CHAZY-TYPE ASYMPTOTICS AND HYPERBOLIC SCATTERING FOR THE n-BODY PROBLEM

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Abstract. We study solutions of the Newtonian n-body problem which tend to infinity hyperbolically, that is, all mutual distances tend to infinity with nonzero speed as $t \to +\infty$ or as $t \to -\infty$. In suitable coordinates, such solutions form the stable or unstable manifolds of normally hyperbolic equilibrium points in a boundary manifold "at infinity". We show that the flow near these manifolds can be analytically linearized and use this to give a new proof of Chazy's classical asymptotic formulas. We also address the scattering problem, namely, for solutions which are hyperbolic in both forward and backward time, how are the limiting equilibrium points related? After proving some basic theorems about this scattering relation, we use perturbations of our manifold at infinity to study scattering "near infinity", that is, when the bodies stay far apart and interact only weakly.

1. Introduction and Overview

Let $q(t) = (q_1(t), q_2(t), \dots, q_n(t)) \in \mathbb{E}$, $t \in \mathbb{R}$ be a solution of the Newtonian n-body problem. Here $\mathbb{E}$ denotes the n-body configuration space with center of mass fixed at the origin. If $q(t)$ has positive energy h, then it is unbounded: some of its interbody distances will tend to infinity with time. Following Chazy [**?**], we call a solution "hyperbolic" if *all* its interbody distances tend to infinity with time, and do so asymptotically linearly. Chazy [**?**, eq. (27)] established the first few terms of a convergent asymptotic expansion for hyperbolic solutions,

$$q(t) = At + B \log |t| + C + \cdots .$$

One can show that $B = B(A)$. The coefficients $A, C \in \mathbb{E}$ determine the entire series and can be thought of as initial conditions at infinity. We will call these two coefficients the scattering parameters. The leading coefficient A represents the asymptotic velocities of the bodies. The mass weighted norm of A satisfies $\|A\|^2 = 2h$. The normalized vector $A/\|A\|$ represents the limiting shape of the configuration $q(t)$. The interpretation of the coefficient C is less straightforward.

Bi-hyperbolic solutions, solutions that are hyperbolic in both time directions, sweep out an open subset of the phase space and allow us to define a *scattering map*

$$F : (A, C) \mapsto (A', C')$$

which sends the scattering parameters associated to the infinite past of a solution to its scattering parameters associated to the infinite future. We then say that A and A' are related by hyperbolic scattering and write $A \to A'$. In other words, $A \to A'$ means that there is some bi-hyperbolic orbit with asymptotic behavior described by

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A in the past and by A' in the future. *The main problem of hyperbolic scattering can now be stated : For a given scattering parameter A at time minus infinity, which parameters A' at time infinity are related to it by hyperbolic scattering ?* Since the energy is constant along solutions, we must have $\|A\| = \|A'\|$, so this question is really a question of connecting the two asymptotic shapes $A/\|A\|$ and $A'/\|A'\|$.

For example, consider the planar two-body problem with the center of mass $m_1q_1 + m_2q_2$ fixed at the origin. Then the vector $\vec{r}(t) = q_2(t) - q_1(t)$ connecting the two-bodies satisfies the Kepler problem and for positive energies, it sweeps out a hyperbola in the plane (see Figure 1). Suppose the hyperbola has semimajor axis $a > 0$ and eccentricity $e > 1$. Then the Chazy parameters A, A' for this orbit are vectors with equal length $\sqrt{2h}$ pointing along the asymptotes of the hyperbola as in Figure 1. ($\mathbb{E}$ can be parameterized by $q_2 - q_2$.) The vectors C, C' are orthogonal with equal lengths proportional to $a\sqrt{e^2-1}$ (see Example 4.1 below). Together, A, C uniquely determine the hyperbola and then $(A', C') = F(A, C)$ are also uniquely determined. The asymptotic shape vectors $A/\|A\|$ and $A'/\|A'\|$ encode the incoming and outgoing directions and the main problem reduces to: For a fixed incoming direction, which outgoing directions are possible ? A little thought about hyperbolas shows that the outgoing unit vector can be anything other than minus the incoming one. To put it another way, all possible "scattering angles" are possible, except π. Of course, the situation for the n-body problem with $n > 2$ is much more complicated.

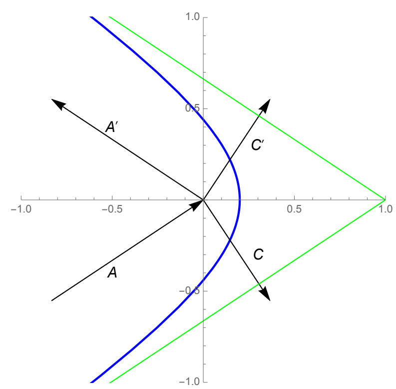

FIGURE 1. Hyperbolic Scattering for the Kepler Problem

We approach scattering by constructing a kind of McGehee regularization which partially compactifies n-body phase space by adding a boundary at infinity. Let r be the square root of the moment of inertia of the bodies with respect to its center of mass, which represents the overall size of the configuration. Then the boundary at infinity is given by $r = +\infty$ or equivalently, $\rho := 1/r = 0$. The regularized dynamics extends analytically to this infinity manifold. The infinity manifold is an invariant manifold for the extended flow and contains a submanifold $\hat{\mathcal{E}}$ of equilibrium points which gives the escaping scattering solutions a place to go. (The hat over $\mathcal{E}$ indicates that we have removed collisions from a slightly larger equilibrium set $\mathcal{E}$ at infinity.) $\hat{\mathcal{E}}$ is normally hyperbolic, has half the dimension of the phase space and falls into two components, $\hat{\mathcal{E}}_-$ and $\hat{\mathcal{E}}_+$, one for the infinite past, the other for the infinite future, with $\hat{\mathcal{E}}_-$ repelling and $\hat{\mathcal{E}}_+$ attracting. Every future-hyperbolic solution has for its ω-limit set a single point of $\hat{\mathcal{E}}_+$. Every past-hyperbolic solution has for its α-limit set a single point of $\hat{\mathcal{E}}_-$. These sets of restpoints $\hat{\mathcal{E}}_\pm$ can be identified with the configuration space $\hat{\mathbb{E}}$ where the hat symbol means that we have deleted the collision configurations. With this identification, specifying a restpoint $p \in \hat{\mathcal{E}}_-$ is equivalent to specifying the Chazy parameter A at time minus infinity. and similarly $q \in \hat{\mathcal{E}}_+$ corresponds to a Chazy parameter A' at time infinity. From this point of view, the scattering parameters parameters C, C' specify particular orbits in the stable and unstable manifolds of the restpoints determined by A, A'. For the blown-up flow, the bi-hyperbolic solutions are exactly the heteroclinic orbits connecting $\hat{\mathcal{E}}_-$ to $\hat{\mathcal{E}}_+$. If $p \in \hat{\mathcal{E}}_-$ and $q \in \hat{\mathcal{E}}_+$ are connected by a bi-hyperbolic orbit we will write $p \to q$ which is equivalent to saying $A \to A'$ for the corresponding Chazy parameters, as above.

By the *image* of A under the scattering relation we will mean the set $\mathcal{HS}(A) = \{A' : A \to A'\}$. Then the main problem of scattering can be stated as: "what is the nature of this image set?". Besides preserving the energy levels, there seem to be no other obvious restrictions and indeed, for $n = 2$, almost all energetically possible scatterings do occur. Nevertheless, for $n \geq 3$, it is challenging to prove anything at all.

Here is an outline of the paper. Section 2 sets up the equations of motion and introduces the blown-up coordinates at infinity. In Section 3 we study the resulting flow on the invariant manifold at infinity. At the end of section 3 we formulate our key analytic linearization result, Theorem 3.1, which is needed in Section 4 to obtain a new proof of Chazy's asymptotic formulae. We defer the proof of this analytic linearization result to the appendix where its proof becomes a corollary of the more general Theorem A.1.

Section 5 forms the heart of our investigation. There we define the scattering map and the scattering relation and prove some of their basic properties. Then we focus on scattering near infinity, that is, on the behavior of bi-hyperbolic solutions whose overall size r remains large throughout. Our main result about the main problem shows that using heteroclinic orbits near infinity, we can connect almost every initial A to at least an open subset of the energetically possible A':

Theorem. *For generic $A \in \hat{\mathbb{E}}$, the image $\mathcal{HS}(A)$ has nonempty interior in the sphere $\{A' : \|A'\| = \|A\|\}$.*

Related Work. The literature on classical scattering is vast. See the somewhat encyclopaedic [**?**]. However, there is very little rigorous work that we are aware of on global properties of Newtonian N-body scattering for $N > 2$. Chazy [**?**] is the seminal piece. Quite recently Maderna and Venturelli [**?**] posted an elegant variationally inspired article which is closely related but complementary to this present work. Their main result is that given any asymptotic shape, an "$s = A/\|A\|$" in Chazy language, and any configuration $q_0 \in \mathbb{E}$, there is a hyperbolic orbit passing through q_0 at time $t = 0$ and whose asymptotic shape is s as $t \to +\infty$. In other words, instead of trying to connect a given state at $t = -\infty$ to one at $t = +\infty$ they connecting finite states at $t = 0$ to states at $t = +\infty$.

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2. Change of variables extending the flow to infinity

Let $m_i > 0, q_i \in \mathbb{R}^d$ and $\xi_i \in \mathbb{R}^d$ denote the masses, positions and velocities of the bodies and let $q = (q_i)_{i=1}^n \in \mathbb{R}^{nd}$ and $\xi = (\xi_i)_{i=1}^n \in \mathbb{R}^{nd}$. The Newtonian potential function (the negative potential energy) is

$$U(q) = \sum_{1 \le i < j \le n} \frac{m_i m_j}{|q_i - q_j|}$$

and the kinetic energy is

$$K(\xi) = \frac{1}{2} \sum_{i=1}^{n} m_i |\xi_i|^2 = \frac{1}{2} \langle \xi, M\xi \rangle$$

where $M = \mathrm{diag}(m_1, \ldots, m_1, m_2, \ldots, m_2, \ldots, m_n, \ldots, m_n)$ is the $nd \times nd$ mass matrix with d copies of each mass along the diagonal and $\langle \cdot, \cdot \rangle$ is the Euclidean metric on $\mathbb{R}^{nd}$.

Introduce the *mass inner product*, also called the *kinetic energy inner product*:

$$\langle\langle v, w \rangle\rangle = \langle v, Mw \rangle. \tag{1}$$

If we write the norm of a vector v relative to this inner product by $\|v\|^2$ then $K(\xi) = \frac{1}{2}\|\xi\|^2$ and $\|q\|^2$ is the moment of inertia with respect to the origin.

Newton's equations can be written

$$\ddot{q} = M^{-1} \nabla_{\mathrm{euc}} U(q) = \nabla U(q)$$

where ∇_{euc} is the Euclidean gradient and $\nabla = M^{-1}\nabla_{\mathrm{euc}}$ is the gradient with respect to the mass inner product. Newton's equations preserve the total energy

$$H(q, \xi) = K(\xi) - U(q) = h,$$

which we assume to be positive throughout the paper.

Without loss of generality, we may fix the center of mass at the origin, and insist that the total linear momentum $\Sigma_{i=1}^n m_i \xi_i$ is zero. Then $q, \xi \in \mathbb{E}$ where

$$\mathbb{E} = \{q \in \mathbb{R}^{nd} : \sum_{i=1}^{n} m_i q_i = 0\} \tag{2}$$

This $\mathbb{E}$ is a subspace of $\mathbb{R}^{nd}$ of dimension $D = d(n-1)$. Denote the collision set by

$$\Delta = \{q : q_i = q_j \text{ for some } i \neq j\}.$$

Then the configuration space for the n-body problem is $\hat{\mathbb{E}} = \mathbb{E} \setminus \Delta$ and its phase space is the tangent bundle $T\hat{\mathbb{E}} = \hat{\mathbb{E}} \times \mathbb{E}$.

Set $S = \{q \in \mathbb{E} : \|q\| = 1\}$ and introduce spherical variables $(r, s) \in (0, +\infty] \times S$ on $\mathbb{E}$ according to:

$$q = rs, \text{ where } r = \|q\|, \;\; s = q/r \in S,$$

where S is a sphere of dimension $D-1$. Set $\hat{S} := S \setminus \Delta$.

Decompose the velocity vector $\xi := \dot{q}$ as

$$\xi = vs + w \qquad \text{where } v = \langle\langle s, \xi \rangle\rangle \text{ and } \langle\langle s, w \rangle\rangle = 0,$$

so that $v \in \mathbb{R}$ is the radial velocity component while $w \in \mathbb{E}$ is the tangential velocity.

Make the change of independent (time) variable

$$dt = r d\tau, \tag{3}$$

writing $' = \frac{d}{d\tau}$. Then Newton's equations become

$$\begin{aligned} r' &= vr \\ s' &= w \\ v' &= \|w\|^2 - \frac{1}{r}U(s) \\ w' &= \frac{1}{r}\nabla U(s) + \frac{1}{r}U(s)s - vw - \|w\|^2 s = \frac{1}{r}\tilde{\nabla} U(s) - vw - \|w\|^2 s \end{aligned} \tag{4}$$

where

$$\tilde{\nabla} U(s) = \nabla U(s) + U(s)s$$

is the tangential component of $\nabla U(s)$, since U is homogeneous of degree -1. The energy equation becomes

$$\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 - \frac{1}{r}U(s) = h. \tag{5}$$

The changes of variables and timescale leading to (4) are a variation on the McGehee blow-up method [?, ?] first introduced for studying triple collision. In that case, the change of timescale was $dt = r^{\frac{3}{2}} d\tau$. The factor of r here is more appropriate for studying hyperbolic orbits near infinity.

As is well-known, any nonsingular solution with $h > 0$ must tend to infinity:

Proposition 2.1. *Let $q(t)$ be a collision-free solution of the N-body problem, with energy $h > 0$, defined for all $t \in [0, +\infty)$. Let $\tau = \tau(t)$ be the new time parameter defined as (3) with*

$$\tau(0) = 0, \quad \tau_0 = \lim_{t \to +\infty} \tau(t).$$

Then $r(t) \to +\infty$ as $t \to +\infty$ and $r(\tau) \to +\infty$ as $\tau \to \tau_0$. Moreover, if $U(q(t))$ is bounded, then $\tau_0 = +\infty$. Similarly for solutions defined for all $t \in (-\infty, 0]$.

Proof. Let $I = r^2$ be the moment of inertia. Using the usual timescale we have

$$\frac{1}{2}\ddot{I} = r\ddot{r} + \dot{r}^2 = v' + v^2$$

where we used the fact that $\dot{r} = v$ and the definition of the new timescale. Using the energy equation and the differential equation (4) for v' we derive

$$v' + v^2 = 2h + \frac{1}{r}U(s) = 2h + U(q),$$

so that

$$\frac{1}{2}\ddot{I} \geq 2h.$$

Integrating twice with respect to t gives

$$I(t) \geq 2ht^2 + kt + l$$

where $k = \dot{I}(0), l = I(0)$. Since $h > 0$ and $I(t) = r(t)^2$, we have $r(t) \to +\infty$ as $t \to +\infty$. In fact

$$r(t) \geq ct$$

for t sufficiently large, where $c > 0$ is a suitable constant.

If we have an upper bound $U(q(t)) \leq K$ we get upper estimates

$$\frac{1}{2}\ddot{I} \leq 2h + K \qquad r(t) \leq Ct$$

for t sufficiently large and some $C > 0$. Then

$$\frac{d\tau}{dt} = \frac{1}{r(t)} \geq \frac{1}{Ct}$$

Hence $\tau(t) \to +\infty$ as $t \to +\infty$ and the rescaled solution exists for $\tau \in [0, +\infty)$. □

For studying behavior near infinity, introduce

$$\rho = 1/r.$$

which transforms equations (4) into

$$\begin{aligned} \rho' &= -v\rho \\ s' &= w \\ v' &= \|w\|^2 - \rho U(s) \\ w' &= \rho\tilde{\nabla}U(s) - vw - \|w\|^2 s \end{aligned} \tag{6}$$

with energy equation

$$\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 - \rho U(s) = h.$$

Define the infinity manifold to be

$$\Sigma = \{(\rho, v, s, w) : \rho = 0,\ v \in \mathbb{R},\ s \in S,\ w \in \mathbb{E},\ \langle\langle s, w\rangle\rangle = 0,\ h > 0\}.$$

Σ is an invariant manifold of (6) which represents the behavior of the n-body problem "at infinity". Topologically, Σ is a manifold of dimension $2D-1$, diffeomorphic to $S \times (\mathbb{R}^D \setminus \{0\})$. To see its topology, use $\xi = w + vs \in \mathbb{R}^D$ as above, and note that when $\rho = 0$ the positive energy is given by $h = \frac{1}{2}\|\xi\|^2$. Fixing this energy to be the positive number h defines a $(2D-2)$-dimensional submanifold $\Sigma_h \subset \Sigma$ which is the product of two $(D-1)$-spheres.

