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Integrable space-time shifted nonlocal nonlinear equations

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ABSTRACT

In 1974 Ablowitz, Kaup, Newell, Segur (AKNS) put forward a theoretical framework whereby one can construct evolution equations that are (i) integrable in the sense of existence of infinite number of conservation laws and (ii) solvable by the inverse scattering transform. In subsequent years, many physically important integrable evolution equations were identified and the focus of the subject shifted towards methods to find special solutions and enhancing the underlying analysis. The discovery of a new reduction of the original AKNS system and the PT symmetric integrable nonlocal nonlinear Schrödinger (NLS) equation more than forty years later was surprising. Subsequently, additional nonlocal integrable reductions were found allowing nonlocality to be manifested in the time domain as well. This paper reports on yet another novel set of integrable reductions for the original AKNS system and associated new space-time nonlocal NLS type equations with space and time shifts. Integrability and inverse scattering transform are established along with soliton solutions. Their unique properties are discussed along with detailed comparison with the respective standard (*non shifted*) PT and reverse space-time symmetric NLS equations.

1. Introduction

For several decades, integrable systems have been at the forefront in numerous research areas in the mathematical sciences and theoretical physics [1–7]. In part, this is due to the quest for exactly solvable models and solutions of physical relevance as well as their elegant mathematical structure. Integrability theory originally emerged in the study of classical Hamiltonian mechanics with few numbers of degrees of freedom. One of its main pillars being the existence of a canonical transformation to action-angle variables that linearize the underlying governing nonlinear equations [8,9].

While the theory of integrable finite-dimensional Hamiltonian dynamics had substantial influence in mathematics and physics, its reach to a broad class of infinite-dimensional extended systems remained elusive for a long time. Amongst the earliest infinite-dimensional integrable systems (albeit dissipative) is the well-known viscous Burgers' equation. Upon applying the so-called Cole-Hopf [10,11] transformation, it reduces to the linear heat equation. Thus, integrability manifests itself by an exact linearization.

About fifteen years later, Zabusky and Kruskal [12] made their seminal discovery of solitons that eventually paved the way to extend the theory of Hamiltonian integrable systems with finite number of degrees of freedom to infinite-dimensions. Indeed, shortly afterwards, Gardner, Greene, Kruskal and Miura [13] established the integrability (linearization) of the celebrated Korteweg-de Vries (KdV) equation [14] by linking it to the linear Schrödinger equation (where the solution of the KdV equation plays the role of a potential) and introduced the method of inverse scattering and obtained the solution to its Cauchy problem with decaying data.

Following these important results, Lax [15] showed that solvable nonlinear evolution equations resulted from a compatibility condition between two linear systems or "linear pair". This formulation was used by Zakharov and Shabat [16] to find the solution of the nonlinear Schrödinger equation (NLS) with decaying data. Indeed, the NLS equation is an infinite dimensional integrable Hamiltonian system.

In 1974 Ablowitz, Kaup, Newell, Segur (AKNS) [17] formulated a general theory leading to a class of integrable nonlinear evolution equations that are solvable by the inverse scattering transform.

These corner stone discoveries have led to an intense and rapid research activities in the general area of integrable systems. Among the various research directions and developments in this field include numerous exactly integrable physically significant nonlinear wave equations including the KdV, scalar and coupled NLS, sine-Gordon, and three-wave equations; multidimensional models

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such as the Kadomtsev-Petviashvili and Davey-Stewartson equations; discrete integrable systems including the Toda lattice [18] and discrete integrable NLS models [19,20], quantum integrable spin systems and field theories [21] to name a few.

Towards the end of the past century it was thought that most of the physically important nonlinear evolution equations had been identified. In turn, interest in integrable systems shifted towards finding special solutions and important mathematical analysis such as long time asymptotics and formulations with different boundary conditions. Unexpectedly, Ablowitz and Musslimani [22] reported in 2013 on a new and unusual integrable symmetry reduction to the AKNS scattering theory. It is of the parity-time (PT) symmetric type (invariance under the combined action of parity operation P and time-reversal symmetry T) and leads to the PT symmetric nonlocal NLS equation. What is unusual about this reduction is that it relates function values at point x in space to its function values at a mirror-reflection space point $(-x)$. Interestingly enough, prior to this discovery, there were no known similar examples of such nonlocality in the general area of integrable systems. We point out that PT symmetric and non-hermitian physics has been the subject of an intense research for the last decade most notably in classical optics, quantum mechanics and topological photonics [23–44].

