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Complete Kähler–Einstein Metric on Stein Manifolds With Negative Curvature

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We show the existence of complete negative Kähler–Einstein metric on Stein manifolds with holomorphic sectional curvature bounded from above by a negative constant. We prove that any Kähler metrics on such manifolds can be deformed to the complete negative Kähler–Einstein metric using the normalized Kähler–Ricci flow.

1 Introduction

In [26], Wu and Yau proved that if a complete noncompact Kähler manifold supports a complete bounded curvature Kähler metric with holomorphic sectional curvature bounded from above by a negative constant, then it supports a complete negative Kähler–Einstein metric with bounded curvature. This extended the previous work [22, 27] to the non-compact case. Using the Kähler–Ricci flow approach, this was recovered by Tong [23], see also the compact case in [17]. A natural question is to ask if the curvature boundedness assumption is necessary in order to obtain the existence of Kähler–Einstein without bounded curvature conclusion. In [11], the author together with the collaborators showed that the curvature boundedness assumption can be weakened to the existence of an exhaustion function with bounded gradient and complex hessian. The main ingredient is the construction of long-time Kähler–Ricci flow solution using Chern–Ricci flow approximation technique introduced by the author and Tam in [15].

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In this note, we continue to study the existence problem using the Kähler–Ricci flow approach. We focus on the special case when M is a Stein manifold. Using the idea originated in [5–7, 24], we show the existence of complete negative Kähler–Einstein on M by establishing a long-time solution to the Kähler–Ricci flow on M. More precisely, we showed that on the negatively curved Stein manifold M, the Kähler–Ricci flow has a long-time solution starting from *any* Kähler metric, which can be incomplete and have unbounded curvature.

Theorem 1.1. Suppose M^n is a Stein manifold and h is a complete Kähler metric on M with holomorphic sectional curvature $H_h \le -\kappa$ for some $\kappa > 0$. Then any Kähler metric g_0 on M admits a longtime solution to the Kähler–Ricci flow on $M \times [0, +\infty)$ such that $g(t)$ is instantaneously complete for $t > 0$.

On general Kähler manifold, the short-time existence of the Kähler–Ricci flow was first studied by Shi [20, 21] when the initial metric g_0 is complete with bounded curvature. Without curvature condition, the general existence theory was still being unclear except the surface case. In the case of surface, the existence of the Ricci flow starting from incomplete metric has been studied in details by Giesen and Topping [5–7, 24] where the notion of instantaneously complete Ricci flow solution was first appeared in the work of Topping in [24]. This can be viewed as a partial generalization of their result in higher dimension.

By combining the existence with the convergence result in [11], we have the existence of the complete negative Kähler–Einstein metric.

Corollary 1.1. Suppose M is a Stein manifold and h is a complete Kähler metric on M with holomorphic sectional curvature $H_h \le -\kappa$ for some $\kappa > 0$, then M admits a unique complete Kähler–Einstein metric with negative scalar curvature.

Using Corollary 1.1, we know that the following situation admit complete negative Kähler–Einstein metric. Part of it has also been discussed in a recent paper by Wu–Yau [28].

Example 1.1. Using a result of Wu [25] and Corollary 1.1, any complete simply connected Kähler manifold (M, g) with sectional curvature bounded from above by -1 admits a complete negative Kähler–Einstein metric.

Example 1.2. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$, then any proper embedded complex submanifold Σ of $\mathbb{B}$ admits a complete negative Kähler–Einstein metric. To see this, $\mathbb{B}$

admits the Bergman metric h with constant holomorphic sectional curvature. By the decreasing property of holomorphic curvature and properness, the pull-back metric of h on Σ is a complete Kähler metric with holomorphic sectional curvature bounded from above by negative value. Then the existence followed from Corollary 1.1.

Example 1.3. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ associated with the Poincaré metric. Let Σ be a proper embedded complex sub-manifold Σ of $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$. The product metric on $\mathbb{D}^n$ is complete with holomorphic sectional curvature bounded from above by negative value. Then Σ admits a complete negative Kähler–Einstein metric using the argument in previous example.

2 Apriori Estimates for the Kähler–Ricci Flow

Let M^n be a complex manifold. The Kähler–Ricci flow on M starting from initial metric g_0 is a family of Kähler metric satisfying $g(t)$, which satisfies

$$\begin{cases} \partial_t g_{i\bar{j}} & = -R_{i\bar{j}}; \\ g(0) & = g_0 \end{cases} \tag{2.1}$$

on $M \times [0, T]$ for some $T > 0$. We say that $g(t)$ is a complete solution if $g(t)$ is a complete metric on M for each $t \in [0, T]$.

