A Comment on and Correction to: Opinion dynamics in the presence of increasing agreement pressure

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Abstract

We identify counter-examples to the consensus result given in [J. Semonsen et al. Opinion dynamics in the presence of increasing agreement pressure. *IEEE Trans. Cyber.*, 49(4): 1270-1278, 2018]. We resolve the counter-examples by replacing Lemma 5 in the given reference with a novel variation of the Banach Fixed Point theorem which explains both the numerical results in the reference and the counter-example(s) in this note, and provides a sufficient condition for consensus in systems with increasing peer-pressure. This work is relevant for other papers that have used the proof technique from Semonsen et al. and establishes the veracity of their claims assuming the new sufficient condition.

1 Introduction

In this technical note we correct and clarify a consensus result given in [1]. This correction is relevant not only to the general literature but in particular to [2], which uses the same proof technique as [1] and [3–9], which cite [1]. We show this proof method is incomplete due to the use of a lemma drawn from outside the consensus literature. We provide a complete result and use this to prove a corrected version of Theorem 2 in [1].

In [1], the authors study a consensus problem (see e.g., [10–48]) under an increasing peer-pressure function, which seems to drive system consensus. Note that recent work on consensus is extensive and the cited work provides only a small sample of this large body of work.

[1] assumes N agents are arranged on a weighted graph with weighted adjacency matrix $\mathbf{A} \in \mathbb{R}^{N\times N}$, where self-weights are all 0. Each agent has a time-varying state $x^{(i)} \in [0,1]$ (though any inputs in $\mathbb{R}$ would suffice) and the vector $\mathbf{x}_k \in \mathbb{R}^N$ is the vector of agent states at time k. Each agent also has a *stubbornness* coefficient $s^{(i)}$ (also used in [2]) and a preferred state $x^{+^{(i)}}$, which defines a fixed vector $\mathbf{x}^+ \in \mathbb{R}^{N\times N}$. Define a diagonal matrix $\mathbf{S}$ containing the $s^{(i)}$ and a diagonal matrix $\mathbf{D}$ of row-sums of $\mathbf{A}$. The the update function studied in [1] is [1] given by:

$$\mathbf{x}_{(k)} = (\mathbf{S} + \rho_k\mathbf{D})^{-1}\left(\mathbf{S}\mathbf{x}^+ + \rho_k\mathbf{A}\mathbf{x}_{(k-1)}\right). \tag{1}$$

Here ρ_k is a time-varying peer-pressure value. This time-varying term incrementally increases the weight a vertex places on its neighbors' strategies as compared to its own stubbornness. Additional details are provided in [1]. Let:

$$f_k(\mathbf{x}) = (\mathbf{S} + \rho_k\mathbf{D})^{-1}\left(\mathbf{S}\mathbf{x}^+ + \rho_k\mathbf{A}\mathbf{x}_{(k-1)}\right). \tag{2}$$

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[1]Note we use subscripts for time in this paper, while [1] uses superscripts. The difference is only notational.

In Lemma 3 of [1] it is shown that $f_k(\mathbf{x})$ is a contraction with a fixed point:

$$\mathbf{x}_k^* = (\mathbf{S} + \rho_k \mathbf{L})^{-1} \mathbf{S}\mathbf{x}^+, \tag{3}$$

where: $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian. It is noted in Theorem 1 of [1] that:

$$\lim_{k\to\infty} \mathbf{x}_k^* = \frac{\sum_{i=1}^N s_i x_i^+}{\sum_{i=1}^N s_i} \mathbf{1}. \tag{4}$$

That is the fixed points of the individual contractions converge to the stubbornness weighted mean of the agents' preferred states. The authors state Lemma 5, taken directly from [49, 50]:

Lemma 1.1 (Theorem 1 of [49] & Theorem 2 of [50])**.** *Let $\{f_n\}$ be a sequence of analytic contractions in a domain D with $f_n(D) \subseteq E \subseteq D_0 \subseteq D$ for all n. Then $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$ converges uniformly in D_0 and locally uniformly in D to a constant function $F(z) = c \in E$. Furthermore, the fixed points of f_n converge to the constant c.* □

[1] then uses this to argue (in Theorem 2) that when:

$$G_k(\mathbf{x}) = (f_k \circ \cdots \circ f_1)(\mathbf{x}),$$

if $\rho_k \to \infty$, then:

$$\lim_{k\to\infty} G_k(\mathbf{x}_0) = \mathbf{x}^*.$$

That is the iteration of the f_k with increasing ρ_k (i.e., increasing peer-pressure) leads to a consensus point.

