$

by the definition of χ_{ext}^α , while the pointwise estimate

$$|\nabla \nabla w_{\text{p.h.}}| \lesssim \frac{\sqrt{||\mu||_{L^\infty}}}{l_{\text{wr}}} \left(1 \vee \frac{l_{\text{wr}}^2}{\delta_{\text{ext}}^2} \vee \frac{l_{\text{wr}}^2}{\delta_{\text{int}}^2} \right) \quad \text{on } \Omega_{\text{wall}} \quad (3.54)$$

follows from (3.45). Note also that

$$\sum_{\alpha \in \mathcal{I}} a_0(\mu_\alpha, Q_\alpha) \leq \sum_{\alpha} \text{tr } \mu_\alpha \cdot |Q_\alpha| + C ||\mu||_{L^\infty} \frac{l_{\text{wr}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{sh}}}{l_{\text{avg}}} \right) |\cup_{\alpha \in \mathcal{I}} Q_\alpha| \quad (3.55)$$

as a result of Lemma 3.3. Using (3.53), (3.54) and the bending part of Lemma 3.2 we deduce that

$$\begin{aligned} \int_{\Omega} |\nabla \nabla w_{\text{p.h.}}|^2 &= \left(\int_{\Omega_{\text{wall}}} + \int_{\Omega_{\text{bulk}}} \right) |\nabla \nabla w_{\text{p.h.}}|^2 \\ &= \int_{\Omega_{\text{wall}}} |\nabla \nabla w_{\text{p.h.}}|^2 + \sum_{\alpha \in \mathcal{I}} \int_{\Omega_{\text{bulk}} \cap Q_\alpha} |\nabla \nabla w_{\text{herr}}^\alpha|^2 \\ &\leq C \frac{||\mu||_{L^\infty}}{l_{\text{wr}}^2} \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{ext}}^4} \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) |\Omega_{\text{wall}}| \\ &\quad + \sum_{\alpha \in \mathcal{I}} \frac{a_0(\mu_\alpha, Q_\alpha) + C ||\mu||_{L^\infty} a_1(\mu_\alpha, Q_\alpha)}{l_{\text{wr}}^2} \\ &\leq \frac{1}{l_{\text{wr}}^2} \sum_{\alpha \in \mathcal{I}} \text{tr } \mu_\alpha \cdot |Q_\alpha| + C \frac{||\mu||_{L^\infty}}{l_{\text{wr}}^2} \left[\left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{ext}}^4} \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) |\Omega_{\text{wall}}| \right. \\ &\quad \left. + \frac{\delta_{\text{int}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{wr}}^4}{\delta_{\text{int}}^4} \right) \left(1 \vee \frac{l_{\text{sh}}}{l_{\text{avg}}} \right) |\cup_{\alpha \in \mathcal{I}} Q_\alpha| \right] \\ &\leq \frac{A_0 + C ||\mu||_{L^\infty} A_1}{l_{\text{wr}}^2} \end{aligned}$$

where we applied (3.52) and (3.55) to pass from the second line to the third.

We finish with the substrate energy. Note that

$$|w_{\text{p.h.}}| \leq |w_{\text{herr}}^\alpha| \quad \text{on } \Omega \cap Q_\alpha \quad (3.56)$$

as there always holds $\chi_{\text{ext}}^\alpha \leq 1$. Using (3.56) and the substrate part of Lemma 3.2, there follows

$$\begin{aligned} \int_{\Omega} |w_{\text{p.h.}}|^2 &\leq \sum_{\alpha \in \mathcal{I}} \int_{\Omega \cap Q_\alpha} |w_{\text{herr}}^\alpha|^2 \leq \sum_{\alpha \in \mathcal{I}} a_0(\mu_\alpha, Q_\alpha) l_{\text{wr}}^2 \\ &\leq l_{\text{wr}}^2 \sum_{\alpha} \text{tr } \mu_\alpha \cdot |Q_\alpha| + C l_{\text{wr}}^2 ||\mu||_{L^\infty} \frac{l_{\text{wr}}}{l_{\text{sh}}} \left(1 \vee \frac{l_{\text{sh}}}{l_{\text{avg}}} \right) |\cup_{\alpha \in \mathcal{I}} Q_\alpha| \\ &\leq (A_0 + C ||\mu||_{L^\infty} A_1) l_{\text{wr}}^2. \end{aligned}$$

We used (3.55) to pass to the second line, and the definitions of A_0 and A_1 from (3.47) and (3.48) at the end. \square

Next, we identify an energetically optimal version of the piecewise herringbone by minimizing over the free parameters l_{avg} , l_{sh} , l_{wr} , δ_{int} , and δ_{ext} from (3.38). To simplify the presentation, and as it turns out to be consistent with optimality, we shall impose the additional constraints

$$l_{\text{wr}} \ll l_{\text{sh}} \ll l_{\text{avg}} \ll \text{diam } \Omega, \quad l_{\text{wr}} \lesssim \delta_{\text{int}}, \quad l_{\text{sh}} \lesssim \delta_{\text{ext}} \quad (3.57)$$

below. We require the asymptotic behavior of the quantities A_0 , A_1 , and A_2 from (3.47)–(3.49).

Lemma 3.5. *We have that*

$$A_0 \rightarrow \int_{\Omega} \text{tr } \mu \, dx, \quad A_1 \lesssim \frac{\delta_{\text{int}}}{l_{\text{sh}}} |\Omega| + \frac{\delta_{\text{ext}}}{l_{\text{avg}}} (\text{diam } \Omega)^2, \quad A_2 = l_{\text{avg}}^2 |\Omega|$$

in any limit satisfying (3.57).

Proof. The claim regarding A_0 follows from its definition, since

$$\cup_{\alpha \in \mathcal{I}} Q_{\alpha} \rightarrow \Omega \quad \text{as } l_{\text{avg}} \rightarrow 0.$$

The claim regarding A_2 is clear. Now we address A_1 . First, note that

$$|\cup_{\alpha \in \mathcal{I}} \overline{Q_{\alpha}}| \lesssim |\Omega|$$

for all small enough l_{avg} . Now recall the definition of Ω_{wall} from (3.46). Each square Q_{α} has perimeter $\lesssim l_{\text{avg}}$, and their δ_{ext} -thickenings have area $\lesssim \delta_{\text{ext}} \cdot l_{\text{avg}}$. The total number of squares is eventually $\lesssim \left(\frac{\text{diam } \Omega}{l_{\text{avg}}}\right)^2$. Hence,

$$|\Omega_{\text{wall}}| \lesssim \left(1 \vee \frac{(\text{diam } \Omega)^2}{l_{\text{avg}}^2}\right) \cdot (\delta_{\text{ext}} \cdot l_{\text{avg}}) = \frac{\delta_{\text{ext}}}{l_{\text{avg}}} \left(1 \vee \frac{l_{\text{avg}}^2}{(\text{diam } \Omega)^2}\right) \cdot (\text{diam } \Omega)^2.$$

Setting these estimates into (3.48) and appealing to (3.57) we see that

$$A_1 \lesssim |\Omega_{\text{wall}}| + \frac{\delta_{\text{int}}}{l_{\text{sh}}} |\cup_{\alpha \in \mathcal{I}} Q_{\alpha}| \lesssim \frac{\delta_{\text{ext}}}{l_{\text{avg}}} (\text{diam } \Omega)^2 + \frac{\delta_{\text{int}}}{l_{\text{sh}}} |\Omega|.$$

\square

We are ready to optimize over the piecewise herringbone patterns adapted to μ . Given (3.57), the estimates

$$\int_{\Omega} |e(v_{\text{p.h.}}) + \frac{1}{2} \nabla w_{\text{p.h.}} \otimes \nabla w_{\text{p.h.}} - \frac{1}{2} \mu|^2 \lesssim_{\Omega, \mu} l_{\text{avg}}^2 \vee \frac{\delta_{\text{int}}}{l_{\text{sh}}} \vee \frac{\delta_{\text{ext}}}{l_{\text{avg}}}, \quad (3.58)$$

$$\begin{aligned} & \frac{b}{2} \int_{\Omega} |\nabla \nabla w_{\text{p.h.}}|^2 + \frac{k}{2} \int_{\Omega} |w_{\text{p.h.}}|^2 \\ & \leq \left(\frac{b}{2} \frac{1}{l_{\text{wr}}^2} + \frac{k}{2} l_{\text{wr}}^2\right) \left(\int_{\Omega} \text{tr } \mu + o(1) + C(\Omega, \mu) \frac{\delta_{\text{int}}}{l_{\text{sh}}} \vee \frac{\delta_{\text{ext}}}{l_{\text{avg}}}\right) \end{aligned} \quad (3.59)$$

follow from Lemmas 3.4 and 3.5. Balancing the dominant terms in (3.58) and (3.59) yields

$$\frac{b}{l_{\text{wr}}^2} = kl_{\text{wr}}^2 \quad \text{and} \quad l_{\text{avg}}^2 \sim \frac{\delta_{\text{int}}}{l_{\text{sh}}} \sim \frac{\delta_{\text{ext}}}{l_{\text{avg}}},$$

while saturating the last two constraints from (3.57) yields

$$\frac{\delta_{\text{int}}}{l_{\text{wr}}} \sim \frac{\delta_{\text{ext}}}{l_{\text{sh}}} \sim 1.$$

These five relations underlie optimal choices for the five free parameters. Using them in (3.58) and (3.59) and recalling the pointwise estimates from (3.44) and (3.45), we conclude the following result:

Corollary 3.1. *Let Ω be bounded and Lipschitz, and let $\mu : \Omega \rightarrow \text{Sym}_2$ be positive definite and Lipschitz continuous. Let $\{(v_{\text{p.h.}}, w_{\text{p.h.}})\}$ be a sequence of piecewise herringbones as constructed in Section 3.3.1, and suppose their parameters from (3.38) satisfy*

$$l_{\text{wr}} = \left(\frac{b}{k}\right)^{1/4} \ll \text{diam } \Omega, \quad l_{\text{avg}} \sim l_{\text{wr}}^{1/5}, \quad l_{\text{sh}} \sim l_{\text{wr}}^{1/2} l_{\text{avg}}^{1/2}, \\ \delta_{\text{int}} \sim l_{\text{wr}}, \quad \delta_{\text{ext}} \sim l_{\text{sh}}.$$

Such a sequence satisfies the energy estimates

$$\int_{\Omega} |e(v_{\text{p.h.}}) + \frac{1}{2} \nabla w_{\text{p.h.}} \otimes \nabla w_{\text{p.h.}} - \frac{1}{2} \mu|^2 \lesssim_{\Omega, \mu} \left(\frac{b}{k}\right)^{1/10}, \\ \frac{b}{2} \int_{\Omega} |\nabla \nabla w_{\text{p.h.}}|^2 + \frac{k}{2} \int_{\Omega} |w_{\text{p.h.}}|^2 \leq \sqrt{bk} \cdot \int_{\Omega} \text{tr } \mu \, dx + o(\sqrt{bk})$$

as well as the pointwise estimates

$$\|v_{\text{p.h.}}\|_{L^\infty} \lesssim_{\mu} \left(\frac{b}{k}\right)^{3/20}, \quad \|\nabla v_{\text{p.h.}}\|_{L^\infty} \lesssim_{\mu} 1, \quad (3.60)$$

$$\|w_{\text{p.h.}}\|_{L^\infty} \lesssim_{\mu} \left(\frac{b}{k}\right)^{1/4}, \quad \|\nabla w_{\text{p.h.}}\|_{L^\infty} \lesssim_{\mu} 1, \quad \|\nabla \nabla w_{\text{p.h.}}\|_{L^\infty} \lesssim_{\mu} \left(\frac{k}{b}\right)^{1/4}. \quad (3.61)$$

3.4. Recovery Sequences

We are finally ready to prove Proposition 3.1. We take for granted the results of Sections 3.1–3.3.

Proof of Proposition 3.1. Let $u_{\text{eff}} \in BD(\Omega)$ be tension-free. By Proposition 2.1, it suffices to construct

$$(u_{b,k,\gamma}, u_{b,k,\gamma}) \xrightarrow{*} (u_{\text{eff}}, 0) \quad \text{weakly-}* \text{ in } BD(\Omega) \times W^{1,2}(\Omega) \quad (3.62)$$

such that

$$\limsup \frac{E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma})}{2\sqrt{bk} + \gamma} \leq \int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \, ds. \quad (3.63)$$

We begin by applying the results of Section 3.1 to reduce to u_{eff} that are smooth and uniformly tension-free. Due to Lemma 3.1, there exist uniformly tension-free $\{u_n\}_{n \in \mathbb{N}} \subset C^\infty(\bar{\Omega}; \mathbb{R}^2)$ converging to u_{eff} in the intermediate sense. In particular,

$$u_n \xrightarrow{*} u_{\text{eff}} \text{ weakly-* in } BD(\Omega) \text{ and } \int_{\partial\Omega} u_n \cdot \hat{v} \, ds \rightarrow \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \, ds$$

as $n \rightarrow \infty$. Suppose for each fixed n we can produce a recovery sequence $\{(u_{n,m}, w_{n,m})\}_{m \in \mathbb{N}}$ for $(u_n, 0)$, i.e., a sequence satisfying the analog of (3.62) and (3.63) but with u_{eff} replaced by u_n . Then, a straightforward diagonalization argument produces a recovery sequence for $(u_{\text{eff}}, 0)$. Thus, it suffices to achieve (3.62) and (3.63) for u_{eff} that are smooth and uniformly tension-free. We do so via the piecewise herringbone patterns from Section 3.3.

Fix some $u_{\text{eff}} \in C^\infty(\bar{\Omega}; \mathbb{R}^2)$ that is uniformly tension-free. Introduce the (pointwise-defined) target defect

$$\mu(x) = \nabla p \otimes \nabla p(x) - 2e(u_{\text{eff}})(x) \quad x \in \Omega \quad (3.64)$$

and note it is Lipschitz as

$$\|\nabla \mu\|_{L^\infty} \lesssim \|\nabla \nabla u_{\text{eff}}\|_{L^\infty} \vee \|\nabla \nabla p\|_{L^\infty} \|\nabla p\|_{L^\infty} < \infty.$$

It is also uniformly positive definite. Therefore, we may apply the results of Section 3.3 to obtain a family of piecewise herringbones $\{(v_{\text{p.h.}}, w_{\text{p.h.}})\}$ indexed by l_{avg} , l_{sh} , l_{wr} , δ_{int} , and δ_{ext} and that are adapted to μ . Guided by Corollary 3.1, we choose these parameters to depend on b and k (and not on γ) as follows: we take

$$l_{\text{wr}} = \left(\frac{b}{k}\right)^{1/4}, \quad l_{\text{sh}} = l_{\text{wr}}^{1/2} l_{\text{avg}}^{1/2}, \quad l_{\text{avg}} = l_{\text{wr}}^{1/5}, \quad \delta_{\text{int}} = l_{\text{wr}}, \quad \text{and} \quad \delta_{\text{ext}} = l_{\text{sh}}$$

noting that these define a valid piecewise herringbone pattern, according to (3.38), so long as

$$\delta_{\text{int}} < \frac{1}{4} \frac{\lambda}{\Lambda} l_{\text{sh}} \quad \text{and} \quad \delta_{\text{ext}} < \frac{1}{2} l_{\text{avg}}.$$

Here, $\lambda, \Lambda \in (0, \infty)$ are as in (3.39). We must arrange for $l_{\text{wr}}^{2/5} < \frac{1}{4} \frac{\lambda}{\Lambda}$. The quantities λ and Λ are fixed by μ , and hence by p and u_{eff} . Of course, $l_{\text{wr}} \ll 1$ within the given parameter regime in (3.1). The required inequalities are eventually satisfied.

All that remains is to assemble the estimates from Corollary 3.1 to prove (3.62) and (3.63). Calling

$$u_{b,k,\gamma} = u_{\text{eff}} + v_{\text{p.h.}} \quad \text{and} \quad w_{b,k,\gamma} = w_{\text{p.h.}},$$

we see from (3.60) and (3.61) that the desired convergence (3.62) holds. Using the formula (1.10) for the energy, the definition of the target defect in (3.64), and the rest of the estimates in the corollary we conclude that

$$\begin{aligned}
E_{b,k,\gamma}(u_{b,k,\gamma}, w_{b,k,\gamma}) &\leq \frac{1}{2} \int_{\Omega} |e(v_{\text{p.h.}}) + \frac{1}{2} \nabla w_{\text{p.h.}} \otimes \nabla w_{\text{p.h.}} - \frac{1}{2} \mu|^2 + \frac{b}{2} \int_{\Omega} |\nabla \nabla w_{\text{p.h.}}|^2 + \frac{k}{2} \int_{\Omega} |w_{\text{p.h.}}|^2 \\
&\quad + \gamma \left(\int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \right) + \gamma \|v_{\text{p.h.}}\|_{L^1(\partial\Omega)} + C(p) \frac{b}{2} (\|\nabla \nabla w_{\text{p.h.}}\|_{L^2} + 1) \\
&\leq (2\sqrt{bk} + \gamma) \cdot \int_{\Omega} \frac{1}{2} \text{tr } \mu + o(\sqrt{bk}) \\
&\quad + C(\Omega, p, u_{\text{eff}}) \left(\left(\frac{b}{k} \right)^{1/10} + b^{3/4} k^{1/4} + \gamma \left(\frac{b}{k} \right)^{3/20} \right) \\
&= (2\sqrt{bk} + \gamma) \cdot \int_{\Omega} \frac{1}{2} \text{tr } \mu + o(2\sqrt{bk} + \gamma),
\end{aligned}$$

by the definition of the parameter regime in (3.1). Note in the second line we applied the identity

$$\int_{\Omega} \frac{1}{2} \text{tr } \mu(x) dx = \int_{\Omega} \frac{1}{2} |\nabla p|^2 dx - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} ds,$$

which follows from (3.64). The desired inequality (3.63) is proved. \square

Together, Propositions 2.1 and 3.1 prove Theorem 1.1. The rest of the results in Sections 1.2.1 and 1.2.2 follow as explained there.

4. Convex Analysis of the Limiting Problems

Sections 2 and 3 established the role of the limiting minimization problems

$$\min_{\substack{u_{\text{eff}} \in BD(\Omega) \\ e(u_{\text{eff}}) \leq \frac{1}{2} \nabla p \otimes \nabla p dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 dx - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} ds \quad \text{and} \quad \min_{\substack{\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2) \\ -\frac{1}{2} \text{curlcurl } \mu = \det \nabla \nabla p}} \frac{1}{2} \int_{\Omega} |\mu|_1 \quad (4.1)$$

in the asymptotic analysis of the energy $E_{b,k,\gamma}$. In particular, we showed under the assumptions at the start of Section 1.2 that

$$\min E_{b,k,\gamma} = C_1 \cdot (2\sqrt{bk} + \gamma) + o(2\sqrt{bk} + \gamma),$$

where C_1 is the common minimum value of the limiting problems in (4.1). We also established via Γ -convergence a correspondence between the almost minimizers of $E_{b,k,\gamma}$, and optimal u_{eff} and μ solving these problems. The fact that their optimal values are the same follows from the change of variables

$$e(u_{\text{eff}}) + \frac{1}{2} \mu = \frac{1}{2} \nabla p \otimes \nabla p dx$$

and the Saint-Venant compatibility conditions for simply connected domains. We refer the reader to Sections 1.2.1 and 1.2.2 for further discussion.

The remainder of this paper is devoted to the analysis of the limiting problems, and in particular to proving the results from Sections 1.2.3 and 1.2.4. The present section contains, amongst other things, a proof of Theorem 1.2: we establish the asserted duality between the “primal” problems in (4.1) and their “dual” problem

$$\max_{\substack{\varphi: \mathbb{R}^2 \rightarrow \mathbb{R} \\ \varphi \text{ is convex} \\ \varphi = \frac{1}{2}|x|^2 \text{ on } \mathbb{R}^2 \setminus \Omega}} \int_{\Omega} \left(\varphi - \frac{1}{2}|x|^2 \right) \det \nabla \nabla p \, dx \quad (4.2)$$

posed over the given admissible Airy potentials φ (our choice of terminology will soon become clear). This duality holds under the basic assumptions from (1.14a) if Ω is simply connected. Actually, the methods developed here extend with little additional effort to general domains, even as the form of the dual problem changes. The choice of primal must be addressed. Since the (linearized) area problem appearing on the lefthand side of (4.1) is the more general of the two, we take it to be our primal in what follows. We do so also because we expect that it should extend as the Γ -limit of $\frac{1}{2\sqrt{b\bar{k}} + \gamma} E_{b,k,\gamma}$ for general domains. We proceed to state its dual.

We require a certain linear functional L . Consider the vector space of functions $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ that are locally affine exterior to Ω , i.e., that satisfy

$$\nabla \nabla a = 0 \quad \text{on } \mathbb{R}^2 \setminus \overline{\Omega}$$

in the pointwise sense, and define

$$L(a) = \int_{\mathbb{R}^2 \setminus \Omega} a \det \nabla \nabla \bar{p} \, dx \quad (4.3)$$

where $\bar{p} \in W^{2,2}(\mathbb{R}^2) \cap C_c(\mathbb{R}^2)$ is chosen once and for all such that $\bar{p} = p$ on Ω . That the value of $L(a)$ is independent of the choice of extension \bar{p} of p follows from the very weak Hessian identity

$$-\frac{1}{2} \operatorname{curlcurl} \nabla \bar{p} \otimes \nabla \bar{p} = \det \nabla \nabla \bar{p}, \quad (4.4)$$

as will be explained later on in Lemma 4.2. Evidently, it depends only on the values taken on by a exterior to Ω . Given that a is locally affine there, we think of L as a sort of “boundary integral”. Note if p is regular enough, this can be understood using the divergence theorem along with (4.4).

We come now to our general duality result. Recall the formally adjoint operators $\operatorname{curlcurl}$ and $\nabla^\perp \nabla^\perp$ from (1.27) and (1.33). Recall also that a sequence of measures is said to converge *narrowly* in $\mathcal{M}(\Omega; \operatorname{Sym}_2)$ if their integrals against arbitrary elements of $C_b(\Omega; \operatorname{Sym}_2)$ converge.

Proposition 4.1. *Let Ω be bounded and Lipschitz and let $p \in W^{2,2}(\Omega)$. We have the duality*

$$\begin{aligned} & \min_{\substack{u_{\text{eff}} \in BD(\Omega) \\ e(u_{\text{eff}}) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial\Omega} u_{\text{eff}} \cdot \hat{v} \, ds \\ &= \max_{\substack{\varphi: \mathbb{R}^2 \rightarrow \mathbb{R} \\ \varphi \text{ is convex} \\ \nabla \nabla (\varphi - \frac{1}{2} |x|^2) = 0 \text{ on } \mathbb{R}^2 \setminus \bar{\Omega}}} \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p \, dx + L \left(\varphi - \frac{1}{2} |x|^2 \right). \end{aligned} \quad (4.5)$$

Regarding complementary slackness, the following are equivalent for u_{eff} and φ admissible in the above:

1. u_{eff} and φ are optimal;
2. there exist non-negative $\{\mu_n\}_{n \in \mathbb{N}} \subset C^2(\bar{\Omega}; \text{Sym}_2)$ approximating

$$\mu = \nabla p \otimes \nabla p \, dx - 2e(u_{\text{eff}})$$

in that

$$\begin{aligned} \mu_n \, dx &\rightarrow \mu \quad \text{narrowly in } \mathcal{M}(\Omega; \text{Sym}_2), \\ -\frac{1}{2} \operatorname{curl} \operatorname{curl} \mu_n \, dx &\xrightarrow{*} \det \nabla \nabla p \, dx \quad \text{weakly-} * \text{ in } \mathcal{M}(\Omega) \end{aligned} \quad (4.6)$$

as $n \rightarrow \infty$, and for which

$$0 = \lim_{n \rightarrow \infty} \int_{\Omega} \left| \langle \mu_n, \nabla^{\perp} \nabla^{\perp} \varphi \rangle \right| = \lim_{n \rightarrow \infty} \int_{\partial\Omega} |\hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu_n \rangle| \, ds; \quad (4.7)$$

3. the limits in (4.7) vanish for all such approximations $\{\mu_n\}_{n \in \mathbb{N}} \subset C^2(\bar{\Omega}; \text{Sym}_2)$ to the given μ .

Here, $[\nabla \varphi]$ denotes the jump in $\nabla \varphi$ across $\partial\Omega$ in the direction of \hat{v} . It is given by $\nabla \varphi|_{\partial(\mathbb{R}^2 \setminus \bar{\Omega})} - \nabla \varphi|_{\partial\Omega}$.

Some remarks are in order. First, we observe that various other statements of the dual problem from (4.5) can be produced given additional regularity for p . Perhaps the most illuminating one is as follows: if $p \in W^{2,2}(\Omega) \cap C^1(\bar{\Omega})$, the dual problem can be rewritten as

$$\max_{\substack{\sigma \in \mathcal{M}_+(\mathbb{R}^2; \text{Sym}_2) \\ \operatorname{div} \sigma = 0 \\ \sigma = Id \, dx \text{ on } \mathbb{R}^2 \setminus \bar{\Omega}}} \int_{\bar{\Omega}} \left\langle \frac{1}{2} \nabla p \otimes \nabla p, Id \, dx - \sigma \right\rangle. \quad (4.8)$$

As (4.8) is never used in this paper, we leave the details of its proof to the reader, and simply remark that it proceeds via the usual change of variables between an Airy potential φ and its induced “Airy stress”

$$\sigma = \nabla^{\perp} \nabla^{\perp} \varphi$$

with the only slight complication being that, here, σ is considered to be a measure. In fact, the first step in our proof of (4.5) will be to obtain an ill-posed version of (4.8) in which σ is taken to be continuous, and for which a maximizer is not guaranteed.

Our next two remarks concern the proof of Theorem 1.2: we claim that Proposition 4.1 reduces to the duality between (4.1) and (4.2) if Ω is simply connected. To see this, first note that the functional on the righthand side of (4.5) is invariant under the addition of any affine function to φ . Indeed, it follows from its definition and the very weak Hessian identity (4.4) that

$$L(a) = - \int_{\Omega} a \det \nabla \nabla p \, dx \quad \text{if } a \text{ is affine.} \quad (4.9)$$

Now if Ω is simply connected it has only one exterior component, and hence any locally affine function on $\mathbb{R}^2 \setminus \overline{\Omega}$ extends automatically as an affine function on \mathbb{R}^2 . It follows that we may take $\varphi = \frac{1}{2}|x|^2$ on $\mathbb{R}^2 \setminus \Omega$ in (4.5), in which case $L = 0$ and the original dual problem (4.2) results.

We finish by showing how the complementary slackness conditions of Theorem 1.2 follow from the general ones established here. The fact is that the mollifications $\{\mu_\delta\}_{\delta>0}$ from (1.32) approximate the given μ in the sense of (4.6). This is a direct consequence of Lemma 4.6 and the identity

$$-\frac{1}{2} \operatorname{curl} \operatorname{curl} \mu = \det \nabla \nabla p.$$

Proposition 4.1 therefore implies that u_{eff} and φ are optimal if and only if

$$\langle \nabla^\perp \nabla^\perp \varphi, \mu \rangle = 0 \quad \text{in } \Omega \quad \text{and} \quad \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu \rangle = 0 \quad \text{at } \partial \Omega$$

in the regularized sense (meaning that (1.35) holds). As the primal problems in (4.1) are equivalent for simply connected Ω , Theorem 1.2 follows from these remarks.

The rest of this section proves Proposition 4.1. Section 4.1 establishes (4.5). It is there that we explain how to anticipate the form of the general dual problem via a minimax procedure. Section 4.2 proves the complementary slackness conditions by establishing the integration by parts identity

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mu|_1 - \int_{\Omega} \left(\varphi - \frac{1}{2}|x|^2 \right) \det \nabla \nabla p - L \left(\varphi - \frac{1}{2}|x|^2 \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} \langle \mu_n, \nabla^\perp \nabla^\perp \varphi \rangle + \frac{1}{2} \int_{\partial \Omega} \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu_n \rangle \end{aligned} \quad (4.10)$$

if u_{eff} and φ are admissible and $\{\mu_n\}_{n \in \mathbb{N}}$ approximates $\mu = \nabla p \otimes \nabla p \, dx - 2e(u_{\text{eff}})$ in the sense of (4.6). Together, these complete the proof of the general duality result. They also lay the groundwork for Section 5 where we make precise our method of stable lines.

