

# DoA Estimation Using Low-Resolution Multi-Bit Sparse Array Measurements

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**Abstract**—This letter studies the problem of Direction of Arrival (DoA) estimation from low-resolution few-bit quantized data collected by Sparse Linear Array (SLA). In such cases, contrary to the one-bit quantization case, the well known arcsine law cannot be employed to estimate the covariance matrix of unquantized array data. Instead, we develop a novel optimization-based framework for retrieving the covariance matrix of unquantized array data from low-resolution few-bit measurements. The MUSIC algorithm is then applied to an augmented version of the recovered covariance matrix to find the source DoAs. The simulation results show that increasing the sampling resolution to 2 or 4 bits per samples could significantly increase the DoA estimation performance compared to the one-bit sampling regime while the power consumption and implementation costs is still much lower in comparison to the high-resolution sampling implementations.

**Index Terms**—Direction of arrival (DoA) estimation, low-resolution quantization, Sparse linear arrays, few-bit quantization.

## I. INTRODUCTION

Direction of Arrival (DoA) estimation from Uniform Linear Array (ULA) measurements is extensively studied in the literature [1–3]. However, the number of identifiable sources with ULAs is limited to the number of array elements minus one [3, 4]. Deployment of Sparse Linear Arrays (SLAs), e.g. Minimum Redundancy Arrays (MRAs) [5], co-prime arrays [6] and nested arrays [7], allows for transcending this limitation under the assumption of uncorrelated source signals such that the number of identifiable sources can go considerably beyond the number array elements. A detailed study on the performance of DoA estimation via SLAs has been conducted in [8] through an analysis of the Cramér-Rao Bound (CRB). Further, a variety of algorithms for estimating DoAs from SLA data have been presented in the literature [7, 9–15].

Most of the algorithms developed for estimating DoAs from SLA measurements are based on the assumption that quantization errors are negligible as a result of using high-resolution Analog-to-Digital Converters (ADCs). However, employment

of high-resolution ADCs is typically expensive and power-consuming [16]. Hence, to reduce energy consumption and production costs, DoA estimation with binary measurements collected by one-bit ADCs has been recently proposed and discussed in the literature [17–27]. One-bit ADCs represent each sample of the analog array observations with only a single bit offering, an exceedingly high sampling rate at a low production cost and very low power consumption [16]. The analytical performance bounds for DoA estimation from one-bit data have been studied in [28–30]. Further, a number of one-bit DoA estimators have been provided in [23–25, 27], which rest on retrieving the covariance matrix of unquantized array observations using the well-known Bussgang’s Theorem [31].

In this paper, as opposed to the previous works which have studied the problem of DoA estimation under two extreme scenarios for analog-to-digital conversion, i.e., infinite-bit quantization and one-bit quantization, we aim to investigate the problem of estimating DoAs from low-resolution few-bit SLA measurements. In such cases, contrary to the one-bit quantization case, the Bussgang’s Theorem may not directly be employed to retrieve the covariance matrix of array unquantized observations. Instead, we develop a novel optimization-based framework for retrieving the covariance matrix of unquantized array observations from low-resolution multi-bit measurements. Then, we apply the Co-Array-Based MUSIC (CAB-MUSIC) [7, 13] to the recovered covariance matrix to find the DoAs of interest. The simulation results show that increasing the sampling resolution with a few bits per samples could significantly increase the DoA estimation performance compared to the one-bit sampling case while the power consumption and implementation costs are still much lower than the high-resolution sampling scenario.

**Paper organization:** The system model is described in Section II. Section III presents the proposed algorithm for estimating DoAs from few-bit data. Simulation results are shown and discussed in Section IV. Finally, conclusions are drawn in Section V.

**Notation:** : Lightface, lower- and upper-case bold-face letters denote scalars, vectors and matrices, respectively. The conjugate, transpose and Hermitian (conjugate transpose) operations are referred to by the superscripts  $*$ ,  $T$ ,  $H$ , respectively.  $\|\mathbf{A}\|_F$  and  $\text{rank}(\mathbf{A})$  stand for the Frobenius norm and the rank of  $\mathbf{A}$ , respectively.  $[\mathbf{a}]_i$  indicates the  $i^{\text{th}}$  entry of  $\mathbf{a}$ .  $\text{diag}(\mathbf{a})$  is a diagonal matrix made out of entries of  $\mathbf{a}$ .  $\mathbf{I}_M$  denotes an  $M \times M$  identity matrix.  $\Re\{a\}$  and  $\Im\{a\}$  stand for the real and imaginary parts of  $a$ , respectively.

