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We establish interior Schauder estimates for kinetic equations with integrodifferential diffusion. We study equations of the form $f_t + v \cdot \nabla_x f = \mathcal{L}_v f + c$, where $\mathcal{L}_v$ is an integrodifferential diffusion operator of order $2s$ acting in the v-variable. Under suitable ellipticity and Hölder continuity conditions on the kernel of $\mathcal{L}_v$, we obtain an a priori estimate for f in a properly scaled Hölder space.

1. Introduction

We study kinetic equations with integral diffusion of the form

$$f_t + v \cdot \nabla_x f = \int_{\mathbb{R}^d} (f' - f) K(t, x, v, v') \, \mathrm{d}v' + c_0(t, x, v). \tag{1-1}$$

Here, we used the notation from kinetic equations $f = f(t, x, v)$ and $f' = f(t, x, v')$. The main result in this article is a Schauder estimate for equations of the form (1-1) whose kernels $K_{(t,x,v)}$ are elliptic and Hölder continuous.

Note that the integral diffusion term on the right-hand side of (1-1) acts on the velocity variable only. The regularization effect on the x-variable is a consequence of the interaction between the integral diffusion and the transport term. The equation (1-1) should be understood as a Kolmogorov-type hypoelliptic integrodifferential equation. The diffusion is of fractional order. We work with Hölder-like spaces (given in Definition 2.3) that are adapted to the particular scaling of this equation in each direction.

Our methods allow us to consider a very general class of kernels K. This is essential for the eventual applications of our result to the Boltzmann equation, as in [Imbert and Silvestre 2019]. We start by specifying our notion of ellipticity and Hölder continuity for the kernel K. In (1-1), $K(t, x, v, v')$ denotes a function that maps the variables (t, x, v) into a nonnegative Radon measure $K_{(t,x,v)}$ in $\mathbb{R}^d \setminus \{0\}$:

$$K_{(t,x,v)}(w) := K(t, x, v, v + w)$$

such that for all (t, x, v), $K_{(t,x,v)}$ belongs to the following ellipticity class of kernels.

Definition 1.1 (the ellipticity class). Given the order $2s \in (0, 2)$ and ellipticity constants $0 < \lambda < \Lambda$, we say that a nonnegative Radon measure K in $\mathbb{R}^d \setminus \{0\}$ belongs to the ellipticity class $\mathcal{K}$ when the following conditions are met:

- (symmetry) $K(w) = K(-w)$.

- (upper bound) For all $r > 0$

$$\int_{B_r} |w|^2 K(w)\, dw \leq \Lambda r^{2-2s}. \tag{1-2}$$

- (coercivity estimate) For any $R > 0$ and $\varphi \in C^2(B_R)$,

$$\iint_{B_R \times B_R} |\varphi(v) - \varphi(v')|^2 K(v'-v)\, dv'\, dv \geq \lambda \iint_{B_{R/2} \times B_{R/2}} |\varphi(v) - \varphi(v')|^2 |v'-v|^{-d-2s}\, dv'\, dv. \tag{1-3}$$

In case $s < \left(0, \frac{1}{2}\right)$, we add the following nondegeneracy assumption to the kernel:

$$\inf_{|e|=1} \int_{B_r} (w \cdot e)_+^2 K(w)\, dw \geq \lambda r^{2-2s}. \tag{1-4}$$

Remark 1.2. Strictly speaking, (1-2), (1-3) and (1-4) should be written with integrals on balls minus the origin. It is customary to extend the measure $K\, dw$ to have zero point mass at the origin. Other choices do not make any difference since all our integrands equal zero at $w = 0$.

Remark 1.3. We write K to denote a nonnegative measure on $\mathbb{R}^d$. Even though we use the notation $K(v')\, dv'$ as if this measure is absolutely continuous, it does not need to be. We abuse notation in this way because it makes some formulas look simpler. For example, we write $K(v'-v)\, dv'$ to denote the measure in terms of the variables v' and translated by v. Otherwise, for a measure $\mu = K(w)\, dw$, it would typically be written $d(\tau_v \mu(v'))$ which is arguably more confusing.

Remark 1.4. When $s < \frac{1}{2}$ we complement the coercivity estimate (1-3) with the nondegeneracy assumption (1-4). These two assumptions may be redundant. Indeed, for stable-like kernels of the form $K(w) = |w|^{-d-2s} a(w/|w|)$, they are equivalent. This follows easily by computing the Fourier symbol of the operator associated with K; see for example [Samorodnitsky and Taqqu 1994; Ros-Oton and Serra 2016]. For non-stable-like processes the situation is less clear. We do not know any example of a kernel K satisfying the upper bound (1-2) that satisfies one of the assumptions in the coercivity estimate but not the other. The nondegeneracy assumption (1-4) is typically much easier to check than the first coercivity estimate (1-3).

For local equations, a Schauder estimate refers to an estimate in a Hölder space when coefficients of the equations are Hölder continuous. For nonlocal equations, the regularity of the coefficients is replaced with the Hölder continuous dependence of the kernel with respect to the variable $z = (t, x, v)$.

Assumption 1.5 (Hölder continuity of coefficients). *Given $z_1 = (t_1, x_1, v_1)$ and $z_2 = (t_2, x_2, v_2)$, for any $r > 0$ we have*

$$\int_{B_r} |K_{z_1}(w) - K_{z_2}(w)| |w|^2\, dw \leq A_0 r^{2-2s} d_\ell(z_1, z_2)^\alpha,$$

where $d_\ell(z_1, z_2)$ stands for the kinetic distance; see Definition 2.1 below.

We can now state the Schauder estimate for the equations we consider.

Theorem 1.6 (the Schauder estimate). *Let $0 < \gamma < \min(1, 2s)$ and $\alpha = (2s/(1+2s))\gamma$. Let $K(t, x, v, v')$ be a kernel such that the two following conditions hold true*:

- *(ellipticity) For each $z = (t, x, v) \in Q_1$, the kernel $K_z(w) = K(t, x, v, v + w)$ belongs to the class $\mathcal{K}$ described in Definition 1.1.*

- *(Hölder continuity) Assumption 1.5 holds.*

Assume further that $2s + \alpha \notin \{1, 2\}$. If $f \in C_\ell^\gamma([-1, 0] \times B_1 \times \mathbb{R}^d)$ solves (1-1) in Q_1, then the following estimate holds:

$$\|f\|_{C_\ell^{2s+\alpha}(Q_{1/2})} \le C(\|f\|_{C_\ell^\gamma([-1,0]\times B_1 \times \mathbb{R}^d)} + \|c\|_{C_\ell^\alpha(Q_1)}).$$

The constant C depends on d, s, λ, Λ, and A_0.

We use the same notation as in [Imbert and Silvestre 2020]: Q_r denotes the kinetic cylinder $Q_r := (-r^{2s}, 0] \times B_{r^{1+2s}} \times B_r$.

Remark 1.7. The Hölder norms $C_\ell^{2s+\alpha}$ and C_ℓ^α must be appropriately understood. They refer to the usual notion of $C^{2s+\alpha}$ and C^α regularity with respect to the v-variable. The order of regularity in the other directions is adjusted in terms of the invariant structure of the class of equations. On the one hand, the fact that the diffusion is of order $2s$ yields an invariant scaling. On the other hand the equation enjoys Galilean invariance, yielding a Lie group structure. We discuss other choices of distances and their differences in Section 2F. Hölder spaces are introduced in Definition 2.3 below. The subindex ℓ refers to the fact that the Hölder norm is taken with respect to a distance that is left-invariant by the Lie group structure.

Remark 1.8. Note that our theorem holds for $\alpha = (2s/(1+2s))\gamma < \gamma$. The distinction between these two Hölder exponents α and γ comes from technical reasons related to the fact that the class of equations is left-invariant, but not right-invariant with respect to the Lie group structure.

Remark 1.9. Note that the global C_ℓ^γ norm of f cannot be replaced by its global L^∞ norm even in the case of the space-homogeneous parabolic fractional heat equation. A solution $f(t, v)$ to

$$f_t + (-\Delta)_v^s f = 0 \quad \text{in } Q_1,$$

does *not* satisfy the estimate

$$\|f\|_{C_\ell^{2s+\alpha}(Q_{1/2})} \le C\|f\|_{L^\infty([-1,0]\times B_1 \times \mathbb{R}^d)}.$$

This is because f will not be better than Lipschitz in time, even though it will have more regularity in space; see [Chang-Lara and Dávila 2014, Section 2.4.1]. The Hölder space $C_\ell^{2s+\alpha}$ would impose C^1 regularity in time for any $\alpha > 0$.

Remark 1.10. Note that (1-1) does not have a structure compatible with the notion of weak solutions in the sense of distributions. It is not an equation in divergence form. Our result in this paper is an a priori estimate provided that all quantities involved make sense classically. It is possible to define a weaker notion of solution of (1-1) in the *viscosity sense*, and presumably our result in Theorem 1.6 applies to that case as well. However, we do not pursue that direction in this paper since it would add

some technical difficulties obfuscating the proofs. The result as currently stated is what we need for our intended applications to the Boltzmann equation.

