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ABSTRACT

A correspondence is established between measure-preserving, ergodic dynamics of a classical harmonic oscillator and a quantum mechanical gauge theory on two-dimensional Minkowski space. This correspondence is realized through an isometric embedding of the $L^2(\mu)$ space on the circle associated with the oscillator's invariant measure, μ , into a Hilbert space $\mathcal{H}$ of sections of a $\mathbb{C}$ -line bundle over Minkowski space. This bundle is equipped with a covariant derivative induced from an $SO^+(1, 1)$ gauge field on the corresponding inertial frame bundle, satisfying the Yang–Mills equations. Under this embedding, the Hamiltonian operator of a Lorentz-invariant quantum system, constructed as a geometrical Laplace-type operator on bundle sections, pulls back to the generator of the unitary group of Koopman operators governing the evolution of classical observables of the harmonic oscillator, with Koopman eigenfunctions of zero, positive, and negative eigenfrequencies corresponding to quantum eigenstates of zero (“vacuum”), positive (“matter”), and negative (“antimatter”) energies. The embedding also induces a pair of operators acting on classical observables of the harmonic oscillator, exhibiting canonical position–momentum commutation relationships. These operators have the structure of order-1/2 fractional derivatives and therefore display a form of non-locality. In a second part of this work, we study a quantum mechanical representation of the classical harmonic oscillator using a one-parameter family of reproducing kernel Hilbert spaces, $\hat{\mathcal{H}}_\tau$, associated with the time- τ transition kernel of a fractional diffusion on the circle. As shown in recent work, these spaces are unital Banach $*$ -algebras of functions. It is found that the evolution of classical observables in these spaces takes place via a strongly continuous, unitary Koopman evolution group, which exhibits a stronger form of classical–quantum consistency than the $L^2(\mu)$ case. Specifically, for every real-valued classical observable in $\hat{\mathcal{H}}_\tau$, there exists a quantum mechanical observable, whose expectation value is consistent with classical function evaluation. This allows for a description of classical state space dynamics, classical statistics, and quantum statistics of the harmonic oscillator within a unified framework.

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I. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The classical harmonic oscillator, or circle rotation, is arguably among the most widely encountered dynamical systems in science, with a vast range of applications in mechanics, wave propagation, signal processing, and many other areas. From the point of view of ergodic theory, it also provides one of the simplest non-trivial examples of measure-preserving, ergodic dynamics, characterized by a discrete spectrum of frequencies at integer multiples of the oscillator's natural frequency. In this work, we establish a correspondence between this simple classical dynamical system and quantum dynamics of a gauge field theory on two-dimensional Minkowski space, having the proper orthochronous Lorentz group $SO^+(1, 1)$ as the structure group. This correspondence is realized through the operator-theoretic formulation of ergodic theory,^{1,2} which describes dynamical systems through the action of intrinsically linear evolution operators, called Koopman operators,^{3,4} acting on appropriate linear spaces of observables by composition with the dynamics. In particular, for a continuous-time, measure-preserving dynamical flow $\Phi^t: S \rightarrow S$ on a state space S , the Koopman operators $U^t: f \mapsto f \circ \Phi^t$ act by unitary transformations on the $L^2(\mu)$ Hilbert space associated with the invariant measure μ , analogously to the unitary Heisenberg operators of quantum mechanics.

Analogies of this type have been at the focus of a number of recent studies on the connections between quantum mechanics and operator-theoretic approaches to ergodic theory.^{5–13} Among these is a scheme for sequential data assimilation (filtering) of partially observed classical dynamical systems, which maps these systems into abstract quantum systems by means of the Koopman formalism and employs the density matrix formulation of quantum dynamics and measurement to perform sequential statistical inference for the evolution of observables.^{9,10} Building on this framework, our goal is to explore the geometrical and algebraic properties of a classical–quantum correspondence in the specific case of the harmonic oscillator. In particular, we seek to address the question: Can a classical harmonic oscillator of fixed frequency (energy) be naturally mapped into a quantum system with a geometrical Hamiltonian operator?

Our approach is inspired by the following basic observations:

1. The Koopman eigenfrequency spectrum of a classical harmonic oscillator of frequency α consists of all integer multiples $\alpha_j = j\alpha$, $j \in \mathbb{Z}$. In particular, the spacing between two successive eigenfrequencies is constant and equal to the natural frequency, $\alpha_{j+1} - \alpha_j = \alpha$.
2. The energy spectrum of a non-relativistic quantum harmonic oscillator of frequency α takes the form $E_j = (2j + 1)\alpha/2$, $j \in \mathbb{N}_0$ (in units with $\hbar = 1$). That is, similar to the classical harmonic oscillator, $E_{j+1} - E_j$ is equal to α , but the quantum harmonic oscillator has a zero-point energy $E_0 = \alpha/2$, and all energies are positive.
3. A pair of quantum harmonic oscillators, consisting of one oscillator as above and another oscillator of the same frequency α , but sign-inverted Hamiltonian, has the energy spectrum $E_j = j\alpha$, with j now being an arbitrary integer. Such a pair is therefore spectrally isomorphic to the Koopman group of a classical harmonic oscillator.

In what follows, we will see that such a pair of quantum harmonic operators with oppositely signed Hamiltonians arises naturally as a coordinate representation of a connection Laplacian of an $\mathrm{SO}^+(1, 1)$ gauge theory on two-dimensional Minkowski space, where wavefunctions correspond to sections of a $\mathbb{C}$ -line bundle, and the positive and negative parts of the energy spectrum can be interpreted as corresponding to “matter” and “antimatter” states, respectively. Furthermore, by mapping the eigenfunctions of the Koopman operator for the harmonic oscillator (which are Fourier functions on the circle) to the Hermite eigenstates of the quantum harmonic oscillator, with positive (negative) Koopman eigenfrequencies corresponding to positive (negative) energies, we will construct an isometric embedding of the $L^2(\mu)$ space of classical observables of the harmonic oscillator to a Hilbert space of sections associated with the gauge field theory.

This embedding allows one to pull back quantum mechanical observables (self-adjoint operators) on Minkowski space to operators on classical observables of the harmonic oscillator. In particular, it is possible to pull back the quantum mechanical position and momentum operators, and we will see that the resulting operators take the form of fractional derivatives, exhibiting canonical commutation relationships. These results are summarized in the following two theorems:

Theorem 1 (Classical–quantum correspondence based on $L^2(\mu)$). *Given a rotation on the circle with frequency α , there exists a smooth $\mathbb{C}$ -line bundle $E \rightarrow M$ over two-dimensional Minkowski space M , equipped with an $\mathrm{SO}^+(1, 1)$ Yang–Mills connection, and a gauge-covariant Hilbert space homomorphism $\mathcal{U}: L^2(\mu) \rightarrow \mathcal{H}$, where μ is the Haar probability measure on S^1 and $\mathcal{H}$ is a Hilbert space of sections $M \rightarrow E$, such that the following hold:*

- (i) *The skew-adjoint generator V of the unitary Koopman evolution group on $L^2(\mu)$ induced by the circle rotation can be expressed as the pullback under $\mathcal{U}$ of a Hamiltonian operator H on $\mathcal{H}$, i.e.,*

$$V = i\mathcal{U}^* H \mathcal{U}$$

on C^∞ functions. In particular, H has the structure of a Lorentz-invariant, gauge-covariant Laplace-type operator, given by the sum

$$H = \Delta + v$$

of the connection Laplacian Δ associated with the bundle and a quadratic potential v taking negative (positive) values along timelike (spacelike) affine coordinates on M . Moreover, up to multiplication by i , V and H have the same spectra, consisting of integer multiples of the frequency α .

- (ii) *For every $\tau > 0$, there exists a continuous, injective map $\Psi_\tau: S^1 \rightarrow \mathcal{Q}(L^2(\mu))$ from the circle into the space $\mathcal{Q}(L^2(\mu))$ of regular quantum states on $L^2(\mu)$ (i.e., the set of positive, trace-class operators of unit trace), which is compatible with the dynamical flow and the unitary evolution $\Phi^t: \rho \mapsto U^{t*} \rho U^t$ on $\mathcal{Q}(L^2(\mu))$ induced by the Koopman group, in the sense that*

$$\Psi_\tau \circ \Phi^t = \Phi^t \circ \Psi_\tau, \quad \forall t \in \mathbb{R}.$$

As a result, for every $\tau > 0$, there exists an injective map $\theta \mapsto \mathcal{U}^ \Psi_\tau(\theta) \mathcal{U}$ from S^1 into the space of quantum states $\mathcal{Q}(\mathcal{H})$ associated with the gauge theory on Minkowski space, mapping classical states (points in the circle) to pure quantum states evolving periodically under the unitary evolution $Z^t: \rho \mapsto e^{-i\hbar t} \rho e^{i\hbar t}$ generated by H .*

- (iii) *For every $\tau > 0$, there exists a continuous linear map $\Omega_\tau: \mathcal{A}(L^2(\mu)) \rightarrow C_\mathbb{R}(S^1)$ from the Abelian Banach algebra $\mathcal{A}(L^2(\mu))$ of bounded, self-adjoint operators on $L^2(\mu)$ (equipped with the operator norm and a symmetrized operator product for the multiplication operation) and the Banach algebra $C_\mathbb{R}(S^1)$ of real-valued continuous functions on the circle. This map is compatible with the evolution $\tilde{U}^t: A \mapsto U^t A U^{t*}$ on $\mathcal{A}(L^2(\mu))$ induced by the Koopman group on $L^2(\mu)$, i.e.,*

$$\Omega_\tau \circ \tilde{U}^t = U^t \circ \Omega_\tau, \quad \forall t \in \mathbb{R}.$$

As a result, $A \mapsto \Omega_\tau(\mathcal{U}A\mathcal{U}^*)$ is a continuous linear map from the space of bounded observables of the quantum system on Minkowski space, $\mathcal{A}(\mathcal{H})$, to classical observables in $C_\mathbb{R}(S^1)$, which is compatible with the unitary Koopman evolution $f \mapsto f \circ \Phi^t$ on $C_\mathbb{R}(S^1)$ and the Heisenberg evolution $A \mapsto e^{itH} A e^{-itH}$ on $\mathcal{A}(\mathcal{H})$.

Theorem 2 (canonically commuting operators for the circle rotation). *With the notation of Theorem 1, the following hold:*

- (i) *The two creation operators associated with the timelike and spacelike degrees of freedom of H pull back under $\mathcal{U}$ to densely defined operators A_-^+ and A_+^+ on $L^2(\mu)$, which act as creation operators for negative- and positive-frequency Koopman eigenfunctions, respectively. These operators and their corresponding annihilation operators, A_- and A_+ , respectively, have the structure of order-1/2 fractional differentiation operators. Moreover, they induce number operators $N_- = A_-^+ A_-$ and $N_+ = A_+^+ A_+$, exhibiting canonical commutation relationships, leading to a decomposition of the generator as a difference between the positive- and negative-frequency number operators,*

$$V = i\alpha(-N_- + N_+).$$

- (ii) *The densely defined operators $\tilde{X}_\pm$ and $\tilde{P}_\pm$ on $L^2(\mu)$ with*

$$\tilde{X}_\pm = \frac{1}{\sqrt{2\alpha}}(A_\pm + A_\pm^+), \quad \tilde{P}_\pm = -i\sqrt{\frac{\alpha}{2}}(A_\pm - A_\pm^+)$$

satisfy canonical position-momentum commutation relationships, i.e.,

$$[\tilde{X}_-, \tilde{X}_+] = [\tilde{P}_-, \tilde{P}_+] = 0, \quad [\tilde{X}_-, \tilde{P}_+] = [\tilde{X}_+, \tilde{P}_+] = \text{Id}.$$

In particular, $\tilde{X}_-$ and $\tilde{P}_-$ (respectively, $\tilde{X}_+$ and $\tilde{P}_+$) are pullbacks under $\mathcal{U}$ of the position and momentum operators, respectively, associated with the timelike (respectively, spacelike) degrees of freedom of H and generate fractional diffusion semigroups on $L^2(\mu)$. Moreover, the pure quantum states in $\mathcal{Q}(L^2(\mu))$ associated with the Koopman eigenfunctions of the circle rotation satisfy canonical position-momentum uncertainty relationships with respect to these operators.

In Theorem 1, by gauge-covariance for $\mathcal{U}$, we mean that under a gauge transformation, this operator transforms as $\mathcal{U} \mapsto \Xi \mathcal{U}$, where Ξ is a unitary multiplication operator on $\mathcal{H}$ that commutes with H . Moreover, by the connection on $E \rightarrow M$ being an $\text{SO}^+(1, 1)$ Yang-Mills connection, we mean that it is induced from a connection 1-form on an $\text{SO}^+(1, 1)$ frame bundle over M , whose corresponding field strength satisfies the Yang-Mills equations (though its Yang-Mills action is infinite due to non-compactness of M). In particular, $\text{SO}^+(1, 1)$ is an Abelian group that may be identified with the universal covering group of $\text{U}(1)$ so that the gauge theory employed in this work can be thought of as a two-dimensional analog of Maxwell electromagnetism. In Theorem 1(ii) and (iii), the one-parameter families of maps Ψ_τ and Ω_τ are constructed using so-called feature maps¹⁴ associated with the reproducing kernel Hilbert spaces (RKHSs) $\mathcal{H}_\tau$ of functions on the circle induced by the canonical heat kernel at time parameter τ . We recall that the time- τ heat kernel on the circle, $\kappa_\tau : S^1 \times S^1 \rightarrow \mathbb{R}_+$, gives the transition probability for the diffusion semigroup $e^{-\tau\mathcal{L}}$ on $L^2(\mu)$, where $\mathcal{L}$ is the (positive-semidefinite) Laplace-Beltrami operator associated with the standard Riemannian metric on S^1 .¹⁵

Next, for $p \in (0, 1)$, we study the action of the dynamics on a different class of RKHSs, $\hat{\mathcal{H}}_\tau$, $\tau > 0$, which are associated with order- $p/2$ fractional diffusions on the circle generated by $-\mathcal{L}^{p/2}$. As shown in recent work,¹⁶ these spaces have the distinguished property of being unital Banach *-algebras of functions. For this reason, it is natural to refer to them as reproducing kernel Hilbert algebras (RKHA). As with the canonical heat kernel κ_τ , the reproducing kernel $\hat{\kappa}_\tau : S^1 \times S^1 \rightarrow \mathbb{R}$ of $\hat{\mathcal{H}}_\tau$ is a smooth, translation-invariant function for any $\tau > 0$. In particular, while a general RKHS of functions on S^1 need not be invariant under the circle rotation (so that one cannot speak of a Koopman evolution group on an arbitrary RKHS), by virtue of the translation invariance of κ_τ and $\hat{\kappa}_\tau$, the spaces $\mathcal{H}_\tau$ and $\hat{\mathcal{H}}_\tau$ are invariant under the dynamics, and moreover, the corresponding Koopman evolution groups are unitary. In the case of $\hat{\mathcal{H}}_\tau$, the fact that this space is also a Banach *-algebra means that every element $f \in \hat{\mathcal{H}}_\tau$ has a corresponding bounded multiplication operator on $\hat{\mathcal{H}}_\tau$ multiplying by that function (which is not the case for $\mathcal{H}_\tau$). To our knowledge, the properties of Koopman groups on the spaces $\hat{\mathcal{H}}_\tau$ have not been discussed elsewhere in the literature, so this material (stated as Theorem 7 in Sec. VI B) should be of independent interest.

For our purposes, a key property provided by the RKHS structure of $\hat{\mathcal{H}}_\tau$ is that pointwise function evaluation can be carried out by bounded, and thus continuous, linear functionals. This leads to a stronger form of classical-quantum correspondence than Theorem 1, which is compatible with a natural embedding of real-valued functions in $\hat{\mathcal{H}}_\tau$ to self-adjoint multiplication operators on $\hat{\mathcal{H}}_\tau$. As a result, the evolution of every classical observable in $\hat{\mathcal{H}}_\tau$ under the circle rotation can be consistently mapped into evolution of an observable of the quantum system on Minkowski space.

Theorem 3 (Classical-quantum correspondence based on $\hat{\mathcal{H}}_\tau$). *Let $\hat{\kappa}_\tau : S^1 \times S^1 \rightarrow \mathbb{R}_+$ be the time $\tau > 0$ transition probability kernel associated with the fractional diffusion $e^{-\tau\mathcal{L}^{p/2}}$ for some $p \in (0, 1)$ and $\hat{\mathcal{H}}_\tau$ be the corresponding RKHA. Let also $\hat{\mathcal{H}}_{\mathbb{R},\tau}$ be the RKHA formed by the real elements of $\hat{\mathcal{H}}_\tau$. Then, with the notation of Theorem 1, there exists a unitary map $\hat{\mathcal{V}}_\tau : \hat{\mathcal{H}}_\tau \rightarrow L^2(\mu)$ such that the following hold for every $\tau > 0$:*

- (i) *The Koopman operators U^t induced by the circle rotation act as a strongly continuous, unitary evolution group on $\hat{\mathcal{H}}_\tau$.*

- (ii) For every $\tau > 0$, there exists a continuous, injective map $\hat{\Psi}_\tau : S^1 \rightarrow \mathcal{Q}(\hat{\mathcal{H}}_\tau)$ from the circle into the space $\mathcal{Q}(\hat{\mathcal{H}}_\tau)$ of regular quantum states on $\hat{\mathcal{H}}_\tau$ [defined analogously to $\mathcal{Q}(L^2(\mu))$], which is compatible with the dynamical flow and the unitary evolution $\hat{\Phi}^t : \rho \mapsto U^{t*} \rho U^t$ on $\mathcal{Q}(\hat{\mathcal{H}}_\tau)$ induced by the Koopman group; that is,

$$\hat{\Psi}_\tau \circ \hat{\Phi}^t = \hat{\Phi}^t \circ \hat{\Psi}_\tau, \quad \forall t \in \mathbb{R}.$$

As a result, there exists an injective map $\theta \mapsto \hat{\mathcal{W}}_\tau^* \hat{\Psi}_\tau(\theta) \hat{\mathcal{W}}_\tau$, $\hat{\mathcal{W}}_\tau = \mathcal{U} \hat{\mathcal{V}}_\tau$, from S^1 into the space of quantum states $\mathcal{Q}(\mathcal{H})$ associated with the gauge theory on Minkowski space, mapping classical states to pure quantum states evolving periodically under the Heisenberg evolution group associated with H .

- (iii) There exists a continuous linear map $\hat{\Omega}_\tau : \mathcal{A}(\hat{\mathcal{H}}_\tau) \rightarrow \hat{\mathcal{H}}_{\mathbb{R},\tau}$, where $\mathcal{A}(\hat{\mathcal{H}}_\tau)$ is the Abelian Banach algebra of bounded, self-adjoint operators on $\hat{\mathcal{H}}_\tau$ [defined analogously to $\mathcal{A}(L^2(\mu))$], such that $\hat{\Omega}_\tau$ is compatible with the Koopman group on $\hat{\mathcal{H}}_\tau$ and the evolution $\tilde{U}^t : A \mapsto U^t A U^{t*}$ on $\mathcal{A}(\hat{\mathcal{H}}_\tau)$ induced by it, viz.,

$$\hat{\Omega}_\tau \circ \tilde{U}^t = U^t \circ \hat{\Omega}_\tau, \quad \forall t \in \mathbb{R}.$$

As a result, $A \mapsto \hat{\Omega}_\tau(\mathcal{W}_\tau A \mathcal{W}_\tau)$ is a continuous linear map from the space $\mathcal{A}(\mathcal{H})$ of bounded quantum mechanical observables of the quantum system on Minkowski space to classical observables in $\hat{\mathcal{H}}_{\mathbb{R},\tau}$.

- (iv) The map $\hat{\Omega}_\tau$ is a left inverse of the map $\hat{T} : \hat{\mathcal{H}}_{\mathbb{R},\tau} \rightarrow \mathcal{A}(\hat{\mathcal{H}}_\tau)$ mapping $f \in \hat{\mathcal{H}}_{\mathbb{R},\tau}$ to the bounded, self-adjoint operator $\hat{T}_f = (T_f + (T_f)^*)/2$, where $T_f : g \mapsto fg$ is the multiplication operator on $\hat{\mathcal{H}}_\tau$ that multiplies by f . As a result, for every classical observable $f \in \hat{\mathcal{H}}_{\mathbb{R},\tau}$ and state $\theta \in S^1$, pointwise evaluation can be expressed as a quantum mechanical expectation value,

$$f(\theta) = \mathbb{E}_{\rho_\theta} \hat{T}_f := \text{tr}(\rho_\theta \hat{T}_f), \quad \rho_\theta = \hat{\Psi}_\tau(\theta).$$

Equivalently, $f(\theta)$ can be expressed as an expectation value of the observable $A_f = \hat{\mathcal{W}}_\tau \hat{T}_f \hat{\mathcal{W}}_\tau^*$ of the quantum system on Minkowski space, i.e.,

$$f(\theta) = \mathbb{E}_{\sigma_\theta} A_f, \quad \sigma_\theta = \hat{\mathcal{W}}_\tau^* \rho_\theta \hat{\mathcal{W}}_\tau.$$

- (v) The correspondences in (iii) are compatible with dynamical evolution, i.e.,

$$f(\hat{\Phi}^t(\theta)) = \mathbb{E}_{\hat{\Phi}^t(\rho_\theta)} \hat{T}_f = \mathbb{E}_{Z^t(\sigma_\theta)} A_f, \quad \forall t \in \mathbb{R}.$$

The plan of this paper is as follows: In Sec. II, we outline aspects of the Koopman operator formalism for the circle rotation on $L^2(\mu)$, as well as the properties of the heat kernel and corresponding RKHSs. In Sec. III, we describe the construction of our $\text{SO}^+(1, 1)$ gauge theory on two-dimensional Minkowski space, and in Sec. IV, we develop its quantum formulation and correspondence with the dynamics of the circle rotation. Together, these sections constitute a Proof of Theorem 1. In Sec. V, we describe the construction of canonically commuting ladder operators for the classical harmonic oscillator, proving Theorem 2. In Sec. VI, we present the Koopman operator formulation for the circle rotation on the RKHSs associated with fractional diffusions on the circle, as well as the classical–quantum correspondence associated with these spaces, proving Theorem 3. Concluding remarks are stated in Sec. VII. This paper contains three appendices with definitions and technical results on fiber bundles (Appendix A), RKHS theory (Appendix B), and fractional derivatives (Appendix C).

II. ERGODIC DYNAMICS OF THE CLASSICAL HARMONIC OSCILLATOR

In this section, we introduce the Koopman operator formalism for the harmonic oscillator, including recently proposed ladder-like operators acting on Koopman eigenspaces.¹⁷ In addition, we define and outline the basic properties of the heat kernel and corresponding RKHSs, which will be useful in establishing our classical–quantum correspondence. Additional details on Koopman operator theory can be found in one of the many references in the literature, e.g., Refs. 1 and 2. Aspects of RKHS theory can also be found in many references, e.g., Refs. 14, 18, and 19. Appendix B 1 summarizes basic definitions and results from RKHS theory that are pertinent to this work.

A. Koopman operator formalism

We represent the dynamics of a classical harmonic oscillator of angular frequency $\alpha \in \mathbb{R}$ by a translation map $\Phi^t : S^1 \rightarrow S^1$, $t \in \mathbb{R}$, on the circle S^1 , viz.,

$$\Phi^t(\theta) = \theta + \alpha t \quad \text{mod } 2\pi.$$

As a concrete example, in Hamiltonian mechanics, S^1 would be a subset of the cotangent bundle $T^*\mathbb{R} \simeq \mathbb{R}^2$ representing a constant-energy surface in position–momentum space, but, in general, we can consider S^1 as the state space of a classical dynamical system without reference to an embedding. As is well known, Φ^t has a unique Borel, ergodic, invariant probability measure μ (i.e., a Haar measure) corresponding to a normalized arclength.

We use the notations $C^r(S^1)$, $r \in \mathbb{N}_0$, and $L^p(\mu)$, $p \geq 1$, to represent the Banach spaces of complex-valued functions on S^1 with r continuous derivatives and (equivalence classes of) complex-valued functions with μ -integrable p -th power, equipped with the standard norms, respectively. In the Hilbert space case, $L^2(\mu)$, we also use the notations $\langle f, g \rangle_\mu = \int_{S^1} f^* g \, d\mu$ and $\|f\|_\mu = \sqrt{\langle f, f \rangle_\mu}$ for the corresponding inner product and norm, respectively. Furthermore, we abbreviate $C^0(S^1)$ by $C(S^1)$. We shall refer to elements of $C^r(S^1)$ and $L^p(\mu)$ as classical observables.

In this setting, the Koopman group of evolution operators is the strongly continuous group of isometries $U^t : \mathcal{E} \rightarrow \mathcal{E}$, where $\mathcal{E}$ stands for either $C(S^1)$ or $L^p(\mu)$, acting on classical observables by composition with the dynamical flow map, i.e.,

$$U^t f = f \circ \Phi^t, \quad t \in \mathbb{R}. \quad (1)$$

We will mainly focus on the Koopman group on $L^2(\mu)$, which in addition to being isometric is unitary, $U^{t*} = U^{t-1} = U^{-t}$. By Stone's theorem on strongly continuous unitary evolution groups,²⁰ $U^t : L^2(\mu) \rightarrow L^2(\mu)$ is completely characterized by its *generator*; the latter is a skew-adjoint, unbounded operator $V : D(V) \rightarrow L^2(\mu)$ with a dense domain $D(V) \subset L^2(\mu)$, acting on elements in its domain according to the formula

$$Vf = \lim_{t \rightarrow 0} \frac{U^t f - f}{t}$$

and generating the Koopman operator at any time $t \in \mathbb{R}$ by operator exponentiation,

$$U^t = e^{tV}.$$

Ter Elst and Lemańczyk²¹ recently showed that the space $D(V) \cap L^\infty(\mu)$ is a Koopman-invariant algebra on which V acts as a derivation. That is, for any two elements $f, g \in D(V) \cap L^\infty(\mu)$, the pointwise product fg also lies in this space, and the generator satisfies a Leibniz rule,

$$V(fg) = (Vf)g + f(Vg). \quad (2)$$

This is a manifestation of the fact that V behaves as a differential operator on classical observables, which can be viewed as an extension of the vector field $\vec{V}$ on S^1 generating the flow Φ^t and acting on continuously differentiable functions as a derivative operator, $\vec{V}f = \alpha f'$.

Consider now the continuous dual space $C'(S^1)$ to $C(S^1)$, equipped with the operator norm. It is a standard result that $C'(S^1)$ can be canonically identified with the Banach space $\mathcal{M}(S^1)$ of complex Radon measures on S^1 , equipped with the total variation (TV) norm, through a linear isometry mapping $m \in \mathcal{M}(S^1)$ to $J_m \in C'(S^1)$ with $J_m f = \int_{S^1} f \, dm$. The dynamics acts on $\mathcal{M}(S^1)$ through a group of isometries, $\Phi_*^t : \mathcal{M}(S^1) \rightarrow \mathcal{M}(S^1)$, $\Phi_*^t m = m \circ \Phi^{-t}$, known as *Perron–Frobenius* or *transfer* operators. Under the $C'(S^1) \simeq \mathcal{M}(S^1)$ isomorphism, Φ_*^t is identified with the transpose of the Koopman operator, $U^{t*} : C'(S^1) \rightarrow C'(S^1)$, $U^{t*} \varphi = \varphi \circ U^t$; specifically, $U^{t*} \varphi_m = \varphi_{\Phi_*^t m}$. It can be readily verified that if $m \in \mathcal{M}(S^1)$ has a density $\rho = dm/d\mu \in \mathcal{E}$ relative to the invariant measure [where again $\mathcal{E}$ stands for either $C(S^1)$ or $L^p(\mu)$], then $\Phi_*^t m$ has density $U^{-t} \rho$. In particular, for $\rho \in L^2(\mu)$, we have $U^{-t} \rho = U^{t*} \rho$, so we may identify the transfer operator with the adjoint of the Koopman operator.

In what follows, we shall be concerned with the action of both the Koopman and transfer operators on spaces of functions and measures on the circle, respectively, as well as their induced action on operator algebras and their duals. We will continue to overload notation and employ the same symbol U^t to represent the Koopman operator acting on any of the $C^r(S^1)$ and $L^p(\mu)$ spaces (as well as the RKHSs introduced below), as the particular instance will be clear from the context. In addition, we will sometimes consider spaces of real-valued continuous functions, their duals, and real Radon measures, which we will distinguish using the symbols $C_{\mathbb{R}}(S^1)$, $C'_{\mathbb{R}}(S^1)$, and $\mathcal{M}_{\mathbb{R}}(S^1)$, respectively. We also let $\mathcal{P}(S^1) \subset \mathcal{M}_{\mathbb{R}}(S^1)$ be the space of Radon probability measures on the circle, equipped with the TV norm topology. Koopman and transfer operators are then defined on these spaces analogously to the complex case, and we again use the symbols U^t , U^{t*} , and Φ_*^t , respectively, to represent them. We will oftentimes refer to any such instance of U^t and U^{t*}/Φ_*^t as “the” Koopman or transfer operator, respectively.

Before carrying on, we note a mathematical correspondence between the operator-theoretic description of a measure-preserving, classical dynamical system, such as the harmonic oscillator studied here, and quantum mechanics. That is, $-iV$ is a self-adjoint operator analogous to the Hamiltonian H of a quantum system, and the Koopman operator U^t on $L^2(\mu)$ is a unitary operator analogous to the Heisenberg evolution operator e^{itH} generated by H .

B. Koopman eigenfunctions

Next, we turn to the spectral characterization of the unitary Koopman group on $L^2(\mu)$ associated with the classical harmonic oscillator. As can be readily verified, there is a smooth orthonormal basis $\{\phi_j\}_{j=-\infty}^\infty$ of $L^2(\mu)$ consisting of Koopman eigenfunctions, satisfying the equation

$$V\phi_j = i\alpha_j \phi_j,$$

where $\alpha_j = j\alpha$ is an eigenfrequency equal to an integer multiple of the oscillator's frequency and $\phi_j(\theta) = e^{ij\theta}$ is equal to a Fourier function. Moreover, each ϕ_j is also an eigenfunction of the Koopman operator U^t corresponding to the eigenvalue e^{iat} . The latter implies that we can

compute the dynamical evolution of any classical observable $f \in L^2(\mu)$ by first expanding it in the Koopman eigenfunction basis, $f = \sum_{j=-\infty}^{\infty} c_j \phi_j$, and then computing

$$U^t f = \sum_{j=-\infty}^{\infty} e^{ij\alpha t} c_j \phi_j,$$

where the infinite sum over j converges in $L^2(\mu)$ norm. As with any measure-preserving, continuous-time dynamical system on a manifold, the Koopman eigenfrequencies and eigenfunctions have an important algebraic group structure [which can be verified from the Leibniz rule in (2)], namely, they are closed under addition and multiplication, respectively, i.e.,

$$\alpha_j + \alpha_k = \alpha_{j+k}, \quad \phi_j \phi_k = \phi_{j+k}, \quad \forall j, k \in \mathbb{Z}.$$

For completeness, we note that the existence of a complete basis of Koopman eigenfunctions is not generic to arbitrary measure-preserving, ergodic dynamical systems, and in particular, systems exhibiting mixing (chaotic) behavior have a non-empty continuous Koopman spectrum with an associated subspace of $L^2(\mu)$, which does not admit a Koopman eigenfunction basis.²²

C. Ladder-like operators

We now draw another analogy between the operator-theoretic description of the classical harmonic oscillator and quantum mechanics (to our knowledge, first pointed out by Mezić¹⁷), pertaining to ladder operators.²³ In particular, consider the bounded operator $L: L^2(\mu) \rightarrow L^2(\mu)$, which multiplies by the Koopman eigenfunction ϕ_{-1} , i.e., $Lf = \phi_{-1}f$. A direct calculation yields

$$[V, L] = -i\alpha L \quad (3)$$

on $D(V)$, which is analogous to the commutation relation between the Hamiltonian and the lowering operator of the quantum harmonic oscillator (modulo the presence of the imaginary number i in the right-hand side due to skew-adjointness of V). Similarly, we have

$$[V, L^*] = i\alpha L^* \quad (4)$$

so that L^* behaves analogously to the raising operator in the context of the quantum harmonic oscillator. Stated explicitly, the last two equations imply

$$L\phi_j = \phi_{j-1}, \quad L^*\phi_j = \phi_{j+1}, \quad (5)$$

respectively, which shows that L (L^*) lowers (raises) the eigenfrequency of ϕ_j by a unit of α .

Despite these similarities with ladder operators of quantum harmonic oscillators, it is important to note that there is no analog of the ground state in this picture and that $L\phi_j$ and $L^*\phi_j$ are equal to ϕ_{j-1} and ϕ_{j+1} , respectively, without the presence of j -dependent multiplication factors [cf. (56) ahead]. The first of these facts suggests the correspondence put forward in Sec. IV C between the classical harmonic oscillator and a *relativistic* oscillator, which naturally supports negative-energy states. The latter fact implies that L and L^* are bounded operators (unlike the ladder operators of the quantum harmonic oscillator) and that L^*L is not a number operator. In fact, it follows directly from (5) that L is a unitary operator, and therefore,

$$L^*L = LL^* = \text{Id}. \quad (6)$$

In Sec. V, we will employ the correspondence established in Sec. IV C to construct operators that, besides satisfying the commutation relations in (3) and (4), they also give rise to a number operator satisfying the appropriate commutation relations. These operators will turn out to be fractional differentiation operators of order $1/2$.

D. Heat kernel and the associated reproducing kernel Hilbert spaces (RKHSs)

Let $\tilde{\mathcal{L}}: C^\infty(S^1) \rightarrow L^2(\mu)$ be the canonical Laplace–Beltrami operator on the circle, defined here as the operator on $L^2(\mu)$ with the dense domain $C^\infty(S^1)$ such that $\tilde{\mathcal{L}}f = -f''$. It is a standard result from analysis on manifolds¹⁵ that $\tilde{\mathcal{L}}$ is a positive-semidefinite, essentially self-adjoint operator, having the Fourier functions ϕ_j as its eigenfunctions,

$$\tilde{\mathcal{L}}\phi_j = j^2 \phi_j, \quad j \in \mathbb{Z}.$$

The (unique) self-adjoint extension $\mathcal{L}: D(\mathcal{L}) \rightarrow L^2(\mu)$ of $\tilde{\mathcal{L}}$, whose domain $D(\mathcal{L})$ is an order-2 Sobolev space, generates a Markov diffusion semigroup $\{e^{-\tau\mathcal{L}}\}_{\tau \geq 0}$ on $L^2(\mu)$, called the *heat semigroup*, such that for any $\tau > 0$,

$$e^{-\tau\mathcal{L}}f = \int_{S^1} \kappa_\tau(\cdot, \theta') f(\theta') d\mu(\theta').$$

Here, $\kappa_\tau : S^1 \times S^1 \rightarrow \mathbb{R}_+$ is the time- τ *heat kernel* on the circle—the smooth, strictly positive, bivariate function given by

$$\kappa_\tau(\theta, \theta') = \sum_{j=-\infty}^{\infty} e^{-j^2\tau} \phi_j^*(\theta) \phi_j(\theta') = 1 + \sum_{j=1}^{\infty} e^{-j^2\tau} \cos(j(\theta - \theta')) = \sqrt{4\pi\tau} \sum_{j=-\infty}^{\infty} e^{-\tau(\theta - \theta' - 2j\pi)^2}, \quad (7)$$

where the sums over $j \in \mathbb{Z}$ converge in any $C^r(S^1)$ norm, $r \in \mathbb{N}_0$. A continuous kernel admitting a uniformly convergent eigenfunction expansion as in the first line of (7) is known as a *Mercer kernel*.