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## II. SYSTEM MODEL

We consider an SLA with  $M$  elements located at positions  $m_1 \frac{\lambda}{2}, m_2 \frac{\lambda}{2}, \dots, m_M \frac{\lambda}{2}$  with  $m_i \in \mathbb{M}$ . Here  $\lambda$  denotes the wavelength of the incoming signals and  $\mathbb{M}$  is a set of integers with a cardinality of  $M$ . It is assumed that  $K$  narrowband signals with distinct DoAs  $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_K]^T$  impinge on the SLA from far-field. The signal received at the array at time instance  $t$  can be modeled as

$$\mathbf{y}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t) \in \mathbb{C}^{M \times 1}, \quad t = 0, \dots, N-1, \quad (1)$$

where  $\mathbf{s}(t) \in \mathbb{C}^{K \times 1}$  denotes the vector of source signals,  $\mathbf{n}(t) \in \mathbb{C}^{M \times 1}$  is additive noise, and  $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \dots, \mathbf{a}(\theta_K)] \in \mathbb{C}^{M \times K}$  represents the SLA steering matrix with

$$\mathbf{a}(\theta_k) = [e^{j\pi \sin \theta_k m_1}, e^{j\pi \sin \theta_k m_2}, \dots, e^{j\pi \sin \theta_k m_M}]^T, \quad (2)$$

being the SLA manifold vector for the  $i^{\text{th}}$  signal. Further, the following assumptions are made on source signals and noise:

- A1**  $\mathbf{n}(t)$  follows a zero-mean circular complex Gaussian distribution with the covariance matrix  $\mathbb{E}\{\mathbf{n}(t)\mathbf{n}^H(t)\} = \sigma^2 \mathbf{I}_M$ .
- A2** The source signals are modeled as zero-mean *uncorrelated* circular complex Gaussian random variables with covariance matrix  $\mathbb{E}\{\mathbf{s}(t)\mathbf{s}^H(t)\} = \text{diag}(\mathbf{p})$  where  $\mathbf{p} = [p_1, p_2, \dots, p_K]^T \in \mathbb{R}_{>0}^{K \times 1}$  (i.e.,  $p_k > 0, \forall k$ ).
- A3** No temporal correlation is assumed between the snapshots, i.e.,  $\mathbb{E}\{\mathbf{n}(t_1)\mathbf{n}^H(t_2)\} = \mathbb{E}\{\mathbf{s}(t_1)\mathbf{s}^H(t_2)\} = \mathbf{0}$  when  $t_1 \neq t_2$  and  $\mathbf{0}$  is an all-zero matrix of appropriate dimensions.

Based on the above assumptions, the covariance matrix of  $\mathbf{y}(t)$  is given by

$$\mathbf{R} = \mathbb{E}\{\mathbf{y}(t)\mathbf{y}^H(t)\} = \mathbf{A}(\boldsymbol{\theta})\text{diag}(\mathbf{p})\mathbf{A}^H(\boldsymbol{\theta}) + \sigma^2 \mathbf{I}_M \in \mathbb{C}^{M \times M}. \quad (3)$$

It is readily verified that  $\mathbf{R}$  is a structured matrix with only  $2D-1$  free parameters where  $D = |\mathbb{D}|$  with  $\mathbb{D} = \{|m_p - m_q| : m_p, m_q \in \mathbb{M}\}$ . The set  $\mathbb{D}$  is called the difference co-array [8, 13, 15]. Accordingly,  $\mathbf{R}$  can be rewritten as follows

$$\mathbf{R}(\mathbf{u}) = u_0 \mathbf{L}_0 + \sum_{n=1}^{D-1} u_n \mathbf{L}_n + \sum_{n=1}^{D-1} u_n^* \mathbf{L}_n^T, \quad (4)$$