Remark 1.11. If we want our estimates to hold uniformly as $s \to 1$, we would have to replace the constant λ in (1-3) by $(1-s)\lambda$. The results in this article hold in the local case $s = 1$ as well, with considerably simplified proofs. It would apply to an equation of the form

$$f_t + v \cdot \nabla_x f = a_{ij}(t, x, v) \, \partial_{v_i v_j} f + c(t, x, v).$$

Schauder estimates for these and more general equations have been studied before (see the next subsection). Our approach is quite different from earlier works, starting from the fact that we use a different definition of the Hölder norm. In the case $s = 1$, some of the difficulties in the proofs presented here disappear. The ellipticity class in Definition 1.1 would be replaced by the usual uniform ellipticity condition of the coefficients a_{ij}. Assumption 1.5 would translate as the Hölder continuity assumption of these coefficients. The section about weak limits of kernels would be unnecessary since it would be replaced by the simple convergence of the matrix of coefficients. The majorant function defined in (3-3), which plays an important role later in the proof of the Liouville theorem, would be irrelevant since the equation is local. Consequently, the final estimate would be in terms of $\|f\|_{L^\infty(Q_1)}$ instead of $\|f\|_{C^\gamma((-1,0] \times B_1 \times \mathbb{R}^d)}$. Moreover, (4-1) in the Liouville theorem could be written in terms of f directly, instead of introducing the function g. The final result is an interior estimate for the $C_\ell^{2+\alpha}$ norm with $s = 1$. This norm is comparable but contains more explicit information than the norms used previously in the literature; see the norm $\|\cdot\|_{2+\alpha,\Omega}$ in [Polidoro 2004].

1A. *Schauder estimates for kinetic and nonlocal equations.* Linear kinetic equations of second order are a particular instance of the more general theory of ultraparabolic equations of Kolmogorov type. Results involving regularity estimates in Hölder spaces for these equations appeared especially in the late 1990s. See [Shatyro 1971; Manfredini 1997; Lunardi 1997; Eidelman et al. 1998; Di Francesco and Polidoro 2006; Radkevich 2008], and the survey article [Polidoro 2004]. More recently in [Henderson and Snelson 2020; Imbert and Mouhot 2018], Schauder estimates were applied to bootstrap higher-regularity estimates for second-order models in kinetic theory, including the Landau equation with moderately soft potentials. The Boltzmann equation can be written in the form (1-1) for a kernel K depending on the solution itself; see [Silvestre 2016]. It is our intention to use the result of Theorem 1.6 to derive higher-order regularity estimates for the noncutoff Boltzmann equation in [Imbert and Silvestre 2019].

Schauder estimates for integrodifferential equations have been obtained in recent years; see [Mikulevicius and Pragarauskas 2014; Jin and Xiong 2015; Serra 2015a; 2015b; Imbert et al. 2018; Dong et al. 2018]. They have the well-known difficulty that the smoothness of the tails of the integrals outside of the domain of the equation is difficult to control. There are two common workarounds that have been used in the literature. One workaround that works well in the elliptic case is to impose some regularity in the kernels K with respect to the variable of integration w. Another approach, arguably more delicate, imposes extra regularity on the values of the function f outside of the domain. That is the case in [Serra

2015a; 2015b; Dong et al. 2018] and it is also the approach we take here. In our kinetic setting, this restriction goes a bit further by requiring the function $f \in C_\ell^\gamma$ outside of Q_1, with $\gamma > \alpha$.

In this paper, we use some key ideas that originated in [Serra 2015a] and simplify enormously the general procedure to prove the Schauder estimates in the nonlocal setting. The key of the proof of Theorem 1.6 is a combination of a blow-up technique (see Proposition 5.1) with a Liouville theorem (see Theorem 4.1).

1B. *Possible extensions and outstanding questions.* It is most probably possible to extend Theorem 1.6 to higher values of γ such that $\gamma \notin \mathbb{N} + 2s\mathbb{N}$. It would require the extension of Assumption 1.5 for α large, or a version of (4-1) involving higher-order incremental quotients. It would not be obvious how to imply the result of Theorem 1.6 for higher values of γ by simply taking derivatives. Knowing $f \in C_\ell^\gamma$ gives us a clean estimate for $f_t + v \cdot \nabla_x f$ in $C_\ell^{\gamma-2s}$ and $\partial_{v_i} f$ in $C_\ell^{\gamma-1}$. These two derivatives do not solve an equation like (1-1). One might attempt to apply hypoelliptic estimates to derive corresponding Hölder spaces for f_t and $\partial_{v_i} f + t \, \partial_{x_i} f$. But then we would lose a fraction of the Hölder exponent that goes beyond the order of differentiation. This is somehow reflected in the final relation between α and γ in the statement of Theorem 1.6.

In the statement of Theorem 1.6, we make the assumption $\alpha \le (2s/(1+2s))\gamma$. We know that the theorem would not hold with $\alpha > \gamma$. The range $(2s/(1+2s))\gamma < \alpha \le \gamma$ is currently unclear.

1C. *Organization of the article.* The article is organized as follows. In Section 2, Hölder spaces adapted to the study of kinetic equations are introduced. In particular, a kinetic degree of polynomials and differential operators and a kinetic distance are defined. Section 3 is devoted to the study of integral operators associated with the class $\mathcal{K}$ of elliptic kernels from Definition 1.1. We then state and prove a Liouville-type theorem in Section 4. Finally, Section 5 is devoted to the proof of the main theorem. It is done by contradiction through a blowup argument.

2. Kinetic Hölder spaces

In this preliminary section, we mainly introduce the Hölder spaces we need to derive the Schauder estimate for kinetic integrodifferential equations. We first define a kinetic distance, then the kinetic degrees of polynomials and differential operators. The definition of Hölder spaces is then given and an interpolation inequality is proved.

2A. *The kinetic distance.* The following Lie group structure of $\mathbb{R}^{1+2d}$ plays a key role in all our computations. The product is defined as

$$(t_1, x_1, v_1) \circ (t_2, x_2, v_2) = (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2).$$

Note that this product is not commutative. The class of equations we will be working with (as in (3-19)) are left-invariant, in the sense that if $f(z)$ is a solution of (1-1), then $f_0(z) := f(z_0 \circ z)$ is also a solution of a similar equation with a translated right-hand side and a translated kernel in the same ellipticity class.

There is also invariance by scaling. We define $S_R(t, x, v) = (R^{2s}t, R^{1+2s}x, Rv)$. If $f(z)$ solves an equation like (1-1), then $f(S_R z)$ solves a similar equation with a scaled right-hand side and a scaled kernel in the same ellipticity class.

Because of this property, it is good to work with a notion of distance, Hölder norms, degree and differential operators that are homogeneous respect to this kinetic scaling and are left-invariant by the action of the Lie group.

Definition 2.1 (a left-invariant distance). Given two points $z_1 = (t_1, x_1, v_1)$ and $z_2 = (t_2, x_2, v_2)$ in $\mathbb{R}^{1+2d}$, we define the distance function

$$d_\ell(z_1, z_2) := \min_{w \in \mathbb{R}^d} \{\max(|t_1 - t_2|^{1/(2s)}, |x_1 - x_2 - (t_1 - t_2)w|^{1/(1+2s)}, |v_1 - w|, |v_2 - w|)\}.$$

The subindex ℓ stands for left-invariant.

It is convenient to also have a notion of "norm" with the right scale invariance. We define

$$\|(t, x, v)\| = \max\{|t|^{1/(2s)}, |x|^{1/(1+2s)}, |v|\}. \tag{2-1}$$

Note that it is *not* an actual norm in the strictly mathematical sense of the term.

Here are some observations:

• The distance d_ℓ is left-invariant by the Lie group action in the sense that $d_\ell(z \circ z_1, z \circ z_2) = d_\ell(z_1, z_2)$ for any $z, z_1, z_2 \in \mathbb{R}^{1+2d}$.

• It is homogeneous with respect to scaling: $d_\ell(S_R z_1, S_R z_2) = R d_\ell(z_1, z_2)$.

• We will see below in Proposition 2.2 that d_ℓ is indeed a distance when $s \geq \frac{1}{2}$ in the sense that it satisfies the triangle inequality. When $s < \frac{1}{2}$, the function d_ℓ^{2s} is a distance. We will still work with d_ℓ (as opposed to d_ℓ^{2s}) when $s < \frac{1}{2}$ so that we keep a consistent scaling formula (as in the previous bullet point) throughout the paper.

