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Sampled-data based Failure Rate Identification for a Multi-state Reparable System

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Abstract— **This paper focuses on failure rate identification of a multi-state reparable system. The mathematical model is governed by coupled transport and integro-differential equations, which describe the probabilities of the system in good and failure modes. The objective of this work is to identify the failure rates based on the sampled-data of the probability of the system in good mode. Rigorous analysis is presented and numerical tests are conducted to demonstrate the design.**

I. Introduction

Failure rate identification for a multi-state reparable system is considered in this paper. A reparable system is one which can be restored to satisfactory operation by repair actions whenever a failure occurs (cf. [19], [24]). It often arises in problems of product design, inventory systems, computer networking and complex manufacturing processes. Mathematical models governed by distributed parameter systems of coupled transport and integro-differential equations have been widely used to study the reparable systems (cf. [2], [3], [4], [6], [7], [8], [9], [23]). Especially, for given failure rates the well-posedness and asymptotic behavior of this type of models with arbitrarily distributed repair have been well-studied using C_0-semigroup theory in the aforementioned references. Lately, optimal maintenance policies interpreted as control inputs of the system are discussed in (cf. [1], [11], [12]). Our current interest is on the identification of system failure rates, which is crucial in understanding and evaluating system performance.

In this paper, we consider a reparable multi-state system introduced by Chung [2], which represents general features of reparable systems. Consider that there are M modes of failure associated with a device. The state of the device is given by its failure mode number j, $j=1,2,\ldots,M$, and 0 represents the good state. The device is good at time zero and transitions are permitted only between states 0 and j. The failure rates are constants and all statistically independent. The repair time is arbitrarily distributed. The transition diagram for the system is demonstrated by Fig. 1. The precise model of system equations reads

$$\frac{dp_0(t)}{dt} = -\sum_{j=1}^{M}\lambda_j p_0(t) + \sum_{j=1}^{M}\int_0^\infty \mu_j(x)p_j(x,t)\,dx, \tag{1}$$

$$\frac{\partial p_j(x,t)}{\partial t} + \frac{\partial p_j(x,t)}{\partial x} = -\mu_j(x)p_j(x,t), \tag{2}$$

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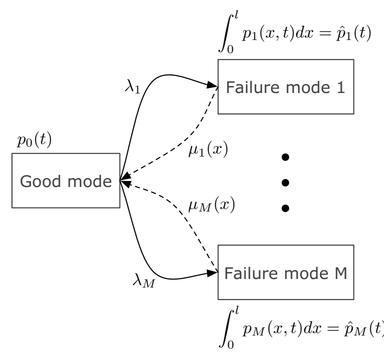

Fig. 1. Transition diagram of the reparable multi-state system

with boundary condition

$$p_j(0,t) = \lambda_j p_0(t), \quad j=1,2,\ldots,M, \quad t>0, \tag{3}$$

and initial conditions

$$p_0(0)=1, \quad p_j(x,0)=0, \quad j=1,2,\ldots,M. \tag{4}$$

Here

1) $p_0(t)$: probability that the device is in good mode 0 at time t. It also represents the pointwise availability of the system;
2) $p_j(x,t)$: probability density (with respect to repair time x) that the failed device is in failure mode j at time t and has an elapsed repair time of x. Let $\hat{p}_j(t)$ denote the probability of the device in failure mode 1 at time t, then $\hat{p}_j(t)$ is given by
$$\hat{p}_j(t) = \int_0^\infty p_j(x,t)\,dx; \tag{5}$$
3) $\lambda_j > 0$: constant failure rate of the device for failure mode j;
4) $\mu_j(x) \geq 0$: repair rate when the device is in failure state at t and has an elapsed repair time of x. Moreover, assume
$$0 \leq \mu_j(x) \leq \bar{\mu}, \quad \int_0^x \mu(s)\,ds < \infty, \ \forall x < \infty,$$
$$\text{and} \quad \int_0^\infty \mu_j(x)\,dx = \infty, \tag{6}$$
where $\bar{\mu} > 0$ is the maximum repair rate.

Due to the coupling of an integro-differential equation in the reparable system, equations (1)–(4) can be essentially formulated as a Volterra integral problem (see Remark 1). The mathematical approach for modeling the reparable system has also been employed in the study of the supply

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chain and queueing network modeling (cf. [5], [7], [8], [22]). In essence, it describes a birth-death process, which shares similar features with the population dynamics described by the first-order hyperbolic equations (cf. [20], [21]). Parameter identification associated with the population dynamics has been discussed in (cf. [14], [15], [16]). In the current work, we aim at using sampled data of the system output measurements to identify its failure rates.