where  $u_0 = \sigma^2 + \sum_{k=1}^K p_k$ ,  $u_n = \sum_{k=1}^K p_k e^{j\pi \sin \theta_k \ell_n}$  and

$$[\mathbf{L}_n]_{p,q} = \begin{cases} 1, & \text{if } m_p - m_q = \ell_n, \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

with  $\ell_n \in \mathbb{D}$ ,  $m_p, m_q \in \mathbb{M}$ ,  $1 \leq p, q \leq M$  and  $0 \leq n \leq D-1$ . A proper design of SLA allows for identifying more uncorrelated source signals than the number of array elements by exploiting the resulting structure of  $\mathbf{R}$  efficiently [6–8, 15]. Fig. 1 illustrates an SLA along with its difference co-array.

We here assume that each array sensor is equipped with a low-resolution multi-bit ADC converting the analog received signal into digital data using  $q$  bits per sample. A generic  $q$ -bit ADC has  $2^q + 1$  threshold levels where  $-\infty = \alpha_1 \leq \alpha_2 \leq \dots \leq 0 \leq \dots \leq \alpha_{2^q} \leq \alpha_{2^q+1} = \infty$ . The  $q$ -bit ADC at the  $m^{\text{th}}$  array element transforms the real and imaginary parts of  $[\mathbf{y}(t)]_m$  into one of the  $2^q + 1$  prescribed quantization levels  $\{\gamma_1, \gamma_2, \dots, \gamma_{2^q+1}\}$  through comparing them individually with

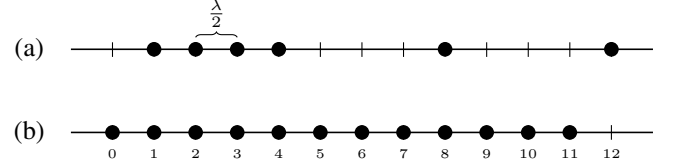


Figure 1. (a) An SLA with  $\mathbb{M} = \{1, 2, 3, 4, 8, 12\}$ ; (b) corresponding difference co-array with  $\mathbb{D} = \{0, 1, \dots, 11\}$ .

the threshold levels. Particularly, the  $q$ -bit quantized output signals at the  $m^{\text{th}}$  array element are expressed as

$$[\mathbf{x}(t)]_m = Q(\Re\{\mathbf{y}(t)\}_m) + jQ(\Im\{\mathbf{y}(t)\}_m), \quad (6)$$

where  $Q(\cdot)$  denotes the  $q$ -bit quantization operation defined as

$$Q(a) = \gamma_h \quad \text{if } \alpha_h \leq a < \alpha_{h+1}. \quad (7)$$

We are interested in estimating source DoAs from  $q$ -bit quantized output signals of the SLA, i.e.,  $\mathbf{X} = [\mathbf{x}(0), \mathbf{x}(1), \dots, \mathbf{x}(N-1)]$ .

## III. MULTI-BIT DOA ESTIMATION WITH SPARSE ARRAYS

In this section, we first formulate an optimization problem whose solution provides us with an estimate of the covariance matrix of  $\mathbf{y}(t)$ , i.e.,  $\mathbf{R}$ , using  $q$ -bit quantized array measurements, i.e.,  $\mathbf{X}$ . Then, we apply the CAB-MUSIC [7, 13] to obtain DoA estimates from the estimate of  $\mathbf{R}$ .

It follows from (4) that  $\mathbf{R}$  is fully described by the complex vector  $\mathbf{u} = [u_0, u_1, \dots, u_{D-1}]^T$ . Hence, for a given  $\mathbf{Y} = [\mathbf{y}(0), \mathbf{y}(1), \dots, \mathbf{y}(N-1)]$ ,  $\mathbf{R}$  can be obtained from the solution of the following optimization problem:

$$\begin{aligned} & \underset{\mathbf{u}}{\text{minimize}} \quad \|\mathbf{R}(\mathbf{u}) - \mathbf{Y}\mathbf{Y}^H\|^2 \\ & \text{subject to} \quad \mathbf{R}(\mathbf{u}) \succeq \mathbf{0}. \end{aligned} \quad (8)$$

