

SCHUBERT VARIETIES AND FINITE FREE RESOLUTIONS OF LENGTH THREE

STEVEN V SAM AND JERZY WEYMAN

Dedicated to Laurent Gruson with thanks for his guidance and friendship.

ABSTRACT. In this paper we describe the relationship between the finite free resolutions of perfect ideals in split format (for Dynkin formats) and certain intersections of opposite Schubert varieties with the big cell for homogeneous spaces G/P where P is a maximal parabolic subgroup.

1. INTRODUCTION

Perfect ideals of codimension three have been investigated for a long time. Linkage theory suggests that it might be possible to classify such ideals. Indeed, if one applies minimal linkage to a perfect ideal of codimension three, one gets an ideal with a minimal free resolution with the same sum of Betti numbers as the original one, and after a double link one obtains an ideal whose free resolution has modules of the same ranks as for the original ideal.

In the paper [W2] the second author constructed the generic rings $\hat{R}_{\text{gen}}$ for resolutions of length three for any format. It turns out that their structure is related to the Kac–Moody Lie algebra of the T -shaped graph $T_{p,q,r}$ where the triple $(p, q, r) = (r_1 + 1, r_2 - 1, r_3 + 1)$ is constructed from the ranks r_1, r_2, r_3 of the differentials d_1, d_2, d_3 of our resolution. We denote the vertices of $T_{p,q,r}$ as follows

$$\begin{array}{ccccccccccc}
 x_{p-1} & - & x_{p-2} & \dots & x_1 & - & u & - & y_1 & \dots & y_{q-2} & - & y_{q-1} \\
 & & & & & & | & & & & & & \\
 & & & & & & z_1 & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & \vdots & & & & & & \\
 & & & & & & z_{r-2} & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & z_{r-1} & & & & & &
 \end{array}$$

This construction suggests that a special role is played by the so-called Dynkin formats for which the graph $T_{p,q,r}$ is Dynkin. In fact, the ring $\hat{R}_{\text{gen}}$ is Noetherian precisely in such cases. One also conjectures (for the case $r_1 = 1$, i.e., the resolutions of cyclic modules R/I) that these are precisely the formats for which every such ideal is licci.

The next idea is to relate the rings $\hat{R}_{\text{gen}}$ to the resolutions of perfect ideals of a given format. The natural idea is to describe an open set U_{CM} (the Cohen–Macaulay locus) in $\text{Spec } \hat{R}_{\text{gen}}$ where the dual of the generic resolution $\mathbf{F}_{\bullet}^{\text{gen}}$ is acyclic. In [CVW] and [LW] a precise conjecture on the form of the subset U_{CM} for Dynkin formats is given. This conjecture

Date: September 14, 2020.

2020 Mathematics Subject Classification. 13D02, 13H10, 14M15.

is based on the observation that for Dynkin formats (except for the format $(1, n, n, 1)$ with n odd), the three top graded components of three critical representations in $\hat{R}_{\text{gen}}$ can be thought of as the differentials in another complex $\mathbf{F}_{\bullet}^{\text{top}}$ of the format (f_0, f_1, f_2, f_3) over $\hat{R}_{\text{gen}}$. The conjecture then says that the open set U_{CM} is equal to the set U_{split} where the complex $\mathbf{F}_{\bullet}^{\text{top}}$ is split exact.

The next task is to find the precise form of the general ideal resolved by a resolution corresponding to a point in U_{split} . This problem is resolved in [CVW] and in [CLKW] for the formats of type D_n and E_6 . It is also proved in [CVW] that for the formats $(1, n, n, 1)$ with n even, and $(1, 4, n, n - 3)$, we have the equality $U_{\text{CM}} = U_{\text{split}}$.

The method employed there consists of identifying the generators of $\hat{R}_{\text{gen}}$ occurring in some critical representations in $\hat{R}_{\text{gen}}$ and calculating these structure theorems for split exact resolutions using the defect variables. The structure theorems corresponding to the top graded components then give the differentials in the complex which is a resolution of a perfect ideal of codimension three which corresponds to a general point in U_{CM} . For the remaining types E_7 and E_8 the analogous method is very difficult to employ because the representations $W(d_3)$, $W(d_2)$, $W(a_2)$ are too complex.

In the present paper we try another approach. We try to construct the relevant resolutions of Dynkin formats by geometric means. A very interesting geometric pattern appears. We prove that for all the Dynkin types there is a certain Schubert variety Ω_{λ} of codimension 3 in a homogeneous space $G(\mathbb{T}_{p,q,r})/P_{x_1}$ ($G(\mathbb{T}_{p,q,r})$ denotes the simply-connected simple group corresponding to the graph $\mathbb{T}_{p,q,r}$, P_{x_1} is a maximal parabolic subgroup corresponding to the vertex x_1 in the above notation), such that the defining ideal of the intersection Y_{λ} of Ω_{λ} with the opposite big open cell Y has exactly the resolution of the format corresponding to the triple (p, q, r) . We expect the singularity of Y_{λ} to be the singularity of the general point in U_{split} (this is established for the types D_n and E_6).

This proves the inclusion $U_{\text{split}} \subseteq U_{\text{CM}}$ and shows that the expected general form of the finite free resolutions of Dynkin formats show a definite pattern. The pattern conjecturally extends to the resolutions of cyclic modules of arbitrary format. This will be taken up in a future paper.

Acknowledgements. SS was partially supported by the NSF grant DMS-1849173. JW was partially supported by the NSF grant DMS 1802067 and by the grant from Narodowa Agencja Wymiany Akademickiej NAWA in Poland.