• There are other equivalent formulas to measure how far apart z_1 and z_2 are. We observe that

$$\begin{aligned} d_\ell(z_1, z_2) &\approx \|z_2^{-1} \circ z_1\| \\ &\approx \|z_1^{-1} \circ z_2\| \\ &\approx \inf_{w \in \mathbb{R}^d} |t_2 - t_1|^{1/(2s)} + |x_1 - x_2 + (t_1 - t_2)w|^{1/(1+2s)} + |v_1 - w| + |v_2 - w|. \end{aligned}$$

None of the three formulas on the right-hand side are proper distances. However, since they give us a good estimate for $d_\ell(z_1, z_2)$ we will use them whenever it is convenient.

• Note that the distance d_ℓ can be reformulated in the following way: $d_\ell(z_1, z_2)$ is the infimum value of $r > 0$ so that both z_1 and z_2 belong to a cylinder $Q_r(z)$ for some $z \in \mathbb{R}^{1+2d}$.

• Our usual definition for the cylinder Q_r would not be affected significantly if we changed it to

$$Q_r = \{(t, x, v) : t \leq 0 \text{ and } d_\ell(0, (t, x, v)) < r\}.$$

Moreover, because of the Lie group invariance, we could also have, for $z_0 = (t_0, x_0, v_0)$,

$$Q_r(z_0) = \{z = (t, x, v) : t \leq t_0 \text{ and } d_\ell(z_0, z) < r\}.$$

Proposition 2.2. *The function $d_\ell : \mathbb{R}^{1+2d} \times \mathbb{R}^{1+2d} \to [0, \infty)$ is a distance when $s \geq \frac{1}{2}$. For $s < \frac{1}{2}$, the function d_ℓ^{2s} is a distance.*

Proof. We start with the case $s \geq \frac{1}{2}$. Because of the invariance by the Lie group action, we only need to prove the triangle inequality when one of the three points is the origin. That is, given any $z_1, z_2 \in \mathbb{R}^{1+2d}$, we must show that $d_\ell(z_1, 0) + d_\ell(0, z_2) \geq d_\ell(z_1, z_2)$.

Let $w_1, w_2 \in \mathbb{R}^d$ be the points where the minimum in Definition 2.1 is achieved for z_1 and z_2 respectively. That is,

$$d_\ell(z_1, 0) = \max(|t_1|^{1/(2s)}, |x_1 + t_1 w_1|^{1/(1+2s)}, |v_1 - w_1|, |w_1|),$$
$$d_\ell(z_2, 0) = \max(|t_2|^{1/(2s)}, |x_2 + t_2 w_2|^{1/(1+2s)}, |v_2 - w_2|, |w_2|).$$

By definition, $d_\ell(z_1, z_2)$ is the minimum over all choices of $w \in \mathbb{R}^d$, so it is less than or equal to the value we get by setting $w = w_1 + w_2$:

$$d_\ell(z_1, z_2) \leq \max(|t_1 - t_2|^{1/(2s)}, |x_1 - x_2 + (t_1 - t_2)(w_1 + w_2)|^{1/(1+2s)}, |v_1 - w_1 - w_2|, |v_2 - w_1 - w_2|).$$

We now analyze every one of the four expressions inside the max.

Clearly, since $1/(2s) \leq 1$, we have

$$|t_1 - t_2|^{1/(2s)} \leq |t_1|^{1/(2s)} + |t_2|^{1/(2s)} \leq d_\ell(z_1, 0) + d_\ell(z_2, 0).$$

Also, simply by the triangle inequality in $\mathbb{R}^d$,

$$|v_1 - w_1 - w_2| \leq |v_1 - w_1| + |w_2| \leq d_\ell(z_1, 0) + d_\ell(z_2, 0).$$

Likewise,

$$|v_2 - w_1 - w_2| \leq |v_2 - w_2| + |w_1| \leq d_\ell(z_2, 0) + d_\ell(z_1, 0).$$

The second argument in the max is the only one that requires a nontrivial analysis. We evaluate

$$\begin{aligned} |x_1 - x_2 + (t_1 - t_2)(w_1 + w_2)|^{1/(1+2s)} &= |(x_1 + t_1 w_1) - (x_2 + t_2 w_2) + t_1 w_2 - t_2 w_1|^{1/(1+2s)} \\ &\leq |d_\ell(z_1, 0)^{1+2s} + d_\ell(z_2, 0)^{1+2s} \\ &\qquad + d_\ell(z_1, 0)^{2s} d_\ell(z_2, 0) + d_\ell(z_2, 0)^{2s} d_\ell(z_1, 0)|^{1/(1+2s)} \\ &\leq d_\ell(z_1, 0) + d_\ell(z_2, 0). \end{aligned}$$

The last inequality follows from the following elementary calculus fact. For any $a, b \geq 0$ and $p \geq 1$, the following inequality holds:

$$a^{1+p} + b^{1+p} + a^p b + ab^p \leq (a + b)^{1+p}. \tag{2-2}$$

Clearly, in the last inequality, we applied (2-2) with $a = d_\ell(z_1, 0)$, $b = d_\ell(z_2, 0)$ and $p = 2s$.

The proof for the case $s < \frac{1}{2}$ goes along the same lines, and we conclude the last inequality also applying (2-2) with $p = 1/(2s)$. $\qquad\square$

2B. *Kinetic degree of polynomials.* We start by defining a modified notion of degree for a polynomial $p \in \mathbb{R}[t, x, v]$. This special degree, which we will call *kinetic degree*, matches the scaling of the equation.

In order to compute $\deg_k p$, every exponent of the variable t should count times $2s$, every exponent of the variables x_i counts times $(1 + 2s)$ and the exponents of the variables v_i count normally. More precisely, if $m \in \mathbb{R}[t, x, v]$ is a monomial, we define its kinetic degree, $\deg_k m$, to be the number κ so that $m(S_R z) = R^\kappa m(z)$.

A polynomial p is always a finite sum of monomials. In general, we define the kinetic degree of a polynomial $p = \sum m_j$ as the maximum of $\deg_k m_j$ for all its monomial terms m_j.

Note that the degree of a polynomial $p \in \mathbb{R}[t, x, v]$ can be any number in the discrete set $\mathbb{N} + 2s\mathbb{N}$.

2C. *Hölder spaces.* We now define a properly scaled version of Hölder spaces.

Definition 2.3 (Hölder spaces). For any $\alpha \in (0, \infty)$, we say that a function $f : D \to \mathbb{R}$ is C_ℓ^α at a point $z_0 \in \mathbb{R}^{1+2d}$ if there exists a polynomial $p \in \mathbb{R}[t, x, v]$ such that $\deg_k p < \alpha$ and for any $z \in D$

$$|f(z) - p(z)| \le C d_\ell(z, z_0)^\alpha.$$

When this property holds at every point z_0 in the domain D, with a uniform constant C, we say $f \in C_\ell^\alpha(D)$. The seminorm $[f]_{C_\ell^\alpha(D)}$ is the smallest value of the constant C so that the inequality above holds for all $z_0, z \in D$. The norm $\|f\|_{C_\ell^\alpha(D)}$ is $[f]_{C_\ell^\alpha(D)} + [f]_{L^\infty(D)}$.

Remark 2.4. Note that d_ℓ, $\deg_k$, and therefore also C_ℓ^α, depend on s implicitly.

Remark 2.5. With the above definition, when $\alpha \in \mathbb{N} + 2s\mathbb{N}$, the C_ℓ^α space corresponds to a Lipschitz-type space instead of the classical C^1 space.

Using the invariance by left translations, we can rephrase the Hölder regularity of f at z_0 in the following way. There exists a polynomial p_0 such that $\deg_k p_0 < \alpha$ and, for all $z \in \mathbb{R}^{1+2d}$ such that $z_0 z \in D$, we have

$$|f(z_0 \circ z) - p_0(z)| \le C\|z\|^\alpha. \tag{2-3}$$

In this case $p_0(z) = p(z_0 \circ z)$, where p is the polynomial of Definition 2.3. If the polynomial p_0 is given by

$$p_0(t, x, v) = a_0 + a_1 t + a_2 \cdot x + a_3 \cdot v + \cdots,$$

it is easy to verify that $a_0 = f(z_0)$, $a_1 = \partial_t f + v_0 \cdot \nabla_x f$, $a_2 = \nabla_x f$ and $a_3 = \nabla_v f$.

As is standard for some proofs of the Schauder estimates, see for example Section 4 in [Gilbarg and Trudinger 1998], we define the adimensional Hölder spaces.

Definition 2.6 (adimensional Hölder spaces). Given a kinetic cylinder $Q \subset \mathbb{R}^{1+2d}$ and any $\alpha > 0$, we define

$$[f]_{C_{\ell,*}^\alpha(Q)} = \sup_{z \in Q} d_z^\alpha [f]_{C_\ell^\alpha(Q_{d_z}(z))},$$

where

$$d_z = d_\ell(\partial_p Q, z). \tag{2-4}$$

Naturally, we also define $\|f\|_{C_{\ell,*}^\alpha(Q)} := \|f\|_{L^\infty(Q)} + [f]_{C_{\ell,*}^\alpha(Q)}$.