4.1. The Dual Problem

We begin by proving (4.5). First we introduce a Lagrange multiplier σ for the tension-free constraint, and apply a minimax procedure to identify the dual problem it should solve. Given $u \in BD(\Omega)$, observe that

$$\begin{aligned} e(u) &\leq \frac{1}{2} \nabla p \otimes \nabla p \, dx \\ \iff \quad \int_{\Omega} \left\langle \sigma, \frac{1}{2} \nabla p \otimes \nabla p \, dx - e(u) \right\rangle &\geq 0 \quad \forall \sigma \in C(\bar{\Omega}; \text{Sym}_2) \text{ with } \sigma \geq 0. \end{aligned}$$

The primal problem on the lefthand side of (4.5) can therefore be rewritten as

$$\begin{aligned} &\min_{\substack{u \in BD(\Omega) \\ e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial\Omega} u \cdot \hat{v} \, ds \\ &= \inf_{u \in BD(\Omega)} \sup_{\substack{\sigma \in C(\bar{\Omega}; \text{Sym}_2) \\ \sigma \geq 0}} \int_{\Omega} \left\langle Id - \sigma, \frac{1}{2} \nabla p \otimes \nabla p \, dx - e(u) \right\rangle. \end{aligned}$$

Now to identify its dual, we reverse the order of operations between \inf and \sup . We do so informally at first, and then again with a rigorous proof in Lemma 4.1. Let $\sigma \in C(\bar{\Omega}; \text{Sym}_2)$. By the divergence theorem,

$$\begin{aligned} \int_{\Omega} \langle Id - \sigma, e(u) \rangle &= 0 \quad \forall u \in BD(\Omega) \\ \iff \quad \text{div } \sigma &= 0 \quad \text{on } \Omega \quad \text{and} \quad \sigma \hat{v} = \hat{v} \quad \text{at } \partial\Omega. \end{aligned}$$

The first condition in the second line is that σ is weakly divergence-free. The second condition holds where the outwards-pointing unit normal \hat{v} is well-defined. It follows that

$$\begin{aligned} &\sup_{\sigma \in C(\bar{\Omega}; \text{Sym}_2)} \inf_{u \in BD(\Omega)} \int_{\Omega} \left\langle Id - \sigma, \frac{1}{2} \nabla p \otimes \nabla p \, dx - e(u) \right\rangle \\ &= \sup_{\substack{\sigma \in C(\bar{\Omega}; \text{Sym}_2) \\ \sigma \geq 0 \text{ and } \text{div } \sigma = 0 \\ \sigma \hat{v} = \hat{v} \text{ at } \partial\Omega}} \int_{\Omega} \left\langle Id - \sigma, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle \, dx, \end{aligned}$$

and this is our candidate dual.

The following result justifies the manipulations above:

Lemma 4.1. *There holds*

$$\begin{aligned} &\min_{\substack{u \in BD(\Omega) \\ e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial\Omega} u \cdot \hat{v} \, ds \\ &= \sup_{\substack{\sigma \in C(\bar{\Omega}; \text{Sym}_2) \\ \sigma \geq 0 \text{ and } \text{div } \sigma = 0 \\ \sigma \hat{v} = \hat{v} \text{ at } \partial\Omega}} \int_{\Omega} \left\langle Id - \sigma, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle \, dx. \end{aligned} \tag{4.11}$$

Equality continues to hold when the boundary condition $\sigma \hat{v} = \hat{v}$ is replaced by the more restrictive one that $\sigma = Id$ at $\partial\Omega$.

Proof. Although the asserted equalities are not yet clear, the inequality

$$\begin{aligned} & \min_{\substack{u \in BD(\Omega) \\ e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 \, dx - \int_{\partial\Omega} u \cdot \hat{v} \, ds \\ & \geq \sup_{\substack{\sigma \in C(\bar{\Omega}; \text{Sym}_2) \\ \sigma \geq 0 \text{ and } \text{div } \sigma = 0 \\ \sigma \hat{v} = \hat{v} \text{ at } \partial\Omega}} \int_{\Omega} \left\langle Id - \sigma, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle \, dx \end{aligned}$$

does follow directly from the minimax argument above (the inf sup of a functional is never less than its sup inf). Eliminating the common term $\int_{\Omega} \frac{1}{2} |\nabla p|^2$, making the change of variables $\sigma = Id - \zeta$, and applying a straightforward inclusion argument, we see it suffices to check that

$$\max_{\substack{u \in BD(\Omega) \\ e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\partial\Omega} u \cdot \hat{v} \, ds = \inf_{\substack{\zeta \in C_c(\Omega; \text{Sym}_2) \\ Id \geq \zeta \text{ and } \text{div } \zeta = 0}} \int_{\Omega} \left\langle Id - \zeta, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle \, dx. \quad (4.12)$$

This can be done via the Fenchel–Rockafeller minimax theorem (see, e.g., [14, Theorem 1.12]), as we explain.

Introduce the vector spaces

$$E = C_c(\Omega; \text{Sym}_2) \quad \text{and} \quad E^* = \mathcal{M}(\Omega; \text{Sym}_2)$$

and equip them respectively with the uniform and dual norms. By Riez–Markov, E^* is the topological dual of E . Define the functionals $\Phi, \Psi : E \rightarrow (-\infty, \infty]$ by

$$\Phi(\zeta) = \begin{cases} \int_{\Omega} \langle Id - \zeta, \frac{1}{2} \nabla p \otimes \nabla p \rangle \, dx & Id \geq \zeta \\ \infty & \text{otherwise} \end{cases} \quad \text{and} \quad \Psi(\zeta) = \begin{cases} 0 & \text{div } \zeta = 0 \\ \infty & \text{otherwise} \end{cases}.$$

Since $\frac{1}{2} \nabla p \otimes \nabla p \, dx \in E^*$, and as zero is bounded uniformly away from Id , the functional Φ is finite and continuous at $\zeta = 0$. Evidently, $\Psi(0) < \infty$. Thus, by the Fenchel–Rockafeller minimax theorem,

$$\max_{\varepsilon \in E^*} -\Phi^*(-\varepsilon) - \Psi^*(\varepsilon) = \inf_{\zeta \in E} \Phi(\zeta) + \Psi(\zeta). \quad (4.13)$$

The Legendre transforms $\Phi^*, \Psi^* : E^* \rightarrow (-\infty, \infty]$ on the lefthand side are given for $\varepsilon \in E^*$ by

$$\Phi^*(\varepsilon) = \sup_{\zeta \in E} \int_{\Omega} \langle \zeta, \varepsilon \rangle - \Phi(\zeta) \quad \text{and} \quad \Psi^*(\varepsilon) = \sup_{\zeta \in E} \int_{\Omega} \langle \zeta, \varepsilon \rangle - \Psi(\zeta).$$

To finish, we must deduce from (4.13) the desired equality (4.12).

It is clear from the definitions that

$$\inf_{\zeta \in E} \Phi(\zeta) + \Psi(\zeta) = \inf_{\substack{\zeta \in C_c(\Omega; \text{Sym}_2) \\ Id \geq \zeta \text{ and } \text{div } \zeta = 0}} \int_{\Omega} \left\langle Id - \zeta, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle \, dx.$$

Thus, the righthand sides of (4.12) and (4.13) agree. To check their lefthand sides, we compute the Legendre transforms of Φ and Ψ . Given $\varepsilon \in E^*$, we claim that

$$\Psi^*(\varepsilon) = \sup_{\substack{\zeta \in E \\ \operatorname{div} \zeta = 0}} \int_{\Omega} \langle \zeta, \varepsilon \rangle = \begin{cases} 0 & \exists u \in BD(\Omega) \text{ s.t. } \varepsilon = e(u) \\ \infty & \text{otherwise} \end{cases}. \quad (4.14)$$

The first equality is clear. To see the second, note by the divergence theorem (the first identity in (1.13)) that

$$\int_{\Omega} \langle \zeta, e(u) \rangle = 0$$

whenever $u \in BD(\Omega)$ and $\zeta \in E \cap C_c^1(\Omega; \operatorname{Sym}_2)$ is divergence-free. By density, it holds for $\zeta \in E$ that are weakly divergence-free. On the other hand, suppose $\varepsilon \in E^*$ but that there does not exist $u \in BD(\Omega)$ such that $\varepsilon = e(u)$. According to [76, Proposition 2.1 and Theorem 2.1], there exists a divergence-free $\zeta \in C_c^\infty(\Omega; \operatorname{Sym}_2)$ for which

$$\int_{\Omega} \langle \zeta, \varepsilon \rangle \neq 0.$$

Making the replacement $\zeta \rightarrow \lambda \zeta$ and sending $\lambda \rightarrow \infty$ or $-\infty$, we deduce (4.14).

Finally, we compute the Legendre transform of Φ . Given $u \in BD(\Omega)$, we see using its definition that

$$\begin{aligned} & \Phi^*(-e(u)) \\ &= \sup_{\substack{\zeta \in E \\ Id \geq \zeta}} \int_{\Omega} \langle \zeta, -e(u) \rangle - \int_{\Omega} \left\langle Id - \zeta, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle dx \\ &= \sup_{\substack{\zeta \in E \\ Id \geq \zeta}} \int_{\Omega} \left\langle \zeta, \frac{1}{2} \nabla p \otimes \nabla p dx - e(u) \right\rangle - \int_{\Omega} \left\langle Id, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle dx \\ &= \begin{cases} \int_{\Omega} \left\langle Id, \frac{1}{2} \nabla p \otimes \nabla p dx - e(u) \right\rangle - \int_{\Omega} \left\langle Id, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle dx & e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p dx \\ \infty & \text{otherwise} \end{cases} \\ &= \begin{cases} - \int_{\partial\Omega} u \cdot \hat{v} ds & e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p dx \\ \infty & \text{otherwise} \end{cases}. \end{aligned}$$

Combining this with (4.14) proves that

$$\max_{\varepsilon \in E^*} -\Phi^*(-\varepsilon) - \Psi^*(\varepsilon) = \max_{u \in BD(\Omega)} -\Phi^*(-e(u)) = \max_{\substack{u \in BD(\Omega) \\ e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p}} \int_{\partial\Omega} u \cdot \hat{v} ds.$$

Thus, the lefthand sides of (4.12) and (4.13) are the same. \square

Lemma 4.1 is a good start, but we much prefer to identify a version of the dual problem for which maximizers are guaranteed. The basic issue is that, while the admissible σ in (4.11) satisfy

$$\int_{\Omega} |\sigma|_1 = \int_{\Omega} \langle \nabla x, \sigma \rangle = \int_{\partial\Omega} x \cdot \hat{v} = \int_{\Omega} \langle \nabla x, Id \rangle = 2|\Omega|$$

so that they are bounded *a priori* in L^1 , no similar control on $\nabla\sigma$ is available (even as it is trace-free). So, while the admissible set in (4.11) is pre-compact in the weak-* topology induced by the injection

$$C(\overline{\Omega}; \text{Sym}_2) \rightarrow \mathcal{M}(\overline{\Omega}; \text{Sym}_2), \sigma \mapsto \sigma dx,$$

it is not closed. Evidently, the boundary conditions $\sigma \hat{v} = \hat{v}$ and *ad hoc* regularity hypothesis that $\sigma \in C$ must be relaxed. Taking into account the low regularity of ∇p , which is not necessarily continuous at the present level of generality, we find it convenient to do so by changing variables to the anticipated potentials φ .

It is well-known that if σ is Sym_2 -valued and divergence-free, there exists a scalar-valued function φ such that

$$\sigma = \nabla^{\perp} \nabla^{\perp} \varphi$$

on any simply connected domain. (See (1.33) for the notation $\nabla^{\perp} \nabla^{\perp}$.) Such functions φ are usually referred to in the literature as “Airy potentials” or “Airy stress functions”, and the divergence-free fields $\nabla^{\perp} \nabla^{\perp} \varphi$ they generate are known as “Airy stresses”. We need not rule out the possibility that Ω is multiply connected. This is because the required change of variables φ can be carried out on \mathbb{R}^2 , as the boundary conditions in (4.11) ensure that σ can be extended there in a divergence-free way. To prepare, we record some useful properties of the functional L from (4.3).

Lemma 4.2. *L is well-defined on the vector space of functions $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ that are locally affine exterior to Ω . It is linear and continuous in any norm for which the restriction map $a \mapsto a|_{\mathbb{R}^2 \setminus \overline{\Omega}}$ is continuous.*

Proof. To check that L is well-defined, we must show that the integral in (4.3) does not depend on the choice of extension \bar{p} . That is, we must prove that

$$\int_{\mathbb{R}^2 \setminus \Omega} a \det \nabla \nabla \bar{p}_1 = \int_{\mathbb{R}^2 \setminus \Omega} a \det \nabla \nabla \bar{p}_2$$

if $\bar{p}_1, \bar{p}_2 \in W^{2,2}(\mathbb{R}^2) \cap C_c(\mathbb{R}^2)$ satisfy $\bar{p}_1 = \bar{p}_2$ on Ω and $\nabla \nabla a = 0$ on $\mathbb{R}^2 \setminus \overline{\Omega}$. By density, it suffices to take \bar{p}_1 and \bar{p}_2 to be smooth. Note we can also take a to be smooth as every locally affine function on $\mathbb{R}^2 \setminus \overline{\Omega}$ admits a smooth extension to \mathbb{R}^2 . Testing (4.4) against a , integrating by parts twice, and subtracting yields that

$$\int_{\mathbb{R}^2} a (\det \nabla \nabla \bar{p}_1 - \det \nabla \nabla \bar{p}_2) = -\frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla^{\perp} \nabla^{\perp} a, \nabla \bar{p}_1 \otimes \nabla \bar{p}_1 - \nabla \bar{p}_2 \otimes \nabla \bar{p}_2 \rangle.$$

The integrand on the right vanishes a.e. by our hypotheses. For the same reason, the integral on the left may be taken over $\mathbb{R}^2 \setminus \Omega$. The desired equality is proved.

Looking back to (4.3), we see that L is a linear functional of $a|_{\mathbb{R}^2 \setminus \bar{\Omega}}$. Since by hypothesis Ω is a bounded, Lipschitz domain, it has finitely many exterior components, i.e.,

$$\mathbb{R}^2 \setminus \bar{\Omega} = \bigcup_{i=1}^N E_i \quad \text{where the sets } E_i \text{ are open and disjoint.}$$

That a is locally affine exterior to Ω is equivalent to the existence of $\{m_i\}_{i=1}^N \subset \mathbb{R}^2$ and $\{b_i\}_{i=1}^N \subset \mathbb{R}$ such that

$$a = m_i \cdot x + b_i \quad \text{on } E_i, \quad \text{for } i = 1, \dots, N.$$

Quotienting out by the equivalence relation that $a_1 \sim a_2$ if $a_1|_{\mathbb{R}^2 \setminus \bar{\Omega}} = a_2|_{\mathbb{R}^2 \setminus \bar{\Omega}}$, there results a finite dimensional vector space on which L is well-defined. The stated continuity now follows from the elementary fact that every linear function of finitely many variables is continuous, regardless of the choice of norm. \square

We are ready to change variables from σ to φ . To help simplify the presentation, and as it does not affect the end result, we use the “restricted” set of admissible σ from Lemma 4.1.

Lemma 4.3. *The restricted sets of admissible stresses and Airy potentials*

$$\begin{aligned} & \{\sigma \in C(\bar{\Omega}; \text{Sym}_2) : \sigma \geq 0, \text{ div } \sigma = 0, \sigma = \text{Id} \text{ at } \partial\Omega\} \\ & \text{and } \{\varphi \in C^2(\mathbb{R}^2) : \nabla \nabla \varphi \geq 0, \nabla \nabla \varphi = \text{Id} \text{ on } \mathbb{R}^2 \setminus \Omega\} \end{aligned}$$

are put into a many-to-one correspondence via the relation

$$\nabla^\perp \nabla^\perp \varphi = \begin{cases} \sigma & \text{on } \Omega \\ \text{Id} & \text{on } \mathbb{R}^2 \setminus \Omega \end{cases}.$$

Under this correspondence,

$$\int_{\Omega} \left\langle \text{Id} - \sigma, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle dx = \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p dx + L \left(\varphi - \frac{1}{2} |x|^2 \right).$$

Proof. The stated correspondence follows from our previous remarks on Airy potentials since \mathbb{R}^2 is simply connected. In particular, when we extend a given σ by setting it equal to Id off of Ω , the resulting Sym_2 -valued function is continuous, non-negative, and weakly divergence-free on \mathbb{R}^2 . Thus, there exists a corresponding φ , which is of course non-unique. The reverse direction is clear.

We turn to prove the stated equality. By Lemma 4.2, we may fix some compactly supported, $W^{2,2}$ extension \bar{p} of p in the definition of L . It follows from the given correspondence and the very weak Hessian identity (4.4) that

$$\begin{aligned} \int_{\Omega} \left\langle \text{Id} - \sigma, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle &= \int_{\Omega} \left\langle \nabla^\perp \nabla^\perp \left(\frac{1}{2} |x|^2 - \varphi \right), \frac{1}{2} \nabla p \otimes \nabla p \right\rangle \\ &= \int_{\mathbb{R}^2} \left\langle \nabla^\perp \nabla^\perp \left(\varphi - \frac{1}{2} |x|^2 \right), -\frac{1}{2} \nabla \bar{p} \otimes \nabla \bar{p} \right\rangle \\ &= \int_{\mathbb{R}^2} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla \bar{p} \\ &= \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p + L \left(\varphi - \frac{1}{2} |x|^2 \right) \end{aligned}$$

as claimed. Note we used (4.3) at the end. \square

Combining Lemmas 4.1 and 4.3 we get that

$$\begin{aligned}
& \min_{\substack{u \in BD(\Omega) \\ e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{v} \\
&= \sup_{\substack{\varphi \in C^2(\mathbb{R}^2) \\ \varphi \text{ is convex} \\ \nabla \nabla \left(\varphi - \frac{1}{2} |x|^2 \right) = 0 \text{ on } \mathbb{R}^2 \setminus \Omega}} \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p + L \left(\varphi - \frac{1}{2} |x|^2 \right).
\end{aligned} \tag{4.15}$$

The last step is to relax the *ad hoc* regularity hypothesis that $\varphi \in C^2$, so as to allow the corresponding σ to be measure-valued. We must check that the supremum remains the same, and that it is achieved.

Lemma 4.4. *The supremum on the righthand side of (4.15) equals the maximum on the righthand side of (4.5). The latter admits an optimal φ .*

Proof. The first part of the result follows immediately once we establish that

$$\int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{v} \geq \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p + L \left(\varphi - \frac{1}{2} |x|^2 \right) \tag{4.16}$$

whenever u and φ are admissible in (4.5). Indeed, the admissible set of φ in (4.15) is a subset of that in (4.5). Enlarging an admissible set can never make the resulting supremum smaller. Now to prove (4.16), we shall make use of the integration by parts identity (4.10) introduced above and proved in Lemma 4.7 below, as well as the properties of the mollifications $\{\mu_\delta\}_{\delta>0}$ from (1.32) to be proved in Lemma 4.6. The reader may check that these results stand independently of the desired inequality.

Let u and φ be admissible in (4.5) and call $\mu = \nabla p \otimes \nabla p \, dx - 2e(u)$. Lemma 4.6 and Lemma 4.7 show that

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} |\mu|_1 - \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p - L \left(\varphi - \frac{1}{2} |x|^2 \right) \\
&= \lim_{\delta \rightarrow \infty} \frac{1}{2} \int_{\Omega} \langle \mu_\delta, \nabla^\perp \nabla^\perp \varphi \rangle + \frac{1}{2} \int_{\partial\Omega} \hat{v} \cdot [\nabla \varphi] \langle \hat{t} \otimes \hat{t}, \mu_\delta \rangle.
\end{aligned}$$

The integrands in the second line are non-negative by admissibility: that $\mu_\delta \geq 0$ follows from our choice to take $\rho \geq 0$ in their definition; that $\nabla \nabla \varphi \geq 0$ in Ω and that $\hat{v} \cdot [\nabla \varphi] \geq 0$ at $\partial\Omega$ are easy consequences of the convexity of φ (see Lemma 4.5 below). Applying the divergence theorem from (1.13) and the fact that $\mu \geq 0$, we find that

$$\begin{aligned}
\frac{1}{2} \int_{\Omega} |\mu|_1 &= \int_{\Omega} \left\langle Id, \frac{1}{2} \mu \right\rangle = \int_{\Omega} \left\langle Id, \frac{1}{2} \nabla p \otimes \nabla p \, dx - e(u) \right\rangle \\
&= \int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{v}.
\end{aligned}$$

The inequality (4.16) is proved. Thus, the optimal values of the maximization problems in (4.5) and (4.15) must be the same, regardless of whether or not they admit any solutions.

We finish by showing the existence of a maximizer for the dual problem in (4.5). We apply the direct method. Let B be a ball of finite radius that contains Ω , and note it suffices to search for a maximizer in the subset

$$\left\{ \varphi \in C(\mathbb{R}^2) : \varphi \text{ is convex, } \nabla \nabla \left(\varphi - \frac{1}{2} |x|^2 \right) = 0 \text{ on } \mathbb{R}^2 \setminus \overline{\Omega} \right\} \cap \left\{ \varphi = \frac{1}{2} |x|^2 \text{ on } \mathbb{R}^2 \setminus B \right\}. \quad (4.17)$$

Indeed, the functional being maximized in (4.5) is unchanged under the addition of any affine function to φ , as was noted in the paragraph surrounding (4.9). Thus, we can take $\varphi = \frac{1}{2} |x|^2$ on the unbounded component of $\mathbb{R}^2 \setminus \overline{\Omega}$. Now observe that (4.17) is compact in the uniform norm topology: it is closed, and using the bounds

$$\|\nabla \varphi\|_{L^\infty(B)} \leq 1 \quad \text{and} \quad \|\varphi\|_{L^\infty(B)} \leq \max_{x \in \overline{B}} \frac{1}{2} |x|^2 + \text{diam } B$$

we deduce from Arzelà–Ascoli that it is pre-compact. That the functional in (4.5) is uniformly continuous follows from our standing assumptions that Ω is bounded and Lipschitz and that $p \in W^{2,2}(\Omega)$. The uniform continuity of $L(\varphi - \frac{1}{2} |x|^2)$ follows from Lemma 4.2. The existence of a maximizing φ is proved. \square

We are ready to prove (4.5).

Proof of the equality part of Proposition 4.1. Combining Lemmas 4.1–4.4 yields the string of equalities

$$\begin{aligned} & \min_{\substack{u \in BD(\Omega) \\ e(u) \leq \frac{1}{2} \nabla p \otimes \nabla p \, dx}} \int_{\Omega} \frac{1}{2} |\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{v} \\ &= \sup_{\substack{\sigma \in C(\overline{\Omega}; \text{Sym}_2) \\ \sigma \geq 0 \text{ and } \text{div } \sigma = 0 \\ \sigma \hat{v} = \hat{v} \text{ at } \partial\Omega}} \int_{\Omega} \left\langle Id - \sigma, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle \\ &= \sup_{\substack{\varphi \in C^2(\mathbb{R}^2) \\ \varphi \text{ is convex} \\ \nabla \nabla \left(\varphi - \frac{1}{2} |x|^2 \right) = 0 \text{ on } \mathbb{R}^2 \setminus \Omega}} \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p + L \\ &= \max_{\substack{\varphi: \mathbb{R}^2 \rightarrow \mathbb{R} \\ \varphi \text{ is convex} \\ \nabla \nabla \left(\varphi - \frac{1}{2} |x|^2 \right) = 0 \text{ on } \mathbb{R}^2 \setminus \overline{\Omega}}} \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p + L \end{aligned}$$

where we have abbreviated the argument $\varphi - \frac{1}{2} |x|^2$ of L . \square

Before moving on to the complementary slackness part of Proposition 4.1, we pause to point out that the admissible Airy potentials from (4.5) can be described using boundary conditions. This was stated in Remark 1.2 in the context of a simply connected domain (where we took $a = 0$), and will be used later on below.

Lemma 4.5. *Upon restriction to Ω , the admissible set of Airy potentials in (4.5) can be equivalently described as those $\varphi \in HB(\Omega)$ for which*

$$\nabla \nabla \varphi \geq 0 \quad \text{on } \Omega \quad (4.18)$$

and such that

$$\varphi = \frac{1}{2}|x|^2 + a \quad \text{and} \quad \hat{v} \cdot \nabla \varphi \leq \hat{v} \cdot (x + \nabla a) \quad \text{at } \partial\Omega \quad (4.19)$$

for some $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ that is locally affine exterior to Ω . These boundary conditions are understood in the sense of trace, i.e., the values of φ and $\nabla \varphi$ at $\partial\Omega$ are taken from Ω , while those of a and ∇a at $\partial\Omega$ are taken from $\mathbb{R}^2 \setminus \overline{\Omega}$.

Proof. The result follows from the identity

$$\nabla \nabla \varphi = \nabla \nabla \varphi|_{\Omega} + Id dx|_{\mathbb{R}^2 \setminus \overline{\Omega}} + \hat{v} \cdot [\nabla \varphi] \hat{v} \otimes \hat{v} \mathcal{H}^1|_{\partial\Omega} \quad \text{on } \mathbb{R}^2 \quad (4.20)$$

which holds for all $\varphi \in HB_{loc}(\mathbb{R}^2)$ such that $\varphi - \frac{1}{2}|x|^2$ is locally affine on $\mathbb{R}^2 \setminus \overline{\Omega}$. Indeed, if $\varphi \in HB(\Omega)$ satisfies (4.18) and (4.19) for some a as in the statement, its extension by $\frac{1}{2}|x|^2 + a$ belongs to $HB_{loc}(\mathbb{R}^2)$ and obeys (4.20). Its Hessian is non-negative, and so it is convex. Therefore it is admissible in (4.5).

On the other hand, if φ is admissible then it is a convex extension of $\frac{1}{2}|x|^2 + a$ from $\mathbb{R}^2 \setminus \overline{\Omega}$ to \mathbb{R}^2 for some a that is locally affine on $\mathbb{R}^2 \setminus \overline{\Omega}$. It belongs to $HB_{loc}(\mathbb{R}^2)$ and satisfies $\nabla \nabla \varphi \geq 0$ on \mathbb{R}^2 . It restricts to an element of $HB(\Omega)$ with trace equal to $\frac{1}{2}|x|^2 + a$, so the first part of (4.19) holds. Testing (4.20) at $\partial\Omega$ and using that

$$[\nabla \varphi] = x + \nabla a|_{\partial(\mathbb{R}^2 \setminus \overline{\Omega})} - \nabla \varphi|_{\partial\Omega} \quad \text{at } \partial\Omega$$

yields the rest of (4.19). Testing it at Ω proves (4.18). \square

4.2. Complementary Slackness Conditions

It remains to prove the complementary slackness part of Proposition 4.1. First, we verify that the mollification procedure from (1.32) can be used to generate the approximations referred to there.

Lemma 4.6. *Let $\mu \in \mathcal{M}(\Omega; \text{Sym}_2)$ have $\text{curl curl } \mu \in \mathcal{M}(\Omega)$. Its mollifications $\{\mu_\delta\}_{\delta>0}$ from (1.32) belong to $C^\infty(\overline{\Omega}; \text{Sym}_2)$ and converge to μ in the following sense:*

$$\begin{aligned} \mu_\delta dx &\rightarrow \mu \quad \text{narrowly in } \mathcal{M}(\Omega; \text{Sym}_2), \\ \text{curl curl } \mu_\delta dx &\xrightarrow{*} \text{curl curl } \mu \quad \text{weakly-} * \text{ in } \mathcal{M}(\Omega), \end{aligned}$$

as $\delta \rightarrow 0$. If in addition $\mu \geq 0$, then $\mu_\delta(x) \geq 0$ for all $x \in \overline{\Omega}$ and $\delta > 0$.

Proof. The last conclusion regarding non-negativity follows from our assumption that the kernel $\rho \geq 0$. We establish the convergences now. Let $\sigma \in C_b(\Omega; \text{Sym}_2)$. Fixing $y \in \Omega$, we see that

$$\int_{\Omega} \rho \left(\frac{1}{\delta} (x - y) \right) \sigma(x) \frac{dx}{\delta^2} \rightarrow \sigma(y) \quad \text{as } \delta \rightarrow 0$$

and also that

$$\left| \int_{\Omega} \rho \left(\frac{1}{\delta} (x - y) \right) \sigma(x) \frac{dx}{\delta^2} \right| \leq \int \mathbb{1}_{\Omega}(y + \delta x) \rho(x) |\sigma(y + \delta x)| dx \leq \|\sigma\|_{L^{\infty}(\Omega)}$$

for all $\delta > 0$. Therefore,

$$\begin{aligned} \int_{\Omega} \langle \sigma, \mu_{\delta} \rangle dx &= \int_{\Omega} \left\langle \sigma(x), \int_{\Omega} \frac{1}{\delta^2} \rho \left(\frac{1}{\delta} (x - y) \right) d\mu(y) \right\rangle dx \\ &= \int_{\Omega} \left\langle \int_{\Omega} \rho \left(\frac{1}{\delta} (x - y) \right) \sigma(x) \frac{dx}{\delta^2}, d\mu(y) \right\rangle \rightarrow \int_{\Omega} \langle \sigma, \mu \rangle \end{aligned}$$

by the bounded convergence theorem. As σ was arbitrary, we conclude the narrow convergence of $\mu_{\delta} dx$ to μ .

Now we show the weak-* convergence of $\text{curlcurl } \mu_{\delta} dx$ to $\text{curlcurl } \mu$. Let $\chi \in C_c^{\infty}(\Omega)$. For small enough $\delta > 0$, we can apply the fact that curlcurl and $\nabla^{\perp} \nabla^{\perp}$ are formally adjoint along with Fubini's theorem to write that

$$\begin{aligned} \int_{\Omega} \chi \text{curlcurl } \mu_{\delta} dx &= \int_{\Omega} \left\langle \int_{\Omega} \frac{1}{\delta^2} \rho \left(\frac{1}{\delta} (x - y) \right) \nabla^{\perp} \nabla^{\perp} \chi(x) dx, \mu(y) \right\rangle \\ &= \int_{\Omega} \left\langle \int_{B_1} \rho(x) \nabla^{\perp} \nabla^{\perp} \chi(\cdot + \delta x) dx, \mu \right\rangle \\ &= \int_{\Omega} \left\langle \nabla^{\perp} \nabla^{\perp} \int_{B_1} \rho(x) \chi(\cdot + \delta x) dx, \mu \right\rangle. \end{aligned}$$

Using that $\text{curl curl } \mu \in \mathcal{M}(\Omega)$, there follows

$$\begin{aligned} &\int_{\Omega} \left\langle \nabla^{\perp} \nabla^{\perp} \int_{B_1} \rho(x) \chi(\cdot + \delta x) dx, \mu \right\rangle \\ &= \int_{\Omega} \left[\int_{B_1} \rho(x) \chi(y + \delta x) dx \right] d\text{curl curl } \mu(y) \\ &\rightarrow \int_{\Omega} \chi d\text{curl curl } \mu \end{aligned}$$

as $\delta \rightarrow 0$. The proof is complete. \square

Next, we establish the integration by parts identity (4.10).

