J. Plasma Phys. (2021), *vol.* 87, 905870307

Published by Cambridge University Press
doi:10.1017/S0022377821000519

Exact conservation laws for gauge-free electromagnetic gyrokinetic equations

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(Received 19 July 2020; revised 28 April 2021; accepted 30 April 2021)

The exact energy and angular momentum conservation laws are derived by the Noether method for the Hamiltonian and symplectic representations of the gauge-free electromagnetic gyrokinetic Vlasov–Maxwell equations. These gyrokinetic equations, which are solely expressed in terms of electromagnetic fields, describe the low-frequency turbulent fluctuations that perturb a time-independent toroidally-axisymmetric magnetized plasma. The explicit proofs presented here provide a complete picture of the transfer of energy and angular momentum between the gyrocentres and the perturbed electromagnetic fields, in which the crucial roles played by gyrocentre polarization and magnetization effects are highlighted. In addition to yielding an exact angular momentum conservation law, the gyrokinetic Noether equation yields an exact momentum transport equation, which might be useful in more general equilibrium magnetic geometries.

Key words: plasma nonlinear phenomena, plasma dynamics

1. Introduction

Nonlinear gyrokinetic theory has been at the forefront of plasma physics research since the pioneering work of Frieman & Chen (1982). In its modern representation (Brizard & Hahm 2007), nonlinear gyrokinetic theory involves a two-step (guiding-centre + gyrocentre) transformation leading to the adiabatic invariance of the gyrocentre magnetic moment and a set of reduced gyrocentre Hamilton equations that are decoupled from the fast gyromotion dynamics. The guiding-centre and gyrocentre dynamical reductions, on the other hand, introduce guiding-centre and gyrocentre polarization and magnetization in the gyrokinetic Maxwell equations, which play crucial roles in the self-consistent evolution of a turbulent magnetized plasma.

The gyrokinetic Vlasov–Maxwell equations presented in this work are based on the gauge-free electromagnetic field gyrokinetic formulation recently introduced by Burby & Brizard (2019) and Brizard (2020), in which only the perturbed electromagnetic fields appear in the gyrokinetic Vlasov–Maxwell equations. This gyrokinetic electromagnetic field formulation, which also facilitates the development of hybrid kinetic particle simulation schemes (Chen & Parker 2009; Chen *et al.* 2019), has been a topic of recent research interest (Chen, Zonca & Chen 2020).

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1.1. *Gyrokinetic energy conservation law*

The energy conservation laws of several sets of gyrokinetic equations have been the topic of active research since the Hamiltonian gyrokinetic work of Dubin *et al.* (1983), where the energy conservation law was derived directly from the electrostatic gyrokinetic Vlasov–Poisson equations. Next, the energy conservation laws for the finite-beta electromagnetic gyrokinetic equations and the fully electromagnetic gyrokinetic equations were constructed by Hahm, Lee & Brizard (1988) and Brizard (1989*b*), respectively, and then proved explicitly by Brizard (1989*a*) for the fully electromagnetic case. Since its inception, the primary use of an exact energy conservation law associated with a gyrokinetic model has been seen as a measure of the accuracy of its numerical implementation using gyrokinetic particle simulation methods (Garbet *et al.* 2010).

The discovery of the Lagrangian (Sugama 2000), Eulerian (Brizard 2000*a*,*b*), and Euler–Poincaré (Squire *et al.* 2013) variational formulations for the nonlinear gyrokinetic equations led to the direct derivations of exact gyrokinetic energy conservation laws by the Noether method (Goldstein, Poole & Safko 2002; Brizard 2005*b*). While the Noether derivation guarantees the existence of an exact energy conservation law, its explicit proof often provides useful insights into the transfer of energy between the perturbed electromagnetic fields, on the one hand, and the gyrocentre Vlasov distribution, on the other hand. This is especially important since polarization and magnetization combine particles and fields at all orders in the gyrocentre perturbation analysis. Using an Eulerian variational formulation, explicit proofs were also presented for several nonlinear gyrokinetic models (Brizard & Chan 1999; Brizard 2010*a*) and reduced-fluid plasma models (Strintzi, Scott & Brizard 2005; Brizard 2005*a*; Brizard *et al.* 2008).

1.2. *Gyrokinetic angular momentum conservation law*

The topic of an exact gyrokinetic momentum conservation law gained crucial importance in the context of the momentum transport (Waltz *et al.* 2007; Parra & Catto 2010*a*; Abiteboul *et al.* 2011; Peeters *et al.* 2011) and intrinsic toroidal rotation (Wang, Peng & Diamond 2018; Stoltzfus-Dueck 2019) in axisymmetric tokamak plasmas. In particular, the phenomenon of intrinsic toroidal rotation, which is observed in the absence of external torque, must be investigated within the context of toroidal angular momentum conservation. Scott & Smirnov (2010) derived an explicit toroidal angular momentum conservation law for the electrostatic gyrokinetic equations by deriving it as a moment of the gyrokinetic Vlasov equation. The same equation was rederived using the Noether method, and explicitly shown to be exact, by Brizard & Tronko (2011), while the Noether derivation of the momentum conservation law was also considered for several reduced plasma fluid models (Brizard 2005*b*, 2010*b*).

1.3. *Previous variational derivations of gyrokinetic conservation laws*

The angular momentum conservation law in gyrokinetic Vlasov–Maxwell models have regained significant interest recently in several works by Hirvijoki *et al.* (2020), Fan, Qin & Xiao (2020) and Sugama *et al.* (2021). By not splitting the magnetic field into equilibrium (time-independent) and perturbed (time-dependent) components, both Fan *et al.* (2020) and Sugama *et al.* (2021) rederived the guiding-centre energy–momentum conservation laws derived earlier by Sugama *et al.* (2016), using a direct moment approach of the drift-kinetic equation, and by Brizard & Tronci (2016), using several equivalent guiding-centre variational principles. Fan *et al.* (2020) generalized earlier results by Pfirsch & Morrison (1985) and Similon (1985) by including higher-order guiding-centre gyrogauge corrections.

In particular, using a hybrid gyrokinetic model that includes gyrokinetic electrostatic fluctuations with a weakly time-dependent guiding-centre Lagrangian, Sugama *et al.* (2021) obtained a symmetric gyrokinetic stress tensor $T^{ji} = T^{ij}$ using a formula, $T^{ij} \equiv \mathcal{L}g^{ij} - 2\partial\mathcal{L}/\partial g_{ij}$, obtained from the general theory of relativity (Landau & Lifshitz 1971), in which partial derivatives of the Lagrangian density $\mathcal{L}$ with respect to components of the symmetric metric tensor $\boldsymbol{g}$ are evaluated (here, a specific choice for the spatial coordinates is not needed). Additional comments about the works of Fan *et al.* (2020) and Sugama *et al.* (2021) are made in § 3.

The work of Hirvijoki *et al.* (2020) presents an Euler–Poincaré variational principle for the drift-kinetic limit of the gauge-free gyrokinetic Vlasov–Maxwell model of Burby & Brizard (2019). In this work, the standard gyrokinetic separation of equilibrium and perturbed components for the electromagnetic field is used, and the asymmetry of the resulting gyrokinetic stress tensor is shown to be driven by electromagnetic field perturbations only. The exact conservation law of toroidal angular momentum, however, will be guaranteed under the assumption of an axisymmetric equilibrium magnetic field. The energy–momentum conservation laws derived by Hirvijoki *et al.* (2020), which will be rederived here from an Eulerian variational principle, will be explicitly proved and expanded in the present work.

1.4. *Organization*

The remainder of the present paper is organized as follows. In § 2, we review the work of Brizard (2008) where the exact energy–momentum conservation laws are given for a generic set of reduced Vlasov–Maxwell equations, in which the electromagnetic fields are not separated into background and perturbed parts. The paradigm set of reduced plasma equations is given by the guiding-centre Vlasov–Maxwell equations, with variational formulations (Pfirsch & Morrison 1985; Similon 1985; Brizard & Tronci 2016) leading to exact reduced energy–momentum conservation laws. While the resulting reduced stress tensor is manifestly asymmetric, as noted by Pfirsch & Morrison (1985) and Similon (1985), we show that the apparent asymmetry of the reduced stress tensor is due to polarization and magnetization effects derived from a ponderomotive Hamiltonian (Cary & Kaufman 1981; Brizard 2009). Since the stress tensor must be explicitly symmetric when the electromagnetic fields $(\boldsymbol{E}, \boldsymbol{B})$ are not split into time-independent (equilibrium) and time-dependent (perturbed) components, we use the guiding-centre Vlasov–Maxwell model of Brizard & Tronci (2016) and show that the explicit expressions of the guiding-centre polarization and magnetization guarantee a symmetric guiding-centre stress tensor.

In § 3, we review the gauge-free gyrocentre Hamiltonian models derived by Burby & Brizard (2019) and Brizard (2020), where the equations of motion are solely expressed in terms of the perturbed electromagnetic fields $(\boldsymbol{E}_1, \boldsymbol{B}_1)$. Here, following the standard gyrokinetic formalism (Brizard & Hahm 2007), the magnetic field $\boldsymbol{B} = \boldsymbol{B}_0 + \epsilon\boldsymbol{B}_1$ is split into the time-independent equilibrium magnetic field $\boldsymbol{B}_0$, which is assumed to be axisymmetric (i.e. $\partial\boldsymbol{B}_0/\partial\varphi = \hat{\boldsymbol{z}} \times \boldsymbol{B}_0$), and the time-dependent magnetic field perturbation $\boldsymbol{B}_1$ (ϵ denotes the magnitude of the perturbation). In addition, we assume that $\boldsymbol{E} = \epsilon\boldsymbol{E}_1$ appears solely as a perturbation electric field in the present work, although a equilibrium electric field may also be considered (Brizard 1995). We note that the choice of the magnetic perturbation $\boldsymbol{B}_1$ is consistent with the source-free perturbed Maxwell equations $\nabla \cdot \boldsymbol{B}_1 = 0$ and $\partial\boldsymbol{B}_1/\partial t = -c\nabla \times \boldsymbol{E}_1$. In § 4, the gyrokinetic Vlasov–Maxwell equations are derived from a variational principle (Brizard 2000*a*,*b*), from which explicit expressions for the gyrocentre polarization and magnetization are obtained for both gauge-free gyrokinetic models. In § 5, the gyrokinetic conservation laws are derived by the Noether method. In the present work, we show that the apparent asymmetry of the gyrokinetic stress tensor, which is only due to electromagnetic field perturbations (since the guiding-centre stress tensor is symmetric), plays a crucial role in establishing an exact toroidal angular momentum conservation law in the presence of a non-uniform (but axisymmetric) equilibrium magnetic field. We also show that an exact gyrokinetic momentum transport equation will be obtained from the gyrokinetic Noether equation, which can be used to study momentum transport in general magnetic geometries.

2. Conservation laws for the reduced Vlasov–Maxwell equations

Although our primary motivation is to discuss exact conservation laws of gyrokinetic systems, we present a brief discussion on the form of these conservation laws for a generic set of reduced Vlasov–Maxwell equations, based on work presented at the 2006 Vlasovia conference (Brizard 2008). In this formal derivation, the electromagnetic fields ($\boldsymbol{E} = -\nabla\Phi - c^{-1}\partial_t\boldsymbol{A}, \boldsymbol{B} = \nabla\times\boldsymbol{A}$) are not split into time-independent background and time-dependent perturbed components and, by using canonical coordinates $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{p}})$, the dependence on the potentials $(\Phi, \boldsymbol{A})$ and the fields $(\boldsymbol{E}, \boldsymbol{B})$ only enter through the reduced Hamiltonian

$$\bar{H}(\bar{\boldsymbol{p}}; \Phi, \boldsymbol{A}, \boldsymbol{E}, \boldsymbol{B}) \equiv m\,|\bar{\boldsymbol{v}}|^2/2 + e\Phi + \bar{\Psi}(\bar{\boldsymbol{v}}; \boldsymbol{E}, \boldsymbol{B}), \tag{2.1}$$

where $\bar{\boldsymbol{v}} \equiv [\bar{\boldsymbol{p}} - (e/c)\boldsymbol{A}]/m$, the potentials and fields $(\Phi, \boldsymbol{A}, \boldsymbol{E}, \boldsymbol{B})$ are evaluated at the reduced position $\bar{\boldsymbol{x}}$ and $\bar{\Psi}(\bar{\boldsymbol{v}}; \boldsymbol{E}, \boldsymbol{B})$ denotes the ponderomotive Hamiltonian (Cary & Kaufman 1981; Brizard 2009). Here, the dependence of the ponderomotive Hamiltonian on the gauge-free term $\bar{\boldsymbol{v}}$ maintains the gauge-transformation property of the reduced Hamiltonian.

2.1. *Reduced Vlasov–Maxwell equations*

Using the reduced Hamiltonian (2.1), the reduced Vlasov–Maxwell equations are now expressed as follows. First, the reduced equations of motion are given in Hamiltonian canonical form as

$$\mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t = \partial\bar{H}/\partial\bar{\boldsymbol{p}} = \bar{\boldsymbol{v}} + \partial\bar{\Psi}/\partial\bar{\boldsymbol{p}}, \tag{2.2}$$

$$\mathrm{d}\bar{\boldsymbol{p}}/\mathrm{d}t = -\bar{\nabla}\bar{H} = -e\bar{\nabla}\Phi + (e/c)\,\bar{\nabla}\boldsymbol{A}\cdot\bar{\boldsymbol{v}} - \bar{\nabla}\bar{\Psi}. \tag{2.3}$$

If the reduced force equation (2.3) is written in terms of $\bar{\boldsymbol{v}}$, we find

$$m\frac{\mathrm{d}\bar{\boldsymbol{v}}}{\mathrm{d}t} = e\boldsymbol{E} + \frac{e}{c}\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t}\times\boldsymbol{B} + \bar{\nabla}\boldsymbol{E}\cdot\bar{\boldsymbol{\pi}} + \bar{\nabla}\boldsymbol{B}\cdot\bar{\boldsymbol{\mu}}, \tag{2.4}$$

where we used (2.2) on the right-hand side and the ponderomotive force

$$-\bar{\nabla}\bar{\Psi} = -\bar{\nabla}\boldsymbol{A}\cdot\left(-\frac{e}{c}\frac{\partial\bar{\Psi}}{\partial\bar{\boldsymbol{p}}}\right) + \bar{\nabla}\boldsymbol{E}\cdot\bar{\boldsymbol{\pi}} + \bar{\nabla}\boldsymbol{B}\cdot\bar{\boldsymbol{\mu}} \tag{2.5}$$

includes the reduced electric and magnetic dipole moments $(\bar{\boldsymbol{\pi}}, \bar{\boldsymbol{\mu}}) \equiv (-\partial\bar{\Psi}/\partial\boldsymbol{E}, -\partial\bar{\Psi}/\partial\boldsymbol{B})$ derived from the ponderomotive Hamiltonian. The reduced Vlasov equation is, therefore, expressed as

$$\frac{\partial \bar{f}}{\partial t} = -\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} \boldsymbol{\cdot} \bar{\nabla}\bar{f} - \frac{\mathrm{d}\bar{\boldsymbol{p}}}{\mathrm{d}t} \boldsymbol{\cdot} \frac{\partial \bar{f}}{\partial \bar{\boldsymbol{p}}}. \tag{2.6}$$

The reduced Maxwell equations, on the other hand, are expressed as

$$\nabla \boldsymbol{\cdot} \boldsymbol{E} = 4\pi\varrho \equiv 4\pi\left(\bar{\varrho} - \nabla \boldsymbol{\cdot} \bar{\mathbb{P}}\right), \tag{2.7}$$

$$\nabla \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{J} \equiv \frac{4\pi}{c}\left(\bar{\boldsymbol{J}} + \frac{\partial \bar{\mathbb{P}}}{\partial t} + c\nabla \times \bar{\mathbb{M}}\right), \tag{2.8}$$

with the source-free Maxwell equations

$$\left.\begin{aligned} \partial \boldsymbol{B}/\partial t + c\nabla \times \boldsymbol{E} = 0, \\ \nabla \boldsymbol{\cdot} \boldsymbol{B} = 0. \end{aligned}\right\} \tag{2.9}$$

In (2.7) and (2.8), the reduced charge and current densities $(\bar{\varrho}, \bar{\boldsymbol{J}})$ and the reduced polarization and magnetization $(\bar{\mathbb{P}}, \bar{\mathbb{M}})$ are derived from the reduced Hamiltonian:

$$\int_{\bar{p}} \bar{f}\,\partial \bar{H}(\bar{\boldsymbol{p}}; \Phi, \boldsymbol{A}, \boldsymbol{E}, \boldsymbol{B}) \equiv \partial\Phi\,\bar{\varrho} - \partial \boldsymbol{A} \boldsymbol{\cdot} \bar{\boldsymbol{J}}/c - \partial \boldsymbol{E} \boldsymbol{\cdot} \bar{\mathbb{P}} - \partial \boldsymbol{B} \boldsymbol{\cdot} \bar{\mathbb{M}}, \tag{2.10}$$

where the notation $\int_{\bar{p}}$ indicates an integral over canonical-momentum space (as well as including a sum over particle species) and ∂ denotes either a space–time partial derivative $(\nabla, \partial/\partial t)$ or an Eulerian variation δ. Specifically, we find the definitions

$$\left(\bar{\rho}, \bar{\boldsymbol{J}}, \bar{\mathbb{P}}, \bar{\mathbb{M}}\right) \equiv \int_{\bar{p}} \bar{f}\left(e, e\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t}, -\frac{\partial \bar{\Psi}}{\partial \boldsymbol{E}}, -\frac{\partial \bar{\Psi}}{\partial \boldsymbol{B}}\right), \tag{2.11}$$

where contributions arise from reduced particles located at the field position (i.e. $\bar{\boldsymbol{x}} = \boldsymbol{x}$).

We also note that the reduced Maxwell equations (2.7) and (2.8) can be written in terms of the reduced Maxwell fields

$$\left.\begin{aligned} \bar{\mathbb{D}} = \boldsymbol{E} + 4\pi\bar{\mathbb{P}}, \\ \bar{\mathbb{H}} = \boldsymbol{B} - 4\pi\bar{\mathbb{M}}, \end{aligned}\right\} \tag{2.12}$$

as

$$\nabla \boldsymbol{\cdot} \bar{\mathbb{D}} = 4\pi\bar{\varrho}, \tag{2.13}$$

$$\nabla \times \bar{\mathbb{H}} - \frac{1}{c}\frac{\partial \bar{\mathbb{D}}}{\partial t} = \frac{4\pi}{c}\bar{\boldsymbol{J}}, \tag{2.14}$$

which guarantees that the reduced charge conservation law

$$\frac{\partial \varrho}{\partial t} + \nabla \boldsymbol{\cdot} \boldsymbol{J} = \frac{\partial}{\partial t}\left(\bar{\varrho} - \nabla \boldsymbol{\cdot} \bar{\mathbb{P}}\right) + \nabla \boldsymbol{\cdot} \left(\bar{\boldsymbol{J}} + \frac{\partial \bar{\mathbb{P}}}{\partial t} + c\nabla \times \bar{\mathbb{M}}\right) = \frac{\partial \bar{\varrho}}{\partial t} + \nabla \boldsymbol{\cdot} \bar{\boldsymbol{J}} = 0 \tag{2.15}$$

follows directly from the charge conservation law.