Here, we used the notation $\partial_p Q_r(t_0, x_0, v_0) := \partial Q_r(t_0, x_0, v_0) \setminus \{t = t_0\}$.

2D. *Hölder norms and differential operators.* The differential operators $(\partial_t + v \cdot \nabla_x)$, ∂_{x_i} and ∂_{v_i} commute with left translations. They do not commute with each other, and they do not keep the class of equations invariant. The operators that commute with the class of equations are ∂_t, ∂_{x_i} and $\partial_{v_i} + t\partial_{x_i}$, which are the ones that commute with right translations (instead of left translations).

It may be convenient to define the *kinetic degree* of a differential operator. We say that the kinetic degree of $\partial_t + v \cdot \nabla_x$ is $2s$, the kinetic degree of ∂_{x_i} is $(1 + 2s)$ and the kinetic degree of ∂_{v_i} is 1.

It is convenient to relate the definition of the Hölder spaces C_ℓ^α with these operators. The following (deceivingly simple) lemma will be used repeatedly.

Lemma 2.7 (derivatives and kinetic Hölder spaces). *Let* $D = \partial_t + v \cdot \nabla_x$, $D = \partial_{x_i}$ *or* $D = \partial_{v_i}$. *Let* f *be a* C_ℓ^α *function in a cylinder* Q *and let* $\deg_k D = \kappa$ *with* $\kappa < \alpha$. *Then* $Df \in C_\ell^{\alpha-\kappa}$ *and*

$$[Df]_{C_\ell^{\alpha-\kappa}(Q)} \lesssim [f]_{C_\ell^\alpha(Q)}.$$

Before proving Lemma 2.7, we prove the following auxiliary lemma about polynomials.

Lemma 2.8. *Let* $p(z)$ *be a polynomial in* t, x *and* v *of kinetic degree* k. *Let us write* p *as*

$$p(t, x, v) = \sum_{j \in \mathbb{N}^{1+2d}} a_j m_j(z),$$

where

$$m_j(z) := t^{j_0} x_1^{j_1} \cdots x_d^{j_d} v_1^{j_{d+1}} \cdots v_d^{j_{2d}}.$$

Assume that

$$\sup_{\|z\| \le r} |p(z)| \le C_0 r^\alpha.$$

Then, for each j, *we have*

$$|a_j| \le CC_0 r^{\alpha - \deg_k m_j},$$

where the constant C *depends on* $\deg_k p$ *and dimension only.*

Proof. We observe that once we establish this lemma for $r = 1$, the other values of r follow by scaling.

The space of polynomials of kinetic degree k in $\mathbb{R}^{1+2d}$ is finite-dimensional. Recall that all norms are equivalent in spaces of finite dimension. The result for $r = 1$ follows easily by comparing the two norms given by

$$\|p\|_1 = \sup_j |a_j| \quad \text{and} \quad \|p\|_2 = \sup_{\|z\| \le 1} |p(z)|. \tag{2-5}$$

This concludes the proof of the technical lemma. $\qquad\square$

Proof of Lemma 2.7. Let z_0 and z_1 be two points in Q. Since $f \in C_\ell^\alpha(Q)$, there exist polynomials q_0 and q_1 of degree less than α so that, for all z so that $z_0 \circ z \in Q$ and $z_1 \circ z \in Q$,

$$|f(z_0 \circ z) - q_0(z)| \lesssim C\|z\|^\alpha,$$
$$|f(z_1 \circ z) - q_1(z)| \lesssim C\|z\|^\alpha,$$

where $C = [f]_{C_\ell^\alpha}$. We used the fact that d_ℓ is left-invariant and $d_\ell(z, 0) \leq \|z\|$. Let $r = d_\ell(z_0, z_1) \approx \|z_0^{-1} \circ z_1\|$ and let us pick any z so that $\|z\| < r$.[1] From the triangle inequality (modified with the power $2s$ when $s < \frac{1}{2}$), we have $\|z_0^{-1} \circ z_1 \circ z\| \approx d_\ell(z_0, z_1 \circ z) \leq d_\ell(z_0, z_1) + d_\ell(z_1, z_1 \circ z) \lesssim r$. We apply the two inequalities above for $z_0^{-1} \circ z_1 \circ z$ and z respectively to obtain

$$|f(z_1 \circ z) - q_0(z_0^{-1} \circ z_1 \circ z)| \lesssim Cr^\alpha,$$

$$|f(z_1 \circ z) - q_1(z)| \lesssim Cr^\alpha.$$

Therefore, for all $\|z\| \leq r$,

$$|q_1(z) - q_0(z_0^{-1} \circ z_1 \circ z)| \lesssim Cr^\alpha.$$

Let us write the coefficients of both polynomials $q_1(z)$ and $q_0(z_0^{-1} \circ z_1 \circ z)$:

$$q_0(z_0^{-1} \circ z_1 \circ z) = a_0(z_1) + a_1(z_1)t + a_2(z_1)x + a_3(z_1)v + \cdots,$$

$$q_1(z) = b_0 + b_1 t + b_2 x + b_3 v + \cdots.$$

The coefficients $a_j(z_1)$ can be computed in terms of the coefficients of q_0 and the value of $z_0^{-1} \circ z_1$. It is not hard to see that each coefficient is a polynomial in $z_0^{-1} \circ z_1$ and thus of z_1 whose degree is not larger than $\deg_k q_0$ minus the degree of the corresponding monomial. Applying Lemma 2.8, we see that

$$|b_j - a_j(z_1)| \leq Cr^{\alpha - k_j}.$$

Here $k_1 = \deg_k t = \deg_k(\partial_t + v \cdot \nabla_x) = 2s$, $k_2 = \deg_k x = \deg_k \partial_{x_i} = 1 + 2s$ and $k_3 = \deg_k v = \deg_k \partial_{v_i} = 1$; i.e., $k_j = \kappa$ in short. Since $b_1 = (\partial_t + v_1 \nabla_x)f(z_1)$, $b_2 = \nabla_x f(z_1)$ and $b_3 = \nabla_v f(z_1)$, i.e., $b_j = Df(z_1)$ in short, we have

$$\sup\{z_1 | Df(z_1) - a_j(z_1)| : d_\ell(z_0, z_1) < r\} \leq Cr^{\alpha - \kappa}. \qquad \square$$

Remark 2.9. Note we can also apply Lemma 2.8 to the other coefficients of the polynomial $q_1(z)$. If p_z is the polynomial expansion of f at z of kinetic degree less than α and it has the form

$$p_z(t, x, v) = \sum_j a_j(z)t^{j_0}x_1^{j_1} \cdots x_d^{j_d}v_1^{j_{d+1}} \cdots v_d^{j_{2d}}.$$

Then, by a direct computation, the coefficients $a_j(z)$ correspond to

$$a_j(z) = \frac{(\partial_t + v \cdot \nabla_x)^{j_0} \partial_{x_1}^{j_1} \cdots \partial_{x_d}^{j_d} \partial_{v_1}^{j_{d+1}} \cdots \partial_{v_d}^{j_{2d}} f(z)}{k_0! \cdots k_{2d}!}.$$

Note that $(\partial_t + v \cdot \nabla_x)$ and ∂_v do not commute.

---

[1] It is slightly problematic when $z_0 \circ z$ or $z_1 \circ z$ fall outside the domain Q. It can be handled as in the proof of Proposition 2.10 by using the equivalent norm in the space of polynomials of degree $< \alpha$ that considers only the points that fall inside the domain.

2E. *Interpolation inequalities.* The usual interpolation estimates for Hölder spaces will hold.

Proposition 2.10 (interpolation). *Given $\alpha_1 < \alpha_2 < \alpha_3$ so that $\alpha_2 = \theta\alpha_1 + (1-\theta)\alpha_3$, for any function $f \in C_\ell^{\alpha_3}(Q_1)$ we have*

$$[f]_{C_\ell^{\alpha_2}(Q_1)} \lesssim [f]_{C_\ell^{\alpha_1}(Q_1)}^\theta [f]_{C_\ell^{\alpha_3}(Q_1)}^{1-\theta} + [f]_{C_\ell^{\alpha_1}(Q_1)}.$$

Also, for any function $f \in C_{\ell,*}^{\alpha_3}(Q_1)$,

$$[f]_{C_{\ell,*}^{\alpha_2}(Q_1)} \lesssim [f]_{C_{\ell,*}^{\alpha_1}(Q_1)}^\theta [f]_{C_{\ell,*}^{\alpha_3}(Q_1)}^{1-\theta} + [f]_{C_{\ell,*}^{\alpha_1}(Q_1)}.$$

Remark 2.11. We classically get from the previous estimates that for all $\varepsilon > 0$

$$[f]_{C_\ell^{\alpha_2}(Q_1)} \lesssim C_\varepsilon [f]_{C_\ell^{\alpha_1}(Q_1)} + \varepsilon [f]_{C_\ell^{\alpha_3}(Q_1)},$$

$$[f]_{C_{\ell,*}^{\alpha_2}(Q_1)} \lesssim C_{*,\varepsilon} [f]_{C_{\ell,*}^{\alpha_1}(Q_1)} + \varepsilon [f]_{C_{\ell,*}^{\alpha_3}(Q_1)}.$$

Proof. We prove the first interpolation inequality. The second one follows as a consequence by scaling. The statement says precisely that the function $\alpha \mapsto \log[f]_{C_\ell^\alpha}$ is convex. This is a local property, so we only need to prove it for α_3 sufficiently close to α_1. Because of this, it is enough to prove the interpolation inequality assuming that $(\mathbb{N} + 2s\mathbb{N}) \cap [\alpha_1, \alpha_3)$ contains at most one element.

