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Optimal bilinear control of a reparable multi-state system

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ABSTRACT
This work addresses an optimal bilinear control design representing the corrective maintenance for a reparable multi-state system. The primary interest is to optimise the availability of the system, which is defined as the probability that the system is operating properly when it is requested for use. The system model considered in our current work is governed by coupled transport and integro-differential equations. A corrective maintenance policy is represented by the repair rate, which depends on the distributed repair time. The objective is to determine an optimal repair rate that maximises the availability of the system in good mode over a given system running period. This essentially leads to a bilinear control problem set in a nonreflexive Banach space using L^1-optimisation. A rigorous proof of existence of an optimal controller and the first-order necessary conditions of optimality are addressed. Numerical experiments are conducted to demonstrate the theoretical results.

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1. Introduction

Reparable systems occur naturally in product design, inventory systems, computer networking and complex manufacturing processes. A reparable system operates under a maintenance strategy that calls for repair actions whenever a failure occurs. These actions revise the overall function of the system. There is an extensive literature on mathematical modelling and analysis of the reparable systems. In this work, we mainly focus on the systems with arbitrarily distributed repair time, governed by distributed parameter systems of coupled transport and integro-differential equations (cf. Chung, 1981; Gupur, 2003, 2011, 2016; Haji & Gupur, 2004; Hu et al., 2007; Xu et al., 2005), where the methods of Markov chain and supplementary variable techniques are used to derive the general mathematical models. Interesting applications can be found in reliability engineering and in the study of supply chain and queueing network modelling (M/M/1 and M/G/1, etc.) (cf. Gupur, 2010, 2011; Gupur et al., 2001; Haji & Radl, 2007a, 2007b; Y. Xin, Li, et al., 2008; Y. H. Xin, Zheng, et al., 2008; Zhao et al., 2009). In particular, we aim at optimising the availability of repairable systems through optimal maintenance strategies. Here availability is defined as the probability that the system is operating properly when it is requested for use (cf. Bazovsky, 2004).

Although the well-posedness and asymptotic behaviour of the mathematical models for reparable systems have been thoroughly addressed by using the C_0-semigroup approach, there are few results, to the best of our knowledge, on the design of optimal maintenance strategies for enhancing the system availability. Moreover, the repair rate is always assumed to be given in the aforementioned literature, which is not realistic for most real-life applications. This work will bring new perspectives to investigate this type of systems. We aim at deriving an optimal repair action for providing practical and efficient maintenance strategies to improve the availability of reparable systems. Numerical schemes will be constructed to implement and validate our design. To demonstrate our idea, we shall focus on the discussion on a multi-state reparable system introduced by Chung (1981), which represents the general features of this type of models. Our theoretical and numerical approaches are in sufficient generality to be applied to address other related problems.

Consider that there are M modes of failure associated with a device. The state of the device is given by its failure mode number j, $j = 1, 2, \dots, M$, and 0 represents the good state. The device is good at time zero and transitions are permitted only between states 0 and j. The failure rates are constant and repair times are arbitrarily distributed. The transition diagram for the system is demonstrated by Figure 1. The following assumptions are associated with the device:

(1) All failures are statistically independent;
(2) Repair is to like-new and it does not cause damage to any other part of the system.
(3) The repair process begins soon after the device is in failure state;
(4) No further failure can occur while the device is down.

The precise model of system equations reads

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -\sum_{j=1}^{M} \lambda_j p_0(t) + \sum_{j=1}^{M} \int_0^\infty \mu_j(x) p_j(x,t)\,\mathrm{d}x, \tag{1}$$

$$\frac{\partial p_j(x,t)}{\partial t} + \frac{\partial p_j(x,t)}{\partial x} = -\mu_j(x) p_j(x,t), \tag{2}$$

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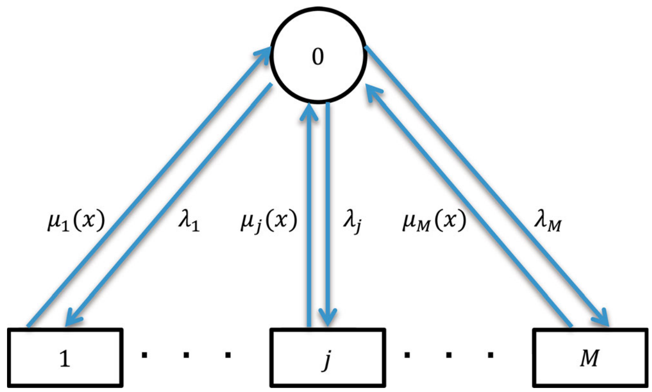

Figure 1. Transition diagram of the reparable two-state system.

with boundary condition

$$p_j(0,t) = \lambda_j p_0(t), \quad j = 1,2,\ldots,M,\ t > 0, \tag{3}$$

and initial conditions

$$p_0(0) = 1, \quad p_j(x,0) = 0, \quad j = 1,2,\ldots,M. \tag{4}$$

Here the involved variables and parameters are defined as follows:

(a) $p_0(t)$: probability that the device is in good mode 0 at time t;
(b) $p_j(x,t)$: probability density (with respect to repair time x) that the failed device is in failure mode j at time t and has an elapsed repair time of x. Let $\hat{p}_j(t)$ denote the probability of the device in failure mode 1 at time t, then $\hat{p}_j(t)$ is given by

$$\hat{p}_j(t) = \int_0^\infty p_j(x,t)\,\mathrm{d}x; \tag{5}$$

(c) $\lambda_j > 0$: constant failure rate of the device for failure mode j;
(d) $\mu_j(x) \geq 0$: repair rate when the device is in failure state j and has an elapsed repair time of x.

The pointwise availability of the system is defined as the probability that the system is in a good state, i.e. $A(t) = p_0(t)$ and the steady-state availability is given by $A^* = \lim_{t\to\infty} p_0(t)$.

The mathematical model (1)–(4) for the reparable system in essence describes a birth-death process, which shares some common features as the population dynamics discussed in (cf. Barbu et al., 2001; Hegoburu et al., 2018; Song, 1980; Song et al., 1988; Webb, 1985; Yu et al., 1999). However, the reparable system considered in our work has the unique property that the system is conservative and the problem is naturally set in an L^1-based nonreflexive Banach space. Moreover, the repair rate $\mu_j(x)$ will play a role as the control input of the system and can be interpreted as a corrective maintenance policy. Mathematically, this leads to a bilinear control problem, which is intrinsically different from those studied in the literature for the population dynamics.

As a starting point to understand the problem of optimal control design for system (1)–(4), we first introduce some basic mathematical results regarding the well-posedness of this model.

1.1 Well-posedness of the model

For given failure and repair rates $\lambda_j > 0$ and $\mu_j(x) \geq 0, j = 1,2,\ldots,M$, the well-posedness and stability issues of system (1)–(4) have been discussed in Hu et al. (2007), Hu (2016) and Xu et al. (2005) by using C_0-semigroup theory. For the convenience of the reader, we recall the following results.

Define the state space $X = \mathbb{R} \times (L^1(0,\infty))^M$ with $\|\cdot\|_X = |\cdot| + \sum_{j=1}^M \|\cdot\|_{L^1(0,\infty)}$. Let $\vec{p} = (p_0, p_1, \ldots, p_M)^{\mathrm{T}} \in X$. The system operator $\mathcal{A}$ and its domain are defined by

$$\mathcal{A}\vec{p} = \begin{bmatrix} -\sum_{j=1}^M \lambda_j p_0 + \sum_{j=1}^M \int_0^\infty \mu_j(x) p_j(x)\,\mathrm{d}x \\ -\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mu_1(x)\right) p_1(x) \\ \vdots \\ -\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mu_M(x)\right) p_M(x) \end{bmatrix} \tag{6}$$

and

$$D(\mathcal{A}) = \left\{ \vec{p} \in X \,\middle|\, \frac{\mathrm{d}p_j(x)}{\mathrm{d}x} \in L^1(0,\infty), \int_0^\infty \mu_j(x) p_j(x)\,\mathrm{d}x < \infty, \text{ and } p_j(0) = \lambda_j p_0,\ j = 1,2,\ldots,M \right\}.$$

System (1)–(4) can be rewritten as an abstract Cauchy problem in Banach space X

$$\dot{\vec{p}}(t) = \mathcal{A}\vec{p}(t), \quad t > 0, \tag{7}$$

$$\vec{p}(0) = \vec{p}_0 = (1,0,\ldots,0)^{\mathrm{T}}. \tag{8}$$

It is proven in Xu et al. (2005) that system operator $\mathcal{A}$ generates a positive C_0-semigroup of contraction, denoted by $S(t),\ t \geq 0$. Thus the solution to (1)–(4) is nonnegative if the initial data are nonnegative. Moreover, 0 is a simple eigenvalue of the system operator and also a unique spectral point on the imaginary axis. Under appropriate assumptions on repair rate $\mu_j(x)$, it can be shown that the time-dependent solution, i.e. the pointwise availability of the system, exponentially converges to the steady-state availability, which is the eigenfunction associated with eigenvalue 0 (Hu, 2016; Hu et al., 2007).

The remainder of this paper is organised as follows. In Section 2, we formulate an optimal control problem to seek for an optimal repair policy. In Section 3, we present a rigorous mathematical proof for the existence of an optimal solution, and then drive the first-order optimality conditions for solving such a solution in Section 4. Finally, we construct numerical algorithms and conduct numerical experiments for implementing and validating our design in Sections 5 and 6.

2. Optimal repair rate design

The present work will mainly focus on optimal maintenance designs for optimising the availability of the system over a finite time interval, i.e. $0 < t \le T < \infty$. Maintenance plays a crucial role in the lifetime behaviour of a reparable system. It revises the system's overall reliability, availability, downtime, cost of operation, etc. (cf. Bazovsky, 2004; Gilardoni & Colosimo, 2007; Moubray, 2001; Sandler, 2012). In general, there are three types of maintenance actions: corrective maintenance, preventive maintenance and inspections (cf. Sandler, 2012). Some preliminary study on the optimal preventive maintenance for the current model can be found in Boardman et al. (2019), Hu and Khong (2017) and Wei et al. (2016), in which the internal (or distributed) linear control design was considered under the assumption that $\mu_j(x)$ are prescribed. In contrast, this work develops an optimal corrective maintenance policy by taking repair functions $\mu_j(x)$, $j = 1, 2, \ldots$, as the control inputs. Corrective maintenance serves to restore a failed system to operational status. This usually involves replacing or repairing the component that is responsible for the failure of the overall system. Since the component's failure time is not known *a priori*, repair is performed at unpredictable (i.e. random) intervals. Our objective of corrective maintenance is to restore the system to be properly operational over a given system running period with an optimal repair cost, as to improve the pointwise availability of the system.

For a given $T > 0$, let $Z = L^\infty(0,T) \times (L^\infty(0,T; L^1(0,T)))^M$. Assume that initially the system is brand new, i.e. initial condition (4) holds.

