

On the uniqueness for the heat equation on complete Riemannian manifolds

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Abstract

We prove some uniqueness result for solutions to the heat equation on Riemannian manifolds. In particular, we prove the uniqueness of L^p solutions with $0 and improves the <math>L^1$ uniqueness result of Li (J Differ Geom 20:447–457, 1984) by weakening the curvature assumption.

Keywords Uniqueness problem · Heat equation on manifolds · Complete noncompact manifolds

1 Introduction

In this article, we consider the uniqueness problem for solutions to the heat equation on complete Riemannian manifolds (M, g):

$$(\partial_t - \Delta)f = 0,$$

where Δ is the Laplace–Beltrami operator with respect to the metric g.

It is well-known that uniqueness may fail in general unless we restrict the solutions on some suitable class of functions. A example is the set of functions bounded from below. In [7], the uniqueness of nonnegative solutions to the heat equation has been established under the quadratic Ricci lower bound assumption

$$Ric(x) \ge -C(r(x) + 1)^2,$$
 (1.1)

where r(x) is the geodesic distance from some fixed point and C is a nonnegative constant. Another typical of class where uniqueness holds is the set of functions with appropriate growth rate in the spirit of [10]. For solutions with L^2 integrals on geodesic balls or

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parabolic cylinders growing under certain rate, the uniqueness was proved in [1, 3]. The same result holds if L^2 is replaced by L^p with 1 , and for a special class of manifolds when <math>p = 1 [8]. These results imply uniqueness for solutions with suitable pointwise growth rate, provided that the manifold has some volume growth constraint. A case of particular interest is for bounded solutions, see [2] for a survey.

Our first theorem is an improvement in the results in [1, 3]. Namely, we allow the integral to be weighted by a positive power of the time variable. We will also demonstrate an example in Sect. 3.

Theorem 1.1 Let M be a complete Riemannian manifold, and let f(x, t) be a nonnegative subsolution to the heat equation on $M \times (0, 1]$ with initial data f(x, 0) = 0 in the sense of $L^2_{loc}(M)$. Suppose for some point $q \in M$, and constant a > 0,

$$\int_0^1 t^a \int_{B_a(r)} f^2 \le e^{L(r)}, \quad \forall r > 0,$$

where L(r) is a positive nondecreasing function satisfying

$$\int_{1}^{\infty} \frac{r}{L(r)} dr = \infty.$$
 (1.2)

Then $f \equiv 0$ on $M \times (0, 1]$.

In [5], Li considered the uniqueness for L^p solutions to the heat equation. When p > 1, the uniqueness holds without further assumption. However when p = 1 the uniqueness may fail on sufficiently negatively curved manifolds, it was proved in [5] that the uniqueness for L^1 solutions holds under the assumption (1.1).

As an application of Theorem 1.1, we prove the following theorem which can be applied to improve the L^1 uniqueness result for the heat equation in [5]. It also implies uniqueness of L^p solutions with $0 . The curvature assumption (1.3) and (1.4) is slightly more general than (1.1) since functions such as <math>r \ln r$ are allowed.

Theorem 1.2 Let (M^n, g) be a complete Riemannian manifold with

$$Ric(x) \ge -k^2(r_q(x)),\tag{1.3}$$

where $r_p(x)$ is the distance function to a fixed point $q \in M$, and k(r) is a positive nondecreasing function satisfying

$$\int_{1}^{\infty} \frac{1}{k(r)} dr = \infty. \tag{1.4}$$

Suppose f is a nonnegative subsolution to the heat equation on $M \times (0, 1]$, with initial data f(0) = 0 in sense of $L^2_{loc}(M)$. If for some 0 ,

$$||f||_{L^p(B_a(r)\times[0,1])} \le e^{Crk(r)}, \text{ for any } r > 0,$$

for some constant C, then $f \equiv 0$ on $M \times (0, 1]$.