2.2. *Reduced energy–momentum conservation laws*

Since the electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ is not split into equilibrium and perturbed components in the reduced Vlasov–Maxwell equations (2.6)–(2.8), the energy–momentum conservation laws derived for the reduced Vlasov equation (2.6) and the reduced Maxwell equations (2.7) and (2.8) (or (2.13) and (2.14)), with the source-free Maxwell equations (2.9), are direct consequences of the Noether theorem. Here, the reduced Noether equation (Brizard 2008) is first expressed as

$$0 = \frac{\partial}{\partial t}\left[\int_{\bar{p}} \bar{f}\delta\bar{\mathcal{S}} - \delta\boldsymbol{A}\cdot\frac{\bar{\mathbb{D}}}{4\pi c} + \frac{\delta t}{8\pi}\left(|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2\right)\right]$$
$$+ \nabla\cdot\left[\int_{\bar{p}} \bar{f}\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t}\delta\bar{\mathcal{S}} - \frac{1}{4\pi}\left(\delta\Phi\bar{\mathbb{D}} + \delta\boldsymbol{A}\times\bar{\mathbb{H}}\right) + \frac{\delta\boldsymbol{x}}{8\pi}\left(|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2\right)\right], \tag{2.16}$$

where the Eulerian variations

$$\left.\begin{aligned} \delta\bar{\mathcal{S}} &= \bar{\boldsymbol{p}}\cdot\delta\boldsymbol{x} - \bar{H}\delta t, \\ \delta\Phi &= \boldsymbol{E}\cdot\delta\boldsymbol{x} - c^{-1}\partial\delta\chi/\partial t, \\ \delta\boldsymbol{A} &= \boldsymbol{E}c\delta t + \delta\boldsymbol{x}\times\boldsymbol{B} + \nabla\delta\chi, \end{aligned}\right\} \tag{2.17}$$

with the gauge term defined as $\delta\chi \equiv \Phi c\delta t - \boldsymbol{A}\cdot\delta\boldsymbol{x}$, are generated by the space–time virtual displacements $(\delta\boldsymbol{x}, \delta t)$. We note that the gauge term $\delta\chi$ appears naturally when the Euler variations $\delta\Phi = -\delta t\partial\Phi/\partial t - \delta\boldsymbol{x}\cdot\nabla\Phi$ and $\delta\boldsymbol{A} = -\delta t\partial\boldsymbol{A}/\partial t - \delta\boldsymbol{x}\cdot\nabla\boldsymbol{A}$ are expressed in terms of the electric field $\boldsymbol{E} = -\nabla\Phi - c^{-1}\partial\boldsymbol{A}/\partial t$ and the magnetic field $\boldsymbol{B} = \nabla\times\boldsymbol{A}$.

We now remove the gauge-dependent terms by using the identity

$$-\frac{\partial}{\partial t}\left(\nabla\delta\chi\cdot\bar{\mathbb{D}}\right) + \nabla\cdot\left(\frac{\partial\delta\chi}{\partial t}\bar{\mathbb{D}} - c\nabla\delta\chi\times\bar{\mathbb{H}}\right)$$
$$= \frac{\partial}{\partial t}\left(\delta\chi\nabla\cdot\bar{\mathbb{D}}\right) - \nabla\cdot\left[\delta\chi\left(\frac{\partial\bar{\mathbb{D}}}{\partial t} - c\nabla\times\bar{\mathbb{H}}\right)\right], \tag{2.18}$$

and, using the reduced Maxwell equations (2.13) and (2.14) and the gauge-independent term $\delta\bar{\mathcal{S}} + (e/c)\delta\chi = m\bar{\boldsymbol{v}}\cdot\delta\boldsymbol{x} - \bar{K}\delta t$, the reduced Noether equation (2.16) yields the reduced energy–momentum conservation law

$$\frac{\partial}{\partial t}\left(\boldsymbol{\mathcal{P}}\cdot\delta\boldsymbol{x} - \mathcal{E}\delta t\right) + \nabla\cdot\left(\mathsf{T}\cdot\delta\boldsymbol{x} - \boldsymbol{S}\delta t\right) = 0. \tag{2.19}$$

Here, the reduced energy–momentum densities

$$\mathcal{E} = \int_{\bar{p}} \bar{f}\bar{K} + \frac{1}{4\pi}\boldsymbol{E}\cdot\bar{\mathbb{D}} - \frac{1}{8\pi}\left(|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2\right), \tag{2.20}$$

$$\boldsymbol{\mathcal{P}} = \int_{\bar{p}} \bar{f}\,\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t}m\bar{\boldsymbol{v}} + \frac{\bar{\mathbb{D}}\times\boldsymbol{B}}{4\pi c} \tag{2.21}$$

both include reduced polarization effects, with (2.21) displaying the Minkowski form $(\bar{\mathbb{D}}\times\boldsymbol{B}/4\pi c)$ for the reduced electromagnetic momentum density. The reduced energy-density flux

$$\boldsymbol{S} = \int_{\bar{p}} \bar{f}\, \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} \bar{K} + \frac{c}{4\pi} \boldsymbol{E} \times \bar{\mathbb{H}}, \tag{2.22}$$

on the other hand, displays the Abraham form $(\boldsymbol{E} \times c\bar{\mathbb{H}}/4\pi)$ for the reduced Poynting flux, while the reduced stress tensor

$$\mathsf{T} = \int_{\bar{p}} \bar{f}\, \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} m\bar{\boldsymbol{v}} + \frac{\boldsymbol{I}}{4\pi} \left[\frac{1}{2} \left(|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2| \right) + \boldsymbol{B} \cdot \bar{\mathbb{H}} \right] - \frac{1}{4\pi} \left(\boldsymbol{B}\bar{\mathbb{H}} + \bar{\mathbb{D}}\boldsymbol{E} \right) \tag{2.23}$$

is composed of the reduced Reynolds stress tensor, which includes the ponderomotive velocity $\partial\bar{\Psi}/\partial\bar{\boldsymbol{p}} = \mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t - \bar{\boldsymbol{v}}$, and the reduced Maxwell stress tensor, which includes polarization and magnetization corrections. We immediately see that the reduced stress tensor (2.23), which can be expressed as

$$\begin{aligned} \mathsf{T} = {} & \frac{\boldsymbol{I}}{8\pi} \left(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2 \right) - \frac{1}{4\pi} (\boldsymbol{E}\boldsymbol{E} + \boldsymbol{B}\boldsymbol{B}) + \int_{\bar{p}} \bar{f} \left[m \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} + \boldsymbol{I} \left(\boldsymbol{B} \cdot \frac{\partial\bar{\Psi}}{\partial\boldsymbol{B}} \right) \right] \\ & - \int_{\bar{p}} \bar{f} \left(m \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} \frac{\partial\bar{\Psi}}{\partial\bar{\boldsymbol{p}}} + \boldsymbol{B} \frac{\partial\bar{\Psi}}{\partial\boldsymbol{B}} - \frac{\partial\bar{\Psi}}{\partial\boldsymbol{E}} \boldsymbol{E} \right), \end{aligned} \tag{2.24}$$

is manifestly asymmetric as a result of ponderomotive, polarization and magnetization effects appearing in the last line of (2.24).

The apparent asymmetry of the reduced stress tensor (2.24) implies that the azimuthal angular momentum may not be conserved:

$$\frac{\partial\mathcal{P}_\varphi}{\partial t} + \nabla \cdot \boldsymbol{T}_\varphi = \mathsf{T}^\top : \nabla(\partial\boldsymbol{x}/\partial\varphi) \equiv \hat{\boldsymbol{z}} \cdot \left(\int_p \bar{f}\bar{\boldsymbol{N}} \right), \tag{2.25}$$

unless the reduced torque $\bar{\boldsymbol{N}}$ vanishes identically. In (2.25), $\mathcal{P}_\varphi \equiv \boldsymbol{\mathcal{P}} \cdot \partial\boldsymbol{x}/\partial\varphi$ is the azimuthal angular momentum density, $\boldsymbol{T}_\varphi \equiv \mathsf{T} \cdot \partial\boldsymbol{x}/\partial\varphi$ is the azimuthal angular momentum-density flux, $\mathsf{T}^\top$ denotes the transpose of T and, since the dyadic tensor $\nabla(\partial\boldsymbol{x}/\partial\varphi)$ is antisymmetric, the reduced torque $\bar{\boldsymbol{N}}$ is expressed as

$$\bar{\boldsymbol{N}} \equiv \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} \times m\bar{\boldsymbol{v}} - (\bar{\boldsymbol{\pi}} \times \boldsymbol{E} + \bar{\boldsymbol{\mu}} \times \boldsymbol{B}), \tag{2.26}$$

which includes contributions from the electric and magnetic torques. The required symmetry of the reduced stress tensor (2.24) must, therefore, introduce constraints on the reduced polarization and magnetization, which force the reduced torque (2.26) to vanish identically.

2.3. *Guiding-centre Vlasov–Maxwell equations*

The apparent asymmetry of the guiding-centre stress tensor was independently noted by Pfirsch & Morrison (1985) and Similon (1985), using different variational formulations. It was recently shown by Brizard & Tronci (2016) and Sugama *et al.* (2016), however, that the guiding-centre stress tensor is indeed explicitly symmetric. Here, we use the variational formulation of the guiding-centre Vlasov–Maxwell model of Brizard & Tronci (2016) to show that the guiding-centre torque, derived from the generic reduced torque (2.26), vanishes identically.

In the work of Brizard & Tronci (2016), which considers the simplest case $\boldsymbol{E} = 0$, the guiding-centre canonical momentum is defined as $\bar{\boldsymbol{p}} = (e/c)\boldsymbol{A} + \bar{p}_\parallel\,\hat{\boldsymbol{b}}$, which implies that $\bar{\boldsymbol{v}} = (\bar{p}_\parallel/m)\,\hat{\boldsymbol{b}}$ and the guiding-centre electric and magnetic dipole moments are

$$\left.\begin{aligned}\bar{\boldsymbol{\pi}} &= (e\hat{\boldsymbol{b}}/\Omega)\times \mathrm{d}\bar{\boldsymbol{x}}/\mathrm{d}t = (e\hat{\boldsymbol{b}}/\Omega)\times \partial\bar{\Psi}/\partial\bar{\boldsymbol{p}},\\ \bar{\boldsymbol{\mu}} &= -\bar{\mu}\hat{\boldsymbol{b}} + \bar{\boldsymbol{\pi}}\times(\bar{p}_\parallel\hat{\boldsymbol{b}}/mc).\end{aligned}\right\} \tag{2.27}$$

Hence, using (2.2) and (2.27), we easily verify that the guiding-centre torque (2.26) vanishes:

$$\bar{\boldsymbol{N}} = \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t}\times m\bar{\boldsymbol{v}} - \left(-\bar{\mu}\hat{\boldsymbol{b}} + \bar{\boldsymbol{\pi}}\times\frac{\bar{p}_\parallel\hat{\boldsymbol{b}}}{mc}\right)\times\boldsymbol{B} = \left(-m\bar{\boldsymbol{v}} + \bar{p}_\parallel\hat{\boldsymbol{b}}\right)\times\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t}\equiv 0, \tag{2.28}$$

and the guiding-centre stress tensor (2.23) is symmetric (Sugama *et al.* 2016; Brizard & Tronci 2016):

$$\begin{aligned}\mathsf{T}_{\mathrm{gc}} &= \frac{1}{4\pi}\left(\frac{\mathbf{I}}{2}|\boldsymbol{B}|^2 - \boldsymbol{B}\boldsymbol{B}\right) + \int_{\bar{p}}\bar{f}\left[m\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t}\left(\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} - \frac{\partial\bar{\Psi}}{\partial\bar{\boldsymbol{p}}}\right) + \mathbf{I}\left(\boldsymbol{B}\cdot\frac{\partial\bar{\Psi}}{\partial\boldsymbol{B}}\right) - \boldsymbol{B}\frac{\partial\bar{\Psi}}{\partial\boldsymbol{B}}\right]\\ &= \frac{1}{4\pi}\left(\frac{\mathbf{I}}{2}|\boldsymbol{B}|^2 - \boldsymbol{B}\boldsymbol{B}\right) + \mathsf{P}_{\mathrm{CGL}} + \int_{\bar{p}}\bar{f}\left[\bar{p}_\parallel\left(\frac{\partial\bar{\Psi}}{\partial\bar{\boldsymbol{p}}}\,\hat{\boldsymbol{b}} + \hat{\boldsymbol{b}}\,\frac{\partial\bar{\Psi}}{\partial\bar{\boldsymbol{p}}}\right)\right],\end{aligned} \tag{2.29}$$

where $\mathsf{P}_{\mathrm{CGL}} = \int_{\bar{p}}\bar{f}[(\bar{p}_\parallel^2/m)\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \bar{\mu}B(\mathbf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})]$ is the symmetric Chew–Goldberger–Low pressure tensor and the ponderomotive velocity $\partial\bar{\Psi}/\partial\bar{\boldsymbol{p}}$, which is assumed to be perpendicular to $\boldsymbol{B}$, represents the magnetic-drift velocity.

3. Gauge-free gyrocentre Lagrangian dynamics

In this section, we present two gauge-free gyrokinetic models whose gyrocentre equations of motion only involve the perturbed electromagnetic fields $(\boldsymbol{E}_1, \boldsymbol{B}_1)$, thereby guaranteeing gauge freedom. Here, the separation of the perturbed magnetic field $\boldsymbol{B}_1$ from the unperturbed (equilibrium) magnetic field $\boldsymbol{B}_0$ satisfies the perturbed Faraday's law $\partial\boldsymbol{B}_1/\partial t \equiv -c\nabla\times\boldsymbol{E}_1$, while the equilibrium magnetic field $\boldsymbol{B}_0$ is assumed to be toroidally axisymmetric, so that $\partial\boldsymbol{B}_0/\partial\varphi \equiv \hat{\boldsymbol{z}}\times\boldsymbol{B}_0$.

Gauge-free electromagnetic gyrokinetic Vlasov–Maxwell models were recently derived in the Hamiltonian representation by Burby & Brizard (2019) and in the symplectic representation by Brizard (2020). The general form of the gauge-free gyrocentre Lagrangian is defined on gyrocentre phase space, with coordinates $(\boldsymbol{X}, p_\parallel, \mu, \zeta)$, as

$$L_{\mathrm{gy}} = \left[\frac{e}{c}\left(\boldsymbol{A}_0^* + \epsilon\boldsymbol{A}_{1\mathrm{gy}}\right) + \boldsymbol{\Pi}_{\mathrm{gy}}\right]\cdot\dot{\boldsymbol{X}} + J\dot{\zeta} - \left(e\epsilon\Phi_{1\mathrm{gy}} + K_{\mathrm{gy}}\right) \equiv \boldsymbol{P}_{\mathrm{gy}}\cdot\dot{\boldsymbol{X}} + J\dot{\zeta} - H_{\mathrm{gy}}, \tag{3.1}$$

where the gyrocentre gyroaction $J \equiv (mc/e)\mu$ (which is canonically conjugate to the gyrocentre gyroangle ζ) is used here only as a matter of convenience, and

$$(e/c)\boldsymbol{A}_0^* \equiv (e/c)\boldsymbol{A}_0 + p_\parallel\,\hat{\boldsymbol{b}}_0 - J\left(\boldsymbol{R}_0 + \frac{1}{2}\nabla\times\hat{\boldsymbol{b}}_0\right) \tag{3.2}$$

is expressed in terms of the unperturbed (equilibrium) magnetic field $\boldsymbol{B}_0 = \nabla\times\boldsymbol{A}_0 = B_0\,\hat{\boldsymbol{b}}_0$, and (3.2) includes the gyrogauge vector field $\boldsymbol{R}_0 \equiv \nabla\hat{\boldsymbol{1}}\cdot\hat{\boldsymbol{2}}$ (where $\hat{\boldsymbol{b}}_0 \equiv \hat{\boldsymbol{1}}\times\hat{\boldsymbol{2}}$) and higher-order guiding-centre corrections (Tronko & Brizard 2015) associated with the guiding-centre electric dipole moment $\boldsymbol{\pi}_{\mathrm{gc}} \equiv (e\hat{\boldsymbol{b}}_0/\Omega_0) \times \dot{\boldsymbol{X}}_{\mathrm{gc}}$. Similar higher-order guiding-centre corrections are retained by Fan *et al.* (2020), with the main difference that, in our work, these guiding-centre terms are explicitly time-independent (and non-variational).

3.1. *Gauge-free gyrocentre models*

In the gyrocentre Hamiltonian model of Burby & Brizard (2019), presented here in the drift-kinetic limit considered by Hirvijoki *et al.* (2020), we find the definitions

$$(\Phi_{1\mathrm{gy}}, \boldsymbol{A}_{1\mathrm{gy}}, \boldsymbol{\Pi}_{\mathrm{gy}}) = (\Phi_1, \boldsymbol{A}_1, 0), \tag{3.3}$$

where the perturbation fields are evaluated at the gyrocentre position $\boldsymbol{X}$ and the gyrocentre kinetic energy is

$$\begin{aligned} K_{\mathrm{gy}} = \frac{p_\|^2}{2m} + \mu\left(B_0 + \epsilon B_{1\|} + \frac{\epsilon^2}{2}\frac{|\boldsymbol{B}_1|^2}{B_0}\right) - \boldsymbol{\pi}_{\mathrm{gc}} \cdot \epsilon\left(\boldsymbol{E}_1 + (p_\|\hat{\boldsymbol{b}}_0/mc) \times \boldsymbol{B}_1\right) \\ - \epsilon^2\frac{mc^2}{2B_0^2}\left|\boldsymbol{E}_1 + (p_\|\hat{\boldsymbol{b}}_0/mc) \times \boldsymbol{B}_1\right|^2, \end{aligned} \tag{3.4}$$

where $B_{1\|} \equiv \hat{\boldsymbol{b}}_0 \cdot \boldsymbol{B}_1$ denotes the parallel component of the perturbed magnetic field $\boldsymbol{B}_1$. We note that the gauge-free model considered by Hirvijoki *et al.* (2020) omits the guiding-centre electric dipole moment $\boldsymbol{\pi}_{\mathrm{gc}}$ in the gyrocentre kinetic energy (3.4) and, thus, the gyrocentre polarization and magnetization derived without this term are incomplete. We explicitly show in § 5, however, that this omission does not jeopardize the energy–momentum conservation laws.