In 2016 a "second wave" of nonlocal integrable symmetry reductions were found where now the nonlocality is no longer exclusively spatial but rather includes time as well [45,46]. The introduction of integrability preserving PT symmetric nonlocality into the NLS equation has created opportunities to investigate nonlinear extended systems where blending nonlocality and general space-time reflection symmetries with integrability theory can lead to unique phenomena that are absent otherwise. Examples include a closed form soliton solution that approaches infinity in finite-time after which, it recovers and, periodically, develops a singularity in finite time periodically [22]; and nonlocal Painlevé equations [46] to name a few. In recent years, much progress has been reported in the general area of PT and space-time symmetric nonlocal integrable systems. Among them are the so-called nonlocal negative AKNS hierarchy [47], nonlocal hydrodynamic type systems [48], Riemann-Hilbert approach to space-time nonlocal mKdV hierarchies [49,50], gauge invariance to magnetic Landau systems [51,52], nonlocal discrete Ablowitz-Ladik type models [46,53], nonlocal breathers and multi-dimensional extensions to name a few [54–74].

In this paper, several novel integrable nonlocal reductions for the AKNS scattering problem are unveiled. Unlike their corresponding (standard) PT symmetric and reverse space-time nonlocal ones, they correspond to a shifted space, or time or space-time nonlocal symmetries. They give rise to a new set of space-time nonlocal NLS-like equations. Among them are the shifted PT symmetric and the shifted time delay nonlocal NLS equations. The latter case, has distinct and unusual properties so far not encountered in integrability theory: It is of a reverse delay differential equation (in time) type. One and two soliton solution for the shifted PT symmetric case are given below.

2. Shifted space-time nonlocal equations

We start the discussion by considering the coupled "(q,r) system"

$$iq_t = q_{xx} - 2q^2 r \,, \tag{1}$$

$$-ir_t = r_{xx} - 2r^2 q \,, \tag{2}$$

where q and r are complex valued functions of the space variable x (the whole real line) and time $t \geq 0$. Here, subscripts denote partial derivatives with respect to either x or t. For notational purposes only, all dependent variables (e.g. q, r) are assumed to be local variables of x, t unless otherwise explicitly noted. Equations (1) and (2) are well known in integrability theory. They were derived by AKNS in 1974 [17] as a compatibility conditions between two linear systems. One is a 2×2 first order Dirac-like scattering problem

$$v_{1x} = -ikv_1 + q(x,t)v_2 \,, \tag{3}$$

$$v_{2x} = ikv_2 + r(x,t)v_1 \,, \tag{4}$$

with q, r playing the role of potentials and an associated time-evolution. Here, k is a complex parameter independent of space and time and v_1, v_2 are complex valued functions that are defined by their plane wave asymptotic structure at infinity. Importantly, Eqns. (1) and (2) were recently derived from asymptotic reductions of a physically relevant prototypical evolution equations such as free-surface ideal water waves, the KdV and cubic nonlinear Klein-Gordon equations [75]. As mentioned earlier, the PT nonlocal, reverse space-time and reverse time-only NLS equations were obtained from Eqns. (1) and (2) under new symmetry reductions [22,46]. Here, we report, a new type of (uncommon) integrable symmetry reductions associated with system (1)-(2). The first is a space shifted PT symmetric nonlocal reduction given by

$$r(x,t) = \sigma q^*(x_0 - x, t) \,, \tag{5}$$

where $\sigma = \mp 1$ and x_0 is an *arbitrary* real parameter. Under this integrable reduction, Eqns. (1) - (2) are compatible and lead to the integrable shifted PT symmetric nonlocal NLS equation:

$$iq_t = q_{xx} - 2\sigma q^2 q^*(x_0 - x, t) \,. \tag{6}$$

Note that Eq. (6) is invariant under the joint transformation of $x \to x_0 - x, t \to -t$ and complex conjugation. In other words, if $q(x,t)$ solves (6), so does $q^*(x_0 - x, -t)$.