II. Sampled-data based Failure Rate Identification

Consider that there are N pointwise measurements of the system output

$$\vec{\mathbf{P}}_N = \mathscr{C} p_0(t,\vec{\lambda}) = [p_0(t_1), p_0(t_2), \dots, p_0(t_N)]^T \in \mathbb{R}^N, \quad (7)$$

where $t_i \in [0,T], i = 1,2,\dots,N$, and

$$\mathscr{C} = [\int_0^T \delta_{t_1}(t)\cdot dt, \int_0^T \delta_{t_2}(t)\cdot dt, \dots, \int_0^T \delta_{t_N}(t)\cdot dt]^T.$$

In particular, let $t_N = T$. The objective of the present work is to identify the failure rate λ_j, $j = 1,2,\dots,M$, of the model based on the sampled data. We formulate this problem as a constrained least squares fit to data. To be more precise, we seek for $\vec{\lambda} = [\lambda_1, \lambda_2, \dots, \lambda_M]^T$ which minimizes:

$$J(\vec{\lambda}) = \frac{1}{2}\sum_{k=1}^{N} |p_0(t_k,\vec{\lambda}) - \tilde{p}_0(t_k)|^2 \quad (P)$$

for a given sampled data set

$$[\tilde{p}_0(t_1), \tilde{p}_0(t_2), \dots, \tilde{p}_0(t_N)]^T,$$

subject to the governing system (1)–(4).

In the remainder of this paper, we let $L^p_x(0,T)$ and $L^p_t(0,T)$ stand for the L^p-spaces, $1 \le p \le \infty$, with respect to x and t.

A. Sensitivity Analysis

For given failure and repair rates, the well-posedness and stability issues of system (1)–(4) have been thoroughly addressed in [9], [10], [23] by using C_0-semigroup theory. To solve problem (P), we first analyze the sensitivity of $\vec{p}$ with respect to $\vec{\lambda}$.

Recall the following basic results regarding the properties of the solution to (1)–(4) given by [1], [12].

Proposition 1: If $\vec{p}(x,t) = (p_0(t), p_1(x,t), \dots, p_M(x,t))^T$ is the solution to (1)–(4), then

$$p_j(x,t) = \begin{cases} \lambda_j p_0(t-x) e^{-\int_0^x \mu_j(s)\,ds}, & x < t, \\ 0, & x \ge t \end{cases} \quad (8)$$

and

$$\begin{aligned} p_0(t) =& e^{-\sum_{j=1}^M \lambda_j t} p_0(0) \\ &+ \sum_{j=1}^{M} \int_0^t e^{-\sum_{j=1}^M \lambda_j (t-\tau)} \int_0^\tau \mu_j(x) p_j(x,\tau)\,dx d\tau. \quad (9) \end{aligned}$$

Moreover, $\vec{p}(x,t) = (p_0(t), p_1(x,t), \dots, p_M(x,t))^T \ge 0$ for any $x,t > 0$,

$$p_0 \in W^{1,\infty}(0,T) \hookrightarrow C[0,T] \quad \text{and} \quad (10)$$

$$\begin{aligned} p_j \in\ & L^\infty(0,T;W^{1,1}(0,T)) \cap W^{1,\infty}(0,T;L^1(0,T)) \\ & \hookrightarrow C([0,T];L^1(0,T)), \quad j = 1,2,\dots,M. \quad (11) \end{aligned}$$

Since $p_j(x,t) = 0$ for $x \ge t$, we have

$$\int_0^T \frac{\partial p_j(x,t)}{\partial x}\,dx = p_j(T,t) - p_j(0,t) = -\lambda_j p_0(t).$$

Now integrating equation (2) for each $j = 1,2,\dots,M$, with respect to x from 0 to T and adding them all to (1) follow

$$\frac{dp_0(t)}{dt} + \sum_{j=1}^{M} \frac{d}{dt}\int_0^T p_j(x,t)\,dx = 0, \quad \forall 0 < t \le T, \quad (12)$$

which together with the initial condition (4) implies

$$p_0(t) + \sum_{j=1}^{M}\int_0^T p_j(x,t)\,dx = p_0(0) + \sum_{j=1}^{M}\int_0^T p_j(x,0)\,dx = 1, \quad (13)$$

for $0 < t \le T$. Therefore, the sum of the probabilities of the system in good and failure modes is always 1. In other words, the system is conservative. Furthermore, based on the nonnegativeity of the solution, (6) and (13), we know that p_0 in (9) satisfies

$$\begin{aligned} p_0(t) &\le e^{-\sum_{j=1}^M \lambda_j t} + \sum_{j=1}^{M}\int_0^t e^{-\sum_{j=1}^M \lambda_j (t-\tau)} \bar{\mu}\,d\tau \\ &= e^{-\sum_{j=1}^M \lambda_j t} + \frac{1 - e^{-\sum_{j=1}^M \lambda_j t}}{\sum_{j=1}^M \lambda_j}\bar{\mu} M, \end{aligned}$$

which converges to zero as $\vec{\lambda}$ converges to infinity.