The following estimates on the lower bound of Kähler–Ricci flow was first considered by [17] under compact Kähler–Ricci flow. The complete non-compact case can be proved using the maximum principle trick originated by Chen [2].

Lemma 2.1. Let $g(t)$ be a complete solution to the Kähler–Ricci flow on $\Omega \times [0, T]$ for $T \leq +\infty$. Suppose h is a Kähler metric on Ω, which can possibly be incomplete such that its holomorphic sectional curvature is bounded from above by $-\kappa$ for some $\kappa > 0$. Then on $\Omega \times (0, T]$, $g(t)$ satisfies

$$\frac{(n+1)\kappa t}{2n} h \leq g(t).$$

Proof. Using parabolic Schwarz Lemma, it gives

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) \operatorname{tr}_g h \leq g^{i\bar{j}} g^{k\bar{l}} \tilde{R}_{i\bar{j}k\bar{l}} \tag{2.2}$$

where $\tilde{R}$ denotes the curvature tensor of metric h. By using a trick of Royden [8, Lemma, p. 552], the bisectional curvature quantities is bounded from above,

$$g^{i\bar{j}}g^{k\bar{l}}\tilde{R}_{i\bar{j}k\bar{l}} \le -\frac{n+1}{2n}\kappa(\mathrm{tr}_g h)^2. \tag{2.3}$$

Note that the inequality also holds when h is not complete. Hence, we have

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)\mathrm{tr}_g h \le -\frac{n+1}{2n}\kappa(\mathrm{tr}_g h)^2. \tag{2.4}$$

If $T < +\infty$, applying [11, Lemma 5.1] with $Q = \mathrm{tr}_g h$ gives

$$\mathrm{tr}_g h \le \frac{2n}{\kappa(n+1)t}$$

on $\Omega \times [0, T]$. If $T = +\infty$, then apply it on $[0, L]$ and followed by letting $L \to +\infty$. ■

We also have a instantaneous lower bound on the scalar curvature.

Lemma 2.2. Let $g(t)$ be a complete solution to the Kähler–Ricci flow on $\Omega \times [0, T]$ for $T \le +\infty$, then the scalar curvature satisfies

$$R_{g(t)} \ge -\frac{n}{t}.$$

Proof. Note that the scalar curvature R satisfies $\left(\frac{\partial}{\partial t} - \Delta_t\right)R \ge \frac{1}{n}R^2$. The proof is similar to the that in Lemma 2.1 by applying [11, Lemma 5.1] with $Q = -R$. ■

We also need the following version of pseudolocality of Ricci flow from [9], see also [18].

Proposition 2.1. For any $m \in \mathbb{N}$ and $v > 0$, there is $\sigma(m, v), \Lambda(m, v) > 0$ such that if $g(t)$ is a complete Ricci flow solution on $M^m \times [0, T]$ with $\sup_{[\tau,T]} |\mathrm{Rm}(g(t))| < +\infty$ for all $\tau > 0$ and satisfies

1. $|\mathrm{Rm}(g(0))| \le r^{-2}$ on $B_{g_0}(p, r)$;
2. $V_{g_0}\left(B_{g_0}(p, r)\right) \ge r^m v$,

for some $p \in M$ and $r > 0$, then we have

$$|\mathrm{Rm}(x, t)| \le \Lambda r^{-2}$$

on $B_{g(t)}(p, \sigma r)$, $t \le \min\{T, \sigma^2 r^2\}$.

Proof. The follows by applying [9, Corollary 3.1] to the rescaled Ricci flow solutions $r^{-2}g(r^2t)$ and then rescale it back. ■

3 Approximation Using Pseudoconvex Domain

In this section, we will construct local Kähler–Ricci flow solution using the Steinness of M. By a result of Grauert [12], there exists a smooth strictly pluri-subharmonic function ρ on M, which is an exhaustion function. For $R > 1$ large, denote its sublevel set to be

$$U_R = \{x \in M : \rho(x) < R\}.$$

This is a bounded strictly pseudo-convex domain in M as defined in [3]. By Sard's Theorem, we may find a sequence of $R_m \to +\infty$ such that $d\rho \neq 0$ on ∂U_{R_m}. By the result in [3], there exists a complete Kähler–Einstein metric $\tilde{\omega}_m = -\mathrm{Ric}(\tilde{\omega}_m)$ with bounded curvature defined on U_{R_m}. Now we will construct approximating sequence of Kähler–Ricci flows using $\{(U_{R_m}, \tilde{\omega}_m)\}_{m\in\mathbb{N}}$. Let ω_0 be the Kähler form of the initial metric g_0. On U_{R_m}, define $\omega_{0,m,\epsilon} = \omega_0 + \epsilon\tilde{\omega}_m$. We will use $g_{0,m,\epsilon}$ to denote the corresponding Kähler metric.

