

REGULARITY OF A GRADIENT FLOW GENERATED BY THE
ANISOTROPIC LANDAU–DE GENNES ENERGY WITH A
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Abstract. In this paper we study a gradient flow generated by the Landau–de Gennes free energy that describes nematic liquid crystal configurations in the space of Q -tensors. This free energy density functional is composed of three quadratic terms as the elastic energy density part, and a singular potential in the bulk part that is considered as a natural enforcement of a physical constraint on the eigenvalues of Q . The system is a non-diagonal parabolic system with a singular potential which trends to infinity logarithmically when the eigenvalues of Q approach the physical boundary. We give a rigorous proof that for rather general initial data with possibly infinite free energy, the system has a unique strong solution after any positive time t_0 . Furthermore, this unique strong solution detaches from the physical boundary after a sufficiently large time T_0 . We also give an estimate of the Hausdorff measure of the set where the solution touches the physical boundary and thus prove a partial regularity result of the solution in the intermediate stage $(0, T_0)$.

Key words. gradient flow, Hausdorff dimension, parabolic PDEs

AMS subject classifications. 35K25, 35K55, 35K67

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1. Introduction. The Landau–de Gennes theory is a continuum theory of nematic liquid crystals [10]. When formulating static or dynamic continuum theories a crucial step is to select an appropriate order parameter that captures the microscopic structure of the rod-like molecule systems. In our framework the order parameter is a matrix-valued function that takes values in the following so-called Q -tensor space

$$(1.1) \quad \mathcal{Q} := \left\{ M \in \mathbb{R}^{3 \times 3} \mid \text{tr}M = 0; M = M^T \right\}.$$

It is considered as a suitably normalized second order moment of the probability distribution function that dictates locally preferred orientations of nematic molecular directors (cf. [2, 21, 22]).

To formulate the problem, let \mathbb{T}^n be the unit box/square in \mathbb{R}^n with $n = 2$ or 3. For each order parameter $Q : \mathbb{T}^n \rightarrow \mathcal{Q}$, the associated free energy functional $\mathcal{E}(Q)$ consists of the elastic and the bulk parts, which reads as

$$(1.2) \quad \mathcal{E}(Q) := \mathcal{G}(Q) + \mathcal{B}(Q) - \alpha \|Q\|_{L^2(\mathbb{T}^n)}^2.$$

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Here \mathcal{G} stands for the anisotropic elastic energy that contains three quadratic terms of ∇Q :

$$(1.3) \quad \mathcal{G}(Q) := \begin{cases} \int_{\mathbb{T}^3} (L_1 \partial_k Q_{ij} \partial_k Q_{ij} + L_2 \partial_j Q_{ik} \partial_k Q_{ij} + L_3 \partial_j Q_{ij} \partial_k Q_{ik}) dx & \text{if } Q \in H^1(\mathbb{T}^n), \\ +\infty & \text{otherwise,} \end{cases}$$

where L_1, L_2, L_3 are material dependent constants. Here and in what follows, $\partial_k Q_{ij}$ denotes the k th spatial partial derivative of the ij th component of Q , and we adopt the Einstein summation convention by summing over repeated Latin letters. Following [19], we assume

$$(1.4) \quad L_1 > 3|L_2 + L_3|,$$

which ensures that (1.3) fulfills the strong Legendre condition.

Further, $\mathcal{B}(Q)$ denotes the bulk energy

$$\mathcal{B}(Q) := \int_{\mathbb{T}^3} \psi(Q) dx,$$

where the integrand $\psi(Q)$ is the singular potential introduced in [5]:

$$(1.5) \quad \psi(Q) := \begin{cases} \inf_{\rho \in \mathcal{A}_Q} \int_{\mathbb{S}^2} \rho(p) \ln \rho(p) dp & \text{if } -\frac{1}{3} < \lambda_i(Q) < \frac{2}{3}, 1 \leq i \leq 3, \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\lambda_i(Q)$ denotes the i th eigenvalue of the matrix Q and \mathcal{A}_Q is the admissible class defined by

$$\mathcal{A}_Q = \left\{ \rho(p) : \mathbb{S}^2 \rightarrow \overline{\mathbb{R}^+} \mid \|\rho\|_{L^1(\mathbb{S}^2)} = 1; \int_{\mathbb{S}^2} \left(p \otimes p - \frac{1}{3} \mathbb{I} \right) \rho(p) dp = Q \right\}.$$

It is noted that the singular potential (1.5) imposes physical constraints on the eigenvalues of Q . Further, $\alpha > 0$ in (1.2) is a temperature dependent constant which characterizes the relative intensity of the molecular Brownian motion and the molecular interaction [5]. We refer interested readers to [2, 5, 4, 3, 15, 20] for detailed discussions of basic analytic properties of ψ , such as convexity, smoothness in its effective domain, blow-up rates near the physical boundary, etc. Meanwhile, various problems in static and dynamic configurations concerning ψ can be found in [7, 11, 13, 14, 15, 16, 27]. Specifically, the free energy in related dynamic problems considered so far in the existing literature [11, 14, 15, 27] only involves the L_1 isotropic term. Therefore, we are motivated to study the dynamic problem whose free energy contains anisotropic L_2, L_3 terms. It is worth pointing out that the presence of such terms is more than a mere technical challenge, since they make it impossible to recover any kind of maximum principle, which was crucial in [27].

This paper is concerned with a rigorous study of the gradient flow generated by $\mathcal{E}(Q)$ in the Hilbert space $L^2(\mathbb{T}^n; \mathcal{Q})$:

$$(1.6) \quad \begin{cases} \partial_t Q(t, \cdot) \in -\partial \mathcal{E}(Q(t, \cdot)), & t > 0, \\ Q(0, x) = Q_0(x), & x \in \mathbb{T}^n, \end{cases}$$

subject to periodic boundary condition

$$(1.7) \quad Q(t, x + e_i) = Q(t, x) \quad \text{for } (t, x) \in \mathbb{R}^+ \times \partial \mathbb{T}^n.$$

Here in (1.6), $\partial\mathcal{E}(Q)$ is formally the variation of the free energy (1.2). However, due to the singular feature of $\psi(Q)$, it should be understood as subdifferential (see Lemma 3.5 for more details).

Parallel to the Q -tensor space (1.1), we introduce the *physical* Q -tensor space by

$$(1.8) \quad \mathcal{Q}_{phy} := \left\{ M \in \mathcal{Q} \mid -\frac{1}{3} < \lambda_1(M) \leq \lambda_2(M) \leq \lambda_3(M) < \frac{2}{3} \right\},$$

where $\lambda_i(M)$ denotes the i th eigenvalue of the matrix M , ordered nondecreasingly. Any element in \mathcal{Q}_{phy} is called a physical Q -tensor.

Our first main result ensures the existence and uniqueness of solutions to the gradient flow (1.6) with rather general initial data.

THEOREM 1.1. *Let $n = 2$ or 3 . For any initial data*

$$(1.9) \quad Q_0 \in \overline{\{Q \in L^2(\mathbb{T}^n; \mathcal{Q}_{phy}) \mid \mathcal{E}(Q) < \infty\}}^{L^2(\mathbb{T}^n)},$$

there exists a unique global solution $Q(t, x) : \mathbb{R}^+ \times \mathbb{T}^n \rightarrow \mathcal{Q}_{phy}$ of (1.6) such that

$$(1.10) \quad \begin{aligned} \partial_t Q_{ij} &= 2L_1 \Delta Q_{ij} + 2(L_2 + L_3) \partial_j \partial_k Q_{ik} - \frac{2}{3}(L_2 + L_3) \partial_k \partial_\ell Q_{\ell k} \delta_{ij} \\ &\quad - \frac{\partial \psi}{\partial Q_{ij}} + \frac{1}{3} \text{tr} \left(\frac{\partial \psi}{\partial Q} \right) \delta_{ij} + 2\alpha Q_{ij} \end{aligned}$$

holds almost everywhere in $(0, \infty) \times \mathbb{T}^n$. And for any fixed $t_0 > 0$, the solution satisfies

$$(1.11) \quad Q \in L^\infty(t_0, \infty; H^1(\mathbb{T}^n)), \quad \partial_t Q \in L^2_{loc}(t_0, \infty; L^2(\mathbb{T}^n)),$$

and the energy dissipative equality

$$(1.12) \quad \int_{t_0}^T \left(\|\partial_t Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 + \|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)}^2 \right) dt = 2\mathcal{E}(Q(t_0)) - 2\mathcal{E}(Q(T))$$

for all $0 < t_0 < T < +\infty$. Further, $Q(t, \cdot)$ is physical in the sense that

$$(1.13) \quad Q(t, x) \in \mathcal{Q}_{phy} \quad \forall t > 0, \text{ a.e. } x \in \mathbb{T}^n.$$

It is worthy to point out that due to the energy dissipative property of the gradient flow as well as the convexity of the singular potential, for any $T > 0$ one can formally establish the a priori estimate of Q in $L^\infty_{loc}(0, T; H^1(\mathbb{T}^n)) \cap L^2_{loc}(0, T; H^2(\mathbb{T}^n))$. As a consequence, existence of weak solutions to (1.6) can be achieved by using two level approximation schemes as in [27], i.e., regularizing the initial data and the singular free energy. However, such arguments involve fairly complicated approximation procedures. Fortunately, Ambrosio, Gigli, and Savaré [1] provide a powerful framework to obtain the solution under very general assumptions of the initial data.

To establish higher regularity, namely a uniform-in-time H^2 bound of the solution Q , essential difficulties arise from the anisotropic terms. Without the $L_2 + L_3$ terms, the convexity of ψ as well as the classical $L^1 - L^\infty$ estimate of heat equation ensure the strict physicality at any positive time (see section 8 in [27] for details), and henceforth the conventional energy method applies. Concerning the gradient flow (1.6), unfortunately the anisotropic terms make such a maximum principle argument invalid. As a consequence, the proof of higher regularity of the solution becomes quite subtle in

the sense that Q might not stay inside any compact subset of \mathcal{Q}_{phy} . To overcome such a difficulty, we need to make a careful exploitation of its gradient flow structure, as well as to combine several results on the gradient flow theory given in [1] and the Gamma-convergence of gradient flows discussed in [26, 25]. These lead to the next theorem, which improves the regularity by establishing the uniform-in-time H^2 bound of the unique solution to the gradient flow (1.6). Further, it can be shown that this unique solution detaches from its physical boundary after a sufficiently large time T_0 .

THEOREM 1.2. *For any $t_0 > 0$, the solution established in Theorem 1.1 enjoys the improved regularity $Q \in L^\infty(t_0, +\infty; H^2(\mathbb{T}^n))$, and for almost every $t \geq t_0$ there holds that*

$$(1.14) \quad \|\Delta Q(t, \cdot)\|_{L^2(\mathbb{T}^n)} \leq C_L \left(e^{4\alpha} \sqrt{\mathcal{E}(Q(t_0)) - \inf \mathcal{E} + 1} + 2\alpha \|Q(t, \cdot)\|_{L^2(\mathbb{T}^n)} \right),$$

where C_L is expressed by

$$(1.15) \quad C_L := \frac{1}{2(L_1 - |L_2 + L_3|)} \sqrt{\frac{L_1 + |L_2 + L_3|}{L_1 + |L_2 + L_3| - 2\sqrt{L_1|L_2 + L_3|}}}.$$

Furthermore, under the stronger assumption

$$(1.16) \quad L_1 - 3|L_2 + L_3| - \alpha C_{\mathbb{T}^n}^2 > 0,$$

where $C_{\mathbb{T}^n} = (2\pi)^n$ is the Poincaré constant in \mathbb{T}^n , there exists $T_0 > 0$ such that the unique solution is strictly physical for all $t \geq T_0$ in the sense that

$$(1.17) \quad -\frac{1}{3} + \kappa \leq \lambda_i(Q(t, x)) \leq \frac{2}{3} - \kappa \quad \forall x \in \mathbb{T}^n$$

for some constant $\kappa \in (0, 1/6)$.

During the period $(0, T_0)$, a partial regularity result of the unique solution can be established, i.e., the Hausdorff dimension of the set where the solution touches the physical boundary $\partial\mathcal{Q}_{phy}$.

THEOREM 1.3. *Let $Q(t, x)$ be the unique strong solution of (1.6) established in Theorem 1.2. Then for a.e. $t \in (0, T_0)$, the contact set*

$$(1.18) \quad \Sigma_t := \{x \in \mathbb{T}^n \mid Q(t, x) \in \partial\mathcal{Q}_{phy}\}$$

has the following estimate:

- $\dim_{\mathcal{H}}(\Sigma_t) \leq 2$ for $n = 3$.
- $\dim_{\mathcal{H}}(\Sigma_t) = 0$ for $n = 2$.