We seek for the optimal repair rates $\mu_j(x)$, $j = 1, 2, \ldots, M$, that maximise the overall performance of the system in good mode for $t \in [0, T]$ as well as at the final time $t = T$, with optimal repair costs. In addition, we utilise the L^1-penalisation for the repair rates to build in the spatial sparsity. Thereby, the objective is to minimise the following cost functional

$$J(\vec{\mu}) = -a\int_0^T p_0(t)\,\mathrm{d}t - bp_0(T) + \alpha\sum_{j=1}^M\int_0^T \mu_j(x)\,\mathrm{d}x + \frac{\beta}{2}\sum_{j=1}^M\int_0^T \mu_j^2(x)\,\mathrm{d}x, \qquad (P)$$

subject to the governing system (1)–(4), where $\vec{\mu}(x) = (\mu_1(x), \ldots, \mu_M(x))^{\mathrm{T}} \ge 0$ and the parameters $a, b \ge 0$ and $\alpha, \beta \ge 0$ are constants and stand for the state and control weight, respectively. Assume that neither a and b nor α and β vanish simultaneously. The penalisation of a quadratic term in J usually helps to improve the regularity of the problem. Other penalisations can be also properly introduced to the cost functional depending on the relevant applications. Now consider the set of admissible controls to be

$$U_{ad} = \{\vec{\mu}(x) = (\mu_1(x), \ldots, \mu_M(x))^{\mathrm{T}} \in (L^\infty(0,T))^M : 0 \le \mu_j(x) \le \bar{\mu},\ j = 1, 2, \ldots, M\}, \qquad (9)$$

where $\bar{\mu}$ is some maximum feasible repair rate. Realistically speaking, although the maximum repair rate of each μ_j may vary with respect to different failure modes, without loss of generality, we assume that their maximum is uniformly bounded by some feasible repair rate $\bar{\mu}$.

Note that our formulation gives rise to a bilinear control problem due to the product terms in (1)–(2). As a consequence, problem (P) is nonconvex, and this creates some technical difficulties in studying the existence and uniqueness of an optimal control. In fact, uniqueness does not hold in general. Furthermore, the optimality conditions must be derived from differentiability arguments, which require some technical analysis. In addition, to solve the resulting optimality system, one has to solve the governing system forward in time, coupled with the adjoint system backward in time together with a nonlinear optimality condition. This results in extremely high computational costs and often intractable problems, especially, when the system is of large scale upon suitable discretisation.

In the sequel, we shall rigorously address these issues and develop numerical schemes specific to the treatment for our problem. We also examine the system responses with respect to different setup of the state and control parameters. Some preliminary theoretical results on the case with $M = 1$ have been presented in Boardman et al. (2019), where no numerical algorithms were developed.

3. Existence of an optimal repair rate

To show the existence of an optimal repair rate to problem (P), it is necessary to understand the regularity of the solution to (1)–(4).

Let $L_x^p(0,T)$ and $L_t^p(0,T)$ stand for the L^p-spaces, $1 \le p \le \infty$, with respect to x and t, and $W^{k,p}(0,T) = \{\phi \in L^p(0,T) \colon D^\alpha\phi \in L^p(0,T),\ \forall\alpha \le k\}$ be the standard Sobolev space. The following results holds.

Proposition 3.1: *If $\vec{p}(x,t) = (p_0(t), p_1(x,t), \ldots, p_M(x,t))^{\mathrm{T}}$ is the solution to (1)–(4), then*

$$\vec{p}(x,t) = \begin{cases} \begin{pmatrix} p_0(0)\,\mathrm{e}^{-\sum_{j=1}^M \lambda_j t} + \sum_{j=1}^M \int_0^t \mathrm{e}^{-\sum_{j=1}^M \lambda_j (t-\tau)} \\ \int_0^\tau \mu_j(x) p_j(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ \lambda_1 p_0(t-x)\,\mathrm{e}^{-\int_0^x \mu_1(s)\,\mathrm{d}s} \\ \vdots \\ \lambda_M p_0(t-x)\,\mathrm{e}^{-\int_0^x \mu_M(s)\,\mathrm{d}s} \end{pmatrix}, & x < t, \\ \begin{pmatrix} p_0(0)\,\mathrm{e}^{-\sum_{j=1}^M \lambda_j t} + \sum_{j=1}^M \int_0^t \mathrm{e}^{-\sum_{j=1}^M \lambda_j (t-\tau)} \\ \int_0^\tau \mu_j(x) p_j(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ 0 \\ \vdots \\ 0 \end{pmatrix}, & x \ge t. \end{cases} \qquad (10)$$

Moreover, $(p_0(t), p_1(x,t), \ldots, p_M(x,t))^{\mathrm{T}} \ge 0$ for any $x, t > 0$,

$$p_0(t) + \sum_{j=1}^M \int_0^T p_j(x,t)\,\mathrm{d}x = 1, \quad \forall t > 0, \qquad (11)$$

and

$$p_0 \in W^{1,\infty}(0,T) \quad \text{and}$$

$$p_j \in L^\infty(0,T;W^{1,1}(0,T)) \cap W^{1,\infty}(0,T;L^1(0,T)),\quad j=1,2,\dots,M. \tag{12}$$

Proof: We first solve $p_j(x,t)$ by using the method of characteristics., Let $\xi = x-t$ and $\tilde{p}_j(t) = p_j(\xi+t,t)$. Then

$$\frac{\mathrm{d}\tilde{p}_j}{\mathrm{d}t} = \frac{\partial p_j}{\partial x} + \frac{\partial p_j}{\partial t} = -\mu_j(\xi+t)p_j(\xi+t,t) = -\mu_j(\xi+t)\tilde{p}_j(t). \tag{13}$$

For $\xi < 0$, i.e. $x<t$, then integrating (13) from $-\xi$ to t and using $\tilde{p}_j(-\xi) = p_j(0,-\xi) = p_j(0,t-x)$, we obtain

$$p_j(x,t) = \tilde{p}_j(t) = \tilde{p}_j(-\xi)\,\mathrm{e}^{-\int_{-\xi}^t \mu(\xi+\tau)\,\mathrm{d}\tau} = p_j(0,t-x)\,\mathrm{e}^{-\int_0^x \mu_j(s)\,\mathrm{d}s}, \tag{14}$$

where $p_j(0,t-x) = \lambda_j p_0(t-x)$ by boundary condition (3). For $\xi \geq 0$, i.e. $x \geq t$, then integrating (13) from 0 to t and using $\tilde{p}_j(0) = p_j(\xi,0) = p_j(x-t,0)$, we have

$$p_j(x,t) = \tilde{p}_j(t) = \tilde{p}_j(0)\,\mathrm{e}^{-\int_0^t \mu_j(\xi+\tau)\,\mathrm{d}\tau} = p_j(x-t,0)\,\mathrm{e}^{-\int_{x-t}^x \mu_j(s)\,\mathrm{d}s} = 0,$$

where $p_j(x-t,0) = 0$ for $x \geq t$ due to initial condition (4).

To solve $p_0(t)$, we first note that $p_j(x,t) = 0$ for $x \geq t$, indicates

$$\int_0^\infty \mu_j(x)p_j(x,t)\,\mathrm{d}x = \int_0^t \mu_j(x)p_j(x,t)\,\mathrm{d}x \tag{15}$$

in (1). Then it is straightforward to derive $p_0(t)$ by using the variation of parameters formula. This completes the proof of (10). The nonnegativity of the solution has been addressed in Section 1.1 due to the positivity of the C_0-semigroup generated by the system operator.

We now proceed to derive (11). Taking the integral of (2) with respect to x from 0 to T, and then adding it to (1) follow

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} + \sum_{j=1}^M \frac{\mathrm{d}}{\mathrm{d}t}\int_0^T p_j(x,t)\,\mathrm{d}x = -\sum_{j=1}^M p_j(T,t) = 0, \quad \forall 0 < t \leq T, \tag{16}$$

which implies

$$p_0(t) + \sum_{j=1}^M \int_0^T p_j(x,t)\,\mathrm{d}x = p_0(0) + \sum_{j=1}^M \int_0^T p_j(x,0)\,\mathrm{d}x = 1, \quad \forall 0 < t \leq T, \tag{17}$$

and hence the summation of the probability of the system in good and failure modes is always 1. In other words, the system is conservative. Consequently,

$$\sup_{t\in[0,T]} |p_0| \leq 1 \quad \text{and} \quad \sum_{j=1}^M \sup_{t\in[0,T]} \|p_j\|_{L^1_x(0,T)} \leq 1. \tag{18}$$

To show the regularity of the solution, with the help of (1), (9), and (18) we have

$$\sup_{t\in[0,T]} \left|\frac{\mathrm{d}p_0}{\mathrm{d}t}\right| \leq \|p_0\|_{L^\infty_t(0,T)} \sum_{j=1}^M \lambda_j + \sum_{j=1}^M \sup_{t\in[0,T]} \int_0^T \mu_j(x)p_j(x,t)\,\mathrm{d}x \leq \sum_{j=1}^M \lambda_j + \bar{\mu}, \tag{19}$$

which yields $p_0 \in W^{1,\infty}(0,T)$. Moreover, for $x<t$, using (10) follows

$$\frac{\partial p_j(x,t)}{\partial x} = -\lambda_j \frac{\mathrm{d}p_0(t-x)}{\mathrm{d}t}\,\mathrm{e}^{-\sum_{j=1}^M \int_0^x \mu_j(s)\,\mathrm{d}s} - \lambda_j p_0(t-x)\,\mathrm{e}^{-\int_0^x \mu_j(s)\,\mathrm{d}s}\mu_j(x). \tag{20}$$

Combining (20) with (18)–(19) yields

$$\begin{aligned}
\sup_{t\in[0,T]} \int_0^T \left|\frac{\partial p_j(x,t)}{\partial x}\right| \mathrm{d}x
&\leq \sup_{t\in[0,T]} \int_0^T \lambda_j \left|\frac{\mathrm{d}p_0(t-x)}{\mathrm{d}t}\right| \mathrm{e}^{-\int_0^x \mu_j(s)\,\mathrm{d}s}\,\mathrm{d}x \\
&\quad + \sup_{t\in[0,T]} \int_0^T \lambda_j p_0(t-x)\,\mathrm{e}^{-\int_0^x \mu_j(s)\,\mathrm{d}s}\mu_j(x)\,\mathrm{d}x \\
&\leq \lambda_j \left(\sup_{t\in[0,T]} \left|\frac{\mathrm{d}p_0}{\mathrm{d}t}\right|\right) \int_0^T \mathrm{e}^{-\int_0^x \mu_j(s)\,\mathrm{d}s}\,\mathrm{d}x \\
&\quad + \lambda_j \left(\sup_{t\in[0,T]} p_0(t)\right) \int_0^T \mathrm{e}^{-\int_0^x \mu_j(s)\,\mathrm{d}s}\mu_j(x)\,\mathrm{d}x \\
&\leq \lambda_j \left(\sum_{j=1}^M \lambda_j + \bar{\mu}\right) T + \lambda_j
\end{aligned}$$

where in the last step we used

$$\int_0^T \mathrm{e}^{-\int_0^x \mu_j(s)\,\mathrm{d}s}\mu_j(x)\,\mathrm{d}x = -\int_0^T \mathrm{d}\mathrm{e}^{-\int_0^x \mu_j(s)\,\mathrm{d}s} = 1 - \mathrm{e}^{-\int_0^T \mu_j(s)\,\mathrm{d}s} < 1.$$

Therefore, $p_1 \in L^\infty(0,T;W^{1,1}(0,T))$.