Lemma 4.7. *Let u and φ be admissible in (4.5) and suppose $\{\mu_n\}_{n \in \mathbb{N}} \subset C^2(\overline{\Omega}; \text{Sym}_2)$ converges to $\mu = \nabla p \otimes \nabla p dx - 2e(u)$ in the sense of (4.6). Then,*

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} |\mu|_1 - \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p dx - L \left(\varphi - \frac{1}{2} |x|^2 \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} \langle \mu_n, \nabla^{\perp} \nabla^{\perp} \varphi \rangle + \frac{1}{2} \int_{\partial\Omega} \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu_n \rangle ds. \end{aligned}$$

Proof. Since $\varphi - \frac{1}{2}|x|^2$ is locally affine on $\mathbb{R}^2 \setminus \bar{\Omega}$, we can find $a \in C^\infty(\mathbb{R}^2)$ such that $\varphi - \frac{1}{2}|x|^2 = a$ there. Integrating by parts twice via (1.13) and recalling that the unit tangent and outwards-pointing unit normal vectors to $\partial\Omega$ were taken to satisfy $\hat{\tau} = \hat{v}^\perp$, we deduce that

$$\begin{aligned} & \int_{\Omega} \left\langle \mu_n, \nabla^\perp \nabla^\perp \left(\varphi - \frac{1}{2}|x|^2 - a \right) \right\rangle \\ &= \int_{\Omega} \left(\varphi - \frac{1}{2}|x|^2 - a \right) \operatorname{curlcurl} \mu_n - \int_{\partial\Omega} \operatorname{curl} \mu_n \cdot \hat{\tau} \left(\varphi - \frac{1}{2}|x|^2 - a \right) \\ &+ \int_{\partial\Omega} \left\langle \mu_n, \nabla^\perp \left(\varphi - \frac{1}{2}|x|^2 - a \right) \otimes \hat{\tau} \right\rangle \\ &= \int_{\Omega} \left(\varphi - \frac{1}{2}|x|^2 - a \right) \operatorname{curlcurl} \mu_n - \int_{\partial\Omega} \hat{v} \cdot [\nabla \varphi] \langle \mu_n, \hat{\tau} \otimes \hat{\tau} \rangle \end{aligned}$$

where in the last line we used that $[\nabla \varphi] = x + \nabla a - \nabla \varphi|_{\partial\Omega}$ is normal to $\partial\Omega$. Thus,

$$\begin{aligned} & \int_{\Omega} \left\langle \mu_n, \nabla^\perp \nabla^\perp \varphi \right\rangle + \int_{\partial\Omega} \hat{v} \cdot [\nabla \varphi] \langle \mu_n, \hat{\tau} \otimes \hat{\tau} \rangle \\ &= \int_{\Omega} \langle Id, \mu_n \rangle + \int_{\Omega} \left(\varphi - \frac{1}{2}|x|^2 - a \right) \operatorname{curlcurl} \mu_n + \int_{\Omega} \left\langle \mu_n, \nabla^\perp \nabla^\perp a \right\rangle. \end{aligned}$$

Taking $n \rightarrow \infty$ and using the given approximation properties we deduce that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} \left\langle \mu_n, \nabla^\perp \nabla^\perp \varphi \right\rangle + \frac{1}{2} \int_{\partial\Omega} \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu_n \rangle \\ &= \frac{1}{2} \int_{\Omega} |\mu|_1 - \int_{\Omega} \left(\varphi - \frac{1}{2}|x|^2 - a \right) \det \nabla \nabla p + \frac{1}{2} \int_{\Omega} \left\langle \nabla^\perp \nabla^\perp a, \mu \right\rangle. \end{aligned} \tag{4.21}$$

The last step is to rewrite the second line above using the definitions of μ and L .

Recall $a \in C^\infty(\mathbb{R}^2)$ and equals $\varphi - \frac{1}{2}|x|^2$ on $\mathbb{R}^2 \setminus \bar{\Omega}$. Observe that

$$\int_{\Omega} \left\langle \nabla^\perp \nabla^\perp a, e(u) \right\rangle = 0. \tag{4.22}$$

Indeed, we can find a compactly supported, *BD* extension of u to \mathbb{R}^2 , and then as $\nabla^\perp \nabla^\perp a$ is divergence-free and vanishes outside of Ω , the desired identity follows by the divergence theorem. Next, we claim that

$$L(a) = - \int_{\Omega} a \det \nabla \nabla p + \left\langle \nabla^\perp \nabla^\perp a, \frac{1}{2} \nabla p \otimes \nabla p \right\rangle. \tag{4.23}$$

To prove it, introduce a compactly supported, $W^{2,2}$ extension \bar{p} of p from Ω to \mathbb{R}^2 , and test the very weak Hessian identity (4.4) against a . The result is that

$$\int_{\mathbb{R}^2} a \det \nabla \nabla \bar{p} = - \int_{\mathbb{R}^2} \left\langle \nabla^\perp \nabla^\perp a, \frac{1}{2} \nabla \bar{p} \otimes \nabla \bar{p} \right\rangle.$$

Breaking up the integral on the left to be over Ω and $\mathbb{R}^2 \setminus \Omega$, and using the definition of L in (4.3), there follows (4.23). Combining (4.21)–(4.23) with our choice to call $\mu = \nabla p \otimes \nabla p dx - 2e(u)$ finishes the proof. \square

We are finally ready to complete the proof of Proposition 4.1.

Proof of the complementary slackness part of Proposition 4.1. Let u_{eff} and φ be admissible in (4.5), let $\mu = \nabla p \otimes \nabla p \, dx - 2e(u_{\text{eff}})$, and let $\{\mu_n\}_{n \in \mathbb{N}}$ be non-negative and converge to μ in the sense of (4.6). Recall from Lemma 4.7 that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mu|_1 - \int_{\Omega} \left(\varphi - \frac{1}{2}|x|^2 \right) \det \nabla \nabla p - L \left(\varphi - \frac{1}{2}|x|^2 \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} \langle \mu_n, \nabla^{\perp} \nabla^{\perp} \varphi \rangle + \frac{1}{2} \int_{\partial \Omega} \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu_n \rangle. \end{aligned} \quad (4.24)$$

Since φ is convex we find, just as in the proof of Lemma 4.5, that

$$\nabla \nabla \varphi \geq 0 \quad \text{on } \Omega \quad \text{and} \quad \hat{v} \cdot [\nabla \varphi] \geq 0 \quad \text{at } \partial \Omega.$$

Thus, the integrals in the second line of (4.24) are non-negative and limit to zero if and only if the difference in the first line vanishes. At the same time, due to (4.5), this difference vanishes if and only if u_{eff} and φ are optimal. It remains to produce an example of an approximating sequence $\{\mu_n\}$. Using Lemma 4.6 and the fact that

$$-\frac{1}{2} \operatorname{curlcurl} \mu = \det \nabla \nabla p,$$

we see the mollification procedure from (1.32) approximates μ as desired. \square

It is natural to wonder if there is some more intrinsic way of stating the complementary slackness conditions, i.e., one that does not make use of *ad hoc* regularizations. The crux of the issue is that one must make sense of the ‘‘Frobenius inner product’’ of two Sym_2 -valued Radon measures, one of which is divergence-free and the other of which has its $\operatorname{curlcurl}$ controlled. Consider, for instance, how to define $\langle \nabla^{\perp} \nabla^{\perp} \varphi, e(u) \rangle$ when $\varphi \in HB$ and $u \in BD$. If for some reason we knew that $\varphi \in C^1$, we could fall back on the identity

$$\langle \nabla^{\perp} \nabla^{\perp} \varphi, e(u) \rangle = 2 \operatorname{curl} (e(u) \nabla^{\perp} \varphi) - \operatorname{curlcurl} (e(u) \varphi)$$

to define the product on the lefthand side as the distribution on the right. This sort of approach goes back at least to [45]. Unfortunately, it is not the case that every optimal Airy potential is C^1 . Lacking a successful distributional approach, we have opted to use regularizations instead. For a related discussion see [4] (however, the functionals there do not appear to allow for one-sided constraints).

5. Wrinkle Patterns by the Method of Stable Lines

We continue our study of the limiting minimization problems in (4.1). Section 4 identified various versions of the dual problem, along with complementary slackness conditions satisfied by optimal primal-dual pairs. There we established a general duality result, applicable even in situations where we do not yet know

the Γ -limit of the rescaled energies $\frac{1}{2\sqrt{bk}+\gamma} E_{b,k,\gamma}$ (such as when Ω is not strictly star-shaped). Our results thus far can be summarized as follows: under the full set of assumptions at the start of Section 1.2 of the introduction, $\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2)$ arises as the defect measure of an almost minimizing sequence for $E_{b,k,\gamma}$ if and only if it satisfies

$$\begin{cases} -\frac{1}{2} \operatorname{curlcurl} \mu = \det \nabla \nabla p & \text{on } \Omega \\ \langle \nabla^\perp \nabla^\perp \varphi, \mu \rangle = 0 & \text{on } \Omega \\ \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu \rangle = 0 & \text{at } \partial \Omega \end{cases} \quad (5.1)$$

where φ solves the dual problem (4.2). The first equation holds in the sense of distributions, while the second and third ones hold in the regularized sense, i.e.,

$$\lim_{\delta \rightarrow 0} \int_{\Omega} \left| \langle \mu_\delta, \nabla^\perp \nabla^\perp \varphi \rangle \right| = 0 \quad \text{and} \quad \lim_{\delta \rightarrow 0} \int_{\partial \Omega} |\hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu_\delta \rangle| ds = 0 \quad (5.2)$$

where $\{\mu_\delta\}_{\delta>0}$ are the mollifications from (1.32). Moreover, the same system (5.1) applies even when only the basic assumptions from (1.14a) hold, so long as we take $\mu = \nabla p \otimes \nabla p dx - 2e(u_{\text{eff}})$ and let u_{eff} and φ be optimal in (4.5). Recall $[\nabla \varphi]$ denotes the jump in $\nabla \varphi$ across $\partial \Omega$ in the direction of \hat{v} .

The purpose of this section is to study (5.1) as a boundary value problem for μ and, in particular, to establish the results from Section 1.2.4 regarding the general formulation of our method of stable lines. Let us briefly outline what we achieve. We begin in Section 5.1 by defining a partition of the shell according to the structure of φ . Included in this partition are the ‘‘stable lines’’ and the ‘‘ordered’’ set O they fill out. We show that

$$\mu = \lambda \hat{\eta} \otimes \hat{\eta} \quad \text{on } O, \quad \text{where } \lambda \geq 0 \text{ and } \hat{\eta} \in R((\nabla \nabla \varphi)_{\text{a.c.}}).$$

The unit vector field $\hat{\eta}$ arises as a suitable choice of normal to the stable lines. In Section 5.2, we justify our assertion that the stable lines are characteristic curves for the PDE

$$-\frac{1}{2} \operatorname{curlcurl}(\hat{\eta} \otimes \hat{\eta} \lambda) = \det \nabla \nabla p \quad \text{on } O$$

implied by (5.1). We do so by producing ODEs for the absolutely continuous and singular parts of λ , which hold on (a.e.) stable line. Finally, in Section 5.3 we show how to derive appropriate boundary data depending on the layout of the stable lines, and how to apply them to conclude (partial) uniqueness, regularity, and explicit solution formulas for λ and μ . The reader wishing to see concrete examples should go forward to Section 6, keeping in mind that we make repeated use of Corollaries 5.1–5.3 there.

A word on assumptions is required: throughout this section, we require that Ω is bounded and Lipschitz and that $p \in W^{2,2}(\Omega)$. We take $\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2)$, let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be convex, and assume they satisfy (5.1) (it will not be necessary for our present purposes to assume they are optimal). Importantly, we must also assume that

$$\text{there exists a non-empty open subset of } \Omega \text{ on which } \varphi \in W^{2,2}. \quad (5.3)$$

As was addressed briefly in Section 1.2.5, this last assumption will allow us to apply the theory of $W^{2,2}$ developable surfaces from [35, 36, 59]. Various further assumptions on φ will be introduced as needed below.

5.1. Stable Lines

Our first task is to explain how the structure of φ constrains that of μ solving (5.1). Guided by the second equation there, we define a partition of Ω by writing

$$\Omega = \Sigma \cup F \cup O \cup U \quad (5.4)$$

where the sets Σ , F , O , and U are as follows:

- the *singular* set Σ is the smallest closed subset of Ω such that $\varphi \in W_{\text{loc}}^{2,2}(\Omega \setminus \Sigma)$;
- the *flattened* set F is the largest open subset of $\Omega \setminus \Sigma$ on which both of the eigenvalues of $(\nabla \nabla \varphi)_{\text{a.e.}}$ are locally uniformly positive a.e.;
- the *ordered* set O is the largest open subset of $\Omega \setminus \Sigma$ on which one of the eigenvalues of $(\nabla \nabla \varphi)_{\text{a.e.}}$ is zero a.e. and the other eigenvalue is locally uniformly positive a.e.;
- the *unconstrained* set U is the complement of $\Sigma \cup F \cup O$ with respect to Ω .

To be clear, we say that a function ζ is *locally uniformly positive a.e.* on a (measurable) set A if for all $x \in A$ there exists $c > 0$ and a relatively open neighborhood $V \subset A$ of x on which $\zeta \geq c$ a.e. Note $\Sigma \neq \Omega$ due to (5.3).

Next, we explain what we mean by “stable lines”. Recall from Section 1.2.4 that a curve parallel to $N(\nabla \nabla \varphi)$ in O was (preliminarily) called a stable line. This definition is no longer suitable in the present, measure-theoretic context. It generalizes as follows: henceforth, we refer to a curve belonging to O as a *stable line* of φ if it is a maximally contained open line segment on which φ is affine.

Lemma 5.1. *Every $x \in O$ belongs to a unique stable line ℓ_x , and $\partial \ell_x \subset \partial O$. The map $x \mapsto \ell_x$ is locally Lipschitz from O to the projective space \mathbb{P}^1 . In particular, there exists $\hat{\eta} \in \text{Lip}_{\text{loc}}(O; S^1)$ that is constant along and perpendicular to the stable lines, i.e.,*

$$\ell_x = \ell_y \implies \hat{\eta}(x) = \hat{\eta}(y) \quad \forall x, y \in O \quad \text{and} \quad \hat{\eta}(x) \perp \ell_x \quad \forall x \in O. \quad (5.5)$$

Remark 5.1. In general, $\hat{\eta}$ can fail to be Lipschitz on O . This can happen for various reasons, such as when distinct stable lines share a common boundary point. Examples of this appear throughout Panel (b) of Fig. 2.

Proof. We require some facts about developable surfaces. In the smooth setting, a *developable surface* is one whose Gaussian curvature vanishes identically. Any such surface can be decomposed into two disjoint parts: a *ruled* part consisting of disjoint open line segments that pass between boundary points—the surface’s *ruling lines*—and a locally planar part. In [59], this decomposition is shown to hold for $W^{2,2}$ developable surfaces, i.e., ones whose Gaussian curvature vanishes a.e. By a covering argument, it holds for $W_{\text{loc}}^{2,2}$ developable surfaces as well.

These facts allow to characterize the graph of φ over O . By definition, this is the image of the mapping $O \rightarrow \mathbb{R}^3$, $(x_1, x_2) \mapsto (x_1, x_2, \varphi(x))$. Looking back to the definition of O immediately after (5.4), we see that

$$\varphi \in W_{\text{loc}}^{2,2}(O) \quad \text{and} \quad \text{rank } \nabla \nabla \varphi = 1 \quad \text{a.e. on } O.$$

Therefore, φ describes a $W_{\text{loc}}^{2,2}$ developable surface over O , consisting entirely of ruling lines. Stable lines are easily produced: the stable line ℓ_x through x arises from the projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ of the unique ruling line through $(x_1, x_2, \varphi(x))$ to the plane. Indeed, φ is affine along ℓ_x , and it is maximally extended in O .

It remains to choose the normal $\hat{\eta}$. Of course, we can take it to satisfy (5.5). That it can be chosen to be locally Lipschitz follows from the known fact [36, 44] that the map $x \mapsto \ell_x$ is locally Lipschitz from O to the projective space $\mathbb{P}^1 \simeq S^1 / \{\hat{n} \sim -\hat{n}\}$. Let us explain. Without loss of generality, we can take O to be connected after passing to its components. In order that $\ell_x \perp \ell_y$ it must be that $|x - y| > d_{\partial O}(x) \vee d_{\partial O}(y)$ as stable lines never intersect. So, once we decide that

$$\hat{\eta}(x) \cdot \hat{\eta}(y) > 0 \quad \text{when} \quad |x - y| \leq d_{\partial O}(x) \vee d_{\partial O}(y)$$

there remain exactly two choices for $\hat{\eta} : O \rightarrow S^1$ (in general, the number of choices depends on the number of connected components). Fixing $\hat{\eta}(x)$ at some $x \in O$ determines it throughout. It now follows that

$$|\hat{\eta}(x) - \hat{\eta}(y)| \lesssim \frac{|x - y|}{d_{\partial O}(x) \vee d_{\partial O}(y)} \quad \forall x, y \in O$$

from a worst case analysis of how stable lines may meet at ∂O . \square

Having defined the stable lines of φ , we now use them to characterize the structure of μ .

Lemma 5.2. *Let Ω be partitioned as in (5.4). Any solution $\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2)$ of (5.1) must satisfy*

$$\mu = 0 \quad \text{on } F \quad \text{and} \quad \mu = \lambda \hat{\eta} \otimes \hat{\eta} \quad \text{on } O \tag{5.6}$$

for some $\lambda \in \mathcal{M}_+(O)$ and $\hat{\eta} \in \text{Lip}_{\text{loc}}(O; S^1)$ satisfying (5.5).

Proof. We combine the previous result with the complementary slackness part of (5.1). In particular, we make use of the first part of (5.2), which implies here that

$$0 = \lim_{\delta \rightarrow 0} \int_F \left| \langle \mu_\delta, (\nabla^\perp \nabla^\perp \varphi)_{\text{a.c.}} \rangle \right| dx = \lim_{\delta \rightarrow 0} \int_O \left| \langle \mu_\delta, (\nabla^\perp \nabla^\perp \varphi)_{\text{a.c.}} \rangle \right| dx. \tag{5.7}$$

Note it follows from their definition that $\mu_\delta dx \xrightarrow{*} \mu$ weakly-* on F and O . See Lemma 4.6.

Consider the flattened set F . We claim that every $x \in F$ is contained in a neighborhood where $\mu = 0$. Indeed, by its definition we can always find a relatively open set $V \subset F$ such that $x \in V$ and

$$(\nabla \nabla \varphi)_{\text{a.c.}} \gtrsim Id \quad \text{a.e. on } V.$$

Upon passing to the limit in the first part of (5.7), we deduce that

$$0 = \lim_{\delta \rightarrow 0} \int_V |(Id, \mu_\delta)| dx = |\mu|(V).$$

Compact subsets of F are covered by finitely many such V . Hence, $\mu = 0$ on F .

The ordered set O requires a bit more care. Recall φ is affine along its stable lines, which run perpendicularly to $\hat{\eta} \in \text{Lip}_{\text{loc}}(O; S^1)$ from Lemma 5.1. So, there exists $\zeta \in L^1(O; (0, \infty)) \cap L^2_{\text{loc}}(O)$ such that

$$(\nabla \nabla \varphi)_{\text{a.e.}} = \zeta \hat{\eta} \otimes \hat{\eta} \quad \text{a.e. on } O.$$

Actually, the definition of O gives a bit more: for each $x \in O$ there exists a relatively open neighborhood $V \subset O$ of x on which $\zeta \gtrsim 1$ a.e. Using this, we can pass to the limit in the second part of (5.7) to find that

$$0 = \lim_{\delta \rightarrow 0} \int_V \left| \langle \hat{\eta}^\perp \otimes \hat{\eta}^\perp, \mu_\delta \rangle \right| dx = \left| \langle \hat{\eta}^\perp \otimes \hat{\eta}^\perp, \mu \rangle \right|(V).$$

The choice of $x \in O$ was arbitrary, so $\langle \hat{\eta}^\perp \otimes \hat{\eta}^\perp, \mu \rangle = 0$ on O . \square

The possibilities for μ at ∂O and ∂F are less clear, as the relevant eigenvalue(s) of $\nabla \nabla \varphi$ may degenerate there. We leave the detailed study of this to future work, and turn to describe the role that stable lines play in our solution of (5.1).

5.2. The Method of Characteristics

The previous section described the relation between the structure of φ and that of μ solving (5.1). Following Lemma 5.2, we continue to let $\lambda \in \mathcal{M}_+(O)$ and $\hat{\eta} \in \text{Lip}_{\text{loc}}(O; S^1)$ satisfy

$$\mu = \lambda \hat{\eta} \otimes \hat{\eta} \quad \text{on } O,$$

where $\hat{\eta}$ is constant along and perpendicular to the stable lines $\{\ell_x\}$ of φ . By the first equation in (5.1),

$$-\frac{1}{2} \text{curlcurl}(\hat{\eta} \otimes \hat{\eta} \lambda) = \det \nabla \nabla p \quad \text{on } O \tag{5.8}$$

in the sense of distributions. We now claim that (5.8) can be solved using the method of characteristics with stable lines as characteristic curves. It is not difficult to understand why this ought to be the case. Denote the first and second directional derivatives along the stable lines by

$$\partial_{\hat{\eta}^\perp} = \hat{\eta}^\perp \cdot \nabla \quad \text{and} \quad \partial_{\hat{\eta}^\perp}^2 = \langle \hat{\eta}^\perp \otimes \hat{\eta}^\perp, \nabla \nabla \rangle. \tag{5.9}$$

Pretending for the moment that λ and $\hat{\eta}$ are smooth (rather than belonging to \mathcal{M}_+ and Lip_{loc}), we apply the product rule along with the identity $\partial_{\hat{\eta}^\perp} \hat{\eta} = 0$ to write

$$\text{curlcurl}(\hat{\eta} \otimes \hat{\eta} \lambda) = \partial_{\hat{\eta}^\perp}^2 \lambda + \frac{2}{\varrho} \partial_{\hat{\eta}^\perp} \varrho \partial_{\hat{\eta}^\perp} \lambda + \frac{1}{\varrho} \partial_{\hat{\eta}^\perp}^2 \varrho \lambda = \frac{1}{\varrho} \partial_{\hat{\eta}^\perp}^2 (\varrho \lambda) \tag{5.10}$$

where $\partial_{\hat{\eta}^\perp} \varrho = \operatorname{div} \hat{\eta}^\perp \varrho$. Thus, the PDE (5.8) can be rewritten (informally, at first) as the family of ODEs

$$-\frac{1}{2\varrho} \partial_{\hat{\eta}^\perp}^2 (\varrho \lambda) = \det \nabla \nabla p \quad \text{along the stable lines.}$$

Lemma 5.3 provides a rigorous version of this observation in the original, measure-theoretic setting of (5.1). In brief: whereas the ODE derived above does turn out to govern the absolutely continuous part of λ along \mathcal{H}^1 -a.e. stable line, its singular part instead turns out to be affine along a complementary set of stable lines.

The next few paragraphs fix the notation used in the remainder. First, due to the topological difficulties inherent in parameterizing the stable lines $\{\ell_x\}$ —see [36] for a detailed account of the related problem of parameterizing ruling lines—we find it convenient to reduce to certain well-prepared regions of the form

$$\begin{aligned} V &= \cup_{s \in \Gamma} \ell_s \quad \text{where } \Gamma \subset V \text{ is a smooth curve such that} \\ \ell_s = \ell_{s'} &\implies s = s' \quad \forall s, s' \in \Gamma \quad \text{and} \quad T_s \Gamma \nparallel \ell_s \quad \forall s \in \Gamma. \end{aligned} \quad (5.11)$$

Recall by a *smooth curve* we mean a diffeomorphic copy of an open interval $I \subset \mathbb{R}$, i.e., its image under a smooth and one-to-one map. (Later on, we allow Γ to denote other, more general index sets.) The conditions on the second line require that the curve Γ meets each stable line it indexes transversely and exactly once. It follows from Lemma 5.1 that every $x \in O$ admits a neighborhood of this form. Indeed, we may simply choose Γ to pass through x and to remain approximately parallel to $\hat{\eta}$ along its extent.

Next, we introduce the technique of disintegration of measure. The basic facts are as follows (see, e.g., [33] for more details). Given one of the regions $V = \cup_{s \in \Gamma} \ell_s$ from (5.11), we say that

$$\pi : V \rightarrow \Gamma \text{ sends } x \in V \text{ to the unique } s \in \ell_x \cap \Gamma. \quad (5.12)$$

For each $s \in \Gamma$, the fiber $\pi^{-1}(\{s\}) = \ell_s$. Given $\lambda \in \mathcal{M}(V)$ and $\vartheta \in \mathcal{M}(\Gamma)$ with $\pi_\# \lambda \ll \vartheta$, there exists a ϑ -a.e. uniquely determined (Borel) family $\{\lambda_s\}_{s \in \Gamma} \subset \mathcal{M}(V)$ such that

$$\operatorname{supp} \lambda_s \subset \ell_s \quad \forall s \in \Gamma \quad \text{and} \quad \int_V \psi \, d\lambda = \int_\Gamma \left[\int_{\ell_s} \psi \, d\lambda_s \right] d\vartheta(s) \quad \forall \psi \in L^1(V, \lambda).$$

Here, $\pi_\#$ is the pushforward map through π . Thus, λ *disintegrates* into its parts $\{\lambda_s\}_{s \in \Gamma}$ with respect to π and ϑ , a situation we indicate by writing

$$\lambda = \int_\Gamma \lambda_s \, d\vartheta(s).$$

A useful example to keep in mind is the formula for the two-dimensional Lebesgue measure

$$\mathcal{L}^2 = \int_\Gamma \varrho \mathcal{H}^1 \lfloor_{\ell_s} d\mathcal{H}^1(s) \quad \text{on } V. \quad (5.13)$$

This defines the change of measure factor $\varrho : V \rightarrow (0, \infty)$ anticipated in (5.10).

Finally, we define the Sobolev spaces $W^{k,r}(\ell_s)$ for $k \in \mathbb{N}$ and $r \in [1, \infty]$. For each ℓ_s , we say that $f \in L^r(\ell_s, \mathcal{H}^1)$ belongs to $W^{1,r}(\ell_s)$ provided there exists $g \in L^r(\ell_s, \mathcal{H}^1)$ such that

$$\int_{\ell_s} f \partial_{\hat{\eta}^\perp} \chi \, d\mathcal{H}^1 = - \int_{\ell_s} g \chi \, d\mathcal{H}^1 \quad \forall \chi \in C_c^\infty(\ell_s).$$

In such a case, we write that

$$\partial_{\hat{\eta}^\perp(s)} f = g \quad \text{on } \ell_s,$$

and call g the *weak directional derivative* of f in the direction of $\hat{\eta}^\perp(s)$. Thus, $\partial_{\hat{\eta}^\perp(s)} : W^{1,r}(\ell_s) \rightarrow L^r(\ell_s, \mathcal{H}^1)$. Similarly, $W^{k,r}(\ell_s)$ consists of all $f \in L^r(\ell_s, \mathcal{H}^1)$ whose weak directional derivatives $\partial_{\hat{\eta}^\perp(s)}^j f$ of orders $j = 1, \dots, k$ belong to $L^r(\ell_s, \mathcal{H}^1)$. Of course, if f is smooth nearby ℓ_s , these derivatives can be computed using (5.9) along with other, analogous formulas at higher order. Given $f \in W^{k,r}(\ell_s)$ we define its *trace* $f|_{\partial\ell_s}$ as usual, by continuous extension of the restriction map. As each ℓ_s is one-dimensional, $\cdot|_{\partial\ell_s} : W^{k,r}(\ell_s) \rightarrow L^\infty(\partial\ell_s, \mathcal{H}^0)$.

We are ready to make precise our claim that the stable lines of φ are characteristic curves for the PDE (5.8).

Lemma 5.3. *Let $\lambda \in \mathcal{M}_+(O)$ solve (5.8), and let $V = \cup_{s \in \Gamma} \ell_s$ and $\pi : V \rightarrow \Gamma$ be as in (5.11) and (5.12). Then there exist $\lambda_{\text{a.c.}}, \lambda_{\text{sing}} : V \rightarrow [0, \infty)$ such that*

$$\lambda = \lambda_{\text{a.c.}} \, dx + \int_{\Gamma} \lambda_{\text{sing}} \mathcal{H}^1 \lfloor \ell_s \, d\vartheta(s) \quad \text{on } V,$$

where ϑ is the singular part of $\pi_\# \lambda$ with respect to \mathcal{H}^1 . The function $\varrho \lambda_{\text{a.c.}}$ belongs to $W^{2,1}(\ell_s)$ and satisfies

$$-\frac{1}{2\varrho} \partial_{\hat{\eta}^\perp(s)}^2 (\varrho \lambda_{\text{a.c.}}) = \det \nabla \nabla p \quad \text{on } \ell_s \tag{5.14}$$

upon restriction to \mathcal{H}^1 -a.e. ℓ_s . Likewise, λ_{sing} belongs to $W^{2,\infty}(\ell_s)$ and satisfies

$$\partial_{\hat{\eta}^\perp(s)}^2 \lambda_{\text{sing}} = 0 \quad \text{on } \ell_s \tag{5.15}$$

upon restriction to ϑ -a.e. ℓ_s .