The rest of the paper is organized as follows. Some notations and preliminaries are provided in section 2. The proofs of the three main results, namely Theorems 1.1, 1.2, and 1.3, are given in sections 3, 4, and 5, respectively.

2. Preliminaries. We start with a few basic notations in Q -tensor theory. For any $Q \in \mathcal{Q}$, $|Q| := \sqrt{\text{tr}(Q^t Q)}$ represents the Frobenius norm of Q . The gradient of the function $\psi(Q)$ will be abbreviated by $\psi'(Q)$, and its components are denoted by $\psi'_{ij}(Q) := \frac{\partial \psi(Q)}{\partial Q_{ij}}$. Moreover, we denote $L^2(\mathbb{T}^n; \mathcal{Q})$ the Hilbert space endowed with the L^2 metric

$$\|Q\|_{L^2(\mathbb{T}^n)} = \sqrt{\int_{\mathbb{T}^n} \text{tr}(Q^t Q)} = \sqrt{\int_{\mathbb{T}^n} \text{tr}(Q^2)} \quad \text{for } Q : \mathbb{T}^n \rightarrow \mathcal{Q}.$$

Here and after, for brevity, $\|\cdot\|_{L^2(\mathbb{T}^n)}$ will often be written as $\|\cdot\|_{L^2}$, or simply $\|\cdot\|$.

Next we provide some preliminaries of gradient flow theory in a Hilbert space. We start with some basic definitions in a Hilbert space H , with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ (cf. [8, 12]).

DEFINITION 2.1. *A function $f : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is called proper if f is not identically equal to $+\infty$. The effective domain of f is defined by*

$$D(f) = \{u \in H \mid f(u) < +\infty\}.$$

By [5], the effective domain $D(\psi)$ is equivalent to (1.5).

DEFINITION 2.2. *Let $\lambda \in \mathbb{R}$; then a λ -convex function $F : H \rightarrow (-\infty, +\infty]$ is a function satisfying*

$$F((1-t)u + tv) \leq (1-t)F(u) + tF(v) - \frac{\lambda}{2}t(1-t)\|u - v\|^2 \quad \forall u, v \in H.$$

For each $u \in H$, $\partial F[u]$ is defined as the set of $w \in H$ such that

$$F(u) + \langle w, v - u \rangle + \frac{\lambda}{2}\|u - v\|^2 \leq F(v) \quad \forall v \in H.$$

The mapping $\partial F : H \rightarrow 2^H$ is called the subdifferential of F . Further, we say that $u \in D(\partial F)$, the domain of ∂F , provided $\partial F[u]$ is not empty.

DEFINITION 2.3. *We say that $u(t)$ is a gradient flow of F starting from $u_0 \in H$ if it is a locally absolutely continuous curve in $(0, +\infty)$ such that*

$$(2.1) \quad \begin{cases} \partial_t u(t) \in -\partial F(u(t)), & \text{a.e. } t > 0, \\ \lim_{t \rightarrow 0^+} u(t) = u_0. \end{cases}$$

The next result is due to [1, Theorem 4.0.4], which was originally stated under the metric space setting. For the purpose of proving Theorem 1.1, it suffices to rewrite it in the Hilbert space setting.

PROPOSITION 2.4. *Let $\lambda \in \mathbb{R}$, and let $F : H \rightarrow (-\infty, +\infty]$ be a proper, λ -convex, bounded from below, and lower semicontinuous functional. Then for each $u_0 \in \overline{D(F)}$, $u(t) = \lim_{k \rightarrow +\infty} J_{t/k}^k(u_0)$ with J_τ being the resolvent*

$$(2.2) \quad X \in J_\tau(Y) \iff X \in \operatorname{argmin} \left\{ F(\cdot) + \frac{1}{2\tau} \|Y - \cdot\|^2 \right\}$$

satisfies the following:

1. *Variational inequality: u is the unique solution to the evolution variational inequality*

$$(2.3) \quad \frac{1}{2} \frac{d}{dt} \|u(t) - v\|^2 + \frac{\lambda}{2} \|u(t) - v\|^2 + F(u(t)) \leq F(v) \text{ for a.e. } t > 0 \text{ and } v \in D(F),$$

among all the locally absolutely continuous curves such that $u(t) \rightarrow u_0$ as $t \downarrow 0^+$.

2. *Regularizing effect: u is locally Lipschitz regular, and $u(t, \cdot) \in D(\mathcal{E})$ for all $t > 0$.*

Remark 2.5. It is well known that for a λ -convex function $F : H \rightarrow (-\infty, +\infty]$, a locally absolutely continuous curve $u(t)$ in $(0, +\infty)$ satisfies (2.1) if and only if it satisfies the evolution variational inequality (2.3).

Now we turn to the Γ -convergence of gradient flows in a Hilbert space, a theory developed in [25, 26]. Let $\{u_n\}$ be the solution to the gradient flow

$$(2.4) \quad \partial_t u_n = -\nabla E_n(u_n)$$

of a C^1 functional sequence $\{E_n\}$. Assume E_n Γ -converges to a functional F , and there is a general sense of convergence $u_n \xrightarrow{S} u$, relative to which the Γ -convergence of E_n to F holds. We introduce the “energy-excess” along a family of curves $u_n(t)$ with $u_n(t) \xrightarrow{S} u(t)$ by setting

$$\tilde{D}(t) = \limsup_{n \rightarrow \infty} E_n(u_n(t)) - F(u(t)).$$

The main result of [26] is the following.

PROPOSITION 2.6. *Assume E_n and F satisfy a $\Gamma - \liminf$ relation: if $u_n \xrightarrow{S} u$ as $n \rightarrow \infty$, then*

$$\liminf_{n \rightarrow \infty} E_n(u_n) \geq F(u).$$

Assume that the following two additional conditions hold:

1. *(Lower bound on the velocities) If $u_n(t) \xrightarrow{S} u(t)$ for all $t \in [0, T]$, then there exists $f \in L^1(0, T)$ such that for every $s \in [0, T]$,*

$$(2.5) \quad \liminf_{n \rightarrow \infty} \int_0^s \|\partial_t u_n(t)\|_H^2 dt \geq \int_0^s [\|\partial_t u(t)\|_H^2 - f(t)\tilde{D}(t)] dt.$$

2. *(Lower bound for the slopes) If $u_n \xrightarrow{S} u$, then*

$$(2.6) \quad \liminf_{n \rightarrow \infty} \|\nabla E_n(u_n)\|_H^2 dt \geq \|\nabla F(u)\|_H^2 - C\tilde{D},$$

where C is a universal constant, and $\|\nabla F(u)\|$ denotes the minimal norm of the elements in $\partial F(u)$.

Assume $u_n(t)$ is a family of solutions to (2.4) on $[0, T]$ with $u_n(t) \xrightarrow{S} u(t)$ for all $t \in [0, T]$, such that

$$E_n(u_n(0)) - E_n(u_n(t)) = \int_0^t \|\partial_t u_n(s)\|_H^2 ds \quad \forall t \in [0, T].$$

Assume also that

$$\lim_{n \rightarrow \infty} E_n(u_n(0)) = F(u(0));$$

then $u \in H^1(0, T; H)$ and is a solution to $\partial_t u = -\partial F(u)$ on $[0, T]$. Moreover, $\tilde{D}(t) = 0$ for all t (that is, the solutions “remain well-prepared”) and

$$\|\partial_t u_n\|_H \xrightarrow{n \rightarrow \infty} \|\partial_t u\|_H, \quad \|\nabla E_n(u_n)\|_H \xrightarrow{n \rightarrow \infty} \|\nabla F(u)\|_H \quad \text{in } L^2(0, T).$$

3. Proof of Theorem 1.1: Existence of solutions. This section is devoted to the proof of Theorem 1.1. First, with the choices $H = L^2(\mathbb{T}^n; \mathcal{Q})$ and $F = \mathcal{E}$ (defined in (1.2)), we show that the assumptions in Proposition 2.4 are satisfied, so that there exists a unique solution $Q(t, \cdot)$ in variational inequality setting (2.3) (see Proposition 3.4 below). Moreover, since the free energy is -2α convex, the solution achieved in Proposition 3.4 is equivalent to the solution of the gradient flow (1.6) in the subdifferential setting. As a consequence, we compute explicitly the subdifferential of \mathcal{E} and obtain a unique strong solution to (1.10). Finally, we apply two theorems in [1] to show further regularity properties of Q in Theorem 1.1. Since all of the following arguments are valid for both \mathbb{T}^3 and \mathbb{T}^2 with minor modifications, for brevity we discuss the case of \mathbb{T}^3 only.

In this subsection we consider the following settings:

$$(H, \|\cdot\|) = L^2(\mathbb{T}^3; \mathcal{Q}), \quad F = \mathcal{E}(Q).$$

To begin with, we need to verify that all assumptions in Proposition 2.4 are valid, which is given in the following two lemmas.

LEMMA 3.1. *The free energy functional \mathcal{E} is proper, bounded from below, -2α convex, and lower semicontinuous in $L^2(\mathbb{T}^3; \mathcal{Q})$.*

Proof. First we show that the elastic energy \mathcal{G} is nonnegative, convex, and lower semicontinuous in $L^2(\mathbb{T}^3; \mathcal{Q})$. It is proved in [19] that when $L_1 > 0, L_1 + L_2 + L_3 > 0$, \mathcal{G} satisfies the strong Legendre condition, which implies the convexity of \mathcal{G} . It suffices to show that \mathcal{G} is nonnegative when $Q \in H^1(\mathbb{T}^3)$, which follows from the coefficient assumption (1.4), integration by parts, and the Cauchy–Schwarz inequality:

$$\begin{aligned} \mathcal{G}(Q) &\geq \int_{\mathbb{T}^3} \left(L_1 \partial_k Q_{ij} \partial_k Q_{ij} + L_2 \partial_k Q_{ik} \partial_k Q_{ij} + L_3 \partial_j Q_{ij} \partial_k Q_{ik} \right) dx \\ &= \int_{\mathbb{T}^3} \left[L_1 \partial_k Q_{ij} \partial_k Q_{ij} + (L_2 + L_3) \partial_j Q_{ik} \partial_k Q_{ij} \right] dx \\ (3.1) \quad &\geq \int_{\mathbb{T}^3} (L_1 - 3|L_2 + L_3|) |\nabla Q|^2 dx \geq 0. \end{aligned}$$

Besides, since \mathcal{G} is convex and quadratic, it is lower semicontinuous [17, Theorem 8.1].

Next we show that the functional \mathcal{B} is convex, bounded from below, and lower semicontinuous in $L^2(\mathbb{T}^3; \mathcal{Q})$. The convexity of \mathcal{B} follows from [5, 15]. A lower bounded can be derived from the inequality $x \ln x \geq -1/e$ for any $x \geq 0$:

$$(3.2) \quad \mathcal{B} = \int_{\mathbb{T}^3} \psi(Q) dx \geq -4\pi^2 |\mathbb{T}^3|/e.$$

To show the lower semicontinuity of \mathcal{B} , let $Q_n \rightarrow Q$ strongly in $L^2(\mathbb{T}^3)$. If $\liminf_{n \rightarrow \infty} \psi(Q_n) = +\infty$ on a set of positive measure, then the proof is done. Thus upon subsequence we assume

$$(3.3) \quad \liminf_{n \rightarrow \infty} \mathcal{B}(Q_n) = \lim_{n \rightarrow \infty} \mathcal{B}(Q_n) < +\infty, \quad \text{and} \quad Q_n(x) \xrightarrow{n \rightarrow \infty} Q(x) \quad \text{for a.e. } x \in \mathbb{T}^3.$$

Consequently, for all $n \in \mathbb{N}$ sufficiently large and a.e. $x \in \mathbb{T}^3$, all eigenvalues of $Q_n(x)$ are in $(-1/3, 2/3)$. Moreover, the eigenvalues of $Q(x)$ are in $[-1/3, 2/3]$ since convergence of eigenvalues follows from convergence of the matrices (cf. [24]).