Next, in the gyrocentre symplectic model of Brizard (2020), we find

$$\left.\begin{aligned} (\Phi_{1\mathrm{gy}}, \boldsymbol{A}_{1\mathrm{gy}}) = (\langle\Phi_{1\mathrm{gc}}\rangle, \langle\boldsymbol{A}_{1\mathrm{gc}}\rangle), \\ \boldsymbol{\Pi}_{\mathrm{gy}} = \epsilon\left(\langle\boldsymbol{E}_{1\mathrm{gc}}\rangle + (p_\|\hat{\boldsymbol{b}}_0/mc) \times \langle\boldsymbol{B}_{1\mathrm{gc}}\rangle\right) \times e\hat{\boldsymbol{b}}_0/\Omega_0, \end{aligned}\right\} \tag{3.5}$$

where perturbation fields are evaluated at $\boldsymbol{X} + \boldsymbol{\rho}_0$, with $\langle\cdots\rangle$ denoting the standard gyroangle averaging (since the lowest-order guiding-centre gyroradius $\boldsymbol{\rho}_0$ depends on the gyrocentre gyroangle ζ), and the gyrocentre kinetic energy is

$$\begin{aligned} K_{\mathrm{gy}} = \frac{p_\|^2}{2m} + \mu\left(B_0 + \epsilon\langle\langle B_{1\|\mathrm{gc}}\rangle\rangle + \frac{\epsilon^2}{2}\frac{|\boldsymbol{B}_1|^2}{B_0}\right) \\ + \epsilon^2\frac{mc^2}{2B_0^2}\left|\boldsymbol{E}_1 + (p_\|\hat{\boldsymbol{b}}_0/mc) \times \boldsymbol{B}_1\right|^2. \end{aligned} \tag{3.6}$$

In (3.6), the finite-Larmor-radius effects are included only at first order in the perturbation expansion, with $\langle\langle\cdots\rangle\rangle$ denoting the gyrosurface averaging introduced by Porazik & Lin (2011).

Previous symplectic gyrokinetic models considered either the parallel component $\langle A_{1\|\mathrm{gc}}\rangle$ of the perturbed vector potential (Hahm *et al.* 1988; Brizard 2017) or the inclusion of the perturbed $E \times B$ velocity (Wang & Hahm 2010*a*,*b*; Leerink, Parra & Heikkinen 2010), or both (Duthoit, Hahm & Wang 2014). In the present symplectic gyrokinetic model (3.5) and (3.6), the addition of the perturbed magnetic flutter momentum to the $E \times B$ momentum yields a covariant treatment of the electric dipole moment in the gyrocentre polarization and magnetization; see (3.23)–(3.26). In their guiding-centre treatment, Fan *et al.* (2020) considered an extension of the Pfirsch & Morrison (1985) variational formulation by including higher-order guiding-centre corrections, where both electric and magnetic fields ($\boldsymbol{E}, \boldsymbol{B} = B\hat{\boldsymbol{b}}$) are considered as variational fields.

3.2. *Gyrocentre Euler–Lagrange equations*

The gyrocentre Euler–Lagrange equations involving arbitrary variations in $(\boldsymbol{X}, p_\parallel, J)$ are, respectively,

$$0 = e\boldsymbol{E}^*_{\mathrm{gy}} + \frac{e}{c}\dot{\boldsymbol{X}} \times \boldsymbol{B}^*_{\mathrm{gy}} - \dot{p}_\parallel \boldsymbol{b}^*_{\mathrm{gy}}, \tag{3.7}$$

$$0 = \dot{\boldsymbol{X}} \cdot \boldsymbol{b}^*_{\mathrm{gy}} - \partial K_{\mathrm{gy}}/\partial p_\parallel, \tag{3.8}$$

$$0 = \dot{\zeta} + \dot{\boldsymbol{X}} \cdot \partial \boldsymbol{P}_{\mathrm{gy}}/\partial J - \partial H_{\mathrm{gy}}/\partial J, \tag{3.9}$$

where the effective gyrocentre electric field $\boldsymbol{E}^*_{\mathrm{gy}}$ is defined as

$$e\boldsymbol{E}^*_{\mathrm{gy}} \equiv -\nabla H_{\mathrm{gy}} - \frac{\partial \boldsymbol{P}_{\mathrm{gy}}}{\partial t} = \epsilon e\boldsymbol{E}_{1\mathrm{gy}} - \left(\frac{\partial \boldsymbol{\Pi}_{\mathrm{gy}}}{\partial t} + \nabla K_{\mathrm{gy}}\right), \tag{3.10}$$

with $\boldsymbol{E}_{1\mathrm{gy}} \equiv -\nabla \Phi_{1\mathrm{gy}} - c^{-1}\partial \boldsymbol{A}_{1\mathrm{gy}}/\partial t$. The effective gyrocentre magnetic field $\boldsymbol{B}^*_{\mathrm{gy}}$ is defined as

$$\boldsymbol{B}^*_{\mathrm{gy}} \equiv \nabla \times \left(\frac{c}{e}\boldsymbol{P}_{\mathrm{gy}}\right) = \boldsymbol{B}^*_0 + \epsilon \boldsymbol{B}_{1\mathrm{gy}} + \nabla \times \left(\frac{c}{e}\boldsymbol{\Pi}_{\mathrm{gy}}\right), \tag{3.11}$$

with $\boldsymbol{B}^*_0 \equiv \nabla \times \boldsymbol{A}^*_0$ and $\boldsymbol{B}_{1\mathrm{gy}} \equiv \nabla \times \boldsymbol{A}_{1\mathrm{gy}}$, while

$$\boldsymbol{b}^*_{\mathrm{gy}} \equiv \partial \boldsymbol{P}_{\mathrm{gy}}/\partial p_\parallel = \hat{\boldsymbol{b}}_0 + \partial \boldsymbol{\Pi}_{\mathrm{gy}}/\partial p_\parallel. \tag{3.12}$$

We note that the effective gyrocentre electromagnetic fields satisfy the source-free Maxwell equations $\nabla \cdot \boldsymbol{B}^*_{\mathrm{gy}} = 0$ and $\partial \boldsymbol{B}^*_{\mathrm{gy}}/\partial t + c\nabla \times \boldsymbol{E}^*_{\mathrm{gy}} = 0$.

The gyrocentre Euler–Lagrange equations (3.7) and (3.8) can also be written in Hamiltonian form as

$$\dot{\boldsymbol{X}} \equiv \left\{\boldsymbol{X}, H_{\mathrm{gy}}\right\}_{\mathrm{gy}} = \boldsymbol{E}^*_{\mathrm{gy}} \times \frac{c\boldsymbol{b}^*_{\mathrm{gy}}}{B^{**}_{\parallel\mathrm{gy}}} + \frac{\partial K_{\mathrm{gy}}}{\partial p_\parallel}\frac{\boldsymbol{B}^*_{\mathrm{gy}}}{B^{**}_{\parallel\mathrm{gy}}}, \tag{3.13}$$

$$\dot{p}_\parallel \equiv \left\{p_\parallel, H_{\mathrm{gy}}\right\}_{\mathrm{gy}} = e\boldsymbol{E}^*_{\mathrm{gy}} \cdot \frac{\boldsymbol{B}^*_{\mathrm{gy}}}{B^{**}_{\parallel\mathrm{gy}}}, \tag{3.14}$$

where $\{\,,\,\}_{\mathrm{gy}}$ denotes the gyrocentre Poisson bracket and $B^{**}_{\parallel\mathrm{gy}} \equiv \boldsymbol{b}^*_{\mathrm{gy}} \cdot \boldsymbol{B}^*_{\mathrm{gy}}$. We note that (3.13) and (3.14) satisfy the Euler–Lagrange identity:

$$\frac{\partial K_{\mathrm{gy}}}{\partial p_\parallel}\dot{p}_\parallel = e\boldsymbol{E}^*_{\mathrm{gy}} \cdot \frac{\partial K_{\mathrm{gy}}}{\partial p_\parallel}\frac{\boldsymbol{B}^*_{\mathrm{gy}}}{B^{**}_{\parallel\mathrm{gy}}} \equiv e\boldsymbol{E}^*_{\mathrm{gy}} \cdot \dot{\boldsymbol{X}}, \tag{3.15}$$

which will be useful in our discussion of energy conservation. The gyrocentre equations (3.13) and (3.14) also satisfy the Liouville theorem:

$$\begin{aligned}
\frac{\partial B_{\|\mathrm{gy}}^{**}}{\partial t} &= \frac{\partial \boldsymbol{b}_{\mathrm{gy}}^{*}}{\partial t}\boldsymbol{\cdot}\boldsymbol{B}_{\mathrm{gy}}^{*} + \boldsymbol{b}_{\mathrm{gy}}^{*}\boldsymbol{\cdot}\frac{\partial \boldsymbol{B}_{\mathrm{gy}}^{*}}{\partial t} = \frac{\partial}{\partial p_{\|}}\left(\frac{\partial \boldsymbol{P}_{\mathrm{gy}}}{\partial t}\right)\boldsymbol{\cdot}\boldsymbol{B}_{\mathrm{gy}}^{*} - \boldsymbol{b}_{\mathrm{gy}}^{*}\boldsymbol{\cdot}\nabla\times\left(c\boldsymbol{E}_{\mathrm{gy}}^{*}\right)\\
&= -\frac{\partial}{\partial p_{\|}}\left[\nabla\boldsymbol{\cdot}\left(H_{\mathrm{gy}}\boldsymbol{B}_{\mathrm{gy}}^{*}\right) + \dot{p}_{\|}\,B_{\|\mathrm{gy}}^{**}\right] - \frac{\partial \boldsymbol{P}_{\mathrm{gy}}}{\partial t}\boldsymbol{\cdot}\nabla\times\left(\frac{c\boldsymbol{b}_{\mathrm{gy}}^{*}}{e}\right)\\
&\quad - \nabla\boldsymbol{\cdot}\left[B_{\|\mathrm{gy}}^{**}\dot{\boldsymbol{X}} - \frac{\partial}{\partial p_{\|}}\left(H_{\mathrm{gy}}\boldsymbol{B}_{\mathrm{gy}}^{*}\right)\right] - \left(e\boldsymbol{E}_{\mathrm{gy}}^{*} + \nabla H_{\mathrm{gy}}\right)\boldsymbol{\cdot}\nabla\times\left(\frac{c\boldsymbol{b}_{\mathrm{gy}}^{*}}{e}\right)\\
&= -\nabla\boldsymbol{\cdot}\left(B_{\|\mathrm{gy}}^{**}\,\dot{\boldsymbol{X}}\right) - \frac{\partial}{\partial p_{\|}}\left(B_{\|\mathrm{gy}}^{**}\,\dot{p}_{\|}\right),
\end{aligned} \tag{3.16}$$

where we used (3.10).

3.3. *Eulerian field variations of the gyrocentre Lagrangian*

In the next section, we will need the Eulerian field variation of the gyrocentre Lagrangian (3.1) at a field point $\boldsymbol{x}$:

$$\delta L_{\mathrm{gy}} = \left(\frac{e}{c}\epsilon\delta\boldsymbol{A}_{1\mathrm{gy}} + \delta\boldsymbol{\Pi}_{\mathrm{gy}}\right)\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \left(e\epsilon\delta\Phi_{1\mathrm{gy}} + \delta K_{\mathrm{gy}}\right), \tag{3.17}$$

where, in contrast to the works of Sugama *et al.* (2021) and Fan *et al.* (2020), the guiding-centre Lagrangian terms $(e/c)\boldsymbol{A}_0^*\boldsymbol{\cdot}\dot{\boldsymbol{X}} + J\dot{\zeta} - (p_{\|}^2/2m + \mu B_0)$ are invariant in our gyrokinetic formalism. While Sugama *et al.* (2021) considered the simplest guiding-centre representation (with $e\boldsymbol{A}_0^*/c = e\boldsymbol{A}_0/c + p_{\|}\hat{\boldsymbol{b}}$), with field variations easily computed (e.g. $\delta\hat{\boldsymbol{b}} = (\hat{\boldsymbol{b}}\times\delta\boldsymbol{B})\times\hat{\boldsymbol{b}}/B$), the higher-order guiding-centre model used by Fan *et al.* (2020) requires complex expressions for the variations of the gyrogauge vector $\boldsymbol{R}_0 = \nabla\hat{\boldsymbol{1}}\boldsymbol{\cdot}\hat{\boldsymbol{2}}$, for example, in which the functional derivatives of all three unit vectors $(\hat{\boldsymbol{1}}, \hat{\boldsymbol{2}}, \hat{\boldsymbol{b}} = \hat{\boldsymbol{1}}\times\hat{\boldsymbol{2}})$ need to be computed, although they are not explicitly calculated.

Here, the field variations are defined in terms of the generic functional derivatives

$$\begin{pmatrix}\delta\Psi_1(\boldsymbol{X})\\ \langle\delta\Psi_1(\boldsymbol{X}+\boldsymbol{\rho}_0)\rangle\end{pmatrix} \equiv \int_{\boldsymbol{x}}\delta\Psi_1(\boldsymbol{x})\begin{pmatrix}\delta^3(\boldsymbol{X}-\boldsymbol{x})\\ \left\langle\delta^3(\boldsymbol{X}+\boldsymbol{\rho}_0-\boldsymbol{x})\right\rangle\end{pmatrix}, \tag{3.18}$$

where Ψ_1 denotes an arbitrary component of the perturbed electromagnetic potentials or fields. We note that the second expression in (3.18) is valid only if the equilibrium (non-variational) magnetic field appears in the definition of the lowest-order gyroangle-dependent gyroradius $\boldsymbol{\rho}_0$. Hence, we find

$$\left(\frac{\delta\Phi_{1\mathrm{gy}}}{\delta\Phi_1(\boldsymbol{x})}, \frac{\delta A_{1\mathrm{gy}}^i}{\delta A_1^j(\boldsymbol{x})}, \frac{\delta E_{1\mathrm{gy}}^i}{\delta E_1^j(\boldsymbol{x})}, \frac{\delta B_{1\mathrm{gy}}^i}{\delta B_1^j(\boldsymbol{x})}\right) = \begin{cases}\left(\delta^3,\, \delta_j^i\delta^3,\, \delta_j^i\delta^3,\, \delta_j^i\delta^3\right)\\ \left(\langle\delta_{\mathrm{gc}}^3\rangle,\, \delta_j^i\langle\delta_{\mathrm{gc}}^3\rangle,\, \delta_j^i\langle\delta_{\mathrm{gc}}^3\rangle,\, \delta_j^i\langle\delta_{\mathrm{gc}}^3\rangle\right)\end{cases} \tag{3.19}$$

and

$$\left(\frac{\delta B_{1\|}(\boldsymbol{X})}{\delta\boldsymbol{B}_1(\boldsymbol{x})}, \frac{\delta\langle\langle B_{1\|\mathrm{gc}}\rangle\rangle}{\delta\boldsymbol{B}_1(\boldsymbol{x})}\right) = \left(\delta^3\hat{\boldsymbol{b}}_0,\, \langle\langle\delta_{\mathrm{gc}}^3\rangle\rangle\hat{\boldsymbol{b}}_0\right), \tag{3.20}$$

with $\delta^3 \equiv \delta^3(\boldsymbol{X}-\boldsymbol{x})$ and $\delta_{\mathrm{gc}}^3 \equiv \delta^3(\boldsymbol{X}+\boldsymbol{\rho}_0-\boldsymbol{x})$ used in the gyrocentre models of Burby & Brizard (2019) and Brizard (2020), respectively, and δ_j^i denotes the standard Kronecker delta.

In the gyrocentre model (3.4) of Burby & Brizard (2019), we find

$$\epsilon^{-1}\frac{\delta K_{\rm gy}}{\delta \boldsymbol{E}_1(\boldsymbol{x})} = -\delta^3\left[\boldsymbol{\pi}_{\rm gc} + \epsilon\frac{mc^2}{B_0^2}\left(\boldsymbol{E}_1 + \frac{p_\parallel\hat{\boldsymbol{b}}_0}{mc}\times\boldsymbol{B}_1\right)\right] \equiv -\delta^3\left(\boldsymbol{\pi}_{\rm gc} + \epsilon\,\boldsymbol{\pi}_2\right), \tag{3.21}$$

$$\epsilon^{-1}\frac{\delta K_{\rm gy}}{\delta \boldsymbol{B}_1(\boldsymbol{x})} = \delta^3\mu\left(\hat{\boldsymbol{b}}_0 + \epsilon\frac{\boldsymbol{B}_1}{B_0}\right) - \delta^3\left(\boldsymbol{\pi}_{\rm gc} + \epsilon\,\boldsymbol{\pi}_2\right)\times\frac{p_\parallel\hat{\boldsymbol{b}}_0}{mc}, \tag{3.22}$$

where the gyrocentre electric dipole moment $\boldsymbol{\pi}_{\rm gc} + \epsilon\,\boldsymbol{\pi}_2$ includes the guiding-centre contribution $\boldsymbol{\pi}_{\rm gc}$ and its first-order gyrocentre correction $\boldsymbol{\pi}_2$ (derived from the second-order gyrocentre Hamiltonian), while the intrinsic gyrocentre magnetic dipole moment $-\mu(\hat{\boldsymbol{b}}_0 + \epsilon\boldsymbol{B}_1/B_0)$ is accompanied by the moving gyrocentre electric dipole moment contribution $(\boldsymbol{\pi}_{\rm gc} + \epsilon\,\boldsymbol{\pi}_2)\times p_\parallel\hat{\boldsymbol{b}}_0/mc$.

In the gyrocentre model (3.5) and (3.6) of Brizard (2020), on the other hand, we find

$$\epsilon^{-1}\frac{\delta \boldsymbol{\Pi}_{\rm gy}}{\delta \boldsymbol{E}_1(\boldsymbol{x})}\cdot\dot{\boldsymbol{X}} = \langle\delta^3_{\rm gc}\rangle\,\frac{e\hat{\boldsymbol{b}}_0}{\Omega_0}\times\dot{\boldsymbol{X}} \equiv \langle\delta^3_{\rm gc}\rangle\,\boldsymbol{\pi}_{\rm gy}. \tag{3.23}$$

$$\epsilon^{-1}\frac{\delta \boldsymbol{\Pi}_{\rm gy}}{\delta \boldsymbol{B}_1(\boldsymbol{x})}\cdot\dot{\boldsymbol{X}} = \langle\delta^3_{\rm gc}\rangle\left(\boldsymbol{\pi}_{\rm gy}\times\frac{p_\parallel\hat{\boldsymbol{b}}_0}{mc}\right), \tag{3.24}$$

and

$$\epsilon^{-1}\frac{\delta K_{\rm gy}}{\delta \boldsymbol{E}_1(\boldsymbol{x})} = \epsilon\,\delta^3\frac{mc^2}{B_0^2}\left(\boldsymbol{E}_1 + \frac{p_\parallel\hat{\boldsymbol{b}}_0}{mc}\times\boldsymbol{B}_1\right) \equiv \epsilon\,\delta^3\boldsymbol{\pi}_2, \tag{3.25}$$

$$\epsilon^{-1}\frac{\delta K_{\rm gy}}{\delta \boldsymbol{B}_1(\boldsymbol{x})} = \mu\left(\langle\langle\delta^3_{\rm gc}\rangle\rangle\hat{\boldsymbol{b}}_0 + \epsilon\,\delta^3\boldsymbol{B}_1/B_0\right) + \epsilon\,\delta^3\left(\boldsymbol{\pi}_2\times\frac{p_\parallel\hat{\boldsymbol{b}}_0}{mc}\right). \tag{3.26}$$

We note that the gyrocentre polarization and magnetization derived from (3.21) and (3.22) for the gyrokinetic model of Burby & Brizard (2019) are explicitly truncated at first order in the perturbation amplitudes of the electric and magnetic fields $(\boldsymbol{E}_1, \boldsymbol{B}_1)$. Because the gyrocentre velocity (3.13) appears in the expressions (3.23) and (3.24) for the gyrokinetic model of Brizard (2020), however, the corresponding gyrocentre polarization and magnetization contain contributions at higher orders in perturbation amplitude.