This collaboration started during the August 2019 workshop organized by the first author at UCSD. We thank Shrawan Kumar for patiently answering several questions on Schubert varieties. We also benefited from many discussions with Ela Celikbas, Lars Christensen, Witold Kraszkiewicz, Jai Laxmi, Kyu-Hwan Lee, and Oana Veliche.

2. GENERAL SETUP

Throughout we work over a field $\mathbf{k}$. (Our constructions will be insensitive to the choice of field and in fact will be defined over $\mathbf{Z}$, but we keep a field to avoid making extra remarks throughout.)

Our goal is to describe the general complex from the open set U_{split} of $\text{Spec}(\hat{R}_{\text{gen}})$ for a Dynkin format. This seems to be related to Schubert varieties in homogeneous spaces for groups of type $\mathbb{T}_{p,q,r}$ and goes as follows. Consider the Dynkin formats of free resolutions of

length 3 of cyclic modules, i.e., assume the rank is $r_1 = 1$. This means that the corresponding graph $T_{p,q,r}$ is $T_{2,q,r}$. We also assume this graph is of Dynkin type.

We fix root data for our group G and look at the homogeneous space G/P where $P = P_{x_1}$ is the maximal parabolic subgroup corresponding to the vertex on the arm of length 1 corresponding to p . We let B and B^- denote the Borel subgroup and opposite Borel subgroup, respectively. We use basic facts on Schubert varieties, described for example in [LR, §3.3]. Our notation is for opposite Schubert varieties, i.e., an element of length ℓ indexes a variety of codimension ℓ . The opposite Schubert varieties in G/P are described by the cosets W/W_P where W is the Weyl group corresponding to $T_{p,q,r}$ and W_P is the Weyl group corresponding to smaller root system we get from $T_{p,q,r}$ by removing x_1 . In our case we can identify W/W_P with the W -orbit of the fundamental weight ω_{x_1} , and we now describe how this works.

In what follows, we represent weights as a sum of fundamental weights, and represent these by labeling the vertices of $T_{p,q,r}$ by the corresponding coefficients. We deal with the Weyl group action on the weight

$$\sigma_0 = \omega_{x_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}.$$

Acting by elements of length ℓ in W of minimal length in their coset, we get opposite Schubert varieties of codimension ℓ . Thus for length 0 we get σ_0 . For length 1 we get

$$\sigma_1 = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}$$

and for length 2

$$\sigma_2 = \begin{pmatrix} 0 & -1 & 1 & 0 & \cdots & 0 \\ 1 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}.$$

Finally for length 3 we get two possibilities

$$\sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ -1 & & & & & \\ 1 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}, \quad \sigma'_3 = \begin{pmatrix} 0 & 0 & -1 & 1 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}.$$

This picture says that in $G(T_{2,q,r})/P_{x_1}$ there is one opposite Schubert variety of codimension 1, one of codimension 2, and two of codimension 3. Define $\Omega_P^w = \overline{B^- w P}/P$ in G/P . We consider the opposite Schubert varieties $\Omega_P^{\sigma_3}$ and $\Omega_P^{\sigma'_3}$ and their intersections Y_P^w with the big cell Y . The cell Y is the unique open B -orbit. The varieties $Y_{\sigma_3}, Y_{\sigma'_3}$ are known to be normal Cohen–Macaulay subvarieties of codimension 3 (see [KS, Proposition 3.4]).

But the same picture represents the extremal Plücker coordinates of $G(T_{2,q,r})/P_{x_1}$ (see [LR, §3.3.1]) embedded in $\mathbf{P}(V_{x_1})$ where V_λ denotes the irreducible highest weight module of highest weight λ . Each element $w \in W/W_P$ determines a unique linear function p_w (defined up to scalar) on $\mathbf{P}(V_{x_1})$ and the defining equations of Y_{σ_3} and $Y_{\sigma'_3}$ are the Plücker coordinates p_τ such that τ is not $\leq$ than σ_3 (resp. σ'_3) in the Bruhat order. This follows from the identification of the linear functions on Ω_P^w with Demazure modules (see [LR, §§3.3.1, 3.3.3]). In this setup, Y is given by the condition $p_{\text{top}} \neq 0$ where p_{top} is the extremal Schubert coordinate corresponding to the coset of maximal length $\ell = \dim G(T_{2,q,r})/P_{x_1}$.

In what follows, working with the parabolic P_{x_1} we give two interpretations of the varieties $Y_{\sigma_3}, Y_{\sigma'_3}$: one in terms of Plücker coordinates, the other in equivariant form related to the grading on $\mathfrak{g}(E_m)$ ($m = 6, 7, 8$) corresponding to the simple root α_{x_1} . The open cell Y is identified with the unipotent radical of the opposite parabolic subgroup. The functions on this space can be identified with the positive portion of the Lie algebra $\mathfrak{g}(E_m)$.

3. TYPE E_6

The Plücker coordinates vanishing on Y_{σ_3} , where

$$\sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & -1 & & \\ & & 1 & \end{pmatrix}$$

are the ones corresponding to the elements

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & & & \\ 0 & & & \end{pmatrix}.$$

There are also 5 Plücker coordinates vanishing on $Y_{\sigma'_3}$, symmetrically with respect to the two arms of length 3. The intersection of both ideals is a complete intersection given by the three Plücker coordinates

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & & & \\ 0 & & & \end{pmatrix}$$

that are common in both ideals.