However,  $\mathbf{Y}$  is unknown here, and instead, we only have access to  $q$ -bit quantized values of it, i.e.,  $\mathbf{X}$ . It follows from (6) and (7) that each element of the observation matrix  $\mathbf{X}$  determines a lower and an upper bound for the real and imaginary parts of the corresponding element in  $\mathbf{Y}$ . Putting all of these lower and upper bound into the matrices  $\mathbf{\Gamma}_l$  and  $\mathbf{\Gamma}_u$ , an optimization problem for joint estimation of  $\mathbf{u}$  and  $\mathbf{Y}$  can be cast as follows:

$$\begin{aligned} & \underset{\mathbf{u}, \mathbf{Y}}{\text{minimize}} \quad \|\mathbf{R}(\mathbf{u}) - \mathbf{Y}\mathbf{Y}^H\|_F^2 \\ & \text{subject to} \quad \mathbf{R}(\mathbf{u}) \succeq \mathbf{0}, \\ & \quad \text{vec}(\Re\{\mathbf{Y}\} - \Re\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\ & \quad \text{vec}(\Im\{\mathbf{Y}\} - \Im\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\ & \quad -[\text{vec}(\Re\{\mathbf{Y}\} - \Re\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}, \\ & \quad -[\text{vec}(\Im\{\mathbf{Y}\} - \Im\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}. \end{aligned} \quad (9)$$

where the last four constraints in (9) aim to enforce the consistency of  $\mathbf{Y}$  with the  $q$ -bit measurements by ensuring that the elements of  $\mathbf{Y}$  lie in the regions determined by the observation matrix  $\mathbf{X}$ . The above optimization problem is non-convex as its objective is a quartic function with respect to  $\mathbf{Y}$ . In what follows, we first present an equivalent reformulation for (9), which paves the way for iteratively solving this non-convex optimization problem.

**Theorem 1.** Consider slack variables  $\mathbf{G} \in \mathbb{C}^{(M+N) \times M}$ ,  $\mathbf{W} \in \mathbb{C}^{M \times M}$  and  $\xi \in \mathbb{R}$ . The optimization problem (9) is equivalent to

$$\begin{aligned}
& \underset{\mathbf{u}, \mathbf{Y}, \mathbf{W}, \mathbf{G}, \xi}{\text{minimize}} && \|\mathbf{R}(\mathbf{u}) - \mathbf{W}\|_F^2 + \eta\xi \\
& \text{subject to} && \mathbf{R}(\mathbf{u}) \succeq \mathbf{0}, \\
& && \text{vec}(\Re\{\mathbf{Y}\} - \Re\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\
& && \text{vec}(\Im\{\mathbf{Y}\} - \Im\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\
& && -[\text{vec}(\Re\{\mathbf{Y}\} - \Re\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}, \\
& && -[\text{vec}(\Im\{\mathbf{Y}\} - \Im\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}, \\
& && \mathbf{T} \succeq \mathbf{0}, \\
& && \xi \mathbf{I}_M - \mathbf{G}^H \mathbf{T} \mathbf{G} \succeq \mathbf{0}, \\
& && \mathbf{G}^H \mathbf{G} = \mathbf{I}_M, \\
& && \xi \geq 0,
\end{aligned} \tag{10}$$

where  $\mathbf{T} = \begin{bmatrix} \mathbf{I}_N & \mathbf{Y}^H \\ \mathbf{Y} & \mathbf{W} \end{bmatrix} \in \mathbb{C}^{(M+N) \times (M+N)}$  and  $\eta$  is a regularization parameter.