For the proof of Theorem 1.2, we use mean value inequality to get a pointwise bound for the solution, which is non-uniform and blows up as $t \to 0$, and we can verify that the assumptions in Theorem 1.1 are satisfied.

To prove the uniqueness of solutions to the heat equation, we can consider a solution starting with 0 initial data and apply Theorems 1.1 or 1.2 to its absolute value which is a nonnegative subsolution.

Furthermore, the above results imply the maximum principle. For instances, if u is a subsolution to the heat equation with $u(0) \le 0$, then one can apply Theorems 1.1 or 1.2 to $(u-0)_+$ to show that $u(t) \le 0$ provided that the assumptions are met.

2 Proof

Proof of Theorem 1.1 The proof is a modification of the arguments due to Karp–Li [3] and Grigor'yan [1]. Define the function

$$\xi(x,t) = -\frac{(r(x) - R)_+^2}{4(T - t)},$$

where r(x) is the distance function to the fixed point q, then it is a direct calculation to check

$$\partial_t \xi + |\nabla \xi|^2 \le 0.$$

For any R > 0, let $\psi(r)$ be a nonincreasing cut-off function with $|\psi'| \le \frac{4}{R}$ and

$$\psi(r) = \begin{cases} 1, \ r \le \frac{3}{2}R; \\ 0, \ r > 2R. \end{cases}$$

Let $\phi(x) = \psi^m(r(x))$ where m > 0 is some large number to be chosen later, then

$$|\nabla \phi|^2 \le \frac{100m^2}{R^2} \phi^{2-2/m}.$$

For any $0 < \tau < T \le 1$, we have

$$\begin{split} &\int_{\tau}^{T} t^{-1} \int \phi^{2} e^{\xi} f^{2} \partial_{t} \xi + 2 \phi^{2} e^{\xi} f \partial_{t} f \\ &\leq \int_{\tau}^{T} t^{-1} \int \phi^{2} e^{\xi} f^{2} \partial_{t} \xi + 2 \phi^{2} e^{\xi} f \Delta f \\ &= \int_{\tau}^{T} t^{-1} \int \phi^{2} e^{\xi} f^{2} \partial_{t} \xi - 2 \phi^{2} e^{\xi} f \langle \nabla \xi, \nabla f \rangle - 2 \phi^{2} e^{\xi} |\nabla f|^{2} - 4 \phi e^{\xi} f \langle \nabla \phi, \nabla f \rangle \quad (2.1) \\ &\leq \int_{\tau}^{T} t^{-1} \int \phi^{2} e^{\xi} f^{2} (\partial_{t} \xi + |\nabla \xi|^{2}) + 4 |\nabla \phi|^{2} e^{\xi} f^{2} \\ &\leq 4 \int_{\tau}^{T} t^{-1} \int |\nabla \phi|^{2} e^{\xi} f^{2}. \end{split}$$

By the choice of ϕ , using similar arguments as in [4], we can apply the Hölder inequality and Young's inequality to show the following.



$$\int |\nabla \phi|^{2} e^{\xi} f^{2} \leq \frac{100m^{2}}{R^{2}} \int_{spt(\nabla \phi)} \phi^{2-2/m} e^{\xi} f^{2}
\leq \frac{100m^{2}}{R^{2}} \left(\int_{spt(\nabla \phi)} \phi^{2} e^{\xi} f^{2} \right)^{(1-1/m)} \left(\int_{spt(\nabla \phi)} e^{\xi} f^{2} \right)^{\frac{1}{m}}
\leq \frac{1}{4t} \left(\int_{spt(\nabla \phi)} \phi^{2} e^{\xi} f^{2} \right) + \frac{C(m)t^{m-1}}{R^{2m}} \left(\int_{spt(\nabla \phi)} e^{\xi} f^{2} \right),$$
(2.2)

where $C(m) = 400^{2m-1} m^{2m}$ and $spt(\nabla \phi)$ is the compact support of $\nabla \phi$. Combines with the previous inequality, we obtain