Remark 5.2. If $p \in W^{2,2r}$ so that $\det \nabla \nabla p \in L^r$, $\varrho \lambda_{\text{a.c.}} \in W^{2,r}$ on \mathcal{H}^1 -a.e. ℓ_s .

Remark 5.3. It is straightforward to check that the following unique continuation-type result holds: if $V_1 = \cup_{s \in \Gamma_1} \ell_s$ and $V_2 = \cup_{s \in \Gamma_2} \ell_s$ satisfy $\Gamma_1 \cap V_2 = \Gamma_2 \cap V_1$, the functions provided above must have $\lambda_{\text{a.c.}}^1 = \lambda_{\text{a.c.}}^2$ Lebesgue a.e. on $V_1 \cap V_2$, and $\lambda_{\text{sing}}^1 = \lambda_{\text{sing}}^2$ upon restriction to ϑ -a.e. ℓ_s in $V_1 \cap V_2$. We use this later on in the proof of Corollary 5.3.

Proof. We start by finding $\{\lambda_s^{\text{a.c.}}\}_{s \in \Gamma}, \{\lambda_s^{\text{sing}}\}_{s \in \Gamma} \subset \mathcal{M}_+(V)$ such that

$$\text{supp } \lambda_s^{\text{a.c.}}, \text{supp } \lambda_s^{\text{sing}} \subset \ell_s \quad \forall s \in \Gamma \quad \text{and} \quad \lambda = \int_{\Gamma} \lambda_s^{\text{a.c.}} d\mathcal{H}^1(s) + \int_{\Gamma} \lambda_s^{\text{sing}} d\vartheta(s) \quad \text{on } V. \quad (5.16)$$

By disintegration of measure, there exist $\{\lambda_s\}_{s \in \Gamma} \subset \mathcal{M}_+(V)$ such that

$$\text{supp } \lambda_s \subset \ell_s \quad \forall s \in \Gamma \quad \text{and} \quad \lambda = \int_{\Gamma} \lambda_s d\pi_{\#}\lambda \quad \text{on } V.$$

Note the Lebesgue decomposition

$$\pi_{\#}\lambda = \frac{d\pi_{\#}\lambda}{d\mathcal{H}^1} \mathcal{H}^1 + \vartheta \quad \text{with} \quad \mathcal{H}^1 \perp \vartheta.$$

Taking

$$\lambda_s^{\text{a.c.}} = \frac{d\pi_{\#}\lambda}{d\mathcal{H}^1}(s) \lambda_s \quad \text{for } \mathcal{H}^1\text{-a.e. } s \quad \text{and} \quad \lambda_s^{\text{sing}} = \lambda_s \quad \text{for } \vartheta\text{-a.e. } s$$

we arrive at (5.16).

Having disintegrated λ into its parts $\{\lambda_s^{\text{a.c.}}\}_{s \in \Gamma}, \{\lambda_s^{\text{sing}}\}_{s \in \Gamma}$ we proceed to establish the desired ODEs. We will make use of the PDE (5.8) or, more precisely, its distributional version

$$\int_V -\frac{1}{2} \partial_{\hat{\eta}^{\perp}}^2 \psi d\lambda = \int_V \psi \det \nabla \nabla p dx \quad \forall \psi \in C_c^{\infty}(V). \quad (5.17)$$

The argument splits into two steps. The first step is to prove that (5.17) holds not only for the test functions above, but also for ones of the form

$$\psi = \chi \Psi \circ \pi \quad \text{where} \quad \chi \in C_c^{\infty}(V) \text{ and } \Psi \in C_c^{\infty}(\Gamma), \quad (5.18)$$

and where in place of $\partial_{\hat{\eta}^{\perp}}^2 \psi$ we write $\partial_{\hat{\eta}^{\perp}}^2 \chi \Psi \circ \pi$. To see this, fix $\chi \in C_c^{\infty}(V)$ and let $W \subset V$ be an open and compactly contained neighborhood of its support. We claim there exists a sequence $\{\pi_k\}_{k \in \mathbb{N}} \subset C^{\infty}(W; \Gamma)$ of smooth approximations to π such that

$$\pi_k \rightarrow \pi, \quad \partial_{\hat{\eta}^{\perp}} \pi_k \rightarrow 0, \quad \text{and} \quad \partial_{\hat{\eta}^{\perp}}^2 \pi_k \rightarrow 0 \quad \text{uniformly on } W \quad (5.19)$$

as $k \rightarrow \infty$. Postponing their construction to Lemmas 5.4 and 5.5 below, we define $\{\psi_k\}_{k \in \mathbb{N}} \subset C_c^{\infty}(V)$ by

$$\psi_k = \chi \Psi \circ \pi_k \quad (5.20)$$

and note using the product rule that

$$\partial_{\hat{\eta}^{\perp}}^2 \psi_k = \partial_{\hat{\eta}^{\perp}}^2 \chi \Psi \circ \pi_k + 2\partial_{\hat{\eta}^{\perp}} \chi \partial_{\hat{\eta}^{\perp}} (\Psi \circ \pi_k) + \chi \partial_{\hat{\eta}^{\perp}}^2 (\Psi \circ \pi_k).$$

Due to (5.19), only the first term on the right survives in the limit. That is,

$$\partial_{\hat{\eta}^{\perp}}^2 \psi_k \rightarrow \partial_{\hat{\eta}^{\perp}}^2 \chi \Psi \circ \pi \quad \text{uniformly on } W.$$

Setting (5.20) into (5.17) and passing to the limit finishes the first step. See Lemmas 5.4 and 5.5 for the construction of the approximations π_k .

We just showed that (5.17) holds for all test functions of the form (5.18). Equivalently, by the disintegration formulas (5.13) and (5.16), we have that

$$0 = \int_{\Gamma} \left[\int_{\ell_s} \frac{1}{2} \partial_{\hat{\eta}^{\perp}}^2 \chi \, d\lambda_s^{\text{a.c.}} + \int_{\ell_s} \chi \varrho \det \nabla \nabla p \, d\mathcal{H}^1 \right] \Psi(s) \, d\mathcal{H}^1(s) \\ + \int_{\Gamma} \left[\int_{\ell_s} \frac{1}{2} \partial_{\hat{\eta}^{\perp}}^2 \chi \, d\lambda_s^{\text{sing}} \right] \Psi(s) \, d\vartheta(s) \quad (5.21)$$

for all $\chi \in C_c^{\infty}(V)$ and $\Psi \in C_c^{\infty}(\Gamma)$. The next step is to show that the bracketed terms vanish, i.e.,

$$\int_{\ell_s} -\frac{1}{2} \partial_{\hat{\eta}^{\perp}(s)}^2 \chi \, d\lambda_s^{\text{a.c.}} = \int_{\ell_s} \chi \varrho \det \nabla \nabla p \, d\mathcal{H}^1, \\ \int_{\ell_s} \partial_{\hat{\eta}^{\perp}(s)}^2 \chi \, d\lambda_s^{\text{sing}} = 0 \quad (5.22)$$

for all $\chi \in C_c^{\infty}(\ell_s)$ up to \mathcal{H}^1 - and ϑ -negligible sets. By an extension argument it suffices to take $\chi \in C_c^{\infty}(V)$. Let $\{\chi_k\}_{k \in \mathbb{N}} \subset C_c^{\infty}(V)$ be C^2 -dense. Setting χ_k into (5.21) and recalling that $\mathcal{H}^1 \perp \vartheta$, we see that

$$\int_{\ell_s} -\frac{1}{2} \partial_{\hat{\eta}^{\perp}(s)}^2 \chi \, d\lambda_s^{\text{a.c.}} = \int_{\ell_s} \chi \varrho \det \nabla \nabla p \, d\mathcal{H}^1 \quad \text{for } \mathcal{H}^1\text{-a.e. } s, \\ \int_{\ell_s} \partial_{\hat{\eta}^{\perp}(s)}^2 \chi \, d\lambda_s^{\text{sing}} = 0 \quad \text{for } \vartheta\text{-a.e. } s$$

where at this stage the exceptional sets are allowed to depend on k . Intersecting over k removes this dependence and yields (5.22). In other words, we have the ODEs

$$-\frac{1}{2} \partial_{\hat{\eta}^{\perp}(s)}^2 \lambda_s^{\text{a.c.}} = \varrho \det \nabla \nabla p \quad \text{on } \mathcal{H}^1\text{-a.e. } \ell_s, \\ \partial_{\hat{\eta}^{\perp}(s)}^2 \lambda_s^{\text{sing}} = 0 \quad \text{on } \vartheta\text{-a.e. } \ell_s. \quad (5.23)$$

These hold in the sense of distributions on the specified stable lines.

The rest of the proof is straightforward. From (5.23) we see that $\lambda_s^{\text{a.c.}}, \lambda_s^{\text{sing}} \ll \mathcal{H}^1|_{\ell_s}$ and that their densities satisfy the same ODEs. By hypothesis, $\det \nabla \nabla p \in L^1(V)$. It follows from (5.13) and Fubini's theorem that $\varrho \det \nabla \nabla p \in L^1(\ell_s, \mathcal{H}^1)$, so that $\frac{d\lambda_s^{\text{a.c.}}}{d\mathcal{H}^1|_{\ell_s}} \in W^{2,1}(\ell_s)$ on \mathcal{H}^1 -a.e. ℓ_s . Evidently, $\frac{d\lambda_s^{\text{sing}}}{d\mathcal{H}^1|_{\ell_s}} \in W^{2,\infty}(\ell_s)$ on ϑ -a.e. ℓ_s as it is affine upon restriction to those stable lines. Setting

$$\lambda_{\text{a.c.}} = \frac{1}{\varrho} \frac{d\lambda_s^{\text{a.c.}}}{d\mathcal{H}^1|_{\ell_s}} \quad \text{on } \mathcal{H}^1\text{-a.e. } \ell_s \quad \text{and} \quad \lambda_{\text{sing}} = \frac{d\lambda_s^{\text{sing}}}{d\mathcal{H}^1|_{\ell_s}} \quad \text{on } \vartheta\text{-a.e. } \ell_s$$

and using (5.13) and (5.16) once more, we conclude that

$$\lambda = \int_{\Gamma} \varrho \lambda_{\text{a.c.}} \mathcal{H}^1|_{\ell_s} \, d\mathcal{H}^1(s) + \int_{\Gamma} \lambda_{\text{sing}} \mathcal{H}^1|_{\ell_s} \, d\vartheta(s) \\ = \lambda_{\text{a.c.}} \, dx + \int_{\Gamma} \lambda_{\text{sing}} \mathcal{H}^1|_{\ell_s} \, d\vartheta(s).$$

The desired ODEs (5.14) and (5.15) follow from (5.23). \square

Left over from the proof above is a result ensuring that the map $\pi : V \rightarrow \Gamma$ from (5.12), whose fibers are the stable lines $\{\ell_s\}_{s \in \Gamma}$, can be approximated by smooth maps $\{\pi_k\}_{k \in \mathbb{N}}$ with fibers converging to the stable lines (the precise sense of approximation is in (5.19)). Similar results appear in the proof that smooth developable surfaces are $W^{2,2}$ -dense [35, 36, 59]. There, the authors replace the surface's ruling lines with smoothly varying ones; here, per Lemma 5.1, the stable lines of φ are the planar projection of the ruling lines of its graph (over O). The main difference is the choice of topology—we need that π_k and certain of its derivatives converge uniformly, rather than only a.e. Nevertheless, the argument from the references can be adapted to produce the desired result. We follow [35, 36].

Let $V = \cup_{s \in \Gamma} \ell_s$ be as in (5.11). The first step is to define coordinates adapted to the stable lines. Recall we took Γ to be diffeomorphic to an open interval $I \subset \mathbb{R}$. Let $\gamma : I \rightarrow \Gamma$ be smooth map such that

$$V = \cup_{s \in I} \ell_{\gamma(s)}, \quad s \mapsto \ell_{\gamma(s)} \text{ is one-to-one}, \quad \gamma' \cdot \hat{\eta} \circ \gamma > 0, \quad |\gamma'| = 1. \quad (5.24)$$

Note the slight redundancy in the usage of s . In a minor modification of [35, 36], we define $\Phi_{\hat{\eta}} : I \times \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$\Phi_{\hat{\eta}}(s, t) = \gamma(s) + t\hat{\eta}^\perp(s) \quad (5.25)$$

for $\hat{\eta} : I \rightarrow S^1$. Unlike the references, we do not require that $\gamma' \parallel \hat{\eta}$, although we will eventually prevent them from being perpendicular. If $\hat{\eta}$ is differentiable then so is $\Phi_{\hat{\eta}}$, in which case

$$\det \nabla \Phi_{\hat{\eta}} = \gamma' \cdot \hat{\eta} - t\kappa_{\hat{\eta}} \quad \text{where } \kappa_{\hat{\eta}} = \hat{\eta}' \cdot \hat{\eta}^\perp. \quad (5.26)$$

Taking $\hat{\eta} = \hat{\eta} \circ \gamma$ leads to the desired coordinates. We refer to $\Phi_{\hat{\eta}}$ and $\kappa_{\hat{\eta}}$ in place of $\Phi_{\hat{\eta} \circ \gamma}$ and $\kappa_{\hat{\eta} \circ \gamma}$. Since $\hat{\eta}$ is locally Lipschitz, $\Phi_{\hat{\eta}}$ is as well. We claim that it admits a locally Lipschitz inverse on V . That it is invertible there is a clear consequence of the disjointness of the stable lines. Note that

$$\det \nabla \Phi_{\hat{\eta}} = \varrho \circ \Phi_{\hat{\eta}} > 0 \quad \text{a.e. on } \Phi_{\hat{\eta}}^{-1}(V), \quad (5.27)$$

where $\varrho : V \rightarrow (0, \infty)$ is as in (5.13). That $|\det \nabla \Phi_{\hat{\eta}}| = \varrho \circ \Phi_{\hat{\eta}}$ follows from the area formula for Lipschitz maps (see, e.g., [55]). Its positivity is due to the given orientation in (5.24). Continuing, we define $t_V^\pm : I \rightarrow \mathbb{R}$ such that

$$t_V^- < 0 < t_V^+ \quad \text{and} \quad \ell_{\gamma(s)} = \Phi_{\hat{\eta}}(\{s\} \times (t_V^-(s), t_V^+(s))) \quad \forall s \in I.$$

Combining (5.26) and (5.27) shows that

$$\frac{1}{t_V^-(s)} \leq \frac{\kappa_{\hat{\eta}}(s)}{\gamma' \cdot \hat{\eta} \circ \gamma(s)} \leq \frac{1}{t_V^+(s)} \quad \text{for a.e. } s \in I. \quad (5.28)$$

Hence, $\Phi_{\hat{\eta}}^{-1} \in \text{Lip}_{\text{loc}}(V; I \times \mathbb{R})$ by the Lipschitz inverse function theorem [18]. For future reference, note that the functions $\pm t_V^\pm$ are lower semi-continuous as V is open; they are also bounded by its diameter.

All this being said, we now rewrite the map $\pi : V \rightarrow \Gamma$ from (5.12) as

$$\pi = \gamma \circ (\Phi_{\hat{\eta}}^{-1})_1 \quad \text{where } (s, t)_1 = s. \quad (5.29)$$

The plan is clear: look for a way of smoothing $\hat{\eta}$ such that the associated maps remain invertible, at least on a portion of V . Note we avoid ∂V as we do not make any assumptions on its regularity, or on the behavior of the stable lines there (see [35, 36] for more on this point).

Lemma 5.4. *Let $t^\pm \in C_c(I)$ and let $J \subset I$ be an open interval such that*

$$t_V^- < t^- \leq 0 \leq t^+ < t_V^+ \quad \text{and} \quad \sup_J t^- < 0 < \inf_J t^+. \quad (5.30)$$

Define the open sets

$$M_{t^\pm, J} = \cup_{s \in J} \{s\} \times (t^-(s), t^+(s)) \quad \text{and} \quad V_{t^\pm, J} = \cup_{s \in J} \Phi_{\hat{\eta}}(\{s\} \times (t^-(s), t^+(s)))$$

and let $W \subset V_{t^\pm, J}$ be open and compactly contained. For all $\epsilon > 0$ there exists $\delta > 0$ such that if $\hat{n} : I \rightarrow S^1$ is Lipschitz on J and satisfies

$$\begin{aligned} & \|\hat{n} - \hat{\eta} \circ \gamma\|_{L^\infty(J)} < \delta \quad \text{and} \\ & \frac{1}{t^-(s)} + \epsilon \leq \frac{\kappa_{\hat{\eta}}(s)}{\gamma' \cdot \hat{n}(s)} \leq \frac{1}{t^+(s)} - \epsilon \quad \text{for a.e. } s \in J, \end{aligned} \quad (5.31)$$

then $\Phi_{\hat{\eta}}$ admits an inverse on W satisfying $\Phi_{\hat{\eta}}^{-1} \in \text{Lip}(W; M_{t^\pm, J})$ as well as the estimates

$$\begin{aligned} & \|\Phi_{\hat{\eta}}^{-1} - \Phi_{\hat{\eta}}^{-1}\|_{L^\infty(W)} \lesssim_{\Gamma, \hat{\eta}, t^\pm, J} \|\hat{n} - \hat{\eta} \circ \gamma\|_{L^\infty(J)}, \\ & \|\nabla \Phi_{\hat{\eta}}^{-1}\|_{L^\infty(W)} \lesssim_{\Gamma, \hat{\eta}, t^\pm, J} \frac{1}{\epsilon}. \end{aligned}$$

Proof. That $\Phi_{\hat{\eta}}$ is onto W can be checked using the homotopy invariance of degree (see, e.g., [30] for the notation). By the definitions, $\Phi_{\hat{\eta}}$ is a homeomorphism between $M_{t^\pm, J}$ and $V_{t^\pm, J}$. Define the continuous homotopy $[0, 1] \rightarrow C(\overline{M_{t^\pm, J}}), \theta \mapsto \Phi_\theta = \Phi_{(1-\theta)\hat{n} + \theta\hat{\eta}}$ from $\Phi_0 = \Phi_{\hat{\eta}}$ to $\Phi_1 = \Phi_{\hat{\eta}}$. Note that

$$|\Phi_\theta(s, t) - \Phi_1(s, t)| \leq |\hat{n}(s) - \hat{\eta} \circ \gamma(s)| |t|.$$

Since by hypothesis $\overline{W} \subset V_{t^\pm, J}$, there exists $\delta_0 > 0$ such that $W \cap \Phi_\theta(\partial M_{t^\pm, J}) = \emptyset$ whenever $\|\hat{n} - \hat{\eta} \circ \gamma\|_{L^\infty(J)} < \delta_0$ and for all θ . Applying [30, Theorem 2.3] proves that $d(\Phi_{\hat{n}}, M_{t^\pm, J}, p) = d(\Phi_{\hat{\eta}}, M_{t^\pm, J}, p) = 1$ for all $p \in W$. Hence,

$$\|\hat{n} - \hat{\eta} \circ \gamma\|_{L^\infty(J)} < \delta_0 \implies W \subset \Phi_{\hat{n}}(M_{t^\pm, J})$$

as in the first part of the claim.

Next, we show that $\Phi_{\hat{n}}$ can be made one-to-one on $M_{t^\pm, J}$. This part of the proof is modeled more or less directly after [36, Section 5]. Introduce a function $\tau_{\hat{n}} : J \times J \setminus \{(s, s') : s = s'\} \rightarrow \mathbb{R} \cup \{\infty\}$ such that

$$\gamma(s) + \tau_{\hat{n}}(s, s') \hat{n}^\perp(s) = \gamma(s') + \tau_{\hat{n}}(s', s) \hat{n}^\perp(s'). \quad (5.32)$$

If $\hat{n}(s) \nparallel \hat{n}(s')$, $\tau_{\hat{n}}(s, s')$ gives the travel time from $\gamma(s)$ in the direction of $\hat{n}^\perp(s)$ to the line containing $\gamma(s')$ and parallel to $\hat{n}^\perp(s')$. If $\hat{n}(s) \parallel \hat{n}(s')$ we set $\tau_{\hat{n}}(s, s') = \tau_{\hat{n}}(s', s) = \infty$. Looking back to (5.25), we see that $\Phi_{\hat{n}}(s, t) \neq \Phi_{\hat{n}}(s', t')$ for $(s, t), (s', t') \in M_{t^\pm, J}$ if and only if

$$\frac{1}{t^-(s)} \leq \frac{1}{\tau_{\hat{n}}(s, s')} \leq \frac{1}{t^+(s)} \quad \text{or} \quad \frac{1}{t^-(s')} \leq \frac{1}{\tau_{\hat{n}}(s', s)} \leq \frac{1}{t^+(s')} \quad (5.33)$$

and $s \neq s'$. We check that this holds when $\|\hat{n} - \hat{\eta} \circ \gamma\|_{L^\infty(J)}$ is sufficiently small.

Let $s, s' \in J$ be such that $\hat{n}(s) \nparallel \hat{n}(s')$, and let (s, s') denote the open interval with boundary points s and s' . Of course, $(s, s') \subset J$. Dotting $\hat{n}(s')$ into (5.32) and rearranging yields the formula

$$\frac{1}{\tau_{\hat{n}}(s, s')} = \frac{(\hat{n}(s') - \hat{n}(s)) \cdot \hat{n}^\perp(s)}{(\gamma(s') - \gamma(s)) \cdot \hat{n}(s')} \quad (5.34)$$

The righthand side approximates $\kappa_{\hat{n}}/\gamma' \cdot \hat{n}$. In particular, there exists $\delta_1 > 0$ and $c_1 > 0$ such that

$$\left| \frac{(\hat{n}(s') - \hat{n}(s)) \cdot \hat{n}^\perp(s)}{(\gamma(s') - \gamma(s)) \cdot \hat{n}(s')} - \int_{(s, s')} \frac{\kappa_{\hat{n}}}{\gamma' \cdot \hat{n}} \right| \lesssim_{\Gamma, \hat{\eta}, t^\pm, J} |s - s'| \quad (5.35)$$

if $\|\hat{n} - \hat{\eta} \circ \gamma\|_{L^\infty(J)} < \delta_1$ and $|s - s'| < c_1$. It suffices to choose δ_1 and c_1 so that

$$\gamma'(a) \cdot \hat{n}(a') \gtrsim_{\Gamma, \hat{\eta}, J} 1 \quad \forall a, a' \in J \text{ with } |a - a'| < c_1. \quad (5.36)$$

This is possible by (5.24) since J is compactly contained. Note we also used the bound $\|\kappa_{\hat{n}}\|_{L^\infty(J)} \lesssim_{t^\pm, J} 1$, which follows from (5.30) and (5.31). Since t^\pm are continuous, these same assumptions yield $c_2 > 0$ so that

$$\frac{1}{t^-(s)} - \frac{\epsilon}{2} \leq \int_{(s, s')} \frac{1}{t^-} + \epsilon \leq \int_{(s, s')} \frac{\kappa_{\hat{n}}}{\gamma' \cdot \hat{n}} \leq \int_{(s, s')} \frac{1}{t^+} - \epsilon \leq \frac{1}{t^+(s)} - \frac{\epsilon}{2} \quad (5.37)$$

if $|s - s'| < c_2$. Combining (5.34)–(5.37) with (5.33) yields that $\Phi_{\hat{n}}(s, t) \neq \Phi_{\hat{n}}(s', t')$, provided that $\|\hat{n} - \hat{\eta} \circ \gamma\|_{L^\infty(J)} < \delta_1$ and if $(s, t), (s', t') \in M_{t^\pm, J}$ satisfy $0 < |s - s'| < c_1 \wedge c_2 \wedge \frac{\epsilon}{2C}$. The constant $C = C(\Gamma, \hat{\eta}, t^\pm, J)$ is the one implicit in the estimate (5.35). On the other hand, since $\Phi_{\hat{n}}(M_{t^\pm, J}) = V_{t^\pm, J}$, we can write that

$$\begin{aligned} \frac{|s - s'|}{\|\nabla \Phi_{\hat{n}}^{-1}\|_{L^\infty(V_{t^\pm, J})}} &\leq |\Phi_{\hat{n}}(s, t) - \Phi_{\hat{n}}(s', t')| \\ &\leq |\Phi_{\hat{n}}(s, t) - \Phi_{\hat{n}}(s', t')| + |\hat{n}(s) - \hat{\eta} \circ \gamma(s)| |t| \\ &\quad + |\hat{n}(s') - \hat{\eta} \circ \gamma(s')| |t'| \end{aligned}$$

on $M_{t^\pm, J}$. There exists $\delta_2 > 0$ such that $\Phi_{\hat{n}}(s, t) \neq \Phi_{\hat{n}}(s', t')$ if $\|\hat{n} - \hat{\eta} \circ \gamma\|_{L^\infty(J)} < \delta_2$ and if $(s, t), (s', t') \in M_{t^\pm, J}$ satisfy $|s - s'| \geq c_1 \wedge c_2 \wedge \frac{\epsilon}{2C}$. The conclusion is that

$$\|\hat{n} - \hat{\eta} \circ \gamma\|_{L^\infty(J)} < \delta_1 \wedge \delta_2 \implies \Phi_{\hat{n}} \text{ is one-to-one on } M_{t^\pm, J}.$$

The first part of the claim on the invertibility of $\Phi_{\hat{\eta}}$ is proved.

We end with the estimates on $\Phi_{\hat{\eta}}^{-1}$. Recall we arranged, by our choice of δ_0 , for the inclusion $W \subset \Phi_{\hat{\eta}}(M_{t^\pm, J})$ to hold. Let $x \in W$ and produce $(s, t) \in M_{t^\pm, J}$ with $\Phi_{\hat{\eta}}(s, t) = x$. It follows that

$$\begin{aligned} |\Phi_{\hat{\eta}}^{-1}(x) - \Phi_{\hat{\eta}}^{-1}(x)| &= |\Phi_{\hat{\eta}}^{-1} \circ \Phi_{\hat{\eta}}(s, t) - \Phi_{\hat{\eta}}^{-1} \circ \Phi_{\hat{\eta}}(s, t)| \\ &\leq \|\nabla \Phi_{\hat{\eta}}^{-1}\|_{L^\infty(V_{t^\pm, J})} |\hat{\eta}(s) - \hat{\eta} \circ \gamma(s)| |t| \end{aligned}$$

which yields the first estimate in the claim. Continuing, we note that δ_1 was chosen so that (5.36) would hold. In particular, $\gamma' \cdot \hat{\eta} \gtrsim_{\Gamma, \hat{\eta}, J} 1$ on J . It follows from (5.26) and (5.31) that

$$\det \nabla \Phi_{\hat{\eta}} \geq \epsilon (t^+ \wedge |t^-|) \gamma' \cdot \hat{\eta} \quad \text{a.e. on } M_{t^\pm, J}.$$

The bound on $\nabla \Phi_{\hat{\eta}}^{-1}$ now follows from (5.25), (5.30), and the Lipschitz inverse function theorem. \square

Having produced a sufficient condition for the invertibility of $\Phi_{\hat{\eta}}$ away from ∂V , we can now construct the desired approximations π_k of π .

Lemma 5.5. *Let $V = \cup_{s \in \Gamma} \ell_s$ and $\pi : V \rightarrow \Gamma$ be as in (5.11) and (5.12), and let $W \subset V$ be open and compactly contained. There exists $\{\pi_k\}_{k \in \mathbb{N}} \subset C^\infty(W; \Gamma)$ such that $\pi_k \rightarrow \pi$, $\partial_{\hat{\eta}^\perp} \pi_k \rightarrow 0$, and $\partial_{\hat{\eta}^\perp}^2 \pi_k \rightarrow 0$ uniformly as $k \rightarrow \infty$.*

Proof. We apply Lemma 5.4. Carrying over the notation from that result, we let $t^\pm \in C_c(I)$ and $J \subset I$ be an open interval satisfying (5.30), such that $\overline{W} \subset V_{t^\pm, J}$ and $\overline{V_{t^\pm, J}} \subset V$. Since $t_V^- < t^- < 0 < t^+ < t_V^+$ nearby J , there exists $\epsilon > 0$ and a slightly larger open interval $J_0 \subset I$ with $\overline{J} \subset J_0$, $\sup_{J_0} t^- < 0 < \inf_{J_0} t^+$, and such that

$$\frac{1}{t^-(s)} + 2\epsilon \leq \frac{1}{t_V^-(s)} \quad \text{and} \quad \frac{1}{t_V^+(s)} \leq \frac{1}{t^+(s)} - 2\epsilon \quad \forall s \in J_0. \quad (5.38)$$

To be clear, we consider t^\pm , J , ϵ , and J_0 to be fixed depending on Γ , $\hat{\eta}$, W , and V at this stage.