The heat kernel on the circle has the property of being *translation invariant*, i.e., $\kappa_\tau(\theta, \theta')$ depends only on the arclength distance between θ and θ' . As a result, since Φ^t preserves arclength distances, κ_τ is also shift-invariant under the circle rotation, $\kappa_\tau(\Phi^t(\theta), \Phi^t(\theta')) = \kappa_\tau(\theta, \theta')$ for all $t \in \mathbb{R}$ and $\theta, \theta' \in S^1$. In addition, κ_τ is *strictly positive-definite*, meaning that for any collection $\theta_1, \dots, \theta_N$ of distinct points in S^1 , the $N \times N$ kernel matrix $K = [\kappa_\tau(\theta_i, \theta_j)]_{ij}$ is strictly positive.

Associated with the heat kernel κ_τ is a *reproducing kernel Hilbert space (RKHS)* of complex-valued functions on S^1 ; that is, a Hilbert space $\mathcal{H}_\tau$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_\tau}$ and corresponding norm $\|\cdot\|_{\mathcal{H}_\tau}$ such that for every $\theta \in S^1$, (i) the kernel section $\kappa_\tau(\theta, \cdot)$ lies in $\mathcal{H}_\tau$; and (ii) the evaluation functional $\mathbb{V}_\theta : \mathcal{H}_\tau \rightarrow \mathbb{C}$, $\mathbb{V}_\theta f = f(\theta)$ is bounded (and thus, continuous) and satisfies $\mathbb{V}_\theta f = \langle \kappa_\tau(\theta, \cdot), f \rangle_{\mathcal{H}_\tau}$. The latter is known as the *reproducing property* and leads to the inner product relationships $\langle \kappa_\tau(\theta, \cdot), \kappa_\tau(\theta', \cdot) \rangle_{\mathcal{H}_\tau} = \kappa_\tau(\theta, \theta')$ for the kernel sections at any $\theta, \theta' \in S^1$.

It can be shown, e.g., using results in Refs. 14 and 18 that the $\mathcal{H}_\tau$ are subspaces of $C^\infty(S^1)$, forming an increasing sequence as τ decreases to 0. Moreover, for every $\tau > 0$, $\mathcal{H}_\tau$ is a dense subspace of $C(S^1)$, and for any $r \in \mathbb{N}_0$, the inclusion $\mathcal{H}_\tau \hookrightarrow C^r(S^1)$ [and thus $\mathcal{H}_\tau \hookrightarrow L^p(\mu)$] is bounded. It is a consequence of the continuity of κ_τ and density of $\mathcal{H}_\tau$ in $C(S^1)$ that $\mathbb{K}_\tau : \mathcal{P}(S^1) \rightarrow \mathcal{H}_\tau$, with

$$\mathbb{K}_\tau(m) = \int_{S^1} \kappa_\tau(\theta, \cdot) dm(\theta),$$

is an injective, continuous map from Radon probability measures to RKHS functions on the circle. Such a map is oftentimes referred to as an *RKHS embedding* of probability measures. Now, as can be verified by routine calculations, the map $\delta : S^1 \rightarrow \mathcal{P}(S^1)$ sending $\theta \in S^1$ to the Dirac measure $\delta_\theta := \delta(\theta)$ supported at θ is also injective and continuous. As a result, the so-called *feature map*, $F_\tau : S^1 \rightarrow \mathcal{H}_\tau$,

$$F_\tau = \mathbb{K}_\tau \circ \delta, \quad F_\tau(\theta) = \kappa_\tau(\theta, \cdot), \quad (8)$$

is an injective, continuous map, mapping points on the circle to the RKHS functions given by the corresponding kernel sections (also known as *feature vectors*). It also follows from the strict positive-definiteness of κ_τ that $F_\tau(\theta)$ and $F_\tau(\theta')$ are linearly independent whenever θ and θ' are distinct so that the image $\mathcal{F}(\mathcal{H}_\tau) := F_\tau(S^1) \subset \mathcal{H}_\tau$ contains only nonzero functions. See Lemmas 15 and 16 in Appendix B 1 for further details.

A useful correspondence between the Hilbert spaces $\mathcal{H}_\tau$ and $L^2(\mu)$ is provided by the integral operators $K_\tau : L^2(\mu) \rightarrow \mathcal{H}_\tau$,

$$K_\tau f = \int_{S^1} \kappa_\tau(\cdot, \theta') f(\theta') d\mu(\theta'),$$

which can be shown to be well-defined, compact integral operators with a dense range for any $\tau > 0$. Moreover, the adjoint K_τ^* maps injectively $f \in \mathcal{H}_\tau \subset C^\infty(S^1)$ to its corresponding $L^2(\mu)$ equivalence class so that $K_\tau^* : \mathcal{H}_\tau \hookrightarrow L^2(\mu)$ is a compact embedding. It then follows that $e^{-\tau\mathcal{L}} = K_\tau^* K_\tau$ and that $\{\phi_{j,\tau}\}_{j \in \mathbb{Z}}$ with

$$\phi_{j,\tau} = e^{j^2\tau/2} K_\tau \phi_j = e^{-j^2\tau/2} \phi_j$$

is an orthonormal basis of $\mathcal{H}_\tau$. As a result, we can characterize $\mathcal{H}_\tau$ as the space of smooth functions $f : S^1 \rightarrow \mathbb{C}$ admitting (uniformly convergent) Fourier expansions $f(\theta) = \sum_{j=-\infty}^{\infty} \hat{f}_j \phi_j(\theta)$, $\hat{f}_j \in \mathbb{C}$, satisfying

$$\sum_{j=-\infty}^{\infty} e^{j^2\tau} |\hat{f}_j|^2 < \infty. \quad (9)$$

It is worth noting that besides K_τ^* , another natural way of mapping functions in $\mathcal{H}_\tau$ to $L^2(\mu)$ elements is through the unitary operator $\mathcal{V}_\tau : \mathcal{H}_\tau \rightarrow L^2(\mu)$ defined uniquely by $\mathcal{V}_\tau \phi_{j,\tau} = \phi_j$ for all $j \in \mathbb{Z}$. This operator can also be defined in a basis-free manner through the polar decomposition of the integral operator K_τ . Specifically, $\mathcal{V}_\tau$ is the unique operator from $\mathcal{H}_\tau$ to $L^2(\mu)$ such that

$$K_\tau = \mathcal{V}_\tau^* e^{-\tau\mathcal{L}/2}. \quad (10)$$

III. GAUGE FIELD THEORY ON TWO-DIMENSIONAL MINKOWSKI SPACE

In this section, we describe the construction of the gauge field theory into which we will embed the classical harmonic oscillator in Sec. IV. We will follow a geometrical approach, where the two key objects are a principal bundle of the proper, orthochronous Lorentz group over two-dimensional Minkowski space and an associated $\mathbb{C}$ -line bundle. The gauge field is then understood as a connection 1-form on the principal bundle, inducing a covariant derivative acting on sections of the associated bundle. The space of sections is also endowed with a Hilbert space structure, providing the foundation for defining quantum mechanical observables as self-adjoint operators. Among these, the Hamiltonian operator generates the quantum dynamics of the theory and will take the form of a connection Laplacian when expressed in terms of the covariant derivative.

In what follows, we will present the steps of this construction sequentially, starting from the basic properties of Minkowski space and adding the necessary structure to arrive at the connection 1-form and its associated covariant derivatives and curvature. Readers familiar with gauge theory may wish to skip to Sec. III F, where these objects are explicitly defined. Auxiliary results, as well as an overview of basic concepts from fiber bundle theory, are included in Appendix A. We refer the reader to one of the many references in the literature (e.g., Refs. 24–26) for detailed expositions on these topics.

A. Notation

We will use the notation $E \xrightarrow{\pi} M$ to represent a smooth fiber bundle with base space M , total space E , and projection map $\pi: E \rightarrow M$. Moreover, $\Gamma(E)$ will denote the space of smooth sections of $E \xrightarrow{\pi} M$. If $E \xrightarrow{\pi} M$ is a vector bundle with real fibers, $\Omega^k(M, E)$ will be the space of sections of the vector bundle over M with total space $\wedge^k T^*M \otimes_{\mathbb{R}} E$, where $\wedge^k T^*M$ is the k th exterior power of the cotangent bundle T^*M and $\otimes_{\mathbb{R}}$ is the tensor product of real vector bundles. If F is a vector space over the real numbers, $C^\infty(M, F)$ and $\Omega^k(M, F)$ will denote the space of smooth F -valued functions and k -forms on M , respectively. Note that $C^\infty(M, F) = \Omega^0(M, F)$ and $\Omega^k(M, F) = \Omega^k(M, M \times F)$, where we view $M \times F$ as the total space of the trivial bundle $M \times F \xrightarrow{\pi} M$, with π being the canonical projection map onto the first factor. We will abbreviate $\Omega^k(M, \mathbb{R})$ by $\Omega^k(M)$. If $E \xrightarrow{\pi} M$ has complex fibers and F is a vector space over the complex numbers, $\Omega^k(M, E)$ and $\Omega^k(M, F)$ are defined analogously to the real case, replacing T^*M by the complexified cotangent bundle, $T_{\mathbb{C}}^*M$, and $\otimes_{\mathbb{R}}$ by the tensor product $\otimes_{\mathbb{C}}$ of vector bundles over the complex numbers. We will use $\Gamma_c(E) \subset \Gamma(E)$ to denote the space of compactly supported sections of $E \xrightarrow{\pi} M$. When convenient, we will use subscript notation to represent pointwise evaluation of sections; for example, for a section $s \in \Gamma(E)$, we set $s_m = s(m)$.

B. Minkowski space

We are interested in constructing a gauge theory over two-dimensional *Minkowski space*, which has the structure of a metric affine space $(M, \vec{M}, \eta)$. Here, M is a two-dimensional manifold, whose points represent *events*, $\vec{M}$ is a two-dimensional vector space over the real numbers, whose elements represent *translations*, acting on M freely and transitively as an Abelian group, and $\eta: \vec{M} \times \vec{M} \rightarrow \mathbb{R}$ is a symmetric, non-degenerate bilinear form with signature $(-, +)$. In addition, $\vec{M}$ is equipped with a distinguished vector $\vec{\tau} \in \vec{M}$ with $\eta(\vec{\tau}, \vec{\tau}) < 0$, providing a notion of *future direction*.

Given any point $m \in M$, the tangent space $T_m M$ can be canonically identified with $\vec{M}$ (through identification of curves) and thus inherits a pseudo-Riemannian metric tensor $\eta_m: T_m M \times T_m M \rightarrow \mathbb{R}$ from η , called the *Minkowski metric*. Moreover, for every point $o \in M$ and basis $\{\vec{X}_0, \vec{X}_1\}$ of $\vec{M}$, there exists a *Cartesian coordinate chart*, $x: M \rightarrow \mathbb{R}^2$, such that $x(m) = (x^0, x^1)$, where $x^0 \vec{X}_0 + x^1 \vec{X}_1 = \vec{v}$, and $\vec{v}$ is the unique element of $\vec{M}$ such that $m = o + \vec{v}$. Here, the notation $o + \vec{v}$ represents translation of the point $o \in M$ by the vector $\vec{v} \in \vec{M}$.

Due to the existence of global Cartesian coordinate charts, M is diffeomorphic to $\mathbb{R}^2$, but note that the diffeomorphism is not canonical as it depends both on the choice of the point o , which can be thought of as an *origin*, and the basis $\{\vec{X}_0, \vec{X}_1\}$. Nonzero vectors $\vec{v} \in \vec{M}$ for which $\eta(\vec{v}, \vec{v})$ is negative, zero, or positive will be said to be *timelike*, *null*, or *spacelike*, respectively. Two timelike vectors $\vec{v}_1$ and $\vec{v}_2$ with $\eta(\vec{v}_1, \vec{v}_2) < 0$ are said to be *co-oriented*. Timelike, null, spacelike, and co-oriented elements of the tangent spaces $T_m M$ are defined analogously.

Given two points $m_1, m_2 \in M$, we will let $\overrightarrow{m_1 m_2}$ be the unique translation in $\vec{M}$ such that $m_2 = m_1 + \overrightarrow{m_1 m_2}$. Moreover, we will denote the “inverse metric” to η , acting on dual vectors, by η' and use the same symbol to represent the canonical lift of η' to 2-forms. We also let ${}^b: \vec{M} \rightarrow \vec{M}'$ and ${}^\sharp: \vec{M}' \rightarrow \vec{M}$ be the Riemannian isomorphisms between vectors and dual vectors, where $\vec{M}'$ is the dual space to $\vec{M}$, $\vec{v}^b = \eta(\vec{v}, \cdot)$, and $w^\sharp = \eta'(w, \cdot)$.

A map $a: M \rightarrow M$ is said to be *affine* if there exists $A \in \text{GL}(\vec{M})$ such that for all $m \in M$ and $\vec{v} \in \vec{M}$, $a(m + \vec{v}) = a(m) + A\vec{v}$. The linear map A may then be identified with the pushforward map a_* on tangent vectors. The set of all affine maps a on M preserving η , i.e., $\eta(\vec{v}_1, \vec{v}_2) = \eta(A\vec{v}_1, A\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in \vec{M}$, forms a group under composition of maps, called the *Poincaré group*. Note that the Poincaré group contains the translations $\vec{M}$ as a subgroup since for every $\vec{v} \in \vec{M}$, the map $a: M \rightarrow M$ with $a(m) = m + \vec{v}$ is affine with $A = \text{Id}$. In what follows, given a linear map A on $\vec{M}$, A^* will denote its adjoint with respect to η , i.e., the unique linear map satisfying $\eta(\vec{v}_1, A^* \vec{v}_2) = \eta(A \vec{v}_1, \vec{v}_2)$.

Next, let $v \in \Omega^2(M)$ be the volume form associated with η . Together, η and v induce a *Hodge star operator* on forms, $\star: \Omega^k(M) \rightarrow \Omega^{2-k}(M)$, defined uniquely through the requirement that $u \wedge \star w = \eta'(u, w)v$ for all $u, w \in \Omega^k(M)$. The Hodge star operator induces, in turn a map, ${}^\perp: \Gamma(TM) \rightarrow \Gamma(TM)$ on vector fields such that

$$X^\perp f = (\star df)X, \quad \forall f \in C^\infty(M). \quad (11)$$

Intuitively, $X^\perp$ can be thought of as a directional derivative in a perpendicular direction to X , obtained by a positive (“anticlockwise”) rotation with respect to the orientation induced by v . One can readily verify that ${}^\perp$ is compatible with the Poincaré group; that is, for every affine map $a: M \rightarrow M$ and vector field $X \in \Gamma(TM)$, $(a_*X)^\perp = a_*(X^\perp)$, where $a_*: \Gamma(TM) \rightarrow \Gamma(TM)$ is the pushforward map on vector fields associated with a . It should be noted that the fact that M is two-dimensional is important in the definition of ${}^\perp$ and thus will also be important in our definition of a connection 1-form in Sec. III F utilizing this map. In what follows, $L^2(v)$ will denote the Hilbert space of (equivalence classes of) complex-valued functions on M , square-integrable with respect to v . This space is equipped with the inner product $\langle f_1, f_2 \rangle_v = \int_M f_1^* f_2 dv$ and the corresponding norm $\|f\|_v = \sqrt{\langle f, f \rangle_v}$.

Hereafter, we will let x be a Cartesian chart with origin o and basis $\{\vec{X}_0, \vec{X}_1\}$ and further require that $\{\vec{X}_0, \vec{X}_1\}$ be orthonormal, positive-oriented, and has $\vec{X}_0$ future-directed, i.e., $\eta(\vec{X}_0, \vec{X}_0) = -1$, $\eta(\vec{X}_1, \vec{X}_1) = 1$, $\eta(\vec{X}_0, \vec{X}_1) = 0$, $v(\vec{X}_0, \vec{X}_1) = 1$, and $\eta(\vec{\tau}, \vec{X}_0) < 0$. We will denote the dual basis vectors to $\vec{X}_i$ by $\hat{X}^i$, where $\vec{X}^i \cdot \hat{X}_j = \delta^i_j$. For convenience, we will abbreviate the coordinate vector fields $\frac{\partial}{\partial x^0}$ and $\frac{\partial}{\partial x^1}$ on M by X_0 and X_1 , respectively. Note that because η does not have $(+, +)$ signature, the dual basis vectors are not all equal to their Riemannian duals; in particular, $\hat{X}^0 = -X_0^\flat$ and $\hat{X}^1 = X_1^\flat$. We also note the relationships $X_0^\perp = -X_1$ and $X_1^\perp = -X_0$, which indicate that $\{X_{0,m}^\perp, X_{1,m}^\perp\}$ is a negatively oriented, past-directed, orthonormal basis of $T_m M$ at every $m \in M$. For brevity, we shall refer to any Cartesian chart with the properties listed above as an *inertial chart with origin o* . Moreover, when there is no risk of confusion about the choice of origin, we will simply refer to x as an *inertial chart*.

C. Lorentz group

In what follows, we will construct principal and associated bundles over M having as their structure group the proper, orthochronous *Lorentz group*, defined as the subgroup $G \subset \text{GL}(\vec{M})$, whose every element Λ satisfies the conditions (i) $\eta(\Lambda \vec{v}_1, \Lambda \vec{v}_2) = \eta(\vec{v}_1, \vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in \vec{M}$, (ii) $\det \Lambda = 1$, and $\eta(\vec{\tau}, \Lambda \vec{\tau}) < 0$. The group G is a one-dimensional, connected, Abelian Lie group and is isomorphic to the matrix group $\text{SO}^+(1, 1)$, consisting of the real 2×2 matrices

$$\Lambda_\theta = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

Specifically, given an inertial chart $x: M \rightarrow \mathbb{R}^2$, each element $\Lambda \in G$ can be smoothly identified with a matrix $\Lambda_\theta \in \text{SO}^+(1, 1)$, with elements $\hat{X}^i \cdot \Lambda \vec{X}_j$. Because G is Abelian, this identification does not depend on the choice of inertial chart x , leading to a canonical global coordinate chart $\vartheta: G \rightarrow \mathbb{R}$ such that $\vartheta(\Lambda) = \theta$ if Λ is identified with Λ_θ . We denote the corresponding coordinate basis vector fields on G by $\Theta = \frac{\partial}{\partial \vartheta}$. The linear transformations $\vec{x} \mapsto \Lambda_\theta \vec{x}$ on $\mathbb{R}^2$ carried out by $\text{SO}^+(1, 1)$ are known as *squeeze mappings*, or *hyperbolic rotations*. We shall denote the identity element of G by I . The action of G on the translation $\vec{M}$ canonically extends to a linear action on the tangent space $T_m M$ at every $m \in M$, which we will also denote by Λ .

The Lie algebra of G , denoted by $\mathfrak{g}$, is isomorphic to the Lie algebra $\mathfrak{so}^+(1, 1)$ of $\text{SO}^+(1, 1)$; the latter consists of the set of symmetric, real 2×2 matrices

$$\lambda_\theta = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}, \quad \theta \in \mathbb{R},$$

where $\exp(\lambda_\theta) = \Lambda_\theta$. In particular, if $\lambda \in \mathfrak{g}$ is such that $\Lambda = \exp(\lambda)$ and $\vartheta(\Lambda) = \theta$, then λ is identified with λ_θ . Equivalently, λ is equal to θu , where u is the basis vector of $\mathfrak{g}$ equal to Θ_I , and we can also write $\theta = u'(\lambda)$, where $u' = d\vartheta_I$. By virtue of these facts, the ϑ chart has the property

$$\vartheta(\Lambda \Lambda') = \vartheta(\Lambda) + \vartheta(\Lambda'), \quad \forall \Lambda, \Lambda' \in G. \quad (12)$$

We equip $\mathfrak{g}$ with a metric $b: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ inherited from the coordinate chart ϑ ; specifically, $b(u, u) = 1$. Note that this metric is canonical, in the sense of being independent of the choice of inertial chart in the construction of ϑ .

As can be seen directly from (12), G and $\mathfrak{g}$ are isomorphic as a Lie group and a Lie algebra to $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot, +)$, respectively, i.e., the Abelian group and Lie algebra of real numbers equipped with the standard addition and multiplication operations. As a result, we may identify G with the universal covering group of unitary maps $U(1)$ on the complex plane.

We denote the left and right multiplication map by $\Lambda \in G$ by $L^\Lambda: G \rightarrow G$ and $R^\Lambda: G \rightarrow G$, respectively, that is, $L^\Lambda \Lambda' = \Lambda \Lambda'$ and $R^\Lambda \Lambda' = \Lambda' \Lambda$ for all $\Lambda' \in G$. Of course, since G is Abelian, the left and right multiplication maps coincide, $L^\Lambda \Lambda' = R^\Lambda \Lambda'$, but we prefer to keep these notions distinct so as to better delineate the correspondence between the analysis that follows with analogous constructions in the non-Abelian setting. It should be noted that a fundamental difference between Abelian and non-Abelian groups is that in the former case, the adjoint map $\text{AD}_\Lambda: G \rightarrow G$ trivially reduces to the identity for all $\Lambda \in G$, i.e., $\text{AD}_\Lambda \Lambda' = \Lambda \Lambda' \Lambda^{-1} = \Lambda'$.

Besides the left action on itself by multiplication, G has a left, affine action $L_o^\Lambda : M \rightarrow M$ on Minkowski space, defined for a fixed origin $o \in M$ as $L_o^\Lambda(m) = o + \Lambda \vec{om}$. The map L_o^Λ is known as a (proper, orthochronous) *Lorentz transformation*. It can be readily verified that for each $o \in M$, $\Lambda \mapsto L_o^\Lambda$ defines a homomorphism of G into the Poincaré group of M . Moreover, for every $\Lambda \in G$, the map $x' : M \rightarrow \mathbb{R}^2$ with $x' = x \circ L_o^\Lambda$ is an inertial chart, with coordinate basis vectors $X'_j = \sum_{i=0}^1 X_i \Lambda^{-1,i}_j$, where $\Lambda^{-1,i}_j = \hat{X}^i \cdot \vec{X}'_j = dx^i \cdot \Lambda^{-1} X_j$ are the matrix elements of the inverse transformation Λ^{-1} in the x chart. See Lemma 10 in [Appendix A 2](#) for further details.

D. Inertial frame bundle

Given any $m \in M$, we let P_m be the set of future-directed, positively oriented, orthonormal frames (ordered bases) of $T_m M$. That is, every element $p \in P_m$ consists of an ordered pair (p_0, p_1) of tangent vectors in $T_m M$, satisfying $\eta(p_0, p_0) < 0$ (future direction), $v(p_0, p_1) > 0$ (positive orientation), and $[\eta(p_i, p_j)]_{ij} = \text{diag}(-1, 1)$ (orthonormality). We will refer to every such p as an *inertial frame*. On P_m , G has a free, transitive right action, denoted $p \cdot \Lambda$, where $\Lambda \in G$ and

$$p \cdot \Lambda = (\Lambda^{-1} p_0, \Lambda^{-1} p_1).$$

With some abuse of notation, when convenient, we will use $R^\Lambda : P \rightarrow P$ to denote the diffeomorphism induced by right action by the group element $\Lambda \in G$, i.e., $R^\Lambda(p) = p \cdot \Lambda$.

The disjoint union $P = \bigsqcup_{m \in M} P_m$, endowed with an appropriate smooth manifold structure, becomes the total space of a principal G -bundle $P \xrightarrow{\pi} M$, where the projection map π maps $p \in P$ to the underlying base point $m \in M$. In differential geometry, such a bundle is known as an *oriented orthonormal frame bundle*; here, we will refer to $P \xrightarrow{\pi} M$ as the *principal bundle*, or *inertial frame bundle*, for brevity.

The kernel of the pushforward map $\pi_* : T_p P \rightarrow T_{\pi(p)} M$ on tangent vectors is called the *vertical subspace* at $p \in P$, denoted $V_p P \subset T_p P$. A vector field $W \in \Gamma(TP)$ is said to be *vertical* if $W_p \in V_p P$ for all $p \in P$. Given any Lie algebra element $\lambda \in \mathfrak{g}$, we can construct a vertical vector field $W^\lambda \in \Gamma(TP)$ such that

$$W^\lambda f = \lim_{\varepsilon \rightarrow 0} \frac{f \circ R^{\exp(\varepsilon \lambda)} - f}{\varepsilon}, \quad \forall f \in C^\infty(P).$$

Such a vector field is called *fundamental*. In fact, the set of fundamental vector fields is in one-to-one correspondence with $\mathfrak{g}$; that is, the map $\lambda \mapsto W^\lambda$ has a smooth inverse. Moreover, at any point $p \in P$, the vertical subspace $V_p P$ is naturally isomorphic to $\mathfrak{g}$ under the map $\lambda \mapsto W^\lambda_p$. We also note the relationship

$$W^{\theta u} = \theta W^u, \quad \forall \theta \in \mathbb{R}, \quad (13)$$

where u is the canonical unit basis vector of $\mathfrak{g}$ from [Sec. III B](#).

Due to the existence of a global inertial coordinate chart x for M , $P \xrightarrow{\pi} M$ admits a global section $\sigma_x : M \rightarrow P$ induced by the coordinate basis vectors, i.e., $\sigma_x(m) = (X_0, X_1)_m$, $\pi \circ \sigma_x = \text{Id}_M$. This section induces, in turn, a global trivialization, i.e., a diffeomorphism $\iota_{\sigma_x} : M \times G \rightarrow P$, defined as $\iota_{\sigma_x}(m, \Lambda) = \sigma_x(m) \cdot \Lambda$. Note that the inverse map $\iota_{\sigma_x}^{-1}$ satisfies $\iota_{\sigma_x}^{-1}(p) = (m, \Lambda)$, where $m = \pi(p)$ and Λ is the unique element of G such that $\sigma_x(m) \cdot \Lambda = p$. The uniqueness of Λ is a consequence of the fact that the action of G on P is free. The map $\iota_{\sigma_x}^{-1}$ induces, in turn, a map $\gamma_{\sigma_x} : P \rightarrow G$ from the principal bundle into the structure group such that $\Lambda = \gamma_{\sigma_x}(p)$ is the unique element of G satisfying $\iota_{\sigma_x}^{-1}(p) = (m, \Lambda)$. It is then straightforward to verify that γ_{σ_x} is G -equivariant, i.e.,

$$\gamma_{\sigma_x} \circ R^\Lambda = L^\Lambda \circ \gamma_{\sigma_x}, \quad \forall \Lambda \in G. \quad (14)$$

Together, these results lead to the following commutative diagram for any $\Lambda \in G$:

$$\begin{array}{ccccc} G & \xleftarrow{\gamma_{\sigma_x}} & P & \xleftarrow{\iota_{\sigma_x}} & M \times G \\ L^\Lambda \uparrow & & R^\Lambda \uparrow & & \text{Id}_M \times R^\Lambda \uparrow \\ G & \xleftarrow{\gamma_{\sigma_x}} & P & \xleftarrow{\iota_{\sigma_x}} & M \times G \\ & & \downarrow \pi & \swarrow \pi_1 & \\ & & M & & \end{array}$$

where π_1 denotes projection onto the first factor. It is a direct consequence of Lemma 10 that for any $\Lambda \in G$, the trivializing section $\sigma_{x'} : M \rightarrow P$ associated with the Lorentz-transformed chart $x' = x \circ L_o^\Lambda$ satisfies $\sigma_{x'} = R^\Lambda \circ \sigma_x$.

More generally, any local section $\sigma : U \rightarrow P$ (not necessarily associated with an inertial chart) defined on a smooth submanifold $U \subseteq M$ induces a local trivialization $\iota_\sigma : U \times G \rightarrow P$ and a map $\gamma_\sigma : \pi^{-1}(U) \rightarrow G$, with analogous properties to those of ι_{σ_x} and γ_{σ_x} , respectively. By definition, every global section σ corresponds to an assignment of an *inertial frame* of TM , that is, a positively oriented pair (e_0, e_1) of smooth, orthonormal vector fields in $\Gamma(TM)$, of which e_0 is timelike and future-directed, such that $(e_{0,m}, e_{1,m}) = \sigma(m)$ at every $m \in M$. Note, however, that e_0 and e_1 may not be coordinate vector fields; in particular, they may fail to commute.

A coordinate-induced section σ_x also induces a global coordinate chart $y : P \rightarrow \mathbb{R}^3$ on the principal bundle, given by $y(p) = (y^0, y^1, y^2) = (x^0(m), x^1(m), \vartheta(\Lambda))$, where $m = \pi(p)$ and $\Lambda = \gamma_{\sigma_x}(p)$. The corresponding coordinate basis vector fields, $Y_j := \frac{\partial}{\partial y^j}$, $j \in \{0, 1, 2\}$, are then lifts of X_0 , X_1 , and Θ , respectively, i.e., $\pi_* Y_0 = X_0$, $\pi_* Y_1 = X_1$, and $\gamma_{\sigma_x}^* Y_2 = \Theta$. These vector fields are all invariant under the G action on P , i.e., $R_*^\Lambda Y_j = Y_j$, and Y_2 , in particular, is a fundamental field generated by the Lie algebra element u . See Lemma 11 in Appendix A 2 for further details. The section σ_x also induces a metric tensor $\tilde{\eta}_p : T_p P \times T_p P \rightarrow \mathbb{R}$, given as a pullback of the direct sum metric $\eta \oplus b$ on $M \times G$ under $\iota_{\sigma_x}^{-1}$. This metric has the matrix representation $[\tilde{\eta}(Y_i, Y_j)]_{ij} = \text{diag}(-1, 1, 1)$.

In gauge theory, a local section of the principal bundle is known as a *gauge*, and oftentimes, the task is to reconstruct global objects (e.g., connection 1-forms) from their behavior on local sections, as well as to study the behavior of these objects under gauge transformations (to be discussed in Sec. III I). As noted above, in the context of the present work, a gauge corresponds to a smooth assignment of an orthonormal basis to tangent spaces of Minkowski space, which can intuitively be thought of as a local choice of inertial frame. Our task is then to construct a *gauge-covariant* quantum mechanical system, i.e., a system transforming naturally under changes of local inertial frame.

E. Associated $\mathbb{C}$ -line bundle

Let $\text{GL}(1, \mathbb{C})$ be the Lie group of invertible linear maps on the complex plane and $\mathfrak{gl}(1, \mathbb{C})$ be its associated Lie algebra. The group $\text{GL}(1, \mathbb{C})$ is canonically isomorphic to the multiplicative group of nonzero complex numbers, i.e., $w \in \text{GL}(1, \mathbb{C})$ acts linearly on $z \in \mathbb{C}$ by multiplication, $z \mapsto wz$. Meanwhile, the Lie algebra $\mathfrak{gl}(1, \mathbb{C})$ is canonically isomorphic to the vector space of complex numbers, and we have the exponential map $\exp : \mathfrak{gl}(1, \mathbb{C}) \rightarrow \text{GL}(1, \mathbb{C})$, where $\exp(\theta) = e^\theta$ for any $\theta \in \mathfrak{gl}(1, \mathbb{C})$. We equip $\mathbb{C}$ with the standard inner product, $\langle z_1, z_2 \rangle_{\mathbb{C}} = z_1^* z_2$.

To construct the $\mathbb{C}$ -line bundle associated with the principal bundle from Sec. III D, we start from the Lie algebra representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(1, \mathbb{C})$, defined as

$$\rho(\lambda) = i\alpha\vartheta(\lambda)/\sqrt{2}.$$

Here, α is a real parameter (which will be set in Sec. IV C below equal to the frequency of the classical oscillator from Sec. II) so that the representation ρ is skew-adjoint,

$$\rho(\lambda)^* = -\rho(\lambda) = \rho(-\lambda), \quad \forall \lambda \in \mathfrak{g}.$$

Expressed in terms of matrices, ρ acts by extracting from the hollow, symmetric matrix λ_θ representing λ the value $\theta = d\vartheta/\lambda$ in its off-diagonal elements and multiplying that value by the imaginary number $i\alpha/\sqrt{2}$. The factor of $1/\sqrt{2}$ is introduced here for later convenience.

Using ρ , we then construct a group representation $\rho : G \rightarrow \text{GL}(1, \mathbb{C})$, making use of the fact that the exponential map $\exp : \mathfrak{g} \rightarrow G$ for G is actually a diffeomorphism, whose inverse, $\log : G \rightarrow \mathfrak{g}$, allows one to recover the unique Lie algebra element underlying a given group element. Based on these facts, we define the unitary representation

$$\rho = \exp \circ \rho \circ \log, \quad \text{ran } \rho = \text{U}(1),$$

where $\rho(\Lambda) = e^{i\alpha\vartheta(\Lambda)/\sqrt{2}}$, and the differential at the identity recovers ρ ,

$$\rho_{*,I} = \rho.$$

Note that unlike $\exp : \mathfrak{g} \rightarrow G$, the exponential map on $\mathfrak{gl}(1, \mathbb{C})$ is not injective, and therefore, ρ is not a faithful (injective) representation. In fact, viewed as a map from $G \simeq \text{SO}^+(1, 1)$ to $\text{U}(1)$, ρ becomes the universal covering map for $\text{U}(1)$.

The representation ρ induces a left action on $\mathbb{C}$, denoted

$$L^\Lambda z \equiv \Lambda \cdot z = \rho(\Lambda)z = e^{i\alpha\vartheta(\Lambda)/\sqrt{2}}z \quad (15)$$

for any $\Lambda \in G$ and $z \in \mathbb{C}$. Together with the G action on the total space P of the principal bundle, this action induces an equivalence relation $\sim$ on $P \times \mathbb{C}$, whereby $(p, z) \sim (p', z')$ if there exists $\Lambda \in G$ such that $(p', z') = (p \cdot \Lambda, \Lambda^{-1} \cdot z)$. We define E as the set of equivalence classes in $P \times \mathbb{C}$ under this equivalence relation. It can be readily verified that the projection map $\pi_E : E \rightarrow M$, sending equivalence class $[p, z] \in E$ to $\pi(p)$, is well defined. Moreover, letting $E_m = \pi_E^{-1}(\{m\})$ denote the fiber in E over $m \in M$ and p denote an arbitrary point in E_m , it can be shown that the map $\varepsilon_p : \mathbb{C} \rightarrow E_m$ with $\varepsilon_p(z) = [p, z]$ is a bijection, and the property $\varepsilon_{R^\Lambda(p)} = L^{\Lambda^{-1}} \circ \varepsilon_p^{-1}$ holds (see Lemma 8 in Appendix A). The tuple $E \xrightarrow{\pi_E} M$ is therefore an *associated vector bundle* to $P \xrightarrow{\pi} M$ over the complex numbers, with typical fiber $\mathbb{C}$. In particular, for any $p \in P$ and $z, c \in \mathbb{C}$, the scalar multiplication $c[p, z] = [p, cz]$ and complex conjugation $[p, z]^* = [p, z^*]$ are well-defined operations.