This completes the proof. ■

Now we introduce the mapping $\mathcal{S}\colon U_{ad} \to Z$ by $\mathcal{S}(\vec{\mu}) = \vec{p}$, which maps the repair rate $\vec{\mu}$ to the corresponding solution $\vec{p}$ of (1)–(4). Then the range of $\mathcal{S}$, denoted by $\mathcal{R}(\mathcal{S})$, satisfies

$$\mathcal{R}(\mathcal{S}) \subseteq \Big\{\vec{p} = (p_0,p_1,\dots,p_M)^{\mathrm{T}} \in Z\colon p_0 \in W^{1,\infty}(0,T),$$

$$\left.\begin{aligned} &p_j \in L^\infty(0,T;W^{1,1}(0,T)) \text{ and} \\ &\frac{\mathrm{d}p_j}{\mathrm{d}t} \in L^\infty(0,T;L^1(0,T)),\ j=1,2,\ldots,M \end{aligned}\right\}.$$

Corollary 3.2: *The embedding $\mathcal{R}(\mathcal{S}) \hookrightarrow C[0,T] \times (C([0,T];L^1(0,T)))^M$ is compact.*

Proof: Since the embeddings

$$W^{1,\infty}(0,T) \hookrightarrow C[0,T] \quad \text{and} \quad W^{1,1}(0,T) \hookrightarrow L^1(0,T)$$

are compact for $0 < T < \infty$ (cf. Adams & Fournier, 2003, p. 144), by Aubin–Lions–Simon lemma (cf. Boyer & Fabrie, 2012) the embedding

$$L^\infty(0,T;W^{1,1}(0,T)) \cap W^{1,\infty}(0,T;L^1(0,T)) \hookrightarrow C([0,T];L^1(0,T))$$

is compact. This immediately establishes the desired result. ■

The following theorem establishes the existence of an optimal solution to problem (P).

Theorem 3.3: *There exists an optimal repair rate $\vec{\mu}^* = (\mu_1^*,\mu_2^*,\ldots,\mu_M^*)^{\mathrm{T}} \in U_{\mathrm{ad}}$ to problem (P).*

Proof: According to (9), (11) and nonnegativity of $p_0(t)$, it is clear that for $0 < T < \infty$,

$$-aT - b \le J(\vec{\mu}) \le \alpha MT\bar{\mu} + \frac{\beta}{2}MT\bar{\mu}^2.$$

Since J is bounded from below, we may choose a minimising sequence $\{\vec{\mu}_n\} = \{(\mu_{1_n},\mu_{2_n},\ldots,\mu_{M_n})^{\mathrm{T}}\} \subset U_{ad}$ such that

$$\lim_{n\to\infty} J(\vec{\mu}_n) = \inf_{\vec{\mu}\in U_{\mathrm{ad}}} J(\vec{\mu}). \tag{21}$$

Since $0 \le \mu_{n,j} \le \bar{\mu}$ for $j = 1,2,\ldots,M$, $\{\vec{\mu}_n\}$ is uniformly bounded in $(L^\infty(0,T))^M$ and $(L^2(0,T))^M$. Thus, there exists a convergent subsequence, still denoted by $\{\vec{\mu}_n\}$, such that

$$\vec{\mu}_n \to \vec{\mu}^* \quad \text{weak} * \text{ in } (L^\infty(0,T))^M. \tag{22}$$

$$\vec{\mu}_n \to \vec{\mu}^* \quad \text{weakly in } (L^2(0,T))^M. \tag{23}$$

Let sequence $\{\vec{p}_n\} = \{(p_{0_n},p_{1_n},\ldots,p_{M_n})^{\mathrm{T}}\}$ be the solutions corresponding to $\{\vec{\mu}_n\}$ with the same initial condition

$$\vec{p}_n(x,0) = (p_0(0),p_1(x,0),\ldots,p_M(x,0))^{\mathrm{T}}.$$

With the help of Corollary 3.2, we may extract a subsequence, still denoted by $\{\vec{p}_n\}$, such that

$$\vec{p}_n \to \vec{p}^* \quad \text{strongly in} \quad C[0,T] \times (C([0,T];L^1(0,T)))^M. \tag{24}$$

Next, we verify that $\vec{p}^*$ is the solution corresponding to $\vec{\mu}^*$ based on (10). In fact, we shall show that $\sup_{t\in[0,T]} |p_{0_n} - p_0^*| \to 0$ and $\sup_{t\in[0,T]} \|p_{j_n} - p_j^*\|_{L^1_x(0,T)} \to 0$ as $n \to \infty$. To this end, we first check

$$\begin{aligned}
&\sup_{t\in[0,T]} \left| \sum_{j=1}^M \int_0^t \mathrm{e}^{-\sum_{j=1}^M \lambda_j(t-\tau)} \int_0^\tau \mu_{j_n}(x)p_{j_n}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right. \\
&\left. - \sum_{j=1}^M \int_0^t \mathrm{e}^{-\sum_{j=1}^M \lambda_j(t-\tau)} \int_0^\tau \mu_j^*(x)p_j^*(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right| \\
&\le \sup_{t\in[0,T]} \left| \sum_{j=1}^M \int_0^t \mathrm{e}^{-\sum_{j=1}^M \lambda_j(t-\tau)} \int_0^\tau (\mu_{j_n}(x)(p_{j_n}(x,\tau) - p_j^*(x,\tau))\,\mathrm{d}x\,\mathrm{d}\tau \right| \\
&+ \sup_{t\in[0,T]} \left| \sum_{j=1}^M \int_0^t \mathrm{e}^{-\sum_{j=1}^M \lambda_j(t-\tau)} \int_0^\tau (\mu_{j_n}(x) - \mu_j^*(x))p_j^*(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right| \\
&= I_1 + I_2,
\end{aligned}$$

where by (24)

$$I_1 \le \bar{\mu}T \sum_{j=1}^M \sup_{t\in[0,T]} \|p_{j_n} - p_j^*\|_{L^1_x(0,T)} \to 0, \quad \text{as } n \to \infty, \tag{25}$$

and

$$\begin{aligned} I_2 = \sum_{j=1}^M \sup_{t\in[0,T]} &\left| \int_0^t (\mu_{j_n}(x) - \mu_j^*(x)) \right. \\ &\left. \times \int_x^t \mathrm{e}^{-\sum_{j=1}^M \lambda_j(t-\tau)} p_j^*(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}x \right| \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{26}$$

due to (22) and

$$\int_x^t \mathrm{e}^{-\sum_{j=1}^M \lambda_j(t-\tau)} p_j^*(x,\tau)\,\mathrm{d}\tau \in L^1_x(0,T) \quad \text{uniformly in } t. \tag{27}$$

In fact, with the help of (10) and nonnegativity of p_j^* we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \int_0^T \left| \int_x^t \mathrm{e}^{-\sum_{j=1}^M \lambda_j(t-\tau)} p_j^*(x,\tau)\,\mathrm{d}\tau \right| \mathrm{d}x \\
&\le \sup_{t\in[0,T]} \int_0^T \int_x^t p_j^*(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}x \\
&= \sup_{t\in[0,T]} \int_0^T \int_x^t \lambda_j p_0^*(\tau - x)\,\mathrm{e}^{-\int_0^x \mu_j^*(s)\,\mathrm{d}s}\,\mathrm{d}\tau\,\mathrm{d}x \\
&\le \sup_{t\in[0,T]} \int_0^T \lambda_j(t-x)\,\mathrm{d}x \\
&\le \frac{\lambda_j}{2}T^2.
\end{aligned}$$

Thus (27) holds and $\sup_{t\in[0,T]} |p_{0_n} - p_0^*| \to 0$ follows immediately. Next,

$$\sup_{t\in[0,T]} \int_0^T \left| \lambda_j p_{0_n}(t-x)\,\mathrm{e}^{-\int_0^x \mu_{j_n}(s)\,\mathrm{d}s} \right.$$

$$-\lambda_j p_0^*(t-x)\,\mathrm{e}^{-\int_0^x \mu_j^*(s)\,\mathrm{d}s}\Big|\,\mathrm{d}x$$

$$\leq \lambda_j \sup_{t\in[0,T]} \int_0^T |(p_{0_n}(t-x)-p_0^*(t-x))\,\mathrm{e}^{-\int_0^x \mu_{jn}(s)\,\mathrm{d}s}|\,\mathrm{d}x$$

$$+\lambda_j \sup_{t\in[0,T]} \int_0^T |p_0^*(t-x)$$

$$\times(\mathrm{e}^{-\int_0^x \mu_{jn}(s)\,\mathrm{d}s}-\mathrm{e}^{-\int_0^x \mu_j^*(s)\,\mathrm{d}s})|\,\mathrm{d}x$$

$$\leq \lambda_j \sup_{t\in[0,T]} |p_{0_n}(t)-p_0^*(t)| \int_0^T \mathrm{e}^{-\int_0^x \mu_j^*(s)\,\mathrm{d}s}\,\mathrm{d}x$$

$$+\lambda_j \int_0^T |\mathrm{e}^{-\int_0^x \mu_{jn}(s)\,\mathrm{d}s}-\mathrm{e}^{-\int_0^x \mu_{jn}(s)\,\mathrm{d}s}|\,\mathrm{d}x$$

$$\leq \lambda_j T \sup_{t\in[0,T]} |p_{0_n}(t)-p_0^*(t)|$$

$$+\lambda_j T \sup_{x\in[0,T]} |\mathrm{e}^{-\int_0^x \mu_{jn}(s)\,\mathrm{d}s}-\mathrm{e}^{-\int_0^x \mu^*(s)\,\mathrm{d}s}| \tag{28}$$

$$\to 0+0, \quad j=1,2,\ldots,M, \tag{29}$$

where from (28) to (29) we used (22) and (24). This establishes that $\sup_{t\in[0,T]}\|p_{jn}-p_j^*\|_{L_x^1(0,T)}\to 0$.