Next, we let $\delta > 0$ be as in Lemma 5.4 and produce $\{\hat{\eta}_k\}_{k \in \mathbb{N}} \subset C^\infty(I; S^1)$ approximating $\hat{\eta} \circ \gamma$. We will verify that our sequence satisfies (5.31) for large enough k , i.e., there eventually holds

$$\|\hat{\eta}_k - \hat{\eta} \circ \gamma\|_{L^\infty(J)} < \delta \quad \text{and} \quad \frac{1}{t^-(s)} + \epsilon \leq \frac{\kappa_{\hat{\eta}_k}(s)}{\gamma' \cdot \hat{\eta}_k(s)} \leq \frac{1}{t^+(s)} - \epsilon \quad \forall s \in J. \quad (5.39)$$

Fix $\rho \in C_c^\infty((-1, 1))$ with $\rho \geq 0$ and $\int_{-1}^1 \rho \, ds = 1$. For all large enough $k \in \mathbb{N}$, define $\hat{\eta}_k : I \rightarrow S^1$ by taking

$$\hat{\eta}_k(s) = \frac{(\hat{\eta} \circ \gamma)_{1/k}(s)}{|(\hat{\eta} \circ \gamma)_{1/k}(s)|}, \quad (\hat{\eta} \circ \gamma)_{1/k}(s) = \int_{J_0} k \rho(k(s - s')) \hat{\eta} \circ \gamma(s') \, ds' \quad (5.40)$$

for $s \in J$ and smoothly extending it to I . Its values on $I \setminus \bar{J}$ are immaterial; we define it there so as to match the lemma. Note we needed that $|(\hat{\eta} \circ \gamma)_{1/k}| > 0$ on J for (5.40) to make sense. This is not a problem, since for all large enough k there holds

$$1 \geq |(\hat{\eta} \circ \gamma)_{1/k}| \geq 1 - \|\kappa_{\hat{\eta}}\|_{L^\infty(J_0)} \frac{2}{k} \quad \text{on } J. \quad (5.41)$$

Observe that

$$\kappa_{\hat{\eta}_k} = \frac{(\hat{\eta} \circ \gamma)'_{1/k} \cdot (\hat{\eta} \circ \gamma)_{1/k}^\perp}{|(\hat{\eta} \circ \gamma)_{1/k}|^2} \quad \text{on } J, \quad \text{while} \quad \kappa_{\hat{\eta}} = (\hat{\eta} \circ \gamma)' \cdot (\hat{\eta} \circ \gamma)^\perp \quad \text{on } I. \quad (5.42)$$

When combined with the usual properties of mollification, these facts imply that

$$\begin{aligned} \|\hat{\eta}_k - \hat{\eta} \circ \gamma\|_{L^\infty(J)} &\lesssim \|\kappa_{\hat{\eta}}\|_{L^\infty(J_0)} \frac{1}{k}, \\ \|\hat{\eta}'_k\|_{L^\infty(J)} &\lesssim \|\kappa_{\hat{\eta}}\|_{L^\infty(J_0)}, \\ \|\hat{\eta}''_k\|_{L^\infty(J)} &\lesssim k \|\kappa_{\hat{\eta}}\|_{L^\infty(J_0)} + \|\kappa_{\hat{\eta}}\|_{L^\infty(J_0)}^2 \end{aligned} \quad (5.43)$$

for large enough k . We proceed to verify (5.39). That the first condition is eventually satisfied is clear. For the second condition, let $s \in J$ and note that

$$\left| \frac{\kappa_{\hat{\eta}_k}(s)}{\gamma' \cdot \hat{\eta}_k(s)} - \int_{J_0} k \rho(k(s-s')) \frac{\kappa_{\hat{\eta}}(s')}{\gamma' \cdot \hat{\eta} \circ \gamma(s')} ds' \right| \lesssim_{\Gamma, \hat{\eta}, J, J_0} \frac{1}{k} \quad (5.44)$$

for large enough k . Besides (5.41)–(5.43), the proof makes use of the lower bound $\gamma' \cdot \hat{\eta}_k \gtrsim_{\Gamma, \hat{\eta}, J} 1$ on J , which is eventually implied by (5.24) and the fact that J is compactly contained. Note that

$$\frac{1}{t^-} + 2\epsilon \leq \frac{\kappa_{\hat{\eta}}}{\gamma' \cdot \hat{\eta} \circ \gamma} \leq \frac{1}{t^+} - 2\epsilon \quad \text{a.e. on } J_0 \quad (5.45)$$

by (5.28) and (5.38). Combining (5.44), (5.45), our choice to take $\rho \geq 0$, and the fact that $1/t^\pm$ are uniformly continuous on J_0 , we deduce the second condition in (5.39) for large enough k .

The previous paragraphs checked that the hypotheses of Lemma 5.4 hold for the given W and for our choices of t^\pm , J , ϵ , and $\hat{\eta} = \hat{\eta}_k$. We had to take k large enough so that the conditions in (5.39) would hold. The conclusion is that $\Phi_{\hat{\eta}_k} : I \times \mathbb{R} \rightarrow \mathbb{R}^2$ from (5.25) admit inverses on W satisfying $\Phi_{\hat{\eta}_k}^{-1} \in \text{Lip}(W; M_{t^\pm, J})$, for large enough k . Recall $M_{t^\pm, J} = \Phi_{\hat{\eta}}^{-1}(V_{t^\pm, J})$. The estimates

$$\begin{aligned} \|\Phi_{\hat{\eta}_k}^{-1} - \Phi_{\hat{\eta}}^{-1}\|_{L^\infty(W)} &\lesssim_{\Gamma, \hat{\eta}, W, V} \|\hat{\eta}_k - \hat{\eta} \circ \gamma\|_{L^\infty(J)}, \\ \|\nabla \Phi_{\hat{\eta}_k}^{-1}\|_{L^\infty(W)} &\lesssim_{\Gamma, \hat{\eta}, W, V} 1 \end{aligned} \quad (5.46)$$

follow directly from the ones in the lemma. We remind the reader that t^\pm , J , and ϵ (and J_0) were taken to depend on Γ , $\hat{\eta}$, W , and V . Since $\Phi_{\hat{\eta}_k}$ is smooth and $|\det \nabla \Phi_{\hat{\eta}_k}| \gtrsim_{\Gamma, \hat{\eta}, W, V} 1$ on $\Phi_{\hat{\eta}_k}^{-1}(W)$, the inverse function theorem gives that

$\Phi_{\hat{\eta}_k}^{-1} \in C^\infty(W; M_{I^\pm, J})$. We are ready to define the desired approximations π_k . In a direct analogy with the formula for π in (5.29), we define $\pi_k \in C^\infty(W; \Gamma)$ by

$$\pi_k = \gamma \circ (\Phi_{\hat{\eta}_k}^{-1})_1 \quad \text{where } (s, t)_1 = s \quad (5.47)$$

for large enough k . The rest of the proof establishes the convergences in the claim.

That $\pi_k \rightarrow \pi$ uniformly on W is clear. In fact, we read off from (5.29), (5.43), (5.46), and (5.47) that

$$\|\pi_k - \pi\|_{L^\infty(W)} \lesssim_{\Gamma, \hat{\eta}, W, V} \frac{1}{k}, \quad \|\nabla \pi_k\|_{L^\infty(W)} \lesssim_{\Gamma, \hat{\eta}, W, V} 1, \quad \|\nabla \nabla \pi_k\|_{L^\infty(W)} \lesssim_{\Gamma, \hat{\eta}, W, V} k. \quad (5.48)$$

The third inequality follows from the elementary estimates

$$|\nabla \Phi_{\hat{\eta}_k}| \lesssim 1 + |t| |\hat{\eta}'_k|, \quad |\nabla \nabla \Phi_{\hat{\eta}_k}| \lesssim |\gamma''| + |\hat{\eta}'_k| + |t| |\hat{\eta}''_k|$$

along with the inverse function theorem. It is now convenient to consider $\hat{\eta}_k$ as being defined on W instead of I . Abusing notation slightly, we let

$$\hat{\eta}_k(x) = \hat{\eta}_k \circ (\Phi_{\hat{\eta}_k}^{-1})_1(x), \quad x \in W.$$

On the righthand side we use (5.40), noting that $(\Phi_{\hat{\eta}_k}^{-1})_1(W) \subset J$. It is natural to compare against $\hat{\eta} = \hat{\eta} \circ \gamma \circ (\Phi_{\hat{\eta}}^{-1})_1$. The estimates

$$\|\hat{\eta}_k - \hat{\eta}\|_{L^\infty(W)} \lesssim_{\Gamma, \hat{\eta}, W, V} \frac{1}{k}, \quad \|\nabla \hat{\eta}_k\|_{L^\infty(W)} \lesssim_{\Gamma, \hat{\eta}, W, V} 1 \quad (5.49)$$

result from (5.43) and (5.46). We define $\partial_{\eta_k^\perp}$ and $\partial_{\eta_k^\perp}^2$ analogously to $\partial_{\hat{\eta}^\perp}$ and $\partial_{\hat{\eta}^\perp}^2$ from (5.9). Then,

$$\partial_{\eta_k^\perp} \pi_k = 0 \quad \text{and} \quad \partial_{\eta_k^\perp}^2 \pi_k = 0. \quad (5.50)$$

These identities are consequences of the fact that $\partial_t(\pi_k \circ \Phi_{\hat{\eta}_k}) = 0$. Indeed, $\partial_{\eta_k^\perp}$ is a directional derivative along the lines traced out by $t \mapsto \Phi_{\hat{\eta}_k}(s, t)$, per (5.25).

We now have all of the ingredients to prove that $\partial_{\hat{\eta}^\perp} \pi_k \rightarrow 0$ and $\partial_{\hat{\eta}^\perp}^2 \pi_k \rightarrow 0$ uniformly on W . Applying the first identity in (5.50) yields that

$$\partial_{\hat{\eta}^\perp} \pi_k = \partial_{\hat{\eta}^\perp} \pi_k - \partial_{\eta_k^\perp} \pi_k = \partial_{(\hat{\eta} - \hat{\eta}_k)^\perp} \pi_k.$$

Using (5.48) and (5.49) we deduce that

$$\|\partial_{\hat{\eta}^\perp} \pi_k\|_{L^\infty(W)} \leq \|\hat{\eta} - \hat{\eta}_k\|_{L^\infty(W)} \|\nabla \pi_k\|_{L^\infty(W)} \lesssim_{\Gamma, \hat{\eta}, W, V} \frac{1}{k}.$$

Next, we differentiate the first identity in (5.50) to see that

$$0 = \partial_{(\hat{\eta} - \hat{\eta}_k)^\perp} \left(\partial_{\hat{\eta}_k^\perp} \pi_k \right) = \partial_{(\hat{\eta} - \hat{\eta}_k)^\perp} \hat{\eta}_k^\perp \cdot \nabla \pi_k + \left\langle (\hat{\eta} - \hat{\eta}_k)^\perp \otimes \hat{\eta}_k^\perp, \nabla \nabla \right\rangle \pi_k.$$

By the second identity there,

$$\begin{aligned} \partial_{\hat{\eta}^\perp}^2 \pi_k &= \left\langle \hat{\eta}^\perp \otimes \hat{\eta}^\perp - \eta_k^\perp \otimes \eta_k^\perp, \nabla \nabla \right\rangle \pi_k = \left\langle (\hat{\eta} - \hat{\eta}_k)^\perp \otimes (\hat{\eta} - \hat{\eta}_k + 2\hat{\eta}_k)^\perp, \nabla \nabla \right\rangle \pi_k \\ &= \partial_{(\hat{\eta} - \hat{\eta}_k)^\perp}^2 \pi_k - 2\partial_{(\hat{\eta} - \hat{\eta}_k)^\perp} \hat{\eta}_k^\perp \cdot \nabla \pi_k. \end{aligned}$$

Using (5.48) and (5.49) again, we find that

$$\begin{aligned} \|\partial_{\hat{\eta}}^2 \pi_k\|_{L^\infty(W)} &\leq \|\hat{\eta} - \hat{\eta}_k\|_{L^\infty(W)}^2 \|\nabla \nabla \pi_k\|_{L^\infty(W)} \\ &+ 2\|\hat{\eta} - \hat{\eta}_k\|_{L^\infty(W)} \|\nabla \hat{\eta}_k\|_{L^\infty(W)} \|\nabla \pi_k\|_{L^\infty(W)} \lesssim_{\Gamma, \hat{\eta}, W, V} \frac{1}{k}. \end{aligned}$$

The proof is complete. \square

5.3. Three Solution Formulas

Ultimately, we are interested in applying the method of characteristics to deduce (partial) uniqueness and regularity theorems, and even explicit solution formulas for μ . Doing so requires supplementing the ODEs from Section 5.2 with boundary data implied by the original system (5.1). Different data arise depending on the stable lines. Guided by the upcoming examples in Section 6, we treat the three configurations shown in Fig. 5. See Corollary 5.1 for Panel (a), Corollary 5.2 for Panel (b), and Corollary 5.3 for Panel (c). We continue to use the notation from the paragraphs leading up to Lemma 5.3.

Stable lines extending between boundary points. First, we consider Panel (a) of Fig. 5. Recall O denotes the ordered set of φ from (5.4), while $\{\ell_x\}$ are its stable lines. Assume there exists an open set $V \subset O$ of the form

$$V = \bigcup_{s \in \Gamma} \ell_s \quad \text{where} \quad \partial \ell_s \subset \partial \Omega \quad \forall s \in \Gamma. \quad (5.51)$$

As in (5.11), we understand $\Gamma \subset V$ to be a smooth curve (i.e., one that is diffeomorphic to an open interval) that meets each stable line it indexes transversely and exactly once. For simplicity, we suppose that

$$\bigcup_{s \in \Gamma} \partial \ell_s \text{ consists of two Lipschitz curves.} \quad (5.52)$$

By Lemma 5.1 and the definition of O , there exist $\zeta \in L^1(V; (0, \infty)) \cap L^2_{\text{loc}}(V)$ and $\hat{\eta} \in \text{Lip}_{\text{loc}}(V; S^1)$ such that

$$\nabla \nabla \varphi = \zeta \hat{\eta} \otimes \hat{\eta} dx \quad \text{on } V,$$

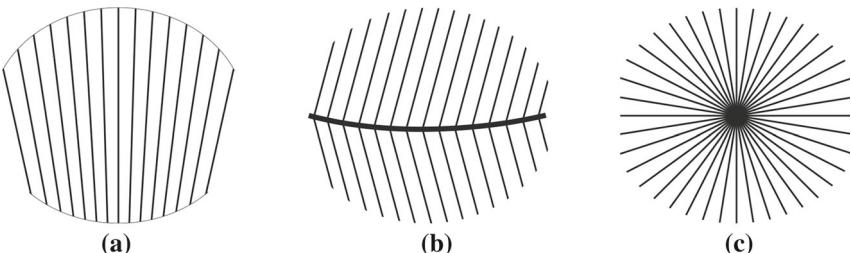


Fig. 5. Three configurations of stable lines. Panel (a) depicts stable lines extending between boundary points. Panel (b) depicts stable lines meeting along an interior curve. Panel (c) shows stable lines meeting at a point. Given suitable non-degeneracy conditions on φ , we prove that μ is uniquely determined on such lines

where $\hat{\eta}$ is constant along and perpendicular to the stable lines. Our next assumption is that

$$\hat{\eta} \in C(\bar{V}). \quad (5.53)$$

In particular, the given stable lines are not allowed to meet at $\partial\Omega$. Finally, we assume there exists $c > 0$ such that

$$\hat{v} \cdot [\nabla\varphi] \geq c \quad \text{and} \quad |\hat{\tau} \cdot \hat{\eta}|_{\partial\Omega} \geq c \quad \mathcal{H}^1\text{-a.e. on } \cup_{s \in \Gamma} \partial\ell_s, \quad (5.54)$$

where \hat{v} and $\hat{\tau}$ are the outwards-pointing unit normal and tangent vectors at $\partial\Omega$. The second part of this last assumption requires that the given stable lines remain (a.e.) uniformly transverse to $\partial\Omega$. Recall the change of measure factor $\varrho : V \rightarrow (0, \infty)$ from (5.13).

Corollary 5.1. *Suppose φ admits some open set $V \subset O$ satisfying (5.51)–(5.54), and let $\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2)$ solve (5.1). Then*

$$\mu = \lambda \hat{\eta} \otimes \hat{\eta} dx \quad \text{on } V$$

where $\lambda : V \rightarrow [0, \infty)$ is the unique weak solution of the two-point boundary value problem

$$\begin{cases} -\frac{1}{2\varrho} \partial_{\hat{\eta}^\perp(s)}^2(\varrho\lambda) = \det \nabla \nabla p & \text{on } \ell_s \\ \varrho\lambda = 0 & \text{at } \partial\ell_s \end{cases}$$

upon restriction to \mathcal{H}^1 -a.e. ℓ_s .

Proof. Combining the second part of (5.6) and Lemma 5.3 yields the formula

$$\mu = \lambda_{\text{a.c.}} \hat{\eta} \otimes \hat{\eta} dx + \int_{\Gamma} \lambda_{\text{sing}} \hat{\eta} \otimes \hat{\eta} \mathcal{H}^1 \lfloor \ell_s d\vartheta(s) \quad \text{on } V \quad (5.55)$$

where $\lambda_{\text{a.c.}}$ and λ_{sing} solve the ODEs (5.14) and (5.15) for \mathcal{H}^1 - and ϑ -a.e. $s \in \Gamma$. Our plan is to use the complementary slackness conditions in the original system (5.1) to deduce the boundary conditions

$$\varrho\lambda_{\text{a.c.}}|_{\partial\ell_s} = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } s \quad \text{and} \quad \lambda_{\text{sing}}|_{\partial\ell_s} = 0 \quad \text{for } \vartheta\text{-a.e. } s. \quad (5.56)$$

It follows immediately from (5.15) and the second of these that $\lambda_{\text{sing}} = 0$. Then, (5.14) and the first condition yield the desired characterization of $\lambda = \lambda_{\text{a.c.}}$. The regularized formulation of the complementary slackness conditions must be applied. Here, we use the second part of (5.2), which implies that

$$0 = \lim_{\delta \rightarrow 0} \int_{\cup_{s \in \Gamma} \partial\ell_s} |\hat{v} \cdot [\nabla\varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu_\delta \rangle| d\mathcal{H}^1 \quad (5.57)$$

due to our assumptions (5.51) and (5.52) (that the domain of integration is measurable follows from the second of these). Recall $\{\mu_\delta\}_{\delta > 0}$ are the mollifications of μ defined in (1.32) using the kernel $\rho \in C_c^\infty(B_1)$. As noted in Remark 1.3 and proved in Section 4, the complementary slackness conditions hold so long as $\rho \geq 0$ and $\int_{B_1} \rho = 1$. We choose it a bit more carefully now: for the purposes of this proof,

we take $\rho > 0$ on a neighborhood of zero. This same choice also appears in the proofs of Corollaries 5.2 and 5.3.

All this being said, we claim that the desired boundary conditions (5.56) hold. Using that μ and ρ are non-negative and applying the disintegration formulas (5.13) and (5.55), we have by Fubini's theorem that

$$\begin{aligned}\mu_\delta(x) &\geq \int_V \frac{1}{\delta^2} \rho\left(\frac{x-y}{\delta}\right) d\mu(y) \\ &= \int_\Gamma \left[\int_{\ell_s} \frac{1}{\delta^2} \rho\left(\frac{x-y}{\delta}\right) \varrho \lambda_{\text{a.c.}} \hat{\eta} \otimes \hat{\eta}(y) d\mathcal{H}^1(y) \right] d\mathcal{H}^1(s) \\ &\quad + \int_\Gamma \left[\int_{\ell_s} \frac{1}{\delta^2} \rho\left(\frac{x-y}{\delta}\right) \lambda_{\text{sing}} \hat{\eta} \otimes \hat{\eta}(y) d\mathcal{H}^1(y) \right] d\vartheta(s)\end{aligned}$$

for all $x \in \overline{\Omega}$. Applying this where the given stable lines meet $\partial\Omega$ yields that

$$\begin{aligned}&\int_{\cup_{s \in \Gamma} \partial \ell_s} \hat{v} \cdot [\nabla \varphi] \langle \hat{\tau} \otimes \hat{\tau}, \mu_\delta \rangle d\mathcal{H}^1 \\ &\geq \int_\Gamma \left[\int_{y \in \ell_s} \int_{x \in \cup_{s \in \Gamma} \partial \ell_s} \hat{v} \cdot [\nabla \varphi](x) |\hat{\tau}(x) \cdot \hat{\eta}(y)|^2 \varrho \lambda_{\text{a.c.}}(y) \right. \\ &\quad \left. \rho\left(\frac{x-y}{\delta}\right) \frac{d\mathcal{H}^1(x) d\mathcal{H}^1(y)}{\delta^2} \right] d\mathcal{H}^1(s) \\ &\quad + \int_\Gamma \left[\int_{y \in \ell_s} \int_{x \in \cup_{s \in \Gamma} \partial \ell_s} \hat{v} \cdot [\nabla \varphi](x) |\hat{\tau}(x) \cdot \hat{\eta}(y)|^2 \lambda_{\text{sing}}(y) \right. \\ &\quad \left. \rho\left(\frac{x-y}{\delta}\right) \frac{d\mathcal{H}^1(x) d\mathcal{H}^1(y)}{\delta^2} \right] d\vartheta(s).\end{aligned}$$

The integral on the lefthand side tends to zero as $\delta \rightarrow 0$ by (5.57). Using Fatou's lemma, we can pass to the limit on the right. Recall from Lemma 5.3 that $\varrho \lambda_{\text{a.c.}}$ and λ_{sing} belong to $W^{2,1}(\ell_s)$ and $W^{2,\infty}(\ell_s)$ respectively for \mathcal{H}^1 - and ϑ -a.e. s . In particular, $\varrho \lambda_{\text{a.c.}}(y)$ and $\lambda_{\text{sing}}(y)$ converge to their traces as $y \rightarrow \partial \ell_s$ along a.e. ℓ_s . Applying the hypotheses (5.53) and (5.54), and making use of our choice to take $\rho > 0$ nearby zero, we conclude that

$$0 = \int_\Gamma \|\varrho \lambda_{\text{a.c.}}\|_{L^\infty(\partial \ell_s, \mathcal{H}^0)} d\mathcal{H}^1(s) = \int_\Gamma \|\lambda_{\text{sing}}\|_{L^\infty(\partial \ell_s, \mathcal{H}^0)} d\vartheta(s)$$

in the limit $\delta \rightarrow 0$. The boundary conditions (5.56) are proved. \square

Stable lines meeting along an interior curve. We turn to Panel (b) of Fig. 5. Again, recall from (5.4) that O denotes the ordered set of φ to which its stable lines $\{\ell_x\}$ belong, while Σ is its singular set. We now let $V \subset O \cup \Sigma$ be an open set such that

$$V \setminus \Sigma = V_- \cup V_+ \quad \text{where} \quad V_\pm = \cup_{s \in \Gamma_\pm} \ell_s \quad \text{are disjoint.} \quad (5.58)$$

Here, Γ_{\pm} are smooth curves belonging to V_{\pm} that meet their indexed stable lines transversely and exactly once. Our second assumption is twofold: we require that

- each indexed stable line ℓ_s passes between Σ and ∂V , and
- for all $z \in V \cap \Sigma$ there exist $s_{\pm} \in \Gamma_{\pm}$ such that $\{z\} = \partial \ell_{s_+} \cap \partial \ell_{s_-}$.

(5.59)

Looking back to Fig. 5 should help make this clear. Continuing, we assume for simplicity that

$$V \cap \Sigma \text{ is a Lipschitz curve.} \quad (5.60)$$

Lemma 5.1 now guarantees the existence of $\zeta \in L^1(V; (0, \infty)) \cap L^2_{\text{loc}}(V \setminus \Sigma)$ and $\hat{\eta} \in \text{Lip}_{\text{loc}}(V \setminus \Sigma; S^1)$ such that

$$\nabla \nabla \varphi = \zeta \hat{\eta} \otimes \hat{\eta} \, dx + \hat{\nu}_{\Sigma} \cdot [\nabla \varphi] \hat{\nu}_{\Sigma} \otimes \hat{\nu}_{\Sigma} \mathcal{H}^1 \lfloor \Sigma \text{ on } V.$$

Here, $\hat{\nu}_{\Sigma}$ denotes a choice of unit normal vector to Σ , which is defined a.e. along the portion of it belonging to V . We take it to point from V_- to V_+ , and write $[\cdot] = \cdot|_{\Sigma_+} - \cdot|_{\Sigma_-}$ for the corresponding jump in a quantity where $\cdot|_{\Sigma_{\pm}}$ are the traces at Σ from V_{\pm} . Our fourth assumption is that

$$\hat{\eta} \text{ restricts to each of } V_{\pm} \text{ as an element of } C(\overline{V_{\pm}}). \quad (5.61)$$

In particular, this implies that the pair s_{\pm} in (5.59) is unique, as no two stable lines on the same side of $V \cap \Sigma$ can meet there. Finally, we suppose there exists $c > 0$ such that

$$\begin{aligned} \zeta &\geq c \quad \mathcal{L}^2\text{-a.e. on } V, \\ \hat{\nu}_{\Sigma} \cdot [\nabla \varphi] &\geq c \quad \text{and} \quad |\hat{\tau}_{\Sigma} \cdot \hat{\eta}|_{\Sigma_{\pm}} \geq c \quad \mathcal{H}^1\text{-a.e. on } V \cap \Sigma \end{aligned} \quad (5.62)$$

where $\hat{\tau}_{\Sigma} = \hat{\nu}_{\Sigma}^{\perp}$. Define the change of measure factor $\varrho : V \rightarrow (0, \infty)$ as in (5.13).

Corollary 5.2. *Suppose φ admits some open set $V \subset O \cup \Sigma$ satisfying (5.58)–(5.62), and let $\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2)$ solve (5.1). Then*

$$\mu = \lambda \hat{\eta} \otimes \hat{\eta} \, dx \text{ on } V,$$

where $\lambda : V \rightarrow [0, \infty)$ is the unique weak solution of the Cauchy problem

$$\begin{cases} -\frac{1}{2\varrho} \partial_{\hat{\eta}^{\perp}(s)}^2(\varrho \lambda) = \det \nabla \nabla p & \text{on } \ell_s \\ \varrho \lambda = \partial_{\hat{\eta}^{\perp}(s)}(\varrho \lambda) = 0 & \text{at } \partial \ell_s \cap \Sigma \end{cases}$$

upon restriction to \mathcal{H}^1 -a.e. ℓ_s .

Remark 5.4. Under the above hypotheses, $\det \nabla \nabla p \leq 0$ a.e. on V . Indeed, it follows from the Cauchy problem that $\varrho \lambda$ and $\det \nabla \nabla p$ take on opposite signs, while $\varrho > 0$ and $\lambda \geq 0$.

Remark 5.5. In each of the examples in Section 6 it will turn out that if Σ is not empty nor a single (smooth) curve, it is nevertheless a tree. At its internal vertices, three or more stable lines will meet, and to achieve the analogous result we will need to show that μ vanishes on these lines. This can be done using the ideas in the proof below. See Example 6.8 for more details.

Proof. The proof is similar in spirit to that of Corollary 5.1, albeit more involved. We start by showing that

$$\mu = 0 \quad \text{on } V \cap \Sigma. \quad (5.63)$$

At the same time, a straightforward application of (5.6) and Lemma 5.3 yields the formula

$$\mu = \lambda_{\text{a.c.}} \hat{\eta} \otimes \hat{\eta} dx + \int_{\Gamma} \lambda_{\text{sing}} \hat{\eta} \otimes \hat{\eta} \mathcal{H}^1 \lfloor \ell_s d\vartheta(s) \quad \text{on } V \setminus \Sigma, \quad (5.64)$$

where $\lambda_{\text{a.c.}}$ and λ_{sing} solve the ODEs (5.14) and (5.15) for \mathcal{H}^1 - and ϑ -a.e. s in the index set $\Gamma = \Gamma_+ \cup \Gamma_-$. Our second step will be to extract the initial conditions

$$\varrho \lambda_{\text{a.c.}}|_{\partial \ell_s \cap \Sigma} = \partial_{\hat{\eta}^\perp(s)} (\varrho \lambda_{\text{a.c.}})|_{\partial \ell_s \cap \Sigma} = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } s, \quad (5.65)$$

$$\lambda_{\text{sing}}|_{\partial \ell_s \cap \Sigma} = \partial_{\hat{\eta}^\perp(s)} \lambda_{\text{sing}}|_{\partial \ell_s \cap \Sigma} = 0 \quad \text{for } \vartheta\text{-a.e. } s \quad (5.66)$$

from the first two equations in (5.1). Combining these with (5.14) and (5.15) proves that $\lambda_{\text{sing}} = 0$, and the desired characterization of $\lambda = \lambda_{\text{a.c.}}$ follows. To accomplish these steps, we shall make use of the following consequences of the (regularized) complementary slackness conditions (5.2), which hold in light of the formula for $\nabla \nabla \varphi$ above:

$$0 = \lim_{\delta \rightarrow 0} \int_{V \setminus \Sigma} \left| \langle \zeta \hat{\eta}^\perp \otimes \hat{\eta}^\perp, \mu_\delta \rangle \right| dx = \lim_{\delta \rightarrow 0} \int_{V \cap \Sigma} |\hat{v}_\Sigma \cdot [\nabla \varphi] \langle \hat{\tau}_\Sigma \otimes \hat{\tau}_\Sigma, \mu_\delta \rangle| d\mathcal{H}^1, \quad (5.67)$$

where $\{\mu_\delta\}_{\delta > 0}$ are the mollified versions of μ from (1.32). Again, we take the kernel $\rho \in C_c^\infty(B_1)$ to satisfy $\rho > 0$ on a neighborhood of zero (see Remark 1.3).