Next, consider the space of smooth sections of E , $\Gamma(E)$. Using any global section $\sigma : M \rightarrow P$ from Sec. III D, we can construct a trivialization of $E \xrightarrow{\pi_E} M$ analogously to that of the principal bundle, viz.,

$$\begin{array}{ccc} E & \xleftarrow{\iota_{\sigma,E}} & M \times \mathbb{C} \\ \downarrow \pi & \swarrow \pi_1 & \\ M & & \end{array}.$$

Here, $\iota_{\sigma,E} : M \times \mathbb{C} \rightarrow E$ is the diffeomorphism with $\iota_{\sigma,E}(m, z) = [\sigma(m), \varepsilon_{\sigma(m)}(z)]$ and inverse $\iota_{\sigma,E}^{-1}([p, z]) = (\pi(p), \varepsilon_{\sigma(\pi(p))}^{-1}([p, z]))$. Moreover, σ induces a Hermitian metric on E , that is, a smooth assignment $m \mapsto g_m$, where $m \in M$ and g_m is a positive-definite sesquilinear form on E_m . Explicitly, given any $m \in M$ and $e_1, e_2 \in E_m$, we have

$$g_m(e_1, e_2) = \langle \varepsilon_{\sigma(m)}(e_1), \varepsilon_{\sigma(m)}(e_2) \rangle_{\mathbb{C}}. \quad (16)$$

The complex structure of E induces a complex structure $J : \Gamma(E) \rightarrow \Gamma(E)$ on sections such that $(Js)(m) = (s(m))^*$.

It can be verified that due to the unitarity of ρ , g_m is independent of the choice of trivializing section σ (see Lemma 9 in Appendix A), i.e., it is *gauge-invariant*. We can therefore define a Hilbert space $\mathcal{H}$ of sections of E (modulo sets of zero Riemannian measure), equipped with the inner product $\langle s_1, s_2 \rangle_{\mathcal{H}} = \int_M g_m(s_1(m), s_2(m)) dv(m)$ and norm $\|s\|_{\mathcal{H}} = \sqrt{\langle s, s \rangle_{\mathcal{H}}}$, in a manner that does not depend on the choice of gauge. We will employ this Hilbert space in Sec. IV A to define quantum mechanical observables as self-adjoint operators acting on it.

The trivializing section σ also induces an isomorphism $\zeta_{\sigma} : C^{\infty}(M) \rightarrow \Gamma(E)$ between smooth, complex-valued functions on M and sections of $\Gamma(E)$ such that

$$\zeta_{\sigma} f(m) = \varepsilon_{\sigma(m)}(f(m)), \quad \zeta_{\sigma}^{-1} s(m) = \varepsilon_{\sigma(m)}^{-1}(s(m)).$$

This extends, in turn, to a Hilbert space isomorphism between equivalence classes of functions in $L^2(\nu)$ and sections in $\mathcal{H}$, which is very convenient for calculational purposes, and will also facilitate establishing the correspondence with Koopman operator theory for the classical harmonic oscillator. In gauge theory, pullbacks of sections of associated bundles, such as $\zeta_{\sigma}^{-1} s$, are known as *matter fields*. As will be discussed in more detail in Sec. III I, matter fields have the distinguished property of being *gauge-covariant*, i.e., they transform naturally under changes of section σ and thus under changes of inertial frame in the present context. In particular, this is contrast to arbitrary $C^{\infty}(M)$ functions that have no natural transformation properties with respect to σ .

Before closing this section, we note another key property of sections in $\Gamma(E)$, namely, that they are in one-to-one correspondence with G -equivariant, $\mathbb{C}$ -valued functions on the principal bundle. In particular, a function $f \in C^{\infty}(P)$ is said to be *G-equivariant* if $f \circ R^{\Lambda} = L^{\Lambda} \circ f$ for any $\Lambda \in G$. We denote the vector space of all such functions by $C_G^{\infty}(P)$. It can be verified that the map $\beta : C_G^{\infty}(P) \rightarrow \Gamma(E)$ with $\beta f(m) = [\pi, f(p)]$, where p is an arbitrary element of $\pi^{-1}(m)$, is well-defined and possesses a smooth inverse given by $\beta^{-1} s(p) = \varepsilon_p^{-1}(s(\pi(p)))$.

F. Connection 1-form and covariant derivative

We now have the necessary ingredients to construct the connection 1-form on the principal bundle for our gauge theory, as well as the induced gauge field on the base space and covariant derivative on sections of the associated bundle.

In the Abelian setting under study, a *connection 1-form* on the principal bundle $P \xrightarrow{\pi} M$ is a Lie-algebra-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ such that for every point $p \in P$, group element $\Lambda \in G$, and fundamental vector field $W^{\Lambda} \in \Gamma(TP)$, the conditions

$$(\omega W^{\Lambda})_p = \lambda, \quad R^{\Lambda*} \omega = \omega \quad (17)$$

hold. That is, ω recovers the Lie algebra elements generating the fundamental vector fields and is also G -invariant. Note that the invariance condition [the second equation in (17)] is specific to Abelian groups and is replaced by a more general equivariance condition in the non-Abelian setting [see (A2) in Appendix A]. Given a section $\sigma : M \rightarrow P$, the pullback $\omega^{\sigma} := \sigma^* \omega$ of the connection 1-form onto the base space M is known as a *gauge field*. A connection 1-form endows the principal bundle, as well as its associated bundles, with important geometrical structure, including the assignment of a horizontal distribution and a corresponding notion of parallel transport of curves from the base space to the total space. See Appendix A 1 and Refs. 24–26 for further details on these topics.

For our purposes, a key implication of the connection 1-form is that it induces covariant derivative operators on the principal bundle, as well as the Minkowski base space. Specifically, associated with a given connection 1-form ω , a vector field $X \in \Gamma(TM)$, a vector field $Y \in \Gamma(TP)$, and a section $\sigma : M \rightarrow P$ are

1. An *exterior covariant derivative* $\mathcal{D}_Y : C^{\infty}(P) \rightarrow C^{\infty}(P)$, acting on $\mathbb{C}$ -valued functions on the principal bundle and preserving the space of G -equivariant functions $C_G^{\infty}(P)$;
2. A *covariant derivative* $\nabla_X : \Gamma(E) \rightarrow \Gamma(E)$, acting on sections of the associated bundle;
3. A *covariant derivative* $\nabla_X^{\sigma} : C^{\infty}(M) \rightarrow C^{\infty}(M)$, acting on $\mathbb{C}$ -valued functions on the base space.

Among these, the exterior covariant derivative of $f \in C^{\infty}(P)$ with respect to Y is given by

$$\mathcal{D}_Y f = df \cdot \text{hor} Y, \quad (18)$$

where $d : C^{\infty}(P) \rightarrow \Omega^1(P)$ is the canonical exterior derivative and $\text{hor} : \Gamma(TP) \rightarrow \Gamma(TP)$ denotes the *horizontal projection map* associated with ω ,

$$\text{hor} Y_p = Y_p - W_p^{(\omega Y)_p}. \quad (19)$$

Setting Y to the *horizontal lift* of X on P (see [Appendix A 1](#)), the covariant derivative operators ∇_X and ∇_X^σ on base space are then obtained by means of the following commutative diagram:

$$\begin{array}{ccc} C_G^\infty(P) & \xrightarrow{\mathcal{D}_Y} & C_G^\infty(P) \\ \beta^{-1} \uparrow \downarrow \beta & & \beta^{-1} \uparrow \downarrow \beta \\ \Gamma(E) & \xrightarrow{\nabla_X} & \Gamma(E) \\ \zeta_\sigma \uparrow \downarrow \zeta_\sigma^{-1} & & \zeta_\sigma \uparrow \downarrow \zeta_\sigma^{-1} \\ C^\infty(M) & \xrightarrow{\nabla_X^\sigma} & C^\infty(M) \end{array}.$$

That is, we have

$$\nabla_X = \beta \circ \mathcal{D}_Y \circ \beta^{-1}, \quad \nabla_X^\sigma = \zeta_\sigma^{-1} \circ \nabla_X \circ \zeta_\sigma. \quad (20)$$

It can be shown that the covariant derivative ∇_X^σ takes the form

$$\nabla_X^\sigma f = df \cdot X + \rho(\omega^\sigma X)f = Xf + \rho(\omega^\sigma X)f, \quad (21)$$

where $\rho(\omega^\sigma X)$ is a shorthand notation for the $\mathfrak{gl}(1, \mathbb{C})$ -valued function $m \mapsto \rho((\omega^\sigma X)_m)$ on M . This expression is particularly useful for calculational purposes as it involves ordinary complex-valued functions on the base space.

It is evident from (18) and (21) that a covariant derivative deviates from the standard exterior derivative through a “correction” that depends on the connection 1-form, and in the case of ∇_X^σ , that correction presents itself through the gauge field ω^σ and Lie algebra representation ρ . It should be kept in mind that it is the triviality of $P \xrightarrow{\pi} M$ that allows us to work with the globally defined gauge fields. More generally, ω^σ would only be defined locally on the domain of definition of σ , and (21) would be valid locally on the same domain with appropriate compatibility conditions fulfilled in the overlap regions between domains of different local sections.

To construct our connection 1-form ω , we begin by fixing an inertial chart $x: M \rightarrow \mathbb{R}^2$ with origin $o \in M$ and introducing a linear map ${}^\odot: T_p P \rightarrow T_p P$, $p \in P$, on the tangent spaces of the principal bundle, characterized uniquely through the relationships

$$Y_{0,p}^\odot = -Y_{1,p}, \quad Y_{1,p}^\odot = -Y_{0,p}, \quad Y_{2,p}^\odot = Y_{2,p}, \quad (22)$$

where Y_j are the basis vector fields of the associated chart $y: P \rightarrow \mathbb{R}^3$ to x . That is, for any $Y \in T_p P$, we have

$$Y^\odot = \sum_{j=0}^2 (dy_p^j \cdot Y) Y_{j,p}^\odot,$$

and it can be verified that this definition is independent of the choice of inertial chart x (see Lemma 12). Note that $\pi_{*,p}(Y^\odot) = (\pi_{*,p}Y)^\perp$ so that ${}^\odot$ can be interpreted as a lift of the $^\perp$ operator from Sec. III B to the principal bundle. In particular, the notation ${}^\odot$ is suggestive of the fact that $Y^\odot$ has perpendicular components to Y in the Y_0 and Y_1 directions, but the same component in the Y_2 direction so that Y_2 can be thought of as an axis of symmetry remaining unchanged under ${}^\odot$.

Next, given the origin $o \in M$ of the inertial chart x , we consider the quadratic form $h: M \rightarrow \mathbb{R}$, defined as

$$h(m) = \eta(\vec{om}, \vec{om})/2. \quad (23)$$

Explicitly, in terms of the coordinates $x(m) = (x^0(m), x^1(m))$, we have

$$h(m) = [-(x^0(m))^2 + (x^1(m))^2]/2,$$

from which it follows that $h(m)$ vanishes for points m with null translations from o and is negative (positive) for points with timelike (spacelike) translations. With the help of the map $\gamma_{\sigma_x}: P \rightarrow G$ associated with the chart, we also introduce the function $\tilde{h}_x: P \rightarrow \mathbb{R}$ on the principal bundle given by

$$\tilde{h}_x(p) = h(\pi(p)) + \vartheta(\gamma_{\sigma_x}(p)).$$

We then define the Lie-algebra-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ with

$$(\omega Y)_p = (Y^\odot \tilde{h}_x)_p u, \quad \forall Y \in \Gamma(TP), \quad \forall p \in P. \quad (24)$$

One can verify that ω satisfies (17) and is thus a connection 1-form on the principal bundle (see Proposition 13 in [Appendix A 2](#)). Moreover, ω does not depend on the choice of inertial chart x with origin o . As will be discussed in more detail in Sec. III I, by virtue of these facts, the corresponding operators ∇_X and ∇_X^σ in (21) are G -covariant, Lorentz-invariant derivatives.

It is also straightforward to check that the action of ω on the coordinate vector fields Y_j of the chart $y : P \rightarrow \mathbb{R}^3$ induced by x on the principal bundle takes the form

$$(\omega Y_0)_p = -y^1(p)u, \quad (\omega Y_1)_p = y^0(p)u, \quad (\omega Y_2)_p = u, \quad (25)$$

leading to the expression

$$\text{hor} Y = (dy^0 \cdot Y)Y_0 + (dy^1 \cdot Y)Y_1 + ((dy^0 \cdot Y)y^1 - (dy^1 \cdot Y)y^0)Y_2 \quad (26)$$

for the associated horizontal projection map on vector fields. Note that, in general, $\text{hor} Y$ has a nonzero component along the y^2 coordinate, despite the fact that Y_2 is a vertical vector field.

Next, pulling back ω to the base space along the section σ_x yields the gauge field ω^{σ_x} , where

$$(\omega^{\sigma_x} X_0)_m = -x^1(m)u, \quad (\omega^{\sigma_x} X_1)_m = x^0(m)u.$$

Correspondingly, the covariant derivative ∇^{σ_x} on complex-valued functions on Minkowski space is found to satisfy

$$\nabla_{X_0}^{\sigma_x} f = X_0 f - i \frac{\alpha}{\sqrt{2}} x^1 f, \quad \nabla_{X_1}^{\sigma_x} f = X_1 f + i \frac{\alpha}{\sqrt{2}} x^0 f.$$

These relationships can be expressed in a coordinate-free manner as

$$\omega^{\sigma_x} X = X^\perp h u, \quad \nabla^{\sigma_x} = d + i \frac{\alpha}{\sqrt{2}} X^\perp h,$$

where it is evident that ω^{σ_x} (and thus ∇^{σ_x}) is independent of σ_x . Henceforth, we will use the notations $\omega^M \equiv \omega^{\sigma_x}$ and $\nabla^M \equiv \nabla^{\sigma_x}$ to highlight that independence. It can also be readily verified that for all $X \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E)$, the property

$$Xg(s_1, s_2) = g(\nabla_X s_1, s_2) + g(s_1, \nabla_X s_2)$$

holds, which implies that ∇_X is a *metric covariant derivative*. Moreover, ∇_X is a formally skew-symmetric operator with respect to the Hilbert space inner product of $\mathcal{H}$; that is, $\langle \nabla_X s_1, s_2 \rangle_{\mathcal{H}} = -\langle s_1, \nabla_X s_2 \rangle_{\mathcal{H}}$ for all $X \in \Gamma(TM)$ and compactly supported sections $s_1, s_2 \in \Gamma_c(E)$. Using the notation ∇_X^+ to represent the formal adjoint of ∇_X with respect to the $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ inner product, we have $\nabla_X^+ = -\nabla_X$.

Before closing this section, we note that by $C^\infty(M)$ -linearity of the dependence $X \mapsto \nabla_X s$, we can lift ∇_X to an operator $\nabla : \Gamma(E) \rightarrow \Omega^1(M, E) = \Gamma(T_C^* M \otimes_{\mathbb{C}} E)$ such that

$$\nabla s \cdot X = \nabla_X s. \quad (27)$$

Note, in particular, that $T_C^* M \otimes_{\mathbb{C}} E \rightarrow M$ is isomorphic as a bundle to the vector bundle $\text{Hom}(T_C M, E) \rightarrow M$ of bundle homomorphisms from $T_C M$ to E . Thus, ∇s can act on a vector field $X \in \Gamma(T_C M)$, giving (27). Similarly, the exterior covariant derivative $\mathcal{D}_Y$ on the principal bundle lifts to an operator $\mathcal{D} : C^\infty(P) \rightarrow \Omega^1(P)$, and for any vector space F , we can define an exterior covariant derivative $\mathcal{D} : C^\infty(P, F) \rightarrow \Omega^1(P, F)$ analogously to (18). In what follows, for simplicity of notation, we will suppress $\mathbb{C}$ subscripts from $T_C M$, $T_C^* M$, and $\otimes_{\mathbb{C}}$.

G. Laplacians on the associated bundle

We now describe the construction of the Laplace-type operator, which will play the role of a quantum mechanical Hamiltonian operator acting on sections in $\Gamma(E)$, and discuss its relationship to the connection Laplacian induced by the connection in Sec. III F.

First, given a vector field $X \in \Gamma(TM)$, we introduce the complex-conjugate adjoint covariant derivative $\bar{\nabla}_X : \Gamma(E) \rightarrow \Gamma(E)$,

$$\bar{\nabla}_X = J \circ \nabla_X^+ \circ J = -J \circ \nabla_X \circ J,$$

where $J : \Gamma(E) \rightarrow \Gamma(E)$ is the complex structure on sections. Given a trivializing section $\sigma \in \Gamma(P)$ of the principal bundle, this operator has a representation $\bar{\nabla}_X^M : C^\infty(M) \rightarrow C^\infty(M)$ as a differential operator on complex-valued functions on M , where $\bar{\nabla}_X^M = \zeta_\sigma^{-1} \circ \bar{\nabla}_X \circ \zeta_\sigma$ and

$$\bar{\nabla}_X^M f = df \cdot X + \rho(\omega^M X)^* f = df \cdot X - \rho(\omega^M X) f.$$

If, in particular, X_0 and X_1 are the coordinate vector fields of an inertial chart x centered at o , we have

$$\bar{\nabla}_{X_0}^M f = X_0 f + i \frac{\alpha}{\sqrt{2}} x^1 f, \quad \bar{\nabla}_{X_1}^M f = X_1 f - i \frac{\alpha}{\sqrt{2}} x^0 f.$$

As with ∇_X , $\bar{\nabla}_X$ can be extended to an operator $\bar{\nabla} : \Gamma(E) \rightarrow \Omega^1(M, E)$ mapping sections to 1-forms.

Next, let $\nabla^{\text{LC}} : \Gamma(TM) \rightarrow \Omega^1(M)$ be the Levi-Civita connection associated with the Minkowski metric η . Together, $\bar{\nabla}$ and ∇^{LC} induce a *tensor covariant derivative* on the tensor product bundle $T^* M \otimes E$, that is, a covariant derivative operator $\bar{\nabla} : \Gamma(T^* M \otimes E) \rightarrow \Omega^1(M, T^* M \otimes E) = \Gamma(T^* M \otimes T^* M \otimes E)$, such that for any two vector fields $X, Y \in \Gamma(TM)$ and any section $T \in \Gamma(T^* M \otimes E)$,

$$(\bar{\nabla}_X T)Y = \bar{\nabla}_X(T \cdot Y) - T(\nabla_X^{\text{LC}} Y). \quad (28)$$

Note that this definition is consistent with the Leibniz rule, $\tilde{\nabla}_X(T \cdot Y) = (\tilde{\nabla}_X T)Y + T(\nabla_X^{\text{LC}} Y)$. The composition of $\tilde{\nabla}$ and ∇ then leads to the *second covariant derivative* $\tilde{H} : \Gamma(E) \rightarrow \Gamma(T^*M \otimes T^*M \otimes E)$, where $\tilde{H} = \tilde{\nabla} \circ \nabla$. It follows from (28) that for any $X, Y \in \Gamma(TM)$ and $s \in \Gamma(E)$, we have

$$\tilde{H}_{X,Y} s \equiv (\tilde{H}s)(X, Y) = (\tilde{\nabla}_X \nabla_Y) \cdot Y = \tilde{\nabla}_X \nabla_Y s - \nabla_{\nabla_X^{\text{LC}} Y} s.$$

Using Riemannian isomorphisms, we also define a second covariant derivative $\tilde{H}' : \Gamma(E) \rightarrow \Gamma(TM \otimes TM^* \otimes E)$ such that for any vector field $X \in \Gamma(TM)$, 1-form $w \in \Gamma(T^*M)$, and section $s \in \Gamma(E)$,

$$\tilde{H}'_{X,w} s = \tilde{H}_{X,w^\sharp} s.$$

Observe now that $TM \otimes TM^* \rightarrow M$ is isomorphic as a bundle to the vector bundle $\text{End}(TM) \rightarrow M$ of endomorphisms of the tangent bundle. In particular, given $s \in \Gamma(E)$ and $m \in M$, $(\tilde{H}'s)_m$ can be viewed as a linear map $L : T_m M \rightarrow T_m M$ such that for any vector field $X \in \Gamma(TM)$ and 1-form $w \in \Gamma(T^*M)$, $w_m(LX_m) = (\tilde{H}'_{X,w} s)_m$. We can therefore reduce $\tilde{H}'$ to an operator $\tilde{\Delta} : \Gamma(E) \rightarrow \Gamma(E)$ on sections by taking the trace of $(\tilde{H}' \cdot)_m$ at every point $m \in M$, i.e.,

$$(\tilde{\Delta}s)_m = \text{tr}(\tilde{H}'s)_m.$$

That is, in any orthonormal frame $\{X_0, X_1\}$ of TM with $\eta(X_0, X_0) = -1$ and $\eta(X_1, X_1) = 1$, we have

$$\tilde{\Delta} = -(\tilde{\nabla}_{X_0} \nabla_{X_0} - \nabla_{\nabla_{X_0}^{\text{LC}} X_0}) + (\tilde{\nabla}_{X_1} \nabla_{X_1} - \nabla_{\nabla_{X_1}^{\text{LC}} X_1}),$$

and if the X_i are coordinate vector fields associated with an affine coordinate chart (as is nominally the case), the covariant derivatives $\nabla^{\text{LC}} X_i$ vanish, leading to the simpler expression

$$\tilde{\Delta} = \tilde{\nabla}_{X_0} \nabla_{X_0} - \tilde{\nabla}_{X_1} \nabla_{X_1}. \quad (29)$$

To characterize this operator more explicitly, it is useful to consider its representation $\tilde{\Delta}^M : C^\infty(M) \rightarrow C^\infty(M)$ on complex-valued functions on M induced by the trivializing section σ_x , i.e., $\tilde{\Delta}^M = \zeta_{\sigma_x}^{-1} \circ \tilde{\Delta} \circ \zeta_{\sigma_x}$. A direct calculation then yields

$$\tilde{\Delta}^M = -\left(-X_0^2 + \frac{\alpha^2}{2}(x^0)^2\right) + \left(-X_1^2 + \frac{\alpha^2}{2}(x^1)^2\right). \quad (30)$$

Noting that the coordinate vector fields X_0 and X_1 are formally skew-symmetric as operators on functions, i.e., $\langle f_1, X_i f_2 \rangle_v = -\langle X_i f_1, f_2 \rangle_v$ for all $f_1, f_2 \in C_c^\infty(M)$, it follows from (30) that $\tilde{\Delta}^M$ (and thus $\tilde{\Delta}$) is formally symmetric, $\langle f_1, \tilde{\Delta}^M f_2 \rangle_v = \langle \tilde{\Delta}^M f_1, f_2 \rangle_v$. Moreover, it is evident that up to a proportionality factor of $1/2$, $\tilde{\Delta}^M$ has the structure of the difference between the Hamiltonians of two one-dimensional quantum harmonic oscillators of frequency α , operating along the x^0 and x^1 coordinates, respectively. We will pursue this correspondence further in Sec. IV C, where we will use a (self-adjoint extension of) $\tilde{\Delta}/2$ as the Hamiltonian of a quantum mechanical theory with $\mathcal{H}$ as its Hilbert space.

For now, we express $\tilde{\Delta}$ in an alternative form, which also exemplifies its relationship with a harmonic oscillator potential. For that, we introduce the *connection Laplacian* $\Delta : \Gamma(E) \rightarrow \Gamma(E)$ on sections of the associated bundle. This operator is defined analogously to $\tilde{\Delta}$ as

$$(\Delta s)_m = \text{tr}(\mathcal{H}'s)_m, \quad \forall m \in M,$$

where $\mathcal{H}' : \Gamma(E) \rightarrow \Gamma(TM \otimes TM^* \otimes E)$ is obtained via Riemannian isomorphisms from the operator $\mathcal{H} : \Gamma(E) \rightarrow \Gamma(TM^* \otimes TM^* \otimes E)$, $\mathcal{H} = \hat{\nabla} \circ \nabla$. Here, $\hat{\nabla} : \Gamma(T^*M \otimes E) \rightarrow \Gamma(T^*M \otimes T^*M \otimes E)$ is a tensor covariant derivative, defined analogously to (28) as

$$(\hat{\nabla}_X T)Y = \nabla_X(T \cdot Y) - T(\nabla_X^{\text{LC}} Y), \quad \forall X, Y \in \Gamma(TM).$$

In other words, the difference between Δ and $\tilde{\Delta}$ is that the former is constructed without using complex conjugation and adjoints.

A calculation analogous to that used to obtain (30) then shows that the function Laplacian $\Delta^M := \zeta_{\sigma_x}^{-1} \circ \Delta \circ \zeta_{\sigma_x}$ induced by σ_x is given by

$$\Delta^M = -\left(-X_0^2 - \frac{\alpha^2}{2}(x^0)^2\right) + \left(-X_1^2 - \frac{\alpha^2}{2}(x^1)^2\right). \quad (31)$$

Comparing (30) and (31), it then follows that

$$\tilde{\Delta}^M = \Delta^M + \alpha^2(-(x^0)^2 + (x^1)^2),$$

or, equivalently,

$$\tilde{\Delta} = \Delta + 2v, \quad v = \alpha^2 h, \quad (32)$$

where h is the quadratic form from (23). Thus, if $\Delta/2$ is interpreted as a “free-particle” Hamiltonian, then $\tilde{\Delta}/2$ corresponds to the Hamiltonian for a particle in a quadratic potential v , with the frequency parameter α . Note the distinguished role that the origin o of the coordinate chart x plays in this definition, as it is the unique saddle point of v .

H. Curvature and Yang-Mills equations

Thus far, we have identified a connection 1-form ω on the inertial frame bundle $P \xrightarrow{\pi} M$, which naturally leads to geometrical Laplace-type operators having the structure of quantum mechanical Hamiltonians with quadratic potentials. Yet, the specific choice of ω in (24) may appear somewhat *ad hoc*, especially given the fact that the space of connections on a principal bundle is an infinite-dimensional (i.e., “large”) space. More specifically, it can be shown that the space of connections, $\mathcal{C}$, of a general principal bundle $P \xrightarrow{\pi} M$ is an infinite-dimensional affine space whose translation group is isomorphic to a vector bundle $\text{Ad}P \xrightarrow{\pi_{\text{Ad}P}} M$ over M , with typical fiber isomorphic to the Lie algebra $\mathfrak{g}$, known as the *adjoint bundle* (see Appendix A 1). That is, the difference $\omega - \omega' \in \Omega^1(P, \mathfrak{g})$ between any two connections in $\mathcal{C}$ can be identified with a unique $\text{Ad}P$ -valued 1-form (equivalently, a section of $TM^* \otimes \text{Ad}P$), and conversely, given any connection $\omega \in \mathcal{C}$, any other connection can be reached by translating from ω by sections in $\Omega^1(M, \text{Ad}P)$.

For an Abelian structure group such as $G \simeq \text{SO}^+(1, 1)$ studied here, $\text{Ad}P \xrightarrow{\pi_{\text{Ad}P}} M$ is canonically isomorphic to the trivial bundle $M \times \mathfrak{g} \xrightarrow{\pi_1} M$ so that every section in $\Gamma(\text{Ad}P)$ can be identified with a unique $\mathfrak{g}$ -valued 1-form on M . Correspondingly, we can express every element of $\mathcal{C}$ as $\omega_o + \pi^* \bar{\omega}$, where $\omega_o \in \mathcal{C}$ is fixed connection 1-form acting as the “origin” and $\bar{\omega} \in \Omega^1(M, \mathfrak{g})$ is a Lie-algebra-valued 1-form on the base space. For instance, choosing ω_o as the trivial connection with coordinate basis values [cf. (25)],

$$\omega_o Y_0 = \omega_o Y_1 = 0, \quad \omega_o Y_2 = u,$$

and the horizontal projection map [cf. (26)],

$$\text{hor}_o Y = (dy^0 \cdot Y) Y_0 + (dy^1 \cdot Y) Y_1,$$

we can express the connection 1-form from (24) as

$$\omega = \omega_o + \pi^* \bar{\omega}, \quad \bar{\omega} X = X^\perp h u.$$

In gauge theories for fundamental physics, the problem of identifying gauge field configurations realized in nature is approached through the *Yang-Mills equations*²⁷ involving a *field strength* associated with the connection 1-form. Given a connection 1-form $\omega \in \mathcal{C}$, the field strength employed in Yang-Mills theory is a 2-form $F^\omega \in \Omega^2(M, \text{Ad}P)$, taking values in the adjoint bundle, which can be constructed by pulling back the *curvature 2-form* associated with ω along trivializing sections of the principal bundle. The latter is defined as the Lie-algebra-valued 2-form $\Omega \in \Omega^2(P, \mathfrak{g})$, given by the exterior covariant derivative of the connection 1-form, i.e., $\Omega = \mathcal{D}\omega$, where

$$\Omega(Y, Z) = d\omega(\text{hor}Y, \text{hor}Z), \quad Y, Z \in \Gamma(TP).$$

Using $\nabla: \Omega^k(M, \text{Ad}P) \rightarrow \Omega^{k+1}(M, \text{Ad}P)$ to denote the covariant derivative induced on the adjoint bundle by ω (constructed analogously to the covariant derivative on $E \xrightarrow{\pi_E} M$ from Secs. III F and III G), the field strength can be shown to satisfy the *Bianchi identity*,

$$\nabla F^\omega = 0. \quad (33)$$

In Yang-Mills theory, F^ω is further required to satisfy

$$\nabla \star F^\omega = 0, \quad (34)$$

where $\star: \Omega^k(M, \text{Ad}P) \rightarrow \Omega^{\dim M - k}(M, \text{Ad}P)$ is the Hodge star operator. Equation (34) is derived by extremizing a quadratic functional of F^ω known as the *Yang-Mills action*. See Appendix A 1 e for further details, and one of the many references in the literature, e.g., Ref. 26, for a complete exposition of Yang-Mills theory.

In the general, non-Abelian, setting, (34) represents a set of nonlinear partial differential equations, known as Yang-Mills equations, whose rigorous characterization, including existence of solutions, is a challenging open problem (a Millennium Prize problem²⁸). However, in the present setting involving an Abelian gauge theory on two-dimensional Minkowski space, (33) and (34) greatly simplify to a linear, first-order differential equation for a single Lie-algebra-valued function,

$$df^\omega = 0, \quad f^\omega \in C^\infty(M, \mathfrak{g}) \quad (35)$$

such that $F^\omega = \star f^\omega$. The reduction to (35) is a consequence of the following three simplifications:

1. For an Abelian structure group such as $G \simeq \text{SO}^+(1, 1)$, the adjoint bundle $\text{Ad}P \xrightarrow{\pi_{\text{Ad}P}} M$ is trivial and canonically isomorphic to $M \times \mathfrak{g} \xrightarrow{\pi_1} M$ (as already stated). As a result, we can canonically identify $F^\omega \in \Omega^2(M, \text{Ad}P)$ with a unique Lie-algebra-valued 2-form $\tilde{F}^\omega \in \Omega^2(M, \mathfrak{g})$. Moreover, because the principal bundle $P \xrightarrow{\pi} M$ over Minkowski space is trivial, $\tilde{F}^\omega$ is given by the pullback $\tilde{F}^\omega = \sigma^* \Omega$ of the curvature 2-form along any global section $\sigma \in \Gamma(P)$.
2. For an Abelian structure group, the covariant derivative on $\text{Ad}P$ reduces to a standard exterior derivative (due to the triviality of the adjoint representation of $\mathfrak{g}$; see Appendix A 1 d). Thus, (34) reduces to the linear equation, $d \star \tilde{F}^\omega = 0$.
3. For a two-dimensional manifold M such as two-dimensional Minkowski space, F^ω is a top form, and thus, the Bianchi identity is trivially satisfied by any element of $\Omega^2(M, \mathfrak{g})$ (even if it is not the field strength induced by a connection). Moreover, $\star \tilde{F}^\omega$ is equal to a Lie-algebra-valued function, $f^\omega \in C^\infty(M, \mathfrak{g})$, giving $\tilde{F}^\omega = \star f^\omega$, since $\star \star = \text{Id}$ for 2-forms in two dimensions. Equation (35) then follows.

The simplified Yang–Mills equation in (35) can be thought of as a two-dimensional analog of Maxwell’s equations for electromagnetism. Since Minkowski space is a connected manifold, the *only* solutions of this equation are constant functions, $f^\omega(m) = \lambda_0$, for some $\lambda_0 \in \mathfrak{g}$. To verify that the connection 1-form from (24) indeed induces a constant f^ω , let $\gamma : P \rightarrow \mathbb{R}^3$ be a coordinate chart on the principal bundle induced by an inertial chart $x : M \rightarrow \mathbb{R}^2$, and consider the matrix representation of the curvature 2-form with respect to the corresponding coordinate vector fields Y_j , i.e., $\Omega = [\Omega(Y_i, Y_j)]$. A direct calculation yields

$$\Omega = \begin{pmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $f \in C^\infty(M, \mathfrak{g})$ is a constant Lie-algebra-valued function, equal to $\lambda_0 = 2u$. As a result, $\tilde{F}^\omega = \sigma_x^* \Omega$ is represented by the matrix

$$F := [\tilde{F}^\omega(X_i, X_j)] = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix},$$

from which we deduce that $f^\omega = \star \tilde{F}^\omega = f$ is a constant function. We therefore conclude that the connection 1-form employed in this work satisfies the Yang–Mills equations.

I. Gauge and Lorentz transformations

In this section, we examine the transformation properties of the gauge theory constructed above under two types of geometrical transformations, namely, gauge and Lorentz transformations.

Starting from the former, a *gauge transformation* is a principal-bundle isomorphism of $P \xrightarrow{\pi} M$, that is, a diffeomorphism $\varphi : P \rightarrow P$ satisfying $\pi \circ \varphi = \pi$ and $\Psi(p \cdot \Lambda) = \Psi(p) \cdot \Lambda$ for every $p \in P$ and $\Lambda \in G$. These properties are represented by the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ R^\Lambda \uparrow & & R^\Lambda \uparrow \\ P & \xrightarrow{\varphi} & P \\ \downarrow \pi & \swarrow \pi & \\ M & & \end{array} \quad (36)$$

The *gauge group* $\mathcal{G}$ of $P \xrightarrow{\pi} M$ is the group formed by all such maps φ with composition of maps as the group multiplication. In the present context, where P represents the collection of all inertial frames over M (see Sec. III D), a gauge transformation can be thought of as a change of the inertial reference frame.

Every gauge transformation $\varphi \in \mathcal{G}$ induces a bundle isomorphism of the associated bundle $E \xrightarrow{\pi_E} M$. Specifically, there is a diffeomorphism $\varphi_* : E \rightarrow E$, defined as $\varphi_*([p, z]) = [\varphi(p), z]$, making the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{\varphi_*} & E \\ \downarrow \pi_E & \swarrow \pi_E & \\ M & & \end{array} \quad .$$

Note that the well-definition of φ_* as a map on G -equivalence-classes in E depends on the G -equivariance of φ .