By construction, all opposite Schubert varieties are invariant under the action of the opposite Borel subgroup B^- . Generally, there is a larger parabolic subgroup that leaves it invariant. This is explained for Schubert varieties in [LMS, Proposition 1.4], but we can use the involution $w \mapsto ww_0^P$ where w_0^P is the minimal length coset representative of the maximal element in W/W_P to get the corresponding statement for opposite Schubert varieties. Namely, the opposite Schubert variety Ω^w is closed under the root subgroups labeled by a simple root α if $\ell(s_\alpha w) > \ell(w)$.

We see that the ideal of $\Omega_P^{\sigma_3}$ is invariant under the action of the root subgroups corresponding to $\alpha_1, \alpha_4, \alpha_5, \alpha_6$. The Levi subgroup of the corresponding parabolic subgroup is isomorphic to $\mathbf{GL}_2 \times \mathbf{GL}_4$. Analogously, $\Omega_P^{\sigma'_3}$ is invariant under the action of the root subgroups corresponding to $\alpha_1, \alpha_3, \alpha_4, \alpha_6$ and the corresponding Levi subgroup is isomorphic to $\mathbf{GL}_4 \times \mathbf{GL}_2$. The first equation corresponding to σ_0 generates an ideal which is invariant under $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$, and the corresponding Levi subgroup is isomorphic to $\mathbf{GL}_6$.

The equivariant picture is as follows. First, G/P is embedded in the fundamental representation $\mathbf{P}(V_{\omega_2})$ which is the adjoint representation. Consider the root space decomposition of $\mathfrak{g}(E_6)$ and coarsen it to a $\mathbf{Z}$ -grading by considering the coefficient of α_1 . Then the grading ranges from -2 to 2 and

$$\mathfrak{g}_0 = \mathfrak{gl}_6, \quad \mathfrak{g}_1 = \bigwedge^3 \mathbf{k}^6, \quad \mathfrak{g}_2 = \bigwedge^6 \mathbf{k}^6.$$

To get the big cell, we set $p_{\text{top}} = 1$ where p_{top} spans $\mathfrak{g}_2$. The ideal of the embedding of G/P into the adjoint representation is generated by quadratic polynomials, which are homogeneous with respect to the α_1 -grading. In particular, given any relation that involves $p_{\text{top}}q$ for some homogeneous element q , we can express q as a sum of product of other elements of smaller α_1 -degree. In particular, a homogeneous element of α_1 -degree d restricts to a homogeneous polynomial of degree $2 - d$ on Y .

The generators for the functions on Y can be identified with $\bigwedge^3 \mathbf{k}^6 \oplus \bigwedge^6 \mathbf{k}^6$ where the first space has degree 1 and the second space has degree 2 (and is a subspace of $\mathfrak{g}_0$). Let A denote the coordinate ring of Y .

Theorem 3.1. *The minimal free resolution of Y_{σ_3} is of the form:*

$$0 \rightarrow A^2(-7) \rightarrow A^6(-5) \rightarrow A(-4) \oplus A^4(-3) \rightarrow A.$$

The minimal free resolution of $Y_{\sigma'_3}$ has the same form. Furthermore, the ideals of these varieties are linked by a complete intersection of length 3. The sum of the two ideals has a resolution of format

$$0 \rightarrow A(-10) \rightarrow A^6(-7) \oplus A(-6) \rightarrow A^{12}(-5) \rightarrow A(-4) \oplus A^6(-3) \rightarrow A.$$

The numerator of the reduced Hilbert series of both $\mathbf{k}[Y_{\sigma_3}]$ and $\mathbf{k}[Y_{\sigma'_3}]$ is

$$2t^4 + 6t^3 + 6t^2 + 3t + 1$$

and hence both Y_{σ_3} and $Y_{\sigma'_3}$ have degree 18. The numerator of the reduced Hilbert series of $\mathbf{k}[Y_{\sigma_3} \cap Y_{\sigma'_3}]$ is

$$t^6 + 4t^5 + 10t^4 + 14t^3 + 10t^2 + 4t + 1$$

and hence the degree of $Y_{\sigma_3} \cap Y_{\sigma'_3}$ is 44.

Proof. We know that the resolution of Y_{σ_3} is of the form

$$0 \rightarrow \mathbf{F}_3 \rightarrow \mathbf{F}_2 \rightarrow A(-4) \oplus A(-3)^4 \rightarrow A.$$

Consider the ideals I_{σ_3} and $I_{\sigma'_3}$ in A and the ideal $(\alpha) = (p_{\text{id}}, p_{s_{x_1}}, p_{s_u s_{x_1}})$ generated by the first three Plücker coordinates. This ideal is radical by [BK, Corollary 2.3.3]. It follows that I_{σ_3} and $I_{\sigma'_3}$ are linked via (α) , as this is true set-theoretically and both ideals $(\alpha) : I_{\sigma_3}$ and $(\alpha) : I_{\sigma'_3}$ are radical. Now using [U, §1] we see that the ideals I_{σ_3} and $I_{\sigma'_3}$ are linked via α , which is a complete intersection generated by polynomials of degrees 3, 3, 4.