The usual phase space $\hat{\mathbb{E}} \times \mathbb{E}$ is coordinatized by those (ρ, s, v, w) for which $\rho > 0$ while $s \notin \Delta$ with the coordinatization being $(\rho, s, v, w) \mapsto (q, \xi) = (\frac{1}{\rho}s, vs + w)$. We will assiduously avoid collisions in our investigations here. To this end set

$$\hat{\Sigma} := \{(0, s, v, w) \in \Sigma :\ s \in \hat{S}\}.$$

Definition 2.1. *We will refer to $(0, +\infty) \times \hat{\Sigma}$ as the "usual" or "original" phase space and $[0, +\infty) \times \hat{\Sigma}$ as the "blown-up" or "extended" phase space.*

Remark 2.1. *Extended phase space is a manifold with boundary, this boundary being the infinity manifold $\hat{\Sigma}$. Equations (6) make sense and are analytic for $\rho < 0$ as well, so we could further extend phase space to $\mathbb{R} \times \hat{\Sigma}$. However $\rho < 0$ has no physical meaning that we are aware of and this extension will rarely, if ever be considered in what follows.*

The idea of introducing an invariant manifold at infinity goes back to McGehee [?, ?, ?]. It has been used many times since then, mainly in the negative or zero energy cases [?, ?, ?], but also in [?] for the positive energy case.

3. Flow on and near the infinity manifold

The differential equations on the infinity manifold Σ arise by setting $\rho = 0$ in (6) and are

$$\begin{aligned} \rho' &= 0 \\ s' &= w \\ v' &= \|w\|^2 \\ w' &= -vw - \|w\|^2 s \end{aligned} \tag{7}$$

The energy equation with $\rho = 0$ becomes

$$v^2 + \|w\|^2 = 2h. \tag{8}$$

Notice that the potential $U(s)$ has completely disappeared from the equations. Σ contains submanifolds of equilibria $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_+$, where

$$\mathcal{E}_\pm = \{(0, s, v, 0) : \ s \in S,\ v = \pm\sqrt{2h},\ h > 0\},$$

These equilibrium points are the alpha (-) and omega (+) limit sets of hyperbolic orbits.

Free particles and the flow at infinity. Since the flow at infinity is independent of the potential, we can understand it by setting the potential to zero, i.e., by considering the motion of free particles. Then each body moves in a straight line at constant speed with respect to the usual timescale, so $q_i(t) = A_i t + C_i$ where $A_i, C_i \in \mathbb{R}^d$ and $m_1 A_1 + \ldots + m_n A_n = m_1 C_1 + \ldots + m_n C_n = 0$, or

$$q(t) = At + C \tag{9}$$

where $A = \dot{q} \in \mathbb{E}$ is the constant velocity vector. The vector C is not unique, since a time translation $t \mapsto t - t_0$ transforms $C \mapsto C - t_0 A$. Using such a translation we may assume $\langle\langle A, C \rangle\rangle = 0$. The corresponding C represents the impact parameter: the closest point on the straight line to the origin. Now clearly (7) is just a rescaled version of this free particle flow.

Consider the asymptotic behavior of a free particle solution (9) as $t \to \pm\infty$ or equivalently $\tau \to \pm\infty$, but written out in our "McGehee" coordinates. We find

$$(\rho(\tau), s(\tau), v(\tau), w(\tau)) \to (0, \pm\frac{A}{\|A\|}, \pm\|A\|, 0) \qquad \tau \to \pm\infty.$$

Thus, in blown-up coordinates, the free particle motion is a heteroclinic orbit connecting restpoint p to restpoint $-p$ where

$$p = (0, -\frac{A}{\|A\|}, -\|A\|, 0) \in \mathcal{E}_-, \qquad -p = (0, \frac{A}{\|A\|}, \|A\|, 0) \in \mathcal{E}_+.$$

When projected onto the sphere, $s(\tau)$ is half of a great circle and connects the antipodal pair $\pm\frac{A}{\|A\|}$. See Figure 2.

The following proposition reformulates the facts just described.

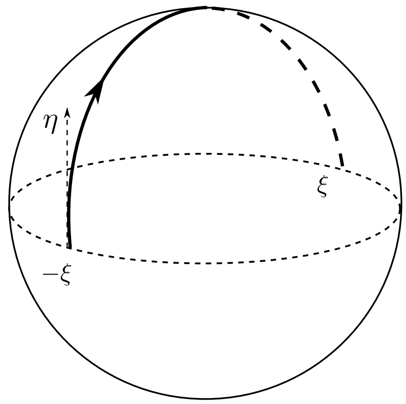

FIGURE 2. A half great circle solution

Proposition 3.1. *For any $\xi, \eta \in S$ satisfying $\langle\langle \xi, \eta \rangle\rangle = 0$, and any $h > 0$, equations (7) and (8) are solved by*

$$\begin{aligned}
\rho(\tau) &= 0 \\
v(\tau) &= \sqrt{2h}\tanh(\sqrt{2h}\tau) \\
s(\tau) &= \xi\sin(\theta(\tau)) + \eta\cos(\theta(\tau)) \\
w(\tau) &= \sqrt{2h}\,\mathrm{sech}(\sqrt{2h}\tau)(\xi\cos(\theta(\tau)) - \eta\sin(\theta(\tau))),
\end{aligned}$$

where $\theta(\tau) = tan^{-1}(\sinh(\sqrt{2h}\tau))$. This solution is heteroclinic, connecting

$$p = (0, -\xi, -\sqrt{2h}, 0) \in \mathcal{E}_- \qquad \text{to } -p = (0, \xi, \sqrt{2h}, 0) \in \mathcal{E}_+.$$

Up to time shift $\tau \mapsto \tau + \tau_0$, every solution to (7) and (8) is of this form.

Proof. The proof is a straightforward computation. Alternatively, convert to the case of a free particle just described. □

Remark 3.1. *Figure 2 and the discussion preceding proposition 3.1 describes the time π geodesic flow for the sphere. Melrose and Zworski* [**?**] *underlined the central role played by this flow for quantum scattering.*

We obtained the limiting equations (7) by setting $\rho = 0$ in (6). However, in the neighborhood of collision points the values of the expressions $\rho U(s)$ and $\rho\tilde{\nabla}U(s)$ are not uniformly small. Some of the "half-great circle" solutions in Proposition 3.1 will pass through collision configurations. In order to correctly extend into the interior $\rho > 0$, where the real solutions lie, we will need to delete these collision solutions. When we impose this restriction we obtain a dynamical system on the open subset of the infinity manifold which is analytic, extends analytically into the interior, and satisfies the usual existence, uniqueness and smoothness properties.

Let $\hat{\mathcal{E}} = \hat{\mathcal{E}}_- \cup \hat{\mathcal{E}}_+$ be the subsets of collision-free equilibrium points in the infinity manifold, where $\hat{\mathcal{E}}_\pm := \mathcal{E}_\pm \setminus \Delta$ and Δ in this context is the extension of collision points to infinity, that is, points when $s_i = s_j, i \neq j$. For $p = (0, s_0, v_0, 0) \in \hat{\mathcal{E}}_\pm$, consider the linearization of (6) at p:

$$\begin{bmatrix}\rho_1\\ s_1\\ v_1\\ w_1\end{bmatrix}' = \begin{bmatrix}-v_0 & 0 & 0 & 0\\ 0 & 0 & 0 & I\\ -U(s_0) & 0 & 0 & 0\\ \tilde{\nabla}U(s_0) & 0 & 0 & -v_0 I\end{bmatrix}\begin{bmatrix}\rho_1\\ s_1\\ v_1\\ w_1\end{bmatrix} \tag{10}$$

where I is the $D \times D$ identity matrix acting on $\mathbb{E}$ and $\rho_1, s_1, \ldots$ represent linearizations of the corresponding variables ρ, s, v, w. The variation vector $(\rho_1, s_1, v_1, w_1) \in \mathbb{R} \times \mathbb{E} \times \mathbb{R} \times \mathbb{E}$ arising in (10) should be restricted to the $2D$-dimensional invariant subspace

$$\mathcal{T}(s_0) := \{\langle\langle s_0, s_1\rangle\rangle = \langle\langle s_0, w_1\rangle\rangle = 0\} \subset \mathbb{R} \times \mathbb{E} \times \mathbb{R} \times \mathbb{E} \tag{11}$$

which is the tangent space at $(0, s_0, v_0, w)$ to our phase space $P \subset \mathbb{R} \times \mathbb{E} \times \mathbb{R} \times \mathbb{E}$. At this point is worth observing that our basic ODE (6) is defined and analytic for all $s, w \in \mathbb{E}$ with $s \notin \Delta$. Within this larger "thickened" phase space $\mathbb{R} \times \hat{\mathbb{E}} \times \mathbb{R} \times \mathbb{E}$ the equilibrium manifold gains a dimension, becoming $\tilde{\mathcal{E}} = 0 \times \hat{\mathbb{E}} \times \mathbb{R} \times 0 \supset \mathcal{E}$. Linearizing the "thickened" flow at any point of this larger equilibrium manifold yields the matrix (10). This full matrix has a $(D+1)$-dimensional eigenspace with eigenvalue $\lambda = 0$ whose eigenvectors form the tangent space to $\tilde{\mathcal{E}}$, which is to say, are of the form

$$\rho_1 = w_1 = 0 \tag{12}$$

Its only other eigenvalue is $\lambda = -v_0$ which has a D-dimensional eigenspace with eigenvectors of the form

$$\rho_1 = v_1 = 0, \qquad w_1 = -v_0 s_1, \tag{13}$$

and a single independent generalized eigenvector

$$\begin{aligned}(\rho_1, s_1, v_1, w_1) &= (1, 0, \frac{1}{v_0}U(s_0), -\frac{1}{v_0}\tilde{\nabla}U(s_0))\\ &= G(s_0, v_0).\end{aligned}$$

Restricting these eigenspaces to $\mathcal{T}(s_0)$ is achieved by simply imposing the constraints defining $\mathcal{T}(s_0)$ yielding $T_{(0,s_0,v_0,0)}\mathcal{E} = \{(0, s_1, 0, w_1); s_1, w_1 \perp s_0\}$ as the eigenspace for 0 and a D-dimensional generalized eigenspace for $-v_0$ consisting of a $(D-1)$-dimensional eigenspace and the generalized eigenvector $G(s_0, v_0)$.

The general solution of the linearized differential equation at an equilibrium point $p_0 = (0, s_0, v_0, 0)$ can be found from the matrix exponential. If $L(p_0)$ denotes the matrix in (10) then we have

$$\exp(\tau L(p_0)) = \begin{bmatrix} u(\tau) & 0 & 0 & 0\\ \tilde{\nabla}U(s_0)\left((1-u(\tau))/v_0^2 - \tau u(\tau)/v_0\right) & I & 0 & I(1-u(\tau))/v_0\\ U(s_0)(u(\tau)-1)/v_0 & 0 & 1 & 0\\ \tilde{\nabla}U(s_0)\tau u(\tau) & 0 & 0 & u(\tau)I\end{bmatrix}$$

where

$$u(\tau) = \exp(-v_0\tau).$$

Restricting the linearized flow to the generalized eigenspace $N_{(s_0,v_0)}$ for $\lambda = -v_0$ and then adding the result to the corresponding equilibrium point yields the linear approximation to the full flow. This generalized eigenspace $N_{(s_0,v_0)}$ consists of all vectors of the form

$$i_{(s_0,v_0)}(s_1,\rho_1) := \rho_1 G(s_0,v_0) + (0, s_1, 0, -v_0 s_1). \tag{14}$$

Apply $\exp(\tau L(p_0))$ to this vector and add the result to the equilibrium point $(0, s_0, v_0, 0)$ we obtain the representation

$$\begin{bmatrix}\rho(\tau)\\ s(\tau)\\ v(\tau)\\ w(\tau)\end{bmatrix} = \begin{bmatrix}0\\ s_0\\ v_0\\ 0\end{bmatrix} + u(\tau)\begin{bmatrix}\rho_1\\ s_1-(\rho_1/v_0)\tilde{\nabla}U(s_0)\tau\\ (\rho_1/v_0)U(s_0)\\ -v_0 s_1-(\rho_1/v_0)\tilde{\nabla}U(s_0)+\rho_1\tilde{\nabla}U(s_0)\tau\end{bmatrix} \tag{15}$$

for this linear flow.

It will help to be explicit about our linearizing change of variables $(\rho, s, v, w) \leftrightarrow (\rho_1, s_0, v_0, s_1)$ which we are advocating for and to give it a name. We call the change of variables J and define it by the equality

$$(\rho, s, v, w) = (0, s_0, v_0, 0) + i_{(s_0,v_0)}(s_1, \rho_1),$$

where $i_{(s_0,v_0)}$ is the linear inclusion (14) above which parameterizes $N_{(s_0,v_0)}$. Thus

$$J : (s_0, v_0, s_1, \rho_1) \mapsto (\rho, s, v, w) \tag{16}$$

has the component expression

$$(\rho, s, v, w) = \left(\rho_1, s_0 + s_1, v_0 + \frac{1}{v_0}U(s_0)\rho_1, -\frac{1}{v_0}\tilde{\nabla}U(s_0, v_0)\rho_1 - v_0 s_1\right)$$

In the new (s_0, v_0, s_1, ρ_1) coordinates, the linearized flow reads:

$$L_\tau : (s_0, v_0, s_1, \rho_1) \mapsto (s_0, v_0, \tilde{s}_1, \tilde{\rho}_1) \tag{17}$$

where

$$\begin{aligned}\tilde{s}_1 &= u(\tau; v_0)s_1 - \tau u(\tau; v_0)\alpha(s_0, v_0)\rho_1\\ \tilde{\rho}_1 &= u(\tau; v_0)\rho_1\end{aligned}$$

with

$$u(\tau; v_0) = \exp(-v_0\tau), \text{ and } \alpha(s_0, v_0) = \frac{1}{v_0}\tilde{\nabla}U(s_0, v_0).$$

The following analytic linearization theorem will be key to the sections that follow. The theorem serves to transfer the (s_0, v_0, s_1, ρ_1) coordinates to the extended phase space, converting the nonlinear flow to the linearized one just described.

Theorem 3.1. *There is an analytic diffeomorphism Φ from a neighborhood of the equilibrium manifold $\hat{\mathcal{E}}$ (collisions deleted!) in extended phase space to a neighborhood of the zero section $\{s_1 = 0, \rho_1 = 0\}$ of the new (s_0, v_0, s_1, ρ_1)-variable space which conjugates the n-body flow ϕ_τ and the linearized flow L_τ (eq (17)):*

$$\Phi(\phi_\tau(p)) = L_\tau(\Phi(p))$$

for as long as τ is small enough so that the curves $\phi_{\tau'}(p), L_{\tau'}(p)$ lie in their corresponding neighborhoods for $0 \le \tau' \le \tau$. Moreover, $\Phi^{-1}(s_0, v_0, s_1, \rho_1) = J(s_0, v_0, s_1, \rho_1) + O(|s_1|^2 + \rho_1^2)$, where J is the fiber-linear change of coordinates defined above by equation (16).

Proof. The proof is found in the Appendix, and follows rather directly from the more general Theorem A.1 there. □

Remark 3.2. *Note in particular that* $(s_0, v_0) \mapsto \Phi^{-1}(s_0, v_0, 0, 0) = (0, s_0, v_0, 0)$ *parametrizes the equilibrium manifold* $\hat{\mathcal{E}}$. *If we fix this equilibrium value* (s_0, v_0) *and vary* (ρ_1, s_1) *then the map* $(\rho_1, s_1) \mapsto \Phi^{-1}(s_0, v_0, s_1, \rho_1)$ *parametrizes the local stable or unstable manifold attached to* $(0, s_0, v_0, 0)$. *Back in the* (ρ, s, v, w) *variables, the flow will be given by (15) up to terms of order* $O(\rho_1^2 + \|s_1\|^2)$.

Remark 3.3. *The alert reader may have noticed something fishy about our change of variables (16). The variable* $s = s_0 + s_1$ *is supposed to lie on the sphere* S, *but if* $s_0 \in S$ *and* $s_1 \in T_{s_0}S = s_0^{\perp}$ *then* $s_0 + s_1 \notin S$ *unless* $s_1 = 0$. *To escape this trap, relax the constraint that* $\|s\| = 1$ *and work on the "thickened phase space" described immediately after (11). Theorem 3.1 holds in this larger context, and that is where we will prove it, restated and somewhat generalized as theorem A.1 and proved in the appendix. As already noted just after (11), the eigenvalue structures of the linearization at an equilibrium for our original phase space and for its thickened version are nearly identical, both having as their only eigenvalues* 0 *and* $-v_0$ *with* 0*'s eigenspace being the tangent space to the equilibrium manifold and* v_0 *enjoying a single nonzero generalized eigenvector and a codimension genuine eigenspace. Theorem 3.1 becomes a corollary of the more general theorem A.1 and an inspection of the proofs of these theorems yields the solution to this "*$s = s_0 + s_1$*" puzzle. In a nutshell, for* s_1 *small the points* $s_0 + s_1$ *of the affine space* $s_0 + T_{s_0}S$ *lies within* $O(|s_1|^2)$ *of the sphere* S. *The* $O(|s_1|^2 + \rho_1^2)$ *error of* $\Phi^{-1} = J + O(|s_1|^2 + \rho_1^2)$ *occurring in the end of the statement of theorem 3.1 (compare with the* N_p *of theorem A.1) includes a projection of the affine space* $s_0 + T_{s_0}S$ *onto the sphere* S.