Proposition 2: The solution $\vec{p}$ to (1)–(4) is absolutely continuous in $\vec{\lambda}$, uniformly in x and t.

Proof: Due to (8), it suffices to show that the probability of the system in good mode $p_0\colon \mathbb{R}^+ \times (\mathbb{R}^+)^M \to \mathbb{R}$ is absolutely continuous in $\vec{\lambda}$, and uniformly in t. With the help of (8), (9) and $p_0(0) = 1$, we get

$$\begin{aligned} p_0(t,\vec{\lambda}) =& e^{-\sum_{j=1}^M \lambda_j t} + \int_0^t \sum_{j=1}^{M} \lambda_j e^{-\sum_{j=1}^M \lambda_j (t-\tau)} \\ & \cdot \int_0^\tau \mu_j(x) p_0(\tau - x) e^{-\int_0^x \mu_j(s)\,ds}\,dx d\tau \\ =& e^{-\sum_{j=1}^M \lambda_j t} + \int_0^t \sum_{j=1}^{M} \lambda_j e^{-\sum_{j=1}^M \lambda_j (t-\tau)} \\ & \cdot \int_0^\tau \mu_j(\tau-\eta) p_0(\eta) e^{-\int_0^{\tau-\eta} \mu_j(s)\,ds}\,d\eta d\tau. \quad (14) \end{aligned}$$

Let $\vec{\lambda}^{(1)} > 0$ and $\vec{\lambda}^{(2)} > 0$. We have

$$\begin{aligned}
&\sup_{t\in[0,T]} |p_0(t,\vec{\lambda}^{(1)}) - p_0(t,\vec{\lambda}^{(2)})| \\
&\leq \sup_{t\in[0,T]} |e^{-\sum_{j=1}^M \lambda_j^{(1)} t} - e^{-\sum_{j=1}^M \lambda_j^{(2)} t}| \\
&+ \sup_{t\in[0,T]} \int_0^t |\sum_{j=1}^M \lambda_j^{(1)} e^{-\sum_{j=1}^M \lambda_j^{(1)}(t-\tau)} - \sum_{j=1}^M \lambda_j^{(2)} e^{-\sum_{j=1}^M \lambda_j^{(2)}(t-\tau)}| \\
&\cdot \int_0^\tau \mu_j(\tau-\eta) p_0(\eta,\vec{\lambda}^{(1)}) e^{-\int_0^{\tau-\eta}\mu_j(s)\,ds} d\eta d\tau \\
&+ \sup_{t\in[0,T]} \int_0^t \sum_{j=1}^M \lambda_j^{(2)} e^{-\sum_{j=1}^M \lambda_j^{(2)}(t-\tau)} \\
&\cdot \int_0^\tau \mu_j(\tau-\eta) e^{-\int_0^{\tau-\eta}\mu_j(s)\,ds} |p_0(\eta,\vec{\lambda}^{(1)}) - p_0(\eta,\vec{\lambda}^{(2)})| d\eta d\tau \\
&= I_1 + I_2 + I_3. \qquad (15)
\end{aligned}$$

It is clear that $I_1 \to 0$ as $|\vec{\lambda}^{(1)} - \vec{\lambda}^{(2)}| \to 0$. For the second term I_2, we get

$$\begin{aligned}
\int_0^\tau & \mu_j(\tau-\eta) p_0(\eta,\vec{\lambda}^{(1)}) e^{-\int_0^{\tau-\eta}\mu_j(s)\,ds} d\eta \\
&\leq \int_0^\tau \mu_j(\tau-\eta) e^{-\int_0^{\tau-\eta}\mu_j(s)\,ds} d\eta \\
&= \int_0^\tau de^{-\int_0^{\tau-\eta}\mu_j(s)} = 1 - e^{-\int_0^\tau \mu_j(s)} < 1,
\end{aligned}$$

for $\tau > 0$. Thus $I_2 \to 0$ as $|\vec{\lambda}^{(1)} - \vec{\lambda}^{(2)}| \to 0$. Furthermore,