By the theory of linkage (see for example [BE, §5]) the dual of the mapping cone of the resolution of the complete intersection to the resolution of Y_{σ_3} gives a resolution for $Y_{\sigma'_3}$ (up to a grading shift). This has the form

$$0 \rightarrow A(-10) \rightarrow \begin{array}{c} A(-10) \oplus \\ A(-7)^4 \oplus \\ A(-6) \end{array} \rightarrow \begin{array}{c} A(-7)^2 \oplus \\ A(-6) \oplus \\ \mathbf{F}_2^*(-10) \end{array} \rightarrow \begin{array}{c} \mathbf{F}_3^*(-10) \oplus \\ A(-4) \oplus \\ A(-3)^2 \end{array} \rightarrow A$$

The last few terms cancel since they identify generators and relations from the original two resolutions, and the result is

$$0 \rightarrow A(-7)^2 \rightarrow \mathbf{F}_2^*(-10) \rightarrow \begin{array}{c} \mathbf{F}_3^*(-10) \oplus \\ A(-4) \oplus \\ A(-3)^2 \end{array} \rightarrow A$$

We know that the ideal of $Y_{\sigma'_3}$ is minimally generated by 4 cubics and 1 quartic, and the differential $\mathbf{F}_2^* \rightarrow \mathbf{F}_3^*$ is minimal, so we conclude that $\mathbf{F}_3 = A(-7)^2$. This tells us that $\text{rank } \mathbf{F}_2 = 6$. Let $d_1 \leq \dots \leq d_6$ be the degrees of its generators. Since Y_{σ_3} has codimension 3, the alternating sum

$$1 - 4t^3 - t^4 + \sum_{i=1}^6 t^{d_i} - 2t^7$$

is divisible by $(1-t)^3$, which translates to its first two derivatives being 0 at $t = 1$:

$$-12 - 4 + \sum_{i=1}^6 d_i - 14 = -24 - 12 + \sum_{i=1}^6 d_i(d_i - 1) - 84 = 0,$$

or equivalently,

$$\sum_{i=1}^6 d_i = 30, \quad \sum_{i=1}^6 d_i^2 = 150.$$

The plane $\sum_{i=1}^6 d_i = 30$ is tangent to the sphere defined by the second equation at the point $(5, 5, 5, 5, 5, 5)$, so this point is the unique solution to both equations.

By [U, Proposition 1.3], the ideal $I_{\sigma_3} + I_{\sigma'_3}$ is Gorenstein. This is the defining ideal of the Schubert variety Y_{σ_4} of codimension 4 which is contained both in Y_{σ_3} and $Y_{\sigma'_3}$. To get the form of its resolution we again appeal to [U, Proposition 1.3] which implies that this resolution is a mapping cone of the map of complexes covering an injection of the canonical module of the coordinate ring of Y_{σ_3} (resp. $Y_{\sigma'_3}$) into the resolution of the coordinate ring of Y_{σ_3} (resp. $Y_{\sigma'_3}$). $\square$

3.1. Linear section. A certain linear section of the first minimal free resolution is described in [CLKW]. The description of the generators given there is a little different, as the present pattern was not observed at that time.

Now we consider the restriction of the ideals to $\text{Sym}(\bigwedge^3 \mathbf{k}^6)$.

Lemma 3.2. *The restriction of p_{σ_3} to $\bigwedge^3 \mathbf{k}^6$ is the unique (up to scalar) $\mathbf{SL}_6$ -invariant Δ of degree 4 on $\bigwedge^3 \mathbf{k}^6$.*

The invariant Δ can be described as a hyperdiscriminant of the representation $\bigwedge^3 \mathbf{k}^6$, i.e., the equation of the hypersurface projectively dual to the Grassmannian (see for example [W1, §9C]).

Proof. p_{σ_3} is a lowest weight vector and hence has α_1 -degree -2 . By what we said above, it becomes a quartic function upon restriction to Y . Next, we also said above that the ideal generated by p_{σ_3} is invariant under $\mathbf{GL}_6$. This implies that p_{σ_3} itself is invariant under the subgroup $\mathbf{SL}_6$. Next, we need to show that this restriction is nonzero. If not, then the quartic in A is h^2 , where h is the generator of $\bigwedge^6 \mathbf{k}^6$, which is non-reduced. $\square$

As explained above, Y_{σ_3} is invariant under the Levi subgroup $\mathbf{GL}_2 \times \mathbf{GL}_4$, so we choose a decomposition $\mathbf{k}^6 = F \oplus G$ where $F = \mathbf{k}^4$, $G = \mathbf{k}^2$ are spanned by the first 4, respectively last 2, coordinate vectors.

So our restricted Schubert variety is in the affine space $\text{Spec}(A')$ where

$$A' = \text{Sym}(\mathfrak{g}_1) = \text{Sym}\left(\bigwedge^3 F \oplus \bigwedge^2 F \otimes G \oplus F \otimes \bigwedge^2 G\right).$$

Lemma 3.3. *The five defining equations of the restriction of Y_{σ_3} to $\text{Spec}(A')$ are Δ , and $\frac{\partial \Delta}{\partial y_i}$, where $\{y_i\}$ is a basis of $(\bigwedge^3 F)^*$.*

Proof. We have already explained the appearance of Δ . The remaining 4 functions have α -degree -1 and hence restrict to cubic functions. By equivariance, these functions are closed under the action of $\mathbf{GL}(F) \times \mathbf{GL}(G)$ and the upper triangular block $G^* \otimes F$, so must span the space $\bigwedge^3 F$. Since the invariant Δ is unique, this space is spanned by its partial derivatives with respect to $(\bigwedge^3 F)^*$. $\square$

The five equations defining the ideal of $Y_{\sigma'_3}$ restricted to $\text{Spec}(A')$ in this language are similar, we just decompose $\mathbf{k}^6 = F' \oplus G'$ with $\dim F' = 2$, $\dim G' = 4$.