We claim that for a.e. $x \in \mathbb{T}^3$ the eigenvalues of $Q(x)$ are in $(-1/3, 2/3)$. To this aim, we argue by contradiction. Assume the opposite, i.e., $E = \{x \in \mathbb{T}^3, \lambda_1(Q(x)) =$

$-1/3\}$ has positive measure. Then it follows from [5] that $\psi(Q_n(x)) \xrightarrow{n \rightarrow \infty} +\infty$ in E , and henceforth Fatou's lemma implies

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathcal{B}(Q_n) &\geq \liminf_{n \rightarrow \infty} \int_E \psi(Q_n) \, dx + \liminf_{n \rightarrow \infty} \int_{\mathbb{T}^3 \setminus E} \psi(Q_n) \, dx \\ &\geq \int_E \liminf_{n \rightarrow \infty} \psi(Q_n) \, dx - \frac{4\pi^2}{e} |\mathbb{T}^3 \setminus E| = +\infty, \end{aligned}$$

which contradicts (3.3). Thus the claim is proved. Since ψ is smooth in $D(\psi) = \mathcal{Q}_{phy}$ (see [15]), we have $\psi(Q_n(x)) \xrightarrow{n \rightarrow \infty} \psi(Q(x))$ for a.e. $x \in \mathbb{T}^3$. Thus Fatou's lemma implies

$$\liminf_{n \rightarrow \infty} \mathcal{B}(Q_n) \geq \int_{\mathbb{T}^3} \liminf_{n \rightarrow \infty} \psi(Q_n) \, dx = \int_{\mathbb{T}^3} \psi(Q(x)) \, dx = \mathcal{B}(Q).$$

To sum up, \mathcal{E} is -2α convex and lower semicontinuous in $L^2(\mathbb{T}^3; \mathcal{Q})$. It remains to show that \mathcal{E} is proper and bounded from below. Clearly, $\mathcal{E}(Q) < +\infty$ provided $Q \in H^1(\mathbb{T}^3)$ and $Q(x) \in D(\psi)$ a.e. $x \in \mathbb{T}^3$, and hence \mathcal{E} is proper. Further, if Q is not physical, then $\mathcal{E}(Q) = +\infty$, while if Q is physical, then $\|Q\|_{L^2(\mathbb{T}^3)}$ is bounded, and hence \mathcal{E} is bounded from below since both \mathcal{G} and ψ are bounded from below. \square

Remark 3.2. It is noted that the coefficient assumption (1.4) is different from the one in [9, 18], which ensures that the elastic energy \mathcal{G} is coercive only.

LEMMA 3.3. *For any $R, P_0, P_1 \in D(\mathcal{E})$, denote $\gamma_t = (1-t)P_0 + tP_1$, $t \in [0, 1]$; then for each $0 < \tau < 1/2\alpha$ the functional*

$$(3.4) \quad \Phi(\tau, R; \gamma_t) := \frac{\|\gamma_t - R\|^2}{2\tau} + \mathcal{E}(\gamma_t)$$

is $(1/\tau - 2\alpha)$ -convex on γ_t in the sense that

$$(3.5) \quad \begin{aligned} \Phi(\tau, R; \gamma_t) &\leq (1-t)\Phi(\tau, R; P_0) + t\Phi(\tau, R; P_1) - \frac{(1-2\alpha\tau)}{2\tau} t(1-t)\|P_1 - P_0\|^2 \quad \forall t \in [0, 1]. \end{aligned}$$

Proof. We infer from the convexity of \mathcal{G} and \mathcal{B} that

$$\begin{aligned} \Phi(\tau, R; \gamma_t) &= \frac{\|(1-t)P_0 + tP_1 - R\|^2}{2\tau} + (\mathcal{G} + \mathcal{B})((1-t)P_0 + tP_1) - \alpha\|(1-t)P_0 + tP_1\|^2 \\ &\leq \frac{(1-t)\|P_0 - R\|^2 + t\|P_1 - R\|^2 - t(1-t)\|P_0 - P_1\|^2}{2\tau} + (1-t)(\mathcal{G} + \mathcal{B})(P_0) \\ &\quad + t(\mathcal{G} + \mathcal{B})(P_1) - \alpha[(1-t)\|P_0\|^2 + t\|P_1\|^2 - t(1-t)\|P_0 - P_1\|^2] \\ &= (1-t)\Phi(\tau, R; P_0) + t\Phi(\tau, R; P_1) - \frac{(1-2\alpha\tau)}{2\tau} t(1-t)\|P_1 - P_0\|^2. \quad \square \end{aligned}$$

To sum up, we manage to verify that all assumptions of Proposition 2.4 are satisfied, which leads to the following proposition.

PROPOSITION 3.4. *Let $n = 3$. For any initial data,*

$$(3.6) \quad Q_0 \in \overline{D(\mathcal{E})} := \overline{\{Q \in L^2(\mathbb{T}^3; \mathcal{Q}_{phy}) \mid \mathcal{E}(Q) < \infty\}}^{L^2(\mathbb{T}^3)}.$$

Let $Q(t) = \lim_{k \rightarrow +\infty} J_{t/k}^k(Q_0)$ with J being the resolvent

$$X \in J_\tau(Y) \iff X \in \operatorname{argmin} \left\{ F(\cdot) + \frac{1}{2\tau} \|Y - \cdot\|^2 \right\}.$$

Then we have the following:

1. *Variational inequality:* Q is the unique solution to the evolution variational inequality

$$(3.7) \quad \frac{d}{dt} \|Q(t, \cdot) - P\|_{L^2(\mathbb{T}^3)}^2 - \alpha \|Q(t, \cdot) - P\|_{L^2(\mathbb{T}^3)}^2 + \mathcal{E}(Q(t, \cdot)) \leq \mathcal{E}(P)$$

for a.e. $t > 0$ and $P \in D(\mathcal{E})$,

among all locally absolutely continuous curves such that $Q(t, \cdot) \rightarrow Q_0$ as $t \downarrow 0^+$.

2. *Regularizing effect:* Q is locally Lipschitz, and $Q(t, \cdot) \in D(\mathcal{E})$ for all $t > 0$. In particular, Q is physical in the sense that

$$(3.8) \quad -\frac{1}{3} < \lambda_i(Q(t, x)) < +\frac{2}{3} \quad \forall t > 0, \text{ a.e. } x \in \mathbb{T}^3.$$

To proceed, note that \mathcal{E} is -2α convex in $L^2(\mathbb{T}^3; \mathcal{Q})$, and hence by Remark 2.5 we know that $Q(t, \cdot)$ constructed in Proposition 3.4 is the unique solution to the gradient flow (1.6). The following lemma computes explicitly the subdifferential of the free energy \mathcal{E} .

LEMMA 3.5. *For any $Q \in D(\partial\mathcal{E})$ and $1 \leq i, j \leq 3$, we have*

$$\begin{aligned} -\partial\mathcal{E}(Q)_{ij} &= 2L_1\Delta Q_{ij} + 2(L_2 + L_3)\partial_k Q_{ik} - \frac{2(L_2 + L_3)}{3}\partial_{k\ell} Q_{k\ell}\delta_{ij} \\ &\quad - \psi'(Q)_{ij} + \frac{\operatorname{tr}(\psi'(Q))}{3}\delta_{ij} + 2\alpha Q_{ij}. \end{aligned}$$

Proof. To begin with, it is immediate to derive

$$(3.9) \quad \partial\mathcal{G}(Q)_{ij} = -2L_1\Delta Q_{ij} - 2(L_2 + L_3)\partial_k\partial_j Q_{ik} + \frac{2}{3}(L_2 + L_3)\partial_\ell\partial_k Q_{k\ell}\delta_{ij}.$$

Next we need to verify that

$$\partial\mathcal{B}(Q) = \left\{ \psi'(Q) - \frac{1}{3}\operatorname{tr}(\psi'(Q))\mathbb{I}_3 \right\}.$$

Case 1: Q is strictly physical.

Let $R \in C_c^\infty(\mathbb{T}^3, \mathcal{Q})$. Then by convexity and smoothness of ψ , any element $\xi \in \partial\mathcal{B}(Q)$ satisfies

$$\begin{aligned} \int_{\mathbb{T}^3} [\psi'(Q) : R] dx &= \lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{B}(Q + \varepsilon R) - \mathcal{B}(Q)}{\varepsilon} \geq \langle \xi, R \rangle_{L^2(\mathbb{T}^3)}, \\ - \int_{\mathbb{T}^3} [\psi'(Q) : R] dx &= \lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{B}(Q - \varepsilon R) - \mathcal{B}(Q)}{\varepsilon} \geq -\langle \xi, R \rangle_{L^2(\mathbb{T}^3)}, \end{aligned}$$

which indicates $\langle \xi, R \rangle_{L^2(\mathbb{T}^3)} = \langle \psi'(Q), R \rangle_{L^2(\mathbb{T}^3)}$. By density,

$$(3.10) \quad \xi = \psi'(Q) - \frac{\operatorname{tr}(\psi'(Q))}{3}\mathbb{I}_3$$

as an element in the Hilbert space $L^2(\mathbb{T}^n; \mathcal{Q})$. Hence $\partial\mathcal{B}(Q) = \{\psi'(Q) - \text{tr}(\psi'(Q))\mathbb{I}_3/3\}$ for any uniformly physical Q .

Case 2: Q is not strictly physical.

We define

$$(3.11) \quad \rho(P) := \min_{1 \leq i \leq 3} \left\{ \lambda_i(P) + \frac{1}{3}, \frac{2}{3} - \lambda_i(P) \right\} \quad \forall P \in \mathcal{Q},$$

and $A_\eta := \{x \in \mathbb{T}^3 : \rho(Q(x)) < \eta\}$ for arbitrarily small $\eta > 0$. Since $Q \in D(\partial\mathcal{E}) \subset D(\mathcal{E})$, we have $\psi(Q) < +\infty$, and thus $|A_\eta| \rightarrow 0$ as $\eta \rightarrow 0^+$. Let

$$T_\eta(Q) := \{R \in L^\infty(\mathbb{T}^3; \mathcal{Q}) : R \equiv 0 \text{ on } A_\eta\}.$$

Let us consider any fixed $\eta > 0$, and let $R \in T_\eta(Q)$. Then it is easy to check that $\rho(Q) \geq \eta$ outside A_η , and for all sufficiently small ε we have $\rho(Q \pm \varepsilon R) \geq \eta/2$ outside A_η ; hence

$$\mathcal{B}(Q + \varepsilon R) = \int_{\mathbb{T}^3 \setminus A_\eta} \psi(Q + \varepsilon R) \, dx + \int_{A_\eta} \psi(Q) \, dx < +\infty.$$

This together with the convexity of ψ implies that for any sufficiently small $\varepsilon > 0$,

$$\begin{aligned} & \varepsilon \int_{\mathbb{T}^3 \setminus A_\eta} [\psi'(Q + \varepsilon R) : R] \, dx \geq \mathcal{B}(Q + \varepsilon R) - \mathcal{B}(Q) \\ &= \int_{\mathbb{T}^3 \setminus A_\eta} [\psi(Q + \varepsilon R) - \psi(Q)] \, dx \geq \varepsilon \int_{\mathbb{T}^3 \setminus A_\eta} [\psi'(Q) : R] \, dx, \end{aligned}$$

and dividing by ε gives

$$(3.12) \quad \liminf_{\varepsilon \rightarrow 0^+} \frac{\mathcal{B}(Q + \varepsilon R) - \mathcal{B}(Q)}{\varepsilon} \geq \int_{\mathbb{T}^3 \setminus A_\eta} [\psi'(Q) : R] \, dx,$$

$$(3.13) \quad \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathcal{B}(Q + \varepsilon R) - \mathcal{B}(Q)}{\varepsilon} \leq \limsup_{\varepsilon \rightarrow 0^+} \int_{\mathbb{T}^3 \setminus A_\eta} [\psi'(Q + \varepsilon R) : R] \, dx.$$

Meanwhile, note that

$$\begin{aligned} \psi' \in L^\infty(\mathbb{T}^3 \setminus A_\eta), \quad & \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{T}^3 \setminus A_\eta} [\psi'(Q + \varepsilon R) : R] \, dx = \int_{\mathbb{T}^3 \setminus A_\eta} [\psi'(Q) : R] \, dx \\ &= \int_{\mathbb{T}^3} [\psi'(Q) : R] \, dx \end{aligned}$$

due to the fact that $R \equiv 0$ on A_η . This together with (3.12), (3.13) yields

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{B}(Q + \varepsilon R) - \mathcal{B}(Q)}{\varepsilon} = \int_{\mathbb{T}^3} [\psi'(Q) : R] \, dx.$$

Further, as discussed in Case 1, for any $R \in \bigcup_{\eta > 0} T_\eta(Q)$ we have

$$\langle \xi, R \rangle_{L^2(\mathbb{T}^3)} = \langle \psi'(Q), R \rangle_{L^2(\mathbb{T}^3)}.$$

By density, it follows that $\xi = \psi'(Q) - \text{tr}(\psi'(Q))\mathbb{I}_3/3$ as elements of the Hilbert space $L^2(\mathbb{T}^3; \mathcal{Q})$. Hence $\partial\mathcal{B}(Q) = \{\psi'(Q) - \text{tr}(\psi'(Q))\mathbb{I}_3/3\}$ even if $Q \subset D(\partial\mathcal{E})$ is not strictly physical.