4. Gyrokinetic variational principle

The gyrokinetic Vlasov–Maxwell equations can be derived from several equivalent variational principles: Low–Lagrange (Sugama 2000); Euler (Brizard 2000*a*,*b*, 2009, 2010*a*, 2017); Hamilton–Jacobi (Correa-Restrepo & Pfirsch 2004); and Euler–Poincaré (Squire *et al.* 2013; Hirvijoki *et al.* 2020). In recent work, Brizard & Tronci (2016) showed how the guiding-centre Vlasov–Maxwell equations (derived without a separation between time-independent equilibrium and variational dynamical plasma fields) can be explicitly derived from many of these equivalent variational principles.

In the present work, the separation of equilibrium and perturbed electromagnetic fields introduces a low-frequency gyrokinetic space–time ordering that assumes that the non-uniform equilibrium magnetic field is time-independent and non-variational. Applications of Noether's theorem, which explicitly take into account the properties of the equilibrium magnetic field, follow most naturally from an Eulerian variational principle. In recent work, Hirvijoki *et al.* (2020) derived the energy–momentum and angular momentum conservation laws (without proof) within an Euler–Poincaré variational formulation for the Vlasov–Maxwell and drift-kinetic Vlasov–Maxwell equations described by the gauge-free gyrocentre model of Burby & Brizard (2019).

We are now ready to derive the gauge-free gyrokinetic Vlasov–Maxwell equations from an Eulerian variational principle $\delta\mathcal{A}_{gy} = 0$, based on the gyrokinetic action functional (Brizard 2000*b*)

$$\mathcal{A}_{gy} \equiv -\int \mathcal{F}_{gy}\mathcal{H}_{gy}\,\mathrm{d}^8\mathcal{Z} + \int \frac{\mathrm{d}^4x}{8\pi}\left(|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2\right), \tag{4.1}$$

where summation over particle species is implicitly assumed in the first term and the infinitesimal extended phase-space volume element $\mathrm{d}^8\mathcal{Z}$ does not include the Jacobian $\mathcal{J}_{gy}$. Instead, the perturbation-field-dependent Jacobian is inserted in the definition of the gyrocentre extended Vlasov density:

$$\mathcal{F}_{gy} \equiv \mathcal{J}_{gy}\mathcal{F} \equiv \mathcal{J}_{gy}F\delta(w - H_{gy}), \tag{4.2}$$

which also includes an energy delta function that enforces the constraint $\mathcal{H}_{gy} = H_{gy} - w \equiv 0$ in extended gyrocentre phase space.

The variation of the gyrokinetic action functional yields

$$\delta\mathcal{A}_{gy} = -\int \left(\delta\mathcal{F}_{gy}\mathcal{H}_{gy} + \mathcal{F}_{gy}\delta\mathcal{H}_{gy}\right)\mathrm{d}^8\mathcal{Z} + \int \frac{\mathrm{d}^4x}{4\pi}\left(\epsilon\delta\boldsymbol{E}_1\cdot\boldsymbol{E} - \epsilon\delta\boldsymbol{B}_1\cdot\boldsymbol{B}\right), \tag{4.3}$$

where the constrained electromagnetic variations

$$\left.\begin{aligned}\delta\boldsymbol{E}_1(\boldsymbol{x}) &\equiv -\nabla\delta\Phi_1(\boldsymbol{x}) - c^{-1}\partial_t\delta\boldsymbol{A}_1(\boldsymbol{x}),\\ \delta\boldsymbol{B}_1(\boldsymbol{x}) &\equiv \nabla\times\delta\boldsymbol{A}_1(\boldsymbol{x})\end{aligned}\right\} \tag{4.4}$$

satisfy the Faraday constraint equation $\nabla\delta\boldsymbol{E}_1 + c^{-1}\partial\delta\boldsymbol{B}_1/\partial t = 0$ and $\nabla\cdot\delta\boldsymbol{B}_1 = 0$, with the equilibrium magnetic field $\boldsymbol{B}_0$ held constant under field variations. The variation of the gyrocentre Hamiltonian

$$\delta\mathcal{H}_{gy} = \epsilon e\delta\Phi_{1gy} + \delta\boldsymbol{E}_1\cdot\frac{\delta K_{gy}}{\delta\boldsymbol{E}_1} + \delta\boldsymbol{B}_1\cdot\frac{\delta K_{gy}}{\delta\boldsymbol{B}_1} \tag{4.5}$$

is expressed in terms of $\delta\Phi_1$ and $(\delta\boldsymbol{E}_1, \delta\boldsymbol{B}_1)$. The variation of the gyrocentre extended Vlasov density $\delta\mathcal{F}_{gy} \equiv \delta\mathcal{J}_{gy}\mathcal{F} + \mathcal{J}_{gy}\delta\mathcal{F}$ is expressed as

$$\begin{aligned}\delta\mathcal{F}_{gy} &= \mathcal{F}\left(\frac{\partial\delta\boldsymbol{P}_{gy}}{\partial p_\parallel}\cdot\frac{e}{c}\boldsymbol{B}^*_{gy} + \boldsymbol{b}^*_{gy}\cdot\nabla\times\delta\boldsymbol{P}_{gy}\right) + \mathcal{J}_{gy}\left(\{\delta\mathcal{S}, \mathcal{F}\}_{gy} + \delta\boldsymbol{P}_{gy}\cdot\{\boldsymbol{X}, \mathcal{F}\}_{gy}\right)\\ &\equiv -\frac{\partial}{\partial\mathcal{Z}^a}\left(\delta\mathcal{Z}^a\mathcal{F}_{gy}\right),\end{aligned} \tag{4.6}$$

where the virtual extended phase-space displacement

$$\delta\mathcal{Z}^a \equiv \{\mathcal{Z}^a, \delta\mathcal{S}\}_{gy} - \delta\boldsymbol{P}_{gy}\cdot\{\boldsymbol{X}, \mathcal{Z}^a\}_{gy} \tag{4.7}$$

is defined in terms of a canonical part generated by $\delta\mathcal{S}$ and a non-canonical part generated by

$$\delta\boldsymbol{P}_{gy} = \epsilon\frac{e}{c}\delta\boldsymbol{A}_{1gy} + \delta\boldsymbol{E}_1\cdot\frac{\partial\boldsymbol{\Pi}_{gy}}{\partial\boldsymbol{E}_1} + \delta\boldsymbol{B}_1\cdot\frac{\partial\boldsymbol{\Pi}_{gy}}{\partial\boldsymbol{B}_1}. \tag{4.8}$$

We note that the final form in (4.6) for the Eulerian variation $\delta\mathcal{F}_{\rm gy}$ is a natural phase-space generalization of the variation $\delta n = -\nabla\boldsymbol{\cdot}(\delta\boldsymbol{x}\, n)$ for the fluid particle density n. In addition, in the Hamiltonian model of Burby & Brizard (2019), the last two terms are absent.

The first two variations in the Vlasov term in (4.3) can be combined:

$$
\begin{aligned}
-\delta(\mathcal{F}_{\rm gy}\mathcal{H}_{\rm gy}) = & -\mathcal{J}_{\rm gy}\{\mathcal{F},\mathcal{H}_{\rm gy}\}_{\rm gy}\delta\mathcal{S} + \mathcal{F}_{\rm gy}\delta L_{\rm gy} \\
& + \frac{\partial}{\partial t}\left(\mathcal{F}_{\rm gy}\delta\mathcal{S}\right) + \nabla\boldsymbol{\cdot}\left(\dot{\boldsymbol{X}}\mathcal{F}_{\rm gy}\delta\mathcal{S}\right) + \frac{\partial}{\partial p_\parallel}\left(\dot{p}_\parallel\,\mathcal{F}_{\rm gy}\delta\mathcal{S}\right),
\end{aligned}
\tag{4.9}
$$

where the variation of the gyrocentre Lagrangian (3.1) is

$$
\begin{aligned}
\delta L_{\rm gy} \equiv & \;\epsilon\left(\frac{e}{c}\delta\boldsymbol{A}_{1\rm gy}\boldsymbol{\cdot}\dot{\boldsymbol{X}} - e\delta\Phi_{1\rm gy}\right) + \delta\boldsymbol{E}_1\boldsymbol{\cdot}\left(\frac{\partial\boldsymbol{\Pi}_{\rm gy}}{\partial\boldsymbol{E}_1}\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \frac{\partial K_{\rm gy}}{\partial\boldsymbol{E}_1}\right) \\
& + \delta\boldsymbol{B}_1\boldsymbol{\cdot}\left(\frac{\partial\boldsymbol{\Pi}_{\rm gy}}{\partial\boldsymbol{B}_1}\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \frac{\partial K_{\rm gy}}{\partial\boldsymbol{B}_1}\right).
\end{aligned}
\tag{4.10}
$$

Using (3.19) and (3.21)–(3.22) or (3.23)–(3.26), the Lagrangian variation term

$$
\int_{\mathcal{Z}}\mathcal{F}_{\rm gy}\delta L_{\rm gy} = \int_{x}\left(\frac{\epsilon}{c}\delta\boldsymbol{A}_1\boldsymbol{\cdot}\boldsymbol{J}_{\rm gy} - \epsilon\delta\Phi_1\varrho_{\rm gy} + \epsilon\delta\boldsymbol{E}_1\boldsymbol{\cdot}\mathbb{P}_{\rm gy} + \epsilon\delta\boldsymbol{B}_1\boldsymbol{\cdot}\mathbb{M}_{\rm gy}\right)
\tag{4.11}
$$

can be expressed in terms of the gyrocentre charge and current densities:

$$
\begin{aligned}
\left(\varrho_{\rm gy}(\boldsymbol{x}),\boldsymbol{J}_{\rm gy}(\boldsymbol{x})\right) \equiv & \int_{\mathcal{Z}}\mathcal{F}_{\rm gy}\left(-\epsilon^{-1}\frac{\delta L_{\rm gy}}{\delta\Phi_1(\boldsymbol{x})},\,\epsilon^{-1}\frac{\delta L_{\rm gy}}{\delta\boldsymbol{A}_1(\boldsymbol{x})}\right) \\
= & \int_{\mathcal{Z}}\mathcal{F}_{\rm gy}\left(e\frac{\delta\Phi_{1\rm gy}}{\delta\Phi_1(\boldsymbol{x})},\,e\frac{\delta\boldsymbol{A}_{1\rm gy}}{\delta\boldsymbol{A}_1(\boldsymbol{x})}\boldsymbol{\cdot}\dot{\boldsymbol{X}}\right)
\end{aligned}
\tag{4.12}
$$

and the gyrocentre polarization and magnetization:

$$
\mathbb{P}_{\rm gy}(\boldsymbol{x}) \equiv \int_{\mathcal{Z}}\mathcal{F}_{\rm gy}\epsilon^{-1}\frac{\delta L_{\rm gy}}{\delta\boldsymbol{E}_1(\boldsymbol{x})} = \int_{\mathcal{Z}}\mathcal{F}_{\rm gy}\left(\epsilon^{-1}\frac{\delta\boldsymbol{\Pi}_{\rm gy}}{\delta\boldsymbol{E}_1(\boldsymbol{x})}\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \epsilon^{-1}\frac{\delta K_{\rm gy}}{\delta\boldsymbol{E}_1(\boldsymbol{x})}\right),
\tag{4.13}
$$

$$
\mathbb{M}_{\rm gy}(\boldsymbol{x}) \equiv \int_{\mathcal{Z}}\mathcal{F}_{\rm gy}\epsilon^{-1}\frac{\delta L_{\rm gy}}{\delta\boldsymbol{B}_1(\boldsymbol{x})} = \int_{\mathcal{Z}}\mathcal{F}_{\rm gy}\left(\epsilon^{-1}\frac{\delta\boldsymbol{\Pi}_{\rm gy}}{\delta\boldsymbol{B}_1(\boldsymbol{x})}\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \epsilon^{-1}\frac{\delta K_{\rm gy}}{\delta\boldsymbol{B}_1(\boldsymbol{x})}\right).
\tag{4.14}
$$

When the gauge-free gyrokinetic models represented by (3.3)–(3.4) and (3.5)–(3.6) are used, the gyrocentre polarization is given for the Burby & Brizard (2019) model (top) and the Brizard (2020) model (bottom) as

$$
\mathbb{P}_{\rm gy}(\boldsymbol{x}) = \int_{\mathcal{Z}}\mathcal{F}_{\rm gy}\left\{\begin{array}{c}\delta^3\left(\boldsymbol{\pi}_{\rm gc}+\epsilon\boldsymbol{\pi}_2\right)\\ \langle\delta^3_{\rm gc}\rangle\boldsymbol{\pi}_{\rm gy} - \epsilon\delta^3\boldsymbol{\pi}_2,\end{array}\right.
\tag{4.15}
$$

where $\boldsymbol{\pi}_2$ and $\boldsymbol{\pi}_{\rm gy}$ are defined in (3.21) and (3.23), respectively, and the gyrocentre magnetization is

$$
\mathbb{M}_{\rm gy}(\boldsymbol{x}) = \int_{\mathcal{Z}}\mathcal{F}_{\rm gy}\left\{\begin{array}{l}\delta^3\left[-\mu\left(\hat{\boldsymbol{b}}_0+\epsilon\boldsymbol{B}_1/B_0\right)+\left(\boldsymbol{\pi}_{\rm gc}+\epsilon\boldsymbol{\pi}_2\right)\boldsymbol{\times}\left(p_\parallel\hat{\boldsymbol{b}}_0/mc\right)\right]\\ \quad -\mu\left(\langle\langle\delta^3_{\rm gc}\rangle\rangle\hat{\boldsymbol{b}}_0+\epsilon\delta^3\boldsymbol{B}_1/B_0\right)\\ \quad +\left(\langle\delta^3_{\rm gc}\rangle\boldsymbol{\pi}_{\rm gy}-\epsilon\delta^3\boldsymbol{\pi}_2\right)\boldsymbol{\times}\left(p_\parallel\hat{\boldsymbol{b}}_0/mc\right).\end{array}\right.
\tag{4.16}
$$

We note, here, that the lowest-order guiding-centre contributions to polarization and magnetization are derived from the first-order gyrocentre Lagrangian, which circumvents the need to consider guiding-centre variations as in the works of Sugama *et al.* (2021) and Fan *et al.* (2020).

The variation of the Maxwell Lagrangian density can be expressed as

$$
\begin{aligned}
\delta \boldsymbol{E}_1 \cdot \boldsymbol{E} - \delta \boldsymbol{B}_1 \cdot \boldsymbol{B} = \delta \boldsymbol{A}_1 \cdot \left(\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} \right) + \delta \varPhi_1 (\nabla \cdot \boldsymbol{E}) \\
- \frac{\partial}{\partial t}\left(\frac{1}{c}\delta \boldsymbol{A}_1 \cdot \boldsymbol{E} \right) - \nabla \cdot (\delta \varPhi_1 \boldsymbol{E} + \delta \boldsymbol{A}_1 \times \boldsymbol{B}) .
\end{aligned} \tag{4.17}
$$

If we now combine (4.9)–(4.17) into the variation of the gyrokinetic action functional (4.3), $\delta \mathcal{A}_{gy} \equiv \int \delta \mathcal{L}_{gy}\, d^4 x$, we obtain the variation of the gyrokinetic Lagrangian density:

$$
\begin{aligned}
\delta \mathcal{L}_{gy} = & - \int_P \mathcal{J}_{gy} \{\mathcal{F}, \mathcal{H}_{gy}\}_{gy} \delta \mathcal{S} + \frac{\epsilon \delta \varPhi_1}{4\pi} (\nabla \cdot \mathbb{D}_{gy} - 4\pi \varrho_{gy}) \\
& + \frac{\epsilon}{4\pi} \delta \boldsymbol{A}_1 \cdot \left(\frac{1}{c}\frac{\partial \mathbb{D}_{gy}}{\partial t} - \nabla \times \mathbb{H}_{gy} + \frac{4\pi}{c} \boldsymbol{J}_{gy} \right) \\
& + \frac{\partial}{\partial t}\left(\int_P \mathcal{F}_{gy} \delta \mathcal{S} - \frac{\epsilon}{4\pi} \delta \boldsymbol{A}_1 \cdot \mathbb{D}_{gy} \right) \\
& + \nabla \cdot \left(\int_P \dot{\boldsymbol{X}} \mathcal{F}_{gy} \delta \mathcal{S} - \frac{\epsilon}{4\pi} \left(\delta \varPhi_1 \mathbb{D}_{gy} + \delta \boldsymbol{A}_1 \times \mathbb{H}_{gy} \right) \right),
\end{aligned} \tag{4.18}
$$

where the gyrocentre macroscopic electromagnetic fields are defined as

$$
\left.
\begin{aligned}
\mathbb{D}_{gy} &\equiv \epsilon \boldsymbol{E}_1 + 4\pi \mathbb{P}_{gy}, \\
\mathbb{H}_{gy} &\equiv \boldsymbol{B}_0 + \epsilon \boldsymbol{B}_1 - 4\pi \mathbb{M}_{gy},
\end{aligned}
\right\} \tag{4.19}
$$

and the variations $(\delta \mathcal{S}, \delta \varPhi_1, \delta \boldsymbol{A}_1)$ are assumed to be arbitrary. Variation with respect to $\delta \mathcal{S}$ yields the gyrokinetic Vlasov equation in extended phase space $\{\mathcal{F}, \mathcal{H}_{gy}\}_{gy} = 0$. If we integrate $\mathcal{J}_{gy}\{\mathcal{F}, \mathcal{H}_{gy}\}_{gy}$ over the energy w coordinate, we find

$$
\begin{aligned}
0 = \int \mathcal{J}_{gy} \{\mathcal{F}, \mathcal{H}_{gy}\}_{gy}\, dw &= \int \frac{\partial}{\partial \mathcal{Z}^a} \left(\mathcal{J}_{gy} \mathcal{F} \dot{\mathcal{Z}}^a \right) dw \\
&= \frac{\partial (\mathcal{J}_{gy} F)}{\partial t} + \nabla \cdot \left(\mathcal{J}_{gy} F \dot{\boldsymbol{X}} \right) + \frac{\partial}{\partial p_\parallel} \left(\mathcal{J}_{gy} F \dot{p}_\parallel \right) \\
&\equiv \mathcal{J}_{gy} \left(\frac{\partial F}{\partial t} + \dot{\boldsymbol{X}} \cdot \nabla F + \dot{p}_\parallel \frac{\partial F}{\partial p_\parallel} \right),
\end{aligned} \tag{4.20}
$$

where we have used the Liouville theorem (3.16) to obtain the last expression in order to recover the gyrokinetic Vlasov equation.