Lemma 3.1. For any $\epsilon > 0$, $\omega_{0,m,\epsilon} = \omega_0 + \epsilon\tilde{\omega}_m$ is a complete Kähler metric on U_{R_m}, which is uniformly equivalent to $\tilde{\omega}_m$.

Proof. The completeness is clear. It remains to establish the upper bound of $\omega_{0,m,\epsilon}$ with respect to $\tilde{\omega}_m$. Since $\rho_{i\bar{j}} > 0$ and U_{R_m} is pre-compact in M, there is $C_m >> 1$ such that

$$\omega_0 \le C_m\sqrt{-1}\partial\bar{\partial}\rho \tag{3.1}$$

on U_{R_m}. On the other hand by [3], $\tilde{\omega}_m$ is uniformly equivalent to the standard metric $-\sqrt{-1}\partial\bar{\partial}\log(R_m - \rho)$ on U_{R_m}. Therefore,

$$\begin{aligned} \omega_0 &\le C_m\sqrt{-1}\partial\bar{\partial}\rho \\ &\le -C_m\sqrt{-1}\partial\bar{\partial}\log(R - \rho) \\ &\le C_m\tilde{\omega}_m. \end{aligned} \tag{3.2}$$

This completes the proof. ■

Using the uniform equivalence of metrics, we can deform the Kähler metric using the Kähler–Ricci flow on each U_{R_m} with long-time existence.

Proposition 3.1. There exists a complete long-time solution $\omega_{m,\epsilon}(t)$ to the Kähler–Ricci flow with initial metric $\omega_{0,m,\epsilon}$ on $U_{R_m} \times [0,+\infty)$ such that $\omega_{m,\epsilon}(t)$ is uniformly equivalent to $\tilde{\omega}_m$ on any $[a,b] \subset [0,+\infty)$.

Proof. From Lemma 3.1, there exists $\delta > 0$ such that

$$\begin{cases} \omega_{0,m,\epsilon} \leq \delta^{-1}\tilde{\omega}_m; \\ \omega_{0,m,\epsilon} - s\mathrm{Ric}(\tilde{\omega}_m) \quad > \delta\tilde{\omega}_m. \end{cases} \tag{3.3}$$

for any $s > 0$. The proposition follows from [1, Theorem 1.4]. ■

4 Proof of Main Theorem

In this section, we will construct a global longtime solution $g(t)$ to the Kähler–Ricci flow with initial metric $g(0) = g_0$ using the idea in [6] adapted in higher dimensional Kähler case. The main goal is to show that $g(t) = \lim_{m\to+\infty} \lim_{\epsilon\to 0} g_{m,\epsilon}(t)$ exists. In [1, 10], it was showed that $g_m(t) = \lim_{\epsilon_i \to} g_{m,\epsilon_i}(t)$ exists smoothly up to $t = 0$ for some subsequence $\epsilon_i \to 0$ using estimates from parabolic Monge–Ampère equation. Since the information of $\mathrm{Ric}(h)$ on M is missing, we need to take an alternative approach to obtain the estimates for compactness when $m \to +\infty$.

Proof of Theorem 1.1. From the discussion in section 3, there is a sequence of bounded strictly pseudo-convex domain U_{R_m} exhausting M. Moreover, there is a sequence of complete Kähler–Ricci flow $g_{m,\epsilon}(t)$ defined on $U_{R_m} \times [0,+\infty)$ with $g_{m,\epsilon}(0) = g_{0,m,\epsilon}$. Because of completeness, we may apply Lemma 2.1 and Lemma 2.2 to deduce that for all $t \in [0,+\infty)$,

$$\begin{cases} \frac{(n+1)\kappa t}{2n} h \leq g_{m,\epsilon}(t); \\ R_{g_{m,\epsilon}(t)} \geq -nt^{-1}. \end{cases} \tag{4.1}$$

Here we regard h as an incomplete metric on the pre-compact set U_{R_m}. Moreover, by the uniform equivalence with a reference Kähler metric of bounded curvature, the Kähler–Ricci flow $g_{m,\epsilon}(t)$ has bounded curvature when $t > 0$, for example, see [1, 16, 19]. It was also showed in [4] that $g_{m,\epsilon}(t)$ has bounded curvature up to $t = 0$.