By Lemma 3.1, \mathcal{G} and \mathcal{B} are proper, convex, and lower semicontinuous, and the intersection $D(\mathcal{G}) \cap \text{int}D(\mathcal{B})$ is nonempty, so we get $\partial(\mathcal{G} + \mathcal{B})(Q) = \partial\mathcal{G}(Q) + \partial\mathcal{B}(Q)$ by [6, Theorem 2.10]. Since the last term $-\alpha\|Q\|_{L^2(\mathbb{T}^3)}^2$ is a C^1 perturbation of the energy, we infer

$$(3.14) \quad \partial\mathcal{E}(Q) = \partial(\mathcal{G} + \mathcal{B})(Q) - 2\alpha Q = \partial\mathcal{G}(Q) + \partial\mathcal{B}(Q) - 2\alpha Q,$$

which concludes the proof. \square

Proposition 3.4 leads to higher regularity of the solution Q . Since the energy \mathcal{E} is -2α -convex, and since the solution $Q(t, \cdot)$ satisfies $Q(t, \cdot) \in D(\mathcal{E})$ for all time $t > 0$, we conclude that for any $t \geq t_0 > 0$ the function $Q(t, \cdot)$ is the gradient flow of \mathcal{E} in $L^2(\mathbb{T}^n; \mathcal{Q})$ with initial datum $Q(t_0, \cdot) \in D(\mathcal{E})$. Thus we can apply [1, Theorem 2.4.15] to obtain the following proposition.

PROPOSITION 3.6. *Let $Q(t, \cdot)$ be the solution given by Theorem 1.1. Then the map*

$$t \mapsto e^{-2\alpha t} \|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)}$$

is nonincreasing and right continuous on $[t_0, +\infty)$ for all $t_0 > 0$.

Finally, combining [1, Corollary 2.4.11] and [1, Theorem 2.3.3] we obtain the energy identity (1.12).

PROPOSITION 3.7. *The solution $Q(t, \cdot)$ given by Theorem 1.1 satisfies the energy equality*

$$\int_{t_0}^T \frac{1}{2} \left(\|\partial_t Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 + \frac{1}{2} \|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)}^2 \right) dt = \mathcal{E}(Q(t_0)) - \mathcal{E}(Q(T))$$

for all $0 < t_0 < T < +\infty$.

In conclusion, the proof of Theorem 1.1 is complete.

Remark 3.8. The nonincreasing property of $e^{-2\alpha t} \|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)}$ in Proposition 3.6 will play an essential role in the proof of Theorem 1.2, which is the main reason that the Ambrosio–Gigli–Savaré gradient flow theory in [1] is adopted in this section rather than the classical Brezis–Pazy theory.

4. Proof of Theorem 1.2: Higher regularity of solutions. This section is devoted to the proof of Theorem 1.2. Since there is only minor modification of arguments between \mathbb{T}^2 and \mathbb{T}^3 , we only discuss the case in \mathbb{T}^3 .

It is noted that the maximum principle argument utilized in [27] fails due to the presence of the anisotropic terms, and hence the solution $Q(t, \cdot)$ is not ensured to stay detached from the physical boundary $\partial\mathcal{Q}$ at any positive time t . To achieve the proof, we have to put together several results in [1, 25, 26] and to make a full exploitation of the gradient flow structure in (1.6).

Our main strategy is as follows. First, to avoid the singular feature of $\partial\mathcal{E}(Q)$ in (1.6), we shall consider a sequence of smooth gradient flows (4.19) that are generated by an approximation sequence $\{\mathcal{E}_n\}$ defined in (4.6) of the free energy \mathcal{E} . Second, we will prove Γ -convergence of $\{\mathcal{E}_n\}$ to \mathcal{E} in Proposition 4.2, which together with energy dissipative equality achieved in Proposition 3.7 can show the “convergence” of the gradient flow sequence (4.19) to (1.6). Next we will show in Proposition 4.6 that the solution sequence $\{Q_n\}$ to the gradient flow sequence (4.19) is in $H^2(\mathbb{T}^3; \mathcal{Q})$ space, and give a corresponding estimate of the H^2 bound of Q in terms of $\|\partial\mathcal{E}(Q)\|$,

which together with Proposition 3.6 leads to the uniform-in-time bound for $\|\partial\mathcal{E}(Q)\|$. Finally, we make use of the convexity of ψ and Sobolev interpolation inequalities to derive strict physicality of the solution at all large times.

Here and after, we denote $\{\psi_n\}$ the sequence of functions that is used in [27] to approximate the Ball-Majumdar bulk potential ψ : first, we introduce the Moreau-Yosida approximations

$$(4.1) \quad \tilde{\psi}_n(Q) := \inf_{A \in \mathcal{Q}} \{n|A - Q|^2 + \psi(A)\}, \quad Q \in \mathcal{Q}.$$

Then using a smooth regularization we define

$$(4.2) \quad \psi_n(Q) = n^5 \int_{\mathcal{Q}} \tilde{\psi}_n(n(Q - R)) \phi(R) dR, \quad Q \in \mathcal{Q}.$$

Here $\phi \in C_c^\infty(\mathcal{Q}, \overline{R^+})$ is of unit mass. Let us recall [27, Proposition 3.1]; for each $n \geq 1$, we have the following:

- (M0) ψ_n is an isotropic function of Q .
- (M1) ψ_n is both smooth and convex in \mathcal{Q} .
- (M2) ψ_n is bounded from below, i.e., $-4\pi^2|\mathbb{T}^3|/e \leq \psi_n(R)$ for all $R \in \mathcal{Q}$, for all $n \geq 1$.
- (M3) $\psi_n \leq \psi_{n+1} \leq \psi$ on \mathcal{Q} for $n \geq 1$.
- (M4) $\psi_n \rightarrow \psi$ in $L_{loc}^\infty(D(\psi))$ as $n \rightarrow \infty$, and ψ_n is uniformly divergent on $\mathcal{Q} \setminus D(\psi)$.
- (M5) $\frac{\partial \psi_n}{\partial Q} \rightarrow \frac{\partial \psi}{\partial Q}$ in $L_{loc}^\infty(D(\psi))$ as $n \rightarrow \infty$.
- (M6) There exist constants $\lambda_n, \Lambda_n > 0$ that may depend on n , such that

$$\lambda_n|R| - \Lambda_n \leq \left| \psi'_n(R) - \frac{1}{3} \text{tr}(\psi'_n(R)) \mathbb{I}_3 \right| \leq \lambda_n|R| + \Lambda_n \quad \forall R \in \mathcal{Q}.$$

Besides the aforementioned properties (M0)–(M6), we need to further derive the following finer estimate of the sequence $\{\psi_n\}$, in order to prove Proposition 4.2 and Lemma 4.3.

LEMMA 4.1. *For any $n \in \mathbb{N}$, there exists a generic constant $C_n > 0$ such that*

$$(4.3) \quad \psi_n(Q) \geq C_n|Q|^2, \quad \text{outside a fixed compact subset in } \mathcal{Q}.$$

Moreover, $C_n \rightarrow +\infty$ as $n \rightarrow +\infty$.

Proof. Since the null matrix $0 \in D(\psi)$, taking $A = 0$ in (4.1) we get the upper bound

$$\tilde{\psi}_n(Q) \leq n|Q|^2 + \psi(0).$$

On the other hand, since such an infimum in (4.1) is finite, there exists a minimizing sequence $A_m \subset D(\psi)$ such that

$$(4.4) \quad \begin{aligned} \tilde{\psi}_n(Q) &= \lim_{m \rightarrow +\infty} (n|A_m - Q|^2 + \psi(A_m)) \\ &\geq \lim_{m \rightarrow +\infty} n|A_m - Q|^2 + \inf \psi \geq n \text{dist}(Q, D(\psi))^2 + \inf \psi. \end{aligned}$$

By the triangle inequality, we have

$$|Q| \leq \text{dist}(Q, D(\psi)) + \text{dist}(0, D(\psi)),$$

and since $0 \in D(\psi)$, we can further get

$$\text{dist}(Q, D(\psi)) \geq |Q| - \text{diam}(D(\psi)).$$

As a consequence, we see that

$$\tilde{\psi}_n(Q) \geq n|Q|^2 - 2n \text{diam}(D(\psi))|Q| + \text{diam}(D(\psi))^2 + \inf \psi \geq n|Q|(|Q| - 2\text{diam}(D(\psi))).$$

Hence we arrive at the uniform quadratic estimate

$$(4.5) \quad \frac{n}{2}|Q|^2 \leq \tilde{\psi}_n(Q)$$

in $\{|Q| > 4\text{diam}(D(\psi))\}$. Then following the mollification of $\tilde{\psi}_n$ in (4.2) one can get (4.3). \square

We introduce the energy sequence $\mathcal{E}_n : \mathcal{Q} \rightarrow \mathbb{R} \cup \{+\infty\}$,

$$(4.6) \quad \mathcal{E}_n(Q) = \begin{cases} \mathcal{G}(Q) + \int_{\mathbb{T}^3} \psi_n(Q) \, dx - \alpha \|Q\|_{L^2(\mathbb{T}^3)}^2 & \text{if } Q \in H^1(\mathbb{T}^3) \\ +\infty & \text{otherwise,} \end{cases}$$

and establish a Γ -convergence result.

PROPOSITION 4.2. *The sequence of energies $\{\mathcal{E}_n\}$ Γ -converges to \mathcal{E} .*

Proof. We first show compactness. Let us assume $\liminf_{n \rightarrow +\infty} \mathcal{E}_n(Q_n) < +\infty$. Upon subsequence, we may assume

$$(4.7) \quad \liminf_{n \rightarrow +\infty} \mathcal{E}_n(Q_n) = \lim_{n \rightarrow +\infty} \mathcal{E}_n(Q_n) < +\infty, \quad \sup_{n \in \mathbb{N}} \mathcal{E}_n(Q_n) < +\infty.$$

We need first to ensure the existence of a strong limit Q . We claim that Q_n is uniformly bounded in $L^2(\mathbb{T}^3)$. Otherwise there exists a subsequence $\{Q_{n_k}\}$, such that $\|Q_{n_k}\|_{L^2(\mathbb{T}^3)} \rightarrow +\infty$. It then follows directly from Lemma 3.1 and (4.3) that $\mathcal{E}_{n_k}(Q_{n_k}) \rightarrow +\infty$, which contradicts the assumption (4.7).

Further, note that

$$\begin{aligned} & (L_1 - 3|L_2 + L_3|) \sup_n \|\nabla Q_n\|_{L^2(\mathbb{T}^3)}^2 \\ & \leq \sup_n \mathcal{G}(Q_n) \leq \sup_n \mathcal{E}_n(Q_n) + \alpha \sup_n \|Q_n\|_{L^2(\mathbb{T}^3)}^2 + \inf_{n,P} \int_{\mathbb{T}^3} |\psi_n(P)| \, dx < +\infty. \end{aligned}$$

Thus Q_n is uniformly bounded in $H^1(\mathbb{T}^3)$, and $Q_n \rightarrow Q$ (up to a subsequence) strongly in $L^2(\mathbb{T}^3)$.

Next we show Γ -lim sup inequality. That is, for any $Q \in \mathcal{Q}$ there exists a recovery sequence Q_n such that

$$(4.8) \quad \limsup_{n \rightarrow +\infty} \mathcal{E}_n(Q_n) \leq \mathcal{E}(Q).$$

Without loss of generality we assume $\mathcal{E}(Q) < +\infty$. Taking $Q_n = Q$, for every $n \geq 1$ we get from (M3) that

$$\mathcal{G}(Q_n) - \alpha \|Q_n\|_{L^2(\mathbb{T}^3)}^2 \equiv \mathcal{G}(Q) - \alpha \|Q\|_{L^2(\mathbb{T}^3)}^2, \quad \int_{\mathbb{T}^3} \psi_n(Q_n) \, dx \leq \int_{\mathbb{T}^3} \psi(Q) \, dx \quad \forall n \geq 1.$$

In all, (4.8) is verified.