In the next section, we show this is not a complete statement and that the system may fail to converge for certain choices of increasing ρ_k. The failure in this case is due to the use of Lemma 1.1 (Lemma 5 of [1]), which appears not to be valid in this case because the containment requirement in the lemma does not ensure that the iterated contractions shrink to a fixed point. We then prove a variation of the Banach fixed-point theorem, which explains our example's failure to converge and provides a correct sufficient condition for convergence, thus completing Theorem 2 of [1].

2 Counter-Example to Consensus

Consider the simple graph K_2 with the following inputs:

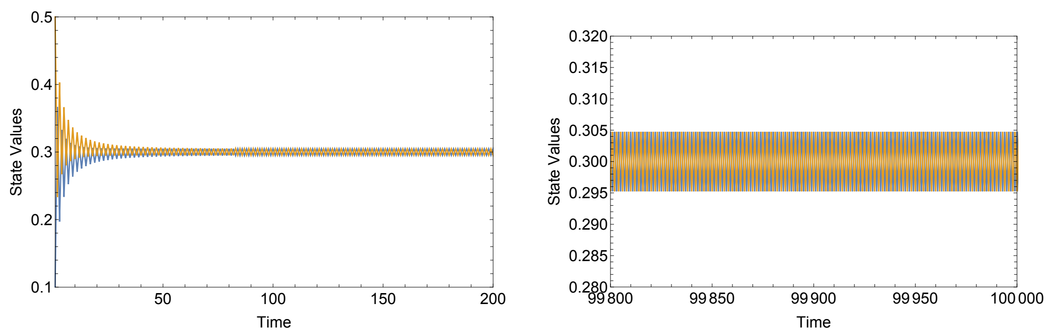
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{S} = \mathbf{D} = \mathbf{I}_2.$$

Let the initial condition and preferred agent states be given by $\mathbf{x}^+ = \langle 0.1, 0.5\rangle$. Assume we define the exponentially increasing peer-pressure function:

$$\rho_k = 2^{\sqrt{k}},$$

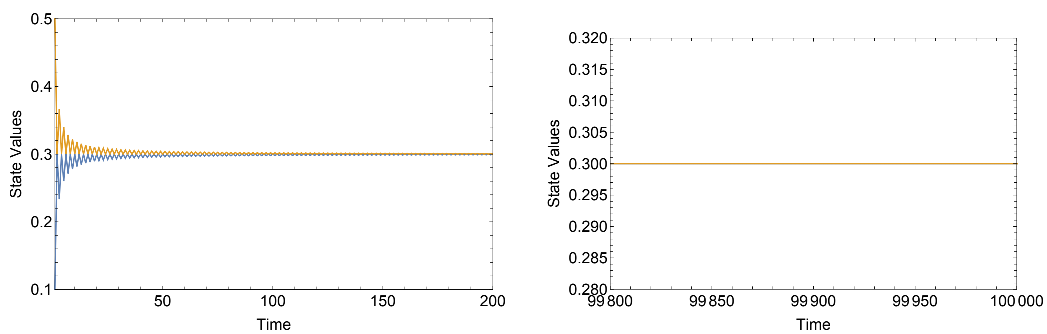
which provides some numerical stability (i.e., does not blow up too quickly) but also shows exponential growth. Simulation of Eq. (1) shows the system does not converge to the expected $\mathbf{x}^* = \langle 0.3, 0.3\rangle$ as given by Eq. (4), but instead oscillates about this point indefinitely. This is shown in Fig. 1. However, if we replace the peer-pressure function with:

$$\rho_k = k,$$



(a) Initial Behavior (b) Long Run Oscillation

Figure 1: Initially fast convergence around the mean point $\mathbf{x}^* = \langle 3, 3 \rangle$ slows and becomes oscillation, showing neutral stability, rather than asymptotic stability.