Lemma 3.4. *The five defining equations of the restriction of $Y_{\sigma'_3}$ to $\text{Spec}(A')$ are Δ and $\frac{\partial \Delta}{\partial z_i}$, where $\{z_i\}$ is a basis of the dual of $\bigwedge^2 F' \otimes G'$.*

The proof is similar to the proof of Lemma 3.3.

4. TYPE E_7

Much of the setup here follows that of the previous section. Rather than re-explain all of the details, we will just list the results of the calculations.

As before, we coarsen the root decomposition of $\mathfrak{g}(E_7)$ by considering the coefficient of α_2 . The non-negative components of the resulting $\mathbf{Z}$ -graded decomposition are as follows:

$$\mathfrak{g}_0 = \mathfrak{gl}_7(\mathbf{k}), \quad \mathfrak{g}_1 = \bigwedge^3 \mathbf{k}^7, \quad \mathfrak{g}_2 = \bigwedge^6 \mathbf{k}^7.$$

We let A denote the coordinate ring of the big cell. We can identify its linear functions with the positive portion of this decomposition.

In this case, V_{x_1} is not the adjoint representation, so we need its $\mathbf{Z}$ -graded decomposition as well (see [LW, §5.3.3]). We do not repeat it here, but just note that the grading goes from 0 to 7 and the top and bottom pieces are 1-dimensional. As before, setting the element in top degree to be nonzero cuts out the big cell.

The ideal of Y_{σ_3} is generated by 6 elements which are Plücker coordinates corresponding to the elements

$$\begin{array}{ccc} \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & & & \end{matrix} & , & \begin{matrix} -1 & 1 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & & & \end{matrix} & , & \begin{matrix} 0 & -1 & 1 & 0 & 0 \\ 1 & & & & \\ 0 & & & & \end{matrix} \\ \\ \begin{matrix} 0 & 0 & -1 & 1 & 0 \\ 1 & & & & \\ 0 & & & & \end{matrix} & , & \begin{matrix} 0 & 0 & 0 & -1 & 1 \\ 1 & & & & \\ 0 & & & & \end{matrix} & , & \begin{matrix} 0 & 0 & 0 & 0 & -1 \\ 1 & & & & \\ 0 & & & & \end{matrix} \end{array}$$

$$\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & & & & & 0 & & & & & 1 & & & & \\ 0 & & & & & 0 & & & & & 0 & & & & \end{array}, \quad \begin{array}{ccccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & & & & & 0 & & & & \\ 1 & & & & & -1 & & & & \end{array}$$
$$0 \rightarrow A^2(-13) \rightarrow A^7(-9) \rightarrow A(-7) \oplus A^5(-6) \rightarrow A.$$
$$0 \rightarrow A^3(-13) \rightarrow A^7(-10) \rightarrow A(-7) \oplus A^4(-6) \rightarrow A$$

The resolution for the sum of the two ideals has the form

The numerator of the reduced Hilbert series of $\mathbf{k}[Y_{\sigma_3}]$ is

and hence Y_{σ_3} has degree 119. The numerator of the reduced Hilbert series of $\mathbf{k}[Y_{\sigma'_3}]$ is

and hence $Y_{\sigma'_3}$ has degree 133. The numerator of the reduced Hilbert series of $\mathbf{k}[Y_{\sigma_3} \cap Y_{\sigma'_3}]$ is

$$t^{15}+4t^{14}+10t^{13}+20t^{12}+35t^{11}+56t^{10}+77t^9+91t^8+91t^7+77t^6+56t^5+35t^4+20t^3+10t^2+4t+1$$

and hence $Y_{\sigma_3} \cap Y_{\sigma'_3}$ has degree 588.

There is an invariant Δ of degree 7 on $\mathfrak{g}_1$, which can be described as a hyperdiscriminant of the representation $\bigwedge^3 \mathbf{k}^7$, i.e., the equation of the hypersurface projectively dual to the Grassmannian (see for example [W1, §9C]).

$$A' = \mathrm{Sym}(\mathfrak{g}_1) = \mathrm{Sym}(\bigwedge^3 F \oplus \bigwedge^2 F \otimes G \oplus F \otimes \bigwedge^2 G).$$

We get a resolution

$$0 \rightarrow A'^2(-13) \rightarrow A'^7(-9) \rightarrow A'(-7) \oplus A'^5(-6) \rightarrow A'.$$

The equivariant form of this resolution is as follows (Here $\mathbf{S}_\lambda$ denotes a Schur functor. This is valid in characteristic 0, and more care is needed in general to describe the correct representations.):

$$\begin{aligned} 0 \rightarrow \mathbf{S}_{6,6,6,6,6} F \otimes \mathbf{S}_{5,4} G \otimes A'(-13) &\rightarrow \left(\begin{array}{c} \mathbf{S}_{4,4,4,4,4} F \otimes \mathbf{S}_{4,3} G \oplus \\ \mathbf{S}_{5,4,4,4,4} F \otimes \mathbf{S}_{3,3} G \end{array} \right) \otimes A'(-9) \rightarrow \\ &\rightarrow \begin{array}{c} \mathbf{S}_{3,3,3,3,3} F \otimes \mathbf{S}_{3,3} G \otimes A'(-7) \oplus \\ \mathbf{S}_{3,3,3,3,2} F \otimes \mathbf{S}_{2,2} G \otimes A'(-6) \end{array} \rightarrow A'. \end{aligned}$$