We start by proving (5.63). First, we note for every $x \in V$ that

$$\mu_\delta(x) \geq \int_{V \cap \Sigma} \frac{1}{\delta^2} \rho \left(\frac{x-y}{\delta} \right) d\mu(y) \quad \text{and} \quad \mu_\delta(x) \geq \int_{V \setminus \Sigma} \frac{1}{\delta^2} \rho \left(\frac{x-y}{\delta} \right) d\mu(y). \quad (5.68)$$

These follow from the non-negativity of ρ and μ . Integrating the first of these and applying Fubini's theorem yields

$$\begin{aligned} & \int_{V \setminus \Sigma} \langle \zeta \hat{\eta}^\perp \otimes \hat{\eta}^\perp, \mu_\delta \rangle dx \\ & \geq \int_{V \cap \Sigma} \left\langle \int_{V \setminus \Sigma} \zeta \hat{\eta}^\perp \otimes \hat{\eta}^\perp(x) \rho \left(\frac{x-y}{\delta} \right) \frac{dx}{\delta^2}, \mu(y) \right\rangle, \\ & \int_{V \cap \Sigma} \langle \hat{v}_\Sigma \cdot [\nabla \varphi] \hat{\tau}_\Sigma \otimes \hat{\tau}_\Sigma, \mu_\delta \rangle d\mathcal{H}^1 \\ & \geq \frac{1}{\delta} \int_{V \cap \Sigma} \left\langle \int_{V \cap \Sigma} \hat{v}_\Sigma \cdot [\nabla \varphi] \hat{\tau}_\Sigma \otimes \hat{\tau}_\Sigma(x) \rho \left(\frac{x-y}{\delta} \right) \frac{d\mathcal{H}^1(x)}{\delta}, \mu(y) \right\rangle. \end{aligned}$$

According to (5.67), the integrals on the left tend to zero as $\delta \rightarrow 0$. Applying (5.61) along with the first part of (5.62), we can pass to the limit in the first inequality above to deduce that

$$0 = \langle \hat{\eta}^\perp \otimes \hat{\eta}^\perp|_{\Sigma_\pm}, \mu \rangle \quad \text{on } V \cap \Sigma.$$

Then, using that μ is non-negative and Sym_2 -valued, we get that

$$\mu = \langle \hat{\eta} \otimes \hat{\eta}|_{\Sigma_\pm}, \mu \rangle \hat{\eta} \otimes \hat{\eta}|_{\Sigma_\pm} \quad \text{on } V \cap \Sigma.$$

Now plug this into the second inequality and send $\delta \rightarrow 0$ again (this is to deal with the case where $\hat{\eta}|_{\Sigma_\pm}$ are parallel on a non-null set). Applying (5.61) along with the second and third parts of (5.62), and recalling that $\rho > 0$ nearby zero, there follows

$$0 = \langle \hat{\eta} \otimes \hat{\eta}|_{\Sigma_\pm}, \mu \rangle \quad \text{on } V \cap \Sigma.$$

Therefore $\mu = 0$ on $V \cap \Sigma$ and (5.63) is proved.

We proceed to control μ on $V \setminus \Sigma$. As explained above, we must establish the initial conditions (5.65) and (5.66) for $\varrho \lambda_{\text{a.c.}}$ and λ_{sing} . We handle their traces first. This part of the proof can be copied almost verbatim from that of Corollary 5.1. Recall the index set $\Gamma = \Gamma_+ \cup \Gamma_-$. Using the second part of (5.68) along with the disintegration formulas (5.13) and (5.64) produces the lower bound

$$\begin{aligned} & \int_{V \cap \Sigma} \langle \hat{v}_\Sigma \cdot [\nabla \varphi] \hat{t}_\Sigma \otimes \hat{t}_\Sigma, \mu_\delta \rangle d\mathcal{H}^1 \\ & \geq \int_{\Gamma} \left[\int_{y \in \ell_s} \int_{x \in V \cap \Sigma} \hat{v}_\Sigma \cdot [\nabla \varphi](x) |\hat{t}_\Sigma(x) \cdot \hat{\eta}(y)|^2 \varrho \lambda_{\text{a.c.}}(y) \right. \\ & \quad \left. \rho \left(\frac{x-y}{\delta} \right) \frac{d\mathcal{H}^1(x) d\mathcal{H}^1(y)}{\delta^2} \right] d\mathcal{H}^1(s) \\ & \quad + \int_{\Gamma} \left[\int_{y \in \ell_s} \int_{x \in V \cap \Sigma} \hat{v}_\Sigma \cdot [\nabla \varphi](x) |\hat{t}_\Sigma(x) \cdot \hat{\eta}(y)|^2 \lambda_{\text{sing}}(y) \right. \\ & \quad \left. \rho \left(\frac{x-y}{\delta} \right) \frac{d\mathcal{H}^1(x) d\mathcal{H}^1(y)}{\delta^2} \right] d\vartheta(s). \end{aligned}$$

According to (5.67), the integral on the lefthand side tends to zero as $\delta \rightarrow 0$. Applying the assumption (5.61), the second and third parts of (5.62), and our choice to take $\rho > 0$ nearby zero, we pass to the limit to deduce that

$$0 = \int_{\Gamma} \varrho \lambda_{\text{a.c.}}|_{\partial \ell_s \cap \Sigma} d\mathcal{H}^1(s) = \int_{\Gamma} \lambda_{\text{sing}}|_{\partial \ell_s \cap \Sigma} d\vartheta(s).$$

Since the integrands are non-negative, they must vanish a.e. The first parts of (5.65) and (5.66) are proved.

The next part of the proof has no analog in that of Corollary 5.1: we must show that $\partial_{\hat{\eta}^\perp}(\varrho \lambda_{\text{a.c.}})$ and $\partial_{\hat{\eta}^\perp} \lambda_{\text{sing}}$ vanish at $V \cap \Sigma$. The argument goes in two steps. The first step is to show that

$$\begin{aligned} 0 & \leq \partial_{\hat{\eta}^\perp(s)}(\varrho \lambda_{\text{a.c.}})|_{\partial \ell_s \cap \Sigma} \quad \text{for } \mathcal{H}^1\text{-a.e. } s \in \Gamma, \\ 0 & \leq \partial_{\hat{\eta}^\perp(s)} \lambda_{\text{sing}}|_{\partial \ell_s \cap \Sigma} \quad \text{for } \vartheta\text{-a.e. } s \in \Gamma. \end{aligned} \tag{5.69}$$

Here, we understand each ℓ_s to be oriented so that it starts at Σ and ends at ∂V , i.e., we take $\hat{\eta}^\perp$ to point away from Σ . (This is possible due to (5.59).) The second step is to deduce from the first equation in (5.1) the matching conditions

$$\begin{aligned} 0 &= \partial_{\hat{\eta}^\perp(s_+)}(\varrho\lambda_{\text{a.c.}})|_{\partial\ell_{s_+} \cap \Sigma} + \partial_{\hat{\eta}^\perp(s_-)}(\varrho\lambda_{\text{a.c.}})|_{\partial\ell_{s_-} \cap \Sigma}, \\ 0 &= \partial_{\hat{\eta}^\perp(s_+)}\lambda_{\text{sing}}|_{\partial\ell_{s_+} \cap \Sigma} + \partial_{\hat{\eta}^\perp(s_-)}\lambda_{\text{sing}}|_{\partial\ell_{s_-} \cap \Sigma} \end{aligned} \quad (5.70)$$

respectively for \mathcal{H}^1 -a.e. and ϑ -a.e. $s_\pm \in \Gamma_\pm$ with $\partial\ell_{s_+} \cap \partial\ell_{s_-} \cap V \cap \Sigma \neq \emptyset$. Note there is a one-to-one correspondence between points $z \in V \cap \Sigma$ and pairs $s_\pm \in \Gamma_\pm$ satisfying $\{z\} = \partial\ell_{s_+} \cap \partial\ell_{s_-}$, due to (5.59) and (5.61). Combining (5.69) and (5.70) yields the remaining parts of (5.65) and (5.66).

First, we handle (5.69). Since $\mu \geq 0$, the densities $\varrho\lambda_{\text{a.c.}}$ and λ_{sing} are non-negative. That is,

$$0 \leq \int_{\ell_s} \psi \varrho \lambda_{\text{a.c.}} d\mathcal{H}^1 \quad \text{for } \mathcal{H}^1\text{-a.e. } s \quad \text{and} \quad 0 \leq \int_{\ell_s} \psi \lambda_{\text{sing}} d\mathcal{H}^1 \quad \text{for } \vartheta\text{-a.e. } s$$

whenever $\psi \geq 0$. Let $\chi \in C_c((1, 2))$ be non-negative and integrate to one, and define $\psi_\delta \in C(\ell_s)$ for $\delta > 0$ by

$$\psi_\delta(x) = \frac{1}{|x - z|} \cdot \frac{1}{\delta} \chi\left(\frac{|x - z|}{\delta}\right) \quad \text{where } \{z\} = \partial\ell_s \cap V \cap \Sigma.$$

Recall from Lemma 5.3 that $\varrho\lambda_{\text{a.c.}} \in W^{2,1}(\ell_s)$ and $\lambda_{\text{sing}} \in W^{2,\infty}(\ell_s)$ for \mathcal{H}^1 - and ϑ -a.e. s , respectively. Above, we proved that they vanish at a.e. $\partial\ell_s \cap \Sigma$. Therefore,

$$\begin{aligned} \int_{\ell_s} \psi_\delta \varrho \lambda_{\text{a.c.}} d\mathcal{H}^1 &\rightarrow \partial_{\hat{\eta}^\perp(s)}(\varrho\lambda_{\text{a.c.}})|_{\partial\ell_s \cap \Sigma} \quad \text{for } H^1\text{-a.e. } s, \\ \int_{\ell_s} \psi_\delta \lambda_{\text{sing}} d\mathcal{H}^1 &\rightarrow \partial_{\hat{\eta}^\perp(s)}\lambda_{\text{sing}}|_{\partial\ell_s \cap \Sigma} \quad \text{for } \vartheta\text{-a.e. } s \end{aligned}$$

and with this the desired inequalities (5.69) follow.

Finally, we prove the matching conditions (5.70). Testing the first part of (5.1) against $\psi \in C_c^\infty(V)$ and applying the disintegration formulas (5.13) and (5.64) yields the identity

$$\begin{aligned} \int_V \psi \det \nabla \nabla p dx &= \int_V \left\langle -\frac{1}{2} \nabla^\perp \nabla^\perp \psi, \mu \right\rangle \\ &= \int_\Gamma \left[\int_{\ell_s} -\frac{1}{2} \partial_{\hat{\eta}^\perp(s)}^2 \psi \varrho \lambda_{\text{a.c.}} d\mathcal{H}^1 \right] d\mathcal{H}^1(s) \\ &\quad + \int_\Gamma \left[\int_{\ell_s} -\frac{1}{2} \partial_{\hat{\eta}^\perp(s)}^2 \psi \lambda_{\text{sing}} d\mathcal{H}^1 \right] d\vartheta(s). \end{aligned}$$

Integrating by parts twice with the ODEs (5.14) and (5.15), there follows

$$\begin{aligned} \int_{\Gamma} \left[\int_{\ell_s} \partial_{\hat{\eta}^{\perp}(s)}^2 \psi \varrho \lambda_{\text{a.c.}} d\mathcal{H}^1 \right] d\mathcal{H}^1(s) &= \int_{\Gamma} (\psi \partial_{\hat{\eta}^{\perp}(s)} (\varrho \lambda_{\text{a.c.}}))|_{\partial \ell_s \cap \Sigma} d\mathcal{H}^1(s) \\ &\quad - 2 \int_V \psi \det \nabla \nabla p dx, \\ \int_{\Gamma} \left[\int_{\ell_s} \partial_{\hat{\eta}^{\perp}(s)}^2 \psi \lambda_{\text{sing}} d\mathcal{H}^1 \right] d\vartheta(s) &= \int_{\Gamma} (\psi \partial_{\hat{\eta}^{\perp}(s)} \lambda_{\text{sing}})|_{\partial \ell_s \cap \Sigma} d\vartheta(s). \end{aligned}$$

Setting these formulas into the identity above and cancelling like terms proves that

$$0 = \int_{\Gamma} (\psi \partial_{\hat{\eta}^{\perp}(s)} (\varrho \lambda_{\text{a.c.}}))|_{\partial \ell_s \cap \Sigma} d\mathcal{H}^1(s) + \int_{\Gamma} (\psi \partial_{\hat{\eta}^{\perp}(s)} \lambda_{\text{sing}})|_{\partial \ell_s \cap \Sigma} d\vartheta(s)$$

for all $\psi \in C_c^{\infty}(V)$. The conditions in (5.70) now follow from the correspondence between $z \in V \cap \Sigma$ and $s_{\pm} \in \Gamma_{\pm}$, as $\mathcal{H}^1 \perp \vartheta$ on $\Gamma = \Gamma_+ \cup \Gamma_-$. \square

Stable lines meeting at an interior point. We end with the possibility in Panel (c) of Fig. 5. Let O be the ordered set of φ from (5.4), and suppose there exists a point $x_0 \in \Omega$ and an open set $V \subset O \cup \{x_0\}$ such that

$$x_0 \in V \quad \text{and} \quad V \setminus \{x_0\} = \cup_{s \in \Gamma} \ell_s \quad \text{where} \quad \ell_s \parallel \hat{e}_r(s) \quad \forall s \in \Gamma. \quad (5.71)$$

We use (r, θ) to denote polar coordinates about x_0 , and $\{\hat{e}_r, \hat{e}_{\theta}\}$ for the associated orthonormal frame. The set Γ is understood to meet each stable line it indexes transversely and exactly once; we take it to be diffeomorphic to the unit circle S^1 . It follows that

$$\nabla \nabla \varphi = \frac{\zeta(\theta)}{r} \hat{e}_{\theta} \otimes \hat{e}_{\theta} dx \quad \text{on } V$$

where $\zeta \in L^2((0, 2\pi))$ is locally uniformly positive a.e. Evidently, x_0 belongs to the singular set Σ of φ , as $1/r$ is not square integrable on any neighborhood of the origin. Denote $\partial_r = \partial_{\hat{e}_r}$.

Corollary 5.3. *Suppose φ admits a point $x_0 \in \Omega$ and some open set $V \subset O \cup \{x_0\}$ satisfying (5.71), and let $\mu \in \mathcal{M}_+(\Omega; \text{Sym}_2)$ solve (5.1). Then*

$$\mu = \lambda \hat{e}_{\theta} \otimes \hat{e}_{\theta} dx \quad \text{on } V,$$

where $\lambda : V \rightarrow [0, \infty)$ is the unique weak solution of the Cauchy problem

$$\begin{cases} -\frac{1}{2r} \partial_r^2(r\lambda) = \det \nabla \nabla p & \text{on } \ell_s \\ r\lambda = \partial_r(r\lambda) = 0 & \text{at } \partial \ell_s \cap \{x_0\} \end{cases}$$

upon restriction to \mathcal{H}^1 -a.e. ℓ_s .

Proof. We follow the same outline as the proof of Corollary 5.2, with the details being modified slightly to make up for the fact that $\nabla\nabla\varphi$ lacks a singular part. The first step will be to show that

$$\mu(\{x_0\}) = 0. \quad (5.72)$$

At the same time, Lemmas 5.2 and 5.3 yield the formula

$$\mu = \lambda_{\text{a.c.}} \hat{e}_\theta \otimes \hat{e}_\theta dx + \int_{\Gamma} \lambda_{\text{sing}} \hat{e}_\theta \otimes \hat{e}_\theta \mathcal{H}^1 \lfloor \ell_s d\vartheta(s) \quad \text{on } V \setminus \{x_0\} \quad (5.73)$$

where $\lambda_{\text{a.c.}}$ and λ_{sing} solve the ODEs (5.14) and (5.15). It is here that we use the unique continuation-type result from Remark 5.3. This allows to prove (5.73) by finding two neighborhoods $V_i = \cup_{\Gamma_i} \ell_s$ of the form (5.11) with $V \setminus \{x_0\} = V_1 \cup V_2$ and $\Gamma \cap V_i = \Gamma_i$, and by applying Lemma 5.3 to each V_i . That the individual disintegration formulas agree on $V_1 \cap V_2$ is assured by the remark. The final step of the proof is to establish the initial conditions

$$r\lambda_{\text{a.c.}}|_{\partial\ell_s}(x_0) = \partial_r(r\lambda_{\text{a.c.}})|_{\partial\ell_s}(x_0) = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } s, \quad (5.74)$$

$$\lambda_{\text{sing}}|_{\partial\ell_s}(x_0) = \partial_r\lambda_{\text{sing}}|_{\partial\ell_s}(x_0) = 0 \quad \text{for } \vartheta\text{-a.e. } s. \quad (5.75)$$

We use the first complementary slackness condition in (5.1), which implies that

$$0 = \lim_{\delta \rightarrow 0} \int_V \left| \left\langle \frac{\zeta(\theta)}{r} \hat{e}_r \otimes \hat{e}_r, \mu_\delta \right\rangle \right| dx. \quad (5.76)$$

Note $\{\mu_\delta\}_{\delta>0}$ are given by (1.32) where the mollifying kernel $\rho \in C_c^\infty(B_1)$ is non-negative and integrates to one. As in the previous two proofs, we take $\rho > 0$ nearby zero (see Remark 1.3).

We start with the proof of (5.72). Given any $x \in V$ we can write that

$$\mu_\delta(x) \geq \rho \left(\frac{x - x_0}{\delta} \right) \mu(\{x_0\}),$$

since ρ and μ are non-negative. It follows that

$$\int_V \left\langle \frac{\zeta}{r} \hat{e}_r \otimes \hat{e}_r, \mu_\delta \right\rangle dx \geq \frac{1}{\delta} \left\langle \int_{B_\delta(x_0)} \zeta \frac{\delta}{r} \hat{e}_r \otimes \hat{e}_r(x) \rho \left(\frac{x - x_0}{\delta} \right) \frac{dx}{\delta^2}, \mu(\{x_0\}) \right\rangle$$

for all sufficiently small $\delta > 0$. The integral on the lefthand side tends to zero by (5.76). Recall ζ is locally uniformly positive a.e., i.e., every $\theta \in (0, 2\pi)$ admits a neighborhood on which $\zeta \geq c(\theta) > 0$ a.e. Recall also that we chose to take $\rho > 0$ nearby zero. Multiplying by δ and sending $\delta \rightarrow 0$ proves that

$$\langle \hat{x} \otimes \hat{x}, \mu(\{x_0\}) \rangle = 0 \quad \forall \hat{x} \in S^1.$$

It follows that $\mu(\{x_0\}) = 0$ as claimed in (5.72).

Next, we determine μ on $V \setminus \{x_0\}$. The formula (5.73) follows as above. We proceed to show the initial conditions (5.74) and (5.75). First, we consider the traces of $r\lambda_{\text{a.c.}}$ and λ_{sing} . Given $x \in V$,

$$\begin{aligned}\mu_\delta(x) &\geq \int_{V \setminus \{x_0\}} \frac{1}{\delta^2} \rho\left(\frac{x-y}{\delta}\right) d\mu(y) \\ &= \int_{\Gamma} \left[\int_{y \in \ell_s} \frac{1}{\delta^2} \rho\left(\frac{x-y}{\delta}\right) \hat{e}_\theta \otimes \hat{e}_\theta(y) r\lambda_{\text{a.c.}}(y) d\mathcal{H}^1(y) \right] d\mathcal{H}^1(s) \\ &\quad + \int_{\Gamma} \left[\int_{y \in \ell_s} \frac{1}{\delta^2} \rho\left(\frac{x-y}{\delta}\right) \hat{e}_\theta \otimes \hat{e}_\theta(y) \lambda_{\text{sing}}(y) d\mathcal{H}^1(y) \right] d\vartheta(s)\end{aligned}$$

by (5.13) and (5.73). It follows from Fubini's theorem that

$$\begin{aligned}\int_V \left\langle \frac{\zeta}{r} \hat{e}_r \otimes \hat{e}_r, \mu_\delta \right\rangle dx &\geq \int_{\Gamma} \left[\int_{y \in \ell_s} \int_{x \in B_\delta(x_0)} \zeta \frac{\delta}{r} (x) |\hat{e}_r(x) \cdot \hat{e}_\theta(y)|^2 r\lambda_{\text{a.c.}}(y) \right. \\ &\quad \left. \rho\left(\frac{x-y}{\delta}\right) \frac{dx d\mathcal{H}^1(y)}{\delta^3} \right] d\mathcal{H}^1(s) \\ &\quad + \int_{\Gamma} \left[\int_{y \in \ell_s} \int_{x \in B_\delta(x_0)} \zeta \frac{\delta}{r} (x) |\hat{e}_r(x) \cdot \hat{e}_\theta(y)|^2 \lambda_{\text{sing}}(y) \right. \\ &\quad \left. \rho\left(\frac{x-y}{\delta}\right) \frac{dx d\mathcal{H}^1(y)}{\delta^3} \right] d\vartheta(s)\end{aligned}$$

for all small enough $\delta > 0$. Again, the lefthand side limits to zero by (5.76). Consider the terms on the right. The function ζ is locally uniformly positive a.e. Even though $\hat{e}_r(x) \cdot \hat{e}_\theta(y) = 0$ when $x = y$, the typical value of $|\hat{e}_r(x) \cdot \hat{e}_\theta(y)|$ is bounded from zero. The kernel $\rho > 0$ nearby zero. Passing to the limit via Fatou's lemma proves that

$$0 = \int_{\Gamma} r\lambda_{\text{a.c.}}|_{\partial\ell_s}(x_0) d\mathcal{H}^1(s) = \int_{\Gamma} \lambda_{\text{sing}}|_{\partial\ell_s}(x_0) d\vartheta(s).$$

As the integrands are non-negative they must vanish a.e. The first parts of (5.74) and (5.75) are proved.

Finally, we check that $\partial_r(r\lambda_{\text{a.c.}})$ and $\partial_r\lambda_{\text{sing}}$ vanish at x_0 . This last part of the proof is directly analogous to that of Corollary 5.2. In fact, it is so similar that we omit most of the details for brevity's sake. First, observe that

$$\begin{aligned}0 &\leq \partial_r(r\lambda_{\text{a.c.}})|_{\partial\ell_s}(x_0) \quad \text{for } \mathcal{H}^1\text{-a.e. } s \in \Gamma, \\ 0 &\leq \partial_r\lambda_{\text{sing}}|_{\partial\ell_s}(x_0) \quad \text{for } \vartheta\text{-a.e. } s \in \Gamma.\end{aligned}\tag{5.77}$$

These hold since the functions $r\lambda_{\text{a.c.}}$ and λ_{sing} are non-negative, while their traces at x_0 were shown to vanish in the previous step. Second, note the matching conditions

$$0 = \int_{\Gamma} \partial_r(r\lambda_{\text{a.c.}})|_{\partial\ell_s}(x_0) d\mathcal{H}^1(s) = \int_{\Gamma} \partial_r\lambda_{\text{sing}}|_{\partial\ell_s}(x_0) d\vartheta(s)\tag{5.78}$$

which arise from the first part of (5.1) as $\hat{\psi}_\theta$ is discontinuous at x_0 . Together, (5.77) and (5.78) yield the remaining parts of (5.74) and (5.75).

We briefly explain the matching conditions. Let $\psi \in C_c^\infty(V)$ have $\psi(x_0) = 1$. Combining the first part of (5.1) with (5.13) and (5.73) yields that

$$\begin{aligned} \int_V \psi \det \nabla \nabla p \, dx &= \int_\Gamma \left[\int_{\ell_s} -\frac{1}{2} \partial_r^2 \psi r \lambda_{\text{a.c.}} \, d\mathcal{H}^1 \right] d\mathcal{H}^1(s) \\ &\quad + \int_\Gamma \left[\int_{\ell_s} -\frac{1}{2} \partial_r^2 \psi \lambda_{\text{sing}} \, d\mathcal{H}^1 \right] d\vartheta(s). \end{aligned}$$

Integrating by parts with the ODEs (5.14) and (5.15) proves the identities

$$\begin{aligned} \int_\Gamma \left[\int_{\ell_s} \partial_r^2 \psi r \lambda_{\text{a.c.}} \, d\mathcal{H}^1 \right] d\mathcal{H}^1(s) &= \int_\Gamma \partial_r(r \lambda_{\text{a.c.}})|_{\partial \ell_s}(x_0) \, d\mathcal{H}^1(s) \\ &\quad - 2 \int_V \psi \det \nabla \nabla p \, dx, \\ \int_\Gamma \left[\int_{\ell_s} \partial_r^2 \psi \lambda_{\text{sing}} \, d\mathcal{H}^1 \right] d\vartheta(s) &= \int_\Gamma \partial_r \lambda_{\text{sing}}|_{\partial \ell_s}(x_0) \, d\vartheta(s). \end{aligned}$$

The desired conditions (5.78) follow. \square

6. Application to Shells with Curvature of Known Sign

This final section combines all of our previous results to deduce the patterns seen in weakly curved, floating shells. In particular, we derive the diagrams in Fig. 2 and use them to demonstrate our method of stable lines. This should serve to complement the general presentation of the method in Section 5. In addition to the basic assumptions in (1.14a), each of the examples we discuss will be subject to the simplifying hypotheses that

Ω is simply connected and $\det \nabla \nabla p$ is a.e. of one sign.

As usual, when we refer to “optimal” μ we mean solutions of the primal problem(s) in (4.1). Equivalently, these are non-negative solutions of the boundary value problem (5.1) where φ solves the dual. Our earlier results show that, under the assumptions at the start of Section 1.2, such μ are nothing other than the defect measures of the almost minimizers of $E_{b,k,\gamma}$. Any reference to almost minimizers is contingent upon the Γ -convergence in Theorem 1.1.

This section is organized as follows. We begin in Section 6.1 with the general task of solving the dual problem

$$\max_{\substack{\varphi: \mathbb{R}^2 \rightarrow \mathbb{R} \\ \varphi \text{ is convex} \\ \varphi = \frac{1}{2}|x|^2 \text{ on } \mathbb{R}^2 \setminus \Omega}} \int_\Omega \left(\varphi - \frac{1}{2}|x|^2 \right) \det \nabla \nabla p \, dx \quad (6.1)$$

under the assumption that the shell is positively or negatively curved. We show how this boils down to finding either the largest or the smallest convex extension φ_+ or

φ_- of $\frac{1}{2}|x|^2$ into Ω , and obtain two more or less explicit geometric optimization procedures for doing so. Thus, we prove Proposition 1.1.

We then go on to the examples. Section 6.2 treats various positively curved shells, including the ones depicted in Panel (a) of Fig. 2. Applying Corollary 5.1, we learn that optimal μ are uniquely determined on the ordered set O of φ_+ , and furthermore that they satisfy

$$\mu = \lambda \hat{\eta} \otimes \hat{\eta} dx \quad \text{on } O, \quad \text{where} \quad \begin{cases} -\frac{1}{2\varrho} \partial_{\hat{\eta}^\perp}^2 (\varrho \lambda) = \det \nabla \nabla p & \text{on } O \\ \varrho \lambda = 0 & \text{at } \partial O \cap \partial \Omega \end{cases} \quad (6.2)$$

Here, $\hat{\eta}$ is a suitable choice of normal to the stable lines of φ_+ . These will turn out to extend between points on $\partial \Omega$ throughout the set O where they are defined.

A parallel discussion of negatively curved shells is in Section 6.3. We show how the stable lines of φ_- follow the paths of quickest exit from Ω , as indicated in Panel (b) of Fig. 2. Such paths meet at the *medial axis*

$$M = \{x \in \Omega : d_{\partial \Omega}(x) = |x - y| \text{ for multiple } y \in \partial \Omega\} \quad (6.3)$$

shown in bold. Apparently, our negatively curved examples are such that their stable lines fill out the given shells. Applying Corollary 5.2 or Corollary 5.3, we consequently show that optimal μ are unique and that they satisfy

$$\mu = \lambda \nabla^\perp d_{\partial \Omega} \otimes \nabla^\perp d_{\partial \Omega} dx \quad \text{on } \Omega, \quad \text{where} \quad \begin{cases} -\frac{1}{2\varrho} \partial_{\nabla d_{\partial \Omega}}^2 (\varrho \lambda) = \det \nabla \nabla p & \text{on } \Omega \setminus \overline{M} \\ \varrho \lambda = \partial_{\nabla d_{\partial \Omega}} (\varrho \lambda) = 0 & \text{at } \overline{M} \end{cases} \quad (6.4)$$

We close with a general conjecture on the (conditional) uniqueness of optimal μ .

To be clear, the systems in (6.2) and (6.4), and that we derive in the examples below, are only abbreviated versions of the ones implied by Corollaries 5.1–5.3: they indicate a situation where $\varrho \lambda$ restricts to \mathcal{H}^1 -a.e. stable line ℓ_s as the unique weak solution of an appropriate two-point boundary value or Cauchy problem (obtained by restoring the s -dependences as in the corollaries). To lighten the notation, we refer to such abbreviated systems throughout.