*Proof.* Consider the slack variables  $\mathbf{W}$  such that  $\mathbf{W} = \mathbf{Y}\mathbf{Y}^H$ . Then it is readily clear that the optimization problem (9) is equivalent to the following one:

$$\begin{aligned}
& \underset{\mathbf{u}, \mathbf{Y}, \mathbf{W}}{\text{minimize}} && \|\mathbf{R}(\mathbf{u}) - \mathbf{W}\|_F^2 \\
& \text{subject to} && \mathbf{R}(\mathbf{u}) \succeq \mathbf{0}, \\
& && \text{vec}(\Re\{\mathbf{Y}\} - \Re\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\
& && \text{vec}(\Im\{\mathbf{Y}\} - \Im\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\
& && -[\text{vec}(\Re\{\mathbf{Y}\} - \Re\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}, \\
& && -[\text{vec}(\Im\{\mathbf{Y}\} - \Im\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}, \\
& && \mathbf{W} = \mathbf{Y}\mathbf{Y}^H.
\end{aligned} \tag{11}$$

It is readily confirmed that  $\mathbf{W} = \mathbf{Y}\mathbf{Y}^H$  if and only if  $\text{rank}(\mathbf{W} - \mathbf{Y}\mathbf{Y}^H) = 0$ . Further,  $\text{rank}(\mathbf{W} - \mathbf{Y}\mathbf{Y}^H) = 0$  can be equivalently expressed as  $\text{rank}(\mathbf{I}_N) + \text{rank}(\mathbf{W} - \mathbf{Y}\mathbf{Y}^H) = N$ . Since  $\mathbf{I}_N$  is positive definite, it follows from the Guttman rank additivity formula [32] that  $\text{rank}(\mathbf{I}_N) + \text{rank}(\mathbf{W} - \mathbf{Y}\mathbf{Y}^H) = \text{rank}(\mathbf{T})$ . Moreover, it follows from  $\mathbf{W} - \mathbf{Y}\mathbf{Y}^H = \mathbf{0}$  and  $\mathbf{I}_N \succ \mathbf{0}$  that  $\mathbf{T}$  has to be positive semi-definite. These imply that the equality constraint in (11) can be replaced with a rank constraint on a semi-definite matrix. Hence, the optimization problem (11), and equivalently (9), can be recast as follows:

$$\begin{aligned}
& \underset{\mathbf{u}, \mathbf{Y}, \mathbf{W}}{\text{minimize}} && \|\mathbf{R}(\mathbf{u}) - \mathbf{W}\|^2 \\
& \text{subject to} && \mathbf{R}(\mathbf{u}) \succeq \mathbf{0}, \\
& && \text{vec}(\Re\{\mathbf{Y}\} - \Re\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\
& && \text{vec}(\Im\{\mathbf{Y}\} - \Im\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\
& && -[\text{vec}(\Re\{\mathbf{Y}\} - \Re\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}, \\
& && -[\text{vec}(\Im\{\mathbf{Y}\} - \Im\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}, \\
& && \mathbf{T} \succeq \mathbf{0}, \\
& && \text{rank}(\mathbf{T}) = N.
\end{aligned} \tag{12}$$

Now, we show that the optimization problem (10) is equivalent to (12). Let  $\rho_1 \leq \rho_2 \leq \dots \leq \rho_{M+N}$  and  $\nu_1 \leq \nu_2 \leq \dots \leq \nu_M$  denote the eigenvalues of  $\mathbf{T}$  and  $\mathbf{G}^H \mathbf{T} \mathbf{G}$ , respectively. From the constraint  $\xi \mathbf{I}_M - \mathbf{G}^H \mathbf{T} \mathbf{G} \succeq \mathbf{0}$ , we have  $\nu_i \leq \xi, i = 1, 2, \dots, M$  for any  $\mathbf{G}$  and  $\mathbf{T}$  in the feasible set of (10). Additionally, it follows from [33, Corollary 4.3.16] that

$0 \leq \rho_i \leq \nu_i, i = 1, 2, \dots, M$  for any  $\mathbf{G}$  and  $\mathbf{T}$  in the feasible set of (10). Hence, we observe that

$$\begin{aligned}
\mathbf{0} & \preceq \text{diag}([\rho_1, \rho_2, \dots, \rho_M]^T) \preceq \text{diag}([\nu_1, \nu_2, \dots, \nu_M]^T) \\
& \preceq \xi \mathbf{I}_M,
\end{aligned} \tag{13}$$

for any  $\mathbf{G}$  and  $\mathbf{T}$  in the feasible set of (10). It is easily observed from (10) and (13) that, by properly selecting  $\eta$ , the optimum value of  $\mathbf{G}$  will be equal to the eigenvectors of  $\mathbf{T}$  corresponding to its  $M$  smallest eigenvalues and the optimum values of  $\xi, \rho_1, \dots, \rho_M, \nu_1, \dots, \nu_M$  will be equal to zero. This implies that the optimum value of  $\mathbf{T}$  in (13) possesses  $N$  nonzero and  $M$  zero eigenvalues. This completes the proof.  $\square$