As a result of (25)–(26) and (29), $\vec{p}^*$ is the solution corresponding to $\vec{\mu}^*$ in light of (10). Lastly, by (23) and the weak lower semicontinuity of norms, we have

$$\int_0^T \mu_j^{*2}\,\mathrm{d}t \leq \varliminf_{n\to\infty} \int_0^T \mu_{jn}^2\,\mathrm{d}t.$$

Combining this with (22) and (24) gives

$$J(\vec{\mu}^*) = -a\int_0^T p_0^*(t)\,\mathrm{d}t - bp_0^*(T) + \alpha\sum_{j=1}^M \int_0^T \mu_j^*\,\mathrm{d}x$$

$$+\beta\sum_{j=1}^M \int_0^T \mu_j^{*2}\,\mathrm{d}x \leq \varliminf_{n\to\infty} J(\vec{\mu}_n) = \inf_{\mu\in U_{\mathrm{ad}}} J(\vec{\mu}^*),$$

which indicates that $\vec{\mu}^*$ is an optimal solution to problem (P). This completes the proof. ■

4. First-order optimality conditions

We now derive the first-order necessary optimality conditions for the problem (P) by using a variational inequality (cf. Mitter & Lions, 2011), that is, if μ^* is an optimal solution of the problem (P), then

$$J'(\vec{\mu})\cdot(\vec{h}-\vec{\mu}^*)$$

$$=\sum_{j=1}^M J_j'(\vec{\mu})\cdot(h_j-\mu_j{}^*)\geq 0, \quad \vec{h}=(h_1,\ldots,h_M)^{\mathrm{T}}\in U_{ad}. \tag{30}$$

Let $z_0=p_0'(\vec{\mu})\cdot\vec{h}=\sum_{j=1}^M p_{0,j}'(\vec{\mu})\cdot h_j$ and $z_j=p_j'(\vec{\mu})\cdot\vec{h}=\sum_{i=1}^M p_{j,i}'(\vec{\mu})\cdot h_i, j=1,2,\ldots,M$, be the Gâteaux derivatives of p_0 and p_j, respectively, with respect to $\vec{\mu}$ in every direction $\vec{h}$ in U_{ad}. Note that $\mu_j'(\vec{\mu})\cdot\vec{h}=\sum_{i=1}^M \mu_{j,i}'(\vec{\mu})\cdot h_i=h_j$. Then by (1)–(4) and (15), $\vec{z}=(z_0,z_1,\ldots,z_M)^{\mathrm{T}}$ satisfies

$$\frac{\mathrm{d}z_0(t)}{\mathrm{d}t} = -\sum_{j=1}^M \lambda_j z_0(t) + \sum_{j=1}^M \int_0^t \mu_j(x)z_j(x,t)\,\mathrm{d}x$$

$$+\sum_{j=1}^M \int_0^t h_j(x)p_j(x,t)\,\mathrm{d}x, \tag{31}$$

$$\frac{\partial z_j(x,t)}{\partial t}+\frac{\partial z_j(x,t)}{\partial x} = -\mu_j(x)z_j(x,t)-h_j(x)p_j(x,t), \tag{32}$$

with boundary condition

$$z_j(0,t)=\lambda_j z_0(t), \quad j=1,2,\ldots,M,\ t>0, \tag{33}$$

and initial conditions

$$z_0(0)=0, \quad z_j(x,0)=0, \quad j=1,2,\ldots,M. \tag{34}$$

For $\vec{h}\in U_{ad}$,

$$J'(\vec{\mu})\cdot\vec{h} = -a\int_0^T z_0\,\mathrm{d}t - bz_0(T) + \alpha\sum_{j=1}^M \int_0^T h_j\,\mathrm{d}x$$

$$+\beta\sum_{j=1}^M \int_0^T \mu_j h_j\,\mathrm{d}x. \tag{35}$$

Before introducing the adjoint system associated with (1)–(4), we recall the duality between nonreflexive Banach spaces. Let $X=\mathbb{R}\times(L_x^1(0,T))^M$. Then its dual space is given by $X^*=\mathbb{R}\times(L_x^\infty(0,T))^M$ and the duality between X and X^* is defined by

$$(\vec{p},\vec{q})_{X,X^*} = p_0q_0 + \sum_{j=1}^M \int_0^T p_jq_j\,\mathrm{d}x,$$

for $\vec{p}=(p_0,p_1,\ldots,p_M)^{\mathrm{T}}\in X$ and $\vec{q}=(q_0,q_1,\ldots,q_M)^{\mathrm{T}}\in X^*$.

If $\vec{q}$ is the adjoint state associated with $\vec{p}$ solving (1)–(4), using a duality argument between $\vec{z}$ and $\vec{q}$, we can show that $\vec{q}$ satisfies

$$\frac{\mathrm{d}q_0(t)}{\mathrm{d}t} = \sum_{j=1}^M \lambda_j(q_0(t)-q_j(0,t))+a, \tag{36}$$

$$\frac{\partial q_j(x,t)}{\partial t}+\frac{\partial q_j(x,t)}{\partial x} = -\mu_j(x)(q_0(t)-q_j(x,t)), \tag{37}$$

with boundary conditions

$$q_j(T,t)=0, \quad j=1,2,\ldots,M,\ t>0, \tag{38}$$

and final conditions

$$q_0(T)=-b, \quad q_j(x,T)=0, \quad j=1,2,\ldots,M. \tag{39}$$

Moreover, it is easy to verify that $\vec{q}\in L^\infty(0,T)\times(L^\infty(0,T;L^\infty(0,T)))^M$.

The following theorem establishes the optimality conditions for characterising the optimal repair rate for problem (P).

Theorem 4.1: *Let $\vec{\mu}^* = (\mu_1^*, \mu_2^*, \dots, \mu_M^*)^{\mathrm{T}} \in U_{ad}$ be the optimal solution to (P).*

(1) *If $\beta > 0$, then*

$$\mu_j^*(x) = \max\left\{0, \min\left\{\frac{1}{\beta}\left(\int_0^T p_j(x,t)(q_j(x,t) - q_0(t))\,\mathrm{d}t - \alpha\right), \bar{\mu}\right\}\right\} =: \mathbb{P}_{[0,\bar{\mu}]}\left\{\frac{1}{\beta}\left(\int_0^T p_j(x,t)(q_j(x,t) - q_0(t))\,\mathrm{d}t - \alpha\right)\right\}, \quad (40)$$

for $j = 1, 2, \dots, M$ and $\beta > 0$, where p_j, q_0 and q_j are the solutions to the governing system (1)–(4) and its adjoint system (36)–(39), respectively. Here, for real numbers $c \le d$, $\mathbb{P}_{[c,d]}$ denotes the projection of $\mathbb{R}$ onto $[c,d]$, that is, $\mathbb{P}_{[c,d]}(f) := \min\{d, \max\{c, f\}\}$.

(2) *If $\beta = 0$, then*

$$\mu_j^*(x) = \begin{cases} 0, & \text{if } \int_0^T p_j(x,t)(q_0(t) - q_j(x,t))\,\mathrm{d}t > -\alpha, \\ \bar{\mu}, & \text{if } \int_0^T p_j(x,t)(q_0(t) - q_j(x,t))\,\mathrm{d}t < -\alpha. \end{cases}$$

If there exists $x^* \in [0,T]$ such that $\int_0^T p_j(x^*,t)(q_0(t) - q_j(x^*,t))\,\mathrm{d}t = -\alpha$, then $\mu_j^*(x^*)$ can any value in $[0,\bar{\mu}]$.

Proof: First taking the duality between $(z_0, z_1, \dots, z_M)^{\mathrm{T}}$ and $(q_0, q_1, \dots, q_M)^{\mathrm{T}}$ and then integrating with respect to t from 0 to T yield

$$\int_0^T \frac{\mathrm{d}z_0}{\mathrm{d}t} q_0\,\mathrm{d}t = -\int_0^T \sum_{j=1}^M \lambda_j z_0 q_0\,\mathrm{d}t + \int_0^T \left(\sum_{j=1}^M \int_0^t \mu_j z_j\,\mathrm{d}x\right) q_0\,\mathrm{d}t + \int_0^T \left(\sum_{j=1}^M \int_0^t h_j p_j\,\mathrm{d}x\right) q_0\,\mathrm{d}t,$$

and

$$\int_0^T \left(\frac{\partial z_j}{\partial t}, q_j\right)\mathrm{d}t + \int_0^T \left(\frac{\partial z_j}{\partial x}, q_j\right)\mathrm{d}t = \int_0^T (-\mu_j z_j - h_j p_j, q_j)\,\mathrm{d}t, \quad j = 1, 2, \dots, M.$$

Using integration by parts together with (34) and (38)–(39), we get

$$z_0(T)q_0(T) - \left(z_0, \frac{\mathrm{d}q_0}{\mathrm{d}t}\right) = -\int_0^T \sum_{j=1}^M \lambda_j z_0 q_0\,\mathrm{d}t + \int_0^T \left(\sum_{j=1}^M \int_0^t \mu_j z_j\,\mathrm{d}x\right) q_0\,\mathrm{d}t + \int_0^T \left(\sum_{j=1}^M \int_0^t h_j p_j\,\mathrm{d}x\right) q_0\,\mathrm{d}t, \quad (41)$$

and

$$-\int_0^T \left(z_j, \frac{\partial q_j}{\partial t}\right)\mathrm{d}t - \lambda_j \int_0^T z_0 q_j(0,t)\,\mathrm{d}t - \int_0^T \left(z_j, \frac{\partial q_j}{\partial x}\right)\mathrm{d}t = \int_0^T (z_j, -\mu_j q_j)\,\mathrm{d}t - \int_0^T (h_j p_j, q_j)\,\mathrm{d}t, \quad j = 1, 2, \dots, M. \quad (42)$$

Now taking the summation of both sides of (42) with respect to j from 1 to M and then adding the resulting equation to (41) give us

$$z_0(T)q_0(T) = a\int_0^T z_0\,\mathrm{d}t + \int_0^T \left(\sum_{j=1}^M \int_0^t h_j p_j\,\mathrm{d}x\right) q_0\,\mathrm{d}t - \int_0^T \sum_{j=1}^M \left(\int_0^T h_j p_j q_j\,\mathrm{d}x\right)\mathrm{d}t,$$

where we utilised (36)–(37). Further invoking (39) follows

$$-a\int_0^T z_0\,\mathrm{d}t - bz_0(T) = \int_0^T \left(\sum_{j=1}^M \int_0^t h_j p_j\,\mathrm{d}x\right) q_0\,\mathrm{d}t - \int_0^T \sum_{j=1}^M \left(\int_0^T h_j p_j q_j\,\mathrm{d}x\right)\mathrm{d}t, \quad (43)$$

where $\int_0^t h_j p_j\,\mathrm{d}x = \int_0^T h_j p_j\,\mathrm{d}x$ due to (14). Next replacing $-a\int_0^T z_0\,\mathrm{d}t - bz_0(T)$ in (35) by the right hand of (43), we derive that if $\vec{\mu}^*$ is the optimal solution to problem (P), then for any $\vec{u} = (u_1, u_2, \dots, u_M)^{\mathrm{T}} \in U_{ad}$,

$$J'(\vec{\mu}^*) \cdot (\vec{u} - \vec{\mu}^*) = \sum_{j=1}^M J_j'(\vec{\mu}^*) \cdot (u_j - \mu_j^*) \quad (44)$$

$$= \int_0^T \left(\sum_{j=1}^M \int_0^T (u_j - \mu_j^*) p_j\,\mathrm{d}x\right) q_0\,\mathrm{d}t - \int_0^T \sum_{j=1}^M \left(\int_0^T (u_j - \mu_j^*) p_j q_j\,\mathrm{d}x\right)\mathrm{d}t + \alpha \sum_{j=1}^M \int_0^T (u_j - \mu_j^*)\,\mathrm{d}x + \beta \sum_{j=1}^M \int_0^T (u_j - \mu_j^*)\mu_j^*\,\mathrm{d}x$$

$$= \sum_{j=1}^M \left(\int_0^T p_j(q_0 - q_j)\,\mathrm{d}t + \alpha, u_j - \mu_j^*\right) + \beta \sum_{j=1}^M (\mu_j^*, u_j - \mu_j^*) \ge 0, \quad (45)$$

which indicates that the optimality condition (40) holds if $\beta > 0$.