Next, the map φ_* induces a map $\tilde{\varphi}_* : \Gamma(E) \rightarrow \Gamma(E)$, acting on sections from the left,

$$\tilde{\varphi}_* s = \varphi_* \circ s. \quad (37)$$

Given a section $\sigma : M \rightarrow P$ of the principal bundle, $\tilde{\varphi}_*$ induces, in turn, a map $\tilde{\varphi}_*^\sigma : C^\infty(M) \rightarrow C^\infty(M)$ on $\mathbb{C}$ -valued functions on M such that

$$\tilde{\varphi}_*^\sigma = \zeta_\sigma^{-1} \circ \tilde{\varphi}_* \circ \zeta_\sigma. \quad (38)$$

Henceforth, we will abbreviate $\tilde{\varphi}_*$ and $\tilde{\varphi}_*^\sigma$ by φ_* and φ_*^σ for simplicity of notation. It can then be shown that for any $X \in \Gamma(TM)$, the covariant derivative ∇_X is *gauge-covariant*, i.e., $\nabla_X \circ \varphi_* = \varphi_* \circ \nabla_X$. The latter implies that the Laplace-type operators Δ and $\tilde{\Delta}$ are also gauge-covariant,

$$\Delta \circ \varphi_* = \varphi_* \circ \Delta, \quad \tilde{\Delta} \circ \varphi_* = \varphi_* \circ \tilde{\Delta}.$$

Similarly, the corresponding operators induced by the section σ on $\mathbb{C}$ -valued functions on M satisfy $\nabla_X^M \circ \varphi_*^\sigma = \varphi_*^\sigma \circ \nabla_X^M$, $\Delta^M \circ \varphi_*^\sigma = \varphi_*^\sigma \circ \Delta^M$, and $\tilde{\Delta}^M \circ \varphi_*^\sigma = \varphi_*^\sigma \circ \tilde{\Delta}^M$, respectively.

The above highlight an important difference between arbitrary complex-valued functions on M on the one hand and sections $s \in \Gamma(E)$ and their associated functions $s^\sigma = \zeta_\sigma^{-1} s$ on the other hand, namely, that the former have no canonical behavior under gauge

transformations [as indicated by the commutative diagram in (36), where there is no action on M], whereas the latter transform non-trivially according to (37) and (38). In fact, as we now discuss, gauge transformations characterize the behavior of gauge fields, matter fields, and other geometrical objects defined on the base space M under changes of section $\sigma: M \rightarrow P$ of the principal bundle, which in the present context corresponds to a choice of inertial frame. Intuitively, all matter fields s^σ corresponding to the same underlying section s represent the same intrinsic physical configuration, and it is precisely the transformation properties in (38) that ensure the mutual consistency of these representations.

To examine this in more detail, we introduce the *nonlinear adjoint bundle* $\text{ADP} \xrightarrow{\pi_{\text{ADP}}} M$, which is a fiber bundle over M associated with the principal bundle, with typical fiber G (see Appendix A 1). The gauge group $\mathcal{G}$ can then be identified with sections in $\Gamma(\text{ADP})$. In the present setting involving the Abelian group $G \simeq \text{SO}^+(1, 1)$, ADP is a trivial bundle canonically isomorphic to $M \times G$, and every gauge transformation $\varphi \in \mathcal{G} \simeq \Gamma(\text{ADP})$ can be identified with a unique G -valued function ξ on M , sometimes referred to as a *gauge map*, such that $\varphi(p) = p \cdot \xi(\pi(p))$. Conversely, every smooth function $\xi: M \rightarrow G$ induces a gauge transformation φ defined by the same formula. It then follows from the definition of the left action in (15) that the induced transformation φ_* on the associated bundle $E \xrightarrow{\pi_E} M$ acts by multiplication by a spatially dependent phase factor, viz.,

$$\varphi_* e = e^{i\alpha\vartheta(\xi(m))/\sqrt{2}} e, \quad m \in M, \quad e \in E_m. \quad (39)$$

See Lemma 14 in Appendix A 2 for additional details.

Let now $\sigma, \sigma': M \rightarrow P$ be sections of the principal bundle and $\xi: M \rightarrow G$ be the unique gauge map such that $\sigma'(m) = \sigma(m) \cdot \xi(m)$ for all $m \in M$ (note that the uniqueness of ξ is a consequence of the fact that the action of G on P is free). Associated with ξ is a gauge transformation $\varphi: P \rightarrow P$, constructed as described above, such that $\sigma' = \varphi \circ \sigma$. This transformation represents a change of inertial frame from σ to σ' . Given a section $s \in \Gamma(E)$, it is a direct consequence of (39) (see also Lemma 14) that the corresponding matter fields $s^\sigma, s^{\sigma'} \in C^\infty(M)$ transform according to

$$s^{\sigma'}(m) = \varphi_*^{-1} s^\sigma(m) = e^{-i\alpha\vartheta(\xi(m))/\sqrt{2}} s^\sigma(m). \quad (40)$$

Equation (40) shows that matter fields transform in a contravariant (“inverse”) manner under the gauge transformation describing the change of section from σ to σ' . This behavior is analogous to the transformation rule for the components of vectors in Euclidean space under a change of basis, which acts by the inverse of the transformation of the basis vectors (the analogs of the inertial frames studied here). As stated above, we view s^σ and $s^{\sigma'}$ as representations of the same intrinsic physical configuration, i.e., the section s , which should transform according to (40) for consistency.

It should be noted that the transformation in (40) is analogous to the transformation of the quantum mechanical wavefunction of a nonrelativistic charged particle moving in an electromagnetic field under gauge transformations of the scalar and vector potential, e.g., Ref. 23 (Sec. 2.6). This should be of no surprise, since electromagnetism is a $U(1)$ gauge theory, and the representation $\rho: G \rightarrow U(1)$ used to construct $E \xrightarrow{\pi_E} M$ acts as the universal cover of $U(1)$.

Next, we turn to Lorentz transformations of the base space, represented by the affine maps $L_o^\Lambda: M \rightarrow M$ from Sec. III B. Here, it is important that the origin $o \in M$ is the same as in the definition of the connection 1-form ω in Sec. III F. This type of transformation is distinct from gauge transformations, in the sense that the latter describe how matter or gauge fields (e.g., sections in $\mathcal{H}$) present themselves under different choices of inertial frame, whereas the former describe how the structure of the theory itself (“physical laws”) depends on the choice of inertial coordinate chart. Here, the key structural object that could be potentially affected by the choice of inertial chart is the connection 1-form. However, as shown in Proposition 13 and already stated in Sec. III F, ω is, in fact, independent of the choice of inertial chart, so long as that chart is centered at o . Thus, the gauge theory employed in this work is invariant under Lorentz transformations with that point as the origin. Clearly, due to the distinguished role that o plays, the theory is not invariant under translations in the Poincaré group.

IV. QUANTUM DYNAMICS

In this section, we construct a quantum mechanical system associated with the gauge theory on Minkowski space from Sec. III. Then, we describe an embedding of the Koopman operator formulation of the classical harmonic oscillator from Sec. II into the quantum system, thereby proving Theorem 1(i). Our construction of the quantum system on Minkowski space will be based on the standard density operator formulation of quantum mechanics, e.g., Ref. 23. This choice is, in part, motivated from the fact that the density operator formalism provides a useful model for statistical inference in measure-preserving deterministic systems.⁹ In particular, here we will not employ the canonical quantization approach employed in the context of quantum field theory, whereby one would treat solutions $\hat{\Delta}s = 0$ associated with the Laplacian in (29) as being “classical” and then promoting each of the normal mode expansion coefficients of these solutions to ladder operators for a quantum field. Instead, we will treat a (self-adjoint extension of) $\hat{\Delta}$ directly as a quantum Hamiltonian without reference to an underlying classical system. When there is no risk of confusion with the usage of that term in other contexts, we will refer to the quantum system on Minkowski space as a quantum harmonic oscillator.

A. Quantum dynamics on Minkowski space

Let $\mathcal{S}(M, E)$ be the Schwartz space of rapidly decreasing sections in $\Gamma(E)$, defined using any inertial chart $x : M \rightarrow \mathbb{R}^2$ and corresponding trivializing section $\sigma_x \in \Gamma(P)$ as

$$\mathcal{S}(M, E) = \left\{ s \in \Gamma(E) : \sup_{m \in M} |x(m)^\alpha \partial_{x(m)}^\beta (s \circ \zeta_{\sigma_x}^{-1} \circ x)| < \infty, \forall \alpha, \beta \in \mathbb{N}^n, n \in \mathbb{N}_0 \right\}.$$

The operator $\tilde{H} : \mathcal{S}(M) \rightarrow \mathcal{H}$

$$\tilde{H} = \frac{1}{2} \tilde{\Delta} = \frac{1}{2} \Delta + v,$$

is then a symmetric operator with a dense domain $\mathcal{S}(M) \subset \mathcal{H}$. Similarly, we let $\mathcal{S}(M) \subset L^2(v)$ be the Schwartz space of complex-valued functions on M and $\tilde{H}^M : \mathcal{S}(M) \rightarrow L^2(v)$ be the densely defined, symmetric operator equal to $\tilde{\Delta}^M/2$. As in Sec. III, we always consider that x is centered at $o \in M$ so that the various coordinate-based formulas for covariant derivatives and Laplace operators from Sec. III F apply.

It can then be readily verified from (29) and (30) that $\tilde{H}$ is diagonalizable in an orthonormal basis of eigensections associated with tensor products of Hermite functions in the x^0 and x^1 coordinates. Specifically, let $p_j \in C^\infty(\mathbb{R})$ be the Hermite polynomial of degree $j \in \mathbb{N}_0$ and $\chi_j \in C^\infty(\mathbb{R})$ be the Hermite function with

$$\chi_j(z) = \frac{1}{\sqrt{2^j j!}} \left(\frac{\alpha}{\pi} \right)^{1/4} e^{-\alpha^2 z^2 / 2} p_j(\alpha^{1/2} z).$$

As is well known, the set $\{\chi_j\}_{j=0}^\infty$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$ associated with the Lebesgue measure on $\mathbb{R}$, consisting of eigenfunctions of the quantum harmonic oscillator Hamiltonian.^{23,29} That is,

$$H_0 \chi_j = E_j \chi_j, \quad E_j = (2j + 1)\alpha/2, \quad j \in \mathbb{N}_0,$$

where $H_0 : \mathcal{S}(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is defined on the Schwartz space $\mathcal{S}(\mathbb{R})$ as

$$H_0 f(z) = \frac{1}{2} f''(z) + \frac{\alpha^2}{2} z^2 f(z).$$

It then follows from (29) and the fact that $L^2(v) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ that the functions $\chi_{jk} = \chi_j \otimes \chi_k$ form an orthonormal basis $\{\chi_{jk}\}_{j,k=0}^\infty$ of $L^2(v)$ consisting of eigenfunctions of $\tilde{H}^M$, corresponding to the eigenvalues

$$E_{jk} = (k + j)\alpha.$$

Correspondingly, $\{\psi_{jk}\}_{j,k=0}^\infty$, with $\psi_{jk} = \zeta_\sigma \chi_{jk}$, is an orthonormal basis of $\mathcal{H}$ consisting of eigensections of $\tilde{H}$ at the same corresponding eigenvalues E_{jk} .

Based on the above, we can conclude that $\tilde{H}$ is a densely defined, symmetric, diagonalizable operator and therefore has a unique self-adjoint extension $H : D(H) \rightarrow \mathcal{H}$ defined on the domain $D(H) \supset \mathcal{S}(M, E)$ with

$$D(H) = \left\{ \sum_{j,k=0}^\infty c_{jk} \psi_{jk} \in \mathcal{H} : \sum_{j,k=0}^\infty E_{jk}^2 |c_{jk}|^2 < \infty \right\}.$$

This operator acts as the Hamiltonian of a quantum system, generating a strongly continuous, unitary group of Heisenberg evolution operators $W^t : \mathcal{H} \rightarrow \mathcal{H}$, given by

$$W^t = e^{itH}. \quad (41)$$

Following the standard formulation of quantum mechanics, we will consider the states of this quantum system to be non-negative, trace-class operators on $\mathcal{H}$ with unit trace. Specifically, the set of *regular quantum states* on $\mathcal{H}$ is the closed, convex subset of the Banach space $B_1(\mathcal{H})$ of trace-class operators on $\mathcal{H}$, equipped with the trace norm, $\|A\|_1 = \text{tr}|A|$, given by

$$\mathcal{Q}(\mathcal{H}) = \{\rho \in B_1(\mathcal{H}) : \rho \geq 0, \text{tr} \rho = 1\}.$$

We recall that a state $\rho \in \mathcal{Q}(\mathcal{H})$ is said to be *pure* if there exists $s \in \mathcal{H}$ such that $\rho = \langle \cdot, s \rangle_{\mathcal{H}} s$. That is, every pure state is a rank-1, orthogonal projection on $\mathcal{H}$, $\rho^2 = \rho$, and we have $\|\rho\|_1 = \|\rho\| = 1$, where $\|\cdot\|$ denotes the $\mathcal{H}$ -operator norm. The states in $\mathcal{Q}(\mathcal{H})$ that are not pure are said to be *mixed*.

The unitary evolution group W^t acts on states in $\mathcal{Q}(\mathcal{H})$ through the conjugation map $Z^t : \mathcal{Q}(\mathcal{H}) \rightarrow \mathcal{Q}(\mathcal{H})$, defined as

$$Z^t(\rho) = W^{t*} \rho W^t. \quad (42)$$

As usual, the observables of the quantum system will be the (bounded and unbounded) self-adjoint operators on $\mathcal{H}$. In what follows, we will typically restrict attention to bounded observables in the set

$$\mathcal{A}(\mathcal{H}) = \{A \in B(\mathcal{H}) : A^* = A\},$$

where $B(\mathcal{H})$ is the Banach space of bounded operators on $\mathcal{H}$, equipped with the operator norm, $\|\cdot\|$. It can be readily verified that $\mathcal{A}(\mathcal{H})$ is closed under addition of operators, scalar multiplication by real numbers, and the multiplication operation,

$$A \cdot B = B \cdot A = (AB + BA)/2.$$

Furthermore, we have $\|A \cdot B\| \leq \|A\| \|B\|$ so that $\mathcal{A}(\mathcal{H})$ has the structure of an Abelian Banach algebra of quantum observables over the real numbers. On $\mathcal{A}(\mathcal{H})$, the quantum system has an induced linear action $\tilde{W}^t : \mathcal{A}(\mathcal{H}) \rightarrow \mathcal{A}(\mathcal{H})$,

$$\tilde{W}^t A = W^t A W^{t*}, \quad (43)$$

which preserves the operator norm. We also let $\mathcal{A}'(\mathcal{H})$ be the Banach space of continuous linear functionals on $\mathcal{A}(\mathcal{H})$, equipped with the standard (operator) norm.

The space of quantum states $\mathcal{Q}(\mathcal{H})$ embeds naturally into $\mathcal{A}'(\mathcal{H})$ through the continuous map $\iota : \mathcal{Q}(\mathcal{H}) \rightarrow \mathcal{A}'(\mathcal{H})$ such that $\iota(\rho) := \mathbb{E}_\rho$ is equal to the quantum mechanical expectation functional given by

$$\mathbb{E}_\rho A = \text{tr}(\rho A). \quad (44)$$

Note that the continuity of $\mathbb{E}_\rho$ follows from the fact that

$$|\mathbb{E}_\rho A| \leq \|\rho\|_1 \|A\| = \|A\|.$$

In fact, it can be readily verified that ι is a contraction, i.e., $\|\iota(\rho)\| \leq \|\rho\|_1 = 1$, where the equality is saturated by pure states. We shall refer to the elements of $\mathcal{A}'(\mathcal{H})$ as *generalized quantum states*.

Intuitively, $A \mapsto \mathbb{E}_\rho A$ can be thought of as “evaluation” of the quantum mechanical observable A at the state ρ . It follows directly from (42) and (43) that the property $\mathbb{E}_{Z^t(\rho)} = \mathbb{E}_\rho \circ \tilde{W}^t$ holds for all $t \in \mathbb{R}$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Q}(\mathcal{H}) & \xrightarrow{Z^t} & \mathcal{Q}(\mathcal{H}) \\ \downarrow \iota & & \downarrow \iota \\ \mathcal{A}'(\mathcal{H}) & \xrightarrow{\tilde{W}^t} & \mathcal{A}'(\mathcal{H}), \end{array}$$

where $\tilde{W}^t : \mathcal{A}'(\mathcal{H}) \rightarrow \mathcal{A}'(\mathcal{H})$ is the composition map by $\tilde{W}^t$, acting isometrically on $\mathcal{A}'(\mathcal{H})$ according to the formula

$$\tilde{W}^t J = J \circ \tilde{W}^t. \quad (45)$$

This completes the construction of our quantum mechanical system on Minkowski space. It should be noted that the dynamics of this system, generated by H , differ from a spherical quantum harmonic oscillator of frequency α in two-dimensional Euclidean space in two key ways as follows:

1. The energy levels E_{jk} range from $-\infty$ to ∞ , as opposed to the Euclidean case, where the energy levels are strictly positive, and have the ground-state eigenvalue α .
2. The eigenspaces corresponding to any energy level E_{jk} are infinite-dimensional, whereas eigenspaces of the harmonic oscillator in Euclidean space are all finite-dimensional.

The evolution group generated by H also has a fundamental difference from the Koopman operator group U^t for the classical harmonic oscillator generated by V . That is, unlike U^t , which is realized through the deterministic flow Φ^t on the circle via (1), the Heisenberg operators W^t in (41) are *not* the realization of a flow on Minkowski space, i.e., there exists no measurable flow $\hat{\Phi}^t : M \rightarrow M$ such that $W^t f = f \circ \hat{\Phi}^t$. Indeed, a necessary condition for that to happen is that H acts as a derivation on an algebra of bounded sections contained in $D(H)$ (Ref. 21) (Theorem 1.1), and this is clearly not the case since H is a second-order operator that does not obey a Leibniz rule [cf. (2)].

Yet, as alluded to in Sec. I, the Koopman and Heisenberg evolution groups studied here do have an important structural similarity, namely, that, up to multiplication by i , the spectra $\{i\alpha_j = ij\alpha\}_{j=-\infty}^{\infty}$ and $\{E_{jk} = (k-j)\alpha\}_{j,k=0}^{\infty}$ of their generators, respectively, are identical. In Sec. IV C ahead, we will take advantage of this spectral equivalence to construct an isometric embedding of the Hilbert space $L^2(\mu)$ associated with the classical harmonic oscillator into the Hilbert space of sections $\mathcal{H}$ that maps the generator V to the Hamiltonian H . This construction will be based on a quantum mechanical representation of the periodic dynamics of the circle rotation, which we describe next.

B. Quantum dynamics of the circle rotation

Following Ref. 9, we assign quantum states to the classical harmonic oscillator as positive, unit-trace operators on $L^2(\mu)$, that is, operators lying in the set

$$\mathcal{D}(L^2(\mu)) = \{\rho \in B_1(L^2(\mu)) : \rho \geq 0, \text{tr} \rho = 1\},$$

on which the Koopman group acts by the conjugation map $\tilde{\Phi}^t : \mathcal{D}(L^2(\mu)) \rightarrow \mathcal{D}(L^2(\mu))$,

$$\tilde{\Phi}^t(\rho) = U^{t*} \rho U^t.$$

Note that we use the symbol $\tilde{\Phi}^t$ to indicate that the dynamics on the quantum states $\mathcal{D}(L^2(\mu))$ is induced by the classical flow Φ^t on the circle [cf. Z^t from (42), where no such correspondence exists].

Next, we assign quantum observables as self-adjoint operators on $L^2(\mu)$. Proceeding as in Sec. IV A, we will focus on the space

$$\mathcal{A}(L^2(\mu)) = \{A \in B(L^2(\mu)) : A^* = A\}$$

of bounded self-adjoint operators, which becomes an Abelian Banach algebra over the real numbers analogously to $\mathcal{A}(\mathcal{H})$. We define the continuous dual $\mathcal{A}'(L^2(\mu))$ of $\mathcal{A}(L^2(\mu))$, as well as the expectation functionals $\mathbb{E}_\rho \in \mathcal{A}'(L^2(\mu))$, evolution maps $\tilde{U}^t : \mathcal{A}(L^2(\mu)) \rightarrow \mathcal{A}(L^2(\mu))$, $\tilde{U}^{t*} : \mathcal{A}'(L^2(\mu)) \rightarrow \mathcal{A}'(L^2(\mu))$, and the inclusion map $\iota : \mathcal{D}(L^2(\mu)) \rightarrow \mathcal{A}'(L^2(\mu))$ analogously to their counterparts for $\mathcal{A}(\mathcal{H})$ and $\mathcal{A}'(\mathcal{H})$ in Sec. IV.

We now describe an embedding of the classical rotation on the circle to the quantum system introduced above, at the level of both states and observables. For that, consider first the Abelian Banach algebra $C_{\mathbb{R}}(S^1)$ consisting of continuous, real-valued functions on the circle, equipped with the maximum norm. This algebra embeds homomorphically into $\mathcal{A}(L^2(\mu))$ through the mapping $T : C_{\mathbb{R}}(S^1) \rightarrow \mathcal{A}(L^2(\mu))$ that sends the continuous, real-valued function $f \in C_{\mathbb{R}}(S^1)$ to the self-adjoint multiplication operator $T_f := T f \in \mathcal{A}(L^2(\mu))$ that multiplies by that function, i.e., $T_f g = f g$ for all $g \in L^2(\mu)$. It is then straightforward to verify that

$$\tilde{U}^t \circ T = T \circ U^t, \quad \forall t \in \mathbb{R}. \quad (46)$$

The above equation shows that continuous, real-valued classical observables evolve consistently under the action of the Koopman operator with their corresponding quantum observables (multiplication operators in $\mathcal{A}(L^2(\mu))$). Similarly, the transpose map $T' : \mathcal{A}'(L^2(\mu)) \rightarrow C'_{\mathbb{R}}(S^1)$, given by $T' J = J \circ T$, satisfies

$$T' \circ \tilde{U}^{t*} = U^{t*} \circ T', \quad \forall t \in \mathbb{R}. \quad (47)$$

Clearly, one could carry out a similar construction for the quantum system on Minkowski space M , replacing $C_{\mathbb{R}}(S^1)$ with the Banach space of real-valued, continuous functions on M vanishing at infinity. In the case of the circle rotation, however, additional correspondences can be derived with the help of the heat kernel (see Sec. II D).

Proposition 4. Let $F_\tau : S^1 \rightarrow \mathcal{F}(\mathcal{H}_\tau) \subset \mathcal{H}_\tau$ be the feature map associated with the heat kernel on the circle at time parameter $\tau > 0$. Let also $\Pi : C(S^1) \setminus \{0\} \rightarrow \mathcal{D}(L^2(\mu))$ map the nonzero classical observable f to the pure quantum state $\Pi(f) = \rho_f$ given by $\rho_f = \langle f, \cdot \rangle_\mu f / \|f\|_\mu^2$. Then, the following hold for every $\tau > 0$ and $t \in \mathbb{R}$:

- (i) $\Psi_\tau : S^1 \rightarrow \mathcal{D}(L^2(\mu))$, $\Psi_\tau = \Pi \circ F_\tau$, is an injective, continuous map with respect to the trace-norm topology of $\mathcal{D}(L^2(\mu))$, and the following dynamical compatibility relationships hold:

$$F_\tau \circ \Phi^t = U^{-t} \circ F_\tau, \quad \Pi \circ U^{-t} = \Phi_\tau^t \circ \Pi, \quad \Psi_\tau \circ \Phi^t = \tilde{\Phi}^t \circ \Psi_\tau.$$

- (ii) The induced linear map $\Omega_\tau : \mathcal{A}(L^2(\mu)) \rightarrow C_{\mathbb{R}}(S^1)$, mapping quantum observable A to the classical observable $f_A = \Omega_\tau A$, with $f_A(\theta) = \mathbb{E}_{\Psi_\tau(\theta)} A$, is well-defined (i.e., f_A is a continuous, real-valued function), satisfying

$$\Omega_\tau \circ \tilde{U}^t = U^t \circ \Omega_\tau.$$

- (iii) The transpose map $\Omega'_\tau : C'_{\mathbb{R}}(S^1) \rightarrow \mathcal{A}'(L^2(\mu))$, defined as $\Omega'_\tau J = J \circ \Omega_\tau$ satisfies

$$\tilde{U}^{t*} \circ \Omega'_\tau = \Omega'_\tau \circ U^{t*}.$$

A Proof of Proposition 4 is included in Appendix B 2. Figure 1 depicts the quantum–classical relationships from the proposition and (46) and (47) in the form of commutative diagrams. These diagrams indicate that the time- τ heat kernel on the circle induces, through its corresponding RKHS feature map (which underpins the maps Ψ_τ , Ω_τ , and Ω'_τ) two types of classical–quantum correspondence associated with the circle rotation, namely:

1. A continuous, one-to-one mapping of classical states (points) in the circle and regular quantum states on $L^2(\mu)$, carried out by Ψ_τ , as well as a continuous one-to-one mapping with the larger space of functionals in $\mathcal{A}'(L^2(\mu))$, carried out by Ω'_τ .
2. A continuous mapping of bounded quantum mechanical observables on $L^2(\mu)$ to classical observables (continuous, real-valued functions) on the circle, carried out by Ω_τ .

Moreover, both of these mappings are consistent (equivariant) with the classical and quantum dynamics, in the sense of the commutative diagrams in Fig. 1. Together, these facts prove the claims about Ψ_τ and Ω_τ in Theorem 1(ii) and (iii), respectively, and will be sufficient to complete the proof of the theorem in Sec. IV C.

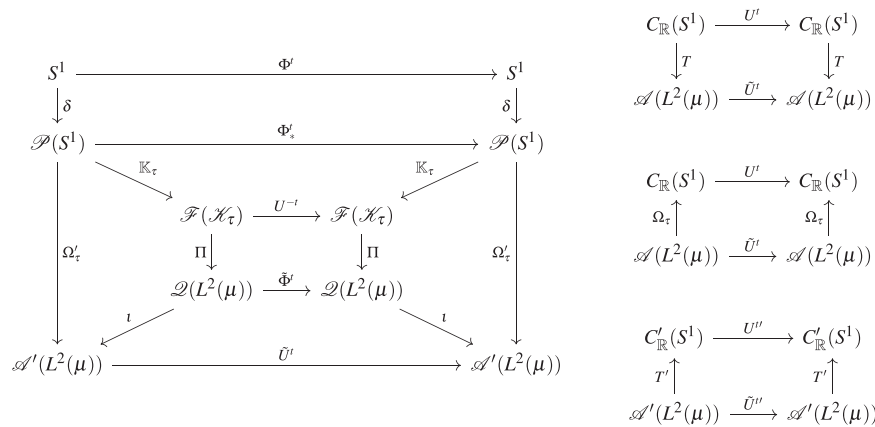


FIG. 1. Commutative diagrams illustrating the correspondence between classical states/observables of the harmonic oscillator and quantum states/observables on the $L^2(\mu)$ space associated with the invariant measure. The diagram on the left shows how classical states (points in the circle S^1) embed injectively into Radon probability measures $\mathcal{P}(S^1)$ via the map δ mapping into Dirac measures, which, in turn, map injectively into nonzero RKHS functions under the embedding $\mathbb{K}_\tau$ and then into quantum states and generalized quantum states on $L^2(\mu)$ via the maps Π and ι , respectively. The composition $\mathbb{K}_\tau \circ \delta$ corresponds to the RKHS feature map F_τ (not shown). These embeddings are all equivariant with the dynamical evolution maps at each level, as depicted in the diagram. The top diagram in the right shows the dynamical equivariance properties of the Banach algebra homomorphism T , mapping real-valued continuous functions on S^1 to bounded, self-adjoint multiplication operators in $\mathcal{A}(L^2(\mu))$. The middle diagram shows the equivariance properties of $\Omega_\tau : \mathcal{A}(L^2(\mu)) \rightarrow C_\mathbb{R}(S^1)$, which is not an algebra homomorphism. The bottom diagram shows the dynamical equivariance properties of the transpose map $T' : \mathcal{A}'(L^2(\mu)) \rightarrow C'_\mathbb{R}(S^1)$.

We close this section by noting that despite their equivariance properties, the classical–quantum correspondences identified thus far fail to preserve an important structure of both the classical and quantum representations of the circle rotation, namely, the Abelian algebraic structure of classical observables in $C_\mathbb{R}(S^1)$ and continuous multiplication operators in $\mathcal{A}(L^2(\mu))$. That is, apart from special cases, starting from a classical observable, mapping it to a multiplication operator in $\mathcal{A}(L^2(\mu))$ via T and mapping the result back to $C_\mathbb{R}(S^1)$ through Ω_τ does not recover the original observable. In other words, Ω_τ is not a left inverse of T , and similarly, Ω'_τ is not a right inverse of T' . Had these properties been true, $\Omega_\tau \circ \tilde{U}^t \circ T$ would be equal to U^t and $T' \circ \tilde{U}^{t'} \circ \Omega'_\tau$ would be equal to $U^{t'}$, which would then imply that the evolution of any classical observable in $C_\mathbb{R}(S^1)$ and Radon measure in $C'_\mathbb{R}(S^1)$ can be, respectively, understood as the evolution of a quantum observable in $\mathcal{A}(L^2(\mu))$ and a generalized quantum state in $\mathcal{A}'(L^2(\mu))$. We will improve upon this inconsistency in Sec. VI by considering quantum states and algebras of quantum observables associated with the RKHAs $\hat{\mathcal{H}}_\tau$, as opposed to the $L^2(\mu)$ space.

C. Isometric embedding

We now have the necessary ingredients to construct an isometric embedding (Hilbert space homomorphism) of $L^2(\mu)$ into the Hilbert space of sections $\mathcal{H}$, thereby inducing embeddings of the classical and quantum states of the circle rotation into the states of the quantum harmonic oscillator on Minkowski space, as well as corresponding pullbacks from quantum observables of the latter system into quantum and classical observables of the circle rotation.

We begin by introducing the following three Hilbert subspaces of $\mathcal{H}$:

$$\begin{aligned}\mathcal{H}_0 &= \overline{\text{span}\{\psi_{00}\}}, \\ \mathcal{H}_- &= \overline{\text{span}\{\psi_{j0} : j > 0\}}, \\ \mathcal{H}_+ &= \overline{\text{span}\{\psi_{0j} : j > 0\}},\end{aligned}\tag{48}$$

where overlines denote closure with respect to the $\mathcal{H}$ norm. Because they are spanned by eigenfunctions of H corresponding to distinct eigenvalues, these subspaces are mutually orthogonal, and each of them is invariant under the Heisenberg operator W^t for all $t \in \mathbb{R}$. Moreover, if a section s lies in $\mathcal{H}_0$, $\mathcal{H}_- \cap D(H)$, or $\mathcal{H}_+ \cap D(H)$, then $\langle s, Hs \rangle_{\mathcal{H}}$ is zero, strictly negative, or strictly positive, respectively. Due to that, we intuitively interpret pure states $\rho = \langle s, \cdot \rangle_{\mathcal{H}}$ with s in $\mathcal{H}_0$, $\mathcal{H}_-$, or $\mathcal{H}_+$, as “vacuum,” “antimatter,” or “matter” states, respectively. Defining $\tilde{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_- \oplus \mathcal{H}_+$, we note that for pure states with s in the orthogonal complement of $\tilde{\mathcal{H}}$ in $\mathcal{H}$, $\langle s, Hs \rangle_{\mathcal{H}}$ is not sign-definite.

With these considerations in mind and in analogy with (48), we define the Hilbert subspaces of $L^2(\mu)$ given by

$$\begin{aligned} L_0^2(\mu) &= \text{span}\{\phi_0\}, \\ L_-^2(\mu) &= \overline{\text{span}\{\phi_j : j < 0\}}, \\ L_+^2(\mu) &= \overline{\text{span}\{\phi_j : j > 0\}}, \end{aligned} \quad (49)$$

where ϕ_j are the Koopman eigenfunctions from Sec. II B. These subspaces are all invariant under the Koopman operator U^t for any $t \in \mathbb{R}$. Moreover, $\langle f, Vf \rangle_\mu$ is zero, strictly negative, or strictly positive whenever f lies in $L_0^2(\mu)$, $L_-^2(\mu) \cap D(V)$, or $L_+^2(\mu) \cap D(V)$, so we interpret the corresponding pure states $\rho = \langle f, \cdot \rangle_\mu f$ as being zero-, negative-, or positive-frequency states, respectively.

Motivated by the similarity between (48) and (49), we define the linear operator $\mathcal{U} : L^2(\mu) \rightarrow \mathcal{H}$, characterized completely through the relationships

$$\mathcal{U}\phi_j = \begin{cases} \psi_{00}, & j = 0, \\ \psi_{-j,0}, & j < 0, \\ \psi_{0j}, & j > 0. \end{cases} \quad (50)$$

It then follows directly from its definition that $\mathcal{U}$ is an isometric embedding of $L^2(\mu)$ into $\mathcal{H}$, i.e., it is an injective operator satisfying

$$\langle \mathcal{U}f_1, \mathcal{U}f_2 \rangle_{\mathcal{H}} = \langle f_1, f_2 \rangle_\mu, \quad \forall f_1, f_2 \in L^2(\mu),$$

and $\mathcal{U}^* \mathcal{U}$ is equal to the identity on $L^2(\mu)$. Moreover, $\mathcal{U}$ clearly maps $L_0^2(\mu)$ to $\mathcal{H}_0$, $L_-^2(\mu)$ to $\mathcal{H}_-$, and $L_+^2(\mu)$ to $\mathcal{H}_+$. In other words, $\mathcal{U}$ maps the zero-, negative-, and positive-frequency states of the classical harmonic oscillator map to the vacuum, matter, and antimatter states of the quantum harmonic oscillator on Minkowski space. The range of $\mathcal{U}$ is equal to $\tilde{\mathcal{H}}$, and as a result, $\mathcal{U}\mathcal{U}^*$ is equal to the identity operator on that space.

The operator $\mathcal{U}$ provides the isometric embedding of the Koopman operator formulation of the classical harmonic oscillator into the quantum harmonic oscillator on Minkowski space stated in Theorem 1. One of its key properties is that for every $j \in \mathbb{Z}$, $\mathcal{U}\phi_j$ is an eigensection of H at the same eigenvalue $j\alpha$ as the eigenfrequency corresponding to ϕ_j . As a result, $\mathcal{U}$ pulls back the Hamiltonian H to the generator V ,

$$\mathcal{U}^* H \mathcal{U} = V/i,$$

and we have the dynamical correspondence

$$\mathcal{U}^* e^{itH} \mathcal{U} = e^{itV}, \quad \forall t \in \mathbb{R}. \quad (51)$$

It should also be noted that the dual operation, i.e., $V \mapsto \mathcal{U}V\mathcal{U}^*$, maps the generator (up to multiplication by i) to a projected Hamiltonian,

$$\mathcal{U}V\mathcal{U}^* = i\Pi_{\tilde{\mathcal{H}}} H \Pi_{\tilde{\mathcal{H}}}$$

where $\Pi_{\tilde{\mathcal{H}}} : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ is the orthogonal projection operator mapping into $\tilde{\mathcal{H}}$.

Turning now to the spaces of quantum mechanical observables and states, $\mathcal{U}$ induces a surjective linear map $\tilde{\mathcal{U}} : \mathcal{A}(\mathcal{H}) \rightarrow \mathcal{A}(L^2(\mu))$, an isometry $\tilde{\mathcal{U}}^* : \mathcal{D}(L^2(\mu)) \rightarrow \mathcal{D}(\mathcal{H})$, and a linear isometry $\tilde{\mathcal{U}}' : \mathcal{A}'(L^2(\mu)) \rightarrow \mathcal{A}'(\mathcal{H})$, where

$$\tilde{\mathcal{U}}A = \mathcal{U}^* A \mathcal{U}, \quad \mathcal{U}\tilde{\mathcal{U}}^*(\rho) = \mathcal{U}\rho\mathcal{U}^*, \quad \tilde{\mathcal{U}}'J = J \circ \tilde{\mathcal{U}}.$$

It is then straightforward to verify using Proposition 4 and (51) that the equivariance properties

$$\tilde{\mathcal{U}} \circ \tilde{W}^t = \tilde{U}^t \circ \tilde{\mathcal{U}}, \quad \tilde{\mathcal{U}}^* \circ \tilde{Z}^t = \tilde{\Phi}^t \circ \tilde{\mathcal{U}}^*, \quad \tilde{\mathcal{U}}' \circ \tilde{U}^{t'} = \tilde{W}^{t'} \circ \tilde{\mathcal{U}}'$$

hold for every $t \in \mathbb{R}$.