6.1. Optimal Airy Potentials and Their Stable Lines

We begin by proving Proposition 1.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded, Lipschitz domain that is simply connected, and let $p \in W^{2,2}(\Omega)$ be such that $\det \nabla \nabla p$ is a.e. of one sign. Define the functions $\varphi_{\pm} : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\begin{aligned} \varphi_+(x) &= \sup \left\{ \varphi(x) : \varphi \text{ is convex on } \mathbb{R}^2 \text{ and equals } \frac{1}{2}|\cdot|^2 \text{ on } \mathbb{R}^2 \setminus \Omega \right\}, \\ \varphi_-(x) &= \inf \left\{ \varphi(x) : \varphi \text{ is convex on } \mathbb{R}^2 \text{ and equals } \frac{1}{2}|\cdot|^2 \text{ on } \mathbb{R}^2 \setminus \Omega \right\} \end{aligned}$$

for all $x \in \mathbb{R}^2$. Clearly, if φ is admissible in the dual problem (6.1) then

$$\varphi_-(x) \leq \varphi(x) \leq \varphi_+(x) \quad \forall x. \quad (6.5)$$

Lemma 6.1. *The functions φ_+ and φ_- are convex, and are equal to $\frac{1}{2}|x|^2$ on $\mathbb{R}^2 \setminus \Omega$. Therefore, they are the largest and smallest convex extensions of $\frac{1}{2}|x|^2$ into Ω . Furthermore, the formulas (1.43) and (1.44) hold: given $x \in \Omega$,*

$$\varphi_+(x) = \min_{\{y_i\} \subset \partial\Omega} \sum_{i=1}^3 \theta_i \frac{1}{2}|y_i|^2 \quad (6.6)$$

where the minimization is taken over all pairs and triples $\{y_i\} \subset \partial\Omega$ satisfying

$$x = \sum_i \theta_i y_i \quad \text{where } \{\theta_i\} \subset (0, 1) \text{ satisfies } \sum_i \theta_i = 1;$$

also

$$\varphi_-(x) = \frac{1}{2}|x|^2 - \frac{1}{2}d_{\partial\Omega}^2(x) \quad \text{where } d_{\partial\Omega}(x) = \min_{y \in \partial\Omega} |x - y|. \quad (6.7)$$

Proof. The convexity of φ_+ is clear, as the pointwise supremum of convex functions is convex. It is also clear that $\varphi_+ = \frac{1}{2}|x|^2$ outside of Ω . The formula (6.6) now follows from the dual characterization of the convex envelope of a function as the infimum of convex combinations of its graph (see, e.g., [21, Theorem 2.35]). Indeed, we recognize from its definition that φ_+ is the largest convex function bounding the function $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}$ equal to ∞ on Ω and $\frac{1}{2}|x|^2$ on $\mathbb{R}^2 \setminus \Omega$ from below. That is, it is the convex envelope of \mathcal{U} . Applying the dual characterization, we get that

$$\begin{aligned} \varphi_+(x) &= \inf \left\{ \sum_{i=1}^3 \theta_i \mathcal{U}(y_i) : y_i \in \mathbb{R}^2 \text{ and } \theta_i \in [0, 1] \text{ for } i = 1, 2, 3, \right. \\ &\quad \left. x = \sum_{i=1}^3 \theta_i y_i, \sum_{i=1}^3 \theta_i = 1 \right\} \\ &= \inf \left\{ \sum_{i=1}^3 \theta_i \frac{1}{2}|y_i|^2 : y_i \in \mathbb{R}^2 \setminus \Omega \text{ and } \theta_i \in [0, 1] \text{ for } i = 1, 2, 3, \right. \\ &\quad \left. x = \sum_{i=1}^3 \theta_i y_i, \sum_{i=1}^3 \theta_i = 1 \right\}. \end{aligned}$$

The minimization can be parameterized by $\{y_i\}$, as once these have been chosen $\{\theta_i\}$ are determined. And, as $\frac{1}{2}|x|^2$ is strictly convex, the minimizing $\{y_i\} \subset \partial\Omega$ whenever $x \in \Omega$. This proves (6.6).

Now we discuss φ_- . We proceed in the opposite order, showing first that the function $\mathcal{L} : \mathbb{R}^2 \rightarrow \mathbb{R}$ equal to $\frac{1}{2}|x|^2 - \frac{1}{2}d_{\partial\Omega}^2$ on Ω and $\frac{1}{2}|x|^2$ on $\mathbb{R}^2 \setminus \Omega$ is convex, and then checking that it equals to φ_- . Note by its definition that

$$\mathcal{L}(x) = \frac{1}{2}|x|^2 - \frac{1}{2} \min_{y \notin \Omega} |x - y|^2 = \max_{y \notin \Omega} x \cdot y - \frac{1}{2}|y|^2.$$

Thus, \mathcal{L} is convex as it is the pointwise supremum of affine functions. To finish, we show that $\varphi \geq \mathcal{L}$ whenever φ is a convex extension of $\frac{1}{2}|x|^2$ into Ω . Clearly

this holds for $x \notin \Omega$, so fix some $x \in \Omega$. Let $y \in \partial\Omega$ be a point closest to x , and let z be on the line segment from x to y . Calling $t = |z - x|$, we obtain the bound

$$\varphi \left(x + t \frac{y - x}{d_{\partial\Omega}(x)} \right) \geq \frac{(y - x) \cdot y}{d_{\partial\Omega}(x)} (t - d_{\partial\Omega}(x)) + \frac{1}{2} |y|^2 = \frac{(y - x) \cdot y}{d_{\partial\Omega}(x)} t + \mathcal{L}(x).$$

Note in the last step we used that $\mathcal{L}(x) = x \cdot y - \frac{1}{2} |y|^2$. Setting $t = 0$ yields the desired inequality $\varphi(x) \geq \mathcal{L}(x)$. It follows from its definition that $\varphi_- = \mathcal{L}$. \square

Corollary 6.1. *The functions φ_+ and φ_- solve the dual problem (6.1) respectively when $\det \nabla \nabla p \geq 0$ and ≤ 0 a.e. Furthermore, if either of these inequalities is strict a.e., then (6.1) is only solved by the corresponding φ_+ or φ_- .*

Proof. We give the proof in the case that $\det \nabla \nabla p \geq 0$, as the other case is the same. Rearranging (6.5) shows that $\varphi_+ - \varphi \geq 0$ whenever φ is admissible for the dual. It follows that

$$\int_{\Omega} \left(\varphi_+ - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p \geq \int_{\Omega} \left(\varphi - \frac{1}{2} |x|^2 \right) \det \nabla \nabla p.$$

As φ_+ is admissible by Lemma 6.1, it is a maximizer. On the other hand, if φ is a maximizer it must be that

$$\int_{\Omega} (\varphi_+ - \varphi) \det \nabla \nabla p = 0.$$

If, in addition, $\det \nabla \nabla p > 0$ a.e., $\varphi = \varphi_+$ so that no other maximizer exists. \square

Proposition 1.1 is proved. Note we also established the uniqueness in Remark 1.5.

6.2. Positively Curved Shells

We are ready to solve for the patterns in Fig. 2. Here, we treat the positively curved shells shown in Panel (a), i.e., we let $p \in W^{2,2}(\Omega)$ satisfy

$$\det \nabla \nabla p \geq 0 \quad \text{a.e.}$$

We begin each example by producing the largest convex extension φ_+ of $\frac{1}{2} |x|^2$ into the given Ω . We identify its singular, flattened, ordered, and unconstrained sets Σ , F , O , and U as well as its stable lines $\{\ell_x\}$ following the definitions in Section 5.1. Then, we show how to apply Corollary 5.1 to characterize optimal μ . The end result is a proof that optimal μ are unique upon restriction to O , and that they satisfy a version of (6.2).

Our first example is the positively curved ellipse in Panel (a) of Fig. 2.

Example 6.1. (positively curved ellipse) Let $0 < b < a$ and take as the domain the ellipse

$$E = \left\{ (x_1, x_2) : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1 \right\}.$$

We claim that

$$\varphi_+(x) = \frac{1}{2} \left(b^2 + \left(1 - \frac{b^2}{a^2} \right) x_1^2 \right), \quad x \in E. \quad (6.8)$$

It is straightforward to check using Lemma 4.5 that (6.8) defines a convex extension of $\frac{1}{2}|x|^2$ into E . It satisfies

$$\nabla \nabla \varphi_+ = \left(1 - \frac{b^2}{a^2} \right) \hat{e}_1 \otimes \hat{e}_1 \, dx \quad \text{on } E$$

so that its Hessian is non-negative and uniformly bounded, and it equals to $\frac{1}{2}|x|^2$ at ∂E . Since

$$\hat{v} \cdot [\nabla \varphi_+] = \frac{\left(\frac{x_1}{a^2}, \frac{x_2}{b^2} \right)}{\sqrt{\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}}} \cdot \left(x_1 - \left(1 - \frac{b^2}{a^2} \right) x_1, x_2 \right) = b^2 \sqrt{\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}} > 0 \quad \text{at } \partial E, \quad (6.9)$$

we conclude that (6.8) is admissible. Now, we must verify it is the largest convex extension. Given $x \in E$ there is a unique line containing it and parallel to \hat{e}_2 . That line intersects ∂E at two points $\{(x_1, \pm x_2(x_1))\}$, and

$$x = \theta(x_1, x_2(x_1)) + (1 - \theta)(x_1, -x_2(x_1)) \quad \text{for some } \theta \in (0, 1).$$

If φ is any convex extension of $\frac{1}{2}|x|^2$ into E it follows that

$$\varphi(x) \leq \theta \frac{1}{2}|(x_1, x_2(x_1))|^2 + (1 - \theta) \frac{1}{2}|(x_1, -x_2(x_1))|^2 = \frac{1}{2} \left(x_1^2 + b^2 \left(1 - \frac{x_1^2}{a^2} \right) \right).$$

Thus (6.8) is indeed the formula for φ_+ on E .

Having obtained φ_+ , we note it partitions E according to

$$O = E \quad \text{and} \quad \Sigma = F = U = \emptyset.$$

In particular, the entire ellipse is ordered. Its stable lines $\{\ell_x\}$ are the lines referred to above or, rather, the portion of them within E . We finish by applying Corollary 5.1 to characterize optimal μ . Given any sufficiently small $\delta > 0$, we claim the hypotheses (5.51)–(5.54) hold with $\varphi = \varphi_+$ and for

$$V = \{x \in E : -a + \delta < x_1 < a - \delta\}, \quad \zeta = 1 - \frac{b^2}{a^2}, \quad \hat{\eta} = \hat{e}_1.$$

That $\hat{v} \cdot [\nabla \varphi_+] \geq c > 0$ follows from (6.9). The uniform transversality condition $|\hat{t} \cdot \hat{e}_1| \geq c(\delta) > 0$ is clear. Note we introduced the cutoff length δ to deal with the fact that $|\hat{t} \cdot \hat{e}_1| \rightarrow 0$ as $x_1 \rightarrow \pm a$. As the stable lines are parallel, the change of measure factor ϱ from (5.13) remains constant and non-zero along each of them; it is a function of x_1 whose exact form is immaterial and depends on the choice of indexing curve Γ . Applying Corollary 5.1 and taking $\delta \rightarrow 0$ we conclude that optimal μ are unique, and that they satisfy

$$\mu = \lambda \hat{e}_1 \otimes \hat{e}_1 \, dx \quad \text{on } E, \quad \text{where} \quad \begin{cases} -\frac{1}{2} \partial_2^2 \lambda = \det \nabla \nabla p & \text{on } E \\ \lambda = 0 & \text{at } \partial E \end{cases}.$$

This completes our discussion of the positively curved ellipse.

Next, we consider the positively curved disc. The corresponding entry in Fig. 2 is blank, indicating a complete lack of stable lines. As a result, we will be able to construct infinitely many optimal μ .

Example 6.2. (positively curved disc) Let $a > 0$ and take as the domain the disc

$$D = \{(x_1, x_2) : x_1^2 + x_2^2 < a^2\}.$$

Since ∂D is a level set of $\frac{1}{2}|x|^2$, it follows immediately from (6.6) that

$$\varphi_+(x) = \frac{1}{2}a^2, \quad x \in D.$$

Hence

$$\nabla \nabla \varphi_+ = 0 \quad \text{on } D$$

and the corresponding partition of D degenerates into

$$U = D \quad \text{and} \quad \Sigma = F = O = \emptyset.$$

The entire disc is unconstrained. The ordered set is empty and there are no stable lines. Nevertheless, optimal μ can still be characterized as non-negative solutions of (5.1), which degenerates into the system

$$\begin{cases} -\frac{1}{2} \operatorname{curl} \operatorname{curl} \mu = \det \nabla \nabla p & \text{on } D \\ \langle \hat{\tau} \otimes \hat{\tau}, \mu \rangle = 0 & \text{at } \partial D \end{cases}. \quad (6.10)$$

Note the PDE holds in the sense of distributions, while the boundary conditions hold in the regularized sense, i.e.,

$$0 = \lim_{\delta \rightarrow 0} \int_{\partial D} |\langle \hat{\tau} \otimes \hat{\tau}, \mu_\delta \rangle| \, ds \quad (6.11)$$

where $\{\mu_\delta\}_{\delta > 0}$ are as in (1.32). We used that $\hat{\tau} \cdot [\nabla \varphi_+] = a > 0$ at ∂D .

As a boundary value problem, (6.10) is severely underdetermined. Here is an example of the non-uniqueness it permits: given any decomposition of D into a disjoint family of open line segments with boundary points on ∂D , and letting $\hat{\eta} \in \operatorname{Lip}_{\operatorname{loc}}(D; S^1)$ be constant along and perpendicular to the segments, we claim that the measure

$$\mu = \lambda \hat{\eta} \otimes \hat{\eta} \, dx \quad \text{on } D, \quad \text{where} \quad \begin{cases} -\frac{1}{2\varrho} \partial_{\hat{\eta}^\perp}^2 (\varrho \lambda) = \det \nabla \nabla p & \text{in } D \\ \varrho \lambda = 0 & \text{at } \partial D \end{cases} \quad (6.12)$$

is a solution of (6.10). Hence, it is optimal. Note $\varrho : D \rightarrow (0, \infty)$ is defined via (5.13). That μ is indeed a solution can be checked using the methods of Section 5. Recall the system in (6.12) stands for a family of two-point boundary value problems indexed by the given segments (just as in Corollary 5.1). The PDE in (6.10) now follows more or less directly from the notation introduced in Section 5.2 above Lemma 5.3. Establishing the boundary conditions takes a bit more work: we achieve

them by reversing the proof of Corollary 5.1. Start by indexing the segments, which we still call $\{\ell_s\}$, with a curve $\Gamma \subset D$ such that $D = \cup_{s \in \Gamma} \ell_s$ where $s \mapsto \ell_s$ is one-to-one and $s \in \ell_s$. It will suffice to take Γ to be Lipschitz. Rewriting (6.11) using disintegration of measure, we must check that

$$0 = \lim_{\delta \rightarrow 0} \int_{s \in \Gamma} \left[\int_{x \in \partial D} \int_{y \in \ell_s} |\hat{\tau}(x) \cdot \hat{\eta}(y)|^2 \varrho \lambda(y) \rho \left(\frac{x-y}{\delta} \right) \frac{d\mathcal{H}^1(x) d\mathcal{H}^1(y)}{\delta^2} \right] d\mathcal{H}^1(s),$$

where $\rho \in C_c^\infty(B_1)$ is non-negative and integrates to one. We do so via the dominated convergence theorem.

The bracketed integrals tend to zero \mathcal{H}^1 -a.e. due to the boundary conditions in (6.12). Indeed, $\varrho \lambda \in W^{2,1}(\ell_s)$ upon restriction to a.e. ℓ_s , and the corresponding traces at $\partial \ell_s$ vanish. Integrating the ODEs from (6.12) along a.e. ℓ_s yields that $|\varrho \lambda(y)| \lesssim d(y, \partial \ell_s) \|\varrho \det \nabla \nabla p\|_{L^1(\ell_s, \mathcal{H}^1)}$ for $y \in \ell_s$. If $(x, y) \in \partial D \times \ell_s$ satisfies $|x-y| < \delta$, then $d(y, \partial \ell_s) \lesssim_a (\delta/\mathcal{H}^1(\ell_s)) \wedge \mathcal{H}^1(\ell_s)$ and $|\hat{\tau}(x) \cdot \hat{\eta}(y)| \lesssim_a \mathcal{H}^1(\ell_s) \vee \delta$. Hence,

$$\int_{\substack{(x,y) \in \partial D \times \ell_s \\ |x-y| < \delta}} |\hat{\tau}(x) \cdot \hat{\eta}(y)|^2 \varrho \lambda(y) \frac{d\mathcal{H}^1(x) d\mathcal{H}^1(y)}{\delta^2} \lesssim_a \|\varrho \det \nabla \nabla p\|_{L^1(\ell_s, \mathcal{H}^1)}$$

for a.e. s . The righthand side is integrable by (5.13) since $p \in W^{2,2}$. Sending $\delta \rightarrow 0$ completes the proof.

Even though the segments used above may remind of stable lines, we prefer not to call them as such. For one, the decomposition $D = \cup_{s \in \Gamma} \ell_s$ is not unique. Each such decomposition gives rise to a different optimal μ and, correspondingly, to a different sequence of almost minimizers of $E_{b,k,\gamma}$ under the assumptions at the start of Section 1.2. In some asymptotic sense, this is the opposite of stability. Also, these are not the only optimal μ . In particular, the set of solutions of (6.10) is convex. Taking convex combinations of the measures in (6.12), we deduce the existence of optimal μ that are everywhere rank two. The corresponding almost minimizers feature two-dimensional patterns instead of one-dimensional, wrinkling-like ones. We wonder if the disordered positively curved discs from [78] can be understood using suitable solutions of (6.10). If no such μ represents the observed patterns, it would necessarily follow that they cannot be modeled as almost minimizers of $E_{b,k,\gamma}$, as least in the parameter regime (1.11).

The previous examples set the extremes: whereas a positively curved ellipse is totally ordered, a positively curved disc is totally unconstrained (save for boundary data). Our next two examples sit somewhere in-between. They address the triangle, square, and rectangle from Panel (a) of Fig. 2. What distinguishes the former shapes from the latter is the fact that triangles and squares admit inscribed circles, whereas rectangles do not.

Example 6.3. (positively curved tangential polygons) A *tangential polygon* is a convex polygon that admits an inscribed circle, known as its *incircle*. Every regular polygon is tangential; more generally, a convex polygon is tangential if and only if its angle bisectors intersect at a distinguished point. This point is called the *incenter*, being the center of the incircle just defined. Given a tangential polygon P , we call

its *contact polygon* P' the convex polygon whose vertices are the points of contact of the incircle with P . Thus, P decomposes into a disjoint union of its contact polygon and finitely many leftover isosceles triangles, one for each vertex.

Now let P be a tangential polygon with vertices $a_1, \dots, a_n \in \mathbb{R}^2$ and interior angles $\alpha_1, \dots, \alpha_n$. Let its incenter be at the origin, and call the radius of its incircle a . Let P' be the contact polygon defined above. The remainder $P \setminus P'$ divides into n isosceles triangles, which we label as T_i for $i = 1, \dots, n$. The labeling is such that the i th vertex of the original polygon a_i is a vertex of the i th triangle T_i . For use in what follows, we take P' to be closed (relative to P) and let each T_i be open. After a fairly straightforward but somewhat lengthy argument, one finds that

$$\varphi_+(x) = \begin{cases} \frac{1}{2}a^2 & x \in P' \\ \frac{1}{2} \left((x \cdot \hat{a}_i)^2 + \tan^2\left(\frac{\alpha_i}{2}\right) (|a_i| - x \cdot \hat{a}_i)^2 \right) & x \in T_i, \quad i = 1, \dots, n \end{cases}$$

for $x \in P$. From this it follows that

$$\nabla \nabla \varphi_+ = \sum_{i=1}^n \left(1 + \tan^2\left(\frac{\alpha_i}{2}\right) \right) \hat{a}_i \otimes \hat{a}_i \mathbb{1}_{T_i} dx \quad \text{on } P.$$

The absolute continuity follows from the fact that P' is the contact polygon of P . Regarding the partition implied by φ_+ , we find that

$$O = \cup_i T_i, \quad U = P', \quad \sigma = F = \emptyset.$$

The triangles T_i are ordered, whereas P' is unconstrained. The stable lines belonging to the i th triangle T_i lie perpendicular to \hat{a}_i and extend from ∂P to ∂P .

Having identified φ_+ , we can apply Corollary 5.1 to characterize optimal μ . Note (5.51)–(5.53) holds with $\varphi = \varphi_+$ and for

$$V = T_i, \quad \zeta = 1 + \tan^2\left(\frac{\alpha_i}{2}\right), \quad \hat{\eta} = \hat{a}_i$$

for $i = 1, \dots, n$. The uniform transversality condition from (5.54) is easily checked, as $|\hat{\tau} \cdot \hat{a}_i| \geq c > 0$. Since

$$\nabla \varphi_+ = x \cdot \hat{a}_i \hat{a}_i - \tan^2\left(\frac{\alpha_i}{2}\right) (|a_i| - x \cdot \hat{a}_i) \hat{a}_i, \quad x \in T_i$$

we see that

$$\hat{v} \cdot [\nabla \varphi_+] = x \cdot \hat{a}_i^\perp \hat{v} \cdot \hat{a}_i^\perp + \tan^2\left(\frac{\alpha_i}{2}\right) (|a_i| - x \cdot \hat{a}_i) \hat{v} \cdot \hat{a}_i \geq 0 \quad \text{at } \partial T_i \cap \partial P.$$

Evidently, the first part of (5.54) fails for V as it allows for $x \rightarrow a_i$. However, this is easy to fix: as in Example 6.1, we can introduce a small cutoff length $\delta > 0$ and modify V such that $\hat{v} \cdot [\nabla \varphi_+] \geq c(\delta) > 0$. Note φ is constant along the stable lines as they remain parallel within each T_i . Applying Corollary 5.1 and taking $\delta \rightarrow 0$ yields that

$$\mu = \lambda_i \hat{a}_i \otimes \hat{a}_i dx \quad \text{on } T_i, \quad \text{where} \quad \begin{cases} -\frac{1}{2} \partial_{\hat{a}_i^\perp}^2 \lambda_i = \det \nabla \nabla p & \text{on } T_i \\ \lambda = 0 & \text{at } \partial T_i \cap \partial P \end{cases}$$

for $i = 1, \dots, n$. In particular, any two optimal μ agree upon restriction to $\cup_i T_i$. Much less is known at present regarding μ on P' .

Most polygons do not admit an inscribed circle, i.e., they fail to be tangential. Nevertheless, the arguments appearing in the previous example can be adapted to handle a more general case.

Example 6.4. (positively curved rectangle) Let $0 < b < a$ and consider the rectangle

$$R = \{(x_1, x_2) : -a < x_1 < a, -b < x_2 < b\}.$$

Although R does not admit an inscribed circle, there does exist a one-parameter family of maximally contained circles given by

$$C_t = \{(x_1, x_2) : (x_1 - t)^2 + x_2^2 = b^2\}, \quad -(b - a) \leq t \leq b - a.$$

The left and rightmost circles

$$C_l = C_{-(b-a)} \quad \text{and} \quad C_r = C_{b-a}$$

play a role analogous to that of the incircle above. Both C_l and C_r touch R at exactly three points: call the triangles formed by these points T_l and T_r . The remainder is made up of four $45 - 45 - 90$ triangles T_{nw} , T_{sw} , T_{se} , and T_{ne} and one sub-rectangle R_c . The subscripts nw and so on indicate location as on a compass. It will probably be helpful to look at Fig. 2. There, T_l and T_r are indicated in blank, whereas the remaining triangles T_{nw} , T_{sw} , T_{se} , T_{ne} and sub-rectangle R_c are drawn with stripes. For use with what follows, we take each of these to be open with the exception of T_l and T_r , which we take to be closed relative to R .

All this being said, we claim that the largest convex extension φ_+ of $\frac{1}{2}|x|^2$ into R is given by solving

$$\begin{aligned} \partial_{(1,-1)}^2 \varphi_+ &= 0 \quad \text{on } T_{\text{ne}} \cup T_{\text{sw}}, & \partial_{(1,1)}^2 \varphi_+ &= 0 \quad \text{on } T_{\text{se}} \cup T_{\text{nw}}, \\ \partial_2^2 \varphi_+ &= 0 \quad \text{on } R_c, & \nabla \nabla \varphi_+ &= 0 \quad \text{on } T_l^\circ \cup T_r^\circ \end{aligned} \quad (6.13)$$

with the boundary data

$$\varphi_+ = \frac{1}{2}|x|^2 \quad \text{at } \partial R. \quad (6.14)$$

The function defined by (6.13) and (6.14) is piecewise quadratic, and is affine upon restriction to T_l and T_r . Clearly, it is a convex extension of $\frac{1}{2}|x|^2$ into R . To see it is the largest one, we must show it yields an upper bound on any other convex extension φ . Evidently $\varphi \leq \varphi_+$ on T_l and T_r since they are equal at their vertices. Now let $x \in R \setminus (T_l \cup T_r)$ and consider the largest open line segment containing x on which φ_+ is affine (see the corresponding entry in Fig. 2). The boundary of each such segment consists of two points $y_1, y_2 \in \partial R$. By convexity,

$$\varphi(x) \leq \theta \frac{1}{2}|y_1|^2 + (1 - \theta) \frac{1}{2}|y_2|^2 = \varphi_+(x),$$

where $x = \theta y_1 + (1 - \theta) y_2$. Hence, (6.13) and (6.14) indeed define the largest convex extension of $\frac{1}{2}|x|^2$ into R .

Moving on, we see that

$$\begin{aligned}\nabla \nabla \varphi_+ &= (1, 1) \otimes (1, 1) \mathbb{1}_{T_{\text{ne}} \cup T_{\text{sw}}} dx + (1, -1) \otimes (1, -1) \mathbb{1}_{T_{\text{se}} \cup T_{\text{nw}}} dx \\ &\quad + (1, 0) \otimes (1, 0) \mathbb{1}_{R_c} dx \quad \text{on } R.\end{aligned}$$

Hence, R is partitioned by φ_+ according as

$$O = \left(\bigcup_{\substack{\alpha \in \{\text{n}, \text{s}\} \\ \beta \in \{\text{e}, \text{w}\}}} T_{\alpha\beta} \right) \cup R_c, \quad U = T_{\text{l}} \cup T_{\text{r}}, \quad \Sigma = F = \emptyset.$$

The stable lines $\{\ell_x\}$ are parallel to $(1, -1)$ on $T_{\text{ne}} \cup T_{\text{sw}}$, $(1, 1)$ on $T_{\text{se}} \cup T_{\text{nw}}$, and $(0, 1)$ on R_c , and they extend between pairs of boundary points. So, we can apply Corollary 5.1 to characterize optimal μ on O . We leave the details to the reader this time, and simply point out that $\hat{v} \cdot [\nabla \varphi_+] \geq 0$ at ∂R , and that this bound degenerates only at the vertices of R . Uniform transversality is clear; also, φ is constant along each stable line as they are parallel within the connected components of O . The conclusion of Corollary 5.1 is that

$$\begin{aligned}\mu &= \lambda_{\text{nesw}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \otimes \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) dx \quad \text{on } T_{\text{ne}} \cup T_{\text{sw}}, \\ \mu &= \lambda_{\text{senw}} \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) \otimes \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) dx \quad \text{on } T_{\text{se}} \cup T_{\text{nw}}, \\ \mu &= \lambda_c (1, 0) \otimes (1, 0) dx \quad \text{on } R_c,\end{aligned}$$

where

$$\begin{aligned}\begin{cases} -\frac{1}{2} \partial^2_{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)} \lambda_{\text{nesw}} = \det \nabla \nabla p & \text{in } T_{\text{ne}} \cup T_{\text{sw}} \\ \lambda_{\text{nesw}} = 0 & \text{at } \partial(T_{\text{ne}} \cup T_{\text{sw}}) \cap \partial R \end{cases}, \\ \begin{cases} -\frac{1}{2} \partial^2_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} \lambda_{\text{senw}} = \det \nabla \nabla p & \text{in } T_{\text{se}} \cup T_{\text{nw}} \\ \lambda_{\text{senw}} = 0 & \text{at } \partial(T_{\text{se}} \cup T_{\text{nw}}) \cap \partial R \end{cases}, \\ \begin{cases} -\frac{1}{2} \partial^2_2 \lambda_c = \det \nabla \nabla p & \text{in } R_c \\ \lambda_c = 0 & \text{at } \partial R_c \cap \partial R \end{cases}.\end{aligned}$$

Optimal μ are uniquely determined on $T_{\text{ne}} \cup T_{\text{sw}}$, $T_{\text{se}} \cup T_{\text{nw}}$, and R_c .

In the previous examples, stable lines ended up being parallel within each connected component of O . Our last positively curved example exhibits non-parallel stable lines. It is the half-disc from Panel (a) of Fig. 2.

Example 6.5. (positively curved half-disc) Consider a disc with radius $a > 0$ and center $(0, a)$, and let

$$D_+ = \left\{ (x_1, x_2) : x_1^2 + (x_2 - a)^2 < a^2, x_2 > a \right\}.$$

Let (r, θ) denote polar coordinates about 0. Given $x \in D_+$, consider the ray parallel to $\hat{e}_r(x)$ that passes through x and begins at the origin. This ray intersects ∂D_+ at

two points, which we label as $p(x)$ and $q(x)$ where $|p| < |q|$. Note there always holds $|p||q| = 2a^2$. Using this, it is easy to check the identities

$$|p| = \frac{a}{\sin \theta} \quad \text{and} \quad |q| = 2a \sin \theta, \quad (6.15)$$

which will come in handy below.