(a) Initial Behavior (b) Long Run Convergence

Figure 2: Asymptotic convergence to the point $\mathbf{x}^* = \langle 3, 3 \rangle$ is illustrated. As noted in [1] this convergence is linear.

then we see the system converges as expected from Theorem 1 of [1]. This is shown in Fig. 2. It is clear from this example that the issue with Theorem 1 is not the statement of the theorem, but its lack of qualification on the growth of ρ_k. This stems directly from Lemma 5 of [1] (or Theorem 1 of [49] & Theorem 2 of [50]), which also does not qualify the analytic contraction to be used. However, it is clear that this pathology is an example of an even easier example.

Consider the family of functions $f : \mathbb{R} \to \mathbb{R}$.

$$f_k(x) = \left(1 - \frac{1}{10^n}\right) x. \tag{5}$$

Each $f_k(x)$ has a fixed point $x_k^* = 0$, thus the fixed points x_k^* converge to $x^* = 0$ (tautologically). Moreover, each function contracts any interval containing $x = 0$ into itself. However, computing:

$$G_k(x) = (f_k \circ f_{k-1} \circ \cdots \circ f_1)(x) = \left(\prod_{i=1}^{k} \left(1 - \frac{1}{10^n}\right)\right) x,$$

we see that:

$$\lim_{k\to\infty} G_k(x) = x \cdot \lim_{k\to\infty} \prod_{n=1}^{k} \left(1 - \frac{1}{10^n}\right) = x \cdot \phi\left(\tfrac{1}{10}\right).$$

Here $\phi(\cdot)$ is Euler's function derived from the q-Pochhammer symbol. We note that:

$$\phi\left(\tfrac{1}{10}\right) \approx 0.89001.$$

Therefore, for $x \neq 0$, $G_\infty(x) \approx 0.89001x$, rather than 0 as would be expected from the statement of Lemma 5 of [1]. By contrast, if we consider the family of functions:

$$f_k(x) = \frac{k-1}{k} x, \tag{6}$$

then:

$$G_k(x) = \left(\prod_{n=1}^{k} \frac{n-1}{n}\right) x,$$

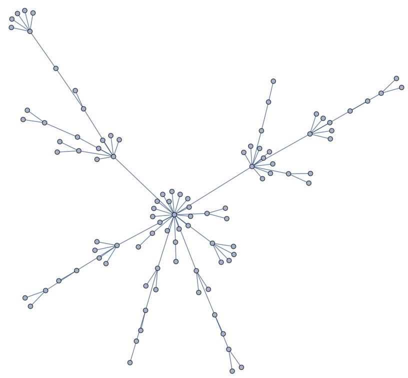
and

$$\lim_{k\to\infty} G_k(x) = x \cdot \lim_{k\to\infty} \left(\prod_{n=1}^{k} \frac{n-1}{n}\right) = 0,$$

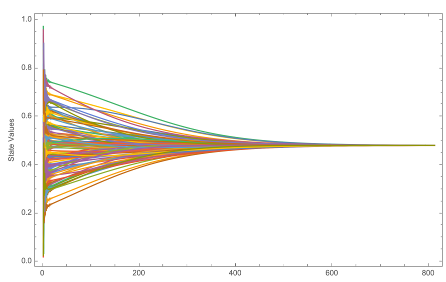
as expected. We note this counter-example can be extended to the complex plane (the domain used in Lemma 1.1). These examples suggest that any future use of Lemma 1.1 should be done with care since these examples can be extended to analytic contractions in the complex plane that appear to satisfy the inclusion requirement given in the Lemma 1.1 as taken from [49, 50].