The second integrable symmetry reduction for system (1) - (2) is of *reverse time-delay* type:

$$r(x,t) = \sigma q(x, t_0 - t) \,, \tag{7}$$

where, as before, $\sigma = \mp 1$ and t_0 is an *arbitrary* real parameter. This integrable reduction makes Eqns. (1) and (2) self-consistent leading to a single integrable shifted reverse time only nonlocal NLS equation

$$iq_t = q_{xx} - 2\sigma q^2 q(x, t_0 - t) \,. \tag{8}$$

What is unusual about the symmetry reduction (7) as well as Eq. (8) is that when viewed as a reduction of the coupled system (1) and (2) with initial conditions $q(x,0), r(x,0)$, it appears to require the (unknown) data $q(x,t_0)$ in order to be able to determine the value of $q(x,t)$. To highlight the intricacies and challenges one faces when dealing with such circumstances we consider two simple linear problems: $\dot{q} = q(t) + q(t - t_0)$ and $\dot{q} = q(t) + q(t_0 - t)$. Solution to the former equation can be sought of in the form $q(t) = e^{st}$ with s, t_0 obeying the constraint $s = 1 + e^{-st_0}$. On the other hand, the same ansatz is not useful in the latter case.

The third symmetry reduction for the AKNS system (1) - (2) involves shifts in both space and time. It is given as

$$r(x,t) = \sigma q(x_0 - x, t_0 - t) \,, \tag{9}$$

for any real constants x_0, t_0. Equation (9) gives rise to the space-time shifted nonlocal integrable NLS equation:

$$iq_t = q_{xx} - 2\sigma q^2 q(x_0 - x, t_0 - t) \,. \tag{10}$$

All the above NLS cases reduce back to their respective "standard" PT and reverse space-time nonlocal NLS limits when x_0 and t_0 are set to zero. With this in mind, we shall later highlight major differences between these two scenarios and emphasize the role x_0, t_0 play.

We also remark that for suitable equations the integrable reduction given in Eq. (9) can be extended to allow for complex conjugation of the function q. This can be achieved by considering the "(q, r) system" associated with the mKdV equation (it belongs to the same hierarchy as for the NLS equation, i.e., it has the same scattering problem as above: Eqns. (3)-(4), but different associated time evolution):

$$q_t + q_{xxx} - 6qrq_x = 0 \,, \tag{11}$$

$$r_t + r_{xxx} - 6qrr_x = 0 \,. \tag{12}$$

With this at hand, it can be seen that under the complex integrable symmetry reduction

$$r(x,t) = \sigma q^*(x_0 - x, t_0 - t) \,, \tag{13}$$

the coupled evolution Eqns. (11) and (12) are compatible and give rise to the shifted complex reverse space-time mKdV equation

$$q_t + q_{xxx} - 6\sigma qq^*(x_0 - x, t_0 - t)q_x = 0 \,. \tag{14}$$

It is clear that the real reduction (without complex conjugation) also holds true and leads to a real space-time shifted mKdV equation

$$q_t + q_{xxx} - 6\sigma qq(x_0 - x, t_0 - t)q_x = 0 \,. \tag{15}$$

We close this section by making an observation regarding the "(q, r) system" and its classical integrable symmetry reduction $r(x,t) = \sigma q^*(x,t)$. Also interesting about the latter is that it has no shifted counter part, i.e., $r(x,t) = \sigma q^*(x_0 + x, t)$ is *not* a reduction to system (1) and (2) with real parameter x_0.