4. Chazy-type Asymptotics

In this section, we reinterpret our results about the flow near infinity using the Newtonian timescale, and so rederive Chazy's asymptotic expansion with error terms. We focus on the case $t \to +\infty$ in most of this section. At the end we describe the various adjustments needed for the $t \to -\infty$ case. The Chazy expansion, discussed in the beginning of the paper, is

$$q(t) = At + B\log(t) + C + f(t; A, C), \text{ as } t \to +\infty \tag{18}$$

where $A, B, C \in \mathbb{R}^{nd}$ are constant vectorial parameters. Chazy claimed that the error term had the form $f(t; A, C) = Q(1/t, \log t/t; A, C)$ where $Q(u, v; A, C)$ is a two-variable power series in u and v having no constant term, absolutely convergent in a neighborhood of $u = v = 0$ and whose coefficients depend analytically on A and C. Note that in particular such an f is $O(\log t/t)$.

The necessity of the $\log(t)$ term in (18) is seen by trying out the ansatz $At+C$ in Newton's equations. The equations fail to hold at $O(1/t^2)$ as $t \to +\infty$. Assuming instead $At + B\log(t) + C$ and plugging into Newton's equations yields $B = B(A)$ as per:

$$B = -\nabla U(A) \tag{19}$$

in order for the solution to hold up to $O(1/t^2)$.

The parameters $A, C \in \mathbb{R}^{nd}$ are fixed "scattering parameters" and determine the analytic function f. The parameter A simultaneously encodes the asymptotic velocities and asymptotic shape since

$$A = \lim_{t\to+\infty} q(t)/t = \lim_{t\to+\infty} \dot{q}(t).$$

In keeping with our hyperbolicity assumption, we insist that A be collision-free: $A \in \hat{\mathbb{E}}$. The parameter C is not uniquely determined, since a time shift $q(t) \mapsto q(t - t_0)$ alters C. By shifting the origin of time, we can find a unique C with $\langle\langle A, C\rangle\rangle = 0$. In analogy with the free particle case, C can be viewed as an "impact parameter" – a measure of how close the straight line $At + C$ misses total collision. But for the orbits in the Newtonian problem, this interpretation of C is less clear.

Chazy [?, eq. (22)–(27)] established his expansion (18) and its analyticity for hyperbolic solutions. See also [?]. We will derive his result from the local behavior of solutions near the stable/unstable manifold at infinity as investigated in the previous sections. We summarize the result now.

Theorem 4.1. *[Case $t, \tau \to +\infty$] Let $q(\tau)$ be the position vector as a function of time τ for any solution lying in the stable manifold of the equilibrium point $(0, s_0, v_0, 0) \in \hat{\mathcal{E}}_+$ and having linearized data ρ_1, s_1 with $\langle\langle s_0, s_1\rangle\rangle = 0$ as per equation (15). Then, reexpressed in terms of Newtonian time t, as $t \to +\infty$ the solution $q(t)$ satisfies Chazy's asymptotics (18) with coefficients given by*

$$\begin{aligned} A &= v_0 s_0 \\ B &= -\nabla U(v_0 s_0) = -\nabla U(A) \\ C &= \frac{s_1}{\rho_1} - \frac{\log(\rho_1 v_0)}{v_0^2}\tilde{\nabla} U(s_0). \end{aligned} \tag{20}$$

The error term is of the form $f(t) = Q(1/t, \log t/t)$ where Q denotes a two-variable power series beginning with no constant term depending analytically on the parameters s_0, v_0, s_1, ρ_1 and converging when the arguments are sufficiently small.

Sketch of Proof. From the analytic linearization Theorem 3.1 we have that ρ, s, v, w are analytic functions of τ depending parametrically in an analytic way on the limiting values s_0, v_0 and linearizing parameters ρ_1, s_1. In particular $r = 1/\rho$ and s are analytic functions of τ whose expansions we can work out. Multiplying $r(\tau)$ and $s(\tau)$ yields the position vector $q(\tau) = r(\tau)s(\tau)$ as an analytic function whose expansion we can work out and which depends parametrically on s_0, v_0 and ρ_1, s_1. To finish off the derivation we require analytic expansion of τ as a function of Newtonian time t.

Proof in Detail. From the analytic linearization theorem and from (15) we have that ρ, s and v are analytic functions of τ, depending analytically on the limiting equilibrium values s_0, v_0, and analytic linearization variables s_1, ρ_1, and having first few terms,

$$\begin{aligned} \rho(\tau) &= \rho_1 u(\tau) + d_2 u(\tau)^2\tau^2 + d_1 u(\tau)^2\tau + \rho_2 u(\tau)^2 + f_3(u, \tau u) \\ s(\tau) &= s_0 + s_1 u(\tau) - \rho_1 \tilde{\nabla} U(s_0) u(\tau)\tau/v_0 + f_2(u, \tau u) \\ v(\tau) &= v_0 + \rho_1 U(s_0) u(\tau)/v_0 + g_2(u, \tau u) \end{aligned} \tag{21}$$

where here and in what follows, expressions like $f_p, g_p, \ldots$ denote unspecified convergent power series which begin with terms of order p and converge when the arguments are sufficiently small. In this section we always set

$$u(\tau) = u(\tau; v_0) = \exp(-v_0\tau).$$

The value of the coefficient ρ_2 will be needed later. Plug the above $\rho(\tau)$ and $v(\tau)$ into the first equation of (6), we get

$$d_1 = d_2 = 0, \quad \rho_2 = \left(\frac{\rho_1}{v_0}\right)^2 U(s_0). \tag{22}$$

Using these we can get asymptotic estimates for $q(\tau) = r(\tau)s(\tau)$. First consider $r(\tau)$.

Lemma 4.1. *The size variable satisfies*

$$r(\tau) = \frac{1}{\rho_1} e^{v_0\tau} - \frac{U(s_0)}{v_0^2} + u^{-1} h_2(u, \tau u) \tag{23}$$

Proof. Recall the differential equation $r' = vr$. Using (21) we have

$$\begin{aligned} r(\tau) &= r(0)\exp(v_0\tau)\exp\left(\int_0^\tau \rho_1 U(s_0)u(\lambda)/v_0\, d\lambda\right)\exp\left(\int_0^\tau g_2(u(\lambda), \lambda u(\lambda))\, d\lambda\right) \\ &= \frac{1}{\rho_1}\exp(v_0\tau)\exp\left(\int_{+\infty}^\tau \rho_1 U(s_0)u(\lambda)/v_0\, d\lambda\right)\exp\left(\int_{+\infty}^\tau g_2(u(\lambda), \lambda u(\lambda))\, d\lambda\right) \end{aligned}$$

where we used the fact that $\lim_{\tau\to+\infty} r(\tau)\exp(-v_0\tau) = 1/\rho_1$ which follows from the first equation of (21). The second exponential integral factor evaluates to

$$\exp\left(-\rho_1 U(s_0)u(\tau)/v_0^2\right) = 1 - \rho_1 U(s_0)u(\tau)/v_0^2 + p_2(u)$$

and the third is of the form

$$\exp\left(G_2(u(\tau), \tau u(\tau)\right) = 1 + r_2(u, \tau u).$$

We have used the fact that the integral of a series of the form $f_p(u(\tau), \tau u(\tau))$ is an expression of the same form. Multiplying the factors and using the fact that $\exp(v_0\tau) = u^{-1}(\tau)$ gives the required formula. □

Multiplying out $q(\tau) = r(\tau)s(\tau)$ we now find

$$q(\tau) = \frac{s_0}{\rho_1} e^{v_0\tau} - \frac{\tilde{\nabla} U(s_0)}{v_0}\tau + \left(\frac{s_1}{\rho_1} - \frac{U(s_0)}{v_0^2} s_0\right) + u^{-1} k_2(u(\tau), \tau u(\tau)). \tag{24}$$

We will rewrite this expansion for q using t as a time parameter in place of τ. For this we need the following lemma.

Lemma 4.2. *Along any solution in the stable manifold of the equilibrium point $(0, s_0, v_0, 0)$, $v_0 > 0$, the Newtonian time t and our new time scale τ are related, for all sufficiently large t, τ by an invertible analytic change of variables of the form*

$$t = \frac{1}{\rho_1 v_0} e^{v_0\tau} - \frac{U(s_0)}{v_0^2}\tau + c + u^{-1} p_2(u, \tau u), \tag{25}$$

where c is an arbitrary constant. Moreover its analytic inverse is characterized by

$$e^{v_0\tau} = \rho_1 v_0 t + \frac{\rho_1}{v_0^2} U(s_0)\log(t) + \frac{\rho_1}{v_0^2} U(s_0)\log(\rho_1 v_0) - \rho_1 v_0 c + t P_2(1/t, \log t/t), \tag{26}$$

$$\tau = \frac{1}{v_0}\log(t) + \frac{1}{v_0}\log(\rho_1 v_0) + P_1(1/t, \log t/t) \tag{27}$$

Proof. Using (23), we get (25) by integrating both sides of the identity $dt = r d\tau$. For simplicity, let

$$a = (\rho_1 v_0)^{-1}, \quad b = U(s_0)/v_0^2. \tag{28}$$

so that (25) becomes

$$t = ae^{v_0\tau} - b\tau + c + u^{-1}k_2(u, \tau u) = au^{-1}\left[1 - b\tau u/a + cu/a + l_2(u, \tau u)\right] \tag{29}$$

Taking logs gives

$$\log t = v_0\tau + \log(a) - b\tau u/a + cu/a + m_2(u, \tau u) \tag{30}$$

Now we claim that any convergent power series of the from $f_p(u, \tau u)$ can be rewritten as a convergent power series $g_p(\frac{1}{t}, \frac{\log t}{t})$. To see this, we view u and $w = \tau u$ as independent variables and similarly for $s = 1/t$ and $\sigma = (\log t)/t$. The formulas above can be used to express (s, σ) as power series in (u, w)

$$s = \frac{u}{a} + \dots \qquad \sigma = v_0 w + \frac{\log a}{a} u + \dots.$$

Since the matrix of linear terms is nondegenerate, the two-dimensional inverse function theorem shows that we can invert the series to get series for (s, σ) is terms of (u, w).

Using this information, we can replace all of the higher order terms in equations (29) and (30) by series in $1/t, (\log t)/t$. For example, replacing all but the first two terms in (30) and solving for τ gives

$$\tau = \frac{1}{v_0}\log t - \frac{\log a}{v_0} + P_1(1/t, \log t/t)$$

which agrees with (27). Using this equation to replace τ in (29) and replacing the term $u^{-1}k_2(u, \tau u)$ by $tp_2(1/t, (\log t)/t)$ we can solve for $e^{v_0\tau}$ to find

$$\exp(v_0\tau) = a^{-1}t + \frac{b}{av_0}\log t - \frac{\log a}{av_0} - a^{-1}c + tP_2(1/t, \log t/t)$$

as claimed in (26). Here we use the fact that a term of the form $f_1(1/t, (\log t)/t)$ is a special case of a term $tf_2(1/t, (\log t)/t)$. □

Completion of the Proof of Theorem 4.1. The error term in (24) can be replaced by $tf_1(1/t, (\log t)/t)$ as in the proof of the last lemma. Then plugging (26) and (27) into (24), we get

$$q(t) = v_0 s_0 t - \frac{\nabla U(s_0)}{v_0^2}\log t + C + tQ_2(1/t, \log t/t).$$

with

$$C = \frac{s_1}{\rho_1} - \frac{\log(\rho_1 v_0)}{v_0^2}\tilde{\nabla} U(s_0) + \left(\frac{U(s_0)}{v_0^2}(\log(\rho_1 v_0) - 1) - cv_0\right)s_0.$$

Here we have used the fact that $\tilde{\nabla} U(s_0) = U(s_0)s_0 + \nabla U(s_0)$. By choosing the constant c, which amounts to shifting the origin of time, we can eliminate the last term from C. The new value satisfies $\langle\langle A, C\rangle\rangle = 0$. This finishes our proof of the formulas for A, B, C, once the reader notices

$$\nabla U(A) = \nabla U(v_0 s_0) = \nabla U(s_0)/v_0^2.$$

It remains to consider the error term. The series $tQ_2(1/t, \log t/t)$ differs from the required series $Q(1/t, \log t/t)$ only in that it allows the presence of terms of the form $\frac{(\log t)^{k+1}}{t^k}$, $k \geq 1$. We will now show that such terms do not occur. Assume that $q(t) = At + B\log t + C + tg(s,\sigma)$ where $g(s,\sigma)$ is a convergent power series in $s = 1/t$ and $\sigma = \log t/t$. Note that $q(t)$ can be factorized as $q(t) = t(A + B\sigma + Cs + g(s,\sigma))$. Then by the homogeneity of the Newtonian potential,

$$\nabla U(q(t)) = s^2 \nabla U(A + B\sigma + Cs + g(s,\sigma)).$$

It follows that $\ddot{q}(t)$ must also admit a factor of s^2. Using the fact that $\dot{s} = -s^2$ and $\dot{\sigma} = s^2 - s\sigma$ we find

$$\ddot{q}(t) = s^2 h(s,\sigma) + s\sigma^2 g_{\sigma\sigma}(s,\sigma)$$

where we have gathered together several terms which have a factor of s^2. It follows that $g_{\sigma\sigma}(s,\sigma)$ must be divisible by s, so the series $g(s,\sigma)$ cannot contain any terms involving monomials $s^0\sigma^k$, $k \geq 2$. In other words every monomial in $Q_2(1/t, \log t/t)$ has at least one factor of $1/t$ so $tQ_2(1/t, \log t/t)$ can be written in the form $Q(1/t, \log t/t)$ as required.

Negative time asymptotics. Deriving the negative time asymptotics is not as simple as replacing $t \to +\infty$ by $t \to -\infty$ in the basic expansion (18). Indeed $\log(t)$ for $t < 0$ is problematic, either being a purely imaginary number or undefined. Instead we will use (18) together with time reversal symmetry to derive the correct formula.

Let $q(t)$ be any solution which is hyperbolic as $t \to -\infty$. Then $\tilde{q}(t) = q(-t)$ is a solution hyperbolic as $t \to +\infty$ and we can apply what we already proved to see that there is an asymptotic expansion

$$\begin{aligned}
\tilde{q}(t) &= \tilde{A}t + \tilde{B}\log t + \tilde{C} + \tilde{f}(1/t, \log t/t) \qquad t \to +\infty \\
\tilde{A} &= \tilde{v}_0\tilde{s}_0 \\
\tilde{B} &= -\nabla U(\tilde{v}_0\tilde{s}_0) = -\nabla U(\tilde{A}) \\
\tilde{C} &= \frac{\tilde{s}_1}{\tilde{\rho}_1} - \frac{\log(\tilde{\rho}_1\tilde{v}_0)}{\tilde{v}_0^2}\tilde{\nabla}U(\tilde{s}_0).
\end{aligned}$$

By definition $q(t) = \tilde{q}(-t)$ so we automatically get

$$q(t) = \tilde{A}(-t) + \tilde{B}\log(-t) + \tilde{C} + \tilde{f}(1/(-t), \log(-t)/(-t)) \qquad t \to -\infty.$$

which we will write in the form

$$q(t) = At + B\log|t| + C + f(1/t, \log|t|/t) \qquad t \to -\infty \tag{31}$$

with $A = -\tilde{A}$, $B = \tilde{B}$, $C = \tilde{C}$ and $f(1/t, \log|t|/t) = \tilde{f}(1/(-t), \log(-t)/(-t))$.

It remains to express the coefficients A, B, C in terms of s_0, v_0, s_1, ρ_1.

Lemma 4.3. *The parameters s_0, v_0, s_1, ρ_1 of the solution $q(\tau)$ are related to the parameters $\tilde{s}_0, \tilde{v}_0, \tilde{s}_1, \tilde{\rho}_1$ of $\tilde{q}(\tau)$ by*

$$s_0 = \tilde{s}_0 \qquad v_0 = -\tilde{v}_0 \qquad s_1 = \tilde{s}_1 \qquad \rho_1 = \tilde{\rho}_1.$$

Using this we have

$$\begin{aligned}
A &= v_0 s_0 \\
B &= \nabla U(v_0 s_0) = \nabla U(A) \\
C &= \frac{s_1}{\rho_1} - \frac{\log(\rho_1(-v_0))}{v_0^2}\tilde{\nabla}U(s_0) = \frac{s_1}{\rho_1} - \frac{\log(\rho_1|v_0|)}{v_0^2}\tilde{\nabla}U(s_0).
\end{aligned} \tag{32}$$

As a corollary we have:

Proposition 4.1. *The Chazy parameter A as $t \to \pm\infty$ is related to the corresponding equilibrium equation (s_0, v_0) as $\tau \to \pm\infty$ by $A = v_0 s_0$ regardless of whether we are considering the asymptotics as $t, \tau \to +\infty$ or $t, \tau \to -\infty$.*

Proof of Lemma 4.3. We have

$$s_0 = \lim_{t\to-\infty} \frac{q(t)}{|q(t)|} = \tilde{s}_0 \qquad v_0 = \lim_{t\to-\infty} \frac{q(t)\cdot\dot{q}(t)}{|q(t)|} = -\tilde{v}_0$$

but s_1, ρ_1 require a bit more effort.