Fix a compact set $\Omega \Subset M$, since $g_{0,m,\epsilon}$ converges to g_0 uniformly in any C^k as $\epsilon \to 0$. For any $m \geq m_0(\Omega)$, we can find $r > 0$ and $v > 0$ small enough such that for all $x \in \Omega$, $m \geq m_0$ and $\epsilon_0(n, m, g_0, \Omega) > \epsilon > 0$, we have

(a) $B_{g_{0,m,\epsilon}}(x, r) \Subset U_{R_m}$;
(b) $|\mathrm{Rm}(g_{0,m,\epsilon})| \leq r^{-2}$ on $B_{g_{0,m,\epsilon}}(x, r)$;
(c) $V_{g_{0,m,\epsilon}}\left(B_{g_{0,m,\epsilon}}(x, r)\right) \geq vr^{2n}$.

By Proposition 2.1, for all $m \geq m_0$ and $\epsilon < \epsilon_0(n, m, g_0, \Omega)$, we have

$$\sup_{\Omega} |\mathrm{Rm}(g_{m,\epsilon})(x, t)| \leq \Lambda(n, g_0, \Omega) \tag{4.2}$$

for $0 \leq t \leq \sigma$ for some $\sigma(n, g_0, \Omega) > 0$. In particular, for all $(x, t) \in \Omega \times [0, \sigma]$,

$$e^{-\Lambda\sigma} g_{0,m,\epsilon} \leq g_{m,\epsilon}(t) \leq e^{\Lambda\sigma} g_{0,m,\epsilon}. \tag{4.3}$$

For $t > \sigma$, we make use of (4.1). Using the evolution of volume form with the 2nd inequality in (4.1), we have

$$\frac{\partial}{\partial s}\left(\log \frac{\det g_{m,\epsilon}(s)}{\det h}\right) = -R_{g_{m,\epsilon}(t)} \leq \frac{n}{s}. \tag{4.4}$$

By integrating it from σ to t, we have

$$\begin{aligned} \det g_{m,\epsilon}(t) &\leq \left(\frac{t}{\sigma}\right)^n \det g_{m,\epsilon}(\sigma) \\ &\leq e^{n\Lambda\sigma}\left(\frac{t}{\sigma}\right)^n \det g_{0,m,\epsilon}. \end{aligned} \tag{4.5}$$

Combining (4.5), (4.1), and the elementary inequality $\mathrm{tr}_h\, g \leq \frac{\det g}{\det h}(\mathrm{tr}_g\, h)^{n-1}$, we deduce that for all $(x, t) \in \Omega \times [\sigma, +\infty)$,

$$\frac{(n+1)\kappa t}{2n} h \leq g_{m,\epsilon}(t) \leq \left(\frac{e^{\Lambda\sigma} t}{\sigma}\right)^n \left[\frac{2n}{(n+1)\kappa t}\right]^{n-1} \left(\frac{\det g_{0,m,\epsilon}}{\det h}\right) h. \tag{4.6}$$

Using (4.3), (4.6), and the fact that $g_{0,m,\epsilon}$ converges to g_0 locally uniformly, in conclusion, we have shown that for any $T < \infty$ and compact set $\Omega \Subset M$, there is $\lambda > 1$ such that for all m sufficiently large, ϵ sufficiently small (depending also on m) and $t \in [0, T]$,

$$\lambda^{-1} h \leq g_{m,\epsilon}(t) \leq \lambda h.$$

By applying the higher order estimate of the Kähler–Ricci flow [19] on local charts, we have all higher order estimates of $g_{m,\epsilon}(t)$ on $\Omega \times [0, T]$, which is independent of $m >> 1$ and $\epsilon < \epsilon_0(n, m, g_0, \Omega)$.

By Ascoli–Arzelà Theorem, for each m large we may first let $\epsilon_i \to 0$ for some subsequence ϵ_i to obtain a Kähler–Ricci flow solution $g_m(t), t \in [0, T]$ on each U_{R_m}. Then by diagonal subsequence argument and Ascoli–Arzelà Theorem again, we may let $m_i \to +\infty$ and followed by $T \to +\infty$ to obtain a global solution $g(t)$ of the Kähler–Ricci flow on $M \times [0, +\infty)$. The instantaneous completeness of $g(t)$ followed by passing (4.1) to the limiting solution $g(t)$. ■

Proof of Corollary 1.1. By Theorem 1.1, there is a long-time solution to the Kähler–Ricci flow starting from h. The existence of complete negative Kähler–Einstein followed from [11, Theorem 5.1]. ■

Remark 4.1. It is clear from the proof of Theorem 1.1 and Corollary 1.1 that the reference metric h can be a incomplete metric. Then the resulting Kähler–Ricci flow solution and Kähler–Einstein metric may be an incomplete metric.

Remark 4.2. Since h only served as a reference metric in the Schwarz Lemma, the existence of Kähler metric h can be replaced by the existence of a Hermitian metric with *real bisectional curvature* bounded from above by negative constant, see [13, 14, 29].

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