We proceed to show Γ -liminf inequality. That is, for any sequence $Q_n \rightarrow Q$ strongly in $L^2(\mathbb{T}^3)$, it holds that

$$(4.9) \quad \liminf_{n \rightarrow +\infty} \mathcal{E}_n(Q_n) \geq \mathcal{E}(Q).$$

Without loss of generality we assume $\liminf_{n \rightarrow +\infty} \mathcal{E}_n(Q_n) < +\infty$. Again upon subsequence, we may assume

$$\liminf_{n \rightarrow +\infty} \mathcal{E}_n(Q_n) = \lim_{n \rightarrow +\infty} \mathcal{E}_n(Q_n) < +\infty, \quad \sup_{n \in \mathbb{N}} \mathcal{E}_n(Q_n) < +\infty.$$

As discussed earlier we have $Q_n \rightarrow Q$ strongly in $L^2(\mathbb{T}^3)$, and upon further extracting a subsequence we may assume $Q_n \rightarrow Q$ a.e. in \mathbb{T}^3 . Hence together with the lower semicontinuity of \mathcal{G} achieved in Lemma 3.1 it yields

$$(4.10) \quad \lim_{n \rightarrow +\infty} \|Q_n(\cdot)\|_{L^2(\mathbb{T}^3)}^2 = \|Q(\cdot)\|_{L^2(\mathbb{T}^3)}^2, \quad \liminf_{n \rightarrow +\infty} \mathcal{G}(Q_n) \geq \mathcal{G}(Q).$$

It remains to prove

$$(4.11) \quad \liminf_{n \rightarrow +\infty} \int_{\mathbb{T}^3} \psi_n(Q_n) \, dx \geq \int_{\mathbb{T}^3} \psi(Q) \, dx.$$

To proceed, we denote

$$D := \{x \in \mathbb{T}^3 : Q(x) \in D(\psi)\},$$

and we distinguish between two cases.

Case 1: $\mathcal{B}(Q) < +\infty$. For any sufficiently small $\varepsilon > 0$, let us define

$$(4.12) \quad \mathbb{T}_\varepsilon^3 = D \cap \left\{ x \in \mathbb{T}^3 : -\frac{1}{3} + \varepsilon \leq \lambda_i(Q(x)) \leq \frac{2}{3} - \varepsilon, 1 \leq i \leq 3 \right\}.$$

Since $|\mathbb{T}^3 \setminus D| = 0$, we get

$$(4.13) \quad \begin{aligned} \int_{\mathbb{T}^3} [\psi_n(Q_n) - \psi(Q)] \, dx &= \int_D [\psi_n(Q_n) - \psi(Q)] \, dx \\ &= \int_{D \setminus \mathbb{T}_\varepsilon^3} [\psi_n(Q_n) - \psi(Q)] \, dx + \int_{\mathbb{T}_\varepsilon^3} [\psi_n(Q_n) - \psi(Q)] \, dx. \end{aligned}$$

On the one hand, for any $n \geq 1$,

$$\int_{D \setminus \mathbb{T}_\varepsilon^3} \psi_n(Q_n) \, dx \geq \lim_{\varepsilon \rightarrow 0} (\inf_{n, P} \psi_n(P)) |D \setminus \mathbb{T}_\varepsilon^3| = 0,$$

and since $\psi \in L^1(\mathbb{T}^3)$, as $|D \setminus \mathbb{T}_\varepsilon^3| \rightarrow 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{D \setminus \mathbb{T}_\varepsilon^3} \psi(Q) \, dx = 0.$$

Hence for all $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$, such that

$$(4.14) \quad \int_{D \setminus \mathbb{T}_\varepsilon^3} [\psi_n(Q_n) - \psi(Q)] \, dx > -\frac{\delta}{2} \quad \forall n \geq 1 \quad \text{whenever } \varepsilon \leq \varepsilon_0.$$

On the other hand, it follows from **(M5)** that $\psi_n(Q(x)) \rightarrow \psi(Q(x))$, $\psi'_n(Q(x)) \rightarrow \psi'(Q(x))$ as $n \rightarrow +\infty$ in $L^\infty(\mathbb{T}_\varepsilon^3)$. Thus it implies that for any fixed $\varepsilon \leq \varepsilon_0$ it holds that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}_\varepsilon^3} [\psi_n(Q_n) - \psi(Q)] \, dx = 0.$$

Specifically, for $\varepsilon = \varepsilon_0$ there exists $N_0 \in \mathbb{N}$, such that

$$(4.15) \quad \int_{\mathbb{T}_{\varepsilon_0}^3} [\psi_n(Q_n) - \psi(Q)] \, dx > -\frac{\delta}{2} \quad \forall n \geq N_0.$$

Combining (4.14) and (4.15) we finish the proof of (4.11) by the arbitrariness of δ .

Case 2: $\mathcal{B}(Q) = +\infty$. In this case it suffices to check

$$(4.16) \quad \liminf_{n \rightarrow \infty} \int_{\mathbb{T}^3} \psi_n(Q_n) \, dx = +\infty.$$

If $|\mathbb{T}^3 \setminus D| > 0$, since $Q_n \rightarrow Q$ a.e., it follows from Egorov's theorem that there exists a set $F \subset (\mathbb{T}^3 \setminus D)$, $|F| > 0$, such that $Q_n \rightarrow Q$ uniformly on F . Note that $Q(x) \in \mathcal{Q} \setminus D(\psi)$ for all $x \in F$. Hence the uniform convergence of Q_n to Q on F implies there exists a sequence $\varepsilon_n \searrow 0^+$, such that

$$\lambda_i(Q_n(x)) \leq -\frac{1}{3} + \varepsilon_n \text{ or } \lambda_i(Q_n(x)) \geq \frac{2}{3} - \varepsilon_n \quad \forall 1 \leq i \leq 3.$$

Then Fatou's lemma and **(M4)** yield

$$\liminf_{n \rightarrow +\infty} \int_F \psi_n(Q_n) \, dx \geq \int_F \liminf_{n \rightarrow +\infty} \psi_n(Q_n) \, dx = +\infty.$$

Therefore, (4.16) is verified in this case.

Alternatively, if $|\mathbb{T}^3 \setminus D| = 0$, then using an argument similar to that in Case 1 we have

$$\int_{\mathbb{T}^3} \psi_n(Q_n) \, dx = \int_D \psi_n(Q_n) \, dx = \int_{D \setminus \mathbb{T}_\varepsilon^3} \psi_n(Q_n) \, dx + \int_{\mathbb{T}_\varepsilon^3} \psi_n(Q_n) \, dx,$$

where \mathbb{T}_ε^3 is given in (4.12). On one hand,

$$(4.17) \quad \int_{D \setminus \mathbb{T}_\varepsilon^3} \psi_n(Q_n) \, dx \geq \left(\inf_{n, P} \psi_n(P) \right) |\mathbb{T}^3 \setminus \mathbb{T}_\varepsilon^3| \geq -\frac{4\pi|\mathbb{T}^3|^2}{e} \quad \forall \varepsilon > 0.$$

On the other hand, since $\mathcal{B}(Q) = +\infty$ and $\mathbb{T}_\varepsilon^3 \nearrow D$, we infer that for all $M > 0$, there exists $\varepsilon_0 > 0$, such that

$$\int_{\mathbb{T}_\varepsilon^3} \psi(Q) \, dx > M + 1 + \frac{4\pi|\mathbb{T}^3|^2}{e} \quad \forall \varepsilon \leq \varepsilon_0.$$

Meanwhile, **(M5)** indicates that $\psi_n(Q(x)) \rightarrow \psi(Q(x))$, $\psi'_n(Q(x)) \rightarrow \psi'(Q(x))$ in $L^\infty(\mathbb{T}_{\varepsilon_0}^3)$, which gives

$$\int_{\mathbb{T}_{\varepsilon_0}^3} \psi_n(Q_n) \, dx \rightarrow \int_{\mathbb{T}_{\varepsilon_0}^3} \psi(Q) \, dx.$$

Thus there exists $N \in \mathbb{N}$, such that

$$(4.18) \quad \int_{\mathbb{T}_{\varepsilon_0}^3} \psi_n(Q_n) dx > \int_{\mathbb{T}_{\varepsilon_0}^3} \psi(Q) dx - 1 > M + \frac{4\pi|\mathbb{T}^3|^2}{e} \quad \forall n \geq N.$$

Putting together (4.17) and (4.18), we conclude that (4.16) is valid. Hence the proof is complete. \square

Since in Theorem 1.1 we allow the initial data $Q_0 \in \overline{D(\mathcal{E})}$, one may not expect any further regularity at time $t = 0$. Thus we only aim to prove the regularity of $Q(t, \cdot)$ for positive times $t > 0$.

Let $t_0 > 0$ be arbitrarily given. Consider the gradient flow sequence

$$(4.19) \quad \begin{cases} \partial_t Q_n \in -\partial\mathcal{E}_n(Q_n), & t \geq t_0, \\ Q_n(t_0, x) = Q(t_0, x), \end{cases}$$

subject to periodic boundary conditions (1.7). Here we let time start from t_0 only as a matter of convenience, in order to avoid an (unnecessary) time shifting. Note that the energies \mathcal{E}_n are also -2α -convex, proper, lower semicontinuous, and uniformly bounded from below. Thus we can apply [1, Theorem 4.0.4] to get the same conclusions as in Theorem 1.1 and Propositions 3.6 and 3.7.

As a consequence, we manage to prove the following lemma.

LEMMA 4.3. *Let Q_n be the solution of (4.19). Then for every $t_0 \in (0, T)$*

$$\begin{aligned} \|\partial\mathcal{E}_n(Q_n(t, \cdot))\|_{L^2(\mathbb{T}^3)} &\rightarrow \|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)} && \text{in } L^2(t_0, T), \\ \|\partial_t Q_n(t, \cdot)\|_{L^2(\mathbb{T}^3)} &\rightarrow \|\partial_t Q(t, \cdot)\|_{L^2(\mathbb{T}^3)} && \text{in } L^2(t_0, T), \end{aligned}$$

up to a subsequence.

Proof. By Proposition 4.2, we have $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$. We aim to show that we are under the hypotheses of Proposition 2.6, where the general sense of convergence is considered to be the strong convergence in $L^2(\mathbb{T}^3; \mathcal{Q})$. Let us first check that the conditions (2.5), (2.6) are valid for the solutions Q_n of (4.19).

By Proposition 3.7 and **(M3)**, we have

$$\mathcal{E}_n(Q_n)(t) \leq \mathcal{E}_n(Q(t_0)) \leq \mathcal{E}(Q(t_0)) \quad \forall t > t_0, n \in \mathbb{N}.$$

Hence following a similar argument in the proof of compactness part in Proposition 4.2 we get

$$(4.20) \quad \{Q_n\} \text{ is uniformly bounded in } L^\infty(t_0, T; H^1(\mathbb{T}^3)).$$

Next, by Proposition 3.7 and **(M3)**, the energy equality

$$(4.21) \quad \begin{aligned} \int_{t_0}^T (\|\partial_t Q_n(t, \cdot)\|^2 + \|\partial\mathcal{E}_n(Q_n(t, \cdot))\|^2) dt &= 2\mathcal{E}_n(Q(t_0)) - 2\mathcal{E}_n(Q_n(T)) \\ &\leq 2\mathcal{E}(Q(t_0)) - 2\inf \mathcal{E}_n \end{aligned}$$

holds for all $t_0 < T < +\infty$. Hence

$$(4.22) \quad \{\partial_t Q_n\} \text{ is uniformly bounded in } L^2(t_0, T; L^2(\mathbb{T}^3))$$

and we can apply the Aubin–Lions lemma to yield

$$(4.23) \quad Q_n \rightarrow \tilde{Q} \text{ strongly in } C([t_0, T]; L^2(\mathbb{T}^3)),$$

and (4.21) implies that

$$(4.24) \quad \partial_t Q_n \rightarrow \partial_t \tilde{Q} \text{ weakly in } L^2(0, T; L^2(\mathbb{T}^3)).$$

Hence the condition (2.5) is satisfied by taking $f = 0$. Since each \mathcal{E}_n is -2α convex, by [23, Proposition 13] we know that the condition (2.6) is also satisfied by taking $C = 0$.

Finally, note that the initial data are “well-prepared” in the sense of $\mathcal{E}_n(Q(t_0)) \rightarrow \mathcal{E}(Q(t_0))$, and thus all the assumptions in Proposition 2.6 are satisfied, which in turn gives that \tilde{Q} is a solution of the gradient flow

$$\begin{cases} \partial_t \tilde{Q} \in -\partial \mathcal{E}(\tilde{Q}), \\ \tilde{Q}(t_0) = Q(t_0). \end{cases}$$

Since Q is already a solution, and by Theorem 1.1 we know that such a solution is unique, we infer $\tilde{Q} = Q$, and the proof is complete. \square

Now we turn to the H^2 -regularity of the sequence $\{Q_n\}$. The following technical lemma implies the coercivity of $\partial \mathcal{G}$.