The optimization problem (10) can be solved iteratively by alternating between  $\mathbf{G}$  and the other parameters, i.e.,  $\mathbf{u}, \mathbf{Y}, \mathbf{W}$  and  $\xi$ . Let  $\mathbf{G}^{(k)}, \mathbf{u}^{(k)}, \mathbf{Y}^{(k)}, \mathbf{W}^{(k)}$  and  $\xi^{(k)}$  be the values of the parameters  $\mathbf{G}, \mathbf{u}, \mathbf{Y}, \mathbf{W}$  and  $\xi$  at the  $k$ -th iteration, respectively. Given  $\mathbf{G}^{(k-1)}$ , the optimization problem with respect to  $\mathbf{u}, \mathbf{Y}, \mathbf{W}$  and  $e$  at the  $k$ -th iteration becomes

$$\begin{aligned}
& \underset{\mathbf{u}^{(k)}, \mathbf{Y}^{(k)}, \mathbf{W}^{(k)}, \xi^{(k)}}{\text{minimize}} && \|\mathbf{R}(\mathbf{u}^{(k)}) - \mathbf{W}^{(k)}\|_F^2 + \eta\xi^{(k)} \\
& \text{subject to} && \mathbf{R}(\mathbf{u}^{(k)}) \succeq \mathbf{0}, \\
& && \text{vec}(\Re\{\mathbf{Y}^{(k)}\} - \Re\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\
& && \text{vec}(\Im\{\mathbf{Y}^{(k)}\} - \Im\{\mathbf{\Gamma}_l\}) \geq \mathbf{0}, \\
& && -[\text{vec}(\Re\{\mathbf{Y}^{(k)}\} - \Re\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}, \\
& && -[\text{vec}(\Im\{\mathbf{Y}^{(k)}\} - \Im\{\mathbf{\Gamma}_u\})] \geq \mathbf{0}, \\
& && \mathbf{T}^{(k)} \succeq \mathbf{0}, \\
& && \xi^{(k)} \mathbf{I}_M - \mathbf{G}^{(k-1)H} \mathbf{T}^{(k)} \mathbf{G}^{(k-1)} \succeq \mathbf{0}, \\
& && \xi^{(k)} \geq 0,
\end{aligned} \tag{14}$$

where  $\mathbf{T}^{(k)} = \begin{bmatrix} \mathbf{I}_N & \mathbf{Y}^{(k)H} \\ \mathbf{Y}^{(k)} & \mathbf{W}^{(k)} \end{bmatrix}$ . Once  $\mathbf{T}^{(k)}$  is found,  $\mathbf{G}^{(k)}$  can be obtained from eigenvalue decomposition of  $\mathbf{T}^{(k)}$ . Indeed, at each iteration of the proposed algorithm, we need to solve a Semi-Definite Program (SDP), which can be solved efficiently, followed by an Eigenvalue Decomposition (ED). The alternating optimization procedure is repeated until either the objective or the optimization variables converge to a constant value. Algorithm 1 summarizes the steps of the aforementioned iterative approach to solving (9). Further, to initialize the algorithm,  $\mathbf{G}^{(0)}$  can be found through the ED of  $\mathbf{T}^{(0)}$  obtained from solving (12) without considering the rank constraint. We note that the proposed algorithm, which is based on alternating optimization method, is guaranteed to converge to at least a local minimum of (10) [34]. Once  $\mathbf{R}$  is retrieved from Algorithm 1, the CAB-MUSIC [7, 13] is applied to the retrieved  $\mathbf{R}$  to estimate DoAs.