In addition, if $\beta = 0$, then $\mu_j^*(x) = 0$ for $\int_0^T p_j(x,t)(q_0(t) - q_j(x,t))\,dt > -\alpha$ and $\mu_j^*(x) = \bar{\mu}$ for $\int_0^T p_j(x,t)(q_0(t) - q_j(x,t))\,dt < -\alpha$. If there exists $x^* \in [0,T]$ such that $\int_0^T p_j(x^*,t)(q_0(t) - q_j(x^*,t))\,dt = -\alpha$, then $\mu_j^*(x^*)$ can take any value between 0 and $\bar{\mu}$. This results in a bang-bang type of control. The proof is complete. ■

In the reminder of the paper, we mainly discuss the case that the system has one failure mode, i.e. $M = 1$, which will be sufficient to capture the essence of the problem and to demonstrate the design of our numerical algorithms.

4.1 A simple example: $M = 1$ with constant repair rate

As a simple example to understand the relation between the optimal repair rate and the system state, we consider the system with one failure mode and the repair rate is constant, i.e. $M = 1$ and $\mu_1(x) = \mu_1 \geq 0$. Then (1)–(4) are reduced to an ODE system

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -\lambda_1 p_0(t) + \mu_1 \hat{p}_1(t) \tag{46}$$

$$\frac{\mathrm{d}\hat{p}_1(t)}{\mathrm{d}t} = \lambda_1 p_0(t) - \mu_1 \hat{p}_1(t), \tag{47}$$

with initial conditions

$$p_0(0) = 1, \quad \hat{p}_1(0) = 0, \tag{48}$$

where $\hat{p}_1$ is defined by (5) for $j = 1$. The objective functional J now becomes

$$J(\mu_1) = -a\int_0^T p_0\,\mathrm{d}t - bp_0(T) + \alpha\mu_1 T + \frac{\beta}{2}\mu_1^2 T. \tag{49}$$

Recall $\hat{p}_1 = 1 - p_0$. We can further reduce (46)–(47) to

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -\lambda_1 p_0(t) + \mu_1(1 - p_0(t)), \tag{50}$$

and hence $z_0 = p_0'(\mu_1) \cdot h_1$ satisfies where z_0 satisfies

$$\frac{\mathrm{d}z_0(t)}{\mathrm{d}t} = -\lambda_1 z_0(t) + h_1 - h_1 p_0(t) - \mu_1 z_0(t), \quad z_0(0) = 0. \tag{51}$$

Accordingly, the adjoint equation is given by

$$-\frac{\mathrm{d}q_0(t)}{\mathrm{d}t} = -\lambda_1 q_0(t) - \mu_1 q_0(t) - a, \tag{52}$$

with final condition

$$q_0(T) = -b. \tag{53}$$

Furthermore,

$$\left(\frac{\mathrm{d}z_0(t)}{\mathrm{d}t}, q_0\right) = z_0(T)q_0(T) - \left(z_0, \frac{\mathrm{d}q_0(t)}{\mathrm{d}t}\right) \tag{54}$$

$$= (z_0, -\lambda_1 q_0) + (h_1, q_0) - (h_1 p_0, q_0) - (z_0, \mu_1 q_0) \tag{55}$$

which follows

$$-(z_0, a) - bz_0(T) = (h_1, q_0) - (h_1 p_0, q_0) = (1 - p_0, q_0)h_1, \tag{56}$$

and thus

$$\begin{aligned} J'(\mu_1) \cdot (u_1 - \mu_1) &= -a\int_0^T z_0\,\mathrm{d}t - bz_0(T) + \alpha T(u_1 - \mu_1) \\ &\quad + \beta\mu_1 T(u_1 - \mu_1) \\ &= \left(\int_0^T (1 - p_0)q_0\,\mathrm{d}t + \alpha T\right)(u_1 - \mu_1) \\ &\quad + \beta\mu_1 T(u_1 - \mu_1) \geq 0, \end{aligned} \tag{57}$$

for $0 < u_1 \leq \bar{\mu}$. The optimality condition becomes

$$\mu_1^* = \max\left\{0, \min\left\{\frac{1}{\beta T}\int_0^T (p_0 - 1)q_0\,\mathrm{d}t - \frac{\alpha}{\beta}, \bar{\mu}\right\}\right\}. \tag{58}$$

If $\beta = 0$, then $\mu_1^* = 0$ for $\frac{\int_0^T (p_0(t)-1)q_0(t)\,\mathrm{d}t}{T} < \alpha$ and $\mu_1^* = \bar{\mu}$ for $\frac{\int_0^T (p_0(t)-1)q_0(t)\,\mathrm{d}t}{T} > \alpha$. If $\frac{\int_0^T (p_0(t)-1)q_0(t)\,\mathrm{d}t}{T} = \alpha$, then μ_1^* can take any value between 0 and $\bar{\mu}$.

Since p_0 and q_0 can be solved explicitly in this simple case, we are able to further investigate the properties of μ_1^*. Solving (50) with (48) and (52) with (53), receptively, gives

$$p_0(t) = \frac{\lambda_1}{\lambda_1 + \mu_1}\mathrm{e}^{-(\lambda_0 + \mu_1)t} + \frac{\mu_1}{\lambda_1 + \mu_1}, \tag{59}$$

and

$$q_0(t) = \left(\frac{a}{\lambda_1 + \mu_1} - b\right)\mathrm{e}^{-(\lambda_1 + \mu_1)(T - t)} - \frac{a}{\lambda_1 + \mu_1}. \tag{60}$$

It is evident that $\frac{\mu_1}{\lambda_1 + \mu_1} < p_0 \leq 1$ and $q_0 \leq 0$. Moreover, from (59) we get

$$\mathrm{e}^{-(\lambda_1 + \mu_1)(T - t)} = p_0(T - t)\frac{\lambda_1 + \mu_1}{\lambda_1} - \frac{\mu_1}{\lambda_1},$$

and hence,

$$\begin{aligned} q_0(t) &= \left(\frac{a}{\lambda_1 + \mu_1} - b\right)\left((p_0(T - t)\frac{\lambda_1 + \mu_1}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \\ &\quad - \frac{a}{\lambda_1 + \mu_1} \\ &= \frac{a - b(\lambda_1 + \mu_1)}{\lambda_1}p_0(T - t) - \frac{a - b\mu_1}{\lambda_1} \end{aligned}$$

Now if $\alpha = 1$ and $\beta > 0$, we can remove the upper bound for μ_1, then (58) becomes

$$\mu_1^* = \frac{1}{\beta T}\int_0^T (p_0 - 1)q_0\,\mathrm{d}t, \tag{61}$$

and thus

$$\mu_1^* = \frac{1}{\beta T}\int_0^T (p_0 - 1)$$

$$\times \left(\frac{a - b(\lambda_1 + \mu_1)}{\lambda_1} p_0(T-t) - \frac{a - b\mu_1}{\lambda_1} \right) \mathrm{d}t$$

which depends on p_0 only. In particular, if $a = 1$ and $b = 0$, we have

$$\mu_1^* = \frac{1}{\beta T \lambda_1} \int_0^T (1 - p_0(t))(1 - p_0(T-t))\, \mathrm{d}t \geq 0.$$

5. A projected gradient descent algorithm

To implement the theoretical results established in the previous sections, we will use a projected gradient descent algorithm (De los Reyes, 2015) based on the temporal method of lines (Schiesser, 2012), for solving the spatially semi-discretised forward state equations and backward adjoint state equations, respectively. Without loss of generality we set $M = 1$ and let $\mu = \mu_1$. The generalisation to the case $M > 1$ is straightforward, but the computational cost will be scaled up for a larger M.

In the spatial domain $[0, T]$, we define a uniform mesh $\{x_i = ih\}_{i=0}^N$ with the step size $h = T/N$. Let $p_{1,i}(t) = p_1(x_i, t)$, $q_{1,i}(t) = q_1(x_i, t)$ and $\mu_i = \mu(x_i)$. We use the (right) rectangular rule to approximate the integral term in (1) and apply the upwind scheme (LeVeque, 2007) to discretise the spatial first-order partial derivative term in (2), which lead to the semi-discretised state equations

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -\lambda_1 p_0(t) + h \sum_{i=1}^N \mu_i p_{1,i}(t), \tag{62}$$

$$\frac{\mathrm{d}p_{1,i}(t)}{\mathrm{d}t} = -\frac{p_{1,i}(t) - p_{1,i-1}(t)}{h} - \mu_i p_{1,i}(t), \quad 1 \leq i \leq N, \tag{63}$$

marching forward with the boundary condition $p_{1,0} = \lambda_1 p_0(t)$ and the initial conditions $p_0(0) = 1, p_{1,i}(0) = 0$. By defining

$$\boldsymbol{p}(t) = \begin{bmatrix} p_0(t) \\ p_{1,1}(t) \\ p_{1,2}(t) \\ \vdots \\ p_{1,N}(t) \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix},$$

and

$$A_h(\boldsymbol{\mu}) = \begin{bmatrix} -\lambda_1 & h\mu_1 & h\mu_2 & h\mu_3 & h\mu_4 & \cdots & h\mu_N \\ \lambda_0/h & -\mu_1 - 1/h & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1/h & -\mu_2 - 1/h & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1/h & -\mu_{N-1} - 1/h & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1/h & -\mu_N - 1/h \end{bmatrix},$$

the above scheme (62)–(63) can be formulated into an initial value problem of (stiff) ODE system

$$\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} = A_h(\boldsymbol{\mu})\boldsymbol{p}(t), \tag{64}$$

$$\boldsymbol{p}(0) = [1, 0, \ldots, 0]^{\mathrm{T}}. \tag{65}$$

It can be shown that the solution to (64)–(65) strongly convergences to (7)–(8) as $N \to \infty$ by using Trotter–Kato Theorem (Xu & Hu, 2013). Moreover, (64)–(65) can be efficiently solved by MATLAB's ODE solvers (e.g.`ode15s`) for any given control $\boldsymbol{\mu}$.