First consider ρ_1. From formula (15), we should have

$$\rho_1 = \lim_{\tau\to-\infty} \exp(v_0\tau)\rho(\tau) \qquad \frac{1}{\rho_1} = \lim_{\tau\to-\infty} \exp(-v_0\tau)r(\tau)$$

Using the known formula (23) for $\tilde{q}$ and $s_0 = \tilde{s}_0, v_0 = -\tilde{v}_0$ to see that

$$\begin{aligned} r(\tau) = \tilde{r}(-\tau) &= \frac{1}{\tilde{\rho}_1}\exp(\tilde{v}_0(-\tau)) - \frac{U(\tilde{s}_0)}{\tilde{v}_0^2} + \dots \\ &= \frac{1}{\tilde{\rho}_1}\exp(v_0\tau) - \frac{U(s_0)}{v_0^2} + \dots. \end{aligned}$$

Multiplying by $\exp(-v_0\tau)$ and taking the limit shows $1/\rho_1 = 1/\tilde{\rho}_1$.

Moving on to s_1, note that from formula (15), we should have

$$\begin{aligned} \tilde{s}_1 &= \lim_{\tau\to+\infty} \exp(\tilde{v}_0\tau)(\tilde{s}(\tau) - \tilde{s}_0 + (\tilde{\rho}_1/\tilde{v}_0)\tilde{\nabla}U(\tilde{s}_0)\tau) \\ s_1 &= \lim_{\tau\to-\infty} \exp(v_0\tau)(s(\tau) - s_0 + (\rho_1/v_0)\tilde{\nabla}U(s_0)\tau). \end{aligned}$$

Then we have

$$\begin{aligned} s_1 &= \lim_{\tau\to-\infty} \exp(v_0\tau)(s(\tau) - s_0 + (\rho_1/v_0)\tilde{\nabla}U(s_0)\tau) \\ &= \lim_{\tau\to+\infty} \exp(v_0(-\tau))(s(-\tau) - s_0 + (\rho_1/v_0)\tilde{\nabla}U(s_0)(-\tau)) \\ &= \lim_{\tau\to+\infty} \exp(\tilde{v}_0\tau)(\tilde{s}(\tau) - \tilde{s}_0 + (\tilde{\rho}_1/\tilde{v}_0)\tilde{\nabla}U(\tilde{s}_0)\tau) \\ &= \tilde{s}_1. \end{aligned}$$

The second line is just changing the variable from τ to $-\tau$ and the limit direction. The third line uses the definition $s(-\tau) = \tilde{s}(\tau)$ and what we already know about s_0, v_0, ρ_1. Two minus signs in the first factor and last term cancel out. □

The following proposition summarizes the relation between the parameters of the asymptotic expansions of a forward hyperbolic solution and its time reversal using the notation T for the operation of time reversal. It follows immediately from the formulas $A = -\tilde{A}, C = \tilde{C}$ and Lemma 4.3.

Proposition 4.2. *If a solution $q(t)$ has Chazy asymptotic parameters (A, C) in the distant future, then its time reversed solution $Tq(t) = q(-t)$ has Chazy asymptotic parameters $(-A, +C)$ for the distant past. If q, written in blown-up variables, tends to the equilibrium $(0, s_0, v_0, 0) \in \hat{\mathcal{E}}_+$ as $\tau \to +\infty$, then Tq, written in blown-up variables, tends to the equilibrium $(0, s_0, -v_0, 0) \in \hat{\mathcal{E}}_-$ as $\tau \to -\infty$.*

Example 4.1. *For the planar two-body problem, $d = n = 2$, we can use $q = q_2 - q_1 \in \mathbb{R}^2 \setminus 0$ to parametrize $\hat{\mathbb{E}}$ which has dimension $D = 2$. The mass norm is* *proportional to the ordinary Euclidean norm:*

$$\|q\| = \sqrt{\mu}|q| \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

The timescale change $dt = \|q\|\, d\tau$ *is almost the same as the one usually employed to solve the Kepler problem as a function of the eccentric anomaly, so we modify the standard formulas for the solution, for example, those in* [**?**]. *For the hyperbolic case, we have*

$$\begin{aligned} r(\tau) &= \|q(\tau)\| = a\sqrt{\mu}(e\cosh(\omega\tau) - 1) \\ q(\tau) &= (ae - a\cosh(\omega\tau), a\sqrt{e^2-1}\sinh(\omega\tau)) \\ t(\tau) &= \frac{a}{\omega}(e\sinh(\omega\tau) - \omega\tau) \end{aligned}$$

where $\omega = \sqrt{2h}$, $a = \frac{m_1 m_2}{2h}$ *is the semimajor axis, and* e *is the eccentricity. For simplicity, we have rotated the hyperbola so that the perihelion occurs on the positive* x*-axis, as in Figure 1.*

It's interesting to compute the scattering parameters A, C, A', C' *for these solutions, as shown in the figure. Taking the limits as* $\tau \to \pm\infty$ *of* $s(\tau) = q(\tau)/r(\tau)$ *gives the asymptotic shapes in backward and forward time*

$$s_0 = \frac{1}{e\sqrt{\mu}}(-1, -\sqrt{e^2-1}) \qquad s_0' = \frac{1}{e\sqrt{\mu}}(-1, \sqrt{e^2-1}).$$

These are unit vectors in the direction of the asymptotes. The limiting values of v *are* $\mp\omega$ *respectively and so*

$$A = \frac{\omega}{e\sqrt{\mu}}(1, \sqrt{e^2-1}) \qquad A' = \frac{\omega}{e\sqrt{\mu}}(-1, \sqrt{e^2-1}).$$

All of the vectors s *have length* $|s| = 1/\sqrt{\mu}$ *and the shape potential* $U(s)$ *is constant. Hence the tangential gradient vanishes and the formula for* C, C' *simplify to*

$$C = \frac{s_1}{\rho_1} \qquad C' = \frac{s_1'}{\rho_1'}.$$

We have

$$\rho_1' = \lim_{\tau\to\infty} \exp\omega\tau\rho(\tau) = \lim_{\tau\to\infty} \frac{\exp\omega\tau}{a\sqrt{\mu}(e\cosh(\omega\tau) - 1)} = \frac{2}{ae\sqrt{\mu}}$$

and a similar computation gives the same result for ρ_1. *Further computation with the formulas above gives*

$$s_1' = \lim_{\tau\to\infty} \exp\omega\tau(s(\tau) - s_0) = \frac{2}{e^2\sqrt{\mu}}(e^2-1, \sqrt{e^2-1})$$

and similarly

$$s_1 = \lim_{\tau\to-\infty} \exp(-\omega\tau)(s(\tau) - s_0) = \frac{2}{e^2\sqrt{\mu}}(e^2-1, -\sqrt{e^2-1}).$$

Hence

$$C = \frac{a}{e}(e^2-1, -\sqrt{e^2-1}) \qquad C' = \frac{a}{e}(e^2-1, \sqrt{e^2-1}).$$

The explicit calculation for $q(\tau)$ *immediately gives the asymptotic expansion as* $\tau \to \infty$

$$q(\tau) = \exp(\omega\tau)(\frac{a}{2}, \frac{a\sqrt{e^2-1}}{2}) + (ae, 0) + \ldots.$$

Using the fact that $\tilde{\nabla}(U(s)) = 0$, a little algebra shows that this agrees with our formula (24) evaluated at $(s_0', \sqrt{2h}, s_1', \rho_1')$.

5. Hyperbolic Scattering

In this section we study orbits which tend to infinity hyperbolically in both forward and backward time. Recall the limiting set of these orbits is the equilibrium manifold $\hat{\mathcal{E}} = \hat{\mathcal{E}}_+ \cup \hat{\mathcal{E}}_-$, where

$$\hat{\mathcal{E}}_\pm = \{(0, s, v, 0) : \; s \in \hat{S}, \; v = \pm\sqrt{2h}, \; h > 0\}.$$

For simplicity, a point $(0, s, v, 0) \in \hat{\mathcal{E}}$ will be written as (s, v) in the rest of this section.

As noted previously, at the infinity manifold the differential equation (7) is independent of the potential function $U(s)$ and all its solutions appear to be nonsingular. But once we move away from it, we will have to insist that our solutions avoid the singular set Δ of the potential. So in what follows we will exclude the singular set Δ from the domain of the flow ϕ_τ of the blown-up differential equations (6). Since collisions are inevitable for the collinear n-body problem, we also require that the space dimension d is at least 2.

Having done this, it follows from the analytic linearization Theorem 3.1 that $\hat{\mathcal{E}}_+$ is an attractor for the flow on the extended phase space $[0, +\infty) \times \hat{\Sigma}$. Moreover, its local stable manifold $W^s_{loc}(\hat{\mathcal{E}}_+)$ is a nonempty open subset of the extended phase space, which is analytically foliated into the individual local stable manifolds $W^s_{loc}(q)$ of the equilibrium points $q \in \hat{\mathcal{E}}_+$. Similarly $\hat{\mathcal{E}}_-$ is a repeller whose local unstable manifold $W^u_{loc}(\hat{\mathcal{E}}_-)$ is a nonempty open subset of the extended phase space analytically foliated into local unstable manifolds $W^u_{loc}(p)$ of its points $p \in \hat{\mathcal{E}}_-$.

Let

$$\mathcal{H}_+ = W^s(\hat{\mathcal{E}}_+) = \{z : \phi_\tau(z) \to \hat{\mathcal{E}}_+ \text{ as } \tau \to +\infty\}$$
$$\mathcal{H}_- = W^u(\hat{\mathcal{E}}_-) = \{z : \phi_\tau(z) \to \hat{\mathcal{E}}_- \text{ as } \tau \to -\infty\}$$

denote the corresponding global stable and unstable manifolds. Note that a point z belongs to the global stable or unstable manifold if and only if there is some finite time such that the orbit enters the corresponding local stable or unstable manifolds, $W^s_{loc}(\hat{\mathcal{E}}_+)$ or $W^u_{loc}(\hat{\mathcal{E}}_-)$. It follows that $\mathcal{H}_\pm$ are also nonempty open subsets of the extended phase space. It follows from the asymptotic estimates in Section 4 that orbits with initial conditions $x \in \mathcal{H}_+$ tend to infinity hyperbolically in forward time in the sense of Chazy. Similarly, if $x \in \mathcal{H}_-$, its orbit tends to infinity hyperbolically in backward time. Finally, let

$$\mathcal{H} = \mathcal{H}_- \cap \mathcal{H}_+.$$

Orbits with initial conditions $z \in \mathcal{H}$ are bi-hyperbolic, that is, they tend to infinity hyperbolically in both time directions.

Proposition 5.1. *$\mathcal{H}$ is a nonempty open subset of the extended phase space. Moreover*

$$\mathcal{H} \cap \big((0, +\infty) \times \hat{\Sigma}\big) \neq \emptyset. \tag{33}$$

Proof. Since $\mathcal{H}_\pm$ are open sets of extended phase space, $\mathcal{H} = \mathcal{H}_- \cap \mathcal{H}_+$ is open. To see that $\mathcal{H}$ is non-empty, recall that the extended phase space includes the boundary manifold at infinity and Proposition 3.1 shows that there are simple solutions in this boundary manifold connecting $\hat{\mathcal{E}}_-$ and $\hat{\mathcal{E}}_+$. Since the space dimension is at least two, the singular set $\Delta \subset S$ has codimension at least two. It follows that most of these connecting solutions from Proposition 3.1 do not encounter Δ. The initial conditions of these orbits are points of $\mathcal{H}$, so $\mathcal{H} \neq \emptyset$. As $\mathcal{H}$ is open, these bi-hyperbolic points on the infinity manifold $\{0\} \times \hat{\Sigma}$ must each be contained in some open ball $\mathcal{B}_p \subset \mathcal{H}$. By a dimension count, for $\mathcal{B}_p$ to be open it cannot be contained in the infinity manifold $\{0\} \times \hat{\Sigma}$. Hence (33) holds. □

Remark 5.1. (33) *shows that there exist bi-hyperbolic solutions of the n-body problem that are close to infinity but do not lie within the infinity manifold. In other words, there are real bi-hyperbolic solutions.*

If $x \in \mathcal{H}_+$, then there is a unique equilibrium point $q = (s_+, v_+) \in \hat{\mathcal{E}}_+$, such that $\phi_\tau(x) \to q$ as $\tau \to +\infty$. This defines an analytic mapping $\pi_+ : \mathcal{H}_+ \to \hat{\mathcal{E}}_+$. Similarly there is an analytic mapping $\pi_- : \mathcal{H}_- \to \hat{\mathcal{E}}_-$ assigning to each $x \in \mathcal{H}_-$ its limiting equilibrium point p as $\tau \to -\infty$.

5.1. The Scattering Relation.

We are interested in the scattering problem: which equilibrium points $p \in \hat{\mathcal{E}}_-$ and $q \in \hat{\mathcal{E}}_+$ can be connected by bi-hyperbolic orbits.

Definition 5.1. *We say that $p \in \hat{\mathcal{E}}_-$ and $q \in \hat{\mathcal{E}}_+$ are* related by hyperbolic scattering *if*

$$\pi_-(x) = p \text{ and } \pi_+(x) = q \text{ for some } x \in \mathcal{H}.$$

We denote this relation by $p \to q$. The hyperbolic scattering relation *is the corresponding subset $\mathcal{HS} \subset \hat{\mathcal{E}}_- \times \hat{\mathcal{E}}_+$. We will also say that x connects p to q.*

For any relation, we can define the image and preimage of particular points. For $p \in \hat{\mathcal{E}}_-$ and $q \in \hat{\mathcal{E}}_+$ we have

$$\mathcal{HS}(p) = \{q \in \hat{\mathcal{E}}_+ : p \to q\} \qquad \mathcal{HS}^{-1}(q) = \{p \in \hat{\mathcal{E}}_- : p \to q\}.$$

So $\mathcal{HS}(p)$ is the set of all final states which can be reached from the initial state p via hyperbolic scattering.

The scattering relation can be reformulated in terms of the Chazy parameter A. Recall that the map $(s, v) \mapsto A = vs$ relates the Chazy parameter of a hyperbolic solution to the corresponding equilibrium $(s, v) \in \hat{\mathcal{E}}$ it converges to, see Proposition 4.1. So far we have only defined Chazy parameter A for hyperbolic solutions of the n-body problem (solutions that are not contained in the infinity manifold in the blow up variables). However we can obviously extend this relation to heteroclinic solutions contained in the infinity manifold.

This map, upon restriction either to $\hat{\mathcal{E}}_+$ or to $\hat{\mathcal{E}}_-$ is a diffeomorphism onto $\hat{\mathbb{E}}$ with inverse $A \mapsto \pm(A/\|A\|, \|A\|)$. Thus we obtain an induced scattering relation on $\hat{\mathbb{E}}$. We continue to use the symbol "$\to$", as it is induced by our original relation. To be more specific, we define $A \to A'$ if and only if

$$p = (-A/\|A\|, -\|A\|) \to q = (A'/\|A'\|, \|A'\|).$$

If γ is a bi-hyperbolic orbit connecting p to q we will also say that γ connects A to A'. The scattering relation, expressed in Chazy parameters, defines a "relation" in the standard sense of set theory: which is to say, a subset of $\hat{\mathbb{E}} \times \hat{\mathbb{E}}$.

We have the following basic theorem concerning this scattering relation on $\hat{\mathbb{E}}$.

Theorem 5.1. *Let $\to$ denote the scattering relation on $\hat{\mathbb{E}}$. For all $A, A' \in \hat{\mathbb{E}}$ we have*

(i). **energy conservation:** $A \to A' \implies \|A\| = \|A'\|$.
(ii). **reflexivity:** $A \to A$.
(iii). **T-symm:** $A \to A' \iff -A' \to -A$.
(iv). **dilation:** *For any* $k > 0$, $A \to A' \iff kA \to kA'$.
(v). **rotation:** *For the diagonal action of any* $R \in O(d)$, $A \to A' \iff RA \to RA'$.
(vi). **reversibility:** $A \to A' \iff A' \to A$
(vii). **openness:** *For generic* $A \in \hat{\mathbb{E}}$ *the image* $\mathcal{HS}(A) := \{A' : A \to A'\}$ *of* A *under the scattering relation contains an open subset of the sphere* $\hat{S}(\|A\|) = \{A' \in \hat{\mathbb{E}} : \|A'\| = \|A\|\}$. *The closure of this open set contains* A.