Now we describe the restriction of $Y_{\sigma'_3}$. We decompose $\mathbf{k}^7 = F' \oplus G'$ where $\dim F' = 3$, $\dim G' = 4$. We get

$$A'' = \text{Sym}(\mathfrak{g}_1) = \text{Sym}\left(\bigwedge^3 F' \oplus \bigwedge^2 F' \otimes G' \oplus F' \otimes \bigwedge^2 G' \oplus \bigwedge^3 G'\right).$$

We get the ideal of $Y_{\sigma'_3}$ generated by five elements $\Delta, \partial\Delta/\partial z_i$, where $\{z_i\}$ is a basis of the dual of $\bigwedge^3 G'$. This gives a resolution of the format

$$0 \rightarrow A''^3(-13) \rightarrow A''^7(-10) \rightarrow A''(-7) \oplus A''^4(-6) \rightarrow A''.$$

The equivariant form of this resolution is (in characteristic 0):

$$\begin{aligned} 0 \rightarrow \mathbf{S}_{7,6,6} F' \otimes \mathbf{S}_{5,5,5,5} G' \otimes A''(-13) &\rightarrow \left(\begin{array}{c} \mathbf{S}_{5,5,4} F' \otimes \mathbf{S}_{4,4,4,4} G' \oplus \\ \mathbf{S}_{5,5,5} F' \otimes \mathbf{S}_{4,4,4,3} G' \end{array} \right) \otimes A''(-10) \rightarrow \\ &\rightarrow \begin{array}{c} \mathbf{S}_{3,3,3} F' \otimes \mathbf{S}_{3,3,3,3} G' \otimes A''(-7) \oplus \\ \mathbf{S}_{3,3,3} F' \otimes \mathbf{S}_{3,2,2,2} G' \otimes A''(-6) \end{array} \rightarrow A''. \end{aligned} \quad \square$$

5. TYPE E_8

Much of the setup here follows that of type E_6 . Rather than re-explain all of the details, we will just list the results of the calculations.

The non-negative portion of the $\mathbf{Z}$ -graded decomposition of $\mathfrak{g}(E_8)$ with respect to α_2 looks like

$$\mathfrak{g}_0 = \mathfrak{gl}_8(\mathbf{k}), \quad \mathfrak{g}_1 = \bigwedge^3 \mathbf{k}^8, \quad \mathfrak{g}_2 = \bigwedge^6 \mathbf{k}^8, \quad \mathfrak{g}_3 = \bigwedge^8 \mathbf{k}^8 \otimes \mathbf{k}^8.$$

We let A denote the coordinate ring of the big cell. We can identify its linear functions with the positive portion of this decomposition.

In this case, V_{x_1} is not the adjoint representation, so we need its $\mathbf{Z}$ -graded decomposition as well (see [LW, §6.3.3]). We do not repeat it here, but just note that the grading goes from 0 to 16 and the top and bottom pieces are 1-dimensional. As before, setting the element in top degree to be nonzero cuts out the big cell.

The ideal of Y_{σ_3} is generated by 7 elements which are the Plücker coordinates corresponding to the elements

$$\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & & & & & & 0 & & & & & & 1 & & & & & & \\ 0 & & & & & & 0 & & & & & & 0 & & & & & & \end{array}, \quad \begin{array}{cccccccc} 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & & & & & & 1 & & & & & & 1 & & & & & & \\ 0 & & & & & & 0 & & & & & & 0 & & & & & & \end{array}.$$

There are 5 Plücker coordinates vanishing on $Y_{\sigma'_3}$, they correspond to weights

$$\begin{array}{cccccccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & & & & & & 0 & & & & & & & 1 & & & & \\ 0 & & & & & & 0 & & & & & & & 0 & & & & \end{array},$$

$$\begin{array}{cccccccccccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & & & & & \\ -1 & & & & & & 0 & & & & & & & & & & & \\ 1 & & & & & & -1 & & & & & & & & & & & \end{array}.$$

Theorem 5.1. *The minimal free resolution of Y_{σ_3} is of the form:*

$$0 \rightarrow A^2(-31) \rightarrow A^8(-21) \rightarrow A(-16) \oplus A^6(-15) \rightarrow A$$

The minimal free resolution of $Y_{\sigma'_3}$ is of the form:

$$0 \rightarrow A^4(-31) \rightarrow A^8(-25) \rightarrow A(-16) \oplus A^4(-15) \rightarrow A$$

Furthermore, the ideals of these varieties are linked by a complete intersection of length 3.

The resolution for the sum of the two ideals has the form

$$0 \rightarrow A(-46) \rightarrow A^8(-31) \oplus A(-30) \rightarrow A^8(-25) \oplus A^8(-21) \rightarrow A(-16) \oplus A^8(-15) \rightarrow A.$$

The numerator of the reduced Hilbert series of $\mathbf{k}[Y_{\sigma_3}]$ is

$$\begin{aligned} & 2t^{28} + 6t^{27} + 12t^{26} + 20t^{25} + 30t^{24} + 42t^{23} + 56t^{22} + 72t^{21} + 90t^{20} + \\ & + 110t^{19} + 124t^{18} + 132t^{17} + 134t^{16} + 130t^{15} + 120t^{14} + 105t^{13} + 91t^{12} + 78t^{11} + \\ & + 66t^{10} + 55t^9 + 45t^8 + 36t^7 + 28t^6 + 21t^5 + 15t^4 + 10t^3 + 6t^2 + 3t + 1 \end{aligned}$$

and hence the degree of Y_{σ_3} is 1640. The numerator of the reduced Hilbert series of $\mathbf{k}[Y_{\sigma'_3}]$ is