This behavior is not limited to trivial examples. In [1] the authors present convergence on a Barabási-Albert graph [51]. We illustrate the non-convergence for ρ_k that grow too quickly using a Barabási-Albert graph with 100 vertices (see Fig. 3). When $\rho_k = 1.01^k$, the system converges to consensus as predicted in [1]. However, when $\rho_k = 10^{\sqrt{k}}$, the peer-pressure increases too quickly and the resulting dynamics do not converge but oscillate just as in the case of Fig. 1, but in a more complex setting.

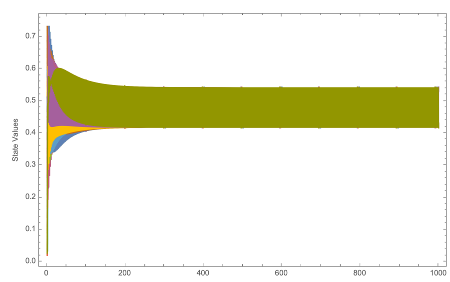
In the next section, we construct a variation of the Banach fixed point theorem that handles the conditions set forth in [1] and predicts the non-convergence of the counter-example(s) using the intuition provided by the simpler cases.



(a) Barabási-Albert Graph



(b) Convergence in the Barabási-Albert Graph



(c) Non-Convergence in the Barabási-Albert Graph

Figure 3: (a) A Barabási-Albert graph with 100 vertices used to illustrate both convergence and non-convergence. (b) An illustration of agent state convergence when $\rho_k = 1.01^k$. (c) An illustration of agent state non-convergence when $\rho_k = 10^{\sqrt{k}}$; i.e., the peer-pressure increases too quickly.

3 A Convergence Theorem

Theorem 3.1. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a family of mappings on a Banach space* X *with norm* $\|\cdot\|$ *so that each mapping* f_i *has a unique fixed point* $\mathbf{x}_i^*$ *satisfying the property:*

$$\|f_i(\mathbf{x}) - \mathbf{x}_i^*\| \leq \alpha_i \|\mathbf{x} - \mathbf{x}_i^*\| \qquad \forall \mathbf{x} \in X,$$

where $0 \leq \alpha_i < 1$ *for all* i*. Furthermore, suppose that:*

$$\lim_{i\to\infty} \mathbf{x}_i^* = \mathbf{x}^* \in X.$$

Then, if $\mathbf{x}_0 \in X$ *and*

$$\lim_{N\to\infty} \prod_{i=1}^{N} \alpha_i = 0,$$

then:

$$\lim_{k\to\infty} (f_k \circ f_{k-1} \circ \cdots \circ f_1)(\mathbf{x}_0) = \mathbf{x}^*$$

Proof. Define:

$$G_k = f_k \circ f_{k-1} \circ \cdots \circ f_1.$$

By assumption:

$$\left\|(f_{k+1} \circ G_k)(\mathbf{x}_0) - \mathbf{x}_{k+1}^*\right\| \leq \alpha_{k+1} \left\|G_k(\mathbf{x}_0) - \mathbf{x}_{k+1}^*\right\| = \alpha_{k+1} \left\|(f_k \circ G_{k-1})(\mathbf{x}_0) - \mathbf{x}_{k+1}^*\right\|.$$

Applying the triangle inequality to the last term we have:

$$\left\|(f_{k+1} \circ G_k)(\mathbf{x}_0) - \mathbf{x}_{k+1}^*\right\| \leq \alpha_{k+1} \left(\|(f_k \circ G_{k-1})(\mathbf{x}_0) - \mathbf{x}_k^*\| + \left\|\mathbf{x}_{k+1}^* - \mathbf{x}_k^*\right\|\right).$$

This is illustrated in Fig. 4.

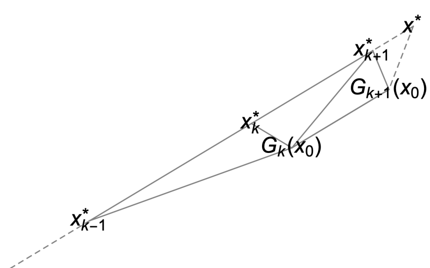


Figure 4: An illustration of some triangle inequalities used in the proof.