$$\frac{1}{T} \int \phi^{2} e^{\xi} f^{2}(T) - \frac{1}{\tau} \int \phi^{2} e^{\xi} f^{2}(\tau) = \int_{\tau}^{T} t^{-1} \int \phi^{2} e^{\xi} f^{2} \partial_{t} \xi + 2\phi^{2} e^{\xi} f \partial_{t} f - \int_{\tau}^{T} t^{-2} \left(\int \phi^{2} e^{\xi} f^{2} \right) \\
\leq \frac{C(m)}{R^{2m}} \int_{\tau}^{T} t^{m-2} \int_{spt(\nabla \phi)} e^{\xi} f^{2}. \tag{2.3}$$

On $spt(\nabla \phi) \subset B_q(2R) \backslash B_q(\frac{3}{2}R)$, we have

$$\xi \le -\frac{R^2}{16(T-\tau)}.$$

Therefore, if we choose m > a + 2, the growth assumption on f will imply

$$\int_{\tau}^{T} t^{m-2} \int_{spt(\nabla \phi)} e^{\xi} f^{2} \le T^{m-a-2} e^{-\frac{R^{2}}{16(T-\tau)} + L(2R)}.$$

Now if we require that

$$(T-\tau) \le \frac{R^2}{16L(2R)},$$

then

$$\frac{1}{T} \int_{B(R)} f^2(T) - \frac{1}{\tau} \int_{B(2R)} f^2(\tau) \le \frac{C(m)T^{m-a-2}}{R^{2m}}.$$
 (2.4)

To proceed, we take an increasing sequence of R_i , and a decreasing sequence of τ_i in the following way. Let $R_i = 2^i R$, $\tau_0 = \tau$ and take τ_{i+1} such that

$$\tau_i - \frac{R_i^2}{16L(2R_i)} \le \tau_{i+1} < \tau_i.$$

Then for any N, apply (2.4) inductively we have



$$\frac{1}{\tau} \int_{B(R)} f^{2}(\tau) = \frac{1}{\tau_{N}} \int_{B(R_{N})} f^{2}(\tau_{N}) + \sum_{i=0}^{N-1} \left(\frac{1}{\tau_{i}} \int_{B(R_{i})} f^{2}(\tau_{i}) - \frac{1}{\tau_{i+1}} \int_{B(R_{i+1})} f^{2}(\tau_{i+1}) \right) \\
\leq \frac{1}{\tau_{N}} \int_{B(R_{N})} f^{2}(\tau_{N}) + \sum_{i=0}^{N-1} \frac{C(m)\tau_{i}^{m-1-a}}{R_{i}^{2m}} \\
\leq \frac{1}{\tau_{N}} \int_{B(R_{N})} f^{2}(\tau_{N}) + \frac{2C(m)\tau^{m-1-a}}{R^{2m}}.$$
(2.5)

By the assumption on L(r) (1.2), we must have

$$\sum_{i=0}^{\infty} \frac{R_i^2}{L(R_i)} = \infty,$$

hence we can choose the sequence $\{\tau_i\}$ such that τ_i becomes zero in finite steps.

To show that the first term in the last line of (2.5) can be dropped, we claim that for any R > 0, we have

$$\lim_{t \to 0^+} \frac{1}{t} \int_{B(R)} f^2(t) = 0.$$

To prove the claim. For any cut-off function ϕ , since $\lim_{t\to 0^+} \int \phi^2 f^2(t) = 0$, we have

$$0 \ge \int_0^t \int \phi^2 f(\partial_t f - \Delta f)$$

$$= \frac{1}{2} \int \phi^2 f^2(t) + \int_0^t \int \phi^2 |\nabla f|^2 + 2 \int_0^t \int \phi f \langle \nabla \phi, \nabla f \rangle$$

$$\ge \frac{1}{2} \int \phi^2 f^2(t) - \int_0^t \int |\nabla \phi|^2 f^2.$$
(2.6)