Let q_z^1, q_z^2 and q_z^3 be the polynomial expansions of f at z of kinetic degrees less than α_1, α_2 and α_3 respectively such that for all $z \circ \xi \in Q_1$

$$|f(z \circ \xi) - q_z^i(\xi)| \le [f]_{C_\ell^{\alpha_i}} \|\xi\|^{\alpha_i} \quad \text{for } i = 1, 2, 3. \tag{2-6}$$

The polynomials q_z^1, q_z^2 and q_z^3 are increasingly higher-order expansions at the same point z. Therefore, q_z^2 contains all the terms in q_z^1 plus perhaps higher-order ones. In the same way, q_z^3 contains all the terms of q_z^2 plus perhaps higher-order ones. Because of our assumption that $(\mathbb{N} + 2s\mathbb{N}) \cap [\alpha_1, \alpha_3)$ has at most one element, which we call $\bar\alpha$, there can be at most one degree of homogeneity in the difference between the polynomials q_z^1 and q_z^3. The polynomial q_z^2 coincides with either q_z^1 or q_z^3 depending on whether $\bar\alpha \ge \alpha_2$ or $\bar\alpha < \alpha_2$. When $(\mathbb{N} + 2s\mathbb{N}) \cap [\alpha_1, \alpha_3) = \varnothing$, we have $q_z^1 = q_z^2 = q_z^3$, which is easier to analyze. Let us consider the case in which there is an $\bar\alpha$ and the polynomials are not equal.

Like in Lemma 2.8, we write

$$q_z^i(\xi) = \sum_{|j| < \alpha_i} a_j(z) m_j(\xi),$$

where each m_j is a monomial.

Subtracting (2-6) for $i = 1, 2$, whenever $z \circ \xi \in Q_1$, we have

$$|q_z^3(\xi) - q_z^1(\xi)| = \left| \sum_{|j| = \bar\alpha} a_j(z) m_j(\xi) \right| \le [f]_{C_\ell^{\alpha_1}} \|\xi\|^{\alpha_1} + [f]_{C_\ell^{\alpha_3}} \|\xi\|^{\alpha_3}. \tag{2-7}$$

From this inequality, we want to infer an estimate for $\|q_z^3 - q_z^1\|$. Let us first make some remarks about the norm of a polynomial. The space of polynomials of kinetic degree less than α_3 is finite-dimensional. So, all norms that we can write are equivalent. A natural choice is perhaps

$$\|q\| = \max\{|q(\xi)| : \|\xi\| \le 1\}.$$

If we change that radius 1 for any other universal constant, we would obtain an equivalent norm. Note that translations of polynomials are also polynomials of the same degree. Therefore, for any two universal constants c and C, $z_1 \in \mathbb{R}^{1+2d}$ and $\deg_k q < \alpha_3$, we deduce that

$$\max\{|q(\xi)| : \|\xi\| \le Q_c(z_1)\} \approx \max\{|q(\xi)| : \|\xi\| \le Q_C(z_1)\}.$$

The factors in $\approx$ depend naturally on c and C.

Coming back to (2-7), for any $N \in (0, 1]$ and $z \in Q_1$, let us pick some point $\xi_1 \in Q_1$ such that:

- $\|\xi_1\| < N$.

- Whenever $d_\ell(\xi_1, \xi) < cN$, then $\|\xi\| \le N$ and $z \circ \xi \in Q_1$. Here c is a universal constant.

It is not hard to see that for any $z \in Q_1$ such a ξ_1 exists (in fact plenty exist).

From (2-7) we get

$$\sup_{\{\xi : d_\ell(\xi_1, \xi) < cN\}} |(q_z^3 - q_z^1)(\xi)| \le [f]_{C_\ell^{\alpha_1}} N^{\alpha_1} + [f]_{C_\ell^{\alpha_3}} N^{\alpha_3}.$$

Since $q_z^3 - q_z^1$ is homogeneous of degree $\bar{\alpha}$,

$$\sup_{\{\xi : d_\ell(S_{1/N}\xi_1, \xi) < c\}} |(q_z^3 - q_z^1)(\xi)| \le [f]_{C_\ell^{\alpha_1}} N^{\alpha_1 - \bar{\alpha}} + [f]_{C_\ell^{\alpha_3}} N^{\alpha_3 - \bar{\alpha}}.$$

Since $\|\xi_1\| \le N$, we have $\|S_{1/N}\xi_1\| \le 1$. From the triangle inequality $\big($modified with power $2s$ when $s < \frac{1}{2}\big)$, there is a universal constant C so that if $\|\xi\| \le 1$, then $\xi \in d_\ell(S_{1/N}\xi_1, \xi) \le C$.

According to the discussion above, the fact that all norms are equivalent in the space of polynomials implies that

$$\|q_z^3 - q_z^1\| = \sup_{\{\xi : \|\xi\| \le 1\}} |(q_z^3 - q_z^1)(\xi)|$$

$$\le \sup_{\{\xi : d_\ell(S_{1/N}\xi_1, \xi) < C\}} |(q_z^3 - q_z^1)(\xi)|$$

$$\lesssim [f]_{C_\ell^{\alpha_1}} N^{\alpha_1 - \bar{\alpha}} + [f]_{C_\ell^{\alpha_3}} N^{\alpha_3 - \bar{\alpha}}.$$

We now optimize for $N \in [0, 1]$ and obtain

$$\|q_z^3 - q_z^1\| \lesssim [f]_{C_\ell^{\alpha_1}}^{\bar{\theta}} [f]_{C_\ell^{\alpha_3}}^{1-\bar{\theta}} + [f]_{C_\ell^{\alpha_1}},$$

where $\bar{\alpha} = \bar{\theta}\alpha_1 + (1 - \bar{\theta})\alpha_2$.

Now we estimate $f(z \circ \xi) - q_z^2(\xi)$ using both $[f]_{C_\ell^{\alpha_1}}$ and $[f]_{C_\ell^{\alpha_3}}$. There are two cases depending on whether $\alpha_2 > \bar{\alpha}$ or $\alpha_2 \le \bar{\alpha}$. The proofs are very similar, so let us do only the latter. In this case $q_z^2 = q_z^1$. We have

$$|f(z \circ \xi) - q_z^2(\xi)| \le \begin{cases} [f]_{C_\ell^{\alpha_1}} \|\xi\|^{\alpha_1}, \\ [f]_{C_\ell^{\alpha_3}} \|\xi\|^{\alpha_3} + C([f]_{C_\ell^{\alpha_1}}^{\bar{\theta}} [f]_{C_\ell^{\alpha_3}}^{1-\bar{\theta}} + [f]_{C_\ell^{\alpha_1}}) \|\xi\|^{\bar{\alpha}}. \end{cases}$$

One can easily verify that the right-hand side is less than $[f]_{C_\ell^{\alpha_1}}^{\theta} [f]_{C_\ell^{\alpha_3}}^{1-\theta} \|\xi\|^{\alpha_2}$ for any value of $\|\xi\|$. Indeed, if $\|\xi\| \ge ([f]_{C_\ell^{\alpha_2}}/[f]_{C_\ell^{\alpha_3}})^{1/(\alpha_3 - \alpha_1)}$, we have

$$[f]_{C_\ell^{\alpha_1}} \|\xi\|^{\alpha_1} \le [f]_{C_\ell^{\alpha_1}}^{\theta} [f]_{C_\ell^{\alpha_3}}^{1-\theta} \|\xi\|^{\alpha_2}.$$

Otherwise, we have

$$[f]_{C_\ell^{\alpha_3}}\|\xi\|^{\alpha_3} + ([f]_{C_\ell^{\alpha_1}}^{\bar\theta}[f]_{C_\ell^{\alpha_3}}^{1-\bar\theta} + [f]_{C_\ell^{\alpha_1}})\|\xi\|^{\bar\alpha} \lesssim [f]_{C_\ell^{\alpha_1}}^{\theta}[f]_{C_\ell^{\alpha_3}}^{1-\theta}\|\xi\|^{\alpha_2} + [f]_{C_\ell^{\alpha_1}}\|\xi\|^{\bar\alpha}.$$

If $\alpha_2 > \bar\alpha$, we would have $q_z^2 = q_z^3$ and the term $|q_z^3 - q^1(z)| \leq ([f]_{C_\ell^{\alpha_1}}^{\bar\theta}[f]_{C_\ell^{\alpha_3}}^{1-\bar\theta} + [f]_{C_\ell^{\alpha_1}})\|\xi\|^{\bar\alpha}$ would appear on the first line of the inequality. If $(\mathbb{N} + 2s\mathbb{N}) \cap [\alpha_1, \alpha_3) = \varnothing$, we have $q_z^1 = q_z^2 = q_z^3$ and the extra term involving $\bar\alpha$ would not be there. $\qquad\square$

2F. *Discussion about choices of distance.* We use the distance d_ℓ, which is invariant by left translations and by the scaling S_R. Let us analyze the consequences of this choice and compare with the other possible choices.