Next, the variation with respect to the electromagnetic potentials $(\delta\Phi_1, \delta\boldsymbol{A}_1)$ yields the macroscopic gyrokinetic Maxwell equations:

$$\nabla \cdot \mathbb{D}_{gy} = 4\pi \varrho_{gy}, \tag{4.21}$$

$$\nabla \times \mathbb{H}_{gy} = \frac{1}{c}\frac{\partial \mathbb{D}_{gy}}{\partial t} + \frac{4\pi}{c}\boldsymbol{J}_{gy}, \tag{4.22}$$

which can also be expressed as the microscopic Maxwell equations:

$$\nabla \cdot \epsilon \boldsymbol{E}_1 = 4\pi \left(\varrho_{gy} - \nabla \cdot \mathbb{P}_{gy}\right), \tag{4.23}$$

$$\nabla \times (\boldsymbol{B}_0 + \epsilon \boldsymbol{B}_1) = \frac{\epsilon}{c}\frac{\partial \boldsymbol{E}_1}{\partial t} + \frac{4\pi}{c}\left(\boldsymbol{J}_{gy} + \frac{\partial \mathbb{P}_{gy}}{\partial t} + c\nabla \times \mathbb{M}_{gy}\right). \tag{4.24}$$

These equations are complemented by Faraday's law

$$\frac{\partial \boldsymbol{B}_1}{\partial t} + c\nabla \times \boldsymbol{E}_1 = 0 \tag{4.25}$$

and $\nabla \cdot \boldsymbol{B}_1 = 0$. Now that the gyrokinetic Vlasov–Maxwell equations (4.20)–(4.22) have been derived from a variational principle, we use the remaining part of the gyrokinetic Lagrangian density variation (4.18) to derive exact conservation laws.

5. Gyrokinetic Noether equation and conservation laws

The variational derivation of the reduced Vlasov–Maxwell equations guarantees that these reduced equations satisfy exact energy–momentum conservation laws (Pfirsch & Morrison 1985; Similon 1985; Correa-Restrepo & Pfirsch 2004; Brizard 2008). In particular, the exact conservation of the gyrokinetic Vlasov–Maxwell energy (Brizard 1989*b*, 2010*a*) has played an important role in the numerical implementation of the energy-conserving gyrokinetic equations (Garbet *et al.* 2010).

For this purpose, the remaining terms in (4.18) are combined to yield the gyrokinetic Noether equation:

$$\begin{aligned}\delta \mathcal{L}_{gy} = {} & \frac{\partial}{\partial t}\left(\int_P \mathcal{F}_{gy}\delta\mathcal{S} - \frac{\epsilon}{4\pi c}\delta\boldsymbol{A}_1 \cdot \mathbb{D}_{gy}\right) \\ & + \nabla \cdot \left[\int_P \dot{\boldsymbol{X}}\mathcal{F}_{gy}\delta\mathcal{S} - \frac{\epsilon}{4\pi}\left(\delta\Phi_1\mathbb{D}_{gy} + \delta\boldsymbol{A}_1 \times \mathbb{H}_{gy}\right)\right],\end{aligned} \tag{5.1}$$

where the variations are now explicitly expressed in terms of the space–time displacements $\delta\boldsymbol{x}$ and δt:

$$\left.\begin{aligned}\delta\mathcal{S} &\equiv \boldsymbol{P}_{gy} \cdot \delta\boldsymbol{x} - w\delta t, \\ \delta\Phi_1 &\equiv -\delta\boldsymbol{x} \cdot \nabla\Phi_1 - \delta t\partial\Phi_1/\partial t = \boldsymbol{E}_1 \cdot \delta\boldsymbol{x} - c^{-1}\partial\delta\chi_1/\partial t, \\ \delta\boldsymbol{A}_1 &\equiv -\delta\boldsymbol{x} \cdot \nabla\boldsymbol{A}_1 - \delta t\partial\boldsymbol{A}_1/\partial t = \boldsymbol{E}_1 c\delta t + \delta\boldsymbol{x} \times \boldsymbol{B}_1 + \nabla\delta\chi_1,\end{aligned}\right\} \tag{5.2}$$

with the gauge variation $\delta\chi_1$ defined as $\delta\chi_1 \equiv \Phi_1 c\delta t - \boldsymbol{A}_1 \cdot \delta\boldsymbol{x}$. Upon rearranging the gauge variation $\delta\chi_1$, and using the identity

$$\begin{aligned}& -\frac{\partial}{\partial t}\left(\nabla\delta\chi_1 \cdot \mathbb{D}_{gy}\right) + \nabla \cdot \left(\frac{\partial\delta\chi_1}{\partial t}\mathbb{D}_{gy} - c\nabla\delta\chi_1 \times \mathbb{H}_{gy}\right) \\ & \quad = \frac{\partial}{\partial t}\left(\delta\chi_1\nabla \cdot \mathbb{D}_{gy}\right) - \nabla \cdot \left[\delta\chi_1\left(\frac{\partial \mathbb{D}_{gy}}{\partial t} - c\nabla \times \mathbb{H}_{gy}\right)\right],\end{aligned} \tag{5.3}$$

with the macroscopic gyrokinetic Maxwell equations (4.21)–(4.22), we obtain the gauge-invariant form of the gyrokinetic Noether equation (5.1):

$$\delta\mathcal{L}_{gy} = \partial\delta\mathcal{N}_{gy}/\partial t + \nabla\boldsymbol{\cdot}\delta\boldsymbol{\Gamma}_{gy}, \tag{5.4}$$

where the action-density variation is

$$\delta\mathcal{N}_{gy} = \int_P \mathcal{F}_{gy}\left(\delta\mathcal{S} + \epsilon\frac{e}{c}\delta\chi_{1gy}\right) - \left(\epsilon\boldsymbol{E}_1\delta t + \delta\boldsymbol{x}\boldsymbol{\times}\frac{\epsilon}{c}\boldsymbol{B}_1\right)\boldsymbol{\cdot}\frac{\mathbb{D}_{gy}}{4\pi} \tag{5.5}$$

and the action-density-flux variation is

$$\begin{aligned}\delta\boldsymbol{\Gamma}_{gy} = {} & \int_P \dot{\boldsymbol{X}}\,\mathcal{F}_{gy}\left(\delta\mathcal{S} + \epsilon\frac{e}{c}\delta\chi_{1gy}\right) - \delta\boldsymbol{x}\boldsymbol{\cdot}\left(\frac{\epsilon}{4\pi}\boldsymbol{E}_1\mathbb{D}_{gy}\right) \\ & + \frac{\epsilon}{4\pi}\left(\boldsymbol{E}_1 c\delta t + \delta\boldsymbol{x}\boldsymbol{\times}\boldsymbol{B}_1\right)\times\mathbb{H}_{gy}.\end{aligned} \tag{5.6}$$

Here, the gauge-invariant terms are

$$\begin{aligned}\delta\mathcal{S} + \epsilon\frac{e}{c}\delta\chi_{1gy} &= \left(\boldsymbol{P}_{gy} - \epsilon\frac{e}{c}\boldsymbol{A}_{1gy}\right)\boldsymbol{\cdot}\delta\boldsymbol{x} - \left(w - \epsilon e\Phi_{1gy}\right)\delta t \\ &= \left(\frac{e}{c}\boldsymbol{A}_0^* + \boldsymbol{\Pi}_{gy}\right)\boldsymbol{\cdot}\delta\boldsymbol{x} - \left(K_{gy} - \mathcal{H}_{gy}\right)\delta t.\end{aligned} \tag{5.7}$$

We note that the guiding-centre vector potential $\boldsymbol{A}_0^*$, which yields the unperturbed equilibrium magnetic field $\boldsymbol{B}_0^* = \nabla\boldsymbol{\times}\boldsymbol{A}_0^*$, is not subject to a gauge transformation.

A complete expression for the gyrokinetic Noether equation (5.4) also requires an explicit expression for the Lagrangian variation $\delta\mathcal{L}_{gy}$ on the left-hand side of (5.4). For the derivation of the momentum–energy conservation laws, we consider the specific space–time variations of the gyrokinetic Lagrangian density:

$$\begin{aligned}\delta\mathcal{L}_{gy} = {} & -\left(\delta t\frac{\partial}{\partial t} + \delta\boldsymbol{x}\boldsymbol{\cdot}\nabla\right)\left[\frac{1}{8\pi}\left(\epsilon^2|\boldsymbol{E}_1|^2 - |\boldsymbol{B}|^2\right)\right] \\ & + \delta\boldsymbol{x}\boldsymbol{\cdot}\left[\int_P \mathcal{J}_{gy}F\left(\nabla'\boldsymbol{P}_{gy}\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \nabla' K_{gy}\right) - \nabla B_0\boldsymbol{\cdot}\frac{\boldsymbol{B}}{4\pi}\right],\end{aligned} \tag{5.8}$$

where the gradient operator ∇' only takes into account the non-uniformity of the equilibrium magnetic field, i.e. the first-order fields $(\Phi_1, \boldsymbol{A}_1, \boldsymbol{E}_1, \boldsymbol{B}_1)$ are frozen at a fixed position $\boldsymbol{x}$ so that, for example, $\nabla'\Phi_{1gy} = 0$ and $\nabla'\langle\langle B_{1\|gc}\rangle\rangle = \nabla\hat{\boldsymbol{b}}_0\boldsymbol{\cdot}\langle\langle\boldsymbol{B}_{1gc}\rangle\rangle$. It is in the second line of (5.8) that the Noether theorem draws its full force. In addition, the w-integration was performed to leave the standard gyrocentre Vlasov distribution $F(\boldsymbol{X}, p_\|, \mu, t)$, with $\int_P$ now denoting an integration over $(p_\|, \mu)$.

The final form of the gyrokinetic Noether equation is obtained by equating (5.4) and (5.8), where the virtual space–time displacements $(\delta\boldsymbol{x}, \delta t)$ appear explicitly:

$$\begin{aligned}& \frac{\partial}{\partial t}\left[\delta\mathcal{N}_{gy} + \frac{\delta t}{8\pi}\left(\epsilon^2|\boldsymbol{E}_1|^2 - |\boldsymbol{B}|^2\right)\right] + \nabla\boldsymbol{\cdot}\left[\delta\boldsymbol{\Gamma}_{gy} + \frac{\delta\boldsymbol{x}}{8\pi}\left(\epsilon^2|\boldsymbol{E}_1|^2 - |\boldsymbol{B}|^2\right)\right] \\ & \quad = \delta\boldsymbol{x}\boldsymbol{\cdot}\left[\int_P \mathcal{J}_{gy}F\left(\nabla'\boldsymbol{P}_{gy}\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \nabla' K_{gy}\right) - \nabla B_0\boldsymbol{\cdot}\frac{\boldsymbol{B}}{4\pi}\right].\end{aligned} \tag{5.9}$$

This form of the Noether theorem relies on the constrained variations (4.4), (4.6) and (5.2), which is in contrast to the more traditional formulation based on the connection between conservation laws and symmetries of the Vlasov–Maxwell Lagrangian (see, for example, Hirvijoki *et al.* (2020) and references therein). Here, for each conservation law derived from our gyrokinetic Noether equation (5.9), we also present an explicit proof based on the gyrokinetic Vlasov–Maxwell equations (4.20)–(4.22).

5.1. *Gyrokinetic energy conservation law*

Since the equilibrium magnetic field $\boldsymbol{B}_0$ is time-independent, the total energy associated with the gyrokinetic Vlasov–Maxwell equations (4.20)–(4.22) is conserved. We derive the energy conservation law from the gyrokinetic Noether equation by setting $\delta t \neq 0$ and $\delta \boldsymbol{x} = 0$ in (5.9), which yields the gyrokinetic energy conservation law:

$$\partial \mathcal{E}_{\mathrm{gy}}/\partial t + \nabla \boldsymbol{\cdot} \boldsymbol{S}_{\mathrm{gy}} = 0, \tag{5.10}$$

where the gyrokinetic energy density is

$$\begin{aligned}\mathcal{E}_{\mathrm{gy}} &= \int_P \mathcal{J}_{\mathrm{gy}} F K_{\mathrm{gy}} + \frac{\epsilon}{4\pi}\boldsymbol{E}_1 \boldsymbol{\cdot} \mathbb{D}_{\mathrm{gy}} - \frac{1}{8\pi}\left(\epsilon^2|\boldsymbol{E}_1|^2 - |\boldsymbol{B}|^2\right) \\ &= \int_P \mathcal{J}_{\mathrm{gy}} F \left[\frac{p_\|^2}{2m} + \mu\left(B_0 + \epsilon\langle\langle B_{1\|\mathrm{gc}}\rangle\rangle + \frac{\epsilon^2}{2}\frac{|\boldsymbol{B}_1|^2}{B_0}\right)\right. \\ &\quad \left. + \boldsymbol{E}_1(\boldsymbol{x}) \boldsymbol{\cdot} \left(\frac{\delta \boldsymbol{\Pi}_{\mathrm{gy}}}{\delta \boldsymbol{E}_1(\boldsymbol{x})} - \frac{\delta K_{\mathrm{gy}}}{\delta \boldsymbol{E}_1(\boldsymbol{x})}\right)\right] + \frac{1}{8\pi}\left(\epsilon^2|\boldsymbol{E}_1|^2 + |\boldsymbol{B}|^2\right),\end{aligned} \tag{5.11}$$

while the gyrokinetic energy-density flux is

$$\boldsymbol{S}_{\mathrm{gy}} = \int_P \mathcal{J}_{\mathrm{gy}} F \dot{\boldsymbol{X}} K_{\mathrm{gy}} + \frac{c}{4\pi}\epsilon \boldsymbol{E}_1 \times \mathbb{H}_{\mathrm{gy}}, \tag{5.12}$$

where the polarization and magnetization $(\mathbb{P}_{\mathrm{gy}}, \mathbb{M}_{\mathrm{gy}})$ are defined in (4.13) and (4.14), with $\mathbb{H}_{\mathrm{gy}}$ defined in (4.19). In addition, we note that the gyrokinetic polarization and magnetization $(\mathbb{P}_{\mathrm{gy}}, \mathbb{M}_{\mathrm{gy}})$ include the full gyrocentre velocity $\dot{\boldsymbol{X}}$ defined in (3.13), which is expressed in terms of the effective electric and magnetic fields (3.10)–(3.11). We also note that, as shown by Burby *et al.* (2015), the gyrokinetic Vlasov–Maxwell Hamiltonian functional is naturally derived from the gyrokinetic energy density (5.11).

The explicit proof of gyrokinetic energy conservation, which applies to both gauge-free gyrokinetic models (Burby & Brizard 2019; Brizard 2020) considered here, proceeds as follows. First, we begin with the partial time derivative of the gyrokinetic energy density (5.11):

$$\begin{aligned}\frac{\partial \mathcal{E}_{\mathrm{gy}}}{\partial t} &= \int_P \left[\frac{\partial(\mathcal{J}_{\mathrm{gy}}F)}{\partial t}K_{\mathrm{gy}} + \mathcal{J}_{\mathrm{gy}}F\left(\frac{\partial \boldsymbol{E}_1}{\partial t}\boldsymbol{\cdot}\frac{\partial K_{\mathrm{gy}}}{\partial \boldsymbol{E}_1} + \frac{\partial \boldsymbol{B}_1}{\partial t}\boldsymbol{\cdot}\frac{\partial K_{\mathrm{gy}}}{\partial \boldsymbol{B}_1}\right)\right] \\ &\quad + \frac{\epsilon \boldsymbol{E}_1}{4\pi}\boldsymbol{\cdot}\frac{\partial \mathbb{D}_{\mathrm{gy}}}{\partial t} + \epsilon\frac{\partial \boldsymbol{E}_1}{\partial t}\boldsymbol{\cdot}\mathbb{P}_{\mathrm{gy}} + \frac{\boldsymbol{B}}{4\pi}\boldsymbol{\cdot}\epsilon\frac{\partial \boldsymbol{B}_1}{\partial t},\end{aligned} \tag{5.13}$$

where we expanded the term $\partial K_{\mathrm{gy}}(\boldsymbol{E}_1, \boldsymbol{B}_1)/\partial t$ and used the definition (4.19) for $\mathbb{D}_{\mathrm{gy}}$. Using the phase-space divergence form (4.20) of the gyrokinetic Vlasov equation, the first term on the right-hand side can be expressed as

$$\int_P \frac{\partial(\mathcal{J}_{\mathrm{gy}}F)}{\partial t} K_{\mathrm{gy}} = -\nabla\boldsymbol{\cdot}\left(\int_P \mathcal{J}_{\mathrm{gy}} F K_{\mathrm{gy}} \dot{\boldsymbol{X}}\right) + \int_P \mathcal{J}_{\mathrm{gy}} F\left(\frac{\partial K_{\mathrm{gy}}}{\partial p_\|}\,\dot{p}_\| + \dot{\boldsymbol{X}}\boldsymbol{\cdot}\nabla K_{\mathrm{gy}}\right), \tag{5.14}$$

while, using the definitions (4.13) and (4.14) of the gyrokinetic polarization and magnetization, the gyrokinetic kinetic terms in (5.13) can be expressed as

$$\int_P \mathcal{J}_{\mathrm{gy}} F \frac{\partial K_{\mathrm{gy}}}{\partial t} = -\epsilon\left(\frac{\partial \boldsymbol{E}_1}{\partial t}\boldsymbol{\cdot}\mathbb{P}_{\mathrm{gy}} + \frac{\partial \boldsymbol{B}_1}{\partial t}\boldsymbol{\cdot}\mathbb{M}_{\mathrm{gy}}\right) + \int_P \mathcal{J}_{\mathrm{gy}} F\left(\frac{\partial \boldsymbol{\Pi}_{\mathrm{gy}}}{\partial t}\boldsymbol{\cdot}\dot{\boldsymbol{X}}\right). \tag{5.15}$$