$$\begin{aligned} & 4t^{28} + 12t^{27} + 24t^{26} + 40t^{25} + 60t^{24} + 84t^{23} + 104t^{22} + 120t^{21} + 132t^{20} + \\ & + 140t^{19} + 144t^{18} + 144t^{17} + 140t^{16} + 132t^{15} + 120t^{14} + 105t^{13} + 91t^{12} + 78t^{11} + \\ & + 66t^{10} + 55t^9 + 45t^8 + 36t^7 + 28t^6 + 21t^5 + 15t^4 + 10t^3 + 6t^2 + 3t + 1 \end{aligned}$$

and hence the degree of $Y_{\sigma'_3}$ is 1960. The numerator of the reduced Hilbert series of $\mathbf{k}[Y_{\sigma_3} \cap Y_{\sigma'_3}]$ is

$$\begin{aligned} & t^{42} + 4t^{41} + 10t^{40} + 20t^{39} + 35t^{38} + 56t^{37} + 84t^{36} + 120t^{35} + 165t^{34} + 220t^{33} + 286t^{32} \\ & + 364t^{31} + 455t^{30} + 560t^{29} + 680t^{28} + 808t^{27} + 936t^{26} + 1056t^{25} + 1160t^{24} + 1240t^{23} + 1288t^{22} \\ & + 1304t^{21} + 1288t^{20} + 1240t^{19} + 1160t^{18} + 1056t^{17} + 936t^{16} + 808t^{15} + 680t^{14} + 560t^{13} \\ & + 455t^{12} + 364t^{11} + 286t^{10} + 220t^9 + 165t^8 + 120t^7 + 84t^6 + 56t^5 + 35t^4 + 20t^3 + 10t^2 + 4t + 1 \end{aligned}$$

and hence the degree of $Y_{\sigma_3} \cap Y_{\sigma'_3}$ is 20400.

Remark 5.2. We conjecture that the restriction of Y_{σ_3} and $Y_{\sigma'_3}$ to $\bigwedge^3 \mathbf{k}^8$ remain codimension 3. We have been unable to verify this even computationally; we now describe the restrictions assuming the fact.

There is an invariant Δ of degree 16 on $\mathfrak{g}_1$, which can be described as a hyperdiscriminant of the representation $\bigwedge^3 \mathbf{k}^8$, i.e., the equation of the hypersurface projectively dual to the Grassmannian (see for example [W1, §9C]).

We write $\mathbf{k}^8 = F \oplus G$ with $\dim F = 6$, $\dim G = 2$ with $A' = \text{Sym}(\mathfrak{g}_1)$. Conjecturally, when restricted to A' , the resolution of Y_{σ_3} is

$$0 \rightarrow A'^2(-31) \rightarrow A'^8(-21) \rightarrow A'(-16) \oplus A'^6(-15) \rightarrow A'.$$

We also conjecture that the ideal of Y_{σ_3} is generated by Δ , $\partial\Delta/\partial y_i$, where $\{y_i\}$ is a basis of the dual of $F \otimes \bigwedge^2 G$. As far as we are aware, this does not follow formally from the codimension conjecture in this case.

The equivariant format of this resolution is (in characteristic 0):

$$\begin{aligned} 0 \rightarrow \mathbf{S}_{12,12,12,12,12,12} F \otimes \mathbf{S}_{11,10} G \otimes A'(-31) &\rightarrow \begin{array}{c} \mathbf{S}_{8,8,8,8,8,8} F \otimes \mathbf{S}_{8,7} G \otimes A'(-9) \oplus \\ \mathbf{S}_{9,8,8,8,8,8} F \otimes \mathbf{S}_{7,7} G \otimes A'(-21) \end{array} \rightarrow \\ &\begin{array}{c} \mathbf{S}_{6,6,6,6,6,6} F \otimes \mathbf{S}_{6,6} G \otimes A'(-16) \oplus \\ \mathbf{S}_{6,6,6,6,5} F \otimes \mathbf{S}_{5,5} G \otimes A'(-15) \end{array} \rightarrow A'. \end{aligned}$$

To describe the restriction of $Y_{\sigma'_3}$, we set $\mathbf{k}^8 = F' \oplus G'$ where $\dim F' = 4$, $\dim G' = 4$ and $A'' = \text{Sym}(\mathfrak{g}_1)$. Conjecturally, when restricted to A'' , the resolution is

$$0 \rightarrow A''^4(-31) \rightarrow A''^8(-25) \rightarrow A''(-16) \oplus A''^4(-15) \rightarrow A''.$$

We conjecture that the ideal of $Y_{\sigma'_3}$ is generated by five elements Δ , $\partial\Delta/\partial z_i$, where $\{z_i\}$ is a basis of the dual of $\bigwedge^3 G'$. Again, this does not seem to follow formally from the codimension conjecture.

The equivariant format of this resolution is (in characteristic 0):

$$\begin{aligned} 0 \rightarrow \mathbf{S}_{13,12,12,12} F' \otimes \mathbf{S}_{11,11,11,11} G' \otimes A''(-31) &\rightarrow \begin{array}{c} \mathbf{S}_{10,10,10,10} F' \otimes \mathbf{S}_{9,9,9,8} G' \otimes A''(-9) \oplus \\ \mathbf{S}_{10,10,10,9} F' \otimes \mathbf{S}_{9,9,9,9} G' \otimes A''(-25) \end{array} \rightarrow \\ &\begin{array}{c} \mathbf{S}_{6,6,6,6} F' \otimes \mathbf{S}_{6,6,6,6} G' \otimes A''(-16) \oplus \\ \mathbf{S}_{6,6,6,6} F' \otimes \mathbf{S}_{6,5,5,5} G' \otimes A''(-15) \end{array} \rightarrow A''. \end{aligned} \quad \square$$

6. QUESTIONS

In the Dynkin case the most important problem is to decide whether the five resolutions we get for Dynkin types E_6 , E_7 , E_8 are generic resolutions for perfect ideals of codimension three with resolutions of these formats. This is equivalent to saying that each perfect ideal of codimension 3 with a resolution of Dynkin format has a split complex $\mathbf{F}_{\bullet}^{\text{top}}$.