Of all these properties the openness property (vii) is the most nontrivial one. Proving it will require some serious work. The genericity requirement is as follows. Suppose $A = vs$ for a restpoint (s, v). If s is a planar configuration, that is, all bodies lie in one plane, then the condition is simply that s not be collinear. If s is not planar, we are only able to say that s must not lie in the zero set of a certain determinant. We postpone the proof of openness to Theorems 5.5 and 5.6.

Proof of Theorem 5.1. (i). If $A \to A'$ then there is a bi-hyperbolic orbit γ that goes from $(-A/\|A\|, -\|A\|)$ to $(A'/\|A'\|, \|A'\|)$. Let h be the energy of γ. By continuity, the energy of the α and ω-limit of γ in $\mathcal{E}$ must be h as well. By (8), this means $\|A\|^2 = \|A'\|^2 = 2h$.

(ii). Set $p = (-A/\|A\|, -\|A\|) \in \hat{\mathcal{E}}_-$, by Proposition 3.1 and the proof given in Proposition 5.1, there is an orbit $\gamma \subset \hat{\Sigma}$ entirely contained in the infinity manifold, which connects p to its antipodal equilibrium $-p = (A/\|A\|, \|A\|) \in \mathcal{E}_+$, so that $p \to -p$. (See also the remark on free particles and the flow at infinity immediately preceding Proposition 3.1). Both p and $-p$ have the same Chazy parameter A. So $A \to A$.

(iii). Suppose $A \to A'$ with a connecting bi-hyperbolic orbit γ. The time reversed path $T\gamma(\tau) = \gamma(-\tau)$ of $\gamma(\tau)$ is again a bi-hyperbolic orbit. We saw in proposition 4.2 that $T\gamma$ has past Chazy parameter $-A'$, and an identical argument shows that it has future Chazy parameter $-A$. Thus $-A' \to -A$.

(iv). The n-body problem admits the dilational symmetries δ_λ, $\lambda > 0$, which in Newtonian time is given by

$$q(t) \mapsto (\delta_\lambda q)(t) := \lambda q(\lambda^{-3/2} t),$$

and in the rescaled time by

$$\delta_\lambda \gamma(\tau) = \lambda \gamma(\lambda^{-1/2} \tau).$$

The dilational symmetry can be worked out easily in our blown up variables where it acts on the equilibrium set $\hat{\mathcal{E}}$ by $\delta_\lambda(s, v) = (s, \lambda^{-1/2} v)$ and hence it acts on Chazy parameters by $A \mapsto \lambda^{-1/2} A$. Setting $k = \lambda^{-1/2}$ and suppose that $A \to A'$ via a bi-hyperbolic orbit $\gamma(\tau)$. Then the path $\delta_\lambda \gamma(\tau)$ is also bi-hyperbolic and connects kA to kA' and so $kA \to kA'$. The inverse relation is obtained using the inverse transformation $\delta_\lambda^{-1} = \delta_{1/\lambda}$.

(v). The proof here is almost identical to (iv). Use the fact that the action of $O(d)$ maps solutions to solutions and preserves their hyperbolicity. So if γ connects A to A' then $R\gamma$ connects RA to RA'.

(vi). Write $P = -I_d$ (P for 'parity') for the action of $-I_d$, the negative of the identity. $-I_d \in O(d)$ and acts on configuration space by $q \mapsto -q$. By (v), we have $A \to A' \iff -A \to -A'$. Now apply the time reversal map T of (iii) to get $-A \to -A' \iff A' \to A$.

(vii). See Theorem 5.5 and 5.6.

□

5.2. **The Scattering Map.** To say that the scattering relation $p \to q$ holds is to assert the existence of a connecting orbit lying in the intersection of the unstable manifold of p with the stable manifold of q. By specifying which unstable orbit connects with which stable orbit, we arrive at a refinement of the scattering relation called the *scattering map*, which we now describe.

Recall that the local stable and unstable manifolds $W^s_{loc}(\hat{\mathcal{E}}_+)$ and $W^u_{loc}(\hat{\mathcal{E}}_-)$ are open subsets of the extended phase space which can be parametrized by quadruples (s_0, v_0, ρ_1, s_1) where the first two entries parametrize the equilibrium manifold $\hat{\mathcal{E}}_+$ or $\hat{\mathcal{E}}_-$ and the second two entries parametrize the stable and unstable manifolds of the corresponding equilibrium points. In these coordinates, the flow on the local stable and unstable manifolds keeps the first two entries fixed and is linear on the second two entries as in (15). We will use a separate coordinate system of this type for each of $W^s_{loc}(\hat{\mathcal{E}}_+)$ and $W^u_{loc}(\hat{\mathcal{E}}_-)$.

To parametrize the *orbits* in $W^s_{loc}(\hat{\mathcal{E}}_+)$ and $W^u_{loc}(\hat{\mathcal{E}}_-)$, we should identify quadruples which map to one another under the linearized flows. The scattering parameter C given in (20) and (32) is an orbit invariant. Unfortunately it is not well-defined at $\rho_1 = 0$ so we will have to use a more complicated approach. Instead of the parameters (ρ_1, s_1), introduce the parameter

$$\gamma = (\rho_1, s_1 - \frac{\rho_1 \log(\rho_1 |v_0|)}{v_0^2} \tilde{\nabla} U(s_0)).$$

Note that

$$\gamma = (\rho_1, \rho_1 C), \text{ if } \rho_1 \neq 0.$$

Then one can check that the linearized flow identifies pairs γ, γ' if and only if $\gamma' = k\gamma$ for some constant $k > 0$. Introducing the notation $[\gamma]$ for the corresponding equivalence classes, we have parametrizations of the orbits in $W^s_{loc}(\hat{\mathcal{E}}_+), W^u_{loc}(\hat{\mathcal{E}}_-)$ by triples $(s_0, v_0, [\gamma])$. We call these triples "orbit parameters". Note that an orbit is constant (an equilibrium point) if and only if $\gamma = 0$ and that an orbit lies in the infinity manifold Σ if and only if $\rho_1 = 0, \gamma = (0, s_1)$ for some $s_1 \perp s_0$. The orbit parameters are related to the Chazy scattering parameters A, C by $A = v_0 s_0$ while C is the ratio of the two components of γ provided that the first coordinate ρ_1 of γ is positive. The inverse relation is $(s_0, v_0, [\gamma]) = (\pm A/\|A\|, \pm\|A\|, [(1, C)])$ with the $\pm$ depending on whether we are parameterizing $W^s_{loc}(\hat{\mathcal{E}}_+)$ or $W^u_{loc}(\hat{\mathcal{E}}_-)$.

Now we can introduce a map describing the scattering behavior of bi-hyperbolic orbits.

Definition 5.2. *Let $(s_0, v_0, [\gamma])$ and $(s'_0, v'_0, [\gamma'])$ parametrize the space of non-constant orbits of $W^u_{loc}(\hat{\mathcal{E}}_-)$ and $W^s_{loc}(\hat{\mathcal{E}}_+)$, respectively, so that $\gamma, \gamma' \neq 0$. Denote these orbit spaces by $W^u_{loc}(\hat{\mathcal{E}}_-)^{orb}$ and $W^s_{loc}(\hat{\mathcal{E}}_+)^{orb}$ respectively, and the open* *subspaces of these orbit spaces consisting of those orbits which also lie in* $\mathcal{H}$ *by* $(W^u_{loc}(\hat{\mathcal{E}}_-)\cap\mathcal{H})^{orb}$ *and* $(W^s_{loc}(\hat{\mathcal{E}}_+)\cap\mathcal{H})^{orb}$*. Then the* hyperbolic scattering map *is the map*

$$F : (W^u_{loc}(\hat{\mathcal{E}}_-)\cap\mathcal{H})^{orb} \to (W^s_{loc}(\hat{\mathcal{E}}_+)\cap\mathcal{H})^{orb}$$
$$F(s_0, v_0, [\gamma]) = (s_0', v_0', [\gamma'])$$

which sends the local orbit parameters of a bi-hyperbolic orbit γ *as* $\tau \to -\infty$ *to those of the same orbit as* $\tau \to +\infty$*. Alternatively, if* $\rho_1 > 0$ *for* γ *(and hence* $\rho_1' > 0$ *for* γ'*) we can write the map using Chazy parameters:*

$$F(A, C) = (A', C'), \qquad \rho_1 > 0.$$

It's clear that the scattering map is a refinement of the scattering relation. For $p = (s_0, v_0) \in \hat{\mathcal{E}}_-$ and $q = (s_0', v_0') \in \hat{\mathcal{E}}_+$, we have that $p \to q$ if and only if there exist orbit parameters $P = (s_0, v_0, [\gamma])$ and $Q = (s_0', v_0', [\gamma'])$ with $Q = F(P)$. In particular

$$\mathcal{HS}(p) = \{q : (q, [\gamma']) = F(p, [\gamma]) \text{ for some } [\gamma], [\gamma']\}.$$

Note also that it follows from Theorem 5.1, property (i), that if $F(P) = Q$ with P and Q as just described, then $v_0' = -v_0 > 0$.

Topological Considerations regarding the domain of F.

The equivalence classes $[\gamma] = [\rho_1, w]$ which we have used to parametrize orbits are rays through the origin in the real vector space $\mathbb{R} \oplus T_{s_0}S \cong \mathbb{E}$. The space of all such rays in a real vector space is a sphere, so a copy of S in our case. In this way we see that $W^u_{loc}(\hat{\mathcal{E}}_-)^{orb} \cong \hat{\mathcal{E}}_- \times S$ and $W^s_{loc}(\hat{\mathcal{E}}_+)^{orb} \cong \hat{\mathcal{E}}_+ \times S$ where $W^u_{loc}(\hat{\mathcal{E}}_-)^{orb}$, $W^s_{loc}(\hat{\mathcal{E}}_+)^{orb}$ are as per definition 5.2.

We are not really interested in these entire local orbit spaces, but rather only those rays for which $\rho_1 \geq 0$, since if $\rho_1 < 0$ the corresponding trajectory immediately enters into the non-physical region $\rho < 0$ of our phase space. Insisting $\rho_1 \geq 0$ determines a closed hemisphere within S. We have seen that the equator $\rho_1 = 0$, the boundary of this hemisphere, represents trajectories tangent to the infinity manifold Σ and that F is the identity when restricted to this equator. The Chazy parameter $C \in T_{s_0}S = s_0^\perp \subset \mathbb{E}$ is an affine parameterization of the open upper hemisphere $\rho_1 > 0$ via $C \mapsto [1, C]$. See figure 3. The closed hemisphere itself is topologically a unit disc in $T_{s_0}S$, and represents the "radial" compactification of the Chazy parameters, i.e. of the tangent space $T_{s_0}S$. We will denote this disc by $D(T_{s_0}S)$ and write it as

$$D(T_{s_0}S) = T_{s_0}S \cup \partial_{+\infty}D(T_{s_0}S)$$

with $\partial_{+\infty}D(T_{s_0}S)$ corresponding to the equator, which is to say the directions tangent to Σ, and diffeomorphic to a sphere S^{D-1}. We denote the corresponding disc bundle over the sphere by $D(S) = \bigcup_s D(T_sS)$. It is a real analytic fiber bundle over S. More generally, take any smooth manifold M , form the vector bundle $\mathbb{R} \oplus TM$ over M and then form the corresponding disc bundle $D(M) \to M$ whose fibers are the non-negative rays $\rho_1 \geq 0$ within the vector spaces $\mathbb{R} \oplus T_mM$. In our case we are interested in the disc bundle $D(\hat{S}) \subset D(S)$. The boundary of our disc bundle $\partial D(\hat{S})$ is identified with trajectories at infinity, $\rho_1 = 0$.

Energy considerations. We know the entire scattering map if we know its restriction to states of energy 1/2, which is the same as $\|A\| = 1$, or $|v_0| = 1$. Indeed we have seen that energy conservation implies that $F(A, C) = (A', C') \implies \|A\| = \|A'\|$. One verifies that the space time dilation, $\delta_\lambda q(t) = \lambda q(\lambda^{-3/2}t)$ in

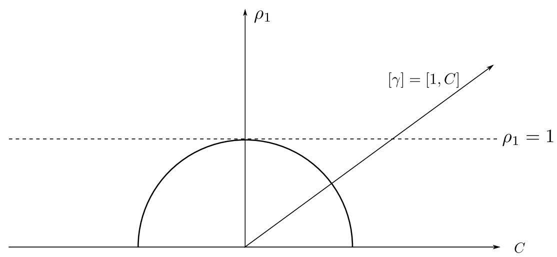

FIGURE 3. Affine coordinates on the rays through a half-space yield a realization of the scattering parameter C.

terms of Newtonian solutions, acts on Chazy parameters according to $\delta_\lambda(A, C) = (\lambda^{-1/2}A, \lambda C)$. It follows that $F(\delta_\lambda(A, C)) = \delta_\lambda(F(A, C))$ when $\rho_1 > 0$.

Theorem 5.2. *The scattering map F is determined by its restriction to those states having energy $1/2$. So restricted, the domain of the scattering map F is an open subset of the disc bundle $D(\hat{S}) \to \hat{S}$ which was obtained by compactifying the tangent bundle $T\hat{S}$ as described above. F is analytic on its domain and equals the identity on the boundary of the disc bundle, $\partial D(\hat{S})$ which corresponds to scattering along the infinity manifold. Moreover, if $T(A, C) = (-A, C)$ is the time reversal operation as it applies to Chazy parameters (see Proposition 4.2) then $T \circ F \circ T \circ F = Id$.*

Remark 5.2. *We expect that the domain of F is not all of $D(\hat{S})$: orbits hyperbolic in the past need not be hyperbolic in the future. Following [?], call an orbit "hyperbolic elliptic" if one or more of the particle pairs $\{i, j\} \subset \{1, \dots, n\}$ are bound in the distant future, meaning the lim sup of their mutual distance r_{ij} is bounded as $t \to +\infty$. According to Alekseev ([?] or Table 1 in p. 61 of [?]) the set of 3-body orbits which are hyperbolic in the distant past and hyperbolic-elliptic in the distant future has positive measure. Such orbits almost certainly sweep out an open subset of phase space. Assuming this openness, we can picture $D(\hat{S})$ as being comprised of two open sets separated by a closed measure zero boundary. The two open sets correspond to the hyperbolic-hyperbolic type orbits which form the domain of F, and the hyperbolic-elliptic type orbits. They are separated by a measure zero set which contains orbits ending in collision, and, more notably perhaps, orbits of hyperbolic-parabolic type: those for which the bound pair motion degenerates to a zero energy Kepler solution, so that $r_{ij} \sim t^{2/3}$. Understanding the two open sets and their boundary could be very interesting, but might be very difficult.*

Proof. Analyticity of F follows from the analytic linearization lemma, and the fact that the flow is analytic. Except for the final assertion regarding $FTFT$, the rest of the theorem was proved in the discussion immediately preceding the theorem's statement.

To verify $FTFT = Id$ use the $(s, v, [\gamma])$ representation. Suppose that a bi-hyperbolic orbit $\gamma \subset \mathcal{H}$ has representation $(s_0, -v_0, [\gamma])$ in the infinite past and $(s_0', v_0, [\gamma'])$ in the infinite future so that $F(s_0, -v_0, [\gamma]) = (s_0', v_0, [\gamma'])$. The time reversal $T\gamma$ of γ is represented by $(s_0', -v_0, [\gamma'])$ in the infinite past, and $(s_0, v_0, [\gamma])$ in the infinite future, so that $F(s_0', -v_0, [\gamma']) = (s_0, v_0, [\gamma])$. (Regarding the action of T, see Proposition 4.2 and the paragraph preceding it.) Now $T(s, \pm v, [\gamma]) =$

$(s, \mp v, [\gamma])$, so we have just shown that $F(T(F(s_0, -v_0, [\gamma]))) = (s_0, v_0, [\gamma]) = T(s_0, -v_0, [\gamma])$, or $FTF = T$. Use $T^2 = Id$ to finish off. □

5.3. **Scattering at Infinity.** Understanding the full hyperbolic scattering map or relation is beyond our grasp right now. It will be useful to restrict ourselves to various subsets. Let Z be any flow-invariant subset of phase space. Consider the smaller scattering map $F_Z = F|_Z$ and relation

$$\mathcal{HS}_Z = \{(p, q) \in \mathcal{E}_- \times \mathcal{E}_+ : \pi_-(x) = p \text{ and } \pi_+(x) = q \text{ for some } x \in \mathcal{H} \cap Z\}.$$

Clearly $\mathcal{HS}_Z \subset \mathcal{HS}$. We also use the notations $\mathcal{HS}_Z(p), \mathcal{HS}_Z^{-1}(q), p \to_Z q$ with the obvious meanings.

Most of what we know about the scattering map arises from our precise understanding of the dynamics along the infinity manifold Σ as summarized by Proposition 3.1. Σ is flow-invariant so we can form the restricted scattering relation $\mathcal{HS}_\Sigma$ and map $F|_\Sigma$.