We could define a distance d_r that is invariant by right translations of the Lie group. It would be given by

$$d_r(z_1, z_2) := \inf_{h\in\mathbb{R}} \max\{|t_2 - h| + |h - t_1|, |x_1 - x_2 + h(v_1 - v_2)|^{2s/(1+2s)}, |v_1 - v_2|^{2s}\}^{1/(2s)}.$$

Moreover, it would be comparable to the following expressions:

$$d_r(z_1, z_2) \approx \begin{cases} \|z_2 \circ z_1^{-1}\|, \\ \|z_1 \circ z_2^{-1}\|, \\ \inf_{h\in\mathbb{R}}\{|t_2 - h|^{1/(2s)} + |h - t_1|^{1/(2s)} + |x_1 - x_2 + h(v_1 - v_2)|^{1/(1+2s)} + |v_1 - v_2|\}. \end{cases}$$

Alternatively, we could ignore the Lie group structure and define a distance d_s that only takes scaling into account:

$$d_s(z_1, z_2) := \|z_1 - z_2\|.$$

Here $\|\cdot\|$ stands for the scaled norm as in (2-1).

The most brutal choice would be to ignore both the Lie group action and scaling and use the plain Euclidean distance in $\mathbb{R}^{1+2d}$:

$$d_e(z_1, z_2) := (|t_1 - t_2|^2 + |x_1 - x_2|^2 + |v_1 - v_2|^2)^{1/2}.$$

The definition of Hölder spaces (Definition 2.3) depends on the choice of distance. We can thus consider the four possible candidates C_ℓ^α, C_r^α, C_s^α and C_e^α. The distances are not equivalent, and these four spaces are all different. Their only equivalence appears when measuring distances from the origin $d_\ell(0, z) \approx d_r(0, z) \approx d_e(0, z)$. Thus $C_\ell^\alpha(0) = C_r^\alpha(0) = C_e^\alpha(0)$ (by $C_{\text{any}}^\alpha(0)$ we mean the functions that are C_{any}^α at the point 0).

The class of equations (1-1) is invariant by left translations. Because of that, the norm d_ℓ is the most appropriate to work with. For example, if we proved an estimate for solutions of (1-1) of the sort $[f]_{C_\ell^\alpha(0)} \lesssim \|f\|_{L^\infty(Q_1)}$, it implies by simple translations that $[f]_{C_\ell^\alpha(Q_{1/2})} \lesssim \|f\|_{L^\infty(Q_1)}$. This implication does not hold true for $C_r^\alpha(Q_{1/2})$, $C_s^\alpha(Q_{1/2})$ or $C_e^\alpha(Q_{1/2})$ (at least not true keeping the same exponent α).

In previous works, people have paid less attention to these distinctions. The results in [Golse et al. 2019; Imbert and Silvestre 2020] are oblivious of the choice of distance. That is because these results are about an estimate in Hölder spaces for an undetermined exponent $\alpha > 0$. For any pair of points z_1 and z_2

in Q_1, the following inequality holds:

$$d_1(z_1, z_2) \leq C d_2(z_1, z_2)^{1/(1+2d)},$$

where d_1 and d_2 are any two choices among d_ℓ, d_r, d_s and d_e. Thus, the main theorems in [Golse et al. 2019; Imbert and Silvestre 2020] hold for the C_ℓ^α norm defined in terms of any of these distances, modulo adjustments of the constants and Hölder exponent α.

For Schauder estimates, the distinction between different distances plays a crucial role. In this case we want to obtain an estimate with the precise exponent $C^{2s+\alpha}$ when the right-hand side is C^α. It seems that such an estimate can only be true with the distance d_ℓ.

For right-invariant Hölder spaces C_r^α in terms of d_r, the corresponding statement of Lemma 2.7 would be in terms of the operators ∂_t, ∂_{x_i} and $\partial_{v_i} + t\partial_{x_i}$. These differential operators have the advantage that they commute with (3-19). For regular Hölder spaces C_s^α or C_e^α, Lemma 2.7 would of course hold with pure derivatives ∂_t, ∂_{x_i} and ∂_{v_i}.

Our Liouville theorem, Theorem 4.1, holds for any choice of distance d_ℓ, d_r or d_s. This is because in Step 1 of the proof we establish that the function is constant in x. After that, we ignore the x-coordinate and the three distances are the same.

In the proof of Lemma 5.2, we select a sequence of functions $\tilde{f}_j$ that are scaled left-translations of a sequence of solutions f_j. If we used a different choice of distance that is not invariant by left translations, we would not be able to conclude anything about their C^β norms.

One needs to be careful throughout this paper to make sure we do not implicitly use the exact triangle inequality for $s < \frac{1}{2}$, we do not commute group operation $\circ$, and that we do not accidentally apply d_r or d_s instead of d_ℓ.

3. Integral operators

This section is devoted to the integral operators

$$\mathcal{L}f(v) = \int_{\mathbb{R}^d} (f(v') - f(v))K(v' - v)\,\mathrm{d}v' \tag{3-1}$$

associated with fixed kernels K from the elliptic class $\mathcal{K}$ given in Definition 1.1. We first explain when these integral operators can be evaluated pointwise. We then turn to limits of kernels and integral operators. We conclude this section by proving Hölder estimates that will be used in the proof of the Schauder estimate.

3A. *Evaluating operators pointwise.* In this subsection, we discuss how to evaluate pointwise operators associated with kernels in the elliptic class $\mathcal{K}$. More precisely, we want to explain the conditions that a function $f : \mathbb{R}^d \to \mathbb{R}$ must meet in order for the integral in (3-1) to be well-defined at the point v_0. On one hand, it must be sufficiently regular so that the integral does not diverge in a neighborhood of v_0. On the other hand, it must also satisfy some growth conditions so that the integral does not diverge at infinity. Let us split the domain of integration accordingly and analyze conditions for convergence of each part:

$$\mathcal{L}f(v_0) = \mathrm{PV}\int_{B_1} (f(v_0 + w) - f(v_0))K(w)\,\mathrm{d}w + \int_{\mathbb{R}^d \setminus B_1} (f(v_0 + w) - f(v_0))K(w)\,\mathrm{d}w.$$

When $s \geq \frac{1}{2}$, the first term must be understood in the principal value sense, even when f is smooth. Using the symmetry condition $K(w) = K(-w)$, we can symmetrize the integral and remove the principal value:

$$\text{PV} \int_{B_1} (f(v_0 + w) - f(v_0)) K(w) \, dw = \frac{1}{2} \int_{B_1} (f(v_0 + w) + f(v_0 - w) - 2f(v_0)) K(w) \, dw.$$

Because of (1-2), this integral is classically computable when $f \in C^{2s+\varepsilon}(B_1(v_0))$ for some $\varepsilon > 0$. Indeed,

$$\frac{1}{2} \int_{B_1} |f(v_0 + w) + f(v_0 - w) - 2f(v_0)| K(w) \, dw \lesssim [f]_{C^{2s+\varepsilon}(B_1(v_0))} \left(\int_{B_1} |w|^{2s+\varepsilon} K(w) \, dw \right)$$

$$\lesssim \Lambda [f]_{C^{2s+\varepsilon}(B_1(v_0))}. \tag{3-2}$$

In order to analyze the tail of the integral, we introduce the function

$$\omega_{v_0}(r) := \sup\{|f(v)| : v \in B_{2r}(v_0) \setminus B_{r/2}(v_0)\}. \tag{3-3}$$

We observe that, because of (1-2), the function ω_{v_0} can be used to bound the tail of the integral. We state the estimate in a lemma for later use.