The pattern with the Plücker coordinates and the pair of Schubert varieties of codimension 3 generalizes beyond Dynkin diagrams. For any $T_{p,q,r}$ with $p = 2$ we get in the homogeneous space $G(T_{p,q,r})/P_{x_1}$ (G is a Kac–Moody group corresponding to $T_{p,q,r}$, P_{x_1} is a parabolic corresponding to a simple root corresponding to vertex x_1) two opposite Schubert varieties Ω_{σ_3} and $\Omega_{\sigma'_3}$ of codimension 3. These are ind-varieties. The opposite Schubert varieties Ω_{σ_3} and $\Omega_{\sigma'_3}$ are normal and Cohen–Macaulay by [KS, Proposition 3.4].

In this case there is no analogue of the big cell Y . Instead we should do the following. Let us denote by p_w the Plücker coordinate corresponding to $w \in W/W_P$. We have open sets U_v ($v \in W/W_P$) in $G(T_{p,q,r})/P_{x_1}$ consisting of points for which $p_v \neq 0$. The sets U_v are infinite dimensional affine spaces. One should look at the sets $Y_{v,\sigma_3} := \Omega_{\sigma_3} \cap U_v$ and $Y_{v,\sigma'_3} := \Omega_{\sigma'_3} \cap U_v$.

In the non-Dynkin cases the main questions are as follows.

- (1) Can we find a sequence of open cells Y_n , cofinal in the Bruhat order on W/W_P , so that for each n , the defining ideals of $Y_n \cap Y_{\sigma_3}$ and of $Y_n \cap Y_{\sigma'_3}$ have the resolutions of the corresponding formats?
- (2) If the answer to the preceding question is yes, one could ask whether the series of resolutions $\mathbf{F}_{\bullet}^{(n)}$ of the cyclic modules given by the defining ideals of $Y_n \cap Y_{\sigma_3}$ and of $Y_n \cap Y_{\sigma'_3}$ could have the versality property with respect to free resolutions of perfect

ideals of this format, i.e., each such resolution comes by a change of rings from the resolution $\mathbf{F}_\bullet^{(n)}$ for some n .

One needs first to deal with the affine cases, i.e., the diagrams $\hat{E}_7 = T_{2,4,4}$ (self-linked format $(1, 6, 8, 3)$) and with $\hat{E}_8 = T_{2,3,6}$ (formats $(1, 8, 9, 2)$ and $(1, 5, 9, 5)$).

REFERENCES

- [BK] Michel Brion, Shrawan Kumar, *Frobenius Splitting Methods in Geometry and Representation Theory*, Progress in Mathematics **231**, Birkhäuser, 2005.
- [CLKW] Ela Celikbas, Jai Laxmi, Witold Kraśkiewicz, Jerzy Weyman, The family of perfect ideals of codimension 3, of type 2 with 5 generators, *Proc. Amer. Math. Soc.* **148** (2020), 2745–2755, [arXiv:1812.09736v1](#).
- [BE] David A. Buchsbaum and David Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, *Amer. J. Math.* **99** (1977), no. 3, 447–485.
- [CVW] Lars Winther Christensen, Oana Veliche, and Jerzy Weyman, The structure of almost complete intersections of codimension three; in preparation.
- [KS] M. Kashiwara, M. Shimozono, Equivariant K-theory of affine flag manifolds and affine Grothendieck polynomials, *Duke Math. J.* **148** (2009), no. 3, 501–538, [arXiv:math/0601563v2](#).
- [LMS] Lakshmi Bai, C. Musili, C. S. Seshadri, Cohomology of line bundles on G/B , *Ann. Scient. Éc. Norm. Sup.* **7** (1974), 89–138.
- [LR] V. Lakshmibai, K.N. Raghavan, *Standard Monomial Theory*, Encyclopaedia of Mathematical Sciences, Vol. 137, Springer-Verlag, Berlin, Heidelberg, 2008.
- [LW] Kyu-Hwan Lee, Jerzy Weyman, Some branching formulas for Kac-Moody Lie algebras, *Commun. Korean Math. Soc.* **34** (2019), no. 4, 1079–1098, [arXiv:1804.10251v2](#).
- [U] Bernd Ulrich, Sums of linked ideals, *Trans. Amer. Math. Soc.* **318** (1990), no. 1, 1–42.
- [W1] Jerzy Weyman, *Cohomology of Vector Bundles and Syzygies*, Cambridge Tracts in Mathematics **149**, Cambridge University Press, 2003.
- [W2] Jerzy Weyman, Generic free resolutions and root systems, *Ann. Inst. Fourier (Grenoble)* **68** (2018) no. 3, 1241–1296, [arXiv:1609.02083v1](#).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT SAN DIEGO, LA JOLLA, CA 92093
Email address: ssam@ucsd.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT 06269
 UNIwersytet Jagielloński, KRAKÓW, POLAND
Email address: jerzy.veyman@uconn.edu, jerzy.veyman@uj.edu.pl