#### IV. SIMULATION RESULTS

In this section, numerical results are provided for assessing the performance of the proposed algorithm for estimating DoAs from low-resolution few-bit SLA output. In all experiments, each simulated point has been computed by 1000 Monte Carlo repetitions. In addition, the  $K$  independent sources with an equal power  $p$  are equally spaced in the angular domain  $[-60^\circ, 60^\circ]$  with respect to a 8-sensor nested array with the following configuration:

$$\mathbf{M} : \{1, 2, 3, 4, 5, 10, 15, 20\}. \tag{15}$$

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**Algorithm 1** Covariance Matrix Estimaiton from Low-Resolution Few-Bit Data

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**Input:** The problem information  $\Gamma_l, \Gamma_u, \eta, \epsilon_1, \epsilon_2, \epsilon_3$  and  $\epsilon_4$ .

**Output:** The estimate of the covariance matrix of the full-precision data.

- 1: **initialization:** Set  $k = 0$  and obtain  $\mathbf{G}^{(0)}$  by dropping the rank constraint.
  - 2: **while**  $\|\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}\|_2 \geq \epsilon_1, \|\mathbf{W}^{(k)} - \mathbf{W}^{(k-1)}\|_F \geq \epsilon_2, \|\mathbf{Y}^{(k)} - \mathbf{Y}^{(k-1)}\|_F \geq \epsilon_3$  and  $\xi^{(k)} \geq \epsilon_4$  **do**
  - 3:   Increase  $k$  by one.
  - 4:   Find  $\mathbf{u}^{(k)}, \mathbf{W}^{(k)}, \mathbf{Y}^{(k)}$  and  $\xi^{(k)}$  by solving (14).
  - 5:   Find  $\mathbf{G}^{(k)}$  by computing the ED of  $\mathbf{T}^{(k)}$ .
  - 6: **end while**
- 

The SNR is also defined as  $10 \log \frac{p}{\sigma^2}$ .

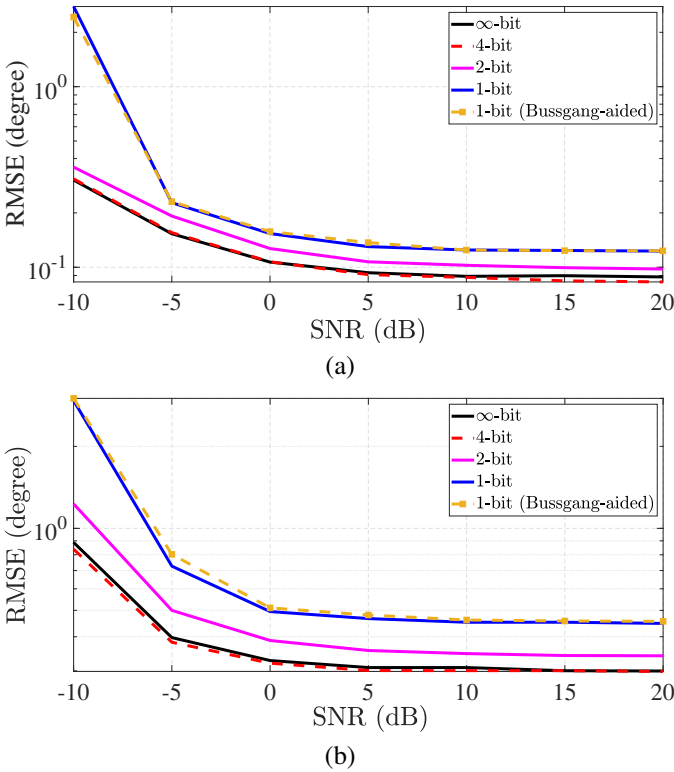


Figure 2. RMSE in degree for  $\theta_2$  versus SNR for a nested array with  $M = 8$  elements and configuration given in (15),  $N = 300$ , and: (a)  $K = 4 < M$ ; (b)  $K = 10 > M$ .