Lemma 3.1. *Let $R > 0$. Then*

$$\int_{\mathbb{R}^d \setminus B_R} f(v_0 + w) K(w) \, dw \lesssim \Lambda \int_{R/2}^{\infty} \omega_{v_0}(r) r^{-1-2s} \, dr.$$

Proof. Using the definition of $\omega_{v_0}(r)$, we can write

$$\int_{\mathbb{R}^d \setminus B_R} f(v_0 + w) K(w) \, dw \leq \int_{\mathbb{R}^d \setminus B_R} \min\left\{ \omega_{v_0}(r) : r \in \left[\frac{|w|}{2}, 2|w| \right] \right\} K(w) \, dw$$

$$\leq \int_{\mathbb{R}^d \setminus B_R} \left\{ \frac{2}{3|w|} \int_{|w|/2}^{2|w|} \omega_{v_0}(r) \, dr \right\} K(w) \, dw$$

$$\lesssim \int_{\mathbb{R}^d \setminus B_R} \left\{ \int_{|w|/2}^{2|w|} \frac{\omega_{v_0}(r)}{r} \, dr \right\} K(w) \, dw$$

$$\lesssim \int_{R/2}^{\infty} \left\{ \int_{B_{2r} \setminus B_{r/2}} K(w) \, dw \right\} \frac{\omega_{v_0}(r)}{r} \, dr.$$

Using (1-2) yields the result. $\qquad \square$

Summarizing, we have the estimate

$$\mathcal{L} f(v_0) \lesssim \Lambda \left([f]_{C^{2s+\varepsilon}(v_0)} + \int_1^{\infty} (|f(v_0)| + \omega_{v_0}(r)) r^{-1-2s} \, dt \right). \tag{3-4}$$

Moreover, the integral expression in (3-1) is classically computable whenever the right-hand side of the inequality is finite.

3B. *Weak limits of kernels.* We now discuss how to pass to the limit in kernels. We first define the notion of weak-$*$ convergence and we then prove that the set $\mathcal{K}$ is compact for the corresponding topology.

Definition 3.2 (weak-$*$ convergence of kernels). We say that a sequence K_j of Radon measures in $\mathbb{R}^d \setminus \{0\}$ converges weakly-$*$ to the Radon measure K_∞ if for any continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$, compactly

supported, whose support does not include the origin, we have

$$\lim_{j \to \infty} \int_{\mathbb{R}^d} \varphi(w) K_j(w) \, \mathrm{d}w = \int_{\mathbb{R}^d} \varphi(w) K_\infty(w) \, \mathrm{d}w.$$

Lemma 3.3 (closeness of $\mathcal{K}$ under weak-$*$ limit). *If the kernels K_j belong to the class $\mathcal{K}$ of Definition 1.1 and K_j converges weakly-$*$ to K_∞, then K_∞ also belongs to the class $\mathcal{K}$.*

Proof. The fact that K_∞ is a nonnegative Radon measure is classical.

As far as the upper bound is concerned, it is enough to consider a cut-off function φ_r valued in $[0, 1]$ with $\varphi_r \equiv 1$ in $B_r \setminus B_\eta$ and whose compact support is contained in $B_{r+\varepsilon} \setminus B_{\eta/2}$ for some $\varepsilon, \eta > 0$. Then we write

$$\int_{\mathbb{R}^d} |w|^2 \varphi_r(w) K_j(w) \, \mathrm{d}w \leq \Lambda (r + \varepsilon)^{2-2s}.$$

Passing to the limit as $j \to +\infty$, we get

$$\int_{B_r \setminus B_\eta} |w|^2 K_\infty(w) \, \mathrm{d}w \leq \int_{\mathbb{R}^d} |w|^2 \varphi_r(w) K_j(w) \, \mathrm{d}w \leq \Lambda (r + \varepsilon)^{2-2s}.$$

Since ε and η are arbitrary, K_∞ satisfies the upper bound.

As far as the coercivity estimate is concerned, let $R > 0$ and $\varphi \in C^2(B_R)$. Since $K_j \in \mathcal{K}$, we have

$$\iint_{B_R \times B_R} |\varphi(v) - \varphi(v')|^2 K_j(v' - v) \, \mathrm{d}v' \, \mathrm{d}v \geq \lambda \iint_{B_{R/2} \times B_{R/2}} |\varphi(v) - \varphi(v')|^2 |v' - v|^{-d-2s} \, \mathrm{d}v' \, \mathrm{d}v.$$

For all $r > 0$, consider a cut-off function Ψ_r valued in $[0, 1]$, $\Psi_r \equiv 1$ in $B_{r/2}$ and $\Psi_r \equiv 0$ outside B_r. Thanks to the uniform upper bound, we have

$$\iint_{B_R \times B_R} \Psi_r(v' - v) |\varphi(v) - \varphi(v')|^2 K_j(v' - v) \, \mathrm{d}v' \, \mathrm{d}v \lesssim \Lambda R^d \|\nabla \varphi\|_{L^\infty(B_R)} r^{2-2s}.$$

Combining the two previous estimates, we get

$$\iint_{B_R \times B_R} (1 - \Psi_r)(v' - v) |\varphi(v) - \varphi(v')|^2 K_j(v' - v) \, \mathrm{d}v' \, \mathrm{d}v$$
$$\geq \lambda \iint_{B_{R/2} \times B_{R/2}} |\varphi(v) - \varphi(v')|^2 |v' - v|^{-d-2s} \, \mathrm{d}v' \, \mathrm{d}v - \mathcal{O}(r^{2-2s}).$$

We can now pass to the limit as $j \to \infty$ and obtain

$$\iint_{B_R \times B_R} |\varphi(v) - \varphi(v')|^2 K_\infty(v' - v) \mathbf{1}_{|v'-v| \geq r/2} \, \mathrm{d}v' \, \mathrm{d}v$$
$$\geq \lambda \iint_{B_{R/2} \times B_{R/2}} |\varphi(v) - \varphi(v')|^2 |v' - v|^{-d-2s} \, \mathrm{d}v' \, \mathrm{d}v - \mathcal{O}(r^{2-2s}).$$

Letting $r \to 0^+$ yields the result. $\qquad\square$

Lemma 3.4 (compactness of $\mathcal{K}$ for weak-$*$ topology). *If K_j is a sequence of kernels in the class $\mathcal{K}$ of Definition 1.1, then it has a weak-$*$ convergent subsequence.*

Proof. We split $\mathbb{R}^d \setminus \{0\}$ into $\bigcup_{k \in \mathbb{Z}} C_k$ with $C_k = B_{2^k} \setminus B_{2^{k-1}}$. The sequence of Radon measures $\{K_j|_{C_k}\}_j$ in C_k is compact because of the Banach–Alaoglu theorem. Thanks to a diagonal argument, we can thus extract a sequence $\{K_{l(j)}\}_j$ converging towards K_∞ on each ring C_k. In particular, this sequence weakly-$*$ converges to K_∞ in the sense of Definition 3.2. $\qquad\square$

3C. *Limits of operators.*

Lemma 3.5 (limits of operators). *Let Ω be an open bounded set of $(-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d$, and let K_j and $f_j : (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be sequences of kernels and functions respectively so that the following conditions hold:*

(1) *Each K_j belongs to the class $\mathcal{K}$.*

(2) *The sequence $K_j \rightharpoonup K_\infty$ weakly-$*$ as $j \to \infty$.*

(3) *The sequence $f_j \to f$ locally uniformly in $(-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d$ as $j \to \infty$.*

(4) *The sequence f_j is uniformly bounded in $C_\ell^{2s+\eta}(\Omega)$ for some $\eta > 0$.*

(5) *There is a function $\omega : [1, \infty) \to \mathbb{R}$ so that $\int_1^\infty \omega(r) r^{-1-2s} \, dr < +\infty$ and for every $j \in \mathbb{N}$,*

$$\text{for all } r \geq 1, \text{ for all } v \in B_r \setminus B_{r/2}, \quad f_j(t, x, v) \leq \omega(r).$$

Then we have

$$\mathcal{L}_j f_j \to \mathcal{L}_\infty f \quad \text{locally uniformly in } \Omega \text{ as } j \to \infty,$$

where $\mathcal{L}_j$ is the integral operator corresponding to K_j; see (3-1).

Proof. Let $\varepsilon > 0$ be arbitrary.