$$\begin{aligned}
I_3 &\leq \sup_{t\in[0,T]} \int_0^t \sum_{j=1}^M \lambda_j^{(2)} e^{-\sum_{j=1}^M \lambda_j^{(2)}(t-\tau)} \\
&\quad \cdot \int_0^\tau \mu_j(\tau-\eta) e^{-\int_0^{\tau-\eta}\mu_j(s)\,ds} d\eta d\tau \\
&\quad \cdot \|p_0(\cdot,\vec{\lambda}^{(1)}) - p_0(\cdot,\vec{\lambda}^{(2)})\|_{L_t^\infty} \\
&\leq \sup_{t\in[0,T]} \int_0^t \sum_{j=1}^M \lambda_j^{(2)} e^{-\sum_{j=1}^M \lambda_j^{(2)}(t-\tau)} \\
&\quad \cdot \int_0^\tau de^{-\int_0^{\tau-\eta}\mu_j(s)} d\tau \|p_0(\cdot,\vec{\lambda}^{(1)}) - p_0(\cdot,\vec{\lambda}^{(2)})\|_{L_t^\infty} \\
&\leq \sup_{t\in[0,T]} \int_0^t \sum_{j=1}^M \lambda_j^{(2)} e^{-\sum_{j=1}^M \lambda_j^{(2)}(t-\tau)} (1 - e^{-\int_0^\tau \mu_j(s)}) d\tau \\
&\quad \cdot \|p_0(\cdot,\vec{\lambda}^{(1)}) - p_0(\cdot,\vec{\lambda}^{(2)})\|_{L_t^\infty} \\
&\leq \sup_{t\in[0,T]} (1 - e^{-\sum_{j=1}^M \lambda_j^{(2)} t}) \|p_0(\cdot,\vec{\lambda}^{(1)}) - p_0(\cdot,\vec{\lambda}^{(2)})\|_{L_t^\infty} \\
&= (1 - e^{-\sum_{j=1}^M \lambda_j^{(2)} T}) \|p_0(\cdot,\vec{\lambda}^{(1)}) - p_0(\cdot,\vec{\lambda}^{(2)})\|_{L_t^\infty}. \qquad (16)
\end{aligned}$$

Combining (15) with (16) yields

$$\begin{aligned}
\|p_0(\cdot,,\vec{\lambda}^{(1)}) &- p_0(\cdot,\vec{\lambda}^{(2)})\|_{L_t^\infty} \leq I_1 + I_2 \\
&+ (1 - e^{-\sum_{j=1}^M \lambda_j^{(2)} T}) \|p_0(\cdot,\vec{\lambda}^{(1)}) - p_0(\cdot,\vec{\lambda}^{(2)})\|_{L_t^\infty}.
\end{aligned}$$

Therefore,

$$\|p_0(\cdot,,\vec{\lambda}^{(1)}) - p_0(\cdot,\vec{\lambda}^{(2)})\|_{L_t^\infty} \leq e^{\sum_{j=1}^M \lambda_j^{(2)} T} (I_1 + I_2) \to 0,$$

as $|\vec{\lambda}^{(1)} - \vec{\lambda}^{(2)}| \to 0$, which establishes the desired result. ■

Remark 1: By changing the order of integration we can further write (14) as a Volterra equation

$$p_0(t,\vec{\lambda}) = f(t,\vec{\lambda}) + \int_0^t K(t-\eta,\vec{\lambda}) p_0(\eta,\vec{\lambda})\, d\eta, \qquad (17)$$

where $f(t,\vec{\lambda}) = e^{-\sum_{j=1}^M \lambda_j t} p_0(0)$ and

$$\begin{aligned}
K(t-\eta,\vec{\lambda}) = &\sum_{j=1}^M \lambda_j e^{-\sum_{j=1}^M \lambda_j (t-\eta)} \\
&\cdot \int_0^{t-\eta} \mu_j(\xi) e^{\sum_{j=1}^M \lambda_j \xi - \int_0^\xi \mu_j(s)\,ds} d\xi.
\end{aligned}$$

Next we discuss the sensitivity of $\vec{p}$ with respect to the failure rate $\vec{\lambda}$. Let $\vec{y}_i = \frac{\delta \vec{p}}{\delta \lambda_i}, i = 1,2,\ldots,M$. Then

$$\vec{y}_i = (y_{0i}, y_{1i}, \ldots, y_{Mi})^T = (\frac{\delta p_0}{\delta \lambda_i}, \frac{\delta p_1}{\delta \lambda_i}, \ldots, \frac{\delta p_M}{\delta \lambda_i})^T$$

satisfies

$$\frac{dy_{0i}(t)}{dt} = -(p_0(t) + \sum_{j=1}^M \lambda_j y_{0i}(t)) + \sum_{j=1}^M \int_0^T \mu_j(x) y_{ji}(x,t)\, dx, \qquad (18)$$

$$\frac{\partial y_{ji}(x,t)}{\partial t} + \frac{\partial y_{ji}(x,t)}{\partial x} = -\mu_j(x) y_{ji}(x,t), \qquad (19)$$

with boundary condition

$$y_{ji}(0,t) = \delta_{ji} p_0(t) + \lambda_j y_{0i}(t), \quad j = 1,2,\ldots,M, \quad t > 0, \qquad (20)$$

and initial conditions

$$y_{0i}(0) = 0, \quad y_{ji}(x,0) = 0, \quad j = 1,2,\ldots,M. \qquad (21)$$

Using the similar procedure as in the proof of Proposition 1, we can verify that y_{0i} satisfies