Fig. 2 depicts the Root-Mean-Squares-Error (RMSE) for  $\theta_2$  in degree versus SNR for different bit-width when  $N = 300$ ,  $M = 8$  and: (a)  $K = 4 < M$ ; (b)  $K = 10 > M$ . Fig. 2 demonstrates that increasing the number of quantization bits from one to two and then to four leads to a considerable performance improvement. Further, it is observed that the RMSE of 4-bit DoA estimation is very close to that of DoA estimates obtained from the unquantized array observations. For instance, when  $K = 4$ , the performance loss arising from quantization, defined as  $10 \log(\text{RMSE}_{\text{quantized}}/\text{RMSE}_{\text{unquantized}})$ , at SNR = 5 dB are about 3.33 and 1.39 dB in case of 1-bit and 2-bit quantization, respectively, while it is almost zero in case of 4-bit quantization. However, the implementation

costs and power consumption of 4-bit and 2-bit ADCs are still much lower compared to high-resolution ADCs. For example, at sampling frequency of 10 MHz, the power consumption of 4-bit and 2-bit are 1 mW and < 1 mW, respectively, while a 14-bit ADC consumes 1 W at the same sampling frequency [35]. Further, it is relatively easy to implement 4-bit and 2-bit ADCs with sampling frequency of 10 MHz while it is not practically feasible to build a 14-bit ADC at the same frequency [35].

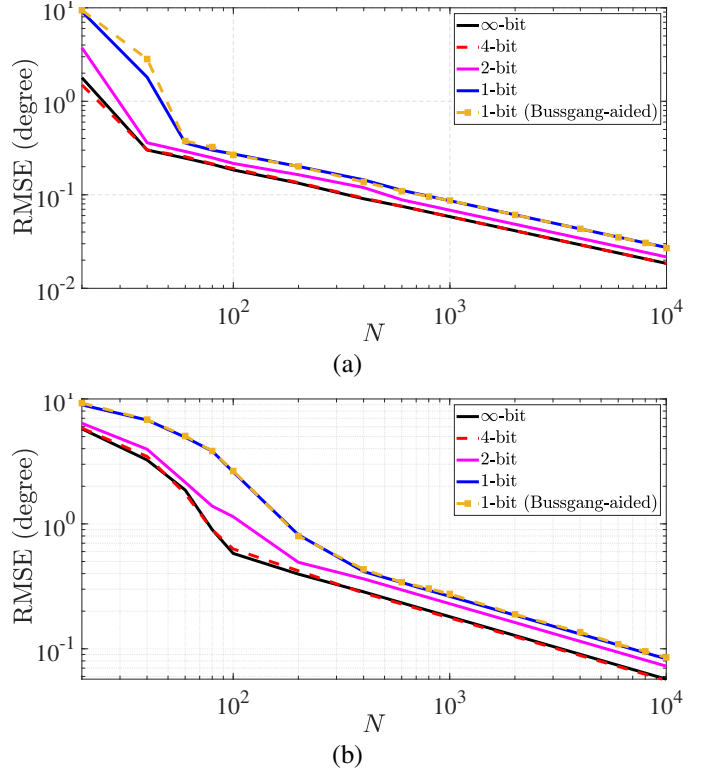


Figure 3. RMSE in degree for  $\theta_2$  versus the number of snapshots for a nested array with  $M = 8$  elements and configuration given in (15), SNR = 0 dB, and: (a)  $K = 4 < M$ ; (b)  $K = 10 > M$ .

Fig. 3 plots the RMSE for  $\theta_2$  in degree versus the number of snapshots for SNR = 0 dB and: (a)  $K = 4 < M$ , and (b)  $K = 10 > M$ . Fig. 3 shows that, to achieve an RMSE of 0.1 for example, infinite-bit, 4-bit, 2-bit and one-bit cases need 300, 300, 500 and 800 samples when  $K = 4$ , respectively. This indicates that the total number of bits required to achieve an RMSE of 0.1 is, respectively, 1200, 1000 and 800 bits for 4-bit, 2-bit and one-bit sampling scenarios.

## V. CONCLUSION

The problem of DoA estimation from low-resolution few-bit SLA data was investigated. Firstly, the covariance matrix of unquantized array observations was retrieved from low-resolution few-bit SLA data by employing an iterative optimization-based algorithm. Then, DoAs were estimated by applying CAB-MUSIC to the recovered covariance matrix of unquantized array observations. The simulation results showed that increasing the sampling resolution to 2 or 4 bits per samples could significantly increase the DoA estimation performance compared to the one-bit sampling case while the

power consumption and implementation costs are still much lower than the high-resolution sampling scenario.

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