The above allow us to augment the commutative diagrams in Fig. 1 to obtain the diagrams in Fig. 2, illustrating the combined relationships between states and observables of the circle rotation (both classical and quantum) and those of the quantum harmonic oscillator on Minkowski space. The following proposition, which can be deduced directly from these diagrams, shows that there exist quantum states of the system on Minkowski space for which the expectation value of any observable evolves consistently as evaluation of a classical observable of the circle rotation.

Proposition 5. For any $\theta \in S^1$ and $\tau > 0$, let $\rho_{\theta,\tau} \in \mathcal{D}(\mathcal{H})$ be the quantum state given by $\sigma_{\theta,\tau} = \tilde{\mathcal{U}}^*(\Psi_\tau(\theta))$. Then, for any quantum mechanical observable $A \in \mathcal{A}(\mathcal{H})$ and $t \in \mathbb{R}$, the relationship

$$\mathbb{E}_{Z^t(\sigma_{\theta,\tau})} A = f_{A,\tau}(\Phi^t(\theta))$$

holds, where $f_{A,\tau} \in C_{\mathbb{R}}(S^1)$ is the classical observable of the circle rotation given by $f_{A,\tau} = \Omega_\tau(\tilde{\mathcal{U}}(A))$. In particular, $t \mapsto \mathbb{E}_{Z^t(\rho_\theta)}$ is periodic with period $2\pi/\alpha$.

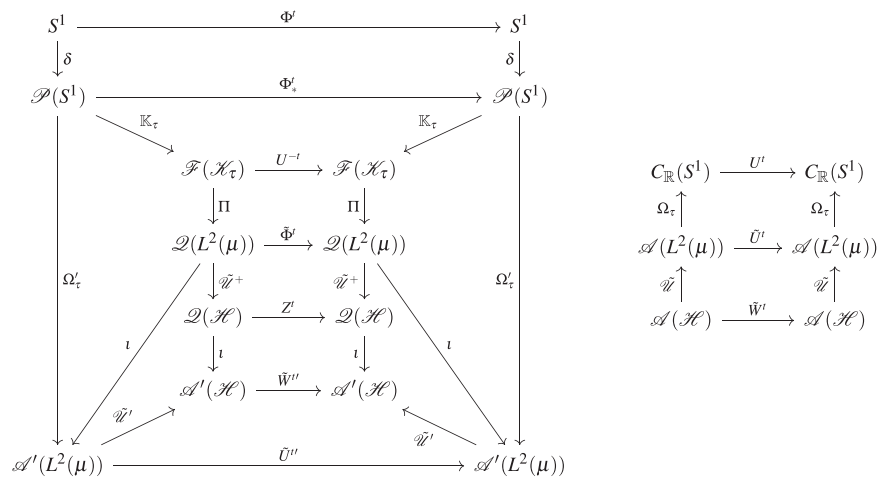


FIG. 2. As in Fig. 1, but for commutative diagrams showing the relationships between the quantum formulations of the circle rotation and the quantum harmonic oscillator on Minkowski space. Note that in the left-hand side diagram, we abuse notation using ι to denote the inclusion maps of both $\mathcal{D}(L^2(\mu))$ into $\mathcal{A}'(L^2(\mu))$ and $\mathcal{D}(\mathcal{H})$ into $\mathcal{A}'(\mathcal{H})$.

It can be readily verified that $\rho_{\theta,\tau}$ takes the form $\rho_{\theta,\tau} = \langle \psi_{\theta,\tau}, \cdot \rangle_{\mathcal{H}} \psi_{\theta,\tau} / \|\kappa_{\tau}(\theta, \cdot)\|_{\mu}^2$, where $\psi_{\theta,\tau} \in \mathcal{H}$ is the section (wavefunction) given by

$$\psi_{\theta,\tau} = \sum_{j=0}^{\infty} e^{-j^2\tau} (\phi_j(\theta) \psi_{j0} + \phi_j^*(\theta) \psi_{0j}). \quad (52)$$

Figure 3 shows plots of the corresponding $\mathbb{C}$ -valued function $\psi_{\theta,\tau}^{\sigma} = \zeta_{\sigma_x}^{-1} \psi_{\theta,\tau}$ associated with an inertial coordinate chart $x = (x^0, x^1)$ centered at o . There, as θ increases from 0 to π , $\psi_{\theta,\tau}^{\sigma}$ is seen to undergo an evolution from a real-valued, diffuse configuration supported mainly in the $x^0, x^1 \geq 0$ coordinate quadrant, to a focused configuration near the origin at $\theta = \pi/2$, and finally to a real-valued, diffuse configuration supported in the $x^0, x^1 \leq 0$ quadrant.

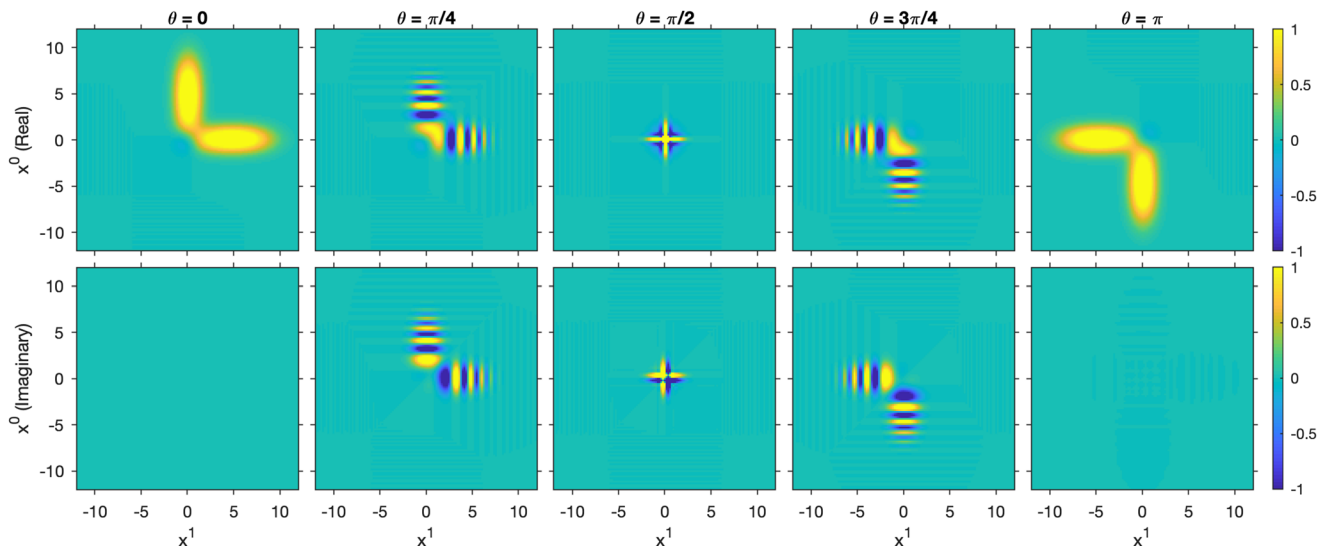


FIG. 3. Wavefunction $\psi_{\theta,\tau}^{\sigma}$ associated with the embedding of the classical states (points) on the circle to states of the quantum harmonic oscillator on two-dimensional Minkowski space. Here, the wavefunction is shown as a function of the inertial coordinates (x^0, x^1) of Minkowski space for $\tau = 10^{-3}$ and representative values of $\theta \in S^1$ in the range $[0, \pi]$, using an arbitrary normalization with respect to the $L^2(\nu)$ norm.

D. Gauge covariance

At this point, we have completed the Proof of Theorem 1 aside from the claimed gauge covariance of the isometry $\mathcal{U} : L^2(\mu) \rightarrow \mathcal{H}$. Specifically, the construction of $\mathcal{U}$ in Sec. IV C employed a particular basis of $\mathcal{H}$ consisting of the eigensections ψ_{jk} , whose construction (in Sec. IV A) depended on a choice of section $\sigma : M \rightarrow P$ of the principal bundle to map Hermite functions into sections. Since every such choice σ corresponds to a choice of inertial frame and Minkowski space does not have a preferred inertial frame, it is important that the transformation imparted to $\mathcal{U}$ by passing from σ to a different section, $\sigma' : M \rightarrow P$, be structure-preserving. In particular, the operator $\mathcal{U}_{\sigma'} : L^2(\mu) \rightarrow \mathcal{H}$ defined as in (50), but using the H eigensections $\psi_{\sigma',jk} = \zeta_{\sigma'} \chi_{jk}$ instead of ψ_{jk} , should be relatable to $\mathcal{U}$ by a unitary map $\Xi : \mathcal{H} \rightarrow \mathcal{H}$ such that $\mathcal{U}_{\sigma'} = \Xi \mathcal{U}$ (otherwise, the Hilbert space structure would not be preserved), and moreover, the dynamical correspondence in (51) should still hold for $\mathcal{U}_{\sigma'}$.

To verify that this is indeed the case, let $\varphi : P \rightarrow P$ be the unique gauge transformation such that $\sigma' = \varphi \circ \sigma$. Then, by Lemma 8(i), $\psi_{\sigma',jk}(m) = e^{ia\vartheta(\xi(m))/\sqrt{2}} \psi_{jk}(m)$ for a G -valued function ξ on M . Therefore, we have $\mathcal{U}_{\sigma'} = \Xi \circ \mathcal{U}$, where Ξ is the unitary multiplication operator on $\mathcal{H}$ by the function $e^{ia\vartheta(\xi(\cdot))}$. Moreover, again by Lemma 8(i), we have $\psi_{\sigma',jk} = \varphi_* \circ \psi_{jk}$, and because the operator $\varphi_* \circ$ commutes with the Laplacian $\bar{\Delta}$ (see Sec. III I), it follows that $\mathcal{U}_{\sigma'}^* H \mathcal{U}_{\sigma'} = \mathcal{U}^* H \mathcal{U}$ so that $\mathcal{U}_{\sigma'}^* e^{itH} \mathcal{U}_{\sigma'} = \mathcal{U}_{\sigma'}^* e^{itH} \mathcal{U}_{\sigma}$. We therefore conclude that the dynamical correspondence in (51) holds for $\mathcal{U}_{\sigma'}$, completing the Proof of Theorem 1.

V. CANONICALLY COMMUTING OPERATORS FOR THE CLASSICAL HARMONIC OSCILLATOR

In this section, we employ the correspondence between the circle rotation and the quantum harmonic oscillator on Minkowski space established in Sec. IV to construct operators acting on classical observables (functions) on the circle exhibiting canonical position–momentum commutation relationships, together with their corresponding ladder operators. This construction will be carried out using the isometry $\mathcal{U} : L^2(\mu) \rightarrow \mathcal{H}$ from (50) to pull back position, momentum, and ladder operators of the quantum harmonic oscillator to densely defined operators on $L^2(\mu)$. These operators will automatically satisfy the appropriate commutation relationships since $\mathcal{U}$ is a Hilbert space homomorphism.

A. Ladder operators

As with the non-relativistic quantum harmonic oscillator, the quantum system on Minkowski space from Sec. IV A admits canonically commuting *position* and *momentum operators* with respect to any inertial chart $x = (x^0, x^1) : M \rightarrow \mathbb{R}^2$ (taken to be centered at $o \in M$ as per our convention). These operators can be constructed as usual by considering the one-parameter unitary groups of operators $\mathcal{R}_j^a : \mathcal{H} \rightarrow \mathcal{H}$ and $\mathcal{P}_j^a : \mathcal{H} \rightarrow \mathcal{H}$, with $j \in \{0, 1\}$ and $a \in \mathbb{R}$, acting on sections $s \in \mathcal{H}$ by phase multiplication and translation, respectively, i.e.,

$$\mathcal{R}_j^a s(m) = e^{iax^j(m)} s(m), \quad \mathcal{P}_j^a s(m) = s(m + a \vec{X}_j).$$

Note that $\mathcal{R}_j^a$ coincides with the action φ_* on sections associated with the gauge transformation $\varphi : P \rightarrow P$ with $\varphi(p) = p \cdot \xi(\pi(p))$, $\xi(m) = \exp(\sqrt{2}ax^j(m)/\alpha)$. We then define the position operators $\hat{X}_j : D(\hat{X}_j) \rightarrow \mathcal{H}$, $D(\hat{X}_j) \subset \mathcal{H}$, and momentum operators $\hat{P}_j : D(\hat{P}_j) \rightarrow \mathcal{H}$, $D(\hat{P}_j) \subset \mathcal{H}$, as the generators of these groups, respectively, times a factor of $1/i$ introduced by convention to render the generators self-adjoint. That is, we have

$$\hat{X}_j s = i^{-1} \lim_{a \rightarrow 0} (\mathcal{R}_j^a s - s)/a, \quad \hat{P}_j s = i^{-1} \lim_{a \rightarrow 0} (\mathcal{P}_j^a s - s)/a, \quad (53)$$

and the domains $D(\hat{X}_j)$ and $D(\hat{P}_j)$ are defined as the dense subspaces of $\mathcal{H}$ where the respective limits in (53) exist with respect to $\mathcal{H}$ norm. It then follows from these definitions that $\hat{X}_j$ is a multiplication operator by the coordinate function x^j , i.e.,

$$\hat{X}_j s = x^j s,$$

while $\hat{P}_j$ is a differentiation operator that behaves as an extension of the coordinate vector field X_j to smooth sections in $\mathcal{H}$,

$$\hat{P}_j s = -iX_j s = -i ds \cdot X_j, \quad \forall s \in D(\hat{P}_j) \cap \Gamma(E).$$

Note now that the Schwartz space $\mathcal{S}(M, E)$ is invariant under all of $\hat{X}_j$ and $\hat{P}_j$, so we can consider the restricted operators $\tilde{X}_j : \mathcal{S}(M, E) \rightarrow \mathcal{H}$ and $\tilde{P}_j : \mathcal{S}(M, E) \rightarrow \mathcal{H}$, where $\tilde{X}_j = \hat{X}_j|_{\mathcal{S}(M, E)}$, $\tilde{P}_j = \hat{P}_j|_{\mathcal{S}(M, E)}$, and $\text{ran} \tilde{X}_j$ and $\text{ran} \tilde{P}_j$ are both subspaces of $\mathcal{S}(M, E)$. It then follows directly from their definition that these operators obey the canonical position–momentum commutation relations,

$$[\tilde{X}_j, \tilde{X}_k] = 0, \quad [\tilde{P}_j, \tilde{P}_k] = 0, \quad [\tilde{X}_j, \tilde{P}_k] = i\delta_{jk}.$$

Moreover, associated with $\tilde{X}_j$ and $\tilde{P}_j$ are the *ladder operators*

$$A_j = \sqrt{\frac{\alpha}{2}} \tilde{X}_j + \frac{i}{\sqrt{2\alpha}} \tilde{P}_j, \quad A_j^+ = \sqrt{\frac{\alpha}{2}} \tilde{X}_j - \frac{i}{\sqrt{2\alpha}} \tilde{P}_j$$

and the *number operators*

$$N_j = A_j^+ A_j.$$

These operators satisfy the standard commutation relations

$$\begin{aligned}[N_j, A_k] &= -A_j \delta_{jk}, & [N_j, A_k^+] &= A_j \delta_{jk}, \\ [A_j, A_k] &= [A_j^+, A_k^+] = [N_j, N_k] = 0,\end{aligned}\quad (54)$$

and moreover, we have

$$\tilde{H} = \alpha(-N_0 + N_1). \quad (55)$$

The following are the formulas for the action of these operators on the ψ_{jk} basis elements of $\mathcal{H}$, which follow from standard results on ladder operators:²³

$$\begin{aligned}A_0 \psi_{jk} &= \sqrt{j} \psi_{j-1,k}, & A_0^+ \psi_{jk} &= \sqrt{j+1} \psi_{j+1,k}, \\ A_1 \psi_{jk} &= \sqrt{k} \psi_{j,k-1}, & A_1^+ \psi_{jk} &= \sqrt{k+1} \psi_{j,k+1}, \\ N_0 \psi_{jk} &= j \psi_{jk}, & N_1 \psi_{jk} &= k \psi_{jk}.\end{aligned}\quad (56)$$

We will now pull back the A_j , A_j^+ , and N_j to operators on classical observables of the circle rotation. For that, let $\mathcal{S}(S^1) = \mathcal{U}^* \mathcal{S}(M, E)$ be the image of the Schwartz space of sections on Minkowski space under $\mathcal{U}^*$. For completeness, we note that $\mathcal{S}(S^1)$ can be identified with a Schwartz space of functions on the circle (see Ref. 30 for definitions), which in this case coincides with $C^\infty(S^1)$ by compactness of S^1 . However, here we will not need that structure.

By construction, $\mathcal{U}$ maps $\mathcal{S}(S^1)$ to $\mathcal{S}(M, E)$, and therefore, the following are well-defined as densely defined operators from $\mathcal{S}(S^1) \subset L^2(\mu)$ to $L^2(\mu)$:

$$\begin{aligned}A_- &= \mathcal{U}^* A_0 \mathcal{U}, & A_-^+ &= \mathcal{U}^* A_0^+ \mathcal{U}, & N_- &= \mathcal{U}^* N_0 \mathcal{U}, \\ A_+ &= \mathcal{U}^* A_1 \mathcal{U}, & A_+^+ &= \mathcal{U}^* A_1^+ \mathcal{U}, & N_+ &= \mathcal{U}^* N_1 \mathcal{U}.\end{aligned}\quad (57)$$

It then follows from the fact that $\mathcal{U} \mathcal{U}^*$ is the identity on $\mathcal{H}$ that these operators satisfy the canonical commutation relationships for ladder operators analogously to (54), viz.,

$$\begin{aligned}[N_-, A_-] &= -A_-, & [N_-, A_-^+] &= A_-^+, \\ [N_+, A_+] &= -A_+, & [N_+, A_+^+] &= A_+^+, \\ [A_-, A_+] &= [A_-^+, A_+^+] = [A_-, A_+^+] = [A_+^+, A_-] = 0, \\ [N_-, N_+] &= 0.\end{aligned}$$

Moreover, the generating vector field $\vec{V}$ of the classical harmonic oscillator, acting on $C^\infty(S^1)$ functions, can be expressed in terms of the number operators as [cf. (55)]

$$\vec{V} = i\alpha(-N_- + N_+) = i\mathcal{U}^* \tilde{H} \mathcal{U}. \quad (58)$$

To characterize these operators more explicitly, consider the order- r fractional derivative operator $\partial^r: D(\partial^r) \rightarrow L^2(\mu)$, $r \geq 0$, associated with standard angle coordinates on the circle (see Appendix C for an explicit definition and additional details). Let also $\Pi_-: L^2(\mu) \rightarrow L^2(\mu)$ and $\Pi_+: L^2(\mu) \rightarrow L^2(\mu)$ be the orthogonal projections mapping into $L_-^2(\mu)$ and $L_+^2(\mu)$, respectively, and define the spectrally truncated derivative operators $\partial_-^r = \partial^r \Pi_-$ and $\partial_+^r = \partial^r \Pi_+$, both of which are defined on the same dense domain $D(\partial^r) \supset \mathcal{S}(S^1)$ as ∂^r . It can be shown (see Proposition 19 in Appendix C) that the ladder operators $A_\pm$ and $A_\pm^+$ take the form

$$\begin{aligned}A_- &= i^{1/2} L^* \partial_-^{1/2}, & A_-^+ &= i^{1/2} \partial_-^{1/2} L, \\ A_+ &= i^{-1/2} L \partial_+^{1/2}, & A_+^+ &= i^{-1/2} \partial_+^{1/2} L^*,\end{aligned}\quad (59)$$

where L and L^* are the ladder-like operators from Sec. II C. Moreover, it can be readily verified that the relationships

$$\partial_-^q \partial_-^r = \partial_-^{q+r}, \quad \partial_+^q \partial_+^r = \partial_+^{q+r}, \quad \partial_-^r + \partial_+^r = \partial^r$$

hold for every $q, r \geq 0$, leading, in conjunction with (6), to the formulas

$$N_- = i\partial_-, \quad N_+ = -i\partial_+$$

for the number operators. Note that the expressions above are consistent with (58), i.e.,

$$\vec{V} = \alpha\partial = \alpha(\partial_- + \partial_+) = i\alpha(-N_- + N_+). \quad (60)$$

In summary, thus far we have seen that pulling back the ladder operators for the quantum harmonic oscillator on Minkowski space under the Hilbert space homomorphism $\mathcal{U}$ leads to operators on classical observables ($\mathbb{C}$ -valued functions) of the circle rotation, which have

the structure of order-1/2 fractional derivatives, composed with the ladder-like operators from Sec. II C. By virtue of the structure-preserving properties of $\mathcal{U}$, the ladder operators for the circle rotation exhibit canonical commutation relationships and lead to the decomposition in (60) of the generator of the Koopman group on $L^2(\mu)$ through their corresponding number operators. This completes the Proof of Theorem 2(i).

B. Position and momentum operators

Next, to construct “position” and “momentum” operators for the circle rotation, we can simply pull back the corresponding operators, $\hat{X}_j$ and $\hat{P}_j$, respectively, for the quantum harmonic oscillator on Minkowski space using $\mathcal{U}$. That is, we define $\hat{X}_\pm : D(\hat{X}_\pm) \rightarrow L^2(\mu)$, $\hat{P}_\pm : D(\hat{P}_\pm) \rightarrow L^2(\mu)$, where $D(\hat{X}_\pm)$ and $D(\hat{P}_\pm)$ are the dense subspaces of $L^2(\mu)$ given by $D(\hat{X}_-) = \mathcal{U}^* D(\hat{X}_0)$, $D(\hat{X}_+) = \mathcal{U}^* D(\hat{X}_1)$, $D(\hat{P}_-) = \mathcal{U}^* D(\hat{P}_0)$, $D(\hat{P}_+) = \mathcal{U}^* D(\hat{P}_1)$, and

$$\begin{aligned}\hat{X}_- &= \mathcal{U}^* \hat{X}_0 \mathcal{U}, & \hat{X}_+ &= \mathcal{U}^* \hat{X}_1 \mathcal{U}, \\ \hat{P}_- &= \mathcal{U}^* \hat{P}_0 \mathcal{U}, & \hat{P}_+ &= \mathcal{U}^* \hat{P}_1 \mathcal{U}.\end{aligned}$$

Restricted to the Schwartz space $\mathcal{S}(S^1)$, these operators can be expressed in terms of the ladder operators from Sec. V A, viz.,

$$\begin{aligned}\tilde{X}_- &= \frac{1}{\sqrt{2\alpha}}(A_- + A_-^\dagger) = \frac{i}{\sqrt{2\alpha}}(L^* \partial_-^{1/2} + \partial_-^{1/2} L), \\ \tilde{P}_- &= -i\sqrt{\frac{\alpha}{2}}(A_- - A_-^\dagger) = -\sqrt{\frac{\alpha}{2}}(L^* \partial_-^{1/2} - \partial_-^{1/2} L), \\ \tilde{X}_+ &= \frac{1}{\sqrt{2\alpha}}(A_+ + A_+^\dagger) = \frac{i}{\sqrt{2\alpha}}(L \partial_+^{1/2} + \partial_+^{1/2} L^*), \\ \tilde{P}_+ &= -i\sqrt{\frac{\alpha}{2}}(A_+ - A_+^\dagger) = -\sqrt{\frac{\alpha}{2}}(L \partial_+^{1/2} - \partial_+^{1/2} L^*),\end{aligned}\tag{61}$$

where $\tilde{X}_- = \hat{X}_-|_{\mathcal{S}(S^1)}$ and similarly for the other operators. In particular, it follows immediately from the commutation relationships for $A_\pm$ and $A_\pm^\dagger$ established in Sec. V A that the operators introduced above exhibit canonical position–momentum commutation relationships, i.e.,

$$\begin{aligned}[\tilde{X}_\pm, \tilde{X}_\pm] &= 0, & [\tilde{P}_\pm, \tilde{P}_\pm] &= 0, \\ [\tilde{X}_-, \tilde{P}_-] &= 1, & [\tilde{X}_+, \tilde{P}_+] &= 1, \\ [\tilde{X}_-, \tilde{P}_+] &= 0, & [\tilde{X}_+, \tilde{P}_-] &= 0.\end{aligned}$$

It is also worth noting that $\tilde{X}_\pm$ and $\tilde{P}_\pm$ can be employed to define “kinetic energy” and “potential energy” operators for the circle rotation, respectively,

$$\mathfrak{K} = \frac{1}{2}(-\tilde{P}_-^2 + \tilde{P}_+^2), \quad \mathfrak{V} = \frac{\alpha^2}{2}(-\tilde{X}_-^2 + \tilde{X}_+^2).$$

One can then verify that up to a multiplication factor of 1/2, $\mathfrak{K}$ is equal to the pullback under $\mathcal{U}$ of the connection Laplacian Δ from Sec. III F, $\mathfrak{K} = \mathcal{U}\Delta\mathcal{U}^*/2$, while $\mathfrak{V}$ is equal to the pullback of the multiplication operator on $\mathcal{H}$ that multiplies by the potential function v , i.e., $\mathfrak{V} = \mathcal{U}^* T_v \mathcal{U}$.

Inspecting the formulas for $\tilde{X}_\pm$ and $\tilde{P}_\pm$ in (61), it is evident that the position and momentum operators for the circle rotation differ fundamentally from their counterparts $\hat{X}_j$ and $\hat{P}_j$ for the quantum harmonic oscillator since (i) unlike $\hat{X}_j$ and $\hat{P}_j$, which are all local operators, $\tilde{X}_\pm$ and $\tilde{P}_\pm$ are all non-local; and (ii) unlike $\hat{X}_j$, which are multiplication operators by the corresponding coordinate functions x^j , $\tilde{X}_\pm$ are not multiplication operators. Here, by an operator A on functions being *local*, we mean that for a given function $f \in D(A)$, the evaluation of the function Af at each point on its domain of definition depends only on the values of f in an arbitrarily small neighborhood of that point. Examples of operators meeting this condition are multiplication operators and derivative operators of integral order, such as $\hat{X}_j$ and $\hat{P}_j$. On the other hand, given a function $f : S^1 \rightarrow \mathbb{C}$ of appropriate smoothness, the result of the fractional derivative $\partial^{1/2}f(\theta)$ depends on the behavior of f at distant points from θ (see Appendix C), and as a result, $\tilde{X}_\pm$ and $\tilde{P}_\pm$ are non-local. In fact, the operators $\tilde{X}_\pm$ and $\tilde{P}_\pm$ generate fractional diffusion semigroups on the circle.

Still, despite these differences, the fact that $\tilde{X}_\pm$ and $\tilde{P}_\pm$ obey canonical commutation relationships means that many results for the quantum harmonic oscillator, which are a consequence of these relationships, carry over to the circle rotation. As an example, we mention here the position–momentum uncertainty relationships, which hold for the $\tilde{X}_\pm$ and $\tilde{P}_\pm$ operators analogously to $\hat{X}_j$ and $\hat{P}_j$. That is, letting $\sigma_{jk} = \langle \psi_{jk}, \cdot \rangle_{\mathcal{H}} \psi_{jk}$ be the pure quantum state in $Q(\mathcal{H})$ associated with eigensection ψ_{jk} , it follows from standard results on quantum harmonic oscillators (e.g., Refs. 23 and 29) that

$$\begin{aligned}\text{var}_{\sigma_0} \hat{X}_0^2 &:= \mathbb{E}_{\sigma_0} \hat{X}_0^2 - (\mathbb{E}_{\sigma_0} \hat{X}_0)^2 = \frac{1}{\alpha} \left(j + \frac{1}{2} \right), \\ \text{var}_{\sigma_0} \hat{P}_0^2 &:= \mathbb{E}_{\sigma_0} \hat{P}_0^2 - (\mathbb{E}_{\sigma_0} \hat{P}_0)^2 = \alpha \left(j + \frac{1}{2} \right),\end{aligned}$$

and similarly that

$$\mathrm{var}_{\sigma_{0j}} \hat{X}_1^2 = \frac{1}{\alpha} \left(j + \frac{1}{2} \right), \quad \mathrm{var}_{\sigma_{0j}} \hat{P}_1^2 = \alpha \left(j + \frac{1}{2} \right),$$

leading to the position–momentum uncertainty relationships

$$(\mathrm{var}_{\sigma_{00}} \hat{X}_0^2)(\mathrm{var}_{\sigma_{00}} \hat{P}_0^2) = (\mathrm{var}_{\sigma_{0j}} \hat{X}_1^2)(\mathrm{var}_{\sigma_{0j}} \hat{P}_1^2) = \left(j + \frac{1}{2} \right)^2. \quad (62)$$

Correspondingly, for the quantum state $\rho_j = \langle \phi_j, \cdot \rangle_\mu \phi_j \in \mathcal{D}(L^2(\mu))$ of the circle rotation with $j \leq 0$, we have

$$\mathrm{var}_{\rho_j} \hat{X}_- = \mathrm{var}_{\rho_j} (\mathcal{U}^* \hat{X}_0 \mathcal{U}) = \mathrm{var}_{\mathcal{U}\rho_j \mathcal{U}^*} \hat{X}_0 = \mathrm{var}_{\sigma_{-j,0}} \hat{X}_0$$

and $\mathrm{var}_{\rho_j} \hat{P}_- = \mathrm{var}_{\sigma_{-j,0}} \hat{P}_0$, while for $j > 0$,

$$\mathrm{var}_{\rho_j} \hat{X}_+ = \mathrm{var}_{\sigma_{0j}} \hat{X}_1, \quad \mathrm{var}_{\rho_j} \hat{P}_+ = \mathrm{var}_{\sigma_{0j}} \hat{P}_1,$$

leading to analogous uncertainty relationships to (62), i.e.,

$$\begin{aligned} (\mathrm{var}_{\rho_j} \hat{X}_-^2)(\mathrm{var}_{\rho_j} \hat{P}_-^2) &= \left(j + \frac{1}{2} \right)^2, \quad j \leq 0, \\ (\mathrm{var}_{\rho_j} \hat{X}_+^2)(\mathrm{var}_{\rho_j} \hat{P}_+^2) &= \left(j + \frac{1}{2} \right)^2, \quad j > 0. \end{aligned} \quad (63)$$

Note that in both (62) and (63) the ground states, ψ_{00} and ϕ_0 , respectively, saturate the corresponding Schrödinger uncertainty inequality.²⁹ With these results, we have completed the Proof of Theorem 2.

VI. CLASSICAL AND QUANTUM DYNAMICS ON REPRODUCING KERNEL HILBERT ALGEBRAS (RKHAS)

What could be considered a shortcoming of the classical–quantum correspondence results established thus far is that the map Ω_τ , mapping quantum observables in $\mathcal{A}(L^2(\mu))$ to classical observables in $C_\mathbb{R}(S^1)$, is not compatible with the natural Banach algebra homomorphism $T : C_\mathbb{R}(S^1) \rightarrow \mathcal{A}(L^2(\mu))$ mapping continuous functions to bounded multiplication operators on $L^2(\mu)$. To address this issue, in this section, we shift attention from the dynamics (either classical or quantum) induced by the circle rotation on $L^2(\mu)$ and consider instead the dynamics on the RKHAS $\hat{\mathcal{K}}_\tau$ associated with fractional diffusions on the circle. After introducing the basic properties of these spaces (Sec. IV A), we consider aspects of classical and quantum dynamics (Secs. VI B and VI C), including the classical–quantum correspondence stated in Theorem 3. In particular, we will see that the reproducing property of $\hat{\mathcal{K}}_\tau$, which has no counterpart in the $L^2(\mu)$ setting, provides an additional structure, ensuring that the analogous operator to Ω_τ is compatible with an appropriate embedding operator analogous to T .

A. RKHAS induced by fractional diffusions

Following Ref. 16, for fixed $p \in (0, 1)$ and any $\tau > 0$, we consider the RKHS $\hat{\mathcal{K}}_\tau$ on the circle associated with the reproducing kernel $\hat{\kappa}_\tau : S^1 \times S^1 \rightarrow \mathbb{R}_+$ (cf. Ref. 7),

$$\begin{aligned} \hat{\kappa}_\tau(\theta, \theta') &= \sum_{j=-\infty}^{\infty} e^{-|j|^p \tau} \phi_j^*(\theta) \phi_j(\theta') \\ &= \sum_{j=-\infty}^{\infty} e^{-|j|^p \tau} e^{ij(\theta' - \theta)}, \end{aligned} \quad (64)$$

where we recognize the $|j|^p$ terms in $e^{-|j|^p \tau}$ as the eigenvalues of the order- $p/2$ fractional Laplacian,

$$\mathcal{L}^{p/2} \phi_j = |j|^p \phi_j.$$

One can verify that the sum over j in (64) converges in any C^r norm, $r \in \mathbb{N}_0$, to a smooth strictly-positive-definite kernel on $S^1 \times S^1$ so that $\hat{\mathcal{K}}_\tau$ is an RKHS of smooth functions. Moreover, $\hat{\kappa}_\tau$ is translation-invariant and exhibits the analogous properties stated for the canonical heat kernel in Lemmas 16 and 17. In particular, for every $\tau > 0$, the feature map $\hat{F}_\tau : S^1 \rightarrow \hat{\mathcal{K}}_\tau$ with $\hat{F}_\tau(\theta) = \hat{\kappa}_\tau(\theta, \cdot)$ is continuous and injective, the image $\mathcal{F}(\hat{\mathcal{K}}_\tau) := \hat{F}_\tau(S^1) \subset \hat{\mathcal{K}}_\tau$ contains only nonzero functions, and $\hat{\mathcal{K}}_\tau$ is dense in $C(S^1)$. In addition, we have the following proposition:

Proposition 6. For every $p \in (0, 1)$ and $\tau > 0$, the RKHS $\hat{\mathcal{K}}_\tau$ is a unital Banach $$ -algebra of functions, i.e., there exists a constant C_τ such that*

$$\|fg\|_{\hat{\mathcal{K}}_\tau} \leq C_\tau \|f\|_{\hat{\mathcal{K}}_\tau} \|g\|_{\hat{\mathcal{K}}_\tau}, \quad \forall f, g \in \hat{\mathcal{K}}_\tau.$$

Moreover, the space $\hat{\mathcal{K}}^\infty = \bigcap_{\tau=1}^\infty \hat{\mathcal{K}}_\tau$ is a dense, unital subalgebra of $\hat{\mathcal{K}}_\tau$, i.e., $fg \in \hat{\mathcal{K}}^\infty$ for all $f, g \in \hat{\mathcal{K}}^\infty$.

Proposition 6 follows from the results in Ref. 16. While we do not reproduce a proof here, it is worth noting that the result relies heavily on the fact that the set of Fourier functions ϕ_j is closed under multiplication, $\phi_j \phi_k = \phi_{j+k}$, and is bounded in $C(S^1)$ norm uniformly with respect to j , $\|\phi_j\|_{C(S^1)} = 1$. Another important element underpinning Proposition 6 is that $|j| \mapsto |j|^p$ is a strictly concave function for every $p \in (0, 1)$.