LEMMA 4.4. *For any $P \in H^2(\mathbb{T}^3; \mathcal{Q})$, the operator $\partial \mathcal{G}(P)$ satisfies*

$$2(L_1 - |L_2 + L_3|) \|\Delta P\|_{L^2(\mathbb{T}^3)}^2 \leq \langle -\partial \mathcal{G}(P), \Delta P \rangle_{L^2} \leq 2(L_1 + |L_2 + L_3|) \|\Delta P\|_{L^2(\mathbb{T}^3)}^2.$$

Proof. Direct computations yield

$$\begin{aligned} \langle -\partial \mathcal{G}(P), \Delta P \rangle_{L^2} &= \int_{\mathbb{T}^3} \left[2L_1 \Delta P_{ij} + 2(L_2 + L_3) \partial_k \partial_j P_{ik} \right. \\ &\quad \left. - \frac{2}{3} (L_2 + L_3) \partial_\ell \partial_k P_{k\ell} \delta_{ij} \right] \Delta P_{ij} \, dx \\ &= 2L_1 \|\Delta P\|_{L^2(\mathbb{T}^3)}^2 + 2(L_2 + L_3) \int_{\mathbb{T}^3} \partial_j \partial_k P_{ik} \Delta P_{ij} \, dx, \end{aligned}$$

where the last term on the right-hand side can be treated by integration by parts and Hölder’s inequality:

$$\begin{aligned} &2(L_2 + L_3) \int_{\mathbb{T}^3} \partial_k \partial_j P_{ik} \Delta P_{ij} \, dx \\ &= 2(L_2 + L_3) \int_{\mathbb{T}^3} \partial_\ell \partial_j P_{ik} \partial_\ell \partial_k P_{ij} \, dx \\ &\leq 2|L_2 + L_3| \int_{\mathbb{T}^3} \sqrt{\sum_{i,j,k,\ell} (\partial_\ell \partial_j P_{ik})^2} \sqrt{\sum_{i,j,k,\ell} (\partial_\ell \partial_k P_{ij})^2} \, dx \\ &= 2|L_2 + L_3| \int_{\mathbb{T}^3} |\nabla^2 P|^2 \, dx = 2(|L_2 + L_3|) \|\Delta P\|_{L^2(\mathbb{T}^3)}^2. \end{aligned} \quad \square$$

Next we recall the notion of angles between two elements in a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Given two nonzero elements $u, v \in H$, the angle between u and v is the unique value

$$\angle(u, v) := \arccos \frac{\langle u, v \rangle}{\|u\| \|v\|} \in [0, \pi], \quad \text{where } \|u\|_H = \sqrt{\langle u, u \rangle}.$$

Then the following triangle inequality is valid.

LEMMA 4.5. *For any three unit vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{S}^2$, it holds that*

$$(4.25) \quad \angle(\vec{u}, \vec{v}) \leq \angle(\vec{u}, \vec{w}) + \angle(\vec{v}, \vec{w}),$$

where $\angle(\vec{u}, \vec{v}) \in [0, \pi]$ stands for the angle between the two vectors \vec{u} and \vec{v} .

Proof. By rotation invariance, we assume the unit vectors \vec{u}, \vec{v} lie on the xy -plane. And in particular, w.l.o.g. we suppose \vec{u} points in the direction of the x -axis:

$$\vec{u} = \langle 1, 0, 0 \rangle, \quad \vec{v} = \langle a, \sqrt{1-a^2}, 0 \rangle, \quad \vec{w} = \langle c_1, c_2, c_3 \rangle \in \mathbb{S}^2, \quad -1 \leq a \leq 1.$$

Correspondingly, we have

$$\begin{aligned} \cos \angle(\vec{u}, \vec{v}) &= a, \quad \cos \angle(\vec{u}, \vec{w}) = c_1, \quad \cos \angle(\vec{v}, \vec{w}) = ac_1 + \sqrt{1-a^2}c_2, \\ \sin \angle(\vec{u}, \vec{v}) &= \sqrt{1-c_1^2}, \quad \sin \angle(\vec{v}, \vec{w}) = \sqrt{1-2a\sqrt{1-a^2}c_1c_2 - a^2c_1^2 - (1-a^2)c_2^2}. \end{aligned}$$

To prove (4.25), it is equivalent to show that

$$\cos \angle(\vec{u}, \vec{v}) \geq \cos(\angle(\vec{u}, \vec{w}) + \angle(\vec{v}, \vec{w})).$$

That is,

$$a \geq c_1(ac_1 + \sqrt{1-a^2}c_2) - \sqrt{1-c_1^2}\sqrt{1-2a\sqrt{1-a^2}c_1c_2 - a^2c_1^2 - (1-a^2)c_2^2},$$

which is equivalent to

$$(4.26) \quad \sqrt{1-c_1^2}\sqrt{1-2a\sqrt{1-a^2}c_1c_2 - a^2c_1^2 - (1-a^2)c_2^2} \geq a(c_1^2 - 1) + \sqrt{1-a^2}c_1c_2.$$

If the right-hand side of (4.26) is nonpositive, the proof is complete. Otherwise, it is equivalent to prove

$$\begin{aligned} (1-c_1^2)[1-2a\sqrt{1-a^2}c_1c_2 - a^2c_1^2 - (1-a^2)c_2^2] \\ \geq a^2(1-c_1^2)^2 + (1-a^2)c_1^2c_2^2 - 2a\sqrt{1-a^2}(1-c_1^2)c_1c_2 \\ \Leftrightarrow (1-c_1^2)(1-a^2c_1^2) - (1-a^2)c_2^2 \geq a^2(1-c_1^2)^2 \\ \Leftrightarrow (1-c_1^2 - c_2^2)(1-a^2) \geq 0, \end{aligned}$$

which is obviously true. \square

Based on Lemmas 4.3 and 4.4, we show that the H^2 -norm of Q_n can be estimated in terms of $\|\partial\mathcal{E}_n(Q_n(t))\|$ uniformly in $n \in \mathbb{N}$.

PROPOSITION 4.6. *For any $n \in \mathbb{N}$, a.e. $t \in (t_0, T)$, it holds that*

$$\|\Delta Q_n(t, \cdot)\|_{L^2(\mathbb{T}^3)} \leq C_L(\|\partial\mathcal{E}_n(Q_n(t, \cdot))\|_{L^2(\mathbb{T}^3)} + 2\alpha\|Q_n(t, \cdot)\|_{L^2(\mathbb{T}^3)}),$$

where C_L is defined in (1.15).

Proof. By Lemma 4.3, we have that (up to a subsequence) $\|\partial\mathcal{E}_n(Q_n(t, \cdot))\|$ is convergent to $\|\partial\mathcal{E}(Q(t, \cdot))\|$ for almost every fixed $t \in [t_0, T]$. Hence for any fixed

$t \in [t_0, T]$ (after removing a set of measure zero), for any $n \in \mathbb{N}$ it holds (recall (4.6)) that we have for every $n \geq 1$ and almost every $t \in (t_0, T)$ that

$$(4.27) \quad \begin{aligned} & \left\| \partial\mathcal{G}(Q_n(t, \cdot)) + \psi'_n(Q_n(t, \cdot)) - \frac{1}{3}\text{tr}(\psi'_n)(Q_n(t, \cdot))\mathbb{I}_3 \right\|_{L^2(\mathbb{T}^3)} \\ & \leq \|\partial\mathcal{E}_n(Q_n(t, \cdot))\|_{L^2(\mathbb{T}^3)} + 2\alpha\|Q_n(t, \cdot)\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

On the other hand, by (M6), and $\{Q_n\} \in L^\infty(0, T; H^1(\mathbb{T}^3))$, we know that

$$\left\| \psi'_n(Q_n(t, \cdot)) - \frac{1}{3}\text{tr}(\psi'_n)(Q_n(t, \cdot))\mathbb{I}_3 \right\|_{L^2(\mathbb{T}^3)} \leq \lambda_n\|Q_n(t, \cdot)\|_{L^2(\mathbb{T}^3)} + \Lambda_n|\mathbb{T}^3| < +\infty,$$

which together with (4.27) implies that

$$\|\partial\mathcal{G}(Q_n(t, \cdot))\|_{L^2(\mathbb{T}^3)} < \infty.$$

By Lemma 4.4, we conclude that $Q_n(t, \cdot) \in H^2(\mathbb{T}^3)$, a.e. $t \geq 0$ for any $n \in \mathbb{N}$.

In the rest of this lemma, for simplicity $Q_n(\cdot, t)$ is abbreviated by Q_n , and w.l.o.g. we assume $\|\Delta Q_n\|_{L^2(\mathbb{T}^3)} > 0$. By (1.4) and Lemma 4.4, we have

$$\frac{\int_{\mathbb{T}^3} (\partial\mathcal{G}(Q_n) : -\Delta Q_n) dx}{\|\partial\mathcal{G}(Q_n)\|_{L^2}\|\Delta Q_n\|_{L^2}} \geq \frac{2(L_1 - |L_2 + L_3|)\|\Delta Q_n\|_{L^2}^2}{2(L_1 + |L_2 + L_3|)\|\Delta Q_n\|_{L^2}^2} \geq \frac{L_1 - |L_2 + L_3|}{L_1 + |L_2 + L_3|} > 0.$$

Hence the angle between $\partial\mathcal{G}(Q_n)$ and $-\Delta Q_n$ is bounded by

$$\angle(\partial\mathcal{G}(Q_n), -\Delta Q_n) \leq \arccos \frac{L_1 - |L_2 + L_3|}{L_1 + |L_2 + L_3|}.$$

Meanwhile, by (M1) we know that

$$\int_{\mathbb{T}^3} (\partial\psi_n(Q_n) : -\Delta Q_n) dx \geq 0,$$

which together with Lemma 4.5 gives

$$\begin{aligned} \angle(\partial\mathcal{G}(Q_n), \partial\psi_n(Q_n)) & \leq \angle(\partial\mathcal{G}(Q_n), -\Delta Q_n) + \angle(\partial\psi_n(Q_n), -\Delta Q_n) \\ & \leq \frac{\pi}{2} + \arccos \frac{L_1 - |L_2 + L_3|}{L_1 + |L_2 + L_3|}. \end{aligned}$$

As a consequence, we obtain that

$$(4.28) \quad \begin{aligned} & \int_{\mathbb{T}^3} (\partial\mathcal{G}(Q_n) : \partial\psi_n(Q_n)) dx \\ & \geq \cos\left(\frac{\pi}{2} + \arccos \frac{L_1 - |L_2 + L_3|}{L_1 + |L_2 + L_3|}\right) \|\partial\mathcal{G}(Q_n)\|_{L^2(\mathbb{T}^3)} \|\partial\psi_n(Q_n)\|_{L^2(\mathbb{T}^3)} \\ & = -\frac{2\sqrt{L_1|L_2 + L_3|}}{L_1 + |L_2 + L_3|} \|\partial\mathcal{G}(Q_n)\|_{L^2(\mathbb{T}^3)} \|\partial\psi_n(Q_n)\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

Note that $2\sqrt{L_1|L_2 + L_3|}/(L_1 + |L_2 + L_3|) < 1$ by (1.4).

In all, after combining Lemma 4.4, (4.27), (4.28), and the Cauchy-Schwarz inequality we conclude that

$$\begin{aligned}
& (\|\partial\mathcal{E}_n(Q_n)\|_{L^2(\mathbb{T}^3)} + 2\alpha\|Q_n\|_{L^2(\mathbb{T}^3)})^2 \\
& \geq \|\partial\mathcal{G}(Q_n)\|_{L^2(\mathbb{T}^3)}^2 + \|\partial\psi_n(Q_n)\|_{L^2(\mathbb{T}^3)}^2 \\
& \quad - \frac{4\sqrt{L_1|L_2+L_3|}}{L_1+|L_2+L_3|} \|\partial\mathcal{G}(Q_n)\|_{L^2(\mathbb{T}^3)} \|\partial\psi_n(Q_n)\|_{L^2(\mathbb{T}^3)} \\
& \geq \frac{L_1+|L_2+L_3|-2\sqrt{L_1|L_2+L_3|}}{L_1+|L_2+L_3|} \|\partial\mathcal{G}(Q_n)\|_{L^2(\mathbb{T}^3)}^2 \\
(4.29) \quad & \geq \frac{L_1+|L_2+L_3|-2\sqrt{L_1|L_2+L_3|}}{L_1+|L_2+L_3|} 4(L_1-|L_2+L_3|)^2 \|\Delta Q_n\|_{L^2(\mathbb{T}^3)}^2.
\end{aligned}$$

The proof is complete. \square

Summing up the results established in this subsection, we are ready to establish the improved regularity properties of the unique solution Q obtained in Theorem 1.1.