Proposition 5.2. *The scattering map induced by the flow at infinity is obtained by restricting F to orbit parameters for which $\rho_1 = 0$, that is, $\gamma = (0, s_1)$. In this case*

$$F_\Sigma(s_0, v_0, [0, s_1]) = (-s_0, -v_0, [0, s_1]).$$

Expressed in partial Chazy parameters, this restricted F is the identity map

$$F(A, [0, s_1]) = (A, [0, s_1]).$$

The corresponding scattering relation is $\mathcal{HS}_\Sigma(p) = -p$.

Proof. The orbits at infinity are the half-circles of Proposition 3.1. We must delete the "singular orbits", i.e., those which hit the collision locus. These orbits connect equilibrium points $p = (s_0, -v_0)$ to $-p = (-s_0, +v_0)$ having the same value of the Chazy parameter A. Since the ambient dimension d is at least 2, not all of the half-circles are singular and it follows that the scattering relation is $\mathcal{HS}_\Sigma(p) = -p$.

To understand the scattering map F_Σ we need to relate the orbit parameters $[\gamma]$ at the two endpoints. These can be found by using the formulas for s_1, ρ_1 as noted in the proof of Lemma 4.3.

Since $\rho(\tau) = 0$ all along the orbit, we have

$$\rho_1 = \lim_{\tau \to \mp\infty} \exp(\pm v_0 \tau) \rho(\tau) = 0$$

at both endpoints. Using this, we have

$$s_1^- = \lim_{\tau \to -\infty} \exp(v_0 \tau)(s(\tau) - s_0) \qquad s_1^+ = \lim_{\tau \to \infty} \exp(-v_0 \tau)(s(\tau) + s_0).$$

Using the formula for $s(\tau)$ in Proposition 3.1 it is easy to see that $s_1^- = s_1^+ = 2\eta$. □

5.4. **Scattering Near Infinity.** What if we "expand" Σ, a bit? Let $Z = Z(R)$ denote the invariant set consisting of all points x such that the orbit $\phi_\tau(x)$ satisfies $\rho(\tau) < 1/R$ for all τ and let $Z(R, K)$ be the set whose orbits also satisfy $U(s(\tau)) < K$ for all τ. If K is sufficiently large, then there will be a nonempty set of points on the infinity manifold in $Z(R, K)$. Using the local structure near the equilibria, it is easy to see that $Z(R, K) \cap \mathcal{H}$ and $Z(R) \cap \mathcal{H}$ are nonempty open sets of phase space. We will denote the restricted scattering maps as $F_R, F_{R,K}$ and the restricted relations by $\mathcal{HS}_R$ and $\mathcal{HS}_{R,K}$. Similarly, we use the notation $p \to_R q$ and $p \to_{R,K} q$. Note that adding restrictions leads to a smaller scattering relation. So, for example,

$\mathcal{HS}_{R,K} \subset \mathcal{HS}_R \subset \mathcal{HS}$. Equivalently, $p \to_{R,K} q$ implies $p \to_R q$ which implies $p \to q$.

Recall that the scattering relation at infinity is $\mathcal{HS}_\Sigma(p) = -p$. The next two results show that scattering via orbits sufficiently near infinity leads to final states near $-p$. The first result also requires a bound on the potential.

Theorem 5.3. *Fix $K > 0$, $p \in \hat{\mathcal{E}}_-$, $\mathcal{V} \subset \hat{\mathcal{E}}_+$ a neighborhood of $-p$. Then for all R sufficiently large we have $\mathcal{HS}_{R,K}(p) \subset \mathcal{V}$.*

Remark 5.3. *If $K < U(p)$ it is possible that $\mathcal{HS}_{R,K}(p)$ is empty. Nevertheless, the theorem still holds.*

Proof. Consider the unstable manifold of p *within the infinity manifold* $\hat{\Sigma}$. We can choose a local cross-section σ to the flow within this manifold which is diffeomorphic to the $D-1$ dimensional sphere S. Let $\sigma_K \subset \sigma$ be the subset whose orbits satisfy $U(s(\tau)) \leq K$ for all τ. The complement of σ_K in σ is open so σ_K is compact. Now for each point $x \in \sigma_K$, we have $\pi_+(x) = -p$ since the orbit through x lies on the infinity manifold and limits to p in the backward direction. Since π_+ is continuous, there is some neighborhood $\mathcal{U}(x)$ of x in the full phase space such that $y \in \mathcal{U}(x)$ implies $\pi_+(y) \in \mathcal{V}$. Since σ_K is compact, there is some neighborhood $\mathcal{U}$ of σ_K itself with this property. We can thicken the local cross section σ_K in $\Sigma = \{\rho = 0\}$ to a cross-section of the form $\sigma_K \times [0, \delta)$ where $u \in [0, \delta)$. For δ sufficiently small, we will have $\sigma_K \times [0, \delta) \subset \mathcal{U}$. Then choose $R = 1/\delta$. If $x \in Z(R, K) \cap \mathcal{H}$ then its orbit meets the cross-section $\sigma_K \times [0, \delta)$ and $\pi_+(x) \in \mathcal{V}$. Since this holds for all such x, $\mathcal{HS}_{R,K}(p) \subset \mathcal{V}$. □

If we want to avoid imposing a fixed upper bound on $U(s)$, we can't guarantee a uniform $1/R$ sufficiently small, but we can still find a neighborhood of the set of nonsingular orbits at infinity starting at p such that scattering via orbits in this neighborhood always leads near $-p$.

Theorem 5.4. *Let $p \in \hat{\mathcal{E}}_-$ and let $\mathcal{V} \subset \hat{\mathcal{E}}_+$ be a neighborhood of $-p$. Then there is a neighborhood Z in phase space of the set of nonsingular orbits in $W^u(p) \cap \hat{\Sigma}$ such that $\mathcal{HS}_Z(p) \subset \mathcal{V}$.*

Proof. Let $Z = \mathcal{H} \cap \pi_+^{-1}(\mathcal{V})$. Then Z is an open, invariant subset of phase space. If $x \in \Sigma$ is a point on any nonsingular orbit at infinity with $\pi_-(x) = p$ then $x \in Z$, so Z is a neighborhood of the nonsingular orbits in $W^u(p) \cap \Sigma$. □

Theorems 5.3 and 5.4 assert a kind of continuity property of the scattering relation when restricted to neighborhoods of the infinity manifold. They can also be viewed as putting an upper bound on the image of p under near-infinity scattering. Next we turn to the more challenging question of lower bounds. From what has been proved so far, it is conceivable that $\mathcal{HS}(p) = \{-p\}$, i.e., that $p \to -p$ is the only scattering possible. We want results showing that $\mathcal{HS}(p)$ is nontrivial. Our main result is the "openness" of $\mathcal{HS}(p)$ described in property (vii) of theorem 5.1. The proof will involve perturbing a simple half-circle orbit at infinity and will actually show the stronger result that the image set $\mathcal{HS}_{R,K}(p)$ contains an open set for certain R, K. In other words, by scattering using bi-hyperbolic orbits which remain near infinity and are bounded away from the singularities, we can reach an open set of final states. First we show that the image contains at least a curve, then we fatten the curve into the required open set.

Fix an equilibrium point $p = (s_0, v_0) \in \hat{\mathcal{E}}_-$. The local unstable manifold of p is parametrized by (s_0, v_0, ρ_1, s_1) as per equation (16). We get the unstable manifold in $\hat{\Sigma}$ in this parameterization by taking $\rho_1 = 0$. Alternatively, the orbit parameter is $[\gamma]$ where $\gamma = (0, s_1)$. For these orbits, we have seen that the scattering map is $F(s_0, v_0, [\gamma]) = (-s_0, -v_0, [\gamma])$. We want to perturb to study orbits with ρ_1 small compared to s_1.

If we normalize $\|s_1\| = 1$, then Proposition 3.1 gives the unstable orbits in $\hat{\Sigma}$ by setting $\xi = -s_0, \eta = s_1, v_0 = -\sqrt{2h}$. Recall that the flow at infinity is really a reparametrization of the free particle flow with $q(t) = At + C$. Thus $A = \dot{q}$, but $\dot{q} = vs + w$ in the variables (ρ, s, v, w) so we have

$$A(s, v, w) = vs + w.$$

The following lemma gives the first-order change in A for perturbations of our unstable-manifold orbit in Σ.

Lemma 5.1. *Consider any nonsingular orbit at infinity, given by Proposition 3.1 with $s_0 = -\xi, v_0 = -\sqrt{2h}, \rho_1 = 0, s_1 = \eta$ and let $(\delta\rho, \delta s, \delta v, \delta w)$ be any solution of the variational differential equation along this orbit. Then the total change in $A(s, v, w) = vs + w$ along the variational orbit is given by*

$$\Delta A(\xi, v_0, \eta) = \frac{\delta\rho(0)}{\sqrt{2h}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nabla U(\xi \sin\theta + \eta \cos\theta)\, d\theta. \tag{34}$$

Proof. From (6) we find $A' = -\rho U(s)s + \rho\tilde{\nabla} U(s) = \rho \nabla U(s)$ and the first order variation along an orbit with $\rho = 0$ is

$$(\delta A)' = \nabla U(s(\tau))\delta\rho$$

where $\delta\rho(\tau)$ satisfies $\delta\rho' = -v\delta\rho$.

Now we will switch to use the variable θ from Proposition 3.1 as the independent variable. We have $v(\theta) = \sqrt{2h}\sin\theta, \theta' = \sqrt{2h}\cos\theta$ and so

$$\frac{d(\delta\rho)}{d\theta} = -\tan\theta\,\delta\rho \qquad \delta\rho(\theta) = \delta\rho(0)\cos\theta.$$

The variational equation for δA becomes

$$\frac{d(\delta A)}{d\theta} = \frac{\delta\rho(0)}{\sqrt{2h}} \nabla U(s(\theta))$$

and (34) follows by integration. □

For later use, we note that the parameter $\delta\rho(0)$ for a solution of the variational equation is related to the parameter ρ_1 in the linearized flow (15) near (s_0, v_0) by

$$\rho_1 = \lim_{\tau\to-\infty} \exp(v_0\tau)\delta\rho(\theta(\tau)) = \lim_{\tau\to-\infty} \exp(v_0\tau)\delta\rho(0)\,\mathrm{sech}(v_0\tau) = 2\delta\rho(0). \tag{35}$$

The integral in (34) is difficult to calculate, except for the following special case.

Example 5.1. *Consider the planar n-body problem and introduce the notation $s^\perp$ for the configuration where each position vector in s is rotated by $\pi/2$. Then taking $\eta = \xi^\perp$ in Proposition 3.1 gives a path*

$$s(\theta) = \xi \sin\theta + \xi^\perp \cos\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

from $s_0 = -\xi$ *to* $s_0' = \xi$ *which consist of rigid rotations of* ξ*. In fact, letting the rotation matrix* $R(\theta) = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ *act diagonally on configuration space, we have* $s(\theta) = R(\theta)\xi$*. The rotational symmetry of the potential gives* $U(R(\theta)\xi) = U(\xi)$ *and*

$$\nabla U(R(\theta)\xi) = R(\theta)\nabla U(\xi) = \nabla U(\xi)\sin\theta + \nabla U(\xi)^\perp \cos\theta.$$

Then (34) gives

$$\Delta A(\xi, -\sqrt{2h}, \xi^\perp) = \frac{2\delta\rho(0)}{\sqrt{2h}} \nabla U(\xi)^\perp.$$

The same proof works when s_0 *is a planar configuration in* $\mathbb{R}^d$*, that is, a configuration such that all of the bodies lie in some fixed plane. Then* $R(\theta)$ *becomes a rotation of that plane which fixes the orthogonal complement.*

This calculation gives some (rather minimal) information about the scattering map and relation.

Proposition 5.3. *Let* $p = (s_0, -\sqrt{2h}) \in \hat{\mathcal{E}}_-$ *where* s_0 *is a planar configuration. Then the image* $\mathcal{HS}(p)$ *under the hyperbolic scattering relation contains an analytic curve of the form* $q(\rho_1) = (s(\rho_1), \sqrt{2h})$*,* $0 \le \rho_1 < \delta$ *for some* $\delta > 0$ *with* $s(0) = -s_0$ *and tangent vector* $s'(0) = \frac{1}{2h}\nabla U(-s_0)^\perp$*. The same holds for* $\mathcal{HS}_R(p)$ *and* $\mathcal{HS}_{R,K}(p)$ *for* $K > U(s_0)$*.*

Proof. Since the planar n-body problem sits within the spatial and higher dimensional n-body problem we can view the solution curve $s(\theta)$ at infinity of the example above as occurring within any d-dimensional n-body problem, $d \ge 2$. So, as above let $\xi = -s_0, \eta = \xi^\perp$. Consider the curve in the local unstable manifold with parameters (ρ_1, s_1), $\rho_1 \in [0, \delta), s_1 = \xi^\perp$ (see Figure 4). If $\delta > 0$ is sufficiently small, the curve will lie in $\mathcal{H}$. Moreover, since $\rho(s(\theta)) = 0$ and $U(s(\theta)) = U(s_0)$ for all θ, the bi-hyperbolic orbits starting at (ρ_1, s_1) will satisfy $\rho(\tau) < 1/R$ and $U(s(\tau)) < K$ for all τ, provided δ is sufficiently small.

Now consider the image curve $\pi_+(s_0, -\sqrt{2h}, \rho_1, \xi^\perp)$. It's an analytic curve in $\hat{\mathcal{E}}_+$ with $\pi_+(s_0, -\sqrt{2h}, 0, \xi^\perp) = (-s_0, \sqrt{2h})$. Along this curve we have

$$A(\pi_+(s_0, -\sqrt{2h}, \rho_1, \xi^\perp)) = \sqrt{2h}s(\rho_1)$$

and, to first order in ρ_1,

$$\sqrt{2h}s'(0)\rho_1 = \Delta A(\xi, -\sqrt{2h}, \xi^\perp) = \frac{2\delta\rho(0)}{\sqrt{2h}}\nabla U(\xi)^\perp = \frac{\rho_1}{\sqrt{2h}}\nabla U(\xi)^\perp$$

which gives the formula for $s'(0)$. □

With more work we can expand this curve into an open set.

Theorem 5.5. *Let* $p = (s_0, -\sqrt{2h}) \in \hat{\mathcal{E}}_-$ *where* s_0 *is a planar, but non-collinear, configuration. Then the image* $\mathcal{HS}(p)$ *under the hyperbolic scattering relation contains an open subset of* $\{(s, v) \in \hat{\mathcal{E}}_+ : v = \sqrt{2h}\}$*. The same holds for* $\mathcal{HS}_R(p)$ *and* $\mathcal{HS}_{R,K}(p)$ *for* $K > U(s_0)$*.*

Proof. The open subset will be a neighborhood of the curve in Proposition 5.3 with its initial endpoint $\rho_1 = 0$ deleted. In other words, around each point of the curve with $0 < \rho_1$ sufficiently small, there is some neighborhood contained in $\mathcal{HS}(p)$.

Let $\xi = -s_0, \eta = \xi^\perp$. Consider the manifold Z of initial conditions in $W^u_{loc}(p)$ of the form $0 \le \rho_1 < \delta$, $\|s_1 - \eta\| < \delta$ with $\langle\langle s_0, s_1\rangle\rangle = 0, \|s_1\| = 1$ (see Figure 4). For

$\delta > 0$ sufficiently small, we have $Z \subset \mathcal{H}$ and we can consider the image $\pi_+(Z) \subset \mathcal{HS}(p) \subset \hat{\mathcal{E}}_+$. Points in this image are of the form $\pi_+(\rho_1, s_1) = (s(\rho_1, s_1), \sqrt{2h})$ for some analytic map $(\rho_1, s_1) \mapsto s(\rho_1, s_1)$ with $s(0, s_1) = -s_0$. Define

$$g(\rho_1, s_1) = 4hs(\rho_1, s_1) = 2\sqrt{2h}A(\rho_1, s_1)$$

where $A = v_0 s = \sqrt{2h}s$ is the Chazy parameter A as per Lemma 5.1. Lemma 5.1 gives the Taylor expansion in ρ_1 for g to be

$$g(\rho_1, s_1) = \rho_1 I(\xi, v_0, s_1) + O(\rho_1^2)$$

where

$$I(\xi, v_0, s_1) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nabla U(\xi \sin\theta + s_1 \cos\theta)\, d\theta.$$

We will show that the derivative of g is nondegenerate at points of the form (ρ_1, η), $\rho_1 > 0$ sufficiently small. Then the inverse function theorem shows that $g(Z)$ contains a neighborhood of $g(\rho_1, \eta)$. Consequently we get a neighborhood of our curve, as claimed.