$$\begin{aligned}
y_{0i}(t,\vec{\lambda}) = &-\int_0^t e^{-\sum_{j=1}^M \lambda_j (t-\tau)} p_0(\tau) d\tau \\
&+ \sum_{j=1}^M \int_0^t e^{-\sum_{j=1}^M \lambda_j (t-\tau)} \int_0^\tau \mu_j(x) y_{ji}(x,\tau)\, dx d\tau
\end{aligned}$$

and y_{ji} satisfies

$$y_{ji}(x,t) = \begin{cases} \delta_{ji} p_0(t-x) + \lambda_j y_{0i}(t-x) e^{-\int_0^x \mu_j(s)\,ds}, & x < t, \\ 0, & x \geq t \end{cases}$$

for $i = 1,2,\ldots,N$ and $j = 1,2,\ldots,M$.

It can be shown that $\vec{y}$ is continuous in $\vec{\lambda}$, thus $\vec{p}$ is continuously differentiable with respect to $\vec{\lambda}$. Due to limited space, we shall leave the detailed proof in our future paper.

To reduce the dimension of computation of the sensitivity equations, we shall employ the adjoint method to derive an efficient way to evaluate $\frac{\delta J(\vec{\lambda})}{\delta \vec{\lambda}}$. To this end, we first introduce the adjoint sensitivity equations. Recall the duality between nonreflexive Banach spaces. Let $X = \mathbb{R} \times (L_x^1(0,T))^M$. Then its dual space is given by $X^* = \mathbb{R} \times (L_x^\infty(0,T))^M$ and the duality between X and X^* is defined by

$$(\vec{p},\vec{q})_{X,X^*} = p_0 q_0 + \sum_{j=1}^M \int_0^T p_j q_j\, dx,$$

for $\vec{p} = (p_0, p_1, \ldots, p_M)^T \in X$ and $\vec{q} = (q_0, q_1, \ldots, q_M)^T \in X^*$.

Let $\vec{q}$ be the adjoint state. It is straightforward to verify that $\vec{q}$ satisfies the adjoint sensitivity equations

$$\frac{dq_0(t)}{dt} = \sum_{j=1}^{M}(\lambda_j + p_0(t))(q_0(t) - q_j(0,t)) - \sum_{k=1}^{N-1}\delta_{t_k}(t)(p_0(t_k) - \tilde{p}_0(t_k)), \tag{22}$$

$$\frac{\partial q_j(x,t)}{\partial t} + \frac{\partial q_j(x,t)}{\partial x} = -\mu_j(x)(q_0(t) - q_j(x,t)), \tag{23}$$

with boundary conditions

$$q_j(T,t) = 0, \quad j = 1,2,\ldots,M, \quad t > 0, \tag{24}$$

and final conditions

$$q_0(T) = p_0(T) - \tilde{p}_0(T), \quad q_j(x,T) = 0, \quad j = 1,2,\ldots,M. \tag{25}$$

Moreover, $\vec{q} \in L^\infty(0,T) \times (L^\infty(0,T;L^\infty(0,T)))^M$.

Theorem 2: If $\vec{\lambda}^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_M^*)^T \in (\mathbb{R}^+)^M$ is the best fit to (P), then $\vec{\lambda}_i^*$ satisfies

$$\int_0^T p_0(t)(q_i(0,t) - q_0(t))\,dt = 0, \tag{26}$$

for $i = 1,2,\ldots,M$, where p_0, $(q_0, q_1, \ldots, q_M)^T$ are the solutions to the governing system (1)-(4) and the adjoint sensitivity equations (22)–(25), respectively.

Proof: Note that if $\vec{\lambda}^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_M^*)^T \in (\mathbb{R}^+)^M$ is an optimal solution to problem (P), it is always an interior point of $(\mathbb{R}^+)^M$, and hence

$$\frac{\delta J(\vec{\lambda})}{\delta \lambda_i}|_{\lambda_i = \lambda_i^*} = \sum_{k=1}^{N}(p_0(t_k, \vec{\lambda}) - \tilde{p}_0(t_k)) y_{0i}(t_k, \lambda_i^*) = 0, \tag{27}$$

for $i = 1,2,\ldots,M$.