Proof of Theorem 1.2.

Part 1: Uniform-in-time bound (1.14) It is proved in (4.23) that (up to a subsequence) $Q_n(t) \rightarrow Q(t, \cdot)$ in $C([t_0, T]; L^2(\mathbb{T}^3))$. Moreover, it follows from Proposition 4.6 that for a.e. $t \in (t_0, T)$, $\|\Delta Q_n(t)\|_{L^2(\mathbb{T}^3)}$ is (up to a subsequence) uniformly bounded in $n \in \mathbb{N}$. Hence $\Delta Q(t, \cdot) \in L^2(\mathbb{T}^3)$ a.e., and from Lemma 4.3 we can further make the following estimates:

$$\begin{aligned}
\|\Delta Q(t, \cdot)\|_{L^2(\mathbb{T}^3)} & \leq \liminf_{n \rightarrow \infty} \|\Delta Q_n(t, \cdot)\|_{L^2(\mathbb{T}^3)} \\
& \leq \liminf_{n \rightarrow \infty} C_L (\|\partial\mathcal{E}_n(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)} + 2\alpha\|Q_n(t, \cdot)\|_{L^2(\mathbb{T}^3)}) \\
(4.30) \quad & = C_L (\|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)} + 2\alpha\|Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}),
\end{aligned}$$

where C_L is defined in (1.15).

W.l.o.g., we assume $T = t_0 + 100$. In view of (4.30), it suffices to provide the L^∞ -bound for $\|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)}^2$. By (1.12), we have

$$(4.31) \quad \int_{t_0}^{+\infty} \|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)}^2 dt \leq \mathcal{E}(Q(t_0)) - \inf \mathcal{E}.$$

Thus for any $n \in \mathbb{N}$ it holds that

$$(4.32) \quad \int_{t_0+n}^{t_0+n+1} \|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)}^2 dt \leq \mathcal{E}(Q(t_0)) - \inf \mathcal{E},$$

and hence there exists a set of positive measure $A_n \subset [t_0+n, t_0+n+1]$ such that

$$(4.33) \quad \|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)}^2 \leq \mathcal{E}(Q(t_0)) - \inf \mathcal{E} + 1 \quad \forall t \in A_n.$$

For any $n \in \mathbb{N}$, $n \leq 99$, let us choose a time $s_n \in A_n$, and for the sake of convenience we set $s_0 = t_0$. Obviously, $\{s_n\}_{n \leq 99}$ is monotone increasing, and $s_{n+1} - s_n \leq 2$.

Let us then consider the gradient flow sequence

$$(4.34) \quad \begin{cases} \partial_t P_n \in -\partial\mathcal{E}(P_n), \\ P_n(0, \cdot) = Q(s_n, \cdot), \end{cases}$$

subject to a periodic boundary condition, whose solution P_n is given by the time shift $P_n(t, \cdot) := Q(s_n + t, \cdot)$ for all $t \geq 0$. By Proposition 3.6, the function

$$t \mapsto e^{-2\alpha t} \|\partial\mathcal{E}(P_n(t))\|_{L^2(\mathbb{T}^3)}$$

is nonincreasing, and thus together with (4.33) we have

$$\begin{aligned} \|\partial\mathcal{E}(Q(t + s_n, \cdot))\|_{L^2(\mathbb{T}^3)}^2 &= \|\partial\mathcal{E}(P_n(t, \cdot))\|_{L^2(\mathbb{T}^3)}^2 \leq e^{4\alpha(s_{n+1} - s_n)} \|\partial\mathcal{E}(P_n(0, \cdot))\|_{L^2(\mathbb{T}^3)}^2 \\ &\leq e^{8\alpha} \|\partial\mathcal{E}(Q(s_n, \cdot))\|_{L^2(\mathbb{T}^3)}^2 \leq e^{8\alpha} (\mathcal{E}(Q(t_0)) - \inf \mathcal{E} + 1) \end{aligned}$$

for a.e. $t \in (0, s_{n+1} - s_n)$. Repeating this argument for all n finally gives

$$\|\partial\mathcal{E}(Q(t, \cdot))\|_{L^2(\mathbb{T}^3)} \leq e^{4\alpha} \sqrt{\mathcal{E}(Q(t_0)) - \inf \mathcal{E} + 1} \quad \text{for a.e. } t \in (t_0, t_0 + 99).$$

In addition, we recall that all eigenvalues of $Q(t, \cdot)$ are in $(-1/3, 2/3)$ for a.e. $t \in (t_0, t_0 + 99)$, and hence $\|Q(t, \cdot)\|_{H^2(\mathbb{T}^3)}$ is uniform-in-time bounded in $(t_0, t_0 + 99)$. Thus (1.14) could be proved by iteration.

Part 2: Strict physicality (1.17). Now that Q is a solution of

$$\partial_t Q = -\partial\mathcal{G}(Q(t, \cdot)) - \psi'(Q(t, \cdot)) + \frac{\text{tr}(\psi'(Q(t, \cdot)))}{3} \mathbb{I}_3 + 2\alpha Q(t, \cdot) \quad \forall t > t_0,$$

with $Q \in L^\infty(t_0, +\infty; H^2)$, let us take the inner product with $-\Delta Q(t, \cdot)$. Then it gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 &= \langle \partial\mathcal{G}(Q(t, \cdot)), \Delta Q(t, \cdot) \rangle_{L^2(\mathbb{T}^3)} + \langle \psi'(Q(t, \cdot)), \Delta Q(t, \cdot) \rangle_{L^2(\mathbb{T}^3)} \\ (4.35) \quad &+ 2\alpha \|\nabla Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2. \end{aligned}$$

Note that by (1.16), Lemma 4.4, and the Poincaré inequality we have

$$\begin{aligned} \langle \partial\mathcal{G}(Q(t, \cdot)), \Delta Q(t) \rangle_{L^2(\mathbb{T}^3)} &\leq -2(L_1 - |L_2 + L_3|) \|\Delta Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 \\ (4.36) \quad &\leq -\frac{2(L_1 - |L_2 + L_3|)}{C_{\mathbb{T}^3}^2} \|\nabla Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \psi'(Q(t, \cdot)), \Delta Q(t, \cdot) \rangle_{L^2(\mathbb{T}^3)} &= \int_{\mathbb{T}^3} \psi'(Q(t, x)) \Delta Q(t, x) \, dx \\ (4.37) \quad &= - \int_{\mathbb{T}^3} \psi''(Q(t, x)) |\nabla Q(t, x)|^2 \, dx \leq 0 \end{aligned}$$

due to the convexity of ψ . Inserting (4.36) and (4.37) into (4.35), we get

$$\frac{d}{dt} \|\nabla Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 \leq 4 \left(-\frac{L_1 - |L_2 + L_3|}{C_{\mathbb{T}^3}^2} + \alpha \right) \|\nabla Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 \quad \forall t \geq t_0.$$

Since $Q(t_0, \cdot) \in H^1(\mathbb{T}^3)$, it follows from Gronwall's inequality that

$$(4.38) \quad \|\nabla Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 \leq \exp \left(4 \left[-\frac{L_1 - |L_2 + L_3|}{C_{\mathbb{T}^3}^2} + \alpha \right] (t - t_0) \right) \|\nabla Q(t_0, \cdot)\|_{L^2(\mathbb{T}^3)}^2$$

for all $t \geq t_0$. Denote by $\bar{Q}(t) := |\mathbb{T}^3|^{-1} \int_{\mathbb{T}^3} Q(t, \cdot) dx$ the mean value of $Q(t, \cdot)$ over \mathbb{T}^3 . Due to the convexity of ψ , one can apply Jensen's inequality to derive

$$\begin{aligned} \psi(\bar{Q}(t)) &= \psi(\bar{Q}(t))|\mathbb{T}^3| \leq \int_{\mathbb{T}^3} \psi(Q(t, x)) dx = \mathcal{E}(Q(t)) - \mathcal{G}(Q(t)) + \alpha \|Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 \\ &\leq \mathcal{E}(Q(t_0)) + \alpha \sup_{s \geq t_0} \|Q(s, \cdot)\|_{L^\infty(\mathbb{T}^3)}^2 |\mathbb{T}^3|, \end{aligned}$$

since $\mathcal{G} \geq 0$ by Lemma 3.1. It is noted that the last bound above is independent of $t \geq t_0$. Thus

$$(4.39) \quad \rho_0 := \inf_{t \geq t_0} \rho(Q(t, \cdot)) > 0,$$

where ρ is defined in (3.11).

Finally, by (4.30) and the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned} \|Q(t, \cdot) - \bar{Q}(t)\|_{L^\infty(\mathbb{T}^3)} &\leq C' \|\nabla Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^{1/2} \|\Delta Q(t, \cdot)\|_{L^2(\mathbb{T}^3)}^{1/2} \\ (4.40) \quad &\leq C' \exp \left(\left[-\frac{L_1 - |L_2 + L_3|}{C_{\mathbb{T}^3}^2} + \alpha \right] (t - t_0) \right) \|\nabla Q(t_0, \cdot)\|_{L^2(\mathbb{T}^3)}^{1/2} \cdot C(t_0)^{1/2} \end{aligned}$$

for some geometric constant C' , and a.e. $t \geq t_0$. This together with (1.16) implies $\|Q(t, \cdot) - \bar{Q}(t)\|_{L^\infty(\mathbb{T}^3)}$ decays uniformly to zero as $t \rightarrow \infty$. Since the convergence in the L^∞ norm implies the uniform convergence of all eigenvalues, due to (4.39) we conclude that there exists some $T_0 > 0$, such that

$$\rho(Q(t, \cdot)) \geq \frac{\rho_0}{2} \quad \forall t \geq T_0.$$

Therefore, the proof of Theorem 1.2 is complete. \square

5. Proof of Theorem 1.3: Size of the contact set. In this section we shall estimate the Hausdorff dimension of the singular set Σ_t where the unique global solution $Q(t, x)$ to (1.6) touches the physical boundary

$$(5.1) \quad \Sigma_t := \{x \in \mathbb{T}^n \mid Q(t, x) \in \partial \mathcal{Q}_{phy}\}.$$

Here $\partial \mathcal{Q}_{phy}$ is the boundary of \mathcal{Q}_{phy} where the smallest eigenvalue of any element equals $-1/3$.

To begin with, we state [20, Theorem 1.2], which provides a lower bound of the blowup rate of $\partial \psi(P)$ as $P \in \mathcal{Q}_{phy}$ approaches its physical boundary.

PROPOSITION 5.1. *For any $P \in \mathcal{Q}_{phy}$, as $\lambda_1(P) \rightarrow -1/3$ it holds that*

$$(5.2) \quad \left| \partial \psi(P) - \frac{1}{3} \text{tr}(\partial \psi(P)) \mathbb{I}_3 \right| \geq \frac{C_1}{\lambda_1(P) + \frac{1}{3}}$$

with the constant C_1 given by

$$(5.3) \quad C_1 = \frac{\sqrt{3}}{9\sqrt{2\pi}e} \cdot \inf_{\xi \geq 0} \frac{e^{-\xi} I_0(\xi)}{e^{\frac{-\xi}{2}} I_0(\frac{\xi}{2})} > 0,$$

where $I_0(\cdot)$ is the zeroth order modified Bessel function of first kind.

Remark 5.2. As was pointed out in [16, Appendix C],

$$d(P, \partial \mathcal{Q}_{phy}) = \frac{\sqrt{6}}{2} \left[\lambda_1(P) + \frac{1}{3} \right]$$

for any $P \in \mathcal{Q}_{phy}$. Together with Proposition 5.1, it is immediate to see that there exists a generic and suitably small constant $\delta_0 > 0$, such that

$$(5.4) \quad \left| \partial\psi(P) - \frac{1}{3} \text{tr}(\partial\psi(P)) \mathbb{I}_3 \right| \geq \frac{\sqrt{6}C_1}{2d(P, \partial \mathcal{Q}_{phy})} \quad \text{whenever } d(P, \partial \mathcal{Q}_{phy}) < \delta_0.$$

The following technical lemma is necessary.

LEMMA 5.3. *For any $s > 0$, there exists a sequence of coverings $V_m = \{B_{i,m}\}$ of Σ_t , where $B_{i,m} = B(x_{i,m}, r_{i,m})$, such that*

$$\lim_{m \rightarrow \infty} \sum_i r_{i,m}^s = \mathcal{H}^s(\Sigma_t) \leq \liminf_{m \rightarrow \infty} \sum_j |5r_{j,m}^*|^s.$$

Here $B_{i,m} = B(x_{i,m}, r_{i,m})$ such that $B_{i,m} \subset \{x \in \mathbb{T}^n \mid d(x, \Sigma_t) \leq \delta\}$ for some $\delta > 0$, and $r_{j,m}^*$ are the radii of the balls $B_{j,m}^*$, i.e., the subcovering given by the Vitali covering lemma.