3. Complex shifts

While all the shifts (i.e., x_0 and t_0) introduced above were considered to be *real*, we remark that the new integrable symmetry reductions given by Eqns. (7) and (9) allow for these shifts to be extended to the *complex* parameter plane. Indeed, under the integrable symmetry reduction

$$r(x,t) = \sigma q(x, t_0 + it_1 - t) \,, \tag{16}$$

the "(q, r) system" given by Eqns. (1) and (2) are indeed compatible and lead to the following integrable complex shift nonlocal reverse time only NLS equations:

$$iq_t = q_{xx} - 2\sigma q^2 q(x, t_0 + it_1 - t) \,, \tag{17}$$

with t_0 and t_1 being real parameters. Similarly, the integrable reduction

$$r(x,t) = \sigma q(x_0 + ix_1 - x, t_0 + it_1 - t) \,, \tag{18}$$

where x_j, t_j, $j = 0, 1$ are all real constants, makes the "(q, r) system" self-consistent giving rise to an integrable complex space-time shifted nonlocal NLS equations

$$iq_t = q_{xx} - 2\sigma q^2 q(x_0 + ix_1 - x, t_0 + it_1 - t) \,. \tag{19}$$

Furthermore, under the same reduction (18) Eqns. (11) and (12) are also compatible leading to the complex space-time shifted mKdV equation:

$$q_t + q_{xxx} - 6\sigma qq(x_0 + ix_1 - x, t_0 + it_1 - t)q_x = 0 \,. \tag{20}$$

This type of AKNS integrable symmetry reductions has not previously appeared in the theory of integrable systems. They would inevitably result in complexification of either time alone or both space and time variables as, for example, Eqns. (17) and (19) seem to imply. In this paper, we shall focus the analysis (integrability, solitons and inverse scattering) on the PT symmetric and space-time nonlocal (real) shifted NLS equations only and leave the complex shifted cases for future study. We point out that complexification of integrable systems by complexification of the independent variables (space and time) has been studied over the years. Examples include the self-dual Yang-Mills equations, Kadomtsev-Petviashvili and Davey-Stewartson equations [3,83–86]. Equations (17)-(19) yield a different type of complexification.

4. Integrability and conservation laws

One of the hallmarks of the AKNS theory is that nonlinear equations derived from a corresponding AKNS scattering problem are integrable. Since the '(q, r) system" given by Eqns. (1) and (2) were obtained from an AKNS scattering problem, any reduction of them would lead to an integrable equation. As such, Eqns. (6), (8), (10), (17) and (19) all form integrable systems in the sense of existence of an infinitely many conservation laws. To keep the discussion concise, we list few global conservation laws for selected integrable systems and leave the full account to a future work.

4.1. Shifted PT nonlocal NLS Eq. (6)

The first few global conservation laws are given by

$$\int_{\mathbb{R}} q(x,t)q^*(x_0 - x, t)dx \,, \tag{21}$$

$$\int_{\mathbb{R}} q_x(x,t)q^*(x_0 - x, t)dx \,, \tag{22}$$

$$\int_{\mathbb{R}} \left[-\sigma q_{xx}(x,t)q^*(x_0 - x, t) + q^2(x,t)q^{*2}(x_0 - x, t) \right] dx \,. \tag{23}$$

4.2. Space-time shifted nonlocal NLS Eq. (10)

Here, we give some constant of motions when the shifts are present both in space and time

$$\int_{\mathbb{R}} q(x,t)q(x_0 - x, t_0 - t)dx \,, \tag{24}$$

$$\int_{\mathbb{R}} q_x(x,t)q(x_0 - x, t_0 - t)dx \,, \tag{25}$$

$$\int_{\mathbb{R}} \left[-\sigma q_{xx}(x,t)q(x_0 - x, t_0 - t) + q^2(x,t)q^2(x_0 - x, t_0 - t) \right] dx \,. \tag{26}$$

Scrutinizing any of the conservation laws given in Eqns. (24)-(26) reveals the various subtleties involved when dealing with reverse time shifts (real or complex). Note that if Eq. (10) is supplemented with initial data at time $t = t_0/2$ (rather than at $t = 0$) that would make the above conserved quantities immediately computable.