Next we apply the adjoint sensitivity equations to further tackle (27). Taking the duality between $\vec{y}$ and $\vec{q}$ and then integrating with respect to t from 0 to T yield

$$\int_0^T \frac{dy_{0i}}{dt} q_0\,dt = -\int_0^T (p_0 + \sum_{j=1}^{M}\lambda_j y_{0i}(t)) q_0\,dt + \int_0^T (\sum_{j=1}^{M}\int_0^T \mu_j y_{ji}\,dx) q_0\,dt$$

and

$$\int_0^T (\frac{\partial y_{ji}}{\partial t}, q_j)\,dt + \int_0^T (\frac{\partial y_{ji}}{\partial x}, q_j)\,dt = \int_0^T (-\mu_j y_{ji}, q_j)\,dt$$

for $j = 1,2,\ldots,M$. Using integration by parts and (24)–(25), we get

$$y_{0i}(T) q_0(T) - \int_0^T y_{0i}\frac{dq_0}{dt}\,dt = -\int_0^T (p_0 + \sum_{j=1}^{M}\lambda_j y_{0i}) q_0\,dt + \int_0^T (\sum_{j=1}^{M}\int_0^t \mu_j y_{ji}\,dx) q_0\,dt \tag{28}$$

and

$$-\int_0^T (y_{ji}, \frac{\partial q_j}{\partial t})\,dt - \int_0^T (\delta_{ji} p_0(t) + \lambda_j y_{0i}(t)) q_j(0,t)\,dt - \int_0^T (y_{ji}, \frac{\partial q_j}{\partial x})\,dt = \int_0^T (y_{ji}, -\mu_j q_j)\,dt, \quad j = 1,2,\ldots,M. \tag{29}$$

Now taking the summation of both sides of (29) with respect to j from 1 to M and adding the resulting equation to (28) give

$$\begin{aligned} & y_{0i}(T)(p_0(T) - \tilde{p}_0(T)) \\ &= -\int_0^T \sum_{k=1}^{N-1}\delta_{t_k}(t) y_{0i}(t)(p_0(t_k) - \tilde{p}_0(t_k))\,dt \\ &\quad - \int_0^T p_0(t) q_0(t)\,dt + \int_0^T p_0(t) q_i(0,t)\,dt \end{aligned}$$

or

$$\sum_{k=1}^{N} y_{0i}(t_k)(p_0(t_k) - \tilde{p}_0(t_k)) = \int_0^T p_0(t)(q_i(0,t) - q_0(t))\,dt,$$

where we utilized equations (22)–(23). As a result, (27) becomes

$$\begin{aligned} \frac{\delta J(\vec{\lambda})}{\delta \lambda_i}|_{\lambda_i = \lambda_i^*} &= \sum_{k=1}^{N}(p_0(t_k, \vec{\lambda}) - \tilde{p}_0(t_k)) y_{0i}(t_k, \vec{\lambda}) \\ &= \int_0^T p_0(t)(q_i(0,t) - q_0(t))\,dt = 0, \end{aligned}$$

for $i = 1,2,\ldots,M$, which completes the proof. ■

III. Numerical Schemes based on the Adjoint Method

To implement the theoretical results established in the previous sections, we will use a projected gradient descent algorithm [18] based on the temporal method of lines, for solving the spatially semi-discretized forward state equations and backward adjoint sensitivity equations, respectively. Below we mainly describe our numerical schemes for the case with $M = 1$.

For the spatial domain $[0,T]$, we define a uniform mesh $\{x_i = ih\}_{i=0}^{n}$ with the step size $h = T/n$. Let $p_{1,i}(t) = p_1(x_i,t)$, $q_{1,i}(t) = q_1(x_i,t)$ and $\mu_i = \mu(x_i)$. We use the (right) rectangular rule to approximate the integral term in (1) and apply the upwind scheme to discretize the spatial first-order partial derivative term in (2), which result in the semi-discretized state equations ($1 \le i \le n$)

$$\begin{aligned} \frac{dp_0(t)}{dt} &= -\lambda p_0(t) + h\sum_{i=1}^{n}\mu_i p_{1,i}(t), \\ \frac{dp_{1,i}(t)}{dt} &= -\frac{p_{1,i}(t) - p_{1,i-1}(t)}{h} - \mu_i p_{1,i}(t), \end{aligned}$$

marching forward with the boundary condition $p_{1,0} = \lambda_1 p_0(t)$ and the initial conditions $p_0(0) = 1, p_{1,i}(0) = 0$. The above scheme (30)–(30) can be formulated into an initial value problem of (stiff) ODE system

$$\begin{aligned} \frac{d\vec{p}(t)}{dt} &= A_h(\lambda)\vec{p}(t), && (30)\\ \vec{p}(0) &= [1, 0, \cdots, 0]^T, && (31) \end{aligned}$$

with

$$\vec{p}(t) = \begin{bmatrix} p_0(t), p_{1,1}(t), p_{1,2}(t), \cdots, p_{1,N}(t) \end{bmatrix}^{\mathsf{T}}$$

and

$$A_h(\lambda) = \begin{bmatrix} -\lambda & h\mu_1 & h\mu_2 & h\mu_3 & \cdots & h\mu_n \\ \lambda/h & -\mu_1 - 1/h & 0 & 0 & \cdots & 0 \\ 0 & 1/h & -\mu_2 - 1/h & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1/h & -\mu_n - 1/h \end{bmatrix},$$

which can be efficiently solved by any ODE solvers.