Remark. In the case of the RKHSs $\mathcal{K}_\tau$ associated with the standard heat kernel, one can verify with explicit counterexamples¹⁶ that the Banach algebra property in Proposition 6 does not hold. Instead, the spaces $\mathcal{K}_\tau$ can be shown to obey a weaker, Hölder-like inequality,

$$\|fg\|_{\mathcal{K}_\tau} \leq \|f\|_{\mathcal{K}_{\tau_1}} \|g\|_{\mathcal{K}_{\tau_2}}, \quad \frac{1}{\tau} \geq \frac{1}{\tau_1} + \frac{1}{\tau_2}.$$

The failure of $\mathcal{K}_\tau$ to have the Banach algebra structure can be traced to the convex (quadratic) dependence of the Laplacian eigenvalues, $j \mapsto j^2$. The latter imposes a stronger constraint on the decay of the expansion coefficients c_j for $f = \sum_{j=-\infty}^\infty c_j \phi_j \in \mathcal{K}_\tau$ than in the case of $f \in \hat{\mathcal{K}}_\tau$, where the eigenvalues $|j|^p$ of $\mathcal{L}^{p/2}$ grow sublinearly. In effect, the decay of the terms $e^{-|j|^p \tau}$ in (64) is fast-enough for $\hat{\mathcal{K}}_\tau$ to contain only smooth functions, yet slow-enough for it to be a Banach algebra. It should be noted that while $\mathcal{K}_\tau$ does not have the structure of a Banach algebra, the space $\mathcal{K}^\infty = \bigcap_{\tau=1}^\infty \mathcal{K}_\tau$ is a unital algebra of functions.

B. Classical dynamics

We now turn attention to the properties of the Koopman operators U^t acting on functions in $\hat{\mathcal{K}}_\tau$. For that, it is worth to begin by noting that while RKHSs have recently received attention in the context of Koopman and transfer operator theory,^{31,32} a general RKHS need not be invariant under a dynamical flow Φ^t , even if that flow is smooth. Intuitively, this is because membership of a function f in an RKHS imposes a stringent condition on the expansion coefficients of f in a natural orthonormal basis for the kernel [e.g., (9)], and as the dynamics can deform the level sets of functions, $f \circ \Phi^t$ need not satisfy those conditions. Nevertheless, the kernels $\hat{\kappa}_\tau$ possess an important special property, namely, that they admit the Mercer representation in (64) in terms of Koopman eigenfunctions. This turns out to be sufficient for $f \circ \Phi^t, f \in \hat{\mathcal{K}}_\tau$, to lie in $\hat{\mathcal{K}}_\tau$ so that one can define groups of Koopman operators $\hat{\mathcal{K}}_\tau \rightarrow \hat{\mathcal{K}}_\tau$, which turn out to be strongly continuous and unitary. Below, we state some of the properties of these groups in the form of a theorem, as we have not seen them stated elsewhere in the literature.

Theorem 7. Let $\Phi^t: S^1 \rightarrow S^1$ be the circle rotation with frequency α , $\hat{\kappa}_\tau: S^1 \times S^1 \rightarrow \mathbb{R}$ be the fractional heat kernel from (64) at the time parameter $\tau > 0$, and $\hat{\mathcal{K}}_\tau$ be the corresponding RKHA of $\mathbb{C}$ -valued functions. Define the Sobolev-like space

$$\hat{\mathcal{K}}_\tau^1 = \left\{ \sum_{j=-\infty}^\infty c_j \phi_j \in \hat{\mathcal{K}}_\tau : \sum_{j=-\infty}^\infty |j|^{2p} e^{|j|^p \tau} |c_j|^2 < \infty \right\}.$$

Then, the following hold for every $\tau > 0$:

- (i) For all $t \in \mathbb{R}$, $\hat{\mathcal{K}}_\tau$, and thus $\hat{\mathcal{K}}^\infty$, are invariant under Φ^t .
- (ii) The group of Koopman operators $U^t: \hat{\mathcal{K}}_\tau \rightarrow \hat{\mathcal{K}}_\tau$, with $t \in \mathbb{R}$ and $U^t f = f \circ \Phi^t$, is a strongly continuous, unitary group.
- (iii) The skew-adjoint generator $V: D(V) \rightarrow \hat{\mathcal{K}}_\tau$ of the Koopman group on $\hat{\mathcal{K}}_\tau$ has the domain $D(V) = \hat{\mathcal{K}}_\tau^1$ and acts as a derivation on $\hat{\mathcal{K}}^\infty$.

A Proof of Theorem 7 can be found in Appendix B 3. Note that the result in Claim (iii) that V acts as a derivation on $\hat{\mathcal{K}}^\infty$ is reminiscent of the result of ter Elst and Lemańczyk²¹ in the L^2 setting, which was stated in Sec. II A. It is also worth noting that analogous results to Theorem 7(i) and (ii) hold for the Koopman groups on $\mathcal{K}_\tau$, and it can also be shown that the generators of these groups have the domain $\mathcal{K}_\tau^1$, defined analogously to $\hat{\mathcal{K}}_\tau^1$. However, our proof that V acts as a derivation on $\hat{\mathcal{K}}^\infty$ makes use of the Banach algebra structure of $\hat{\mathcal{K}}_\tau$, so it does not carry over in an obvious way to show that the generator acts as a derivation on $\mathcal{K}^\infty$ (even though that space is invariant under U^t).

C. Quantum dynamics and classical-quantum correspondence

We now study the quantum dynamics of the circle rotation associated with the RKHAs $\hat{\mathcal{K}}_\tau$ and its correspondence with the classical dynamics described in Sec. VI B. Starting with the basic definitions, as space of classical observables, we consider the RKHA $\hat{\mathcal{K}}_{\tau, \mathbb{R}}$ consisting of the real elements of $\hat{\mathcal{K}}_\tau$ for some $\tau > 0$. Note that $\hat{\mathcal{K}}_{\tau, \mathbb{R}}$ is a dense subalgebra of the Banach algebra $C_{\mathbb{R}}(S^1)$ of classical observables employed in Secs. IV and V, possessing the additional Hilbert space structure. As spaces of regular quantum states, observables, and generalized quantum states, we consider $\mathcal{Q}(\hat{\mathcal{K}}_\tau)$, $\mathcal{A}(\hat{\mathcal{K}}_\tau)$, and $\mathcal{A}'(\hat{\mathcal{K}}_\tau)$, respectively, defined analogously to their counterparts $\mathcal{Q}(L^2(\mu))$, $\mathcal{A}(L^2(\mu))$, and $\mathcal{A}'(L^2(\mu))$ from Sec. IV B. The dynamics on these spaces are governed by unitary operators $U^t: \hat{\mathcal{K}}_{\tau, \mathbb{R}} \rightarrow \hat{\mathcal{K}}_{\tau, \mathbb{R}}$ (the Koopman operators) and isometries $\Phi^t: \mathcal{Q}(\hat{\mathcal{K}}_\tau) \rightarrow \mathcal{Q}(\hat{\mathcal{K}}_\tau)$, $\tilde{U}^t: \mathcal{A}(\hat{\mathcal{K}}_\tau) \rightarrow \mathcal{A}(\hat{\mathcal{K}}_\tau)$, and $\tilde{U}''^t: \mathcal{A}'(\hat{\mathcal{K}}_\tau) \rightarrow \mathcal{A}'(\hat{\mathcal{K}}_\tau)$, defined analogously to $\Phi^t: \mathcal{Q}(L^2(\mu)) \rightarrow \mathcal{Q}(L^2(\mu))$, $\tilde{U}^t: \mathcal{A}(L^2(\mu)) \rightarrow \mathcal{A}(L^2(\mu))$, and $\tilde{U}''^t: \mathcal{A}'(L^2(\mu)) \rightarrow \mathcal{A}'(L^2(\mu))$, respectively.

Next, similar to $\mathbb{K}_\tau : \mathcal{P}(S^1) \rightarrow \mathcal{K}_\tau$ and $\Pi : C(S^1) \setminus \{0\} \rightarrow \mathcal{Q}(L^2(\mu))$, we define an RKHA embedding of probability measures $\hat{\mathbb{K}}_\tau : \mathcal{P}(S^1) \rightarrow \hat{\mathcal{K}}_{\tau,\mathbb{R}}$,

$$\hat{\mathbb{K}}_\tau(m) = \int_{S^1} \hat{\kappa}_\tau(\theta, \cdot) dm(\theta),$$

and a map $\hat{\Pi}_\tau : \hat{\mathcal{K}}_\tau \setminus \{0\} \rightarrow \mathcal{Q}(\hat{\mathcal{K}}_\tau)$,

$$\hat{\Pi}_\tau(f) = \frac{\langle f, \cdot \rangle_{\hat{\mathcal{K}}_\tau} f}{\|f\|_{\hat{\mathcal{K}}_\tau}^2}, \quad (65)$$

mapping nonzero RKHA functions into pure quantum states. The composition $\hat{\Psi}_\tau = \hat{\Pi}_\tau \circ \hat{F}_\tau$ then maps classical states in the circle to pure quantum states in $\mathcal{Q}(\hat{\mathcal{K}}_\tau)$. Note that by the reproducing property of $\hat{\mathcal{K}}_\tau$, $\langle \hat{F}_\tau(\theta), \cdot \rangle_{\hat{\mathcal{K}}_\tau}$ is equal to the evaluation functional $\mathbb{V}_\theta : \hat{\mathcal{K}}_\tau \rightarrow \mathbb{C}$ at θ so that

$$\hat{\Psi}_\tau(\theta) = \frac{\hat{\kappa}_\tau(\theta, \cdot) \mathbb{V}_\theta}{\hat{\kappa}_\tau(\theta, \theta)}.$$

That is, $\hat{\Psi}_\tau(\theta)$ acts on functions in $\hat{\mathcal{K}}_\tau$ by “reading off” their value at θ and multiplying the result by the normalized kernel section $\hat{\kappa}_\tau(\theta, \cdot)/\hat{\kappa}_\tau(\theta, \theta)$. Equipped with this map, we define the linear operator $\hat{\Omega}_\tau : \mathcal{A}(\hat{\mathcal{K}}_\tau) \rightarrow \hat{\mathcal{K}}_{\tau,\mathbb{R}}$ mapping quantum mechanical observables in $\mathcal{A}(\hat{\mathcal{K}}_\tau)$ to classical observables in $\hat{\mathcal{K}}_{\tau,\mathbb{R}}$ according to the formula [cf. Proposition 4(ii)]

$$(\hat{\Omega}_\tau A)(\theta) = \mathbb{E}_{\hat{\Psi}_\tau(\theta)} A.$$

In addition, we have the transpose map $\hat{\Omega}'_\tau : \hat{\mathcal{K}}'_{\tau,\mathbb{R}} \rightarrow \mathcal{A}'(\hat{\mathcal{K}}_\tau)$, $\hat{\Omega}'_\tau J = J \circ \hat{\Omega}_\tau$, where we note that $\hat{\Omega}'_\tau$ can be equivalently defined as a map on $\hat{\mathcal{K}}_{\tau,\mathbb{R}}$ by the canonical isomorphism between Hilbert spaces and their duals.

Defining, further, the unitary map $\hat{\mathcal{V}}_\tau : \hat{\mathcal{K}}_\tau \rightarrow L^2(\mu)$ by polar decomposition of the integral operator $\hat{K}_\tau : L^2(\mu) \rightarrow \hat{\mathcal{K}}_\tau$ associated with $\hat{\kappa}_\tau$ [cf. $\mathcal{V}_\tau$ in (10)], we can proceed as in Sec. IV C to construct an isometric embedding $\hat{\mathcal{W}}_\tau : \hat{\mathcal{K}}_\tau \rightarrow \mathcal{H}$ of the RKHA $\hat{\mathcal{K}}_\tau$ into the Hilbert space $\mathcal{H}$ of sections on Minkowski space, as well as a Banach algebra homomorphism $\hat{\mathcal{W}}_\tau : \mathcal{A}(\mathcal{H}) \rightarrow \mathcal{A}(\hat{\mathcal{K}}_\tau)$ and linear isometries $\hat{\mathcal{W}}_\tau^* : \mathcal{Q}(\hat{\mathcal{K}}_\tau) \rightarrow \mathcal{Q}(\mathcal{H})$ and $\hat{\mathcal{W}}_\tau' : \mathcal{A}'(\hat{\mathcal{K}}_\tau) \rightarrow \mathcal{A}'(\mathcal{H})$. These maps have analogous properties to those established in the $L^2(\mu)$ case, as depicted in Fig. 4 using commutative diagrams.

Several of these properties are stated as claims in Theorem 3(i)–(iii), and we will not repeat their derivation as the arguments are entirely analogous to those in Sec. IV. Instead, we will focus on the properties of the classical–quantum correspondence associated with $\hat{\mathcal{K}}_\tau$, which are not present in the $L^2(\mu)$ setting, namely, Claims (iv) and (v) of Theorem 3. To that end, consider first the map $T : \hat{\mathcal{K}}_\tau \rightarrow B(\hat{\mathcal{K}}_\tau)$ that maps f to the corresponding multiplication operator $T_f : g \mapsto Tg$ on $\hat{\mathcal{K}}_\tau$. Unlike the $L^2(\mu)$ case, T does not map each element of $\hat{\mathcal{K}}_{\tau,\mathbb{R}}$ to a self-adjoint operator.¹⁶ Nevertheless, we can construct a map $\hat{T} : \hat{\mathcal{K}}_{\tau,\mathbb{R}} \rightarrow \mathcal{A}(\hat{\mathcal{K}}_{\tau,\mathbb{R}})$ that maps classical observables in $\hat{\mathcal{K}}_{\tau,\mathbb{R}}$ to self-adjoint operators in

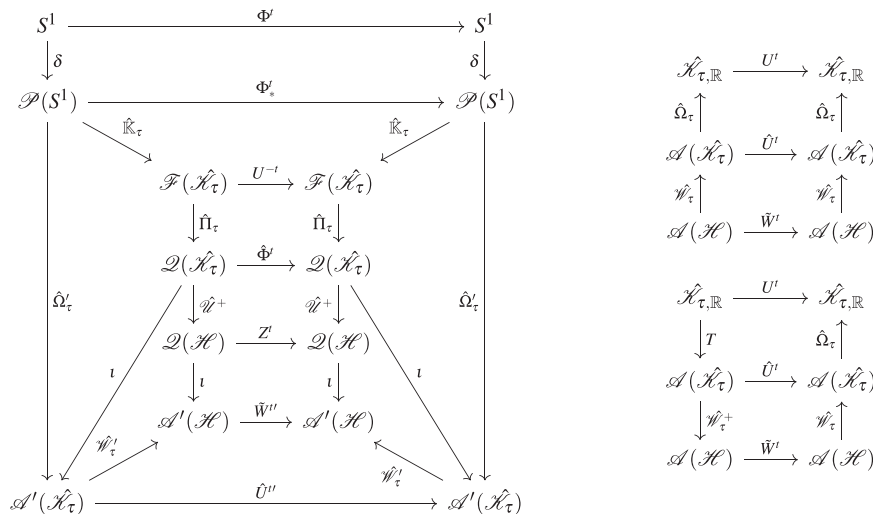


FIG. 4. As in Fig. 2, but for commutative diagrams showing the relationships between the classical/quantum dynamics associated with the RKHAs $\hat{\mathcal{K}}_\tau$ on the circle and the quantum harmonic oscillator on Minkowski space. Note that the commutative diagram in the bottom right cannot be drawn in the $L^2(\mu)$ setting of Fig. 4.

$\mathcal{A}(\hat{\mathcal{H}}_{\tau, \mathbb{R}})$ by symmetrization of T , viz., $\hat{T}f = (T_f + (T_f)^*)/2$. It then follows from standard properties of the trace that for any quantum state $\rho \in \mathcal{Q}(\hat{\mathcal{H}}_{\tau})$,

$$\mathbb{E}_{\rho}(\hat{T}f) = \frac{\text{tr}(\rho T_f) + \text{tr}(\rho (T_f)^*)}{2} = \frac{\text{tr}(\rho T_f) + (\text{tr}(\rho T_f))^*}{2}.$$

In particular,

$$\mathbb{E}_{\rho}(\hat{T}f) = \text{tr}(\rho T_f) \quad (66)$$

whenever f is real.

Next, observe that by (65), for every operator $A \in B(\hat{\mathcal{H}}_{\tau})$, we have

$$\begin{aligned} \text{tr}(\hat{\Psi}_{\tau}(\theta)A) &= \frac{\sum_{j=-\infty}^{\infty} \langle \hat{\phi}_{j,\tau}, \hat{\kappa}_{\tau}(\theta, \cdot) \nabla_{\theta}(A \hat{\phi}_{j,\tau}) \rangle_{\hat{\mathcal{H}}_{\tau}}}{\hat{\kappa}_{\tau}(\theta, \theta)} = \frac{\sum_{j=-\infty}^{\infty} (\nabla_{\theta} \hat{\phi}_{j,\tau})^* (\nabla_{\theta}(A \hat{\phi}_{j,\tau}))}{\hat{\kappa}_{\tau}(\theta, \theta)} \\ &= \frac{\sum_{j=-\infty}^{\infty} \hat{\phi}_{j,\tau}^*(\theta) (A \hat{\phi}_{j,\tau})(\theta)}{\sum_{k=-\infty}^{\infty} \hat{\phi}_{k,\tau}^*(\theta) \hat{\phi}_{k,\tau}(\theta)}. \end{aligned}$$

As a result, if A is a multiplication operator, i.e., $A = T_f$ for some $f \in \hat{\mathcal{H}}_{\tau}$, we have $(A \hat{\phi}_{j,\tau})(\theta) = f(\theta) \hat{\phi}_{j,\tau}(\theta)$, and the expression above simplifies to

$$\text{tr}(\hat{\Psi}_{\tau}(\theta)T_f) = f(\theta). \quad (67)$$

Combining (66) and (67), we conclude that for every classical observable $f \in \hat{\mathcal{H}}_{\tau, \mathbb{R}}$ and state $\theta \in S^1$,

$$f(\theta) = \mathbb{E}_{\hat{\Psi}_{\tau}(\theta)} \hat{T}f = (\hat{\Omega}_{\tau} \hat{T}f)(\theta).$$

That is, $\hat{\Omega}_{\tau} \circ \hat{T} = \text{Id}_{\hat{\mathcal{H}}_{\tau, \mathbb{R}}}$, as stated in Claim (iv) of Theorem 3. The dynamical compatibility condition in Claim (v) then follows directly from the definition of the operators involved and leads to an additional commutative diagram, displayed in the bottom right of Fig. 4.

Figure 5 displays the evolution of the wavefunction $\hat{\psi}_{\theta, \tau}^{\sigma_x}$ on Minkowski space underlying the quantum state $\hat{\sigma}_{\theta, \tau} = \hat{\mathcal{W}}_{\tau} \hat{\Psi}_{\tau}(\theta) \hat{\mathcal{W}}_{\tau}^*$, defined analogously to $\psi_{\theta, \tau}^{\sigma_x}$ from Sec. IV C. Here, we have $\hat{\psi}_{\theta, \tau}^{\sigma_x} = \zeta_{\sigma_x}^{-1} \psi_{\theta, \tau}^{\sigma_x}$, where $\psi_{\theta, \tau} \in \mathcal{H}$ is the section given by [cf. (52)]

$$\hat{\psi}_{\theta, \tau} = \sum_{j=0}^{\infty} e^{-|j|^p \tau / 2} (\phi_j(\theta) \psi_{j0} + \phi_j^*(\theta) \psi_{0j}). \quad (68)$$

The evolution of $\hat{\psi}_{\theta, \tau}^{\sigma_x}$ exhibits a qualitatively similar behavior to the evolution of $\psi_{\theta, \tau}^{\sigma_x}$ in Fig. 3 but with the notable difference that [due to the slower decay of the exponential coefficients $e^{-|j|^p \tau / 2}$ in (68) compared to $e^{-j^2 \tau}$ in (52)] $\hat{\psi}_{\theta, \tau}^{\sigma_x}$ develops smaller-scale oscillatory features than $\psi_{\theta, \tau}^{\sigma_x}$.

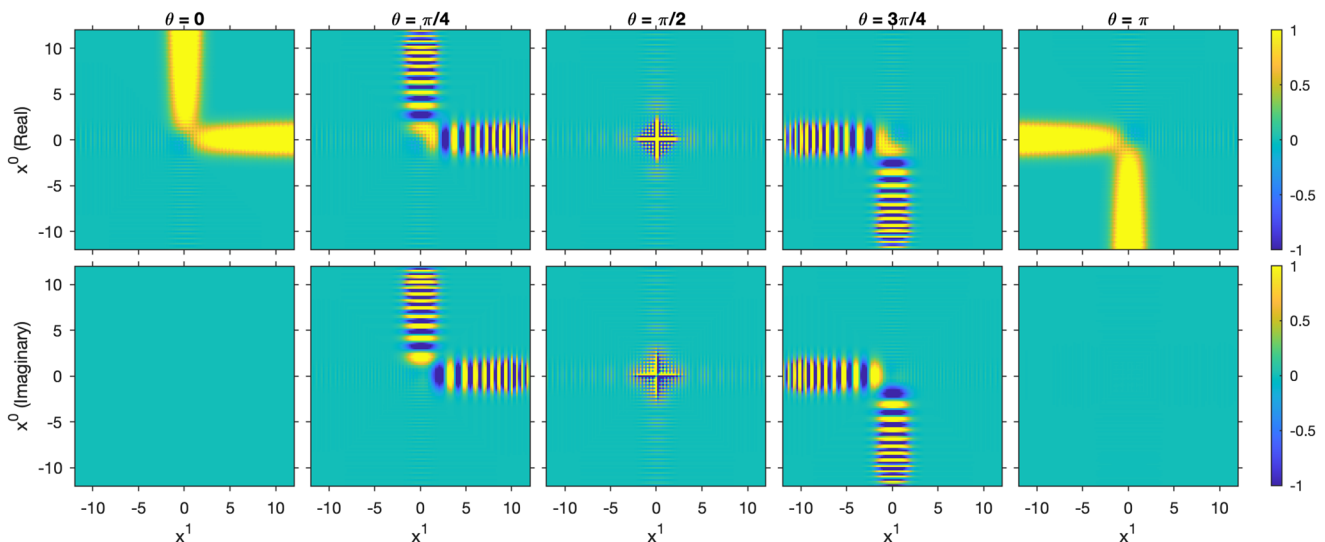


FIG. 5. As in Fig. 3, but for the wavefunction for the RKHA with $p = 1/2$ and $\tau = 10^{-3}$. Note the smaller-scale oscillatory features developing due to the sublinear growth of the eigenvalues of $\mathcal{L}^{p/2}$, as opposed to the quadratic growth of the eigenvalues of $\mathcal{L}$.

VII. CONCLUDING REMARKS

The classical–quantum correspondences described in this paper, culminating with the RKHA-based formulation in Sec. VI, provide a mutually consistent description of the dynamics of the circle rotation at three levels, namely, the dynamical flow $\Phi^t: S^1 \rightarrow S^1$ on state space, the evolution of classical probability measures $\Phi_*^t: \mathcal{P}(S^1) \rightarrow \mathcal{P}(S^1)$, and the evolution of quantum states $\hat{\Phi}^t: \mathcal{Q}(\hat{\mathcal{H}}_\tau) \rightarrow \mathcal{Q}(\hat{\mathcal{H}}_\tau)$ on a reproducing kernel Hilbert space associated with fractional diffusions of order $p/2$, $p \in (0, 1)$, having the additional structure of a Banach $*$ -algebra. Furthermore, these dynamics embed consistently into the evolution $Z^t: \mathcal{Q}(\mathcal{H}) \rightarrow \mathcal{Q}(\mathcal{H})$ of quantum states associated with an $SO^+(1, 1)$ gauge theory for the harmonic oscillator on two-dimensional Minkowski space, whose dynamics are generated by a Laplace-type operator acting on sections of an associated $\mathbb{C}$ -line bundle, induced by a Lorentz-invariant connection satisfying the Yang–Mills equations. In particular, the $SO^+(1, 1)$ action on the associated bundle is through a unitary, $U(1)$, representation so that the gauge theory employed in this work can be thought of as a two-dimensional analog of Maxwell electromagnetism. Using this correspondence, we have constructed ladder operators acting on classical observables of the circle rotation, which take the form of order-1/2 fractional derivatives, factorizing the positive- and negative-frequency components of the Koopman generator through number operators.

The geometrical and algebraic structure of the classical–quantum correspondences identified in this work, occurring for a dynamical system as simple as the circle rotation, motivates further study for more general systems, including systems with quasiperiodic or mixing dynamics, as well as systems with spatial structure (e.g., partial differential equation models), where non-Abelian structure groups may play a role.

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APPENDIX A: FIBER BUNDLES AND GAUGE THEORY

In this appendix, we collect various definitions and results on the theory of fiber bundles and gauge theory supporting the analysis in the main text. We begin by considering standard results on general fiber bundles (Appendix A 1) and then restrict attention to the principal and associated bundles over two-dimensional Minkowski space studied in this work (Appendix A 2). For additional details on this material, we refer the reader to one of the many textbooks in the literature, e.g., Refs. 24–26.

1. General fiber bundles

Throughout this section, G will be a smooth Lie group with Lie algebra $\mathfrak{g}$, $P \xrightarrow{\pi} M$ will be a smooth principal G -bundle, and $E \xrightarrow{\pi_E} M$ will be a smooth associated bundle with typical fiber F . Moreover, $P_m = \pi^{-1}(\{m\})$ and $E_m = \pi_E^{-1}(\{m\})$ will denote the fibers in P and E , respectively, over $m \in M$. As in the main text, we will use the equivalent notations $R^\Lambda(p) \equiv p \cdot \Lambda$ for the right action at $p \in P$ by the group element $\Lambda \in G$, and similarly $L^\Lambda(f) \equiv \Lambda \cdot f$ will represent the left action at $f \in F$.

a. Fiber-wise isomorphisms and metrics

In the main text, we make use of the results stated in the following two lemmas:

Lemma 8. For any $p \in P$ with $\pi(p) = m$, the map $\varepsilon_p: F \rightarrow E_m$, defined by $\varepsilon_p(f) = [p, f]$, is a bijection. Moreover, for every $\Lambda \in G$, the property $\varepsilon_{R^\Lambda(p)}^{-1} = L^{\Lambda^{-1}} \circ \varepsilon_p^{-1}$ holds.

Proof. Suppose that $\tilde{f} \in F$ is such that $[p, f] = [p, \tilde{f}]$. Then, by definition of the $[\cdot, \cdot]$ equivalence classes, there exists $\Lambda \in G$ such that $p \cdot \Lambda = p$ and $\Lambda^{-1} \cdot f = \tilde{f}$. However, because the action of G on P is free, Λ is equal to the identity element of G , and $\tilde{f} = f$. If now $p' \in P_m$ and $f' \in F$ are such that $[p, f] = [p', f']$, then there exists a unique $\Lambda' \in G$ such that $p' = p \cdot \Lambda'$, and therefore, $[p', f'] = [p \cdot \Lambda', f'] = [p, \Lambda' \cdot f']$. It then follows that $\Lambda \cdot f' = f$, meaning that there is a well-defined map from E_m to F mapping $[p', f']$ to the unique f such that $[p', f'] = [p, f]$. It is straightforward to verify that this map is the inverse of ε_p , proving that ε_p is a bijection.

Next, for any $e \in E_m$ and $p \in P_m$, we have $\varepsilon_p^{-1}e = f$, where f is the unique element of F such that $[p, f] = e$. Thus, for any $\Lambda \in G$,

$$\varepsilon_{R^\Lambda(p)}^{-1}(e) = \varepsilon_{R^\Lambda(p)}^{-1}([p, f]) = \varepsilon_{p \cdot \Lambda}^{-1}([p \cdot \Lambda, \Lambda^{-1} \cdot f]) = \Lambda^{-1} \cdot f = L^{\Lambda^{-1}}(\varepsilon_p^{-1}(e)),$$

proving the second claim and completing the proof of the lemma. $\square$

Lemma 9. Suppose that $E \xrightarrow{\pi_E} M$ is a vector bundle, whose typical fiber F is equipped with an inner product $\langle \cdot, \cdot \rangle_F$, and G acts on F via a unitary representation $\rho: G \rightarrow U(F)$. Then, the fiber-wise metric $g_m: E_m \times E_m \rightarrow \mathbb{C}$, constructed analogously to (16) using a local section $\sigma: U \rightarrow P$ whose domain $U \subseteq M$ contains m , is independent of the choice of σ .

Proof. Let $\tilde{\sigma}: \tilde{U} \rightarrow P$ be an arbitrary smooth section whose domain $\tilde{U}$ contains m and $\iota_{\tilde{\sigma}, E}: \tilde{U} \times F \rightarrow E$ be the associated trivializing map, constructed analogously to $\iota_{\sigma, E}$. We must show that for any $e_1, e_2 \in E_m$, $\tilde{g}_m(e_1, e_2) := \langle \varepsilon_{\tilde{\sigma}(m)}(e_1), \varepsilon_{\tilde{\sigma}(m)}(e_2) \rangle_F$ is equal to $g_m(e_1, e_2) = \langle \varepsilon_{\sigma(m)}(e_1), \varepsilon_{\sigma(m)}(e_2) \rangle_F$. For that, observe that $\tilde{g}_m(e_1, e_2) = \langle \tilde{f}_1, \tilde{f}_2 \rangle_F$ and $g_m(e_1, e_2) = \langle f_1, f_2 \rangle_F$, where, by Lemma 8, $\tilde{f}_j$ and f_j are unique elements in F such that $[\tilde{\sigma}(m), \tilde{f}_j] = [\sigma(m), f_j] = e_j, j \in \{1, 2\}$. Because G acts on P freely, there exists a unique $\Lambda \in G$ such that $\tilde{\sigma}(m) = \sigma(m) \cdot \Lambda$, and thus,

$$[\tilde{\sigma}(m), \tilde{f}_j] = [\sigma(m) \cdot \Lambda, \tilde{f}_j] = [\sigma(m), \Lambda \cdot \tilde{f}_j].$$

Using again Lemma 8, it follows that $\rho(\Lambda)\tilde{f}_j = \Lambda \cdot \tilde{f}_j = f_j$, and by unitarity of $\rho(\Lambda)$,

$$\tilde{g}_m(e_1, e_2) = \langle \tilde{f}_1, \tilde{f}_2 \rangle_F = \langle \rho(\Lambda)\tilde{f}_1, \rho(\Lambda)\tilde{f}_2 \rangle_F = \langle f_1, f_2 \rangle_F = g_m(e_1, e_2),$$

as claimed. $\square$

b. Connection 1-forms, covariant derivatives, and curvature tensors

A *connection 1-form* on a principal bundle $P \xrightarrow{\pi} M$ with structure group G and Lie algebra $\mathfrak{g}$ is a Lie-algebra-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, satisfying the following conditions for every point $p \in P$, group element $\Lambda \in G$, Lie algebra element $\lambda \in \mathfrak{g}$, fundamental vector field W^λ , and tangent vector $W \in T_p P$:

$$\omega W_p^\lambda = \lambda, \quad (\text{A1})$$

$$(R^{\Lambda*} \omega)_p W = \text{Ad}_{\Lambda^{-1}} \omega_p W. \quad (\text{A2})$$

In (A2), $R^{\Lambda*}: \Omega^1(P, \mathfrak{g}) \rightarrow \Omega^1(P, \mathfrak{g})$ is the pullback map on $\mathfrak{g}$ -valued 1-forms associated with the right action $R^\Lambda: P \rightarrow P$ by group element $\Lambda \in G$, and $\text{Ad}_{\Lambda^{-1}}: \mathfrak{g} \rightarrow \mathfrak{g}$ is the representative of Λ under the adjoint representation of G . The latter is equal to the pushforward map on tangent vectors to G associated with the map $\text{AD}_\Lambda: G \rightarrow G$, $\text{AD}_{\Lambda^{-1}}\Lambda' = \Lambda\Lambda'\Lambda^{-1}$, evaluated at the identity I , that is, $\text{Ad}_{\Lambda^{-1}} = \text{AD}_{\Lambda^{-1}*I}$. If G is Abelian, $\text{AD}_\Lambda = \text{Id}_G$ and $\text{Ad}_\Lambda = \text{Id}_{\mathfrak{g}}$ for all $\Lambda \in G$, and (A2) reduces to the G -invariance condition in (17).

Every such connection 1-form is equivalent to assigning a *horizontal distribution* on TP , that is, a smooth assignment of a subspace $H_p P \subseteq T_p P$, called *horizontal subspace*, at every $p \in P$ such that $H_p P \oplus V_p P = T_p P$, and the equivariance condition $R_*^\Lambda H_p P = H_{R^\Lambda(p)} P$ holds for every $\Lambda \in G$. At any $p \in P$, $H_p P$ is equal to the kernel of ω_p . Moreover, we have the vertical and horizontal projection maps $\text{ver}_p: T_p P \rightarrow T_p P$ and $\text{hor}_p: T_p P \rightarrow T_p P$, where $\text{ran ver}_p = V_p P$, $\text{ran hor}_p = H_p P$, and

$$\text{ver}_p X = W^{\omega_p X}, \quad \text{hor}_p X = X - \text{ver}_p X.$$

The pointwise maps ver_p and hor_p naturally extend to maps $\text{ver}: \Gamma(E) \rightarrow \Gamma(E)$ and $\text{hor}: \Gamma(E) \rightarrow \Gamma(E)$ on sections, respectively [cf. (19)].

The horizontal distribution also induces a notion of *parallel transport* of curves on the base space M to curves on the total space P . In particular, if $C: (-1, 1) \rightarrow M$ is a smooth curve on the base space passing through $C(0) = m$, then for every point $p \in P_{\pi^{-1}(m)}$, there exists a unique curve $C_p^p: (-1, 1) \rightarrow P$ on the principal bundle such that (i) $C_p^p(0) = p$; (ii) $\pi \circ C_p^p = C$; and (iii) for all $s \in (-1, 1)$, the tangent vector $\dot{C}_{p,p'}^p$ to C_p^p at $p' = C_p^p(s)$ lies in $H_{p'} P$, and $\pi_* \dot{C}_{p,p'}^p = \dot{C}_{m'}$, where $\dot{C}_{m'} \in T_{m'} M$ is the tangent vector to C at $m' = \pi(p')$. Given a vector field $X \in \Gamma(TM)$ with the integral curve $C_m: (-1, 1) \rightarrow M$, $m \in M$, parameterized such that $C_m(0) = m$, the assignment $p \mapsto \dot{C}_{p,p}^p \in T_p P$, $p \in \pi^{-1}(\{m\})$, defines a smooth vector field $Y \in \Gamma(TP)$ with $Y_p = \dot{C}_{p,p}^p$, called the *horizontal lift* of X . By construction, the vector field Y is projectible under π_* , satisfying $\pi_* Y = X$. It should be noted that analogous notions of horizontal subspace, parallel transport, and horizontal lift of vector fields can be defined for any associated bundle to $P \xrightarrow{\pi} M$, but we will not need these concepts here.