Using the similar approach for discretising the backward adjoint state equations follows

$$\frac{\mathrm{d}q_0(t)}{\mathrm{d}t} = \lambda_1(q_0(t) - q_{1,0}(t)) + a, \tag{66}$$

$$\frac{\mathrm{d}q_{1,i}(t)}{\mathrm{d}t} = -\frac{q_{1,i+1}(t) - q_{1,i}(t)}{h} - \mu_i(q_0(t) - q_{1,i}(t)), \quad 0 \leq i \leq N-1, \tag{67}$$

which march backward with the boundary condition $p_{1,N} = 0$, the final conditions $q_0(T) = -b$ and $q_{1,i}(T) = 0$. By defining

$$\boldsymbol{q}(t) = \begin{bmatrix} q_0(t) \\ q_{1,0}(t) \\ q_{1,1}(t) \\ \vdots \\ q_{1,N-1}(t) \end{bmatrix}, \quad \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and

$$B_h(\boldsymbol{\mu}) = \begin{bmatrix} \lambda_1 & -\lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ -\mu_0 & \mu_0 + 1/h & -1/h & 0 & 0 & \cdots & 0 \\ -\mu_1 & 0 & \mu_1 + 1/h & -1/h & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ -\mu_{N-2} & 0 & \cdots & 0 & 0 & \mu_{N-2} + 1/h & -1/h \\ -\mu_{N-1} & 0 & 0 & \cdots & 0 & 0 & \mu_{N-1} + 1/h \end{bmatrix},$$

the above equations (66) and (67) can be written as a final value problem of (stiff) ODE system

$$\frac{\mathrm{d}\boldsymbol{q}(t)}{\mathrm{d}t} = B_h(\boldsymbol{\mu})\boldsymbol{q}(t) + a\boldsymbol{e}_1, \tag{68}$$

$$\boldsymbol{q}(T) = [-b, 0, \ldots, 0]^{\mathrm{T}}, \tag{69}$$

which, upon reversing the time via a change of variable $\tau = T - t$, can also be efficiently solved by MATLAB's ODE solvers (e.g. `ode15s`).

For the time interval $[0, T]$ with a general (possibly nonuniform) mesh $\{t_j\}_{j=0}^N$, we define $p_{1,i}^j = p_{1,i}(t_j) = p_1(x_i, t_j)$, $q_{1,i}^j = q_{1,i}(t_j) = q_1(x_i, t_j)$, and $q_0^j = q_0(t_j)$. With numerical quadrature, the first-order optimality condition (40) of optimal control $\mu_i = \mu(x_i)$ can be pointwisely approximated by

$$\beta\mu_i = \mathbb{P}_{[0,\bar{\mu}]}\left\{h\sum_{j=0}^N w_j p_{1,i}^j q_{1,i}^j - h\sum_{j=i}^N \hat{w}_j p_{1,i}^j q_0^j - \alpha\right\}, \quad 0 \le i \le N, \tag{70}$$

where w_j and $\hat{w}_j$ denote appropriate quadrature weights (e.g. based on the trapezoidal rule).

Algorithm 1 A projected gradient descent (PGD) algorithm:
Input: $T, \alpha, \beta, a, b, \gamma, N, h = T/N, tol, k_{\max}$
Output: optimal control vector $\boldsymbol{\mu}_h = [\mu_0, \mu_1, \cdots, \mu_N]^{\mathrm{T}}$

procedure $\boldsymbol{\mu}_h$=PGD$(T, \alpha, \beta, a, b, \gamma, N, h = T/N, tol, k_{\max})$
  choose an initial guess of control $\boldsymbol{\mu}^{(0)}$;
  for $k = 0$ to $k_{\max}$ **do**
    solve the forward semi-discrete state ODE system:

$$\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} = A_h(\boldsymbol{\mu}^{(k)})\boldsymbol{p}(t), \quad \text{with } \boldsymbol{p}(0) = [1, 0, \cdots, 0]^{\mathrm{T}},$$

    solve the backward semi-discrete adjoint state ODE system:

$$\frac{d\boldsymbol{q}(t)}{dt} = B_h(\boldsymbol{\mu}^{(k)})\boldsymbol{q}(t) + a\boldsymbol{e}_1, \quad \text{with } \boldsymbol{q}(T) = [-b, 0, \cdots, 0]^{\mathrm{T}},$$

    approximate the integral terms appeared in (70):

$$Q_i = h\sum_{j=0}^N w_j p_{1,i}^j q_{1,i}^j, \quad \widehat{Q}_i = h\sum_{j=i}^N \hat{w}_j p_{1,i}^j q_0^j, \quad 0 \le i \le N,$$

    update the control $\boldsymbol{\mu}^{(k+1)}$ based on projected gradient descent iteration

$$\mu_i^{(k+1)} = \mathbb{P}_{[0,\bar{\mu}]}\{\mu_i^{(k)} - \gamma_k(\beta\mu_i^{(k)} - (Q_i - \widehat{Q}_i - \alpha))\}, \quad 0 \le i \le N,$$

    where $\gamma_k > 0$ is the step size computed with the Armijo rule (or other line search rules);
    if $\|\boldsymbol{\mu}^{(k+1)} - \boldsymbol{\mu}^{(k)}\| \le tol$ **then**
      return $\boldsymbol{\mu}_h = \boldsymbol{\mu}^{(k+1)}$;
    end if
  end for
end procedure

Based on the described schemes above, a projection gradient descent (PGD) algorithm for solving the fully coupled optimality system (64)–(65), (68)–(69), and (70), is to construct a fixed point iteration for iteratively updating $\boldsymbol{\mu}$ along the gradient descent direction, which is followed by a projection step. In each iteration, it requires forward and backward time-marching to solve the decoupled state and adjoint state ODEs for $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t)$, respectively. Both ODEs can be accurately and efficiently solved with MATLAB's built-in ODE solvers (i.e.`ode45` and `ode15s`) for a selected tolerance. The complete PGD algorithm is summarised in Algorithm 1, where the stopping tolerance *tol* should be appropriately chosen based on the level of finite difference discretisation and quadrature approximation errors. Such a PGD algorithm can also be easily tailored to handle the simple case with constant repair rate, which will be demonstrated by a numerical example. Based on our following numerical simulations, we have observed a rough linear convergence rate of the proposed PGD algorithm. However, we remark that a rigorous convergence analysis of such a PGD algorithm is beyond the scope of this paper, which will be left as our future work.

6. Numerical examples

In this section, we provide several numerical examples to validate the theoretical results and to demonstrate the efficiency of our proposed PGD algorithm. The approximation errors are measured in the discrete L^∞-norm. All simulations are implemented using MATLAB 2017b on a Dell Precision Workstation with Intel(R) Core(TM) i7-7700K CPU@4.2GHz and 32GB RAM. The CPU time (in seconds) is estimated using the timing functions `tic/toc`.

6.1 Numerical simulation with constant repair rate

The optimal control model with constant repair rate is relatively simple to solve, since both the state and adjoint state equation are described by simple linear first-order ODEs. We mainly focus on the case with $\alpha = 0$ and $\beta = 1$. For numerical verification, it is convenient to reformulate the above optimal control expression μ^* as the zero of the following nonlinear equation

$$F(\mu) := \mu - \frac{1}{\beta T}\int_0^T (p_0 - 1)q_0\,\mathrm{d}t = 0$$

with

$$\frac{1}{\beta T}\int_0^T (p_0 - 1)q_0\,\mathrm{d}t = \frac{\lambda_0}{\beta(\lambda_0 + \mu)^2 T}\left[\frac{2(\mathrm{e}^{-(\lambda_0+\mu)T} - 1)}{\lambda_0 + \mu} + T\,\mathrm{e}^{-(\lambda_0+\mu)T} + T\right],$$

where the above expressions of $p_0(t)$ and $q_0(t)$ are used in deriving the exact integration. Note that $F(\mu)$ is a monotonically increasing function, $F(\mu) \to +\infty$ as $\mu \to +\infty$, and

$$F(0) = -\frac{1}{\beta\lambda_0 T}\left[\frac{2(\mathrm{e}^{-\lambda_0 T} - 1)}{\lambda_0} + T\,\mathrm{e}^{-\lambda_0 T} + T\right] = -\frac{1}{\beta\lambda_0 T}\left[\frac{(2 + \lambda_1 T)\,\mathrm{e}^{-\lambda_0 T} + \lambda_0 T - 2}{\lambda_0}\right] < 0.$$

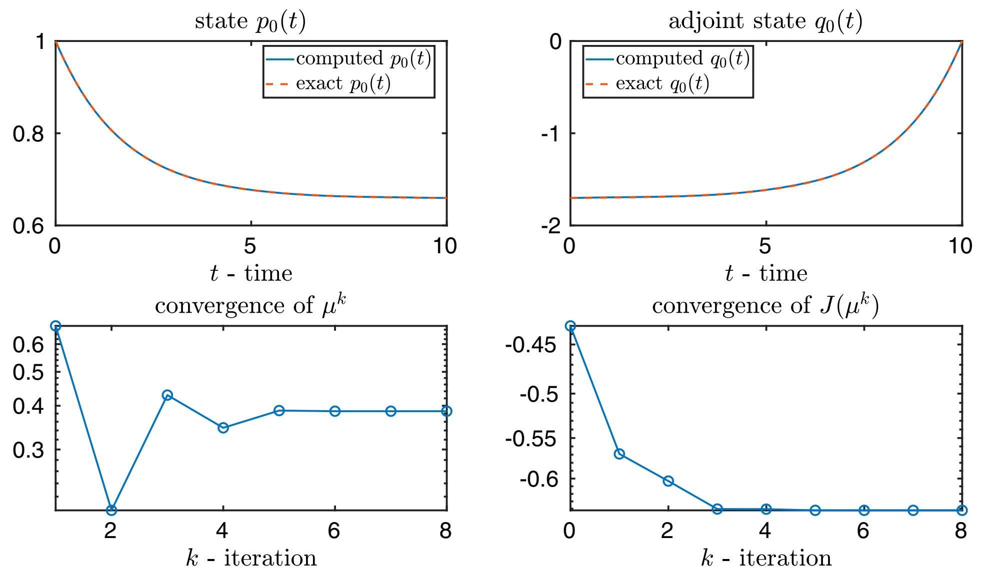

Figure 2. The computed state, adjoint state, and convergence history of our PGD algorithm with $T = 10$.

Table 1. Convergence of the computed optimal repair rate μ_h for a sequence of increasing T with $\lambda_0 = 0.2, \alpha = 0$ and $\beta = 1$.

T	Points	μ_h	μ^*	$\|\mu_h - \mu^*\|_{l^\infty}$	$J(\mu_h)/T$	Iter	CPU
10	148	0.38590082	0.38593398	3.32e−05	−0.642306	8	0.01
10^2	328	0.45370187	0.45370418	2.31e−06	−0.595810	15	0.03
10^3	1760	0.45905576	0.45905589	1.34e−07	−0.591630	11	0.06
10^4	16,072	0.45958368	0.45958098	2.70e−06	−0.591216	13	0.51
10^5	159,172	0.45963309	0.45963339	3.02e−07	−0.591174	11	3.93
10^6	1,590,196	0.45963871	0.45963863	8.38e−08	−0.591170	11	37.97
10^7	15,900,400	0.45963924	0.45963915	8.39e−08	−0.591170	11	383.05

To see this, it suffices to show that

$$(2 + \lambda_1 T)\, e^{-\lambda_0 T} + \lambda_0 T - 2 > 0, \tag{71}$$

for $\lambda_0 T > 0$. In fact, if letting $f(x) = (2 + x)\, e^{-x} + x$, we can easily verify that $f(x) > 2$ for $x > 0$. Therefore, $F(\mu) = 0$ has a unique positive solution.