Choose a cut-off function ϕ similarly as before such that $|\nabla \phi| \le C\phi^{1-1/m}$ for some $m \ge 2$, then the above inequality yields

$$\int \phi^{2} f^{2}(t) \leq C \int_{0}^{t} \int (\phi^{2} f^{2})^{\frac{m-1}{m}} f^{\frac{2}{m}}
\leq C \int_{0}^{t} \left(\int \phi^{2} f^{2} \right)^{\frac{m-1}{m}} \left(\int_{spt(\phi)} f^{2} \right)^{\frac{1}{m}}
\leq C \sup_{s \in (0,t)} \|f(s)\|_{L^{2}(spt(\phi))}^{\frac{1}{m}} \int_{0}^{t} \left(\int \phi^{2} f^{2} \right)^{\frac{m-1}{m}}
\leq C \sup_{s \in (0,t)} \|f(s)\|_{L^{2}(spt(\phi))}^{\frac{1}{m}} \left(\sup_{s \in (0,t)} \int \phi^{2} f^{2}(s) \right)^{\frac{m-1}{m}} t.$$
(2.7)

Since the RHS is nondecreasing in t, we have

$$\left(\sup_{s\in(0,t)}\int \phi^2 f^2(s)\right)^{\frac{1}{m}} \le C\sup_{s\in(0,t)} \|f(s)\|_{L^2(spt(\phi))}^{\frac{1}{m}} t,$$



and hence $\int \phi^2 f^2(t) = o(t^m)$. Since the cut-off function ϕ is chosen for an arbitrary radius, this proves the claim.

By letting $R \to \infty$ in (2.5), we show that $f(\tau) \equiv 0$ for any $\tau \in (0, 1]$. This completes the proof.

Proof of Theorem 1.2 By [9], the curvature assumption (1.3) implies that there is a Sobolev inequality in the following form:

$$\left(\int \phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \frac{R^2 e^{C(n)Rk(R)}}{Vol(B_q(R))^{\frac{2}{n}}} \int (|\nabla \phi|^2 + R^{-2}\phi^2),$$

for any smooth function ϕ compactly supported in the geodesic ball $B_q(R)$. With the Sobolev inequality, we can apply Nash–Moser iteration to prove a mean value inequality for f (see Chapter 19 of [6]), for any $t \in (0, 1]$,

$$||f||_{L^{\infty}(B_{q}(\frac{R}{2})\times[\frac{t}{2},t])} \leq C(n,p)e^{CaRk(R)}\left(\frac{1}{R^{\beta}} + \frac{1}{t}\right)^{\gamma}Vol\left(B_{q}\left(\frac{R}{2}\right)\right)^{-\frac{1}{p}}||f||_{L^{p}(B_{q}(R)\times[0,t])},$$

where α , β , γ are positive constants depending on n and p. Without loss of generality, we can assume R > 2 and hence

$$Vol(B_q(R/2)) \ge v_0 := Vol(B_q(1)).$$

Now the assumption of the theorem implies

$$|f(x,t)| \le \frac{e^{Cr(x)k(2r(x))}}{t^a}, \quad t \in (0,1],$$

for some constants C depending on n, p, v_0 , and the constant a only depends on n and p. By (1.1) and volume comparison theorem, we have the volume growth estimate

$$Vol(B_q(R)) \le C(n)e^{c(n)Rk(R)}$$

for any R > 0, see for example [2]. Hence, we have

$$\int_0^1 t^a \int_{B(R)} f^2 \le e^{L(R)},$$

with

$$L(R) = CRk(2R)$$
,

for some constant C. By (1.4), the function L(R) satisfies (1.2). The result now follows from Theorem 1.1.



3 Example

In this section, we describe the construction of a solution to the heat equation, which belongs to the uniqueness class of Theorem 1.1, but not in that of [1, 3] or [8]. Intuitively, we want to construct a solution which has a sequence of 'spikes' with fast growing heights, while supported on decaying domains so that we have some integral control of the solution locally.