A *covariant derivative*, or *connection*, on a real vector bundle $E \xrightarrow{\pi_E} M$ is a linear map $\nabla: \Gamma(E) \rightarrow \Omega^1(M, E)$ satisfying the Leibniz rule,

$$\nabla(fs) = df \otimes s + f \nabla s, \quad \forall f \in C^\infty(M), \quad \forall s \in \Gamma(E).$$

If $X \in \Gamma(TM)$ is a vector field, ∇ induces a linear map $\nabla_X: \Gamma(E) \rightarrow \Gamma(E)$, also referred to as the covariant derivative, given by $\nabla_X s = (\nabla s)(X)$. This map has the properties

$$\begin{aligned} \nabla_X s &= f \nabla_X s, \\ \nabla_{X+Y} s &= \nabla_X s + \nabla_Y s, \\ \nabla_X(fs) &= X(f)s + f \nabla_X s \end{aligned} \quad (\text{A3})$$

for all $f \in C^\infty(M)$, $Y \in \Gamma(TM)$, and $s \in \Gamma(E)$. It is straightforward to verify that ∇_X from (20) satisfies these properties and is thus a covariant derivative operator. If $E \xrightarrow{\pi_E} M$ is complex, then the covariant derivative is constructed as above, but with X and Y taken to be sections of the complexified tangent bundle $T_{\mathbb{C}} M$.

The covariant derivative $\nabla: \Gamma(E) \rightarrow \Omega^1(M, E)$ on sections induces covariant derivatives $\nabla: \Omega^k(M, E) \rightarrow \Omega^{k+1}(M, E)$ to k -forms with values in E . Specifically, given $s \in \Omega^k(M, E)$ and a collection $X_0, \dots, X_k$ of vector fields in $\Gamma(TM)$, we define

$$(\nabla s)(X_0, \dots, X_k) \equiv \nabla_{X_0, \dots, X_k} s = \sum_{j=0}^k (-1)^j \nabla_{X_j} (s(X_0, \dots, \hat{X}_j, \dots, X_k)) + \sum_{0 \leq j < l \leq k} (-1)^{j+l} s([X_j, X_l], X_0, \dots, \hat{X}_j, \dots, \hat{X}_l, \dots, X_k), \quad (\text{A4})$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields and $\hat{\cdot}$ indicates a missing term. It can then be verified that the Leibniz rule,

$$\nabla(w \wedge s) = dw \wedge s + (-1)^l w \wedge \nabla s,$$

holds for any $s \in \Omega^k(M, E)$ and $w \in \Omega^l(M)$.

If $E \xrightarrow{\pi_E} M$ is an associated vector bundle to a principal bundle $P \xrightarrow{\pi} M$ with typical fiber F , onto which G acts linearly through a representation $\rho: G \rightarrow \text{GL}(F)$ [i.e., $L^\Lambda = \rho(\Lambda)$], the definition for ∇ in (A4) is consistent with first defining an exterior covariant derivative $\mathcal{D}: \Omega_G^k(P, F) \rightarrow \Omega_G^k(P, F)$ on G -equivariant, F -valued k -forms on the principal bundle and pulling back $\mathcal{D}$ to an operator on E -valued k -forms on the base space, analogously to the construction of $\nabla: \Gamma(E) \rightarrow \Omega^1(M, E)$ from $\mathcal{D}: C_G^\infty(P, F) \rightarrow \Omega_G^1(P, F)$ in Sec. III F.

Unlike standard exterior derivatives, ∇^2 and $\mathcal{D}^2$ are, in general, nonzero operators, which play an important role in defining notions of curvature for vector bundles and principal bundles. Focusing, for now, on the vector bundle $E \xrightarrow{\pi_E} M$, it can be shown that $\nabla^2: \Gamma(E) \rightarrow \Omega^2(M, E) = \Gamma(\wedge^2 T^*M \otimes E)$, given according to (A4) by

$$\nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \quad X, Y \in \Gamma(TM),$$

acts as a multiplication operator by a 2-form taking values in the endomorphism bundle $\text{End } E \rightarrow M$ of E . That is, we have

$$\nabla_{X,Y}^2 s = R_{X,Y} s \quad (\text{A5})$$

for some $R \in \Omega^2(M, \text{End } E)$, and despite appearances, ∇^2 is a zeroth-order differential operator since $R_{X,Y}$ is a $(1, 1)$ tensor. The tensor field R is called the *curvature tensor*, or *curvature endomorphism*, associated with the connection ∇ . It can further be shown that R satisfies a *differential Bianchi identity*,

$$\nabla R = 0,$$

where $\nabla: \Omega^2(M, \text{End } E) \rightarrow \Omega^3(M, \text{End } E)$ is a connection induced canonically from $\nabla: \Omega^2(M, E) \rightarrow \Omega^3(M, E)$ on tensor bundles of $E \xrightarrow{\pi_E} M$. If R vanishes, the connection ∇ is said to be *flat*.

It should be noted that ∇^2 is different from the second covariant derivative $H: \Gamma(TM) \rightarrow \Gamma(T^*M \otimes T^*M \otimes E)$ in Sec. III G in that the former takes values in the product bundle of E and the second exterior power of T^*M [consisting of antisymmetric $(0, 2)$ tensors], whereas the latter takes values in the product bundle of E and the second tensor power of T^*M [consisting of general $(0, 2)$ tensors]. Nevertheless, the two operators are related, in the sense that ∇^2 represents the antisymmetric component of H . Specifically, because the Levi-Civita connection is *torsion-free*, i.e.,

$$\nabla_X^{\text{LC}} Y - \nabla_Y^{\text{LC}} X = [X, Y], \quad \forall X, Y \in \Gamma(TM),$$

it follows that

$$\nabla_{X,Y}^2 = H_{X,Y} - H_{Y,X}.$$

c. Adjoint bundles

First, we describe the *nonlinear adjoint bundle* $\text{AD } P \xrightarrow{\pi_{\text{AD } P}} M$ associated with the principal bundle $P \xrightarrow{\pi} M$. $\text{AD } P \xrightarrow{\pi_{\text{AD } P}} M$ is defined as the fiber bundle over M with typical fiber $F = G$ and left group action on F given by $\Lambda \cdot f = \text{AD}_\Lambda f$, where $\Lambda, f \in G$. A section $\varsigma \in \Gamma(\text{AD } P)$ defines a function $\tilde{\xi}: P \rightarrow G$ such that $\tilde{\xi}(p)$ is the unique element of G satisfying $\varsigma(m) = [p, \tilde{\xi}(p)]$. Moreover, for every $\Lambda \in G$, we have

$$[p, \tilde{\xi}(p)] = [p \cdot \Lambda, \Lambda^{-1} \cdot \tilde{\xi}(p)] = [p \cdot \Lambda, \text{AD}_{\Lambda^{-1}} \tilde{\xi}(p)].$$

Because $\varsigma(m) = [p \cdot \Lambda, \tilde{\xi}(p \cdot \Lambda)]$, it follows that $\tilde{\xi}(p \cdot \Lambda) = \text{AD}_{\Lambda^{-1}} \tilde{\xi}(p)$, and thus, $\Lambda \cdot \tilde{\xi}(p \cdot \Lambda) = \tilde{\xi}(p) \cdot \Lambda$. The latter relation implies, in turn, that $\varphi_\varsigma: P \rightarrow P$ with $\varphi_\varsigma(p) = p \cdot \tilde{\xi}(p)$ satisfies the commutative diagram in (36) and is thus a gauge transformation. Conversely, starting from a gauge transformation $\varphi: P \rightarrow P$, one can construct a section $\varsigma_\varphi \in \Gamma(\text{AD } P)$ by reversing the steps leading to φ_ς from ς so that every gauge transformation φ induces a section ς_φ . One can further verify that $\varsigma = \varsigma_{\varphi_\varsigma}$ and $\varphi = \varphi_{\varsigma_\varphi}$ so that the gauge group $\mathcal{G}$ can be identified with $\Gamma(\text{AD } P)$.

If now $e = [p, f]$, with $p \in P$ and $f \in F$, is a point on an associated bundle $E \xrightarrow{\pi_E} M$ to $P \xrightarrow{\pi} M$, lying above $m = \pi(p) = \pi_E(e)$, the action $\varphi_*(e)$ of the pushforward map $\varphi_*: E \rightarrow E$ associated with the gauge transformation φ is given by

$$\varphi_*(e) = [\varphi(p), f] = [p \cdot \tilde{\xi}(p), f] = [p, \tilde{\xi}(p) \cdot f]. \quad (\text{A6})$$

Thus, if $\sigma: U \rightarrow P$ is a trivializing section defined on $U \subseteq M$, we have

$$\varepsilon_{\sigma(m)}^{-1}(\varphi_*(e)) = \tilde{\xi}(\sigma(m))\varepsilon_{\sigma(m)}^{-1}(e). \quad (\text{A7})$$

Note that if G is Abelian, $\text{Ad}_\Lambda = \text{Id}$, and every equivalence class $[p, \Lambda] \in \text{ADP}$ lying above a given point $m \in M$ is characterized by a unique group element $\Lambda \in G$. As a result, ADP is canonically isomorphic to $M \times G$, and every section $\varsigma \in \Gamma(\text{AD } P)$ induces a function $\xi: M \rightarrow G$ such that $\varsigma(m) = [p, \xi(m)]$, where p is an arbitrary point in P_m . In particular, the gauge transformation $\varphi_\varsigma: P \rightarrow P$ associated with ς is given by $\varphi_\varsigma(p) = p \cdot \xi(\pi(p))$, and (A7) becomes

$$\varepsilon_{\sigma(m)}^{-1}(\varphi_*(e)) = \xi(m)\varepsilon_{\sigma(m)}^{-1}(e).$$

We now consider the *adjoint bundle* $\text{Ad}P \xrightarrow{\pi_{\text{Ad}P}} M$, which is a distinct bundle from $\text{ADP} \xrightarrow{\pi_{\text{ADP}}} M$, also playing an important role in gauge theory. In particular, $\text{Ad}P \xrightarrow{\pi_{\text{Ad}P}} M$ is defined as the vector bundle over M associated with $P \xrightarrow{\pi} M$ with typical fiber equal to the Lie algebra of G , $F = \mathfrak{g}$, and left action given by the adjoint representation, $L^\Lambda \lambda = \text{Ad}_\Lambda \lambda$. This bundle is useful for characterizing the space of connection 1-forms on P , denoted by $\mathcal{C}$. That is, it can be shown that $\mathcal{C}$ is an infinite-dimensional affine space with translation group isomorphic to $\Omega^1(M, \text{Ad}P)$, the space of 1-forms on the base space M with values in $\text{Ad}P$. If the structure group G is Abelian, $\text{Ad}_\Lambda = \text{Id}$, and the equivalence classes $[p, \lambda] \in \text{Ad } P$ are characterized by unique Lie algebra elements $\lambda \in \mathfrak{g}$. It then follows that $\text{Ad } P$ is canonically isomorphic to $M \times \mathfrak{g}$ (analogously to the isomorphism of ADP to $M \times G$) and that sections in $\Gamma(\text{Ad}P)$ are canonically identified with $\mathfrak{g}$ -valued functions on M (analogously to the identification of sections in $\Gamma(\text{ADP})$ with G -valued functions).

d. Curvature 2-forms and field strengths

In addition to characterizing the space of connections, the adjoint bundle plays an important role in the context of curvature, as we now describe. First, as already stated in Sec. III H, the *curvature 2-form* associated with a connection 1-form $\omega \in \mathcal{C}$ is the Lie-algebra valued 2-form $\Omega \in \Omega^2(P, \mathfrak{g})$ given by the covariant exterior derivative $\Omega = \mathcal{D}\omega$. It can be shown that Ω obeys the structure equation

$$\Omega(Y, Z) = d\omega(Y, Z) + [\omega Y, \omega Z], \quad (\text{A8})$$

where $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is the Lie algebra commutator. This leads to the *Bianchi identity*,

$$\mathcal{D}\Omega = \mathcal{D}^2\omega = 0, \quad (\text{A9})$$

which is a non-trivial result since, as noted in Appendix A 1 b, $\mathcal{D}$ is, in general, a nonzero operator. Equation (A8) manifestly exhibits the fact that Ω depends nonlinearly on ω if G is non-Abelian, whereas in the Abelian case, that dependence is affine. By examining pullbacks $\sigma^*\Omega \in C^\infty(U, \mathfrak{g})$ along trivializing sections $\sigma: U \subseteq M \rightarrow P$ of the principal bundle, Ω can be identified with a 2-form $F^\omega \in \Omega^2(M, \text{Ad } P)$ taking values in the adjoint bundle. The 2-form F^ω is known as the *gauge field strength* associated with ω .

Let now $E \xrightarrow{\pi_E} M$ be a vector bundle with typical fiber F , acted upon by G through a representation $\rho: G \rightarrow \text{GL}(F)$. Let also $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(F)$ be the Lie algebra representation induced by ρ through its differential at the identity, $\rho = \rho_{*,I}$. Then, if $\nabla: \Gamma(E) \rightarrow \Omega^1(M, E)$ is the covariant derivative associated with ω , the Bianchi identity in (A9) is equivalent to

$$\nabla F^\omega = 0,$$

and the corresponding curvature tensor $R \in \Omega^2(M, \text{End } E)$ from (A5) satisfies

$$(R_{X,Y} s)_m = [p, \rho(\Omega(X, Y))f].$$

Here, m is a point in M , X, Y are vector fields in $\Gamma(TM)$, and s is a section in $\Gamma(E)$ with $s(m) = [p, f]$. Specializing these results to the case of the adjoint bundle, $E = \text{Ad } P$, ρ becomes the adjoint representation of $\mathfrak{g}$, and thus,

$$(R_{X,Y} s)_m = [p, \text{ad}_{\Omega(X,Y)} f].$$

Moreover, on the domain $U \subseteq M$ of any trivializing section $\sigma: M \rightarrow P$, we have [cf. (21)]

$$\nabla_X^\sigma f = df \cdot X + \text{ad}_{\omega^\sigma X} f, \quad f \in C^\infty(U, \mathfrak{g}), \quad X \in \Gamma(TU).$$

From the last two equations, we deduce that if G is Abelian (i.e., $\text{ad}_\lambda = 0$ for all $\lambda \in \mathfrak{g}$), then every connection 1-form $\omega \in \mathcal{C}$ induces a flat connection ∇ on the adjoint bundle, which coincides with the standard exterior derivative.

e. Outline of Yang-Mills theory

In this section, we briefly outline the aspects of Yang-Mills theory leading to the constructions in Sec. III H. As in Appendix A 1 d, let $\omega \in \mathcal{C}$ be a connection 1-form on the principal bundle, and let $\text{Ad}P \xrightarrow{\pi_{\text{Ad}P}} M$ be the adjoint bundle, equipped with the covariant derivative ∇ induced from ω . In Yang-Mills theory, it is assumed that the base space manifold M is equipped with a Riemannian or pseudo-Riemannian metric η with volume form v , and the Lie algebra $\mathfrak{g}$ is similarly equipped with a non-degenerate Ad-invariant sesquilinear form $b: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ (cf. b from Sec. III B). Then, for every $m \in M$, one can define a sesquilinear form g_m on $\text{Ad } P_m$ analogously to (16), as well as pointwise

sesquilinear forms for k -forms in $\Omega^k(M, \text{Ad}P)$ and corresponding norms; the latter is denoted by $\|\cdot\|_m$. The Hodge star operator $\star: \Omega^k(M) \rightarrow \Omega^{\dim M - k}(M)$ also lifts canonically to an operator $\star: \Omega^k(M, \text{Ad}P) \rightarrow \Omega^{\dim M - k}(M, \text{Ad}P)$.

With these definitions, the *Yang–Mills action* associated with a connection ω is given by

$$S_{\text{YM}}(\omega) = \int_M \|F^\omega\|_m^2 dv(m),$$

whenever the integral exists. One can verify that S_{YM} is gauge-invariant.

If M is compact, then $S_{\text{YM}}(\omega)$ exists for all connections in $\mathcal{C}$. In the non-compact case, including the Minkowski space studied here, the domain of definition of S_{YM} is a proper subset of $\mathcal{C}$. The Yang–Mills condition states that ω should be a critical point of this functional; that is, for any section $s \in \Gamma(\text{Ad}P)$,

$$\left. \frac{d}{d\tau} S_{\text{YM}}(\omega + \tau s) \right|_{\tau=0} = 0.$$

This condition leads to the *Yang–Mills equation*,

$$\nabla \star F^\omega = 0,$$

which represents a system of partial differential equations for local gauge fields $\omega^\sigma = \sigma^* \omega$ associated with local trivializing sections $\sigma: U \subseteq M \rightarrow P$ of the principal bundle. Due to the nonlinear dependence of Ω on ω , these equations are nonlinear if G is non-Abelian. Note that if M is non-compact, a connection 1-form ω can satisfy the Yang–Mills equation while having infinite action.

2. Fiber bundles over Minkowski space

We now restrict attention to the setting where $P \xrightarrow{\pi} M$ and $E \xrightarrow{\pi_E} M$ are the principal and associated bundles introduced in Sec. III over two-dimensional Minkowski space, with the structure group $G \simeq \text{SO}^+(1, 1)$. We begin with three lemmas establishing basic properties of Cartesian charts on M and their induced charts on P .

Lemma 10. Let $x: M \rightarrow \mathbb{R}^2$ be an inertial chart with origin o and basis vector fields X_j , and $y: P \rightarrow \mathbb{R}^3$ be the induced chart on the principal bundle. Let also L_o^Λ be a Lorentz transformation associated with group element $\Lambda \in G$. Then, the following hold:

- (i) $x' = x \circ L_o^\Lambda$ is an inertial chart, whose coordinates and basis vector fields satisfy

$$x'^i(m) = x^i(L_o^\Lambda(m)) = \sum_{j=0}^1 \Lambda_j^i x^j(m), \quad X'_i = \sum_{j=0}^1 X_j \Lambda^{-1j}_i,$$

respectively. Here, Λ_j^i are constant coefficients given by

$$\Lambda_j^i = dx^i \cdot \Lambda X_j = \hat{X}^i \cdot \Lambda \vec{X}_j,$$

and the Λ^{-1j}_i are defined similarly with Λ replaced by Λ^{-1} .

- (ii) The trivializing section $\sigma_{x'}: M \rightarrow P$ and maps $\iota_{\sigma_{x'}}: M \times G \rightarrow P$, $\gamma_{\sigma_{x'}}: P \rightarrow G$ associated with x' have the equivariance properties

$$\sigma_{x'} = R^\Lambda \circ \sigma_x, \quad \iota_{\sigma_{x'}} = R^\Lambda \circ \iota_{\sigma_x}, \quad \gamma_{\sigma_{x'}} = L^{\Lambda^{-1}} \circ \gamma_{\sigma_x}.$$

- (iii) The coordinates and basis vector fields of the coordinate chart $y': P \rightarrow \mathbb{R}^3$ induced by x' satisfy

$$y'^i(p) = \sum_{j=0}^1 \Lambda_j^i y^j(p), \quad Y'_i = \sum_{j=0}^1 Y_j \Lambda^{-1j}_i, \quad i \in \{0, 1\},$$

$$y'^2(p) = y^2(p) - \vartheta(\Lambda), \quad Y'_2 = Y_2,$$

where $\vartheta: G \rightarrow \mathbb{R}$ is the coordinate chart on the gauge group from Sec. III B.

Proof. (i) Let $m \in M$ be arbitrary. Then, by the definition of the x chart, $m = o + \vec{v}$, where $\vec{v} = \sum_{j=0}^1 x^j(m) \vec{X}_j$, and

$$L_o^\Lambda(m) = o + \sum_{j=0}^1 x^j(m) \Lambda \vec{X}_j = o + \sum_{j,k=0}^1 \vec{X}_k \Lambda^k_j x^j(m),$$

where $\Lambda^k_j = \hat{X}^k \cdot \Lambda \vec{X}_j$. Therefore,

$$x'^i(m) = x^i(L_o^\Lambda(m)) = \sum_{j=0}^1 \Lambda_j^i x^j(m),$$

which confirms the claimed relationship between the x' and x coordinates. Moreover,

$$\vec{v} = \sum_{j=0}^1 x^j(m) \vec{X}_j = \sum_{i,j=0}^1 \Lambda^{-1j} x'^i(m) \vec{X}_j = \sum_{i=0}^1 x'^i(m) \left(\sum_{j=0}^1 X_j \Lambda^{-1j} \right)_i,$$

which implies that $X'_i = \sum_{j=0}^1 X_j \Lambda^{-1j}_i$, as claimed. The equality of $dx^i \cdot \Lambda X_j$ and $\hat{X}^i \cdot \Lambda \vec{X}_j$ follows by definition of inertial charts.

(ii) The claim about $\sigma_{x'}$ follows directly from the fact that for every $m \in M$,

$$\sigma_{x'}(m) = \{X'_{0,m}, X'_{1,m}\} = \{\Lambda^{-1} X_{0,m}, \Lambda^{-1} X_{1,m}\} = \sigma_x(m) \cdot \Lambda = R^\Lambda(\sigma_x(m)).$$

Moreover, for every $p \in P$, we have $\iota_{\sigma_{x'}}(p) = (m, \gamma_{\sigma_{x'}}(p))$, where $\gamma_{\sigma_{x'}}(p)$ is the unique element of G satisfying $\sigma_{x'}(m) \cdot \gamma_{\sigma_{x'}}(p) = p$. However, by the result just proved, $\sigma_{x'}(m) = \sigma_x(m) \cdot \Lambda$, and therefore, $\sigma_x(m) \cdot \Lambda \gamma_{\sigma_{x'}}(p) = p$. The latter implies that $\Lambda \cdot \gamma_{\sigma_{x'}}(p) = \gamma_{\sigma_x}(p)$, leading to the claim about $\gamma_{\sigma_{x'}}$. To prove the claim about $\iota_{\sigma_{x'}}$, observe that for every $m \in M$ and $\tilde{\Lambda} \in G$,

$$\iota_{\sigma_{x'}}(m, \tilde{\Lambda}) = \sigma_{x'}(m) \cdot \tilde{\Lambda} = \sigma_x(m) \cdot \Lambda \tilde{\Lambda} = \sigma_x(m) \cdot \tilde{\Lambda} \Lambda = \iota_{\sigma_x}(m, \tilde{\Lambda}) \cdot \Lambda,$$

and the claim follows.

(iii) By definition of the y' chart, for any $p \in P$, we have

$$y'(p) = (x' \otimes \vartheta) \iota_{\sigma_{x'}}^{-1}(p) = (x'(\pi(p)), \vartheta(\gamma_{\sigma_{x'}}(p))).$$

Therefore, for $i \in \{0, 1\}$, Claim (i) implies that

$$y'^i(p) = x'^i(\pi(p)) = \sum_{j=0}^1 \Lambda^i_j x^j(\pi(p)) = \sum_{j=0}^1 \Lambda^i_j y^j(p).$$

Moreover, it follows from Claim (ii) and (12) that

$$y'^2(p) = \vartheta(\gamma_{\sigma_{x'}}(p)) = \vartheta(\Lambda^{-1} \gamma_{\sigma_x}(p)) = \vartheta(\Lambda^{-1}) + \vartheta(\gamma_{\sigma_x}(p)) = -\vartheta(\Lambda) + y^2(p),$$

which proves the claimed relation between $y'(p)$ and $y(p)$. Turning to the coordinate basis vector fields, for any $f \in C^\infty(P)$, we have

$$\begin{aligned} Y'_{i,p} f &= \partial_j (f \circ y'^{-1})_{y'(p)} = \partial_j (f \circ y^{-1} \circ y \circ y'^{-1})_{y'(p)} \\ &= \sum_{j=0}^2 \partial_j (f \circ y^{-1})_{y(p)} \partial_i (y^j \circ y'^{-1})_{y'(p)} = \sum_{j=0}^2 (Y_{j,p} f) \partial_i (y^j \circ y'^{-1})_{y'(p)}, \end{aligned}$$

Thus, by the results just established,

$$\partial_i (y^j \circ y'^{-1})_{y'(p)} = \begin{cases} \Lambda^{-1j}_i, & i, j \in \{0, 1\} \\ 1, & i = j = 2 \\ 0, & \text{otherwise,} \end{cases}$$

and we conclude that $Y'_{i,p} f = \sum_{j=0}^1 Y_{j,p} f \Lambda^{-1j}_i$ for $i \in \{0, 1\}$ and $Y'_{2,p} f = Y_{2,p} f$. This completes the proof of the claim and the lemma. $\square$

Lemma 11. Let $y : P \rightarrow \mathbb{R}^3$ be the coordinate chart on the principal bundle induced by an inertial chart $x : M \rightarrow \mathbb{R}^2$, and let Y_j and X_j be the associated coordinate vector fields. Then, for any $\Lambda \in G$, the Y_j are invariant under the right action R^Λ on the principal bundle, i.e., $R^\Lambda_* Y_j = Y_j$. Moreover, Y_2 is a fundamental vector field generated by the Lie algebra basis vector u , i.e., $Y_2 = W^u$.

Proof. Let $p = \{p_0, p_1\} \in P$ be a point on the principal bundle lying above $\pi(p) = m$, and $p' = R^{\Lambda^{-1}}(p)$. We begin by observing that

$$y^0(p') = y^0(p), \quad y^1(p') = y^1(p), \quad y^2(p') = y^2(p) - \vartheta(\Lambda). \quad (\text{A10})$$

Indeed, the first expression in the above equation follows immediately from the facts that $\pi(p') = m$ and

$$y^0(p') = x^0(\pi(p')) = x^0(\pi(p)) = y^0(p).$$

The second expression follows similarly. Meanwhile, the third expression follows from the G -equivariance of γ_{σ_x} , viz.,

$$\gamma^2(p') = \vartheta(\gamma_{\sigma_x}(p')) = \vartheta(\gamma_{\sigma_x}(R^{\Lambda^{-1}}(p))) = \vartheta(L^{\Lambda^{-1}}(\gamma_{\sigma_x}(p))) = \vartheta(\Lambda^{-1}) + \vartheta(\gamma_{\sigma_x}(p)) = -\vartheta(\Lambda) + \gamma^2(p),$$

where the last equality in the first line follows from (12).

To prove that $R_*^\Lambda Y_j = Y_j$, we must show that for any $f \in C^\infty(P)$, $(R_*^\Lambda Y_j)_p f$ is equal to $Y_{j,p} f$. We have

$$(R_*^\Lambda Y_j)_p f = Y_{j,p'}(f \circ R^\Lambda) = \partial_j(f \circ R^\Lambda \circ \gamma^{-1})_{\gamma(p')} = \sum_{k=0}^2 \partial_k(f \circ \gamma^{-1})_{\gamma(p)} \partial_j(\gamma^k \circ R^\Lambda \circ \gamma^{-1})_{\gamma(p')} = \sum_{k=0}^2 Y_{k,p} f \partial_j(\gamma^k \circ R^\Lambda \circ \gamma^{-1})_{\gamma(p')},$$

and by (A10), $\partial_j(\gamma^k \circ R^\Lambda \circ \gamma^{-1})_{\gamma(p')} = \delta_j^k$, leading to $(R_*^\Lambda Y_j)_p f = Y_{j,p} f$.

Finally, to verify that Y_2 is the fundamental vector field generated by u , it is enough to show that for all $p \in P$, $dy^j \cdot W_p^u = \delta_j^2$. Indeed, it follows from (A10) that for $j \in \{0, 1\}$ and any $\varepsilon \in \mathbb{R}$, $y^j(R^{\exp(\varepsilon u)}(p)) = y^j(p)$ so that

$$dy_p^j \cdot W_p^u = \lim_{\varepsilon \rightarrow 0} \frac{y^j(R^{\exp(\varepsilon u)}(p)) - y^j(p)}{\varepsilon} = 0,$$

while $y^2(R^{\exp(\varepsilon u)}(p)) = y^2(p) + \varepsilon$ so that

$$dy_p^2 \cdot W_p^u = \lim_{\varepsilon \rightarrow 0} \frac{y^2(R^{\exp(\varepsilon u)}(p)) - y^2(p)}{\varepsilon} = 1,$$

proving the claim and completing the proof of the lemma. $\square$

Lemma 12. The $^\odot$ operator from (22) is (i) Lorentz-invariant, i.e., independent of the choice of inertial chart x with origin $o \in M$, and (ii) G -equivariant, in the sense that $(R_*^\Lambda Y)^\odot = R_*^\Lambda(Y^\odot)$ for any $\Lambda \in G$, $p \in P$, and $Y \in T_p P$.

Proof. (i) Let $x' : M \rightarrow \mathbb{R}^2$ be an inertial chart centered at o , and Λ be the unique element of the structure group G such that $x' = x \circ L_o^\Lambda$. Let also $y' : P \rightarrow \mathbb{R}^3$ be the associated coordinate chart on the principal bundle, with coordinate vector fields Y'^j . For any $p \in P$, define the operator $^\square : T_p P \rightarrow T_p P$ such that

$$Y'_{0,p}{}^\square = -Y'_{1,p}, \quad Y'_{1,p}{}^\square = -Y'_{0,p}, \quad Y'_{2,p}{}^\square = Y'_{2,p}.$$

We must show that for any $Y \in T_p P$, $Y^\square = Y^\odot$. For that, observe first that $^\odot$ and $^\square$ have the same matrix elements in their respective defining bases, i.e.,

$$A^j_i := dy_p^j \cdot Y_{i,p}^\odot = dy_p^j \cdot Y_{i,p}^\square,$$

where

$$A = [A^j_i] = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It then follows from these facts and Lemma 10(iii) that

$$\begin{aligned} Y^\square &= \sum_{i=0}^2 (dy_p^{i,j} \cdot Y) Y_{i,p}^\square = \sum_{i,j=0}^2 (dy_p^{i,j} \cdot Y) A^j_i Y'_{j,p} = \sum_{i,j=0}^1 (dy_p^{i,j} \cdot Y) A^j_i Y'_{j,p} + (dy_p^{2,2} \cdot Y) Y'_{2,p} \\ &= \sum_{i,j,k,l=0}^1 (dy_p^k \cdot Y) \Lambda^{-1,l}{}_j A^j_i \Lambda^i{}_k Y_{l,p} + (dy_p^2 \cdot Y) Y_{2,p} = \sum_{k,l=0}^1 (dy_p^k \cdot Y) A^l{}_k Y_{l,p} + (dy_p^2 \cdot Y) Y_{2,p} = Y^\odot, \end{aligned}$$

proving Claim (i).

(ii) By the R_*^Λ -invariance of the Y_j established in Lemma 11 and the pointwise definition of $^\odot$ in (22), it follows that

$$(R_*^\Lambda Y_{j,p})^\odot = (Y_{j,R^\Lambda(p)})^\odot = Y_{j,R^\Lambda(p)}^\odot.$$

Therefore, using again Lemma 11, we obtain

$$\left(R_*^\Lambda Y\right)^\odot = \sum_{j=0}^2 \left(dy_{R^\Lambda(p)}^j \cdot R_*^\Lambda Y\right) Y_{j,R^\Lambda(p)}^\odot = \sum_{j=0}^2 \left(R^{\Lambda*} dy_{R^\Lambda(p)}^j \cdot Y\right) \left(R_*^\Lambda Y_{j,p}\right)^\odot = \sum_{j=0}^2 \left(dy_p^j \cdot Y\right) \left(R_*^\Lambda Y_{j,R^\Lambda(p)}\right)^\odot = R_*^\Lambda \left(Y^\odot\right),$$

where the second-to-last line follows from the $R^{\Lambda*}$ -invariance of the dual vector fields dy^j (which is, in turn, a direct consequence of the R_*^Λ -invariance of the Y_j). This proves the claim and the lemma. $\square$

We now turn to the construction of the connection 1-form on the principal bundle, described in Sec. III F.

Proposition 13. *The 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ defined in (24) satisfies (17), i.e., it is a connection 1-form on the principal bundle $P \xrightarrow{\pi} M$. Moreover, ω does not depend on the choice of inertial chart $x : M \rightarrow \mathbb{R}^2$ with origin o .*

Proof. First, note that in the induced coordinate chart $y : P \rightarrow \mathbb{R}^3$ on the principal bundle, with $y(p) = (y^0(p), y^1(p), y^2(p)) = (x^0(\pi(p)), x^1(\pi(p)), \vartheta(\gamma_{\sigma_x}(p)))$, the function $\tilde{h}_x$ has the representation

$$\tilde{h}_x(p) = \frac{-(y^0(p))^2 + (y^1(p))^2}{2} + y^2(p).$$

As a result, $Y_0 \tilde{h}_x = -y^0$, $Y_1 \tilde{h}_x = y^1$, and $Y_2 \tilde{h}_x = 1$, leading, in conjunction with (22) and (24), to (25). Moreover, by (A10), for any $\Lambda \in G$ and $p \in P$, we have

$$\left(\tilde{h}_x \circ R^\Lambda\right)(p) = \tilde{h}_x(p) + \vartheta(\Lambda). \quad (\text{A11})$$

That is, $\tilde{h}_x \circ R^\Lambda$ differs from $\tilde{h}_x$ by a constant on P , namely, $\vartheta(\Lambda)$, so that $Y(\tilde{h}_x \circ R^\Lambda) = Y \tilde{h}_x$ for any $Y \in T_p P$.

Next, by (13), every fundamental vector field W^λ associated with $\lambda \in \mathfrak{g}$ satisfies $W^\lambda = W^{\vartheta(\lambda)u} = \vartheta(\lambda)W^u = \vartheta(\lambda)Y_2$. Thus, by (25),

$$(\omega W^\lambda)_p = \vartheta(\lambda)(\omega Y_2)_p = \vartheta(\lambda)u = \lambda,$$

proving the first condition in (17). To verify the second condition, it must be shown that for every $p \in P$, $Y \in T_p P$, and $\Lambda \in G$, $(R^{\Lambda*} \omega)_p Y = \omega_p Y$. Indeed, by Lemma 11 and (A11), we obtain

$$\left(R^{\Lambda*} \omega\right)_p Y = \omega_{R^\Lambda(p)} R_{*p}^\Lambda Y = R_{*p}^\Lambda Y^\odot \tilde{h}_x = Y_p^\odot \left(\tilde{h}_x \circ R^\Lambda\right) = Y_p^\odot \tilde{h}_x = \omega_p Y,$$

as claimed.

Next, to verify that ω is independent of the choice of inertial chart x with origin o , note that (24) can be equivalently expressed as

$$(\omega Y)_p = (\pi_{*,p} Y)^\perp h u + (dy^2 \cdot Y)_p Y_{2,p}(\vartheta \circ \gamma_{\sigma_x}).$$

So long as o is kept fixed, the first term in the right-hand side is independent of x by the definition of h in (23) and Poincaré invariance of the $^\perp$ operator (see Sec. III B). It thus suffices to show that the second term is x -independent. For that, note that if $x' : M \rightarrow \mathbb{R}^2$ is the inertial chart with $x' = x \circ L^\Lambda$ for some $\Lambda \in G$, then by Lemma 10, $\gamma_{\sigma_{x'}} = L^{\Lambda^{-1}} \circ \gamma_{\sigma_x}$ and $Y'_2 = Y_2$ so that

$$(dy'^2 \cdot Y)_p Y'_{2,p}(\vartheta \circ \gamma_{\sigma_{x'}}) = (dy^2 \cdot Y)_p Y_{2,p}(\vartheta \circ L^{\Lambda^{-1}} \circ \gamma_{\sigma_x}) = (dy^2 \cdot Y)_p Y_{2,p}(\vartheta \circ \gamma_{\sigma_x} \circ R^{\Lambda^{-1}}) = (dy^2 \cdot Y)_p Y_{2,p}(\vartheta \circ \gamma_{\sigma_x}).$$

Note that to obtain the second-to-last line, we used the equivariance property of γ_{σ_x} in (14), and to obtain the last line, we made use of the fact that $\vartheta \circ \gamma_{\sigma_x} \circ R^{\Lambda^{-1}}$ differs from $\vartheta \circ \gamma_{\sigma_x}$ by a constant shift of $-\vartheta(\Lambda)$ [in accordance with (12)], which is annihilated upon application of the tangent vector $Y_{2,p}$. Since Y and p were arbitrary, this shows that ω is independent of the inertial chart x with origin o , completing the proof of the proposition. $\square$

As a final result in this appendix, we establish how points on the associated vector bundle $E \xrightarrow{\pi_E} M$ transform under gauge transformations.