Applying similar logic, we see that:

$$\left\|(f_{k+1} \circ G_k)(\mathbf{x}_0) - \mathbf{x}_{k+1}^*\right\| \leq \alpha_{k+1}\alpha_k \|G_{k-1}(\mathbf{x}_0) - \mathbf{x}_k^*\| + \alpha_{k+1} \left\|\mathbf{x}_{k+1}^* - \mathbf{x}_k^*\right\|.$$

Repeating this argument, we see:

$$\left\|(f_{k+1} \circ G_k)(\mathbf{x}_0) - \mathbf{x}_{k+1}^*\right\| \leq \alpha_{k+1}\alpha_k\alpha_{k-1} \left\|G_{k-2}(\mathbf{x}_0) - \mathbf{x}_{k-1}^*\right\| + \alpha_{k+1}\alpha_k \left\|\mathbf{x}_k^* - \mathbf{x}_{k-1}^*\right\| + \alpha_{k+1} \left\|\mathbf{x}_{k+1}^* - \mathbf{x}_k^*\right\|.$$

We can continue in this way until we see that:

$$\left\|(f_{k+1} \circ G_k)(\mathbf{x}_0) - \mathbf{x}^*_{k+1}\right\| \leq \left(\prod_{i=1}^{k+1} \alpha_i\right) \|f_1(\mathbf{x}_0) - \mathbf{x}^*_1\| + \sum_{j=1}^{k} \left(\prod_{i=j+1}^{k+1} \alpha_i\right) \left\|\mathbf{x}^*_{j+1} - \mathbf{x}^*_j\right\|. \quad (7)$$

We assume that the fixed points $\mathbf{x}^*_i$ converge and therefore for any $\epsilon > 0$ there is an $N > 0$ so that:

$$\left\|\mathbf{x}^*_{N+1} - \mathbf{x}^*_N\right\| < \epsilon. \quad (8)$$

Before proceeding, recall:

$$\lim_{N\to\infty} \prod_{i=1}^{N} \alpha_i = 0, \quad (9)$$

and

$$\lim_{N\to\infty} \left\|\mathbf{x}^*_{N+1} - \mathbf{x}^*_N\right\| = 0. \quad (10)$$

Suppose we are given an $\epsilon > 0$, choose N so that:

1. $$\left(\prod_{i=1}^{k+1} \alpha_i\right) \|f_1(\mathbf{x}_0) - \mathbf{x}^*_1\| < \frac{\epsilon}{2(N+1)},$$

   which is possible because of Eq. (9).

2. For each j:
   $$\left(\prod_{i=j+1}^{N+1} \alpha_i\right) \left\|\mathbf{x}^*_{j+1} - \mathbf{x}^*_j\right\| < \frac{\epsilon}{2(N+1)}.$$

   This is possible because of the combination of Eqs. (9) and (10).

3. Finally assume:
   $$\left\|\mathbf{x}^*_{N+1} - \mathbf{x}^*\right\| < \frac{\epsilon}{2}.$$

   This is possible from the convergence of the fixed points.

Then from Eq. (7) we have:

$$\left\|G_{n+1}(\mathbf{x}_0) - \mathbf{x}^*_{n+1}\right\| < \frac{\epsilon}{2}.$$

By one more application of the triangle inequality, we have:

$$\|G_{n+1}(\mathbf{x}_0) - \mathbf{x}^*\| \leq \left\|\mathbf{x}^*_{N+1} - \mathbf{x}^*\right\| + \left\|G_{n+1}(\mathbf{x}_0) - \mathbf{x}^*_{n+1}\right\| < \epsilon.$$

This completes the proof. □

This theorem explains, as special cases, the families of functions defined in Eq. (5) and Eq. (6). In the next section, we use it to resolve the counter-examples discussed in Section 2.