5. Other space-time shifted nonlocal NLS

In this section we derive other type of space-time shifted nonlocal NLS equations that emerge from a vectorial generalization of the "(q, r) system" defined by Eqns. (1) and (2). The starting point is the representation

$$i\mathbf{Q}_t = \mathbf{Q}_{xx} - 2\mathbf{QRQ}\,, \tag{27}$$

$$-i\mathbf{R}_t = \mathbf{R}_{xx} - 2\mathbf{RQR}\,, \tag{28}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are row and column vectors respectively of size n. Under the "classical" integrable symmetry reduction

$$\mathbf{R}(x,t) = \sigma \mathbf{Q}^H(x,t)\,, \tag{29}$$

with superscript H denoting matrix transpose and complex conjugation, the system (27) and (28) are compatible and lead to the vectorial NLS equation

$$i\mathbf{Q}_t = \mathbf{Q}_{xx} - 2\sigma ||\mathbf{Q}||^2\mathbf{Q}\,, \tag{30}$$

where $||\cdot||$ denotes a vector norm. For the simple and physically relevant two component case ($n = 2$), we arrive at the coupled system of NLS equations for the function $\mathbf{Q}(x,t) \equiv (q^{(1)}(x,t), q^{(2)}(x,t))$

$$iq_t^{(1)} = q_{xx}^{(1)} - 2\sigma\left(|q^{(1)}|^2 + |q^{(2)}|^2\right)q^{(1)}\,, \tag{31}$$

$$iq_t^{(2)} = q_{xx}^{(2)} - 2\sigma\left(|q^{(1)}|^2 + |q^{(2)}|^2\right)q^{(2)}\,. \tag{32}$$

The coupled equations (31) and (32) was derived and shown to be integrable by Manakov in 1974 [76]. Physically speaking, one application describes the dynamics of two incoherently coupled spatial beams propagating in Kerr-type media [77–79]. Other applications can be found in temporal optics where $q^{(j)}, j = 1, 2$ represent two orthogonal polarizations [80–82]. System (31) and (32) was used in [55] to derive various PT symmetric and space-time nonlocal Manakov-type equations [55]. In this section, we use the Manakov system (31) and (32) to derive several new shifted nonlocal equations that share the same symmetries as the ones obtained so far. If we choose

$$q^{(2)}(x,t) = e^{i\theta_0}q^{(1)}(x_0 - x, t)\,, \tag{33}$$

with arbitrary real and constant θ_0, then system (31) and (32) are compatible and leads to a single evolution equation for the "polarization" $q^{(1)}$ (dropping the super script):

$$iq_t = q_{xx} - 2\sigma\left(|q|^2 + |q(x_0 - x, t)|^2\right)q\,. \tag{34}$$

The shifted reverse time only is obtained by letting

$$q^{(2)}(x,t) = e^{i\theta_0}q^{(1)*}(x, t_0 - t)\,, \tag{35}$$

which, after substitution in Eqns. (31) and (32) leads to (dropping the super script)

$$iq_t = q_{xx} - 2\sigma\left(|q|^2 + |q(x, t_0 - t)|^2\right)q\,. \tag{36}$$

The third reduction to system (31) and (32) is of the space-time shifted nonlocal type

$$q^{(2)}(x,t) = e^{i\theta_0}q^{(1)*}(x_0 - x, t_0 - t)\,, \tag{37}$$

giving rise to the following space-time shifted nonlocal NLS-like equation (dropping the super script):

$$iq_t = q_{xx} - 2\sigma\left(|q|^2 + |q(x_0 - x, t_0 - t)|^2\right)q\,. \tag{38}$$

Due to the nature of the reduction (37) the shifts must be in the real domain.

6. Inverse scattering: Riemann-Hilbert approach

So far we have introduced several novel space-time shifted nonlocal integrable NLS type equations. They arise from a reduction of the AKNS scattering theory in which case they are solvable by the inverse scattering transform. In this section, we shall outline the solution method for the shifted PT NLS Eq. (6); the remaining cases are left for future work. The approach to solving Eq. (6) follows three major steps. The first is concerned with the analysis of the direct scattering problem. Primarily, this entails establishing the analytic and symmetry properties of the eigenfunctions governed by equations (3)-(4) as well as the underlying scattering data and asymptotic behavior. The second step involves determining the time evolution of the scattering data which, in essence, encodes the time dependence of the general solution and solitons. Lastly, the solution $q(x,t)$ is recovered by applying the inverse scattering transform via a Riemann-Hilbert formulation. With these steps at hand, the most general solution to Eq. (6) can be obtained which, for the sake of simplicity, we write below for the case where no continuous spectrum is present and discrete soliton eigenvalues located on the imaginary (k) axis only:

$$q(x,t) = 2i\sigma\sum_{\ell=1}^{J} C_\ell^*(t,x_0)F^*(x_0 - x, k_\ell)e^{-2ik_\ell^*(x_0-x)}\,, \tag{39}$$

where $F(x, k_\ell)$ satisfies the following linear system

$$F(x,k_j) = 1 + \sum_{\ell=1}^{\bar{J}}\sum_{\ell'=1}^{J}\frac{\bar{C}_\ell C_{\ell'}e^{2i(k_{\ell'}-\bar{k}_\ell)x}F(x,k_{\ell'})}{(k_j - \bar{k}_\ell)(\bar{k}_j - k_{\ell'})}\,, \tag{40}$$

and $k_j \equiv i\eta_j, \bar{k}_j \equiv -i\bar{\eta}_j$ are the soliton eigenvalues with $\eta_j, \bar{\eta}_j$ being arbitrary *positive* and *real*. Furthermore, $C_j(t) = C_j(0,x_0)e^{-4ik_j^2t}$ and $\bar{C}_j(t) = \bar{C}_j(0,x_0)e^{4i\bar{k}_j^2t}$ are the so-called norming constants (in space). They encode the time-dependence of the soliton or the general solution. Note that throughout the paper, complex conjugation is indicated by a star and not by bar. The inverse scattering theory taking into account complex eigenvalues $k_j, \bar{k}_j$ has been also worked out but, in order to keep the presentation compact, the details will be left for future paper. Nonetheless, the one and two soliton solutions will be discussed.

7. Soliton solutions

In this section we discuss a one and two soliton solution for the shifted PT symmetric NLS Eq. (6) only. Solutions to all other space-time shifted nonlocal cases will be reported in future paper. The one soliton solution is characterized by a single isolated and simple eigenvalue $k_1, \bar{k}_1$ located at the imaginary axis in the complex k plane. By setting $J = 1, \bar{J} = 1$ and $k_1 \equiv i\eta_1$, $\bar{k}_1 \equiv -i\bar{\eta}_1$ (with *real* and *positive* $\eta_1, \bar{\eta}_1$) we find the time evolution of the norming constants when $\sigma = -1$ to be $\bar{C}_1(t) = -i(\eta_1 + \bar{\eta}_1)e^{\bar{\eta}_1 x_0}e^{i\bar{\theta}_1}e^{-4i\bar{\eta}_1^2t}$ and $C_1(t) = i(\eta_1 + \bar{\eta}_1)e^{\eta_1 x_0}e^{i\theta_1}e^{4i\eta_1^2t}$ where $\theta_1, \bar{\theta}_1$ are arbitrary real constants. Substituting these quantities back into Eq. (39) and (40); we then find the most general one soliton solution to Eq. (6)

$$q(x,t) = -\frac{2(\eta_1 + \bar{\eta}_1)e^{-4i\eta_1^2t}e^{-\eta_1 x_0}e^{-i\theta_1}e^{2\eta_1 x}}{1 - e^{-(\eta_1+\bar{\eta}_1)x_0}e^{-i(\theta_1+\bar{\theta}_1)}\,e^{4i(\bar{\eta}_1^2-\eta_1^2)t}e^{2(\bar{\eta}_1+\eta_1)x}} \tag{41}$$

This one soliton solution has four free parameters: $\eta_1, \bar{\eta}_1; \theta_1$ and $\bar{\theta}_1$ accounting for two free real soliton eigenvalues and two phases (not magnitude) of the complex norming constants C_1 and $\bar{C}_1$. This count is consistent with its classical NLS counter part where one complex eigenvalue and one complex norming constant contribute