Upon solving $F(\mu) = 0$ to a very high accuracy with any standard nonlinear solver (Kelley, 2003) (e.g. `fsolve` in MATLAB) would provide an accurate benchmark approximation to μ^*, which will be used as reference to estimate the approximation accuracy of our implemented PGD algorithm. We point out that approximately solving the nonlinear equation $F(\mu) = 0$ does not introduce any discretisation errors in treating the ODEs and integral terms.

We first check the convergence of our PGD algorithm for a fixed final time $T = 10$ and $\lambda_0 = 0.2$. Figure 2 shows the convergence history of our PGD algorithm and the corresponding computed optimal state and adjoint state, where the initial guess of control is set to be zero. As expected, we observe a linear convergence rate of our PGD algorithm based on the used `ode45` solver (with default tolerance 10^{-6}). Smaller errors can be obtained if we choose to use tighter tolerance in the ODE solvers and the PGD stopping condition, which, however, will take more iterations and time stepping points.

Next, we check the influence of the final time T on the computed optimal control μ_h. Table 1 reports the convergence of the approximated optimal control μ_h and the PGD algorithm as T increases, where the error term $\|\mu_h - \mu^*\|_{L^\infty}$ indicates the computed optimal control μ_h by our PGD algorithm converges to (with the given tolerance) the exact optimal control μ^* that satisfies $F(\mu^*) = 0$. The number of iterations used in our PGD algorithm seems to be very robust with respect to the larger values of T. It is also interesting to observe the convergence behaviours of μ^* and $J(\mu_h)/T$ as T increases. The experimental data are shown in Table 1.

6.2 Numerical simulation with distributed repair rate

Let $\lambda_0 = 0.2$ and $\bar{\mu} = 1$. To validate our algorithm, we first apply it to an extended optimality system with an exact solution. The detailed construction and accuracy are presented in the Appendix. We compare the objective functional with various choice of parameters: a, b, α, and β.

In Figures 3 and 4, we compare the computed state $p_0(t)$ and optimal control $\mu(x)$ over a time interval $[0, T]$ with or without optimising $p_0(T)$, respectively, by testing different choice of weight parameters. Again we set $T = 10$ and $N = 400$. In our PGD algorithm, the initial guess of optimal control is taken as the straight line connecting $(0, 1)$ and $(T, 0)$, i.e. $\boldsymbol{\mu}_i^{(0)} = (T - x_i)/T$. The optimal solution seems to be insensitive to the other choices of initial guess. Figures 5 and 6 show the adjoint state and the convergence of the objective functional corresponding to the selected parameters. Clearly, we observed the expected facts that $0 \le p_1 \le 1$ and $q_1 \le 0$.

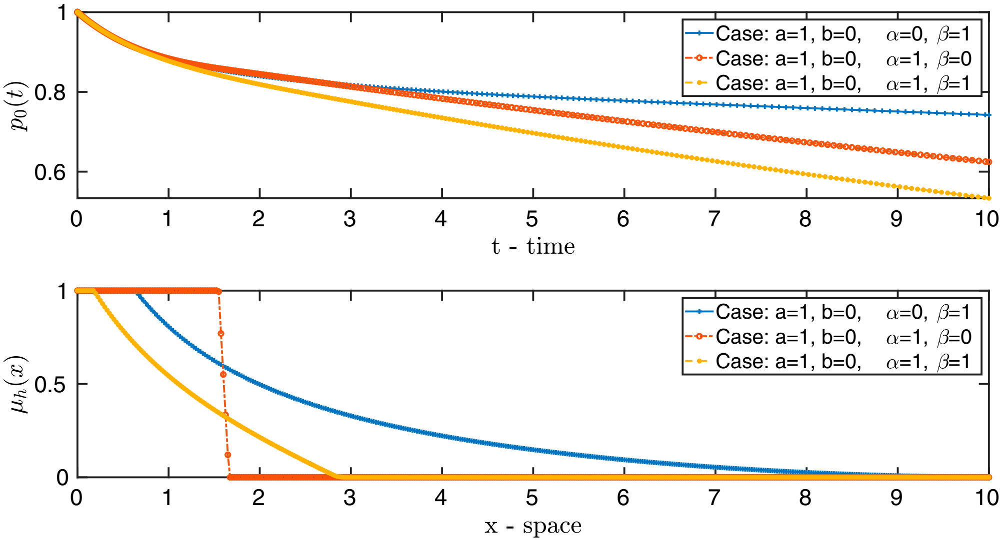

Figure 3. The computed state and optimal control by our PGD algorithm for different models with $T = 10$.

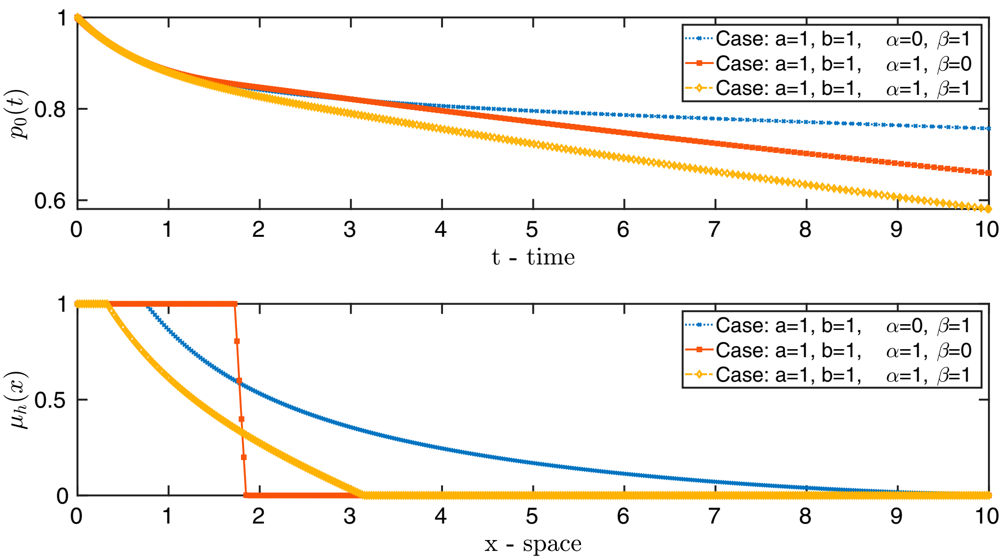

Figure 4. The computed state and optimal control by our PGD algorithm for different models with $T = 10$.

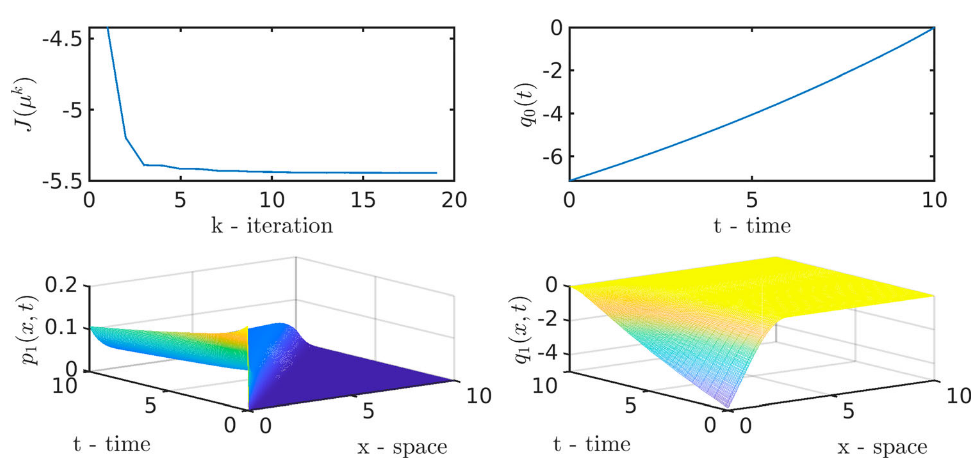

Figure 5. The computed state, adjoint state and objective functional for a selected case with $T = 10$.

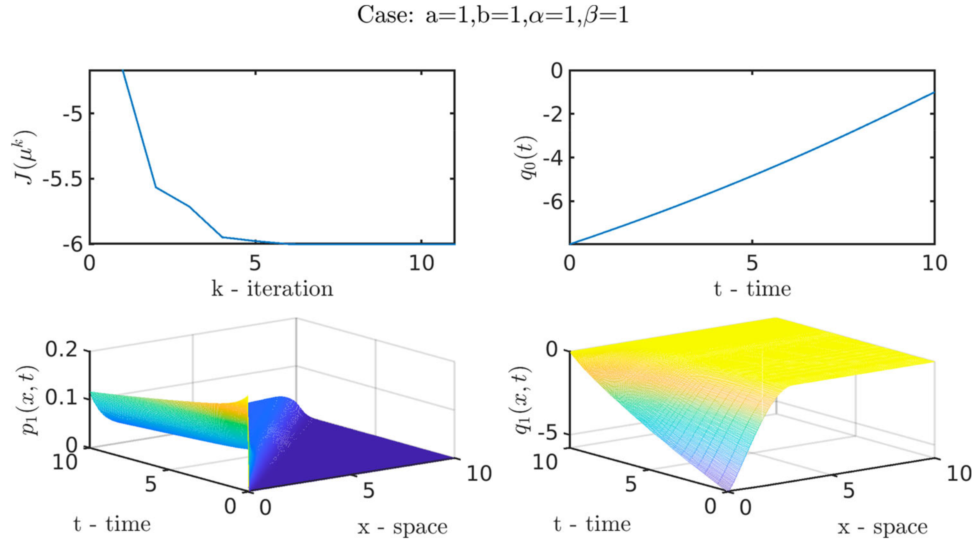

Figure 6. The computed state, adjoint state and objective functional for a selected case with $T = 10$.

6.2.1 Numerical tests for $a = 1$ and $b = 0$

Figure 3 demonstrates the results for $a = 1$ and $b = 0$. In this setup, we only consider to maximise the probability distribution of the system in good mode in term of maximising the L^1-norm of $p_0(t)$ over $[0, T]$ over $[0, T]$ without taking into account its final state. We conducted three experiments by testing different combinations of the control weight α and β, with $(\alpha, \beta) = (0, 1), (1, 0)$ and $(1, 1)$. Among the three cases tested, the best performance is given by the blue line, where only L^2-regularisation of the control is employed, i.e. $(\alpha, \beta) = (0, 1)$. However, the optimal repair function has a long decreasing nonzero tail and the repair effort is the greatest as shown in Figure 3. In contrast, including an L^1 -penalisation, either $(\alpha, \beta) = (1, 0)$ or $(\alpha, \beta) = (1, 1)$, the control function gains sparsity over repair intervals, as illustrated by the yellow and maroon lines, where sharp cuts are presented. In particular, using L^1-penalisation only, i.e. $(\alpha, \beta) = (1, 0)$, leads to a bang-bang type of control, which suggests putting the maximum efforts in repair in the initial stage of the system running period and doing nothing when certain degree of deterioration is met. However, once repair stops, i.e. $\mu = 0$, we see from the yellow and maroon lines that the system deteriorates faster than the case with nonzero repair rate represented by the blue line, which agrees with the real-life applications.