4 Resolution of the Counter Examples

Returning to the example in Section 2, let $\mathbf{x}^+ = \langle a, b\rangle$, thus generalizing the initial condition. In this case, Eq. (1) can be written as:

$$f_k(\mathbf{x}) = \begin{bmatrix} \frac{(\rho+1)(a+\rho x_2)}{\rho^2+2\rho+1} \\ \frac{(\rho+1)(b+\rho x_1)}{\rho^2+2\rho+1} \end{bmatrix}. \tag{11}$$

For each $f_k(\mathbf{x})$ the explicit fixed point is given by:

$$\mathbf{x}_k^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \frac{a+\rho(a+b)}{2\rho+1} \\ \frac{b+\rho(a+b)}{2\rho+1} \end{bmatrix}. \tag{12}$$

Since $\langle s_1, s_2\rangle = \langle 1, 1\rangle$, we see that:

$$\lim_{k\to\infty} \mathbf{x}_k^* = \mathbf{x}^* = \begin{bmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{bmatrix},$$

as expected.

Because this example is particularly simple, we can compute (see Appendix A):

$$\|F(\mathbf{x}) - \mathbf{x}_k^*\|^2 = \left(\frac{\rho_k}{1+\rho_k}\right)^2 \|\mathbf{x} - \mathbf{x}_k^*\|^2, \tag{13}$$

for any $\mathbf{x}$. In Theorem 3.1, we now have:

$$\alpha_k = \frac{\rho_k}{1+\rho_k}.$$

When $\rho_k = 2^{\sqrt{k}}$, then:

$$\prod_{k=1}^{\infty} \alpha_k \approx 0.0310128 > 0,$$

which implies (as expected) that the system may not converge to the fixed point (see Fig. 1). On the other hand, when $\rho_k = k$,

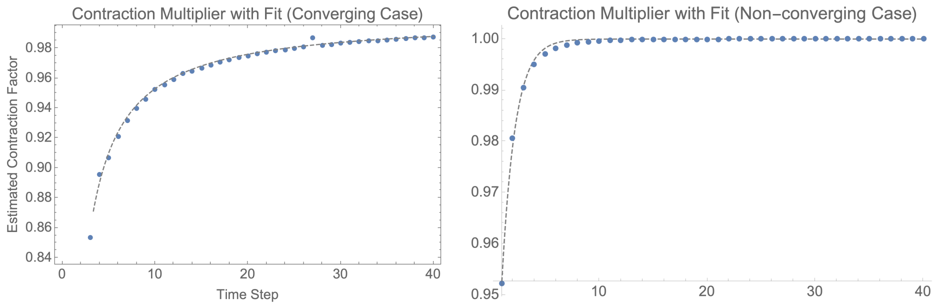
$$\prod_{k=1}^{\infty} \alpha_k = 0,$$

ensuring that the system will converge to the weighted average of $\mathbf{x}^+$.

We can likewise resolve the counter-example using the Barabási-Albert graph. It is difficult to compute an exact contraction factor in this case. However, given an initial value $\mathbf{x}^+$ and a peer-pressure function ρ_k, it is possible to approximate an upper-bound on the contraction at each step by solving the optimization problem:

$$\begin{cases} \max\limits_{\mathbf{x}} & \dfrac{\|f_k(\mathbf{x}) - \mathbf{x}_k^*\|_\infty}{\|\mathbf{x} - \mathbf{x}_k^*\|_\infty} \\ s.t. & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}, \end{cases}$$

where $f_k(\mathbf{x})$ is defined in Eq. (2) and $\mathbf{x}_k^*$ is defined in Eq. (3). Here we use the ∞-norm because the matrix norm used to show f_k is a contraction in Lemma 3 of [1] is the infinity norm. For the



(a) Converging Contraction Factor

(b) Converging Contraction Factor

Figure 5: (a) Illustration of the computed contraction factor and its non-linear fit in the case of convergence in the Barabási-Albert graph. (b) Illustration of the computed contraction factor and its non-linear fit in the case of non-convergence in the Barabási-Albert graph. Notice it increases toward 1 much faster than the converging case.

counter-example using the 100 vertex Barabási-Albert graph, we plot the bounds on the contraction factor in Fig. 5. In the case when $\rho_k = 1.01^k$, the contraction factor grows according to:

$$\alpha_k \sim \frac{k}{0.493345 + k}.$$

Using this analysis we can see the product of the fit for α_k approaches zero explaining why this case seems to converges to effective consensus (i.e., consensus to a numerical tolerance). On the other hand, when $\rho_k = 10^{\sqrt{k}}$, we see that:

$$\alpha_k \sim 1 - 10^{-0.355178x - 0.969752}.$$

In this case, the product of the fit for α_k converges to ~ 0.917, explaining why the process does not converge to consensus. This numerical approximation can be used as a rule of thumb to determine whether a system is likely to converge (numerically).