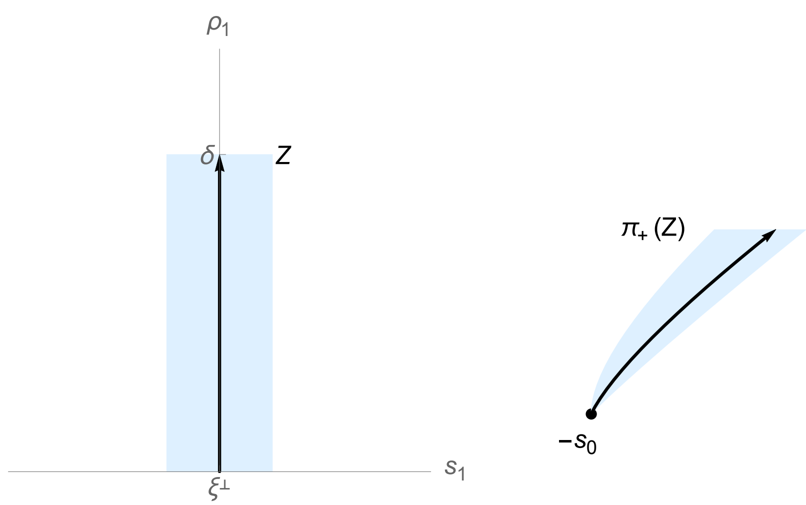

FIGURE 4. The domain Z parametrizing the thickened curve and its image, the thickened curve. $Z \subset W^u_{loc}(s_0, v_0)$ and the image is contained in $\mathcal{E}_+$.

We have

$$g_{\rho_1}(\rho_1, \eta) = I(\xi, v_0, \eta) + O(\rho_1) = 2\nabla U(\xi)^\perp + O(\rho_1)$$

and

$$\begin{aligned} g_{s_1}(\rho_1, \eta) &= \rho_1 \frac{\partial}{\partial s_1} I(\xi, v_0, s_1)|_{s_1=\eta} + O(\rho_1^2) \\ &= \rho_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D\nabla U(\xi \sin\theta + \eta\cos\theta)\, \cos\theta\, d\theta + O(\rho_1^2). \end{aligned}$$

Case $d = 2$**.** We first give the proof for the case of the planar n-body problem and later extend to the case of a planar configuration in $\mathbb{R}^d$. Then the right hand side of this last formula is a $2n \times 2n$ matrix which we will now evaluate. The rotational symmetry of the potential gives

$$D\nabla U(\xi \sin\theta + \eta\cos\theta) = D\nabla U(R(\theta)\xi) = R(\theta)D\nabla U(\xi)R(\theta)^{-1}$$

where $R(\theta)$ acts on the 2×2 blocks of $D\nabla U(\xi)$.

It is straightforward to calculate the $2n \times 2n$ matrix $D\nabla U(\xi)$ with the result

$$D\nabla U(\xi) = \begin{bmatrix} D_{11} & D_{12} & \dots & D_{1n} \\ D_{21} & D_{12} & \dots & D_{2n} \\ \vdots & & & \vdots \end{bmatrix} \tag{36}$$

where the 2×2 blocks are

$$D_{ij} = \frac{m_j}{r_{ij}^3}\left(I - 3u_{ij}u_{ij}^T\right), \quad u_{ij} = \frac{\xi_i - \xi_j}{r_{ij}} \qquad \text{for } i \neq j$$

and

$$D_{ii} = -\sum_{j\neq i} D_{ij}$$

where $r_{ij} = |\xi_i - \xi_j|$. The action of the rotation just conjugates each 2×2 block D_{ij} by $R(\theta)$. One can check that the result of conjugating each block by $R(\theta)$, multiplying by $\cos\theta$ and then integrating over $[-\pi/2, \pi/2]$ is to replace the blocks D_{ij}, D_{ii} by

$$\bar{D}_{ij} = \frac{-2m_j}{r_{ij}^3} v_{ij}v_{ij}^T \qquad \bar{D}_{ii} = -\sum_{j\neq i}\bar{D}_{ij}$$

where $v_{ij} = u_{ij}^{\perp}$. Writing $\bar{D}$ for this integrated matrix, we have

$$g_{s_1}(\rho_1, \eta) = \rho_1\bar{D} + O(\rho_1^2).$$

If $\beta \in \mathbb{R}^{2n}$ then one can check that

$$\beta^T M\bar{D}\beta = 2\sum_{i<j}\frac{m_i m_j}{r_{ij}^3}(\beta_{ij}\cdot v_{ij})^2.$$

Regarding the null space of this quadratic form, which is the same as the kernel of $\bar{D}$, we have the following lemma, whose proof will be postponed for the moment.

Lemma 5.2. *The kernel of $\bar{D}$ has dimension 3 and can be spanned by three independent vectors $\beta = \xi$, $\beta = (1, 0, 1, 0 \dots)$ and $\beta = (0, 1, 0, 1, \dots)$.*

Now we should restrict $g_{s_1}(\rho_1, \eta)$ and $\bar{D}$ to vectors β with center of mass 0 and with $\langle\langle \beta, s_0\rangle\rangle = \langle\langle \beta, \eta\rangle\rangle = 0$ since, in Z, we are perturbing away from $s_1 = \eta$ while keeping $\langle\langle s_0, s_1\rangle\rangle = 0$ and $\|s_1\| = 1$. Lemma 5.2 shows that none of these vectors are in the kernel of $\bar{D}$. On the other hand, the vector

$$g_{\rho_1}(\rho_1, \eta) = 2\nabla U(\xi)^{\perp} + O(\rho_1)$$

and this is equal to $\bar{D}\eta$. Namely, the i-th pair of components of $\bar{D}\eta$ is

$$(\bar{D}\eta)_i = (\bar{D}\xi^{\perp})_i = 2\sum_{j\neq i}\frac{m_j}{r_{ij}^3}(\xi_i^{\perp} - \xi_j^{\perp}) = (2\nabla U(\xi)^{\perp})_i.$$

Lemma 5.2 shows that η is also not in the kernel of $\bar{D}$. Therefore, taking the vector $\bar{D}\eta$ together with the image under $g_{s_1}(\rho_1, \eta)$ of all relavant vectors β, we find that for ρ_1 sufficiently small, the image of the derivative map of g at (ρ_1, η) has dimension $2n - 3 = D - 1$. Since this is the dimension of Z and of $\{(s, v) \in \hat{\mathcal{E}}_+ : v = \sqrt{2h}\}$, the derivative map is indeed nondegenerate for ρ_1 sufficiently small and the proof is complete.

Case $d > 2$. When $d > 2$ we think of the planar n-body problem as it embeds in the higher-dimensional n-body problem, writing $\mathbb{R}^d = \mathbb{R}^2 \oplus \mathbb{R}^{d-2}$ as an orthogonal decomposition. We continue to perturb off the same analytic curve of hyperbolic solutions as per proposition 5.3, with the inducing hyperbolic scattering curve now viewed as lying in $(\mathbb{R}^2)^n \subset (\mathbb{R}^d)^n$. To show that the derivative along this curve is invertible we split each component variation β_i into $\beta_i = \beta_i^{||} + \beta_i^{\perp}$, $\beta_i^{||} \in \mathbb{R}^2, \beta_i^{\perp} \in \mathbb{R}^{d-2}$. The formulae for D_{ij} still holds. Since $u_{ij} \in \mathbb{R}^2$ so we have that each D_{ij} itself splits into block-diagonal form : $D_{ij}\beta_j = D_{ij}^{||}\beta_j^{||} + D_{ij}^{\perp}\beta_j^{\perp}$ with $D_{ij}^{||} : \mathbb{R}^2 \to \mathbb{R}^2$ being exactly the same as the 2 by 2 block D_{ij} for the $d = 2$ case, and with $D_{ij}^{\perp} : \mathbb{R}^{d-2} \to \mathbb{R}^{d-2}$ equal to $(m_j/r_{ij}^3)I_{d-2}$, a multiple of the identity independent of θ. The conjugation by $R(\theta)$ result still holds, with $R(\theta)$ acting trivially on the orthogonal "$\perp$" blocks. Multiplying by $\cos\theta$ and integrating gives a block matrix $\bar{D}$ with a 2×2 block $\bar{D}^{||}$ as above and a $(d-2) \times (d-2)$ block $\bar{D}^{\perp}$ which is just the original block $D^{\perp}$ multiplied by a factor of 2.

The end result is that

$$\beta^T M \bar{D} \beta = 2 \sum_{i<j} \frac{m_i m_j}{r_{ij}^3} (\beta_{ij}^{||} \cdot v_{ij})^2 - 2 \sum_{i<j} \frac{m_i m_j}{r_{ij}^3} |\beta_i^{\perp} - \beta_j^{\perp}|^2.$$

Recall that Lemma 5.2 implies, the first term, $\bar{D}^{||}$ has corank 3. The second term has a $d-2$-dimensional kernel consisting of vectors with all $\beta_i^{\perp}$ equal. It is nondegenerate on the space of vectors $\beta^{\perp}$ with center of mass zero. Then for ρ_1 sufficiently small, the image of the derivative map of g at (ρ_1, η) has dimension $dn - 3 - (d-2) = d(n-1) - 1 = D - 1$ and the implicit function theorem applies as before to complete the proof. □

Proof of Lemma 5.2. Let $\beta = (\beta_i)_{i=1}^n$ be a vector from the kernel of $\bar{D}$. Then $\langle v_{ij}, \beta_{ij} \rangle = 0$, for all pairs of i, j with $i \neq j$, where $\beta_{ij} = \beta_i - \beta_j$. Recall that $v_{ij} = u_{ij}^{\perp}$ and $u_{ij} = \frac{\xi_i - \xi_j}{r_{ij}}$. This means either $\beta_{ij} = 0$ or $\beta_{ij} \neq 0$ and is parallel to $\xi_{ij} = \xi_i - \xi_j$. Notice that $\xi_{ij} \neq 0$, for any $i \neq j$, as ξ is a non-singular configuration.

We claim if there exists a pair of $i \neq j$, such that $\beta_{ij} = 0$, then $\beta_k = \beta_i$, for all $1 \leq k \leq n$. In this case, β is in the span of $(1, 0, \dots), (0, 1, \dots)$. To prove the claim, choose a $k \neq i$ or j. First let's assume $\xi_k \notin \overline{\xi_{ij}}$, where $\overline{\xi_{ij}}$ represents the straight line pass through ξ_i and ξ_j. Since ξ is non-singular, $\xi_i \neq \xi_j$, and this line is unique. The non-singularity of ξ further implies $\xi_{ik} \neq \xi_{jk}$ and neither is zero. Meanwhile $\beta_{ij} = 0$ implies $\beta_{ik} = \beta_{jk}$. This means β_{ik} must be zero, as otherwise, β_{ik} is parallel to both ξ_{ik} and ξ_{jk}, which is absurd.

Now assume $\xi_k \in \overline{\xi_{ij}}$. Since ξ is non-collinear, there must exist another index ℓ, such such $\xi_\ell \notin \overline{\xi_{ij}}$. By the previous result, $\beta_{i\ell} = 0$. Meanwhile $\xi_k \notin \overline{\xi_{i\ell}}$ and by repeating the previous argument, we can show $\beta_{ik} = 0$. This finished our proof of the claim.

With the above claim, we may assume $\beta_{ij} \neq 0$, for all pairs of $i \neq j$, as otherwise all β_{ij} must be zero. Since ξ is non-collinear, there exist at least three different indices i, j and k, such that ξ_i, ξ_j and ξ_k are the vertices of a non-degenerate triangle. Meanwhile by the assumption, β_i, β_j and β_k are the vertices of a non-degenerate triangle as well. Since β is in the kernel of $\bar{D}$, the corresponding sides of the two triangles are parallel to each other. Therefore the two triangles must be similar to each other and there is a constant $\lambda > 0$, such that

$$\beta_{ij} = \lambda\xi_{ij}, \quad \beta_{jk} = \lambda\xi_{jk}, \quad \beta_{ki} = \lambda\xi_{ki}. \tag{37}$$

Hence after translating β by a constant vector, we have

$$\beta_i = \lambda\xi_i, \quad \beta_j = \lambda\xi_j, \quad \beta_k = \lambda\xi_k. \tag{38}$$

Now for any index ℓ different from i, j, k, we can always find two vectors from ξ_i, ξ_j and ξ_k, such that together with ξ_ℓ, they form the vertices of a non-degenerate triangle. Without loss of generality, we assume ξ_i and ξ_j are such two vectors. Then by a similar argument as above we can show the two non-degenerate triangles form by ξ_i, ξ_j, ξ_ℓ and $\beta_i, \beta_j, \beta_\ell$ correspondingly are similarly to each other, and there is a constant $\mu > 0$, such that

$$\beta_{ij} = \mu\xi_{ij}, \quad \beta_{j\ell} = \mu\xi_{j\ell}, \quad \beta_{\ell i} = \mu\xi_{\ell i}.$$

Combining the first identity above with the first identity in (37), we get $\mu = \lambda$. Then the last identity above implies

$$\beta_\ell - \beta_i = \lambda\xi_\ell - \lambda\xi_i.$$

Combining this with the first identity in (38), we get $\beta_\ell = \lambda\beta_\ell$. As a result we have proven that up to a translation $\beta = \lambda\xi$. This means that β is in the span of the three vectors given in the lemma. □

Finishing off the proof of Theorem 5.1. To finish off the proof of the openness property (vii) of Theorem 5.1 we must extend the results just achieved for planar configurations to generic non-planar ones when $d > 2$. We will use an analyticity argument.

Theorem 5.6. *There are real-valued continuous functions f, K defined on an open, dense subset of S with f real analytic and non-constant, such that if $p = (s_0, -v_0) \in \hat{\mathcal{E}}_-$ and $f(s_0) \neq 0$, then the subsets $\mathcal{HS}(p)$, $\mathcal{HS}_R(p)$ and $\mathcal{HS}_{R,K}(p)$ for $K > K(s_0)$ all satisfy the openness condition, condition (vii) of theorem 5.1: each set contains a non-empty open subset of the sphere $\hat{\mathcal{E}}_+ \cap \{v = +v_0\}$ whose closure contains $-p$.*

Proof. When s_0 is planar and non-collinear, the proofs we just gave above that the sets $\mathcal{HS}(p), \dots$ satisfy condition (vii) boiled down to showing a certain linear operator had full rank, this rank being $D-1$. This operator was the restriction of $\bar{D}$ to the $D-1$-dimensional linear space $V(s_0)$ of vectors with zero center of mass and orthogonal to s_0. Once bases are chosen in the domain and range of the operator, this rank condition could have been verified by showing a determinant is nonzero. When s_0 is not planar, the linear space $V(s_0)$ still has dimension $D-1$, the same as the dimension of the target space $\{(s, v) \in \hat{\mathcal{E}}_+ : v = \sqrt{2h}\} \cong S$. Nondegeneracy of a corresponding operator will allow us to apply the inverse function theorem and conclude that the sets again satisfy condition (vii).

$\bar{D}$ was defined as the integral

$$\bar{D}(s_0) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D\nabla U(\xi \sin\theta + \eta\cos\theta)\, \cos\theta \, d\theta$$

where $\xi = -s_0$ and $\eta = \xi^\perp$ (obtained by rotating all of the ξ_i by 90 degrees in the plane). This construction of an operator can be extended to almost all nonplanar s_0 by a choosing an $\eta = \eta(s_0)$ in the following (rather artificial) way.

Let $\xi = -s_0$ where s_0 is not necessarily planar. Project all of the ξ_i into $\mathbb{R}^2 \times 0$ and let η be obtained by rotating all of these projections in that plane. As long as not all of the projections are $(0,0)$ this gives a nonzero vector η which is orthogonal to ξ and which can be normalized so $||\eta|| = 1$ as in Proposition 3.1. Call this vector $\eta(s_0)$. This $\eta(s_0)$ is well defined and analytic on an open and dense subset of $\mathcal{E}_-$. We define $\bar{D}(s_0)$ to be the integral above, an $nd \times nd$ matrix depending analytically on s_0, and defined for all s_0 in the open, dense set $\mathcal{U}$ of s_0's satisfying the projection condition above and the additional condition that that the great circle path $s(\theta, s_0) = -s_0 \sin\theta + \eta(s_0)\cos\theta$ misses the collision locus. The operator continues to represent the linearized flow along the corresponding great circle, 'pushed' out infinitesimally into the 'bulk' $\rho > 0$ so the arguments relating the nondegeneracy of $\bar{D}(s_0)$ to condition (vii) continue to hold. To see that $\mathcal{U}$ can be specified by the nonvanishing of an analytic function g, observe that the projection condition, being a linear independence condition, can be written as the non-vanishing of products of minors built from the analytic vector functions $-s_0$ and $\eta(s_0)$, while the non-collision condition can be written as the nonvanishing of the pullback of the product of the algebraic functions r_{ij}^2's by the geodesic flow on the sphere, which is analytic.

Regarding the function K, note that although our paths $s(\theta, s_0)$ avoid the poles of U, the potential $U(s(\theta, s_0)))$ is no longer constant along them. For this reason, the constant $U(s_0)$ in the planar theorem should be replaced by $K(s_0) = \max_\theta U(s(\theta, s_0)))$.

Now let $\mathcal{S}(s_0)$ be the $D-1$ dimensional subspace of vectors with zero center of mass and orthogonal to s_0. If we can choose a basis for $\mathcal{S}(s_0)$ in an analytic way, then the nondegeneracy becomes the requirement that some $(D-1)\times(D-1)$ determinant be nonzero. Multiply this by determinant by g we get the required f. There is no problem choosing a basis for the zero center of mass subspace (for example the basis implicit in Jacobi coordinates). Choosing a basis for the spaces orthogonal to s_0 is not possible globally on S (unless $D-1 = 3$ or 7) but since we have deleted the singular points, the resulting normal bundle is trivial and such a basis can be found. We now have our function f.