By combining these expressions, (5.13) becomes

$$\begin{aligned}\frac{\partial \mathcal{E}_{\mathrm{gy}}}{\partial t} &= -\nabla\boldsymbol{\cdot}\left(\int_P \mathcal{J}_{\mathrm{gy}} F K_{\mathrm{gy}}\dot{\boldsymbol{X}}\right) + \frac{\mathbb{H}_{\mathrm{gy}}}{4\pi}\boldsymbol{\cdot}\epsilon\frac{\partial \boldsymbol{B}_1}{\partial t} + \frac{\epsilon \boldsymbol{E}_1}{4\pi}\boldsymbol{\cdot}\frac{\partial \mathbb{D}_{\mathrm{gy}}}{\partial t}\\ &\quad + \int_P \mathcal{J}_{\mathrm{gy}} F\left[\frac{\partial K_{\mathrm{gy}}}{\partial p_\|}\,\dot{p}_\| + \dot{\boldsymbol{X}}\boldsymbol{\cdot}\left(\nabla K_{\mathrm{gy}} + \frac{\partial \boldsymbol{\Pi}_{\mathrm{gy}}}{\partial t}\right)\right],\end{aligned} \tag{5.16}$$

where we introduced the definition (4.19) for $\mathbb{H}_{\mathrm{gy}}$. Next, we use Faraday's law (4.25) to write

$$\frac{\mathbb{H}_{\mathrm{gy}}}{4\pi}\boldsymbol{\cdot}\epsilon\frac{\partial \boldsymbol{B}_1}{\partial t} = -\frac{c\mathbb{H}_{\mathrm{gy}}}{4\pi}\boldsymbol{\cdot}\nabla\times\epsilon\boldsymbol{E}_1 = -\nabla\boldsymbol{\cdot}\left(\frac{c}{4\pi}\epsilon\boldsymbol{E}_1\times\mathbb{H}_{\mathrm{gy}}\right) - \frac{\epsilon\boldsymbol{E}_1}{4\pi}\boldsymbol{\cdot}c\nabla\times\mathbb{H}_{\mathrm{gy}}, \tag{5.17}$$

so that (5.16) becomes

$$\begin{aligned}\frac{\partial \mathcal{E}_{\mathrm{gy}}}{\partial t} + \nabla\boldsymbol{\cdot}\boldsymbol{S}_{\mathrm{gy}} &= -\frac{\epsilon\boldsymbol{E}_1}{4\pi}\boldsymbol{\cdot}\left(c\nabla\times\mathbb{H}_{\mathrm{gy}} - \frac{\partial \mathbb{D}_{\mathrm{gy}}}{\partial t}\right)\\ &\quad + \int_P \mathcal{J}_{\mathrm{gy}} F\left[\frac{\partial K_{\mathrm{gy}}}{\partial p_\|}\,\dot{p}_\| + \dot{\boldsymbol{X}}\boldsymbol{\cdot}\left(\nabla K_{\mathrm{gy}} + \frac{\partial \boldsymbol{\Pi}_{\mathrm{gy}}}{\partial t}\right)\right],\end{aligned} \tag{5.18}$$

where we reconstructed the gyrokinetic energy-density flux (5.12) on the left-hand side of (5.18). Lastly, we use the identity derived from (3.10):

$$\nabla K_{\mathrm{gy}} + \frac{\partial \boldsymbol{\Pi}_{\mathrm{gy}}}{\partial t} = e\left(\epsilon\boldsymbol{E}_{1\mathrm{gy}} - \boldsymbol{E}^*_{\mathrm{gy}}\right), \tag{5.19}$$

and we use the macroscopic gyrokinetic Maxwell equation (4.22), with

$$-\frac{\epsilon\boldsymbol{E}_1}{4\pi}\boldsymbol{\cdot}\left(c\nabla\times\mathbb{H}_{\mathrm{gy}} - \frac{\partial \mathbb{D}_{\mathrm{gy}}}{\partial t}\right) = -\epsilon\boldsymbol{E}_1\boldsymbol{\cdot}\boldsymbol{J}_{\mathrm{gy}} = -\int_P \mathcal{J}_{\mathrm{gy}} F\left(\epsilon e\boldsymbol{E}_{1\mathrm{gy}}\boldsymbol{\cdot}\dot{\boldsymbol{X}}\right), \tag{5.20}$$

to obtain

$$\frac{\partial \mathcal{E}_{\mathrm{gy}}}{\partial t} + \nabla\boldsymbol{\cdot}\boldsymbol{S}_{\mathrm{gy}} = \int_P \mathcal{J}_{\mathrm{gy}} F\left(\frac{\partial K_{\mathrm{gy}}}{\partial p_\|}\,\dot{p}_\| - \dot{\boldsymbol{X}}\boldsymbol{\cdot}e\boldsymbol{E}^*_{\mathrm{gy}}\right). \tag{5.21}$$

Using the Euler–Lagrange identity (3.15), the right-hand side of (5.21) is shown to vanish identically and we readily recover the exact gyrokinetic energy conservation law.

5.2. *Gyrokinetic Noether momentum equation*

Because the equilibrium magnetic field $\boldsymbol{B}_0$ considered in standard gyrokinetic Vlasov–Maxwell theory is spatially non-uniform (i.e. it serves to magnetically confine charged particles in accordance with the guiding-centre approximation), a general gyrokinetic Vlasov–Maxwell momentum conservation law does not exist. Indeed, according to the Noether theorem, the gyrokinetic Vlasov–Maxwell momentum is conserved only in directions corresponding to symmetries of the equilibrium magnetic field. Before we derive the gyrokinetic angular momentum conservation law associated with an axisymmetric equilibrium magnetic field, we wish to show that the gyrokinetic Noether momentum equation, from which our exact angular momentum conservation law will be derived, is consistent with the gyrokinetic Vlasov–Maxwell equations (4.20)–(4.22).

We begin with the gyrokinetic Noether momentum equation derived by setting $\delta t = 0$ and $\delta \boldsymbol{x} \neq 0$ in (5.9):

$$\frac{\partial \boldsymbol{\mathcal{P}}^*_{\mathrm{gy}}}{\partial t} + \nabla \boldsymbol{\cdot} \boldsymbol{T}^*_{\mathrm{gy}} = \int_P \mathcal{J}_{\mathrm{gy}} F \left(\frac{e}{c} \nabla \boldsymbol{A}^*_0 \boldsymbol{\cdot} \dot{\boldsymbol{X}} + \nabla' \boldsymbol{\Pi}_{\mathrm{gy}} \boldsymbol{\cdot} \dot{\boldsymbol{X}} - \nabla' K_{\mathrm{gy}} \right) - \nabla \boldsymbol{B}_0 \boldsymbol{\cdot} \frac{\boldsymbol{B}}{4\pi}, \quad (5.22)$$

where the gyrokinetic canonical momentum density is defined as

$$\boldsymbol{\mathcal{P}}^*_{\mathrm{gy}} = \int_P \mathcal{J}_{\mathrm{gy}} F \left(\frac{e}{c} \boldsymbol{A}^*_0 + \boldsymbol{\Pi}_{\mathrm{gy}} \right) + \frac{\mathbb{D}_{\mathrm{gy}}}{4\pi c} \times \epsilon \boldsymbol{B}_1 \quad (5.23)$$

and the gyrokinetic canonical stress tensor is defined as

$$\begin{aligned} \boldsymbol{T}^*_{\mathrm{gy}} = & \int_P \mathcal{J}_{\mathrm{gy}} F \dot{\boldsymbol{X}} \left(\frac{e}{c} \boldsymbol{A}^*_0 + \boldsymbol{\Pi}_{\mathrm{gy}} \right) - \frac{\epsilon}{4\pi} \left(\mathbb{D}_{\mathrm{gy}} \boldsymbol{E}_1 + \boldsymbol{B}_1 \mathbb{H}_{\mathrm{gy}} \right) \\ & + \boldsymbol{I} \left[\frac{1}{8\pi} \left(\epsilon^2 |\boldsymbol{E}_1|^2 - |\boldsymbol{B}|^2 \right) + \frac{\epsilon}{4\pi} \boldsymbol{B}_1 \boldsymbol{\cdot} \mathbb{H}_{\mathrm{gy}} \right], \end{aligned} \quad (5.24)$$

where $\boldsymbol{I}$ denotes the identity matrix. We note that, while the gyrokinetic stress tensor (5.24) is manifestly not symmetric, the conservation of the gyrokinetic angular momentum will follow exactly from (5.22).

We would now like to show that (5.22) is an exact consequence of the gyrokinetic Vlasov–Maxwell equations (4.20)–(4.22). We begin with the partial time derivatives of the first two terms in the gyrokinetic canonical momentum density (5.23):

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_P \mathcal{J}_{\mathrm{gy}} F \frac{e}{c} \boldsymbol{A}^*_0 \right) = & -\nabla \boldsymbol{\cdot} \left(\int_P \mathcal{J}_{\mathrm{gy}} F \dot{\boldsymbol{X}} \frac{e}{c} \boldsymbol{A}^*_0 \right) \\ & + \int_P \mathcal{J}_{\mathrm{gy}} F \left(\frac{e}{c} \nabla \boldsymbol{A}^*_0 \boldsymbol{\cdot} \dot{\boldsymbol{X}} + \dot{p}_\parallel \hat{\boldsymbol{b}}_0 - \frac{e}{c} \dot{\boldsymbol{X}} \times \boldsymbol{B}^*_0 \right), \end{aligned} \quad (5.25)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_P \mathcal{J}_{\mathrm{gy}} F \boldsymbol{\Pi}_{\mathrm{gy}} \right) = & -\nabla \boldsymbol{\cdot} \left(\int_P \mathcal{J}_{\mathrm{gy}} F \dot{\boldsymbol{X}} \boldsymbol{\Pi}_{\mathrm{gy}} \right) + \int_P \mathcal{J}_{\mathrm{gy}} F \left(\nabla \boldsymbol{\Pi}_{\mathrm{gy}} \boldsymbol{\cdot} \dot{\boldsymbol{X}} - \nabla K_{\mathrm{gy}} \right) \\ & + \int_P \mathcal{J}_{\mathrm{gy}} F \left[\dot{p}_\parallel \frac{\partial \boldsymbol{\Pi}_{\mathrm{gy}}}{\partial p_\parallel} + e \left(\epsilon \boldsymbol{E}_{1\mathrm{gy}} - \boldsymbol{E}^*_{\mathrm{gy}} \right) - \dot{\boldsymbol{X}} \times \nabla \times \boldsymbol{\Pi}_{\mathrm{gy}} \right], \end{aligned} \quad (5.26)$$

where we used the phase-space divergence form (4.20) of the gyrokinetic Vlasov equation, followed by integrations by parts, and used (5.19) to write $\partial \boldsymbol{\Pi}_{\mathrm{gy}} / \partial t$. By combining these two expressions, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left[\int_P \mathcal{J}_{gy}F\left(\frac{e}{c}\boldsymbol{A}_0^* + \boldsymbol{\Pi}_{gy}\right)\right] &= -\nabla\boldsymbol{\cdot}\left[\int_P \mathcal{J}_{gy}F\dot{\boldsymbol{X}}\left(\frac{e}{c}\boldsymbol{A}_0^* + \boldsymbol{\Pi}_{gy}\right)\right] \\ &\quad + \int_P \mathcal{J}_{gy}F\left[\left(\frac{e}{c}\nabla\boldsymbol{A}_0^* + \nabla\boldsymbol{\Pi}_{gy}\right)\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \nabla K_{gy}\right] \\ &\quad + \int_P \mathcal{J}_{gy}F\left[\epsilon\left(e\boldsymbol{E}_{1gy} + \frac{e}{c}\dot{\boldsymbol{X}}\times\boldsymbol{B}_{1gy}\right)\right] \\ &\quad + \int_P \mathcal{J}_{gy}F\left[\dot{p}_\parallel \boldsymbol{b}_{gy}^* - \left(e\boldsymbol{E}_{gy}^* + \frac{e}{c}\dot{\boldsymbol{X}}\times\boldsymbol{B}_{gy}^*\right)\right], \end{aligned} \tag{5.27}$$

where the last line vanishes as a result of the gyrocentre Euler–Lagrange equation (3.7). Next, we take the partial time derivative of the third term in (5.23):

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\mathbb{D}_{gy}}{4\pi c}\times\epsilon\boldsymbol{B}_1\right) &= \frac{1}{4\pi}\left(\frac{1}{c}\frac{\partial\mathbb{D}_{gy}}{\partial t}\times\epsilon\boldsymbol{B}_1 + \mathbb{D}_{gy}\times\frac{\epsilon}{c}\frac{\partial\boldsymbol{B}_1}{\partial t}\right) \\ &= (\nabla\times\mathbb{H}_{gy})\times\frac{\epsilon\boldsymbol{B}_1}{4\pi} - \frac{\mathbb{D}_{gy}}{4\pi}\times(\nabla\times\epsilon\boldsymbol{E}_1) - \int_P \mathcal{J}_{gy}F\left(\frac{e}{c}\dot{\boldsymbol{X}}\times\epsilon\boldsymbol{B}_{1gy}\right) \\ &= \nabla\boldsymbol{\cdot}\left[\frac{\epsilon}{4\pi}(\boldsymbol{B}_1\mathbb{H}_{gy} + \mathbb{D}_{gy}\boldsymbol{E}_1) - \frac{\boldsymbol{I}}{4\pi}\left(\epsilon\boldsymbol{B}_1\boldsymbol{\cdot}\mathbb{H}_{gy} + \frac{\epsilon^2}{2}|\boldsymbol{E}_1|^2 - \frac{1}{2}|\boldsymbol{B}|^2\right)\right] \\ &\quad - \epsilon\left(\nabla\boldsymbol{E}_1\boldsymbol{\cdot}\mathbb{P}_{gy} + \nabla\boldsymbol{B}_1\boldsymbol{\cdot}\mathbb{M}_{gy}\right) - \nabla\boldsymbol{B}_0\boldsymbol{\cdot}\frac{\boldsymbol{B}}{4\pi} \\ &\quad - \int_P \mathcal{J}_{gy}F\epsilon\left(e\boldsymbol{E}_{1gy} + \frac{e}{c}\dot{\boldsymbol{X}}\times\boldsymbol{B}_{1gy}\right). \end{aligned} \tag{5.28}$$

When we combine (5.27) and (5.28), we obtain

$$\begin{aligned}\frac{\partial\boldsymbol{\mathcal{P}}_{gy}^*}{\partial t} + \nabla\boldsymbol{\cdot}\boldsymbol{T}_{gy}^* &= \int_P \mathcal{J}_{gy}F\left[\left(\frac{e}{c}\nabla\boldsymbol{A}_0^* + \nabla\boldsymbol{\Pi}_{gy}\right)\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \nabla K_{gy}\right] \\ &\quad - \epsilon\left(\nabla\boldsymbol{E}_1\boldsymbol{\cdot}\mathbb{P}_{gy} + \nabla\boldsymbol{B}_1\boldsymbol{\cdot}\mathbb{M}_{gy}\right) - \nabla\boldsymbol{B}_0\boldsymbol{\cdot}\frac{\boldsymbol{B}}{4\pi}, \end{aligned} \tag{5.29}$$

where

$$\begin{aligned}-\epsilon\left(\nabla\boldsymbol{E}_1\boldsymbol{\cdot}\mathbb{P}_{gy} + \nabla\boldsymbol{B}_1\boldsymbol{\cdot}\mathbb{M}_{gy}\right) &= -\int_P \mathcal{J}_{gy}F\left(\nabla\boldsymbol{E}_1\boldsymbol{\cdot}\frac{\delta\boldsymbol{\Pi}_{gy}}{\delta\boldsymbol{E}_1} + \nabla\boldsymbol{B}_1\boldsymbol{\cdot}\frac{\delta\boldsymbol{\Pi}_{gy}}{\delta\boldsymbol{B}_1}\right)\boldsymbol{\cdot}\dot{\boldsymbol{X}} \\ &\quad + \int_P \mathcal{J}_{gy}F\left(\nabla\boldsymbol{E}_1\boldsymbol{\cdot}\frac{\delta K_{gy}}{\delta\boldsymbol{E}_1} + \nabla\boldsymbol{B}_1\boldsymbol{\cdot}\frac{\delta K_{gy}}{\delta\boldsymbol{B}_1}\right) \\ &\equiv -\int_P \mathcal{J}_{gy}F\left[\left(\nabla\boldsymbol{\Pi}_{gy} - \nabla'\boldsymbol{\Pi}_{gy}\right)\boldsymbol{\cdot}\dot{\boldsymbol{X}} - \left(\nabla K_{gy} - \nabla' K_{gy}\right)\right]. \end{aligned} \tag{5.30}$$

By inserting these terms in (5.29), we recover the gyrokinetic Noether momentum equation (5.22).

We note that, while the gyrokinetic Noether momentum equation (5.22) is not a gyrokinetic conservation law, it can be used directly to obtain a gyrokinetic momentum transport equation (e.g. in the parallel direction) by taking its projection in the desired direction. For example, the gyrokinetic canonical parallel-momentum transport equation is expressed as

$$\frac{\partial \mathcal{P}^{*}_{\mathrm{gy}\|}}{\partial t} + \nabla\cdot\left(\boldsymbol{T}^{*}_{\mathrm{gy}}\cdot\hat{\boldsymbol{b}}_0\right) = \boldsymbol{T}^{*\top}_{\mathrm{gy}} : \nabla\hat{\boldsymbol{b}}_0 - \hat{\boldsymbol{b}}_0\cdot\nabla\boldsymbol{B}_0\cdot\boldsymbol{B}/4\pi + \int_P \mathcal{J}_{\mathrm{gy}}F\,\hat{\boldsymbol{b}}_0\cdot\left(\frac{e}{c}\nabla\boldsymbol{A}^{*}_0\cdot\dot{\boldsymbol{X}} + \nabla'\boldsymbol{\varPi}_{\mathrm{gy}}\cdot\dot{\boldsymbol{X}} - \nabla' K_{\mathrm{gy}}\right), \tag{5.31}$$

where $\mathcal{P}^{*}_{\mathrm{gy}\|} \equiv \boldsymbol{\mathcal{P}}^{*}_{\mathrm{gy}}\cdot\hat{\boldsymbol{b}}_0$ and $\boldsymbol{T}^{*\top}_{\mathrm{gy}}$ denotes the transpose of the gyrokinetic stress tensor (5.24). The gyrokinetic canonical parallel-momentum transport equation (5.31) can be transformed into a simpler form as the $p_\|$-moment of the gyrokinetic Vlasov equation:

$$\frac{\partial}{\partial t}\left(\int_P \mathcal{J}_{\mathrm{gy}}F\,p_\|\right) + \nabla\cdot\left(\int_P \mathcal{J}_{\mathrm{gy}}F\dot{\boldsymbol{X}}p_\|\right) = \int_P \mathcal{J}_{\mathrm{gy}}F\dot{p}_\|, \tag{5.32}$$

where the gyrocentre parallel force $\dot{p}_\|$ is defined by (3.14). See Brizard & Tronko (2011) for the explicit transformation from (5.31) to (5.32) for the case of the gyrokinetic Vlasov–Poisson equations. We note that the parallel contraction of the gyrokinetic stress tensor $\boldsymbol{T}^{*}_{\mathrm{gy}}\cdot\hat{\boldsymbol{b}}_0$ on the left-hand side of (5.31) contains the gyrokinetic Maxwell stress tensor term $-\mathbb{D}_{\mathrm{gy}}\epsilon E_{1\|}/4\pi$, which plays a central role in the electrostatic gyrokinetic Vlasov–Poisson model of McDevitt *et al.* (2009) in discussing toroidal rotation driven by the gyrocentre polarization $\mathbb{P}_{\mathrm{gy}}$. In particular, McDevitt *et al.* (2009) show how this polarization contribution can be retrieved from a perturbation expansion (up to fourth order) of the right-hand side of (5.32) through a δF-decomposition of the gyrocentre Vlasov distribution. Our gyrokinetic canonical parallel-momentum transport equation (5.31), in contrast, explicitly exhibits the complete gyrocentre polarization and magnetization effects in a full-F gyrokinetic Vlasov–Maxwell theory.