Lemma 14. *Let $\varphi : P \rightarrow P$ be a gauge transformation. Then, for every point $e \in E_m$ in the associated bundle lying above $m \in M$, we have $\varphi_* e = e^{i\vartheta(\xi(m))/\sqrt{2}} e$, where $\vartheta : G \rightarrow \mathbb{R}$ is the coordinate chart from (12) and $\xi : M \rightarrow G$ is the unique function satisfying $\varphi(p) = p \cdot \xi(\pi(p))$. Moreover, if $\sigma \in \Gamma(P)$ is a section of the principal bundle, the following hold:*

- (i) For any function $f \in C^\infty(M)$, the corresponding sections $f_\sigma = \zeta_\sigma f$ and $f_{\varphi \circ \sigma} = \zeta_{\varphi \circ \sigma} f$ satisfy $f_{\varphi \circ \sigma} = \varphi_* \circ f_\sigma$, and

$$f_{\varphi \circ \sigma}(m) = e^{i\alpha\vartheta(\xi(m))/\sqrt{2}} f_\sigma(m), \quad \forall m \in M.$$

- (ii) For any section $s \in \Gamma(E)$ of the associated bundle, the corresponding $C^\infty(M)$ functions (matter fields) $s^\sigma = \zeta_\sigma^{-1} s$ and $s^{\varphi \circ \sigma} = \zeta_{\varphi \circ \sigma}^{-1} s$ satisfy $s^{\varphi \circ \sigma} = (\varphi_*^\sigma)^{-1} \circ s^\sigma$, and

$$s^{\varphi \circ \sigma}(m) = e^{-i\alpha\vartheta(\xi(m))/\sqrt{2}} s^\sigma(m), \quad \forall m \in M.$$

Proof. First, the existence and uniqueness of ξ was established in [Appendix A 1](#). Let $e = [p, z]$, where $p \in P_m$ and $z \in \mathbb{C}$. To prove the claim about $\varphi_* e$, we use [\(A6\)](#) and the fact that G acts on the fiber $F = \mathbb{C}$ by multiplication according to [\(15\)](#), leading to

$$\varphi_*(e) = [p, \tilde{\xi}(p) \cdot z] = [p, \xi(m) \cdot z] = \left[p, e^{i\alpha\vartheta(\xi(m))/\sqrt{2}} z \right] = e^{i\alpha\vartheta(\xi(m))/\sqrt{2}} [p, z] = e^{i\alpha\vartheta(\xi(m))/\sqrt{2}} e,$$

as claimed.

Next, to verify Claim (i), we have

$$f_{\varphi \circ \sigma}(m) = \zeta_{\varphi \circ \sigma} f(m) = \varepsilon_{(\varphi \circ \sigma)(m)}(f(m)) = [(\varphi \circ \sigma)(m), f(m)] = [\sigma(m) \cdot \xi(m), f(m)] = [\sigma(m), \xi(m) \cdot f(m)] = e^{i\alpha\vartheta(\xi(m))/\sqrt{2}} f_\sigma(m),$$

and the claim follows using the result about φ_* just proved. To verify Claim (ii), we employ the G -equivariance of ε_p^{-1} from Lemma 8 to obtain

$$s^{\varphi \circ \sigma}(m) = \varepsilon_{(\varphi \circ \sigma)(m)}^{-1}(s(m)) = \varepsilon_{\sigma(m) \cdot \xi(m)}^{-1}(s(m)) = \xi(m)^{-1} \cdot \varepsilon_{\sigma(m)}^{-1} s(m) = \xi(m)^{-1} \cdot s^\sigma(m) = e^{-i\alpha\vartheta(\xi(m))/\sqrt{2}} s^\sigma(m).$$

The above equation, in conjunction with Claim (i), leads to

$$s^{\varphi \circ \sigma}(m) = e^{-i\alpha\vartheta(\xi(m))/\sqrt{2}} \zeta_\sigma^{-1} s(m) = \zeta_\sigma^{-1} \left(e^{-i\alpha\vartheta(\xi(\cdot))/\sqrt{2}} s \right)(m) = (\zeta_\sigma^{-1} \circ \varphi_*^{-1} \circ s)(m) = (\zeta_\sigma^{-1} \circ \varphi_*^{-1} \circ \zeta_\sigma) s^\sigma(m) = ((\varphi_*^\sigma)^{-1} \circ s^\sigma)(m),$$

and the claim follows. $\square$

APPENDIX B: RESULTS FROM RKHS THEORY

1. Basic properties of the heat kernel on the circle and the associated RKHSs

We begin by stating some commonly used terminology in the theory of RKHSs on topological spaces. We will only consider compact spaces, as our main interest here is on RKHSs on the circle equipped with the standard-metric topology.

Let then S be a compact topological space, and $\kappa : S \times S \rightarrow \mathbb{C}$ be a Hermitian function, i.e., $\kappa(\theta, \theta') = \kappa(\theta', \theta)^*$ for all $\theta, \theta' \in S$. The function κ is said to be *positive-definite* if for any finite sequence $\theta_1, \dots, \theta_n \in S$ the kernel matrix $K = [\kappa(\theta_i, \theta_j)]_{ij}$ is non-negative. By the Moore–Aronszajn theorem,³³ every positive-definite Hermitian function is the reproducing kernel for a unique RKHS $\mathcal{H}$ of complex-valued functions on S ; that is, a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ such that (i) the kernel sections $\kappa(\theta, \cdot)$ lie in $\mathcal{H}$ for all $\theta \in S$; and (ii) for every $\theta \in S$, the pointwise evaluation functional $\mathbb{V}_\theta : \mathcal{H} \rightarrow \mathbb{C}$, $\mathbb{V}_\theta f = f(\theta)$, is continuous and satisfies $\mathbb{V}_\theta f = \langle \kappa(\theta, \cdot), f \rangle_{\mathcal{H}}$. The latter relation is known as the *reproducing property*. In addition, the kernel κ is said to be^{18,34}

- *strictly positive-definite* if K is a strictly positive matrix whenever the $\theta_1, \dots, \theta_n$ are all distinct;
- *C-universal* if κ is continuous, and $\mathcal{H}$ is a dense subspace of the space of complex-valued, continuous functions on S , equipped with the uniform norm;
- *L^p -universal* if κ is bounded and Borel-measurable, and $\mathcal{H}$ is a dense subspace of $L^p(m)$ for any Borel probability measure m on S , where $p \in [1, \infty]$ and $L^p(m)$ is equipped with the standard norm; and
- *characteristic* if κ is bounded and Borel-measurable, and the map $m \mapsto \int_S \kappa(\cdot, \theta) d\mu(\theta)$ is injective, where m is any Borel probability measure on S .

On a compact space, C -universality and L^p universality of a continuous kernel are equivalent notions.¹⁸ Moreover, every C -universal kernel is strictly positive definite and characteristic, though the converses of these statements are not true. In addition, we have the following lemma:

Lemma 15. Let $\kappa : S \times S \rightarrow \mathbb{C}$ be a Hermitian positive-definite kernel on a metric space S , $\mathcal{H}$ be the corresponding RKHS, and $F : S \rightarrow \mathcal{H}$ be the feature map such that $F(\theta) = \kappa(\theta, \cdot)$. Then, the following hold:

- (i) F is continuous if and only if κ is continuous.
(ii) If κ is strictly positive-definite, then F is injective, and $F(\theta)$ and $F(\theta')$ are linearly independent whenever θ and θ' are distinct.

Proof. (i) The equivalence between continuity of kernels and continuity of feature maps was proved in Ref. 14 (Lemma 2.1).

(ii) It is enough to show that $F(\theta)$ and $F(\theta')$ are linearly independent for any two distinct points θ and θ' , for this property implies injectivity of F . To prove the claim by contradiction, suppose that there exist distinct points $\theta, \theta' \in S^1$ such that $F(\theta)$ and $F(\theta')$ are linearly dependent. Then, there exist coefficients $c, c' \in \mathbb{C}$, at least one of which is nonzero, such that $cF(\theta) + c'F(\theta') = 0$, and thus,

$$\begin{aligned} c\langle \kappa(\theta, \cdot), F(\theta) \rangle_{\mathcal{H}} + c'\langle \kappa(\theta', \cdot), F(\theta') \rangle_{\mathcal{H}} &= 0, \\ c\langle \kappa(\theta', \cdot), F(\theta) \rangle_{\mathcal{H}} + c'\langle \kappa(\theta', \cdot), F(\theta') \rangle_{\mathcal{H}} &= 0. \end{aligned}$$

Then, by the reproducing property,

$$\langle \kappa(\theta, \cdot), F(\theta) \rangle_{\mathcal{H}} = \langle \kappa(\theta, \cdot), \kappa(\theta, \cdot) \rangle_{\mathcal{H}} = \kappa(\theta, \theta),$$

and similarly,

$$\langle \kappa(\theta, \cdot), F(\theta') \rangle_{\mathcal{H}} = \kappa(\theta, \theta'), \quad \langle \kappa(\theta', \cdot), F(\theta') \rangle_{\mathcal{H}} = \kappa(\theta', \theta').$$

It then follows that the kernel matrix

$$K = \begin{pmatrix} \kappa(\theta, \theta) & \kappa(\theta, \theta') \\ \kappa(\theta', \theta) & \kappa(\theta', \theta') \end{pmatrix}$$

is a non-invertible kernel matrix associated with the distinct points θ and θ' , contradicting the fact that κ is strictly positive-definite. $\square$

With these definitions and results in place, we state the following lemma summarizing some of the basic properties of the heat kernel on the circle:

Lemma 16. For any $\tau > 0$, the heat kernel $\kappa_\tau : S^1 \times S^1 \rightarrow \mathbb{R}$ on the circle from (7) is a strictly positive-definite, C -universal, L^p -universal, characteristic kernel. Moreover, the feature map $F_\tau : S^1 \rightarrow \mathcal{H}_\tau$ is an injective, continuous map, mapping distinct points in S^1 to linearly independent functions in $\mathcal{H}_\tau$.

Proof. By compactness of S^1 and continuity of κ_τ , that κ_τ is L^p -universal, strictly positive, and characteristic will follow if it can be shown that it is C -universal. Indeed, by Ref. 35 (Corollary 11), a translation-invariant kernel $\kappa : S^1 \times S^1 \rightarrow \mathbb{R}$ of the form $\kappa(\theta, \theta') = \tilde{\kappa}(|\theta - \theta'|)$, where $\tilde{\kappa} : [0, 2\pi] \rightarrow \mathbb{R}$ is a continuous function and $|\cdot|$ denotes the arclength distance, is C -universal if and only if $\tilde{\kappa}$ admits a pointwise absolutely convergent Fourier expansion $\tilde{\kappa}(\theta) = \sum_{j=0}^{\infty} \hat{\kappa}_j \cos(j\theta)$, where the $\hat{\kappa}_j$ are strictly positive. Clearly, the heat kernel κ_τ from (7) takes this form for any $\tau > 0$, so it is C -universal. The claim about injectivity and linear independence of F_τ applied to distinct points then follows from Lemma 15. $\square$

Lemma 17. For every $\tau > 0$, the $L^2(\mu)$ norm of the kernel section $\kappa_\tau(\theta, \cdot)$ is independent of $\theta \in S^1$.

Proof. By shift-invariance of κ_τ and unitarity of the Koopman group on $L^2(\mu)$, for every $t \in \mathbb{R}$, we have

$$\|\kappa_\tau(\theta, \cdot)\|_\mu = \|\kappa_\tau(\Phi^\tau(\theta), \Phi^\tau(\cdot))\|_\mu = \|\kappa_\tau(\Phi^t(\theta), \cdot) \circ \Phi^t\|_\mu = \|U^t \kappa_\tau(\Phi^t(\theta), \cdot)\|_\mu = \|\kappa_\tau(\Phi^t(\theta), \cdot)\|_\mu.$$

The claim then follows from the fact that for every $\theta, \theta' \in S^1$, there exists $t \in \mathbb{R}$ such that $\theta' = \Phi^t(\theta)$. $\square$

2. Proof of Proposition 4

First, to show that $\Psi_\tau = \Pi \circ F_\tau$ is injective, note that, by Lemma 16, F_τ maps distinct points in S^1 to linearly independent functions in $\mathcal{F}(\mathcal{H}_\tau) \subset C(S^1) \setminus \{0\}$. Moreover, by definition, Π maps each (nonzero) function f its domain to the rank-1 orthogonal projection operator on $L^2(\mu)$ mapping onto $\text{span}\{f\}$. Since $\text{span}\{f\}$ and $\text{span}\{f'\}$ are distinct subspaces of $L^2(\mu)$ whenever f, f' are linearly independent continuous functions, these two facts together imply that Ψ_τ is injective.

To show that Ψ_τ is continuous, observe that

$$\Psi_\tau(\theta) = \langle F_\tau(\theta), \cdot \rangle_\mu F_\tau(\theta) / c,$$

where $c := \|\kappa(\theta, \cdot)\|_\mu^2$ is independent of θ by Lemma 17. Then, for any θ, θ' , we have

$$\begin{aligned} c\|\Psi_\tau(\theta) - \Psi_\tau(\theta')\|_1 &= \|\langle F_\tau(\theta), \cdot \rangle_\mu F_\tau(\theta) - \langle F_\tau(\theta'), \cdot \rangle_\mu F_\tau(\theta')\|_1 \\ &= \|\langle F_\tau(\theta), \cdot \rangle_\mu F_\tau(\theta) - \langle F_\tau(\theta'), \cdot \rangle_\mu F_\tau(\theta) + \langle F_\tau(\theta'), \cdot \rangle_\mu F_\tau(\theta) - \langle F_\tau(\theta'), \cdot \rangle_\mu F_\tau(\theta')\|_1 \\ &\leq \|\langle F_\tau(\theta) - F_\tau(\theta'), \cdot \rangle_\mu F_\tau(\theta)\|_1 + \|\langle F_\tau(\theta'), \cdot \rangle_\mu (F_\tau(\theta) - F_\tau(\theta'))\|_1 \\ &= \|F_\tau(\theta) - F_\tau(\theta')\|_\mu (\|F_\tau(\theta)\|_\mu + \|F_\tau(\theta')\|_\mu) \\ &\leq 2c^{1/2} \|F_\tau(\theta) - F_\tau(\theta')\|_{\mathcal{H}_\tau}, \end{aligned}$$

and the continuity of Ψ_τ follows from continuity of F_τ (Lemma 16). Note that to obtain the equality in the second-to-last line above, we used the fact that the trace norm of a rank-1 operator $\langle f, \cdot \rangle_\mu g$ on $L^2(\mu)$ is equal to $\|f\|_\mu \|g\|_\mu$, and the inequality in the last line follows from the fact that the $L^2(\mu)$ norm is bounded above by the $\mathcal{H}_\tau$ norm for any $\tau > 0$. Next, to check that $F_\tau \circ \Phi^t = U^{-t} \circ F_\tau$, we compute

$$F_\tau(\Phi^t(\theta)) = \kappa_\tau(\Phi^t(\theta), \cdot) = \kappa_\tau(\theta, \Phi^{-t}(\cdot)) = \kappa_\tau(\theta, \cdot) \circ \Phi^{-t} = U^{-t} \kappa_\tau(\theta, \cdot) = U^{-t} F_\tau(\theta),$$

verifying the claim. In addition, for any $f \in C(S^1) \setminus \{0\}$, we have

$$\tilde{\Phi}^t(\Pi(f)) = U^{t*} \Pi(f) U^t = \frac{\langle f, U^t(\cdot) \rangle_\mu U^{t*} f}{\|f\|_\mu^2} = \frac{\langle U^{-t} f, \cdot \rangle_\mu U^{-t} f}{\|U^{-t} f\|_\mu^2} = \Pi(U^{-t} f),$$

showing that $\Pi \circ U^{-t} = \tilde{\Phi}^t \circ \Pi$. That $\Psi_\tau \circ \Phi^t = \tilde{\Phi}^t \circ \Psi_\tau$ then follows from the fact that $\Psi_\tau = \Pi \circ F_\tau$. This completes the proof of Claim (i).

Turning now to the map Ω_τ in Claim (ii), for any $A \in \mathcal{A}(L^2(\mu))$,

$$(\Omega_\tau A)^* = \text{tr}((\Psi_\tau(\cdot))^* A^*) = \text{tr}(\Psi_\tau(\cdot) A) = \Omega_\tau A,$$

where the second-to-last equality follows from the fact that for any $\theta \in S^1$, $\Psi_\tau(\theta)$ is a density operator and thus self-adjoint. Moreover, for any $\theta, \theta' \in S^1$, we have

$$|\Omega_\tau A(\theta) - \Omega_\tau A(\theta')| = |\text{tr}((\Psi_\tau(\theta) - \Psi_\tau(\theta')) A)| \leq \text{tr}((\Psi_\tau(\theta) - \Psi_\tau(\theta')) A) \leq \|\Psi_\tau(\theta) - \Psi_\tau(\theta')\|_1 \|A\|,$$

and the continuity of Ω_τ follows from the continuity of Ψ_τ established in Claim (i). We therefore conclude that Ω_τ is well-defined as an operator mapping into $C_\mathbb{R}(S^1)$. In addition, for any $t \in \mathbb{R}$,

$$\Omega_\tau(\tilde{U}^t A) = \text{tr}(\Psi_\tau(\cdot) U^t A U^{t*}) = \text{tr}(U^{t*} \Psi_\tau(\cdot) U^t A) = \text{tr}(\tilde{\Phi}^t(\Psi_\tau(\cdot)) A) = \text{tr}(\Psi_\tau(\Phi^t(\cdot)) A) = \text{tr}(\Psi_\tau(\cdot) A) \circ \Phi^t = U^t \text{tr}(\Psi_\tau(\cdot) A) = U^t(\Omega_\tau A),$$

proving that $\Omega_\tau \circ \tilde{U}^t = U^t \circ \Omega_\tau$. This completes the proof of Claim (ii).

Finally, Claim (iii) follows directly from the definition of Ω'_τ and Claim (ii).

3. Proof of Theorem 7

Starting from Claims (i) and (ii), it follows from the Mercer representation of the kernel $\hat{\kappa}_\tau$ in (64) that $\{\hat{\phi}_{j,\tau}\}_{j \in \mathbb{Z}}$ with $\hat{\phi}_{j,\tau} = e^{-|j|^2 \tau/2} \phi_j$ is an orthonormal basis of $\hat{\mathcal{H}}_\tau$. Moreover, for every $t \in \mathbb{R}$, $\hat{\phi}_{j,\tau} \circ \Phi^t = e^{i\alpha t} \hat{\phi}_{j,\tau}$ so U^t maps an orthonormal basis of $\hat{\mathcal{H}}_\tau$ to an orthonormal basis. We therefore conclude that $\hat{\mathcal{H}}_\tau$, and since τ is arbitrary, $\hat{\mathcal{H}}_\tau^\infty$ are invariant under U^t , proving Claim (i). Similarly, $U^t : \hat{\mathcal{H}}_\tau \rightarrow \hat{\mathcal{H}}_\tau$ is unitary since $\{U^t \hat{\phi}_{j,\tau}\}_{j \in \mathbb{Z}}$ is an orthonormal basis, and the strong continuity of $\{U^t\}_{t \in \mathbb{R}}$ follows from the fact that $t \mapsto U^t \hat{\phi}_{j,\tau} = e^{i\alpha j t} \hat{\phi}_{j,\tau}$ is a continuous map.

Next, turning to Claim (iii), the basis elements $\hat{\phi}_{j,\tau}$ are clearly eigenfunctions of the generator $V : D(V) \rightarrow \hat{\mathcal{H}}_\tau$, i.e.,

$$V \hat{\phi}_{j,\tau} = \lim_{t \rightarrow 0} \frac{U^t \hat{\phi}_{j,\tau} - \hat{\phi}_{j,\tau}}{t} = \lim_{t \rightarrow 0} \frac{e^{i\alpha j t} - 1}{t} \hat{\phi}_{j,\tau} = i\alpha j \hat{\phi}_{j,\tau},$$

where the first limit is taken with respect to the $\hat{\mathcal{H}}_\tau$ norm. Then, for every $f = \sum_{j=-\infty}^\infty c_j \hat{\phi}_{j,\tau} \in \hat{\mathcal{H}}_\tau^1$ and $l \in \mathbb{N}_0$, the action of V on the partial sum $f_l = \sum_{j=-l}^l c_j \hat{\phi}_{j,\tau}$ is given by $g_l := V f_l = \sum_{j=-l}^l i\alpha j c_j \hat{\phi}_{j,\tau}$, and it follows from the definition of $\hat{\mathcal{H}}_\tau^1$ that g_l is a Cauchy sequence in $\hat{\mathcal{H}}_\tau$. Thus, $\hat{\mathcal{H}}_\tau^1$ is a subspace of $D(V)$. Now, if $f = \sum_{j=-\infty}^\infty c_j \hat{\phi}_{j,\tau} \in \hat{\mathcal{H}}_\tau \setminus \hat{\mathcal{H}}_\tau^1$ for every $C \geq 0$, there exists $l_0 \in \mathbb{N}_0$ such that for all $l \geq l_0$, $\sum_{j=-l}^l |j|^2 |c_j|^2 \geq C^2$. It therefore follows that $g_l = V f_l$ has norm $\|g_l\|_{\hat{\mathcal{H}}_\tau} > \alpha C$, and thus, the sequence g_l is unbounded. This shows that $(\hat{\mathcal{H}}_\tau^1)^c \subseteq D(V)^c$, and therefore, $\hat{\mathcal{H}}_\tau^1 \supseteq D(V)$. The latter, together with the fact that $\hat{\mathcal{H}}_\tau^1 \subseteq D(V)$ just shown, implies that $\hat{\mathcal{H}}_\tau^1 = D(V)$, as claimed.

What remains is to show that V acts on $\hat{\mathcal{H}}^\infty$ as a derivation. For that, it needs to be shown that (i) $\hat{\mathcal{H}}^\infty$ is invariant under V ; and (ii) the Leibniz rule holds, i.e.,

$$V(fg) = (Vf)g + f(Vg), \quad \forall f, g \in \hat{\mathcal{H}}^\infty. \quad (\text{B1})$$

These claims will follow, in turn, from the following useful lemma:

Lemma 18. *For every $\tau > 0$ and $\tau' > \tau$, $\hat{\mathcal{H}}_{\tau'}$ is a subspace of $\hat{\mathcal{H}}_\tau^1$. Moreover, the restriction of the generator $V : D(V) \rightarrow \hat{\mathcal{H}}_\tau$ to $\hat{\mathcal{H}}_{\tau'}$, viewed as an operator $V|_{\hat{\mathcal{H}}_{\tau'}} : \hat{\mathcal{H}}_{\tau'} \rightarrow \hat{\mathcal{H}}_\tau$, is bounded.*

A Proof of Lemma 18 will be given below. Assuming, for now, that it holds, let $f = \sum_{j=-\infty}^\infty \hat{f}_j \phi_j$ and $g = \sum_{j=-\infty}^\infty \hat{g}_j \phi_j$ be arbitrary elements of $\hat{\mathcal{H}}^\infty$, and note that the partial sums $f_l = \sum_{j=-l}^l \hat{f}_j \phi_j$ and $g_l = \sum_{j=-l}^l \hat{g}_j \phi_j$ converge in any $\hat{\mathcal{H}}_{\tau'}$ norm. Moreover, because $\hat{\mathcal{H}}_{\tau'}$ is a Banach algebra, as $l \rightarrow \infty$, $f_l g_l$ converges in $\hat{\mathcal{H}}_{\tau'}$ norm to fg . Choosing, in particular, $\tau' > \tau$, it follows from Lemma 18 that Vf_l , Vg_l , and $V(f_l g_l)$ are Cauchy sequences in $\hat{\mathcal{H}}_\tau$ converging to Vf , Vg , and $V(fg)$, respectively. The Banach algebra property of $\hat{\mathcal{H}}_\tau$ then implies that $(Vf_l)g_l$ and $f_l(Vg_l)$ are also $\hat{\mathcal{H}}_\tau$ Cauchy sequences, converging to $(Vf)g$ and $f(Vg)$, respectively.

Now, because τ was arbitrary, the convergence of Vf_l to Vf in $\hat{\mathcal{H}}_\tau$ norm implies that Vf lies in $\hat{\mathcal{H}}_\tau$ for any $\tau > 0$ and therefore that $\hat{\mathcal{H}}^\infty$ is invariant under V . Moreover, for any $l \in \mathbb{N}_0$,

$$V(f_l g_l) = V\left(\sum_{j,k=-l}^l \hat{f}_j \hat{g}_k \phi_j \phi_k\right) = V\left(\sum_{j,k=-l}^l \hat{f}_j \hat{g}_k \phi_{j+k}\right) = \sum_{j,k=-l}^l i\alpha(j+k) \hat{f}_j \hat{g}_k \phi_{j+k} = \sum_{j,k=-l}^l i\alpha j \hat{f}_j \hat{g}_k \phi_j \phi_k + \sum_{j,k=-l}^l i\alpha k \hat{f}_j \hat{g}_k \phi_j \phi_k = (Vf_l)g_l + f_l(Vg_l),$$

and taking $l \rightarrow \infty$ limits in $\hat{\mathcal{H}}_\tau$ norm, we obtain

$$V(fg) = V\left(\lim_{l \rightarrow \infty} f_l g_l\right) = \lim_{l \rightarrow \infty} V(f_l g_l) = \lim_{l \rightarrow \infty} (Vf_l)g_l + \lim_{l \rightarrow \infty} f_l(Vg_l) = (Vf)g + f(Vg),$$

verifying (B1).

Finally, to prove Lemma 18, observe that $\hat{\mathcal{H}}_{\tau'} \subset \hat{\mathcal{H}}_\tau$ whenever $\tau' > \tau$, and every $f \in \hat{\mathcal{H}}_{\tau'}$ admits an expansion $f = \sum_{j=-\infty}^\infty c_j \hat{\phi}_{j,\tau}$, where $\sum_{j=-\infty}^\infty e^{(\tau'-\tau)|j|^p} |c_j|^2 < \infty$. Since $e^{(\tau'-\tau)|j|^p} > C|j|^2$ for some constant $C > 0$ whenever $\tau' > \tau$, it follows that $\sum_{j=-\infty}^\infty |j|^2 |c_j|^2 < \infty$, which shows that $\hat{\mathcal{H}}_{\tau'} \subseteq \hat{\mathcal{H}}_\tau^1$. Similarly, defining $f_l = \sum_{j=-l}^l c_j \hat{\phi}_{j,\tau}$ with c_j as above, we have

$$\|Vf_l\|_{\hat{\mathcal{H}}_\tau}^2 = \sum_{j=-l}^l \alpha^2 |j|^2 |c_j|^2 \leq \sum_{j=-l}^l \alpha^2 e^{(\tau'-\tau)|j|^p} |c_j|^2 / C = \|f_l\|_{\hat{\mathcal{H}}_{\tau'}}^2 \alpha^2 / C.$$

This shows that $V|_{\hat{\mathcal{H}}_{\tau'}} : \hat{\mathcal{H}}_{\tau'} \rightarrow \hat{\mathcal{H}}_\tau$ is uniformly bounded on the dense subspace of $\hat{\mathcal{H}}_{\tau'}$ consisting of finite sums of the form $\sum_{j=-l}^l c_j \hat{\phi}_{j,\tau}$ and is therefore bounded on the whole of $\hat{\mathcal{H}}_{\tau'}$. $\square$

APPENDIX C: FRACTIONAL DERIVATIVES

In this appendix, we provide definitions and outline the basic properties of fractional derivative operators on functions on the circle. Additional details on these topics can be found, e.g., in Refs. 36 and 37.

To construct the fractional derivative operator ∂^r of order $r \geq 0$, it is convenient to pass to a Fourier representation using the Fourier analysis operator $\mathfrak{F} : L^2(\mu) \rightarrow \ell^2$. The latter is the unitary operator mapping $f = \sum_{j=-\infty}^\infty c_j \phi_j \in L^2(\mu)$ to the ℓ^2 sequence $\hat{f} = \mathfrak{F}f := (c_j)_{j \in \mathbb{Z}}$ consisting of the expansion coefficients of f in the Koopman eigenfunction basis $\{\phi_j\}_{j=-\infty}^\infty$. Note, in particular, that

$$c_j = \langle \phi_j, f \rangle_\mu = \frac{1}{2\pi} \int_0^{2\pi} e^{-ij\theta} f(\theta) d\theta,$$

indicating that the elements of $\hat{f}$ coincide with the standard Fourier expansion coefficients of f . Then, for any $r \geq 0$, we define

$$\partial^r : D(\partial^r) \rightarrow L^2(\mu), \quad \partial^r = \mathfrak{F}^* \hat{\partial}^r \mathfrak{F}, \quad (\text{C1})$$

where $\hat{\partial}^r : D(\hat{\partial}^r) \rightarrow \ell^2$ is the multiplication operator with the dense domain

$$D(\hat{\partial}^r) = \left\{ (c_j)_{j \in \mathbb{Z}} \in \ell^2 : \sum_{j=-\infty}^{\infty} j^{2r} |c_j|^2 < \infty \right\} \subseteq \ell^2,$$

defined as

$$\hat{\partial}^r (c_j)_{j \in \mathbb{Z}} = ((ij)^r c_j)_{j \in \mathbb{Z}}.$$

The domain of ∂^r is then the dense subspace $D(\partial^r)$ of $L^2(\mu)$ given by

$$D(\partial^r) = \left\{ \sum_{j=-\infty}^{\infty} c_j \phi_j \in L^2(\mu) : \sum_{j=-\infty}^{\infty} j^{2r} |c_j|^2 < \infty \right\}.$$

It follows directly from these definitions that if r is a non-negative integer, ∂^r from (C1) coincides with the weak derivative operator with respect to standard angle coordinates θ on the circle. It can also be verified that the restriction $\partial^r : W^{p,2}(\mu) \rightarrow L^2(\mu)$ of ∂^r to the Sobolev space $W^{p,2}(\mu) \subseteq L^2(\mu)$ of order $p = \lceil r \rceil$, where

$$W^{p,2}(\mu) = \left\{ \sum_{j=-\infty}^{\infty} c_j \phi_j \in L^2(\mu) : \sum_{j=-\infty}^{\infty} j^{2p} |c_j|^2 < \infty \right\},$$

is a bounded operator. Here, $W^{p,2}(\mu)$ is equipped with the standard norm,

$$\|f\|_{W^{p,2}(\mu)}^2 = \sum_{q=0}^p \sum_{j=-\infty}^{\infty} j^{2q} |c_j|^2, \quad f = \sum_{j=-\infty}^{\infty} c_j \phi_j.$$

To gain intuition on the behavior of ∂^r in a pointwise (as opposed to Fourier) representation, we state an equivalent definition to (C1), given by the $L^2(\mu)$ limit

$$\partial^r f = \lim_{a \rightarrow 0^+} \frac{1}{a^r} \mathbb{D}_a^r f, \quad f \in D(\partial^r), \quad (\text{C2})$$

where $\mathbb{D}_a^r f$ is the Riemann difference of f ,

$$\mathbb{D}_a^r f(\theta) = \sum_{n=0}^{\infty} (-1)^n \binom{r}{n} f(\theta - an), \quad \binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

In the fractional calculus, the operator in (C2) is known as the Liouville–Grünwald, or Grünwald–Letnikov derivative. From this definition, it is clear that whenever r is a non-negative integer, $\partial^r f(\theta)$ depends on the behavior of f in an infinitesimally small neighborhood of θ , for the Riemann difference $\mathbb{D}_a^r$ contains finitely many terms. On the other hand, for the non-integer r , $\mathbb{D}_a^r$ contains infinitely many terms, and $\partial^r f(\theta)$ depends on the behavior of f on distant points from θ . In other words, ∂^r is a non-local operator whenever r is not an integer.

With these definitions in place, we have the following proposition:

Proposition 19. *The ladder operators $A_{\pm}$ and $A_{\pm}^+$ from (57) satisfy (59).*

Proof. We will prove the claim only for A_- and A_-^+ , as the results for A_+ and A_+^+ follow similarly. For that, note first that the set

$$\left\{ \dots, \frac{\phi_{-2}}{\sqrt{2}}, \frac{\phi_{-1}}{\sqrt{1}}, \phi_0, \frac{\phi_1}{\sqrt{1}}, \frac{\phi_2}{\sqrt{2}}, \dots \right\}$$

is an orthonormal basis of the Sobolev space $W^{1,2}(\mu)$. As a result, because $\partial^{1/2} : W^{1,2}(\mu) \rightarrow L^2(\mu)$ is bounded, it is enough to show that

$$\mathcal{U}^* A_0 \mathcal{U} \phi_j = i^{1/2} L^* \partial_-^{1/2} \phi_j, \quad \mathcal{U}^* A_0^+ \mathcal{U} \phi_j = i^{1/2} \partial_-^{1/2} L \phi_j \quad (\text{C3})$$

for all $j \in \mathbb{Z}$. Indeed, using (56), for any non-positive integer j , we obtain

$$\mathcal{U}^* A_0 \mathcal{U} \phi_j = \mathcal{U}^* A_0 \psi_{-j,0} = \mathcal{U}^* \left(\sqrt{-j} \psi_{-j-1,0} \right) = \sqrt{-j} \phi_{j+1} = i^{1/2} \phi_1 \left(\sqrt{ij} \phi_j \right) = i^{1/2} L^* \partial_-^{1/2} \phi_j = i^{1/2} L^* \partial_-^{1/2} \phi_j$$

and

$$\mathcal{U}^* A_0^+ \mathcal{U} \phi_j = \mathcal{U}^* A_0^+ \psi_{-j,0} = \mathcal{U}^* \left(\sqrt{-j+1} \psi_{-j+1,0} \right) = \sqrt{-j+1} \phi_{j-1} = i^{1/2} \sqrt{i(j-1)} \phi_{-1} \phi_j = i^{1/2} \partial_-^{1/2} L \phi_j = i^{1/2} \partial_-^{1/2} L \phi_j.$$

On the other hand, if j is positive,

$$\begin{aligned} \mathcal{U}^* A_0 \mathcal{U} \phi_j &= \mathcal{U}^* A_0 \psi_{0j} = 0 = i^{1/2} L^* \partial_-^{1/2} \phi_j, \\ \mathcal{U}^* A_0^+ \mathcal{U} \phi_j &= \mathcal{U}^* A_0^+ \psi_{0j} = \mathcal{U}^* \psi_{1j} = 0 = i^{1/2} L^* \partial_-^{1/2} \phi_j. \end{aligned}$$

We therefore conclude that (C3) holds, as claimed. $\square$

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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