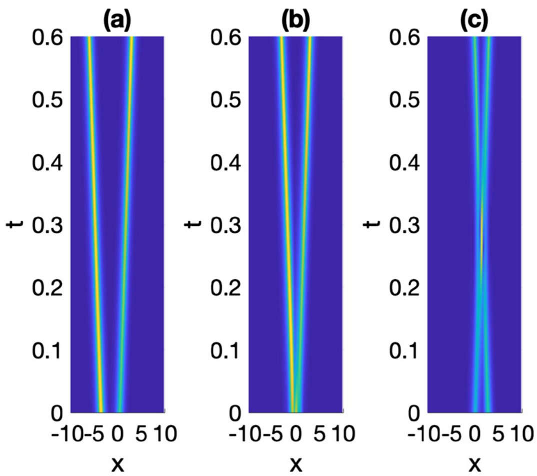

Fig. 1. Two soliton solution for the shifted nonlocal PT symmetric NLS Eq. (6) corresponding to $x_0 = -3$ (a), $x_0 = 0$ (b) and $x_0 = +3$ (c). Soliton parameters are: $\eta = 1, \bar{\eta} = 1.1; \xi = -1, \bar{\xi} = -1.1$.

to the tally of free parameters. Equation (41) reveals an unexpected result: it relates the one soliton solution of Eq. (6) to that of the "standard" (unshifted) PT symmetric NLS equation given by

$$iu_t = u_{xx} + 2u^2 u^*(-x, t)\,. \tag{42}$$

It turns out that if we denote by $q_1(x, t)$ and $u_1(x, t)$ the one soliton solution for Eqns. (6) and (42) respectively, (corresponding to eigenvalues on the imaginary complex k axis) then Eq. (41) implies $q_1(x, t) = u_1(x - x_0/2, t)$. This (nonobvious) fact prompted us to check whether this type of relation would persist at the two soliton solution level.

To that purpose, we have extended the inverse scattering theory associated with Eq. (6) to account for soliton eigenvalues located off the imaginary complex k axis. In this case, the formula for the potential $q(x, t)$ is given as

$$q(x,t) = 2i\sigma \sum_{\ell=1}^{J} C_\ell^*(t, x_0) G^*(x_0 - x, k_\ell) e^{-2ik_\ell^*(x_0 - x)} + 2i\sigma \sum_{\ell=1}^{J} D_\ell^*(t, x_0) G^*(x_0 - x, -k_\ell^*) e^{2ik_\ell(x_0 - x)}\,, \tag{43}$$

where now the soliton eigenvalues $k_j \equiv \xi_j + i\eta_j$ are assumed to be complex ($\xi_j \neq 0$); the norming constants $D_j(t, x_0)$ are given by $D_j(t) = D_j(0, x_0)e^{-4ik_j^{*2}t}$ and $G(x, k_j), G(x, -k_j^*)$ are independent functions that satisfy a set of algebraic equations similar (in spirit) to those in (40) yet too long to write in a compact form. In Fig. 1, a space-time density plot of a two-soliton solution is shown corresponding to three different scenarios: $x_0 = -3$ (a), $x_0 = 0$ (b) and $x_0 = 3$ (c). Sub-figure (b) in fact represents the two-soliton solution for Eq. (42) for $x_0 = 0$. Clearly, these results reveal an unambiguous picture: the two soliton solution for the shifted PT nonlocal NLS equation *cannot* be obtained from knowledge of its corresponding standard PT nonlocal NLS equation by applying a point coordinate transformation. In other words, $q_2(x, t) \neq u_2(x + c(x_0), t)$ where $q_2(x, t)$ and $u_2(x, t)$ the one soliton solution for Eqns. (6) and (42) respectively and $c(x_0)$ is a shift.

8. Conclusions

In this paper, several novel integrable symmetry reductions to the well-known AKNS scattering theory were proposed each of which giving rise to a new kind of PT symmetric or a reverse space-time nonlocal NLS type equations. What is atypical about these integrable systems is that the nonlocality occurs in a remarkably simple, yet very different, way; it relates functions values at (generally speaking) a point (x, t) in space-time domain to its function values at its corresponding shifted and mirror reflected space-time point $(x_0 - x, t_0 - t)$. So far, to our knowledge such a circumstance has not been encountered in integrable systems. The inverse scattering theory for the space shifted and PT symmetric NLS equation was formulated and a one and two soliton solutions were discussed. The inverse scattering and soliton solutions for the other proposed nonlocal equations will be the focus of future work. These results also imply that there are similar symmetry reductions like the ones found here for other integrable equations such as the Ablowitz-Ladik systems and the multi-dimensional Davey-Stewartson equations.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Mark J. Ablowitz reports financial support was provided by National Science Foundation.

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