5.3. *Gyrokinetic angular momentum conservation law*

Assuming now that the equilibrium magnetic field $\boldsymbol{B}_0$ is axisymmetric (i.e. $\partial\boldsymbol{B}_0/\partial\varphi = \hat{\boldsymbol{z}}\times\boldsymbol{B}_0$), we derive the gyrokinetic canonical angular momentum conservation law by taking the scalar product of (5.22) with $\partial\boldsymbol{x}/\partial\varphi$ (i.e. $\delta\boldsymbol{x} = \delta\varphi\,\partial\boldsymbol{x}/\partial\varphi$), where the toroidal angle φ is associated with rotations about the z axis. Hence, the toroidal canonical angular momentum density $\mathcal{P}^{*}_{\mathrm{gy}\varphi} \equiv \boldsymbol{\mathcal{P}}^{*}_{\mathrm{gy}}\cdot\partial\boldsymbol{x}/\partial\varphi$ satisfies the Noether canonical angular momentum equation

$$\frac{\partial \mathcal{P}^{*}_{\mathrm{gy}\varphi}}{\partial t} + \nabla\cdot\left(\boldsymbol{T}^{*}_{\mathrm{gy}}\cdot\frac{\partial\boldsymbol{x}}{\partial\varphi}\right) = \boldsymbol{T}^{*\top}_{\mathrm{gy}} : \nabla\left(\frac{\partial\boldsymbol{x}}{\partial\varphi}\right) - \frac{\partial\boldsymbol{B}_0}{\partial\varphi}\cdot\frac{\boldsymbol{B}}{4\pi} + \int_P \mathcal{J}_{\mathrm{gy}}F\left(\frac{e}{c}\frac{\partial\boldsymbol{A}^{*}_0}{\partial\varphi}\cdot\dot{\boldsymbol{X}} + \frac{\partial'\boldsymbol{\varPi}_{\mathrm{gy}}}{\partial\varphi}\cdot\dot{\boldsymbol{X}} - \frac{\partial' K_{\mathrm{gy}}}{\partial\varphi}\right). \tag{5.33}$$

Under the assumption that the equilibrium magnetic field is axisymmetric, we have $\partial B_0/\partial\varphi \equiv 0$ and we will use the identity $\partial\hat{\boldsymbol{b}}_0/\partial\varphi \equiv \hat{\boldsymbol{z}}\times\hat{\boldsymbol{b}}_0$, so that $\boldsymbol{B}\cdot\partial\boldsymbol{B}_0/\partial\varphi = \epsilon\boldsymbol{B}_1\cdot(\hat{\boldsymbol{z}}\times\boldsymbol{B}_0)$.

Instead of merely assuming that the right-hand side of (5.33) is zero, we now systematically show how the various terms do cancel each other out to yield an exact conservation law. Before we begin, however, we note that the first term vanishes identically if the gyrokinetic stress tensor (5.24) is symmetric (i.e. $\boldsymbol{T}^{*\top}_{\mathrm{gy}} = \boldsymbol{T}^{*}_{\mathrm{gy}}$), which is expected (and required) when there is no separation between dynamical fields and equilibrium fields, e.g. in guiding-centre Vlasov–Maxwell theory (Brizard & Tronci 2016). In the present case, however, the asymmetry of the gyrokinetic stress tensor (5.24) is necessary in order to cancel the additional terms on the right-hand side of (5.33).

We now proceed with the proof that the right-hand side of (5.33) is zero for the gauge-free model of Burby & Brizard (2019), where $\boldsymbol{\Pi}_{\mathrm{gy}} \equiv 0$, and present the results for the gauge-free model of Brizard (2020). First, we note that since the dyadic tensor $\nabla(\partial \boldsymbol{x}/\partial\varphi) = \hat{R}\,\hat{\varphi} - \hat{\varphi}\,\hat{R}$ is antisymmetric (where $R \equiv |\partial \boldsymbol{x}/\partial\varphi|$), only the antisymmetric part of $\mathsf{T}^{*\top}_{\mathrm{gy}}$ contributes in the first term of (5.33):

$$
\begin{aligned}
\mathsf{T}^{*\top}_{\mathrm{gy}} : \nabla\left(\frac{\partial \boldsymbol{x}}{\partial\varphi}\right) &= \hat{\boldsymbol{z}}\boldsymbol{\cdot}\left[\int_P \mathcal{J}_{\mathrm{gy}} F\left(\dot{\boldsymbol{X}} \times \frac{e}{c}\boldsymbol{A}^*_0\right) - \frac{\epsilon}{4\pi}\left(\mathbb{D}_{\mathrm{gy}} \times \boldsymbol{E}_1 + \boldsymbol{B}_1 \times \mathbb{H}_{\mathrm{gy}}\right)\right] \\
&= \hat{\boldsymbol{z}}\boldsymbol{\cdot}\left[\int_P \mathcal{J}_{\mathrm{gy}} F\left(\dot{\boldsymbol{X}} \times \frac{e}{c}\boldsymbol{A}^*_0\right) + \epsilon\boldsymbol{E}_1 \times \mathbb{P}_{\mathrm{gy}} + \epsilon\boldsymbol{B}_1 \times \mathbb{M}_{\mathrm{gy}}\right] \\
&\quad - \frac{\hat{\boldsymbol{z}}}{4\pi}\boldsymbol{\cdot}(\epsilon\boldsymbol{B}_1 \times \boldsymbol{B}_0),
\end{aligned} \tag{5.34}
$$

where we used the dyadic identities $\mathbf{I} : \nabla(\partial \boldsymbol{x}/\partial\varphi) = \nabla\boldsymbol{\cdot}(\partial \boldsymbol{x}/\partial\varphi) = 0$ and $\boldsymbol{V}\boldsymbol{W} : \nabla(\partial \boldsymbol{x}/\partial\varphi) \equiv \hat{\boldsymbol{z}}\boldsymbol{\cdot}(\boldsymbol{W} \times \boldsymbol{V})$, which holds for an arbitrary pair of vectors $(\boldsymbol{V}, \boldsymbol{W})$. Next, the last term is

$$
\frac{\partial' K_{\mathrm{gy}}}{\partial\varphi} = \hat{\boldsymbol{z}}\boldsymbol{\cdot}\left[\epsilon\mu\hat{\boldsymbol{b}}_0 \times \langle\langle \boldsymbol{B}_{1\mathrm{gc}}\rangle\rangle + \frac{p_\parallel \hat{\boldsymbol{b}}_0}{mc} \times \left(\epsilon\boldsymbol{B}_1 \times \boldsymbol{\pi}_{\mathrm{gy}}\right) + \boldsymbol{\pi}_{\mathrm{gy}} \times \epsilon\left(\boldsymbol{E}_1 + \frac{p_\parallel \hat{\boldsymbol{b}}_0}{mc} \times \boldsymbol{B}_1\right)\right], \tag{5.35}
$$

where $\boldsymbol{\pi}_{\mathrm{gy}} \equiv \boldsymbol{\pi}_{\mathrm{gc}} + \epsilon\boldsymbol{\pi}_2$. Lastly, we write $\partial\boldsymbol{A}^*_0/\partial\varphi = \hat{\boldsymbol{z}} \times \boldsymbol{A}^*_0$ and, after some cancellations, (5.33) becomes

$$
\frac{\partial \mathcal{P}^*_{\mathrm{gy}\varphi}}{\partial t} + \nabla\boldsymbol{\cdot}\left(\mathsf{T}^*_{\mathrm{gy}}\boldsymbol{\cdot}\frac{\partial \boldsymbol{x}}{\partial\varphi}\right) = \hat{\boldsymbol{z}}\boldsymbol{\cdot}\epsilon\left(\boldsymbol{E}_1 \times \mathbb{P}_{\mathrm{gy}} + \boldsymbol{B}_1 \times \mathbb{M}_{\mathrm{gy}}\right) - \int_P \mathcal{J}_{\mathrm{gy}} F \frac{\partial' K_{\mathrm{gy}}}{\partial\varphi}, \tag{5.36}
$$

where

$$
\begin{aligned}
\hat{\boldsymbol{z}}\boldsymbol{\cdot}\epsilon\left(\boldsymbol{E}_1 \times \mathbb{P}_{\mathrm{gy}} + \boldsymbol{B}_1 \times \mathbb{M}_{\mathrm{gy}}\right) &= \int_P \mathcal{J}_{\mathrm{gy}} F \hat{\boldsymbol{z}}\boldsymbol{\cdot}\epsilon\left(\boldsymbol{E}_1 \times \boldsymbol{\pi}_{\mathrm{gy}} - \mu\langle\langle \boldsymbol{B}_{1\mathrm{gc}}\rangle\rangle \times \hat{\boldsymbol{b}}_0\right) \\
&\quad + \int_P \mathcal{J}_{\mathrm{gy}} F \hat{\boldsymbol{z}}\boldsymbol{\cdot}\left[\boldsymbol{B}_1 \times \left(\boldsymbol{\pi}_{\mathrm{gy}} \times \frac{p_\parallel \hat{\boldsymbol{b}}_0}{mc}\right)\right].
\end{aligned} \tag{5.37}
$$

Upon further cancellations, (5.36) becomes

$$
\frac{\partial \mathcal{P}^*_{\mathrm{gy}\varphi}}{\partial t} + \nabla\boldsymbol{\cdot}\left(\mathsf{T}^*_{\mathrm{gy}}\boldsymbol{\cdot}\frac{\partial \boldsymbol{x}}{\partial\varphi}\right) = \int_P \mathcal{J}_{\mathrm{gy}} F \hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{N}, \tag{5.38}
$$

where the gyrocentre torque

$$
\boldsymbol{N} \equiv \epsilon\frac{p_\parallel}{mc}\left[\boldsymbol{B}_1 \times \left(\boldsymbol{\pi}_{\mathrm{gy}} \times \hat{\boldsymbol{b}}_0\right) + \boldsymbol{\pi}_{\mathrm{gy}} \times \left(\hat{\boldsymbol{b}}_0 \times \boldsymbol{B}_1\right) + \hat{\boldsymbol{b}}_0 \times \left(\boldsymbol{B}_1 \times \boldsymbol{\pi}_{\mathrm{gy}}\right)\right] \equiv 0 \tag{5.39}
$$

vanishes according to the Jacobi identity

$$
\boldsymbol{U} \times (\boldsymbol{V} \times \boldsymbol{W}) + \boldsymbol{V} \times (\boldsymbol{W} \times \boldsymbol{U}) + \boldsymbol{W} \times (\boldsymbol{U} \times \boldsymbol{V}) \equiv 0 \tag{5.40}
$$

for the double vector product of any three arbitrary vector fields $(\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W})$. For the gauge-free model of Brizard (2020), the gyrocentre torque

$$\boldsymbol{N} \equiv \sum_{i=1}^{3} [\boldsymbol{U}_i \times (\boldsymbol{V}_i \times \boldsymbol{W}_i) + \boldsymbol{V}_i \times (\boldsymbol{W}_i \times \boldsymbol{U}_i) + \boldsymbol{W}_i \times (\boldsymbol{U}_i \times \boldsymbol{V}_i)] \equiv 0 \tag{5.41}$$

also vanishes as a result of the Jacobi vector identity (5.40), where

$$\left.\begin{aligned} (\boldsymbol{U}_1, \boldsymbol{V}_1, \boldsymbol{W}_1) &= \left(\epsilon\langle \boldsymbol{B}_{1\mathrm{gc}}\rangle, \boldsymbol{\pi}_{\mathrm{gy}}, p_\parallel \hat{\boldsymbol{b}}_0/mc\right), \\ (\boldsymbol{U}_2, \boldsymbol{V}_2, \boldsymbol{W}_2) &= \left(\epsilon \boldsymbol{B}_1, -\epsilon\boldsymbol{\pi}_2, p_\parallel \hat{\boldsymbol{b}}_0/mc\right), \\ (\boldsymbol{U}_3, \boldsymbol{V}_3, \boldsymbol{W}_3) &= \left(\dot{\boldsymbol{X}}, \epsilon\langle \boldsymbol{E}_{1\mathrm{gc}}\rangle + p_\parallel \hat{\boldsymbol{b}}_0/mc \times \epsilon\langle \boldsymbol{B}_{1\mathrm{gc}}\rangle, e\hat{\boldsymbol{b}}_0/\Omega_0\right). \end{aligned}\right\} \tag{5.42}$$

5.4. *Gyrokinetic angular momentum conservation in axisymmetric tokamak plasmas*

Hence, we have explicitly proved that the gyrokinetic canonical angular momentum conservation law

$$\frac{\partial \mathcal{P}^*_{\mathrm{gy}\varphi}}{\partial t} + \nabla\boldsymbol{\cdot}\left(\boldsymbol{T}^*_{\mathrm{gy}}\boldsymbol{\cdot}\frac{\partial \boldsymbol{x}}{\partial\varphi}\right) = 0 \tag{5.43}$$

follows exactly from the gyrokinetic Vlasov–Maxwell equations. We now evaluate this equation in axisymmetric tokamak geometry, in which the tokamak magnetic field is $\boldsymbol{B}_0 = B_{0\varphi}(\psi)\nabla\varphi + \nabla\varphi \times \nabla\psi$, where ψ denotes the magnetic poloidal flux and the toroidal component $B_{0\varphi}(\psi)$ is a flux function. In (5.43), the total toroidal angular momentum density

$$\mathcal{P}^*_{\mathrm{gy}\varphi} = \int_P \mathcal{J}_{\mathrm{gy}} F\left(P^*_{\mathrm{gc}\varphi} + \boldsymbol{\Pi}_{\mathrm{gy}}\boldsymbol{\cdot}\frac{\partial \boldsymbol{x}}{\partial\varphi}\right) + \frac{\mathbb{D}_{\mathrm{gy}}}{4\pi c} \times \epsilon \boldsymbol{B}_1 \boldsymbol{\cdot}\frac{\partial \boldsymbol{x}}{\partial\varphi} \tag{5.44}$$

is the sum of three groups of terms.

The first group in (5.44) is defined as the gyrocentre moment of the guiding-centre toroidal angular momentum:

$$P^*_{\mathrm{gc}\varphi} \equiv \frac{e}{c}\boldsymbol{A}^*_0\boldsymbol{\cdot}\frac{\partial \boldsymbol{x}}{\partial\varphi} = -\frac{e}{c}\psi + p_\parallel\, b_{0\varphi} - J\left[2b_{0z} + \nabla\boldsymbol{\cdot}\left(\frac{1}{2B_0}\nabla\psi\right)\right], \tag{5.45}$$

which contains higher-order guiding-centre corrections (Tronko & Brizard 2015). In a careful numerical analysis of the exact particle orbits of energetic ions in a tokamak magnetic field, Belova, Gorlenkov & Cheng (2003) have shown that the higher-order guiding-centre corrections to the lowest-order guiding-centre toroidal angular momentum $P^*_{\mathrm{gc}\varphi} = -(e/c)\psi + p_\parallel b_{0\varphi} + \cdots$ play a crucial role in the guiding-centre toroidal angular momentum law (i.e. in the absence of electromagnetic field perturbations). We note that it is a common practice to extract the dominant guiding-centre contribution from $-(e/c)\psi$ using the identity

$$-\frac{\partial}{\partial t}\left(\int_P \mathcal{J}_{\mathrm{gy}} F\frac{e}{c}\psi\right) - \nabla\boldsymbol{\cdot}\left(\int_P \mathcal{J}_{\mathrm{gy}}\dot{\boldsymbol{X}} F\frac{e}{c}\psi\right) = -\int_P \mathcal{J}_{\mathrm{gy}} F\frac{e}{c}\dot{\psi} \equiv -\frac{1}{c}J^{\psi}_{\mathrm{gy}}, \tag{5.46}$$

where the radial velocity $\dot{\psi} \equiv \dot{\boldsymbol{X}}\boldsymbol{\cdot}\nabla\psi$ is expressed in terms of the gyrocentre velocity $\dot{\boldsymbol{X}}$:

$$\dot{\psi} = \nabla\psi\boldsymbol{\cdot}\left(\boldsymbol{E}^*_{\mathrm{gy}} \times \frac{c\boldsymbol{b}^*_{\mathrm{gy}}}{B^{**}_{\mathrm{gy}\parallel}} + \frac{\partial K_{\mathrm{gy}}}{\partial p_\parallel}\frac{\boldsymbol{B}^*_{\mathrm{gy}}}{B^{**}_{\mathrm{gy}\parallel}}\right). \tag{5.47}$$

Hence, we may now define $P_{gc\varphi} \equiv P^{*}_{gc\varphi} + (e/c)\psi$, and thus (5.43) becomes

$$\frac{\partial \mathcal{P}_{gy\varphi}}{\partial t} + \nabla\cdot\left(\mathsf{T}_{gy}\cdot\frac{\partial \boldsymbol{x}}{\partial\varphi}\right) = \frac{1}{c}J^{\psi}_{gy}, \tag{5.48}$$

where the toroidal angular momentum density (5.44) is now defined with $P_{gc\varphi}$.