Before concluding, it is worth noting that if we used $\rho_k = 1.01^k$ with the graph K_2 (the graph from the first counter-example), we would see that:

$$\prod_{k=1}^{\infty} \alpha_k = \prod_{k=1}^{\infty} \frac{1.01^k}{1 + 1.01^k} \approx 1.79 \times 10^{-36},$$

implying that the dynamics may not truly converge to perfect consensus. However numerically we may not be able to distinguish this as illustrated with the Barabási-Albert graph. Thus it may be sufficient to ensure the product of the contraction factors is *small enough* to achieve effective numerical consensus in a real-world system.

4.1 Correction to Theorem 2 of [1]

Using this information, we can correct Theorem 2 of [1] to read:

Theorem 4.1 (Clarification of Theorem 2 of [1])**.** *Let*

$$f_k(\mathbf{x}) = \left(\mathbf{S} + \rho^{(k)}\mathbf{D}\right)^{-1} \left(\mathbf{S}\mathbf{x}^+ + \rho^{(k)}\mathbf{A}\mathbf{x}\right).$$

Define:

$$\mathbf{x}_k = f_k(\mathbf{x}_{k-1}),$$

with $\mathbf{x}_0$ *given (and assumed to be* $\mathbf{x}^+$*). If* $\rho_k \to \infty$ *and*

$$\|f_i(\mathbf{x}) - \mathbf{x}_i^*\| \leq \alpha_i \|\mathbf{x} - \mathbf{x}_i^*\| \qquad \forall \mathbf{x} \in X,$$

so that the resulting contraction constants α_k *of* f_k *satisfy:*

$$\prod_{k=1}^{\infty} \alpha_k = 0,$$

then

$$\lim_{k\to\infty} \mathbf{x}_k = \frac{\sum_i s_i \mathbf{x}_{k_i}}{\sum_i s_i}.$$

□

As a final remark, we note that this result could be anticipated from the convergence rate analysis in [1], which shows that Eq. (1) is an instance of gradient descent. Convergence guarantees for such an algorithm require satisfaction of the Wolfe Conditions and several pathological examples exist in which the step length (governed by ρ_k) is improperly defined leading to oscillation in gradient descent (see [52]). Thus, one might view Theorem 4.1 as a specialized sufficient condition on step length in this gradient descent.

5 Conclusions

In this technical note, we corrected and clarified Theorem 2 of [1]. This correction is important because the proof method has been used by other authors [2]. The correction is based on replacing a lemma (Lemma 5) used in [1] with a new variation on the Banach Fixed Point theorem. The modified theorem(s) now ensure results in [1] and [2] can be used for development of consensus systems or for their further study.

Acknowledgement

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A Computation of the Norms

Computing directly we have:

$$\|f_k(\mathbf{x}) - \mathbf{x}_k^*\|^2 = \left(\frac{a + \rho x_2}{\rho + 1} - \frac{a + \rho(a+b)}{2\rho + 1}\right)^2 + \left(\frac{b + \rho x_1}{\rho + 1} - \frac{b + \rho(a+b)}{2\rho + 1}\right)^2.$$

We also have:

$$\|\mathbf{x} - \mathbf{x}_k^*\|^2 = \left(x_1 - \frac{\rho(a+b) + a}{2\rho + 1}\right)^2 + \left(x_2 - \frac{\rho(a+b) + b}{2\rho + 1}\right)^2.$$

Dividing these expressions into each other and simplifying[2] yields Eq. (13).

[2] Using Mathematica™.

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