We now have condition (vii) holding for the set $\mathcal{V}$ all s_0's such that $f(s_0) \neq 0$. $\mathcal{V}$ is nonempty because it contains the planar nonconlinear s_0's described earlier. $\mathcal{V}$ is clearly open. To see that $\mathcal{V}$ is dense, we will use the fact that the open, dense set where $g(s_0) \neq 0$ is connected. To see its connectedness, note that the projection condition gives a connected set since we can use a homotopy to carry out the projection (without changing η). Now the set of initial conditions ξ, η leading to collision paths $\xi \sin\theta + \eta\cos\theta$ has codimension at least two, so it can be avoided. Thus any configuration s_0 with $g(s_0) \neq 0$ can be connected to a planar, noncollinear one by an analytic curve, say $\gamma(s)$, with $G(\gamma(s)) \neq 0$. Then $f(\gamma(s))$ is an analytic function of one variable which is not identically zero. Therefore its zeros are isolated. □

What's Left to do? Most questions regarding hyperbolic scattering remain open. The broadest is simply

Problem 1. *From a given initial state $p \in \hat{\mathcal{E}}_-$ determine which final states in $\hat{\mathcal{E}}_+$ can be reached by hyperbolic scattering. In other words, determine the image $\mathcal{HS}(p)$ under the hyperbolic scattering relation.*

For the two-body problem in the plane we saw in example 4.1 that every state can in $\mathcal{E}_+$ having the same energy as p can be reached, except for $-p$. If the infinity manifold is added, then its flow takes p to $-p$. Written in terms of Chazy variables, $\mathcal{HS}(A) = S^1$, the full circle of asymptotic states of radius $\|A\|$. Perhaps an analogous result holds for the n-body problem, i.e. it could be that $\mathcal{HS}(A) = \hat{S}$ for all $A \in \hat{S}$.

Since so little is known about the problem with $n \geq 3$, even the following seems to be mostly open.

Problem 2. *From a given initial state $p \in \hat{\mathcal{E}}_-$ give examples of states besides $-p$ which can be reached or of states which cannot be reached by hyperbolic scattering.*

One can pose the same problems with some restrictions on the scattering relation, for example, with $\mathcal{HS}(p)$ replaced by $\mathcal{HS}_R(p)$ or $\mathcal{HS}_{R,K}(p)$. Theorem 5.3 shows that certain states, namely those outside $\mathcal{V}$, cannot be reached using $\mathcal{HS}_{R,K}(p)$, while Proposition 5.3 and Theorems 5.5 and 5.6 give some information in the other direction. But little else seems to be known.

It seems likely that near collisions will lead to very different kinds of scattering, even for orbits near infinity. In particular, the scattering relation $\mathcal{HS}_R$ does not impose a bound on the potential and allows near collisions.

Problem 3. *From a given initial state $p \in \hat{\mathcal{E}}_-$, determine the image $\mathcal{HS}_R(p)$. Calculate $\cap_{R>0}\mathcal{HS}_R(p)$. Is it equal to $\{-p\}$?*

Ideas from [**?**] might help in making progress on these problems.

Appendix A. Analytic Linearizations

Here we prove the analytic linearization theorem, Theorem 3.1 as a corollary to:

Theorem A.1. *Let X be a real analytic vector field defined on an open subset of $\mathbb{V} = \mathbb{R}^m \times \mathbb{R}^k$ and vanishing on $\mathcal{E} = U \times 0$ where U is an open subset of $\mathbb{R}^m$. Suppose that the linearization $L(p) = DX_p : \mathbb{V} \to \mathbb{V}$ of X at each $p \in \mathcal{E}$ has exactly one nonzero eigenvalue $\lambda(p)$ (necessarily real since X is real) of algebraic multiplicity k. Set $N_p = L(p)(\mathbb{V}) \subset \mathbb{V}$ and write $N \to \mathcal{E}$ for the rank k analytic vector bundle over $\mathcal{E}$ whose fiber over p is N_p. Let $X_N : N \to N$ be the fiber-linear vector field given by $X_N(p, v) = (0, L(p)v)$. Then there is an analytic diffeomorphism Φ such that $\Phi^* X = X_N$ where Φ maps a neighborhood of the zero section in N to a neighborhood of $\mathcal{E}$ in $\mathbb{V}$. Finally, if we define $i : N \to \mathbb{V}$ by $i(p, v) = p+v$, then, upon restriction to a neighborhood of the zero section, i is an analytic diffeomorphism onto a neighborhood of $\mathcal{E}$ and Φ agrees with i to 1st order along the zero section.*

Remark A.1. *The kernel of $L(p)$ equals $\mathbb{R}^m \times 0$. This fact follows from the assumptions that U is open and that the multiplicity of $\lambda(p)$ is k. N_p is a complementary subspace to $\mathbb{R}^m \times 0$ but need not be equal to $0 \times \mathbb{R}^k$. $L(p)$ restricted to N_p is a rank k linear map. It is essential for our application, namely theorem 3.1, that this map be allowed to have nonzero Jordan blocks.*

Let us see see how theorem 3.1 follows from theorem A.1.

Proof of theorem 3.1. . Let X^{newt} be our N-body vector field, extended to infinity, as per (6). We saw in the discussion of the linearization structure, immediately following (11) and also remark 3.3, that X^{newt} is the restriction to a codimension

2 analytic subvariety of an analytic vector field X defined on an open subset of $\mathbb{R}\times\mathbb{E}\times\mathbb{R}\times\mathbb{E} = \mathbb{V}$ and whose variables we wrote as (ρ, s, v, w) in eq (6). Rearranging these coordinates in the order (s, v, ρ, w), viewing $(s, v) \in \mathbb{R}^m, (\rho, w) \in \mathbb{R}^k$ (with $k = m$), and referring to the discussion of the linearization structure of X following (11), we see that X satisfies the hypothesis of theorem A.1 with $U = \{(s, v) \in \mathbb{E}\times\mathbb{R} : s \notin \Delta, v \neq 0\}$, and with $\lambda(s, v) = -v$. Theorem A.1 now supplies the analytic conjugation to X_N for X. But theorem 3.1 claims the analytic conjugation for X^{newt} not for X. To finish off, observe that the extended phase space P on which X^{newt} lives is invariant under the flow of X, and hence $\Phi^{-1}(P)$ is invariant under the flow of X_N. Restricting X_N and Φ to $\Phi^{-1}(P)$ yields the claimed conjugacy. □

Proof of A.1. Decompose $\mathcal{E}$ into $\mathcal{E}_+$ and $\mathcal{E}_-$ with $\mathcal{E}_- = \{p \in \mathcal{E} : \lambda(p) < 0\}$ and $\mathcal{E}_+ = \{p \in \mathcal{E} : \lambda(p) > 0\}$. (The + and − subscripts are for forward-time attracting and backward time attracting.) We give the proof for the open component $\mathcal{E}_+$ of $\mathcal{E}$. The proof for $\mathcal{E}_-$ follows a symmetrical proof with stable manifold replaced by unstable and $t \to +\infty$ by $t \to -\infty$.

The proof proceeds in three steps. **Step 1.** Straighten out the local stable manifolds so they are aligned with the fibers of N. **Step 2.** Work within each fiber N_p and linearize the vector field there insuring that the linearization is also analytic in p. **Step 3.** Use uniqueness of the linearization from step 1 and 2 to insure that the linearizing transformation is defined in a whole neighborhood of $\mathcal{E}_+$.

Step 1. [Straightening out the stable manifolds.] The generalized eigenspace for $\lambda(p)$ is $N_p = L(p)(\mathbb{V})$ and must be the tangent space at p to the local stable manifold $W(p) := W^s_{loc}(p)$ passing through $p \in \mathcal{E}_+$. Now $\mathbb{V} = \mathbb{R}^m \oplus N_p$. According to the usual stable manifold theorem, $W(p)$ is the graph of a smooth function $\psi_p : (N_p, 0) \to \mathbb{R}^m$ with $\psi_p(0) = p$ and $d\psi_p(0) = 0$. Here, and in the remainder of this section, the broken arrow notation is used to indicate that the domain of the function is an open neighborhood of p and not all of N_p.

We need to upgrade $v \to \psi_p(v)$ so as to be analytic not just smooth, and analytic in both v and p. To this end, look at [**?**, Chapter 13, Theorem 4.1, also Exercise 4.11], where this work is essentially done. The Perron-style proof of the stable manifold theorem there proceeds by constructing a nonlinear integral operator $F = F_{p,v}$ acting on paths. The path space it acts on is the space of smooth paths $\gamma : [0, \infty) \to \mathbb{V}$ which tend to p as $t \to \infty$ and have initial condition $\gamma(0)$ projecting onto $v \in N_p$. One proves that for $|v|$ small F is a contraction mapping. Its unique fixed point $\gamma(t; p, v)$ lies in $W(p)$. One has, by definition, that $\gamma(0; p, v) = p + \psi_p(v) + v$ - yielding the map $\psi_p : \mathbb{V} \to \mathbb{R}^m$. To get ψ analytic in both p and v, complexify both vectors so that $p \in \mathbb{C}^m$, $v \in \mathbb{N}^{\mathbb{C}}_p = N_p \otimes \mathbb{C}$, and $p + v \in \mathbb{V}^{\mathbb{C}}$. Write $Re(p), Im(p) \in \mathbb{R}^m, Re(v), Im(v) \in N_p$ for their real and imaginary parts. One verifies that all the properties of F are retained in this complexified setting provided $|Im(p)| + |Im(v)| < \delta$ is sufficiently small. One also verifies that the iteration scheme $\gamma^j(t; p, v) = F^j_{(p,v)}(\gamma_0(t)) \in \mathbb{V}^{\mathbb{C}}$ satisfies uniform C^0 bounds in this thickened neighborhood with the γ^j analytic in t, p, v at each step. Since the uniform limit of complex analytic functions is complex analytic, we get that the endpoints, the $\gamma(0; p, v) = p + \psi_p(v) + v$ are complex analytic in p, v in this thickened strip, so, upon restriction to the real parts, are analytic. We have our analytic stable manifold, depending analytically on p.

Now write $\phi_p(v) = p + v + \psi_p(v)$ for $v \in N_p$. Because $W(p)$ is the graph of ψ_p we have that ϕ_p maps a neighborhood of 0 in N_p to a neighborhood of p in $W(p)$. Then $\Phi : N- \to \mathbb{V}$ by $\Phi(p, v) = \phi_p(v)$ is our desired analytic straightening diffeomorphism, with domain a neighborhood of the zero section of N. Recall that N_p is tangent to $W(p)$ at p, and that $\psi_p(0) = 0, d\psi_p(0) = 0$ to see that the derivative of Φ along the zero section $(p, 0)$ of N can be identified with the "identity" $(h, v) \mapsto h + v$ from $\mathbb{R}^m \oplus N_p \to \mathbb{V}$. In other words, the derivative is invertible and Φ agrees with i to first order along the zero section. (By the inverse function theorem, Φ and i are invertible in a neighborhood of the zero section.) Now the vector field X is everywhere tangent to the foliation of a neighborhood of $\mathcal{E}_+$ by the stable manifolds, the $W(p)$'s, so Φ^*X must be everywhere tangent to the inverse image of this foliation by Φ, which is to say, tangent to the fibers of $N \to \mathcal{E}_+$. This means that Φ^*X is a vertical vector field. Finally, due to the nature of the linearization of Φ along the zero section, the linearization of Φ^*X and of X both agree along the zero section, which means that

$$\Phi^*X(p, v) = (0, L(p)v + g(p, v)), g(p, v) = O(|v|^2)$$

with g analytic.

Step 2. [Linearizing fiber-by-fiber]. Invoke the theorem of Brushlinskaya.

Theorem A.2 (Brushlinskaya [**?**])**.** *Consider a family of analytic vector fields X_p on $\mathbb{R}^k$, analytically depending on the parameters $p \in \Omega \subset \mathbb{R}^m$, Ω open. Assume that at 0 and $p = p_0 \in \Omega$ there is an equilibrium with eigenvalues whose real parts all have the same sign. Then, with the aid of a near identity transformation Φ, analytic in a neighborhood of 0 and depending analytically on the parameters p within a small neighborhood of p_0, one can reduce the family X_p to "resonant polynomial form": $\hat{X}_p = \Phi^*X_p$ is a family of polynomial vector fields whose coefficients depend analytically on the parameters p and for which the only monomials occuring in these polynomials are the resonant monomials in the sense of the Poincaré-Dulac theory.*

To use this theorem in our situation we will need to understand the terms "resonant monomials" and "near-identity". Fix p. Use the notation $v^Q = v_1^{q_1} \dots v_k^{q_k}, Q = (q_1, \dots, q_k) \in \mathbb{N}^k$ to describe monomials on $\mathbb{R}^k$. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be the eigenvalues of the linearization of X_p at 0.

Definition A.1. *A resonant monomial v^Q for X_p is a monomial of degree 2 or higher for which $Q \cdot \lambda = \lambda_i$ for some $i = 1, \dots, k$.*

Observe "degree 2" or higher is equivalent to $|Q| := q_1 + \dots + q_k \geq 2$.

Definition A.2. *A near-identity analytic transformation ϕ of $\mathbb{R}^k$ is an analytic transformation defined near 0 and having convergent power series expansion $v \mapsto v + \phi_2(v) + \phi_3(v) + \dots$, with the $\phi_i(v)$ being homogeneous vector-valued polynomials of degree i.*

In the Poincaré-Dulac method, as explained for example in [**?**], one tries to successively kill all the monomials of degree 2 or higher arising in X_p by appropriately choosing the q_i of ϕ. All non-resonant monomials can be killed. The non-resonant ones, being in the kernel of the "cohomological operator" associated to the process remain. (See step 3 below for this cohomological operator.) The essence of the Brushlinskaya theorem then is that this process leads to a convergent power series for the map ϕ with coefficents depending analytically on p.

What does that mean for our specific family? We have that the vector of eigenvectors λ is $\lambda = \lambda(p)(1, 1, \ldots, 1)$ at each point p. The resonance condition of definition A.1 then reads $|Q|\lambda(p) = \lambda(p)$ which is impossible since $|Q| \geq 2$. There are no resonant monomials! The normal form $\hat{X}_p$ is simply the linearization $v \mapsto L(p)v$ of $\tilde{X}_p$ for each p. We now have that X is locally analytically conjugate to its linearizations X_N in neighborhoods U_p of each $p \in \mathcal{E}_+$.

Step 3. [Patching together] We must make sure that all these local analytic conjugacies guaranteed by the last two steps piece together to form a single analytic conjugacy defined in a neighbhorhood of all of $\mathcal{E}_+$.

To this end, suppose that Ψ_1, Ψ_2 are two linearizations, each defined on its own open set $\mathcal{U}_i \subset N$, this open set containing neighborhoods $U_i \subset \mathcal{E}$. For simplicity we will say "Ψ_i is defined over U_i" to encode this information. Then $\Psi_1^* X = X_N$ holds over U_1 while $\Psi_2^* X = X_N$ holds over U_2. The map $\Phi = \Psi_2^{-1} \circ \Psi_1$ is then a near identity transformation defined over $U_1 \cap U_2$ and satisfying $\Phi^* X_N = X_N$ over $U_1 \cap U_2$. Φ must preserve the fibers of $N \to \mathcal{E}_+$, for otherwise upon pull-back it would add "tangential terms" to X_N. Thus we have $\Phi(p, v) = (p, v + h(p, v)))$. Because Ψ_1 and Ψ_2 are both near-identity, so is Φ which means that when expanded as a power series $h(p, v)$ consists entirely of terms quadratic and higher in v. Write out its convergent Taylor series

$$h = h_2 + h_3 + \ldots,$$

with $h_i(p, \cdot) : N_p \to N_p$ a homogeneous degree i vector-valued polynomial. At the heart of the Poincaré-Dulac method is the fact that $\Phi^* X_N = X_N$ is equivalent to the (infinite) system of "cohomological equations" $[L, h_i] = 0$. See for instance [**?**, Chapter 5] for details. But the kernel of this cohomological operator $h \mapsto [h, L]$ is precisely the linear span of the resonant monomials. (To be precise, what we mean by a 'monomial' h is any h of the form $h(v) = v^Q e_i$ with e_i one of the standard basis elements for $\mathbb{R}^k$.) We have seen that in our case there are no resonant monomials: this kernel is zero. Thus $h_i = 0$, $i = 2, 3, \ldots$ and hence $\Phi = Id$ so that $\Psi_1 = \Psi_2$ over $U_1 \cap U_2$. Thus there is a single $\Psi : N- \to \mathbb{V}$, defined over $\mathcal{E}_+$, whose restrictions to the various open sets $U_p \subset \mathcal{E}$ as per step 2 are the analytic near-identity diffeomorphisms guaranteed by Brushlinskaya.

□

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