Proof. First, by recalling the definition of Hausdorff content

$$\mathcal{H}_\delta^s(\Sigma_t) := \inf \left\{ \sum_i r_i^s : \Sigma_t \subset \bigcup_i B_i, \sup_i r_i < \delta \right\}$$

and the Hausdorff measure

$$\mathcal{H}^s(\Sigma_t) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(\Sigma_t),$$

we know that for any $\delta > 0$ we have a sequence of coverings $V_m = \{B_{i,m,\delta}\}$ such that

$$\lim_{m \rightarrow \infty} \sum_i r_{i,m,\delta}^s = \mathcal{H}_\delta^s(\Sigma_t).$$

Therefore, using the standard diagonal argument we can choose a sufficiently large $m = m(\delta)$ such that

$$0 \leq \sum_i r_{i,m,\delta}^s - \mathcal{H}_\delta^s(\Sigma_t) := \varepsilon_\delta \ll 1, \quad \text{where } \varepsilon_\delta \rightarrow 0 \Leftrightarrow m \rightarrow +\infty,$$

which further gives

$$\lim_{m \rightarrow \infty} \sum_i r_{i,m}^s = \mathcal{H}^s(\Sigma_t), \quad \text{where } r_{i,m} := r_{i,m,\delta(m)}.$$

W.l.o.g., we may choose m to be bijective in δ . Thus $V_m = \{B_{i,m}\}$ is an admissible sequence to achieve the Hausdorff measure.

Note that $\{5B_{j,m}^*\}$ is another covering with balls for Σ_t , and hence for each m , we have

$$\sum_j |5r_{j,m}^*|^s \geq \mathcal{H}_{\delta(m)}^s(\Sigma_t) = \sum_i r_{i,m,\delta(m)}^s - \varepsilon_\delta,$$

which implies

$$\liminf_{m \rightarrow \infty} \sum_j |5r_{j,m}^*|^s \geq \liminf_{m \rightarrow \infty} \mathcal{H}_{\delta(m)}^s(\Sigma_t) = \mathcal{H}^s(\Sigma_t).$$

The proof is complete. \square

Using Proposition 5.1 and Lemma 5.3, now we are ready to finish the proof of Theorem 1.3.

Proof of Theorem 1.3. First, for any $t_0 > 0$, by Theorems 1.1 and 1.2, we know that the unique strong solution $Q(t, x)$ to (1.10) satisfies

$$(5.5) \quad Q \in L^\infty(t_0, +\infty; H^2(\mathbb{T}^n)), \quad \partial_t Q \in L^\infty(t_0, T_0; L^2(\mathbb{T}^n)),$$

which together with (1.10) gives

$$\partial\psi(Q) - \frac{1}{3}\text{tr}(\partial\psi(Q))\mathbb{I}_3 \in L^\infty(t_0, T_0; L^2(\mathbb{T}^n)).$$

Here and after, we are only concerned with $t \in (0, T_0)$, which satisfies

$$\left\| \partial\psi(Q) - \frac{1}{3}\text{tr}(\partial\psi(Q))\mathbb{I}_3 \right\|_{L^2(\mathbb{T}^n)} < +\infty,$$

which obviously has a full measure in $(0, T_0)$.

Case 1: $n = 3$.

By Sobolev embedding $H^2(\mathbb{T}^3) \hookrightarrow C^{0, \frac{1}{2}}(\mathbb{T}^3)$ and $Q \in L^\infty(t_0, +\infty; H^2)$, there exists a generic constant $C_H > 0$ that is independent of t , such that

$$(5.6) \quad |Q(t, x) - Q(t, y)| \leq C_H |x - y|^{\frac{1}{2}} \quad \forall x, y \in \mathbb{T}^3.$$

In particular, for any given x , let $x^\perp \in \Sigma_t$ be a projection such that

$$(5.7) \quad |x - x^\perp| = d(x, \Sigma_t) := \text{dist}(x, \Sigma_t),$$

and henceforth we get

$$d(Q(t, x), \partial\mathcal{Q}_{phy}) \leq |Q(t, x) - Q(t, x^\perp)| \leq C_H |x - x^\perp|^{\frac{1}{2}} = C_H \sqrt{\text{dist}(x, \Sigma_t)}.$$

This combined with Proposition 5.1 implies for any ball B and $\delta \ll 1$ that

$$(5.8) \quad \begin{aligned} & \int_{\{x \in B \mid d(x, \Sigma_t) \leq \delta\}} \left| \partial\psi(Q(t, x)) - \frac{1}{3}\text{tr}(\partial\psi(Q(t, x)))\mathbb{I}_3 \right|^2 dx \\ & \geq C_1^2 \int_{\{x \in B \mid d(x, \Sigma_t) \leq \delta\}} d(Q(t, x), \partial\mathcal{Q}_{phy})^{-2} dx \\ & \geq \frac{C_1^2}{C_H^2} \int_{\{x \in B \mid d(x, \Sigma_t) \leq \delta\}} \frac{1}{\text{dist}(x, \Sigma_t)} dx. \end{aligned}$$

To proceed, by applying Lemma 5.3 with $s = 2$, we obtain the existence of a sequence of covering $\{B_{i,m}\}$ of Σ_t with balls $B_{i,m} = B(x_{i,m}, r_{i,m})$ such that $B_{i,m} \subset \{x \in \mathbb{T}^3 \mid d(x, \Sigma_t) \leq \delta\}$ and

$$\lim_{m \rightarrow \infty} \sum_i r_{i,m}^s = \mathcal{H}^s(\Sigma_t).$$

We can assume that each such ball $B_{i,m}$ intersects Σ_t , for otherwise we can just remove it from the covering. By Vitali's covering lemma, for each m we can choose a subcollection of mutually disjoint balls $\{B_{j,m}^*\}$, with $B_{j,m}^* = B(x_{j,m}^*, r_{j,m}^*)$ such that

$$\Sigma_t \subset \bigcup_i B_{i,m} \subset \bigcup_j 5B_{j,m}^*.$$

On each such ball $B_{j,m}^*$, it follows from (5.8) that

$$(5.9) \quad \int_{B_{j,m}^*} \left| \partial\psi(Q(t,x)) - \frac{1}{3} \text{tr}(\partial\psi(Q(t,x))) \mathbb{I}_3 \right|^2 dx \geq \frac{C_1^2}{C_H^2} \int_{B_{j,m}^*} \frac{1}{\text{dist}(x, \Sigma_t)} dx.$$

As each of such ball $B_{j,m}$ intersects Σ_t , we can choose an arbitrary intersection point $y_{j,m} \in B_{j,m}^* \cap \Sigma_t$, and

$$(5.10) \quad \int_{B_{j,m}^*} \frac{1}{\text{dist}(x, \Sigma_t)} dx \geq \int_{B_{j,m}^*} \frac{1}{|x - y_{j,m}|} dx.$$

We claim that the last integral in (5.10) is minimized when $y_{j,m} \in \partial B_{j,m}^*$. To prove this claim, w.l.o.g. we assume $B_{j,m}^*$ is the unit ball B centered at the origin, and we denote $y_{j,m} = y = (y_1, 0, 0)$, $0 \leq y_1 \leq 1$. Hence we consider

$$f(y) := \int_B \frac{1}{|x - y|} dx.$$

Then

$$\frac{\partial f}{\partial y_1} = \int_B \frac{x_1 - y_1}{|x - y|^3} dx, \quad \text{where } x = (x_1, x_2, x_3).$$

If $y_1 = 0$, by symmetry we see that $\frac{\partial f}{\partial y_1} = 0$. If $0 < y_1 \leq 1$, we denote $A = \{x \in \mathbb{R}^3 | x_1 \geq y_1\}$, and A' is the reflection of A across $\{x \in \mathbb{R}^3 | x_1 = y_1\}$. Note that $A \cup A'$ is symmetric with respect to both $\{x \in \mathbb{R}^3 | x_1 = y_1\}$ and the point $y = (y_1, 0, 0)$. Thus we have

$$\frac{\partial f}{\partial y_1} = \int_{B \setminus (A \cup A')} \frac{x_1 - y_1}{|x - y|^3} dx + \underbrace{\int_{A \cup A'} \frac{x_1 - y_1}{|x - y|^3} dx}_{=0} < 0$$

due to the fact that $B \setminus (A \cup A')$ is entirely contained in $\{x \in \mathbb{R}^3 | x_1 \leq y_1\}$. Hence the claim is proved.

As a consequence,

$$(5.11) \quad \int_{B_{j,m}^*} \frac{1}{|x - y_{j,m}|} dx \geq \frac{4\pi}{3} |r_{j,m}^*|^2,$$

which together with (5.9) and (5.10) gives

$$(5.12) \quad \int_{B_{j,m}^*} \left| \partial\psi(Q(t,x)) - \frac{1}{3} \text{tr}(\partial\psi(Q(t,x))) \mathbb{I}_3 \right|^2 dx \geq \tilde{C} |r_{j,m}^*|^2, \quad \text{where } \tilde{C} := \frac{4\pi}{3} \frac{C_1^2}{C_H^2}.$$

Since $\{B_{j,m}^*\}$ are nonoverlapping, after summing up the above inequality over j we obtain

$$(5.13) \quad \begin{aligned} & \left\| \partial\psi(Q(t, x)) - \frac{1}{3} \text{tr}(\partial\psi(Q(t, x))) \mathbb{I}_3 \right\|_{L^2(\mathbb{T}^3)}^2 \\ & \geq \sum_j \int_{B_{j,m}^*} \left| \partial\psi(Q(t, x)) - \frac{1}{3} \text{tr}(\partial\psi(Q(t, x))) \mathbb{I}_3 \right|^2 dx \geq \tilde{C} \sum_j |r_{j,m}^*|^2, \end{aligned}$$

which together with Lemma 5.3 yields

$$(5.14) \quad \left\| \partial\psi(Q(t, x)) - \frac{1}{3} \text{tr}(\partial\psi(Q(t, x))) \mathbb{I}_3 \right\|_{L^2(\mathbb{T}^3)}^2 \geq \tilde{C} \liminf_{m \rightarrow \infty} \sum_j |r_{j,m}^*|^2 \geq \frac{\tilde{C}}{5^2} \mathcal{H}^2(\Sigma_t).$$

Thus we conclude that $\dim_{\mathcal{H}}(\Sigma_t) \leq 2$.

Case 2: $n = 2$.

In the 2D case the Sobolev embedding reads as $H^2(\mathbb{T}^2) \hookrightarrow C^\beta(\mathbb{T}^2)$ for all $\beta \in (0, 1)$. Correspondingly, we have

$$(5.15) \quad \|Q(t, x) - Q(t, y)\| \leq C_\beta |x - y|^\beta \quad \forall x, y \in \mathbb{T}^2,$$

and (5.8) is replaced by

$$(5.16) \quad \begin{aligned} & \int_{\{x \in B \mid d(x, \Sigma_t) \leq \delta\}} \left| \partial\psi(Q(t, x)) - \frac{1}{3} \text{tr}(\partial\psi(Q(t, x))) \mathbb{I}_3 \right|^2 dx \\ & \geq \frac{C_1^2}{C_\beta^2} \int_{\{x \in B \mid d(x, \Sigma_t) \leq \delta\}} \frac{1}{\text{dist}^{2\beta}(x, \Sigma_t)} dx. \end{aligned}$$

As a consequence, applying Lemma 5.3 with $s = 2 - 2\beta$, and using Vitali's covering argument exactly as in Case 1, one may replace (5.14) by

$$(5.17) \quad \begin{aligned} & \left\| \partial\psi(Q(t, x)) - \frac{1}{3} \text{tr}(\partial\psi(Q(t, x))) \mathbb{I}_3 \right\|_{L^2(\mathbb{T}^2)}^2 \\ & \geq \tilde{C}' \liminf_{m \rightarrow \infty} \sum_j |r_{j,m}^*|^{2-2\beta} \geq \frac{\tilde{C}'}{5^{2-2\beta}} \mathcal{H}^{2-2\beta}(\Sigma_t), \quad \text{where } \tilde{C}' = \frac{4\pi C_1^2}{C_\beta^2}. \end{aligned}$$

In conclusion, we obtain $\dim_{\mathcal{H}}(\Sigma_t) \leq 2 - 2\beta$ for any $\beta \in (0, 1)$. The proof is complete by the arbitrariness of $\beta \in (0, 1)$. \square

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