Take $M = \mathbb{R}^n$ with $n \ge 3$, and we will make several assumptions for simplicity; however, the same method can be used to construct more complicated examples.

To start with, let \tilde{u}_0 be a continuous function on \mathbb{R}^n with growth rate slower than $e^{C|x|^2}$. For simplicity, we take $\tilde{u}_0 \ge 0$ and $\tilde{u}_0 \in L^1(\mathbb{R}^n)$.

We will construct a "spiked" initial function u_0 by modifying \tilde{u}_0 : for each positive integer i = 1, 2, 3, ..., choose a geodesic ball

$$B(p_i, r_i) \subset B(0, i+1) \backslash B(0, i),$$

where the radii is chosen to be

$$r_i = \left(\frac{1}{\omega_n i^2 e^{i^3}}\right)^{\frac{1}{n}}.$$

Here ω_n is the volume of the unit ball in \mathbb{R}^n . Denote

$$\tilde{r}_i = \frac{r_i}{2^{1/n}}.$$

We now modify \tilde{u}_0 in each $B(p_i, r_i)$ to obtain the desired initial data u_0 . Define

$$u_0 = \begin{cases} e^{i^3}, & \text{on } B(p_i, \tilde{r}_i), \\ \text{continuous and } \leq e^{i^3}, & \text{on } B(p_i, r_i) \backslash B(p_i, \tilde{r}_i), \\ \tilde{u}_0, & \text{otherwise.} \end{cases}$$

The new function u_0 is a continuous function which is L^1 on the modified region $\bigcup_{i=1}^{\infty} B(p_i, r_i)$.

Solve the Cauchy problem of the heat equation with initial function u_0 by convoluting with the heat kernel:

$$u(x,t) = \int \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy.$$

For each $x \in B(p_i, \tilde{r}_i)$ and t > 0,

$$u(x,t) \ge \frac{1}{(4\pi t)^{n/2}} \int_{B(p_i, \tilde{r}_i)} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy$$

$$\ge \frac{1}{(4\pi t)^{n/2}} e^{-\frac{4\tilde{r}_i^2}{4t}} e^{i\tilde{s}} \omega_n \tilde{r}_i^n$$

$$= \frac{1}{2i^2 (4\pi t)^{n/2}} e^{-\frac{\tilde{r}_i^2}{t}}.$$
(3.1)

Hence



$$\int_{0}^{1} \int_{B(0,i+1)} u(x,t)^{2} \ge \int_{0}^{1} \int_{B(p_{i},\tilde{r}_{i})} u(x,t)^{2}$$

$$\ge \frac{\omega_{n}\tilde{r}_{i}^{n}}{4i^{4}} \int_{0}^{1} (4\pi t)^{-n} e^{-2\tilde{r}_{i}^{2}/t} dt$$

$$= \frac{\omega_{n}}{2^{n+1}(4\pi)^{n}} \frac{1}{i^{2}\tilde{r}_{i}^{n-2}} \int_{1}^{\infty} s^{n-2} e^{-s} ds$$

$$\ge C(n) e^{\frac{n-2}{n}\tilde{t}^{2}}.$$
(3.2)

Thus *u* violates the assumption in either [3] or [1] when $n \ge 3$. For L^p integrals with p > 1 one can compute similarly.

On the other hand, since we assumed u_0 to be L^1 , we have

$$|u(x,t)| \le \frac{\|u_0\|_{L^1(\mathbb{R}^n)}}{(4\pi t)^{n/2}},$$

hence it satisfies the assumption of Theorem 1.1.

To construct examples which are not in L^1 , we can start with $\tilde{u} \equiv 1$ instead of a L^1 function; and to construct examples not bounded from either side, we can add a sequence of "negative spikes" to u_0 sufficiently far away from the positive one.

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