6.2.2 Numerical tests for $a = 1$ and $b = 1$

The inclusion of the final state $p_0(T)$ in the objective functional, i.e. $a = 1$ and $b = 1$, can help boost up the value of $p_0(T)$ with slightly more control costs, as readily seen in Figure 4 and Table 2. Overall, there is a trade-off between system availability and repair/maintenance cost. The sparse structure of optimal repair/maintenance rate can be adjusted by choosing appropriate weight parameter $\alpha > 0$. The other cases are very similar and hence omitted for briefness.

In summary, all the numerical simulation results suggest putting the maximum maintenance efforts over the early running period if the system is brand new initially. In other words, those new machines should receive the best maintenance. In the cases equipped with L^1 -penalisation, the optimal repair policy suggests no repair necessary for those machines with an elapsed repair time over a fixed threshold. For example, $\mu = 0$ when repair time $x > 2$, indicated by the red lines in Figures 3 and 4. This observation may help to provide a practical guidance for making decisions in industrial management.

Table 2. Comparison of optimal control cost and system availability for different cases with $T = 10$.

Case	L_1-norm of $\vec{\mu}^*(x)$	L_2-norm of $\vec{\mu}^*(x)$	L_1-norm of $p_0^*(t)$	$p_0^*(T)$
$a = 1, b = 0, \alpha = 0, \beta = 1$	2.72	1.28	8.04	0.74
$a = 1, b = 0, \alpha = 1, \beta = 0$	1.62	1.27	7.62	0.62
$a = 1, b = 0, \alpha = 1, \beta = 1$	1.26	0.91	7.10	0.53
$a = 1, b = 1, \alpha = 0, \beta = 1$	2.93	1.34	8.11	0.76
$a = 1, b = 1, \alpha = 1, \beta = 0$	1.80	1.33	7.79	0.66
$a = 1, b = 1, \alpha = 1, \beta = 1$	1.45	0.99	7.35	0.58

7. Conclusion

An optimal corrective maintenance design represented by the repair rate is discussed for optimising the availability of a simple reparable system in a nonreflexive Banach space. Sparsity of the optimal repair rate is built in by using L^1 -regularisation. First-order necessary conditions of optimality are derived for characterising the optimal repair rate. A projected gradient descent (PGD) algorithm based on the upwind finite difference scheme is developed for solving the optimality system. The presented numerical results may shed light on the practical applications. However, to be more realistic, reparable systems with time−dependent failure rate and repair rate, and the associated problems such as parameter identification for failure rate based on system performance and nonlinear controllability of the system with respect to repair rate are worth further examination. These topics will be considered in our future work.

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Appendix. A model based on extended optimality system with an exact solution

This example will be constructed from extending the optimality system by adding the free force terms: f_0, f_1, g_0, g_1, so that we can choose the exact solutions for the optimal state, adjoint state, and optimal control, respectively. More specifically, we test our numerical algorithm by solving the following extended optimality system:

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -\lambda_1 p_0(t) + \int_0^T \mu(x) p_1(x,t)\,\mathrm{d}x + f_0(t), \tag{A1}$$

$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -\mu(x) p_1(x,t) + f_1(x,t), \tag{A2}$$

$$\frac{\mathrm{d}q_0(t)}{\mathrm{d}t} = \lambda_1 (q_0(t) - q_1(0,t)) + a + g_0(t), \tag{A3}$$

Table A1. Error results of the computed optimal state, adjoint state, and optimal control with $T = 2$.

N	Error of p_0^h	Error of p_1^h	Error of q_0^h	Error of q_1^h	Error of μ_h	Iter	CPU
40	2.63e−03	1.20e−03	1.32e−02	1.57e−01	6.14e−03	71	7.09
80	1.29e−03	6.34e−04	6.61e−03	8.17e−02	3.16e−03	43	6.29
160	6.39e−04	3.29e−04	3.33e−03	4.20e−02	1.62e−03	40	9.02
320	3.19e−04	1.69e−04	1.52e−03	2.12e−02	8.14e−04	41	20.74
640	1.59e−04	8.61e−05	7.40e−04	1.06e−02	4.02e−04	36	45.71
1280	7.78e−05	4.94e−05	3.14e−04	5.32e−03	2.02e−04	32	139.40

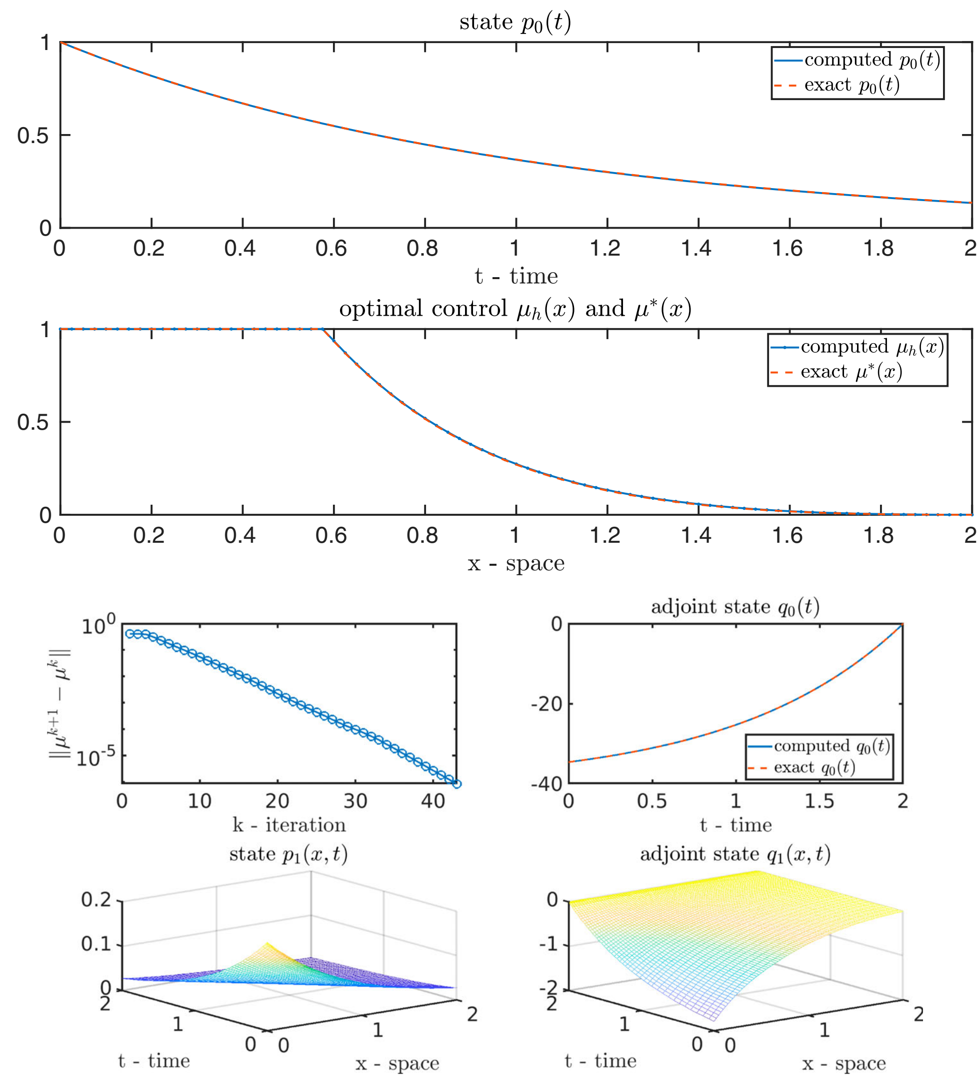

Figure A1. The state, the adjoint state, and the optimal control (in solid lines) by our PGD algorithm with $T = 2$ and $N = 80$. For better comparison, the exact solutions of $p_0(t)$, $q_0(t)$, and $\mu^*(t)$ are also plotted in dashed lines.

$$\frac{\partial q_1(x,t)}{\partial t} + \frac{\partial q_1(x,t)}{\partial x} = -\mu(x)(q_0(t) - q_1(x,t)) + g_1(t,x), \qquad \text{(A4)}$$

$$\mu(x) = \mathbb{P}_{[0,\bar{\mu}]}\left\{\frac{1}{\beta}\left(\int_0^T p_1(x,t)(q_1(x,t) - q_0(t))\,\mathrm{d}t - \alpha\right)\right\}. \qquad \text{(A5)}$$

We highlight that such an extended optimality system is only for verifying the accuracy of our finite difference scheme and the convergence of our PGD algorithm. It may not correspond to the original optimization problem anymore and hence we will not report the values of objective functional.

Inspired by the close-form solution expression for the case with constant repair rate, we choose the following exact solution according to the boundary and initial conditions (with $\lambda_0 = 0.2$):

$$p_0(t) = \mathrm{e}^{-t}, \quad p_1(x,t) = \lambda_1\,\mathrm{e}^{-x-t}, \qquad \text{(A6)}$$

$$q_0(t) = 40(\mathrm{e}^{-(T-t)} - 1), \quad q_1(x,t) = (T-x)(\mathrm{e}^{-x-(T-t)} - \mathrm{e}^{-x}), \quad \text{(A7)}$$

which can be used in (A5) to obtain the expression of exact optimal control $\vec{\mu}^*(x)$. Notice there hold $p_1(0,t) = \lambda_1 p_0(t)$, $p_0(1) = 1$, $q_0(T) = 0$, $q_1(x,T) = 0$, and $q_1(T,t) = 0$, but $p_1(x,0) \neq 0$. To numerically validate our proposed PGD algorithm (based on the stiff ODE solver `ode15s`), we apply it with tolerance $tol = 10^{-6}$ to the above optimality system with the given exact solution.

In Table A1, we report the convergence performance of our PGD algorithm and the approximation errors (in L^∞-norm) of the computed optimal state, adjoint state, and optimal control with $T = 2$ and $\bar{\mu} = 1$ for a sequence of refined meshes. As expected from our used first-order finite difference schemes, the estimated approximation errors (reduced by about half) show about first-order accuracy as the mesh step size h is halved. Moreover, the iteration number used in our PGD algorithm seems to be mesh-independent, which is well-known (Kelley & Sachs, 1992) for gradient projection method. As an illustration, Figure A1 plots the computed optimal state, adjoint state, and optimal control, which are almost overlapped with the given exact solutions. The linear decreasing of the difference $\|\boldsymbol{\mu}^{k+1} - \boldsymbol{\mu}^k\|$ implies a typical linear convergence rate of our PGD algorithm. This constructed extended optimality system indicates our PGD algorithm indeed delivers first-order accurate numerical solution, although its rigorous convergence analysis is not given in this paper.