To numerically solve the adjoint sensitivity equations (22)-(25), we introduce the following change of variables:

$$\widehat{q}_0(t) = q_0(t) + \phi(t) := q_0(t) + \sum_{k=1}^{N-1} H(t - t_k)(p_0(t_k) - \tilde{p}_0(t_k))$$

to eliminate the Dirac delta function within (22) to get

$$\begin{aligned} &\frac{d\widehat{q}_0(t)}{dt} = (\lambda + p_0)(\widehat{q}_0(t) - \phi(t) - q_1(0,t)) \\ &\frac{\partial q_1(x,t)}{\partial t} + \frac{\partial q_1(x,t)}{\partial x} = -\mu(x)(\widehat{q}_0(t) - \phi(t) - q_1(x,t)), \end{aligned}$$

with a modified final condition

$$\widehat{q}_0(T) = p_0(T) - \tilde{p}_0(T) + \phi(T) \quad = \sum_{k=1}^{N} (p_0(t_k) - \tilde{p}_0(t_k)).$$

Using the same upwind scheme, the semi-discretized adjoint sensitivity equations read ($i = 0, 1, 2, \cdots, n-1$)

$$\begin{aligned} \frac{d\widehat{q}_0(t)}{dt} &= (\lambda + p_0(t))(\widehat{q}_0(t) - \phi(t) - q_{1,0}(t)), && (32)\\ \frac{dq_{1,i}(t)}{dt} &= -\frac{q_{1,i+1}(t) - q_{1,i}(t)}{h} \\ &\quad -\mu_i(\widehat{q}_0(t) - \phi(t) - q_{1,i}(t)), && (33) \end{aligned}$$

which march backward with the boundary condition $q_{1,n} = 0$, the final conditions $\widehat{q}_0(T) = \sum_{k=1}^{N}(p_0(t_k) - \tilde{p}_0(t_k))$ and $q_{1,i}(T) = 0$. By defining

$$\begin{aligned} \vec{q}(t) &= \begin{bmatrix} \widehat{q}_0(t), q_{1,0}(t), \cdots, q_{1,n-1}(t) \end{bmatrix}^{\mathsf{T}}, \\ \vec{b}(t) &= \begin{bmatrix} -\lambda - p_0(t), \mu_0,, \cdots, \mu_{N-1} \end{bmatrix}^{\mathsf{T}}, \end{aligned}$$

$$B_h(\lambda, t) = \begin{bmatrix} \lambda + p_0(t) & -\lambda - p_0(t) & 0 & 0 & \cdots & 0 \\ -\mu_0 & \mu_0 + 1/h & -1/h & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1/h \\ -\mu_{N-1} & 0 & 0 & \cdots & 0 & \mu_{N-1} + 1/h \end{bmatrix},$$

the above scheme (32)-(33) can be written as a final value problem of (stiff) time-varying ODE system:

$$\begin{aligned} \frac{d\vec{q}(t)}{dt} &= B_h(\lambda, t)\vec{q}(t) + \phi(t)\vec{b}(t), && (34)\\ \vec{q}(T) &= [\sum_{k=1}^{N}(p_0(t_k) - \tilde{p}_0(t_k)), 0, \cdots, 0]^T, && (35) \end{aligned}$$

which, upon reversing the time via a change of variable $\tau = T - t$, can also be efficiently solved by ODE solvers. After solving $\widehat{q}_0(t)$, we can recover $q_0(t) = \widehat{q}_0(t) - \phi(t)$. Finally, a gradient descent (GD) algorithm (see Algorithm 1) for solving our least square minimization problem can be constructed by approximating the gradient (27) for $M = 1$ by a quadrature rule (e.g., trapezoidal rule).

Algorithm 1 A gradient descent (GD) algorithm:
Input: $T, n, h = T/n, tol, k_{\max}$
Output: estimated failure rate λ_h

choose an initial guess $\lambda^{(0)} > 0$;
for $k = 0$ to $k_{\max}$ **do**
  solve the forward ODEs (30-31) with $\lambda = \lambda^{(k)}$;
  solve the backward ODEs (34-35) with $\lambda = \lambda^{(k)}$;
  compute the gradient (27): use trapezoidal rule

$$\frac{\delta J(\lambda^{(k)})}{\delta \lambda} \approx G_k := \sum_{j=0}^{n} w_j p_0(\tau_j)(q_{1,0}(\tau_j) - q_0(\tau_j)),$$

  where τ_j and w_j are quadrature nodes and weights;
  update $\lambda^{(k+1)}$ along gradient descent iteration

$$\lambda^{(k+1)} = \lambda^{(k)} - \alpha_k G_k,$$

  where $\alpha_k \in (0, 1]$ is the appropriate step size;
  if $\|\lambda^{(k+1)} - \lambda^{(k)}\| \leq tol$ **then**
    return $\lambda_h = \lambda^{(k+1)}$;
  end if
end for

IV. Numerical Results

In this section, we provide numerical examples to validate our proposed GD algorithm. All simulations are implemented using MATLAB. We choose $T = 20$, $n = 200$, and $\lambda_{exact} = 0.618$ in our forward model simulation. The repair rate $\mu(x) = \frac{5}{10}(\frac{x}{10})^{(5-1)}$ is a Weibull distribution. For the GD algorithm, we take the initial guess $\lambda^{(0)} = 0.5$ and the stopping tolerance $tol = 10^{-4}$.