We now claim that

$$\varphi_+(r, \theta) = \frac{1}{2}(|p| + |q|)r - \frac{1}{2}|p||q| \quad \text{on } D_+. \quad (6.16)$$

As we show at the end of this example,

$$\nabla \nabla \varphi_+ = \frac{a}{r \sin^3 \theta} \hat{e}_\theta \otimes \hat{e}_\theta \, dx \quad \text{on } D_+,$$

so that the partition of D_+ is given simply by

$$O = D_+ \quad \text{and} \quad \Sigma = F = U = \emptyset.$$

The half-disc is totally ordered. Its stable lines $\{\ell_x\}$ are the portions of the rays described above within D_+ . To identify μ we apply Corollary 5.1 with $\varphi = \varphi_+$ and with

$$V = D_+, \quad \zeta = \frac{a}{r \sin^3 \theta}, \quad \hat{\eta} = \hat{e}_\theta.$$

The hypotheses (5.51)–(5.53) are not difficult to check, but again we must be careful about (5.54). Note that

$$\nabla \varphi_+ = \frac{1}{2}(|p| + |q|) \hat{e}_r.$$

Hence,

$$\hat{v} \cdot [\nabla \varphi_+] = -\hat{e}_2 \cdot ((x_1, a) - \frac{1}{2}(|p| + |q|) \hat{e}_r) = \frac{1}{2}(|q| - |p|) \sin \theta \geq 0$$

at the bottom part of ∂D_+ , while at the top part

$$\begin{aligned} \hat{v} \cdot [\nabla \varphi_+] &= \left(\frac{x_1}{a}, \frac{x_2}{a} - 1 \right) \cdot \left(r - \frac{1}{2}(|p| + |q|) \right) (\cos \theta, \sin \theta) \\ &= \frac{1}{2}(|q| - |p|) \sin \theta \geq 0. \end{aligned}$$

Both inequalities are strict away from the corners where $|p| = |q|$. Thus, we can introduce a cutoff length $\delta > 0$ to be sent to zero as in the other examples. We get that $\hat{v} \cdot [\nabla \varphi_+] \geq c(\delta) > 0$. The uniform transversality condition $|\hat{\tau} \cdot \hat{e}_\theta| \geq c > 0$ holds. This time, the change of measure factor from (5.13) satisfies $\varrho = c_\Gamma(\theta)r$ where the exact form of c_Γ depends on the choice of indexing curve Γ . This is because the stable lines follow rays. Applying Corollary 5.1 and taking $\delta \rightarrow 0$ proves that optimal μ are unique, and that they satisfy

$$\mu = \lambda \hat{e}_\theta \otimes \hat{e}_\theta \, dx \quad \text{on } D_+, \quad \text{where} \quad \begin{cases} -\frac{1}{2r} \partial_r^2(r\lambda) = \det \nabla \nabla p & \text{in } D_+ \\ \lambda = 0 & \text{at } \partial D_+ \end{cases}$$

for the positively curved half-disc.

We finish by proving that (6.16) is indeed the largest convex extension of $\frac{1}{2}|x|^2$ into D_+ . Making use of convexity along the rays, one easily concludes that $\varphi \leq \varphi_+$ as in the other examples. Here, we focus on checking that the given φ_+ is a convex extension after all. We apply Lemma 4.5. First, note that $\varphi_+ = \frac{1}{2}|x|^2$ for $x \in \partial D_+$. In particular, when $r = |p|$ it follows from (6.16) that

$$\varphi_+(r, \theta) = \frac{1}{2}(|p| + |q|)|p| - \frac{1}{2}|p||q| = \frac{1}{2}|p|^2 = \frac{1}{2}r^2,$$

and similarly for $r = |q|$. That $\hat{v} \cdot [\nabla \varphi_+] \geq 0$ at ∂D_+ was shown above. Finally, we check that $\nabla \nabla \varphi_+ \geq 0$ on D^+ . Note that

$$\begin{aligned} \varphi_+(r, \theta) &= b_1(\theta)r + b_0, \quad \text{where } b_0 = -\frac{1}{2}|p||q| = -a^2 \\ \text{and } b_1 &= \frac{1}{2}(|p| + |q|) = \frac{a}{2} \left(2 \sin \theta + \frac{1}{\sin \theta} \right), \end{aligned}$$

due to (6.15). Differentiating twice yields that

$$\nabla \nabla \varphi_+ = \frac{1}{r}(b_1'' + b_1)\hat{e}_\theta \otimes \hat{e}_\theta = \frac{1}{r} \frac{a}{\sin^3 \theta} \hat{e}_\theta \otimes \hat{e}_\theta \geq 0.$$

The lemma now implies that φ_+ is admissible and the example is complete.

Before moving on to the negatively curved examples, we pause to reflect on the fact that the singular set Σ turned out to be empty in each of the examples above. Of course, this is related to the regularity of φ_+ and, ultimately, to the shape of $\partial \Omega$. As an example of what can be proved, we note that if Ω is uniformly convex and $\partial \Omega \in C^{1,1}$, then the results of [66, 79] on Alexandrov solutions to the Dirichlet problem $\det \nabla \nabla \varphi = 0$ in Ω and $\varphi = \frac{1}{2}|x|^2$ at $\partial \Omega$ imply that $\varphi_+ \in C_{\text{loc}}^{1,1}(\Omega)$. Of course, $\Sigma = \emptyset$ in such a case.

6.3. Negatively Curved Shells

We turn to the patterns in Panel (b) of Fig. 2. These were drawn assuming that $p \in W^{2,2}(\Omega)$ satisfies

$$\det \nabla \nabla p \leq 0 \quad \text{a.e.}$$

For each specified shell we obtain the smallest convex extension φ_- of $\frac{1}{2}|x|^2$ into Ω , which requires solving for the boundary distance function $d_{\partial \Omega}$ from (6.7). Its singular, flattened, ordered, and unconstrained sets Σ , F , O , and U follow, as do its stable lines $\{\ell_x\}$. Again, we refer to Section 5.1 for the relevant definitions.

Let us comment briefly on the role that the medial axis M from (6.3) plays. As our examples will show, the stable lines of φ_- follow the paths of quickest exit from Ω , i.e., they lie parallel to $\nabla d_{\partial \Omega}$ where they exist. Based on this and the formula for φ_- in (6.7), it is reasonable to expect that

$$\begin{aligned} \nabla \nabla \varphi_- &= (1 - d_{\partial \Omega} \Delta d_{\partial \Omega}) \nabla^\perp d_{\partial \Omega} \otimes \nabla^\perp d_{\partial \Omega} dx \\ &\quad + d_{\partial \Omega} |[\nabla d_{\partial \Omega}]| \hat{v}_M \otimes \hat{v}_M \mathcal{H}^1|_M \quad \text{on } \Omega \end{aligned} \quad (6.17)$$

if M is regular enough. (For a general discussion on the regularity of the medial axis, see [50].) We show a version of (6.17) in each example below, and use it to identify the aforementioned partition and the stable lines. Finally, we prove that optimal μ are unique and that they satisfy (6.4) by applying Corollary 5.2 or Corollary 5.3.

We start with the negatively curved disc from Panel (b) of Fig. 2.

Example 6.6. (negatively curved disc) Let

$$D = \left\{ (x_1, x_2) : x_1^2 + x_2^2 < a^2 \right\}$$

and observe its medial axis is the singleton

$$M = \{0\}.$$

Using polar coordinates (r, θ) , the boundary distance function is simply $d_{\partial D} = a - r$ and hence

$$\varphi_- = ar - \frac{1}{2}a^2 \quad \text{on } D.$$

Differentiating yields that

$$\nabla \nabla \varphi_- = \frac{a}{r} \hat{e}_\theta \otimes \hat{e}_\theta \, dx \quad \text{on } D.$$

Even though $\nabla \nabla \varphi_- \ll dx$, its density is not square integrable on any neighborhood of M . Therefore, D is partitioned by φ_- according as

$$\Sigma = \{0\}, \quad O = D \setminus \{0\}, \quad F = U = \emptyset.$$

The stable lines form rays parallel to $\nabla d_{\partial D} = -\hat{e}_r$.

Next, we apply Corollary 5.3 to identify optimal μ . Its hypothesis (5.71) holds with

$$x_0 = 0 \quad \text{and} \quad V = D.$$

We immediately conclude that the unique optimal μ for the negatively curved disc satisfies

$$\mu = \lambda \hat{e}_\theta \otimes \hat{e}_\theta \, dx \quad \text{on } D, \quad \text{where} \quad \begin{cases} -\frac{1}{2r} \partial_r^2(r\lambda) = \det \nabla \nabla p & \text{on } D \setminus \{0\} \\ r\lambda = \partial_r(r\lambda) = 0 & \text{at } 0 \end{cases}.$$

In particular, the almost minimizers of $E_{b,k,\gamma}$ must exhibit an (approximately) azimuthally symmetric response, of course subject to the assumptions at the start of Section 1.2 under which our Γ -convergence results hold.

Before moving on, we note that a similar result can be proved for the case of a flat disc attached to a weakly curved spherical substrate—a model problem that has been the focus of much previous research, including at least [11, 24, 34]. The conclusion is that optimal μ are uniquely determined, absolutely continuous, and parallel to $\hat{e}_\theta \otimes \hat{e}_\theta$ with a density as above. As far as we know, this yields the first mathematically rigorous proof that azimuthal wrinkling is energetically preferred in a problem absent tensile loads. That azimuthal wrinkling should be preferred has

often been explained as a consequence of symmetry (with the notable exception of [11] where it was derived via minimization, albeit for a problem on the borderline between tension- and curvature-driven). Just because a shell is naturally symmetric does not mean that it should remain so when pressed onto a substrate, even if it has the same symmetries as the shell: indeed, Example 6.2 shows that a positively curved disc confined to the plane admits infinitely many non-symmetric optimal μ and, correspondingly, infinitely many non-symmetric almost minimizers. Whether or not global minimizers must exhibit symmetry remains unknown.

Our next example concerns the ellipse in Panel (b) of Fig. 2. In lieu of producing an exact formula for $d_{\partial\Omega}$, we will make use of the following fact: $d_{\partial\Omega}$ is concave if and only if Ω is convex [3]. Note this is an example where the medial axis is strictly smaller than the singular set.

Example 6.7. (negatively curved ellipse) Let

$$E = \left\{ (x_1, x_2) : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1 \right\}$$

where $0 < b < a$. The boundary distance function $d_{\partial E}$ is smooth off of the closure of the medial axis

$$\overline{M} = \left\{ (x_1, 0) : |x_1| \leq a \left(1 - \frac{b^2}{a^2} \right) \right\},$$

and we find that

$$\begin{aligned} \nabla \nabla \varphi_- &= (1 - d_{\partial E} \Delta d_{\partial E}) \nabla^\perp d_{\partial E} \otimes \nabla^\perp d_{\partial E} dx \\ &\quad + d_{\partial E} |[\nabla d_{\partial E}]| \hat{v}_M \otimes \hat{v}_M \mathcal{H}^1[M] \quad \text{on } E \end{aligned}$$

as anticipated in (6.17). The function φ_- partitions E according as

$$\Sigma = \overline{M}, \quad O = E \setminus \overline{M}, \quad F = U = \emptyset$$

and the stable lines $\{\ell_x\}$ run parallel to $\nabla d_{\partial E}$. We now claim that there is a single optimal μ for the negatively curved ellipse, and that it satisfies

$$\begin{aligned} \mu &= \lambda \nabla^\perp d_{\partial E} \otimes \nabla^\perp d_{\partial E} dx \quad \text{on } E, \\ \text{where } &\begin{cases} -\frac{1}{2\varrho} \partial_{\nabla d_{\partial E}}^2(\varrho\lambda) = \det \nabla \nabla p & \text{on } E \setminus \overline{M} \\ \varrho\lambda = \partial_{\nabla d_{\partial E}}(\varrho\lambda) = 0 & \text{at } \overline{M} \end{cases} \end{aligned} \tag{6.18}$$

and where $\varrho : E \setminus \overline{M} \rightarrow (0, \infty)$ is given by (5.13). We proceed in two steps: first, we use Corollary 5.2 to identify μ off of the half-open line segments

$$L_\pm = \left\{ (x_1, 0) \in E : \pm x_1 \geq a \left(1 - \frac{b^2}{a^2} \right) \right\},$$

and then we check that $\mu = 0$ on L_\pm by a separate argument.

Recall ℓ_x denotes the stable line through x , given by $\ell_x = \{\theta z + (1 - \theta)y : \theta \in (0, 1)\}$ for $z \in \overline{M}$ and $y \in \partial E$. For any small enough $\delta > 0$, let

$$M_\delta = \left\{ (x_1, 0) : |x_1| < a \left(1 - \frac{b^2}{a^2} \right) - \delta \right\}$$

and observe the assumptions (5.58)–(5.62) of Corollary 5.2 hold with $\varphi = \varphi_-$ and for

$$V = M_\delta \cup \{x \in E \setminus \overline{M} : \partial \ell_x \cap M_\delta \neq \emptyset\}, \quad \zeta = 1 - d_{\partial E} \Delta d_{\partial E}, \quad \hat{\eta} = \nabla^\perp d_{\partial E}.$$

In particular as E is convex, $d_{\partial E}$ is concave so that $\zeta \geq 1$. Both $|\nabla d_{\partial E}|$ and $|\hat{\tau}_M \cdot \nabla^\perp d_{\partial E}|_{M_\pm}$ are bounded below by some $c(\delta) > 0$ uniformly on M_δ . Applying Corollary 5.2 and sending $\delta \rightarrow 0$ yields that

$$\begin{aligned} \mu &= \lambda \nabla^\perp d_{\partial E} \otimes \nabla^\perp d_{\partial E} dx \quad \text{on } E \setminus \{L_+ \cup L_-\}, \\ \text{where } &\begin{cases} -\frac{1}{2\varrho} \partial_{\nabla d_{\partial E}}^2(\varrho\lambda) = \det \nabla \nabla p & \text{on } E \setminus \{L_+ \cup L_-\} \\ \varrho\lambda = \partial_{\nabla d_{\partial E}}(\varrho\lambda) = 0 & \text{at } M \end{cases}. \end{aligned}$$

This identifies μ uniquely off of the segments L_\pm .

All that remains is to prove that $\mu = 0$ on L_\pm . These segments are the closure, relative to E , of the stable lines between $z_\pm = (\pm a(1 - \frac{b^2}{a^2}), 0)$ and $y_\pm = (\pm a, 0)$. Going back to Lemmas 5.2 and 5.3, we see that

$$\begin{aligned} \mu|_{L_\pm} &= A_\pm \delta_{z_\pm} + \lambda_\pm \nabla^\perp d_{\partial E} \otimes \nabla^\perp d_{\partial E} \mathcal{H}^1|_{L_\pm}, \\ \text{where } &\lambda_\pm(x) = c_\pm + \tilde{c}_\pm(x - z_\pm) \cdot \nabla d_{\partial E} \end{aligned}$$

and where $A_\pm \in \text{Sym}_2$ and $c_\pm, \tilde{c}_\pm \in \mathbb{R}$. The measures $\mu|_{L_\pm}$ are curlcurl-free in the sense of distributions on E , i.e.,

$$\begin{aligned} 0 &= \int_E \left\langle \nabla^\perp \nabla^\perp \psi, \mu|_{L_\pm} \right\rangle = \left\langle A_\pm, \nabla^\perp \nabla^\perp \psi(z_\pm) \right\rangle + \int_{L_\pm} \lambda_\pm \partial_{\nabla d_{\partial E}}^2 \psi d\mathcal{H}^1 \\ &= \left\langle A_\pm, \nabla^\perp \nabla^\perp \psi(z_\pm) \right\rangle + c_\pm \partial_{\nabla d_{\partial E}} \psi(z_\pm) - \tilde{c}_\pm \psi(z_\pm) \end{aligned}$$

for all $\psi \in C_c^\infty(E)$. It follows that $A_\pm = 0$ and $c_\pm = \tilde{c}_\pm = 0$. Thus, $\mu|_{L_\pm} = 0$ and this completes the proof of (6.18).

Several negatively curved convex polygons appear in Panel (b) of Fig. 2. We consider these next. As in the previous example, the solution formulas from Section 5.3 must be supplemented by a separate argument showing that μ is not supported on certain leftover stable lines. This time, the argument involves the complementary slackness conditions from (5.1).

Example 6.8. (negatively curved convex polygons) Let P be a convex polygon with vertices $a_1, \dots, a_n \in \mathbb{R}^2$ labeled in counterclockwise order and sides $S_1 = [a_1, a_2], \dots, S_n = [a_n, a_{n+1}]$ where $a_{n+1} = a_1$. The outwards-pointing unit normal vector to ∂P takes on n distinct values, which we label as

$$\hat{\nu}_i = \hat{\nu}|_{S_i} = \frac{(a_i - a_{i+1})^\perp}{|a_i - a_{i+1}|}, \quad i = 1, \dots, n.$$

The medial axis M is a tree made up of finitely many line segments. Its complement $P \setminus M$ is the disjoint union of n (open) sub-polygons P_1, \dots, P_n . The labels are such that $S_i \subset \partial P_i$. Let $\hat{\tau}_M$ and \hat{v}_M denote unit tangent and unit normal vectors to M . We take them to be locally constant off of its internal vertices.

With the notation set, we can describe φ_- . Observe that $d_{\partial P} = d_{S_i} = d(\cdot, S_i)$ on the i th sub-polygon P_i . Hence,

$$\varphi_- = \frac{1}{2}|x|^2 - \frac{1}{2}d_{S_i}^2 \quad \text{on } P_i$$

for $i = 1, \dots, n$. As each side S_i is a line segment, it follows easily that

$$\nabla \nabla \varphi_- = \sum_{i=1}^n \hat{v}_i^\perp \otimes \hat{v}_i^\perp \mathbb{1}_{P_i} dx + \sum_{1 \leq i < j \leq n} d_{\partial P} |\hat{v}_i - \hat{v}_j| \hat{v}_M \otimes \hat{v}_M \mathcal{H}^1 \lfloor \partial P_i \cap \partial P_j \quad \text{on } P.$$

Thus, the stable lines run parallel to $\nabla d_{\partial P} = \hat{v}_i$ on P_i , and the original polygon P is partitioned by φ_- into the sets

$$\Sigma = M, \quad O = P \setminus M, \quad F = U = \emptyset.$$

All this being said, we now claim that the unique optimal μ is given by

$$\mu = \sum_{i=1}^n \lambda_i \hat{v}_i^\perp \otimes \hat{v}_i^\perp \mathbb{1}_{P_i} dx \quad \text{on } P,$$

where
$$\begin{cases} -\frac{1}{2} \partial_{\hat{v}_i}^2 \lambda_i = \det \nabla \nabla p & \text{on } P_i \\ \lambda_i = \partial_{\hat{v}_i} \lambda_i = 0 & \text{at } \partial P_i \cap M \end{cases} \quad \text{for } i = 1, \dots, n. \quad (6.19)$$

As in the previous example, our plan is as follows: first we apply Corollary 5.2 to identify μ away from an exceptional one-dimensional set L , and then we verify separately that $\mu = 0$ on L .

Corollary 5.2 is built to handle situations where stable lines meet along a curve. Here, stable lines meet along a tree—the medial axis M . Its edges are line segments, its external vertices are given by $\{a_i\}_{i=1}^n \subset \partial P$, and we label its internal vertices as $\{z_k\}_{k=1}^N \subset P$. It will probably be useful to look back at the medial axis (in bold) of the triangle, square, or rectangle in Fig. 2. Each internal vertex z_k belongs to the boundary of finitely many stable lines. Let L_k be the union of z_k and its associated stable lines, and let $L = \bigcup_{k=1}^N L_k$. Now in the same manner as was done for Example 6.7 (and as will be done for Example 6.9 below), we can apply Corollary 5.2 to deduce that

$$\mu = \lambda \nabla^\perp d_{\partial P} \otimes \nabla^\perp d_{\partial P} dx \quad \text{on } P \setminus L,$$

where
$$\begin{cases} -\frac{1}{2} \partial_{\nabla d_{\partial P}}^2 \lambda = \det \nabla \nabla p & \text{on } P \setminus L \\ \lambda = \partial_{\nabla d_{\partial P}} \lambda = 0 & \text{at } \partial(P \setminus L) \cap M \end{cases}$$

thus showing it is uniquely determined there. For brevity's sake we leave the details of this to the reader, and simply note the relevant hypotheses can be checked using that $|\hat{v}_i - \hat{v}_j|$, $\hat{\tau}_M \cdot \hat{v}_i^\perp$, and $\hat{\tau}_M \cdot \hat{v}_j^\perp$ are all non-zero at $\partial P_i \cap \partial P_j$.

It remains to show that $\mu = 0$ on L . Consider the restriction of μ to L_k , which we recall consists of z_k along with the finitely many stable lines $\{\ell_{S_i}\}$ having $z_k \in \partial \ell_{S_i}$. The labeling is such that $\ell_{S_i} \subset P_i$, and we think of it as running from z_k to the i th side S_i . In particular, ℓ_{S_i} is parallel to \hat{v}_i . Lemmas 5.2 and 5.3 prove that

$$\mu|_{L_k} = A\delta_{z_k} + \sum_i \lambda_i \hat{v}_i^\perp \otimes \hat{v}_i^\perp \mathcal{H}^1|_{\ell_{S_i}}, \quad \text{where } \lambda_i(x) = c_i + \tilde{c}_i(x - z_k) \cdot \hat{v}_i \quad (6.20)$$

for $A \in \text{Sym}_2$ and $c_i, \tilde{c}_i \in \mathbb{R}$. Since $\mu|_{L_k}$ is curlcurl-free in the sense of distributions on P , there holds

$$0 = \int_P \langle \nabla^\perp \nabla^\perp \psi, \mu|_{L_k} \rangle = \langle A, \nabla^\perp \nabla^\perp \psi(z_k) \rangle - \sum_i c_i \partial_{\hat{v}_i} \psi(z_k) + \sum_i \tilde{c}_i \psi(z_k)$$

for all $\psi \in C_c^\infty(P)$. It follows immediately that

$$A = 0, \quad \sum_i c_i \hat{v}_i = 0, \quad \sum_i \tilde{c}_i = 0. \quad (6.21)$$

So far, the argument has been more or less the same as in the previous example, and indeed we can already conclude that $\mu(\{z_k\}) = 0$. However, we cannot conclude that $\mu|_{L_k} = 0$ at this point. The trouble is that stable lines belonging to different P_i may be parallel (e.g., for a negatively-curved rectangle). In such a case, at least two of the vectors \hat{v}_i appearing in (6.21) will be parallel, and the desired conclusion that $c_i = 0$ will not follow.

The key is to go back to the first complementary slackness conditions in (5.1), which state here that $\langle \nabla^\perp \nabla^\perp \varphi_-, \mu \rangle = 0$ in the regularized sense. In particular,

$$0 = \lim_{\delta \rightarrow 0} \int_M |d_{\partial P}[\nabla d_{\partial P}]| \langle \hat{\tau}_M \otimes \hat{\tau}_M, \mu_\delta \rangle | d\mathcal{H}^1, \quad (6.22)$$

where $\{\mu_\delta\}_{\delta>0}$ are the mollifications in (1.32). As noted in Remark 1.3, we may take the kernel $\rho > 0$ nearby zero. Given $x \in M$, observe using the non-negativity of μ , the formula (6.20), and Fubini's theorem that

$$\begin{aligned} & \int_M d_{\partial P} |[\nabla d_{\partial P}]| \langle \hat{\tau}_M \otimes \hat{\tau}_M, \mu_\delta \rangle d\mathcal{H}^1 \\ & \geq \int_{y \in \ell_{S_i}} \int_{x \in M \cap \partial P_i} d_{\partial P} |[\nabla d_{\partial P}](x)| |\hat{\tau}_M(x) \cdot \hat{v}_i^\perp|^2 \lambda_i(y) \rho\left(\frac{x-y}{\delta}\right) \frac{d\mathcal{H}^1(x) d\mathcal{H}^1(y)}{\delta^2} \end{aligned}$$

for each i . According to (6.22), the lefthand side tends to zero as $\delta \rightarrow 0$. Regarding the righthand side, note that $d_{\partial P} |[\nabla d_{\partial P}]|$ and $|\hat{\tau}_M \cdot \hat{v}_i^\perp|$ are bounded away from zero within the integral. Also, $\lambda_i(y) \rightarrow c_i$ as $y \rightarrow z_k$ along ℓ_{S_i} . Using that $\rho > 0$ nearby zero, we find upon sending $\delta \rightarrow 0$ that

$$c_i = 0 \quad \forall i. \quad (6.23)$$

With this, we can easily control the remaining $\{\tilde{c}_i\}$. Since $\mu|_{\ell_{S_i}} \geq 0$ there holds $\lambda_i \geq 0$, and then using that $\lambda_i(z_k) = c_i = 0$ we see that $\partial_{\hat{v}_i} \lambda_i(z_k) = \tilde{c}_i \geq 0$. It now follows from the third part of (6.21) that

$$\tilde{c}_i = 0 \quad \forall i. \quad (6.24)$$

Looking back to (6.20) once more, we see that $\mu|L_k = 0$ for each k . Hence, $\mu|L = 0$ and (6.19) is proved.

It is no accident that these last few steps followed along the same lines as the proofs of Corollaries 5.2 and 5.3. Our task was, once again, to show that μ vanishes on certain leftover stable lines. The constraints (6.23) and (6.24) entered as Cauchy data analogous to, e.g., (5.66) in Corollary 5.2. We imagine a similar approach may be used to control μ in other circumstances where the formulas from Section 5.3 do not directly apply.

Our final example is the half-disc in Panel (b) of Fig. 2. It is the only of our examples in which M ends up being curved.

Example 6.9. (negatively curved half-disc) Consider a disc of radius a centered at the origin and let

$$D_+ = \left\{ (x_1, x_2) : x_1^2 + x_2^2 < a^2, x_2 > 0 \right\}.$$

Note $d_{\partial D_+}$ is smooth away from the medial axis

$$M = \left\{ (x_1, x_2) : 2ax_2 = a^2 - x_1^2, x_2 > 0 \right\}$$

which is the unique parabolic arc passing through the corners $(\pm a, 0)$ and $(0, \frac{a}{2})$. Denote the part of D_+ below M by D_{+S} and the part above M by D_{+N} . Then,

$$\begin{aligned} \nabla \nabla \varphi_- &= \hat{e}_1 \otimes \hat{e}_1 \mathbb{1}_{D_{+S}} dx + \frac{a}{r} \hat{e}_\theta \otimes \hat{e}_\theta \mathbb{1}_{D_{+N}} dx \\ &\quad + d_{\partial D_+} |\hat{e}_2 + \hat{e}_r| \hat{\tau}_M \otimes \hat{\tau}_M \mathcal{H}^1|_M \quad \text{on } D_+. \end{aligned}$$

We see that φ_- partitions D_+ according as

$$\Sigma = M, \quad O = D_+ \setminus M, \quad F = U = \emptyset.$$

The stable lines are parallel to $\nabla d_{\partial D_+} = \hat{e}_2$ in D_{+S} and $\nabla d_{\partial D_+} = -\hat{e}_r$ in D_{+N} .

A straightforward application of Corollary 5.2 proves that optimal μ are unique. The conditions (5.58)–(5.62) hold for $\varphi = \varphi_-$ and for

$$V = D_+, \quad \zeta = \begin{cases} 1 & x \in D_{+S} \\ \frac{a}{r} & x \in D_{+N} \end{cases}, \quad \hat{\eta} = \begin{cases} \hat{e}_1 & x \in D_{+S} \\ \hat{e}_\theta & x \in D_{+N} \end{cases}.$$

In particular, the medial axis is smooth and the quantities $|\hat{e}_2 + \hat{e}_r|$, $|\hat{\tau}_M \cdot \hat{e}_1|$, and $|\hat{\tau}_M \cdot \hat{e}_\theta|$ are uniformly positive there. Also, $\zeta > 0$ uniformly on D_+ . Finally, we see that $\varrho = c_S(x_1)$ in D_{+S} and $\varrho = c_N(\theta)r$ in D_{+N} due to the fact that the stable lines describe either parallel lines or rays (the functions c_S and c_N depend on the index set Γ). Applying Corollary 5.2, we conclude that optimal μ satisfy

$$\mu = \lambda_S \hat{e}_2 \otimes \hat{e}_2 \mathbb{1}_{D_{+S}} dx + \lambda_N \hat{e}_\theta \otimes \hat{e}_\theta \mathbb{1}_{D_{+N}} dx \quad \text{on } D_+,$$

where

$$\begin{cases} -\frac{1}{2} \partial_2^2 \lambda_S = \det \nabla \nabla p & \text{in } D_{+S} \\ \lambda_S = \partial_2 \lambda_S = 0 & \text{at } M \end{cases} \quad \text{and} \quad \begin{cases} -\frac{1}{2r} \partial_r^2 (r \lambda_N) = \det \nabla \nabla p & \text{in } D_{+N} \\ \lambda_N = \partial_r \lambda_N = 0 & \text{at } M \end{cases}.$$

These systems determine optimal μ uniquely for the negatively curved half-disc.

We close with a general conjecture on the uniqueness of optimal μ . Although optimal μ turned out to be unique in each of the negatively curved examples above, the reader may yet wonder whether the convexity of Ω is crucial for this, or if it is simply an artifact of our examples. We believe the latter is true. In fact, based on our method of stable lines, we expect optimal μ will be unique whenever there exists an optimal φ that is nowhere locally affine (regardless of the curvature). Here is a more concrete version of this conjecture specialized to simply connected, negatively curved shells. Recall the medial axis M from (6.3) and the change of measure factor ϱ from (5.13).

Conjecture 6.1. Suppose that Ω is simply connected and let $\det \nabla \nabla p \leq 0$ a.e. Assume the paths of quickest exit from Ω do not meet at $\partial\Omega$. Then optimal μ are unique, and moreover satisfy

$$\mu = \lambda \nabla^\perp d_{\partial\Omega} \otimes \nabla^\perp d_{\partial\Omega} dx \quad \text{on } \Omega, \\ \text{where } \begin{cases} -\frac{1}{2\varrho} \partial_{\nabla d_{\partial\Omega}}^2 (\varrho\lambda) = \det \nabla \nabla p & \text{on } \Omega \setminus \overline{M} \\ \varrho\lambda = \partial_{\nabla d_{\partial\Omega}} (\varrho\lambda) = 0 & \text{at } \overline{M} \end{cases}.$$

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