The second group in (5.44), which appears because of the symplectic momentum perturbation $\boldsymbol{\Pi}_{gy}$, contains the toroidal components of the perturbed $E\times B$ velocity and magnetic flutter momentum:

$$\begin{aligned}\boldsymbol{\Pi}_{gy}\cdot\frac{\partial\boldsymbol{x}}{\partial\varphi} &= \epsilon\left(\langle\boldsymbol{E}_{1gc}\rangle\times\frac{e\hat{\boldsymbol{b}}_0}{\Omega_0} + \frac{p_\parallel}{B_0}\langle\boldsymbol{B}_{1\perp gc}\rangle\right)\cdot\frac{\partial\boldsymbol{x}}{\partial\varphi}\\ &= \frac{e}{B_0\Omega_0}\left(\langle\boldsymbol{E}_{1gc}\rangle + \frac{p_\parallel\hat{\boldsymbol{b}}_0}{mc}\times\langle\boldsymbol{B}_{1gc}\rangle\right)\cdot\nabla\psi,\end{aligned} \tag{5.49}$$

which can be expressed in terms of the radial component of the perturbed gyrocentre force, where we used the tokamak identity

$$\boldsymbol{B}_0\times\partial\boldsymbol{x}/\partial\varphi = \nabla\psi. \tag{5.50}$$

The third group in (5.44) contains the toroidal component of the Minkowski electromagnetic momentum (Abiteboul *et al.* 2011):

$$\frac{\mathbb{D}_{gy}}{4\pi c}\times\epsilon\boldsymbol{B}_1\cdot\frac{\partial\boldsymbol{x}}{\partial\varphi} = \frac{1}{4\pi c}\left[\left(\epsilon\boldsymbol{E}_1 + 4\pi\mathbb{P}_{gy}\right)\times\epsilon\boldsymbol{B}_1\right]\cdot\frac{\partial\boldsymbol{x}}{\partial\varphi}. \tag{5.51}$$

The partial time derivative of this term can be directly obtained from the toroidal component of (5.28). We note that, in the electrostatic limit (i.e. in the absence of magnetic field perturbations), we recover the flux-averaged gyrokinetic toroidal angular momentum density previously derived (without guiding-centre corrections, i.e. $P_{gc\varphi} = p_\parallel b_{0\varphi}$) (Hahm *et al.* 2007; Scott & Smirnov 2010; Abiteboul *et al.* 2011; Brizard & Tronko 2011).

Finally, we proceed with a flux-surface average (Brizard & Tronko 2011):

$$[\![\cdots]\!] \equiv \frac{1}{\mathcal{V}}\oint(\cdots)\mathcal{J}_\psi\,\mathrm{d}\vartheta\,\mathrm{d}\varphi, \tag{5.52}$$

where $\mathcal{V}(\psi) \equiv \oint\mathcal{J}_\psi\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$ is the surface integral of the magnetic coordinate Jacobian $\mathcal{J}_\psi \equiv (\nabla\psi\times\nabla\theta\cdot\nabla\varphi)^{-1} = 1/B^{\theta}_0$. The flux-surface average (5.52) satisfies the property

$$[\![\nabla\cdot\boldsymbol{C}]\!] \equiv \frac{1}{\mathcal{V}}\frac{\partial}{\partial\psi}\left(\mathcal{V}[\![\boldsymbol{C}\cdot\nabla\psi]\!]\right) \tag{5.53}$$

for any vector field $\boldsymbol{C}$. In a time-independent axisymmetric tokamak geometry, we note that $\partial/\partial t$ also commutes with magnetic surface averaging. The magnetic surface-averaged gyrokinetic canonical angular momentum conservation law (5.48) becomes

$$\frac{\partial [\![\mathcal{P}_{gy\varphi}]\!]}{\partial t} + \frac{1}{\mathcal{V}}\frac{\partial}{\partial \psi}\left(\mathcal{V}\,[\![T^{\psi}_{gy\varphi}]\!]\right) = \frac{1}{c}\,[\![J^{\psi}_{gy}]\!], \tag{5.54}$$

where $T^{\psi}_{gy\varphi} \equiv \nabla\psi \cdot \boldsymbol{T}^{*}_{gy} \cdot \partial\boldsymbol{x}/\partial\varphi$ is defined as

$$T^{\psi}_{gy\varphi} = \int_{P} \mathcal{J}_{gy} F \dot{\psi}\left(P_{gc\varphi} + \boldsymbol{\Pi}_{gy} \cdot \frac{\partial \boldsymbol{x}}{\partial\varphi}\right) - \frac{\epsilon}{4\pi}\nabla\psi \cdot \left(\mathbb{D}_{gy}\boldsymbol{E}_1 + \boldsymbol{B}_1\mathbb{H}_{gy}\right) \cdot \frac{\partial \boldsymbol{x}}{\partial\varphi}, \tag{5.55}$$

where we have used $\nabla\psi \cdot \partial\boldsymbol{x}/\partial\varphi = 0$ and $\dot{\psi}$ is given in (5.47). We note that, using the tokamak identity (5.50), the third term in (5.55), which contains the polarization term derived by McDevitt *et al.* (2009) in the parallel limit, can be expressed as

$$\nabla\psi \cdot \left(\frac{\epsilon}{4\pi}\mathbb{D}_{gy}\boldsymbol{E}_1\right) \cdot \frac{\partial \boldsymbol{x}}{\partial\varphi} = \frac{\partial \boldsymbol{x}}{\partial\varphi} \cdot \left[\frac{\epsilon}{4\pi}\left(\mathbb{D}_{gy} \times \boldsymbol{B}_0\right)\boldsymbol{E}_1\right] \cdot \frac{\partial \boldsymbol{x}}{\partial\varphi}, \tag{5.56}$$

and similarly for the fourth term. Similar terms have appeared in the toroidal angular momentum transport analysis of Parra & Catto (2010*b*).

6. Summary

The energy–momentum and toroidal angular momentum conservation laws of two gauge-free gyrokinetic Vlasov–Maxwell models were derived by the Noether method under the assumption of a time-independent and axisymmetric equilibrium magnetic field. The explicit proof of these conservation laws highlights the roles played by the equilibrium magnetized plasma and the electromagnetic field fluctuations that perturb it. In addition, we also demonstrated how the gyrokinetic Noether momentum equation (5.22) follows exactly from the gyrokinetic Vlasov–Maxwell equations. Hence, a gyrokinetic parallel-momentum transport equation can be derived explicitly without proceeding through a gyrokinetic Vlasov moment approach.

The proofs presented in § 5 also show how gyrokinetic models can be simplified without jeopardizing the energy–momentum conservation laws. For example, Hirvijoki *et al.* (2020) considered the simplified gyrocentre kinetic energy (3.4) for the Hamiltonian gyrokinetic model (Burby & Brizard 2019) obtained by omitting the guiding-centre electric dipole moment $\boldsymbol{\pi}_{gc}$. This omission yields simplified expressions for the gyrocentre polarization and magnetization (4.13) and (4.14), without sacrificing energy and angular momentum conservation.

In the gauge-free symplectic gyrokinetic model considered by Brizard (2020), it is possible to truncate the gyrocentre kinetic energy (3.6) at first order in ϵ, thereby eliminating the corrections $\boldsymbol{\pi}_2$ and $\mu\boldsymbol{B}_1/B_0$ in the gyrocentre polarization and magnetization (4.13) and (4.14), which arise from functional derivatives of the second-order gyrocentre Hamiltonian.

Finally, we note that an exact toroidal angular momentum conservation for the gyrokinetic Vlasov–Maxwell equations is obtained even though the gyrokinetic stress tensor is manifestly asymmetric. In contrast to the guiding-centre Vlasov–Maxwell equations, where the interplay between ponderomotive, polarization and magnetization effects results in a symmetric guiding-centre stress tensor (as required because the magnetic field is not split into background and perturbed components), the case of the standard gyrokinetic splitting of the magnetic field into background and perturbed components requires an asymmetric gyrokinetic stress tensor, as can be seen from (5.33).

The recent work by Chen *et al.* (2020) and Sugama *et al.* (2021) may pave the way to a nonlinear gyrokinetic theory with full electromagnetic effects without field splitting, from which a symmetric stress tensor will arise (but only a careful analysis of ponderomotive, polarization and magnetization effects is carried out).

Acknowledgements

Part of the work presented here was carried out as part of a collaboration with the ELMFIRE numerical simulation group at Aalto University (Finland). The Author acknowledges support from the National Science Foundation under contract no. PHY-1805164.

Editor William Dorland thanks the referees for their advice in evaluating this article.

Declaration of interests

The author reports no conflict of interest.

REFERENCES

ABITEBOUL, J., GARBET, X., GRANDGIRARD, V., ALLFREY, S.J., GHENDRIH, P., LATU, G., SARAZIN, Y. & STRUGAREK, A. 2011 Conservation equations and calculation of mean flows in gyrokinetics. *Phys. Plasmas* **18**, 082503.

BELOVA, E.V., GORLENKOV, N.N. & CHENG, C.Z. 2003 Self-consistent equilibrium model of low aspect-ratio toroidal plasma with energetic beam ions. *Phys. Plasmas* **10**, 3240–3251.

BRIZARD, A. 1989*a* Gyrokinetic energy conservation and Poisson-bracket formulation. *Phys. Fluids* B **1**, 1381–1384.

BRIZARD, A.J. 1989*b* Nonlinear gyrokinetic Maxwell–Vlasov equations using magnetic coordinates. *J. Plasma Phys*. **41**, 541–559.

BRIZARD, A.J. 1995 Nonlinear gyrokinetic vlasov equation for toroidally rotating axisymmetric tokamaks. *Phys. Plasmas* **2**, 459–471.

BRIZARD, A.J. 2000*a* A new variational principle for Vlasov–Maxwell equations. *Phys. Rev. Lett*. **84**, 5768–5771.

BRIZARD, A.J. 2000*b* Variational principle for nonlinear gyrokinetic Vlasov–Maxwell equations. *Phys. Plasmas* **7**, 4816–4822.

BRIZARD, A.J. 2005*a* Energy-conserving finite-beta electromagnetic drift-fluid equations. *Phys. Plasmas* **12**, 092302.

BRIZARD, A.J. 2005*b* Noether methods for fluids and plasmas. *J. Plasma Phys*. **71**, 225–236.

BRIZARD, A.J. 2008 On the dynamical reduction of the Vlasov equation. *Commun. Nonlinear Sci. Numer. Simul*. **13**, 24–33.

BRIZARD, A.J. 2009 Variational principles for reduced plasma physics. *J. Phys.: Conf. Ser*. **169**, 012003.

BRIZARD, A.J. 2010*a* Exact energy conservation laws for full and truncated nonlinear gyrokinetic equations. *Phys. Plasmas* **17**, 042303.

BRIZARD, A.J. 2010*b* Noether derivation of exact conservation laws for dissipationless reduced fluid models. *Phys. Plasmas* **17**, 112503.

BRIZARD, A.J. 2017 Variational principle for the parallel-symplectic representation of electromagnetic gyrokinetic theory. *Phys. Plasmas* **24**, 081201.

BRIZARD, A.J. 2020 Symplectic gyrokinetic Vlasov–Maxwell theory. arXiv:1907.11204v2.

BRIZARD, A.J. & CHAN, A.A. 1999 Nonlinear relativistic gyrokinetic Vlasov–Maxwell equations. *Phys. Plasmas* **6**, 4548–4558.

BRIZARD, A.J., DENTON, R.E., ROGERS, B. & LOTKO, W. 2008 Nonlinear finite-Larmor-radius effects in reduced fluid models. *Phys. Plasmas* **15**, 082302.

BRIZARD, A.J. & HAHM, T.S. 2007 Foundations of nonlinear gyrokinetic theory. *Rev. Mod. Phys*. **79**, 421–468.

BRIZARD, A.J. & TRONCI, C. 2016 Variational formulations of guiding-center Vlasov–Maxwell theory. *Phys. Plasmas* **23**, 062107.

BRIZARD, A.J. & TRONKO, N. 2011 Exact momentum conservation laws for the gyrokinetic Vlasov–Poisson equations. *Phys. Plasmas* **18**, 082307.

BURBY, J.W. & BRIZARD, A.J. 2019 Gauge-free electromagnetic gyrokinetic theory. *Phys. Lett.* A **383**, 2172–2175.

BURBY, J.W., BRIZARD, A.J., MORRISON, P.J. & QIN, H. 2015 Hamiltonian gyrokinetic Vlasov–Maxwell system. *Phys. Lett.* A **379**, 2073–2077.

CARY, J.R. & KAUFMAN, A.N. 1981 Ponderomotive effects in collisionless plasmas: a Lie transform approach. *Phys. Fluids* **24**, 1238–1250.

CHEN, L., LIN, Y., WANG, X.Y. & BAO, J. 2019 A new particle simulation scheme using electromagnetic fields. *Plasma Phys. Control. Fusion* **61**, 035004.

CHEN, L., ZONCA, F. & CHEN, H. 2020 Unexpanded nonlinear electromagnetic gyrokinetic equations for magnetized plasmas. *Plasma Sci. Tech.* **22**, 102001.

CHEN, Y. & PARKER, S.E. 2009 Particle-in-cell simulation with Vlasov ions and drift kinetic electrons. *Phys. Plasmas* **16**, 052305.

CORREA-RESTREPO, D. & PFIRSCH, D. 2004 New method of deriving local energy- and momentum-conserving Maxwell-collisionless drift-kinetic and gyrokinetic theories: conservation laws and their structures. *J. Plasma Phys.* **70**, 757–797.

DUBIN, D.H.E., KROMMES, J.A., OBERMAN, C. & LEE, W.W. 1983 Nonlinear gyrokinetic equations. *Phys. Fluids* **26**, 3524–3535.

DUTHOIT, F.-X., HAHM, T.S. & WANG, L. 2014 Electromagnetic nonlinear gyrokinetics with polarization drift. *Phys. Plasmas* **21**, 082301.

FAN, P., QIN, H. & XIAO, J. 2020 Discovering exact local energy-momentum conservation laws for electromagnetic gyrokinetic system by high-order field theory on heterogeneous manifolds. arXiv:2006.11039v2.

FRIEMAN, E.A. & CHEN, L. 1982 Nonlinear gyrokinetic equations for low-frequency electromagnetic waves in general plasma equilibria. *Phys. Fluids* **25**, 502–508.

GARBET, X., IDOMURA, Y., VILLARD, L. & WATANABE, T.H. 2010 Gyrokinetic simulations of turbulent transport. *Nucl. Fusion* **50**, 043002.

GOLDSTEIN, H., POOLE, C. & SAFKO, J. 2002 *Classical Mechanics*, 3rd edn. Addison Wesley.

HAHM, T.S., DIAMOND, P.H., GÜRCAN, Ö. D. & REWOLDT, G. 2007 Nonlinear gyrokinetic theory of toroidal momentum pinch. *Phys. Plasmas* **14**, 072302.

HAHM, T.S., LEE, W.W. & BRIZARD, A.J. 1988 Nonlinear gyrokinetic theory for finite-beta plasmas. *Phys. Fluids* **31**, 1940–1948.

HIRVIJOKI, E., BURBY, J.W., PFEFFERLÉ, D. & BRIZARD, A.J. 2020 Energy and momentum conservation in the Euler–Poincaré formulation of local Vlasov–Maxwell-type systems. *J. Phys.* A*: Theor. Math.* **53**, 235204.

LANDAU, L.D. & LIFSHITZ, E.M. 1971 *The Classical Theory of Fields*, 3rd edn. Pergamon.

LEERINK, S., PARRA, F.I. & HEIKKINEN, J.A. 2010 Comment on 'Nonlinear gyrokinetic theory with polarization drift'. *Phys. Plasmas* **17**, 124701.

MCDEVITT, C.J., DIAMOND, P.H., GÜRCAN, Ö.D. & HAHM, T.S. 2009 Toroidal rotation driven by the polarization drift. *Phys. Rev. Lett.* **103**, 205003.

PARRA, F.I. & CATTO, P.J. 2010*a* Transport of momentum in full f gyrokinetics. *Phys. Plasmas* **17**, 056106.

PARRA, F.I. & CATTO, P.J. 2010*b* Turbulent transport of toroidal angular momentum in low flow gyrokinetics. *Plasma Phys. Control. Fusion* **52**, 045004.

PEETERS, A.G., ANGIONI, C., BORTOLON, A., CAMENEN, Y., CASSON, F.J., DUVAL, B., FIEDERSPIEL, L., HORNSBY, W.A., IDOMURA, Y., HEIN, T., *et al.* 2011 Overview of toroidal momentum transport. *Nucl. Fusion* **51**, 094027.

PFIRSCH, D. & MORRISON, P.J. 1985 Local conservation laws for the Maxwell–Vlasov and collisionless guiding-center theories. *Phys. Rev.* A **32**, 1714–1721.

PORAZIK, P. & LIN, Z. 2011 Gyrokinetic simulation of magnetic compressional modes in general geometry. *Commun. Comput. Phys.* **10**, 899–911.

SCOTT, B. & SMIRNOV, J. 2010 Energetic consistency and momentum conservation in the gyrokinetic description of tokamak plasmas. *Phys. Plasmas* **17**, 112302.

SIMILON, P.L. 1985 Conservation laws for relativistic guiding-center plasmas. *Phys. Lett.* A **112**, 33–37.

SQUIRE, J., QIN, H., TANG, W.M. & CHANDRE, C. 2013 The Hamiltonian structure and Euler–Poincaré formulation of the Vlasov–Maxwell and gyrokinetic systems. *Phys. Plasmas* **20**, 022501.

STOLTZFUS-DUECK, T. 2019 Intrinsic rotation in axisymmetric devices. *Plasma Phys. Control. Fusion* **61**, 124003.

STRINTZI, D., SCOTT, B.D. & BRIZARD, A.J. 2005 Nonlocal nonlinear electrostatic gyrofluid equations: a four-moment model. *Phys. Plasmas* **12**, 052517.

SUGAMA, H. 2000 Gykokinetic field theory. *Phys. Plasmas* **7**, 466–480.

SUGAMA, H., MATSUOKA, S., NUNAMI, M. & SATAKE, S. 2021 The eulerian variational formulation of the gyrokinetic system in general spatial coordinates. *Phys. Plasmas* **28**, 022312.

SUGAMA, H., MATSUOKA, S., SATAKE, S. & KANNO, R. 2016 Radially local approximation of the drift kinetic equation. *Phys. Plasmas* **23**, 042502.

TRONKO, N. & BRIZARD, A.J. 2015 Lagrangian and Hamiltonian constraints for guiding-center Hamiltonian theories. *Phys. Plasmas* **22**, 112507.

WALTZ, R.E., STAEBLER, G.M., CANDY, J. & HINTON, F.L. 2007 Gyrokinetic theory and simulation of angular momentum transport. *Phys. Plasmas* **14**, 122507.

WANG, L. & HAHM, T.S. 2010*a* Nonlinear gyrokinetic theory with polarization drift. *Phys. Plasmas* **17**, 082304.

WANG, L. & HAHM, T.S. 2010*b* Response to "Comment on 'Nonlinear gyrokinetic theory with polarization drift' ". *Phys. Plasmas* **17**, 124702.

WANG, L., PENG, S. & DIAMOND, P.H. 2018 Gyrokinetic theory of turbulent acceleration and momentum conservation in tokamak plasmas. *Plasma Sci. Technol.* **20**, 074004.