A. Noise-free case: influence of number of measurements

In Fig. 2, we compare the simulated $p_0(t, \lambda_h)$ based on the estimated failure rate λ_h with the noise-free $N = 1$ and $N = 4$ measurements data (excluding the initial condition). It seems that one measurement provides good estimate of the failure rate, but more measurements help to improve the estimation accuracy and accelerate the convergence rate of the GD algorithm.

B. Noise case: influence of noise level

In Fig. 3, we compare the simulated $p_0(t, \lambda_h)$ based on the estimated failure rate λ_h, from $N = 1, 4, 9$ measurement data with noise. The added random noise is normally distributed with zero mean and standard deviation $\delta = 1\%$ and $\delta = 5\%$, respectively. The approximation accuracy of the estimated failure rate λ_h is comparable to the noise level, which indicates that our least-square minimization model and the designed GD algorithm are very stable.

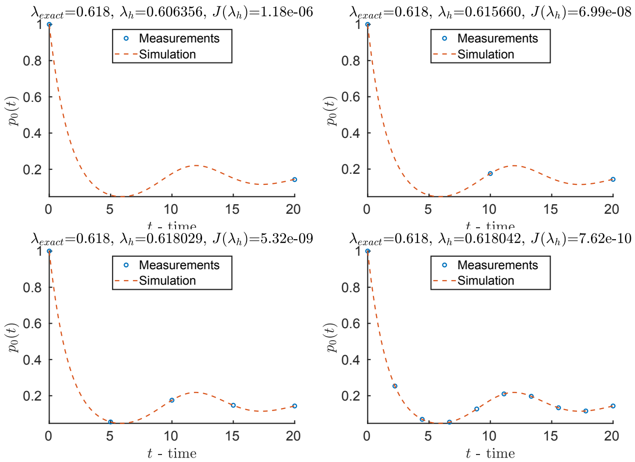

Fig. 2. The estimated failure rate with 1,2,4 and 9 measurements.

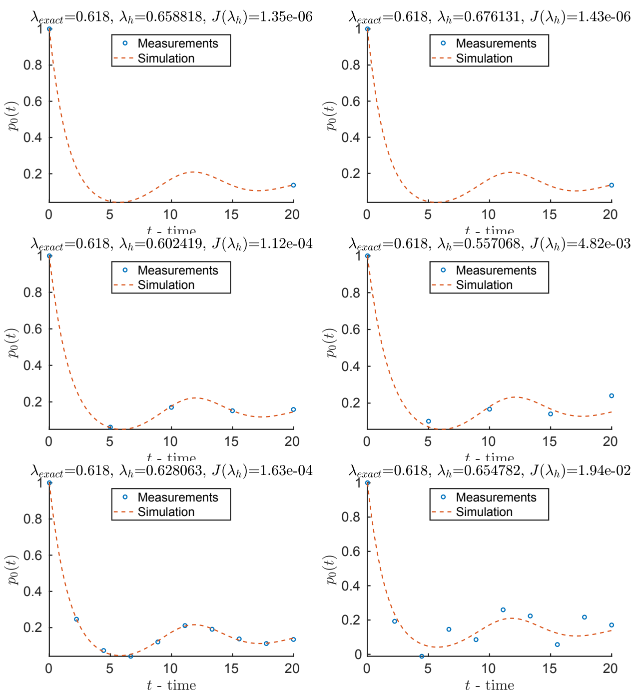

Fig. 3. The estimated failure rate for 1, 4, and 9 measurements with 1% (left) and 5% (right) noise, respectively.

V. Conclusion

We considered the problem of failure rates identification for a multi-state reparable system using the sampled data of the system output measurements. We formulated it as a PDE-constrained least squares problem to find the best fit to data. To numerically solve this problem, we employed a gradient descent algorithm based on the adjoint method, where we established the rigorous sensitivity analysis and the adjoint sensitivity equations. Numerical results for $M = 1$ demonstrated the effectivity of our designd algorithm. However, when increasing the number of failure modes to $M > 1$, we observed that our GD algorithm still converges, but it might converge to a different local minimizer since the uniqueness is unknown. The questions of what conditions or extra measurements are needed to uniquely determine the failure rates and how to provide the optimal repair based on the failure rates will be further investigated in our future work.

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