We use the assumption (4) to bound the part of the integrals in $\mathcal{L}_j f_j$ and $\mathcal{L}f$ around the origin. Thanks to the symmetry assumption of the kernels,

$$\left| \text{PV} \int_{B_\rho} (f_j(v+w) - f_j(v)) K_j(w) \, dw \right| = \left| \frac{1}{2} \int_{B_\rho} (f_j(v+w) + f_j(v-w) - 2f_j(v)) K_j(w) \, dw \right|$$

$$\leq \frac{1}{2} [f_j]_{C_\ell^{2s+\eta}(\Omega)} \int_{B_\rho} |w|^{2s+\eta} K_j(w) \, dw$$

$$\lesssim \Lambda \rho^\eta < \frac{\varepsilon}{8} \quad \text{provided that } \rho \text{ is sufficiently small.} \tag{3-5}$$

We use the assumption (5) to bound the tails of the integrals. Note that for any $(t, x, v) \in \Omega$ and $r > \text{diam}(\Omega)$, we can obtain a common majorant function $\omega_v(r)$ for all functions f_j, as in (3-3), by the formula

$$\omega_v(r) \leq \omega(r - \text{diam}(\Omega)) + \|f_j\|_{L^\infty}.$$

Using Lemma 3.1, for R sufficiently large,

$$\left| \int_{\mathbb{R}^d \setminus B_R} (f_j(v+w) - f_j(v)) K_j(w) \, dw \right| \leq \int_{R/2}^\infty (\omega(r - \text{diam}(\Omega)) + \|f_j\|_{L^\infty}) r^{-1-2s} \, dr < \frac{\varepsilon}{8}. \tag{3-6}$$

Using that $K_j \in \mathcal{K}$ and $f_j \to f$ locally uniformly, for j sufficiently large

$$\int_{B_R \setminus B_\rho} |f_j(v+w) - f_j(v) - f(v+w) + f(v)| K_j(w) \, dw \lesssim \Lambda r^{-2s} \|f_j - f\|_{L^\infty(B_R + \Omega)} < \frac{\varepsilon}{4}. \quad (3\text{-}7)$$

Finally, since $K_j \to K$ weakly-$*$, for j large

$$\left| \int_{B_R \setminus B_\rho} (f(v+w) - f(v))(K_j - K) \, dw \right| < \frac{\varepsilon}{4}. \quad (3\text{-}8)$$

Note that because f is a continuous function on $\mathbb{R}^{1+2d}$, the choice of j can be made uniform with respect to the point $z = (t, x, v) \in \Omega$ by the argument that led to (3-7).

Adding up (3-5), (3-6), (3-7) and (3-8), we get that $|\mathcal{L}_j f - \mathcal{L} f| < \varepsilon$ uniformly in Ω for j sufficiently large. $\qquad\square$

3D. *Consequences of Assumption 1.5.* We gather here some consequences of Assumption 1.5 that will be used in the next subsection when deriving Hölder estimates:

$$\int_{|w| \leq 1} |w|^{2s+\alpha} |K_{z_1}(w) - K_{z_2}(w)| \, dw \lesssim A_0 d_\ell(z_1, z_2)^\alpha, \quad (3\text{-}9)$$

$$\int_{|w| \geq 1} |K_{z_1}(w) - K_{z_2}(w)| \, dw \lesssim A_0 d_\ell(z_1, z_2)^\alpha. \quad (3\text{-}10)$$

Both inequalities are consequences of the fact that Assumption 1.5 implies that for all $r > 0$

$$\int_{B_r \setminus B_{r/2}} |K_{z_1}(w) - K_{z_2}(w)| \, dw \lesssim A_0 r^{-2s} d_\ell(z_1, z_2)^\alpha.$$

To get (3-10), we use dyadic rings $B_{2^{k+1}} \setminus B_{2^k}$ and sum over k.

3E. *Hölder estimates.* We gather here estimates that will be used when proving the main Schauder estimate; see the terms A and B on page 201.

Let us consider a sign-changing kernel K such that $K(w) = K(-w)$ and it satisfies the upper bound, for all $r > 0$,

$$\int_{B_r} |w|^2 |K(w)| \, dw \leq \Lambda r^{2-2s}. \quad (3\text{-}11)$$

Let us study the corresponding integral operator

$$\mathcal{L}_K f(z) = \int (f(z \circ (0, 0, w)) - f(z)) K(w) \, dw.$$

We start with a global estimate.

Lemma 3.6. *Assume $\alpha < \min(1, 2s)$. For any sign-changing symmetric kernel K satisfying* (3-11), *and $f \in C_\ell^{2s+\alpha}(\mathbb{R}^{1+2d})$, we have the estimate*

$$[\mathcal{L}_K f]_{C_\ell^\alpha(\mathbb{R}^{2d+1})} \lesssim \Lambda [f]_{C_\ell^{2s+\alpha}(\mathbb{R}^{2d+1})}.$$

Proof. Let us start by fixing some notation. As usual, we denote by p_z the polynomial expansion of f at z so that $\deg_k p_z < 2s + \alpha$ and, for $z, \xi \in \mathbb{R}^{1+2d}$,

$$|f(z \circ \xi) - p_z(\xi)| \leq [f]_{C_\ell^{2s+\alpha}} \|\xi\|^{2s+\alpha}.$$

Let us also write, for $z, \xi, \zeta \in \mathbb{R}^{1+2d}$,

$$\Delta_\xi f(z) = f(z \circ \xi) - f(z) \quad \text{and} \quad \delta_\zeta p_z(\xi) = p_{z \circ \zeta}(\xi) - p_z(\xi).$$

We must estimate the quantity

$$
\begin{aligned}
\Delta_\xi \mathcal{L}_K f(z) &= \mathcal{L}_K f(z \circ \xi) - \mathcal{L}_K f(z) \\
&= \int_{\mathbb{R}^d} [f(z \circ \xi \circ (0, 0, w)) - f(z \circ (0, 0, w)) - f(z \circ \xi) + f(z)] K(w) \, \mathrm{d}w \\
&= \int_{\mathbb{R}^d} [\Delta_\xi \Delta_{(0,0,w)} f(z)] K(w) \, \mathrm{d}w.
\end{aligned}
$$

Since $\alpha < \min(1, 2s)$, proving an estimate for $[\mathcal{L}_K f]_{C_\ell^\alpha}$ amounts to finding the right upper bound for $|\Delta_\xi \mathcal{L}_K f(z)|$.

We split the integral above into two subdomains: B_R and $\mathbb{R}^d \setminus B_R$. We will later choose $R = \|\xi\|$. Estimating the integral in B_R, we symmetrize using that $K(w) = K(-w)$ and

$$
\begin{aligned}
\left| \int_{B_R} \Delta_{(0,0,w)} f(z) K(w) \, \mathrm{d}w \right| &= \left| \int_{B_R} [f(z \circ (0, 0, w)) - f(z)] K(w) \, \mathrm{d}w \right| \\
&= \frac{1}{2} \left| \int_{B_R} [f(z \circ (0, 0, w)) + f(z \circ (0, 0, -w)) - 2f(z)] K(w) \, \mathrm{d}w \right|.
\end{aligned}
$$

Here we use that

$$|f(z \circ (0, 0, w)) - p_z(0, 0, w)| \leq [f]_{C_\ell^{2s+\alpha}} |w|^{2s+\alpha}.$$

The polynomial p_z has kinetic degree smaller than $2s + \alpha$. The first-order terms in v cancel out by the symmetrization. There may be second-order terms in v if $2s + \alpha > 2$. There cannot be higher-order terms in v with our restrictions on s and α. Any term involving t and x vanishes when evaluating on $(0, 0, w)$. Thus, when $2s + \alpha \leq 2$ we continue, using the assumption (3-11), to get

$$
\begin{aligned}
\left| \int_{B_R} \Delta_{(0,0,w)} f(z) K(w) \, \mathrm{d}w \right| &\leq [f]_{C_\ell^{2s+\alpha}} \int_{B_R} |w|^{2s+\alpha} |K(w)| \, \mathrm{d}w, \\
&\lesssim [f]_{C_\ell^{2s+\alpha}} \Lambda R^\alpha.
\end{aligned}
$$

When $2s + \alpha > 2$, we cannot cancel out the second-order terms in v in the polynomial p_z. Thus, in that case the same computation leads to

$$\left| \int_{B_R} \Delta_{(0,0,w)} f(z) K(w) \, \mathrm{d}w - \partial_{v_i v_j} f(z) \left(\int_{B_R} w_i w_j K(w) \, \mathrm{d}w \right) \right| \lesssim [f]_{C_\ell^{2s+\alpha}} \Lambda R^\alpha.$$

In the estimates above, the value of $z \in \mathbb{R}^{1+2d}$ is arbitrary. The inequalities hold for $z \circ \xi$ just as well. Therefore, applying Δ_ξ we get

$$\left| \int_{B_R} \Delta_\xi \Delta_{(0,0,w)} f(z) K(w) \, \mathrm{d}w \right| \lesssim [f]_{C_\ell^{2s+\alpha}} \Lambda R^\alpha$$
$$+ |\Delta_\xi D^2 f| \Lambda R^{2-2s} \quad \text{(only relevant if } 2s + \alpha > 2\text{)}$$
$$\lesssim [f]_{C_\ell^{2s+\alpha}} (\Lambda R^\alpha$$
$$+ \|\xi\|^{2s+\alpha-2} \Lambda R^{2-2s}) \quad \text{(only relevant if } 2s + \alpha > 2\text{)}.$$

In the last inequality we used Lemma 2.7 for the case $2s + \alpha > 2$. Note that for any value of $2s + \alpha$, when we choose $R = \|\xi\|$ we will get

$$\left| \int_{B_R} \Delta_\xi \Delta_{(0,0,w)} f(z) K(w) \, \mathrm{d}w \right| \lesssim [f]_{C_\ell^{2s+\alpha}} \Lambda \|\xi\|^\alpha.$$

Now we move on to estimate the part of the integral in $\mathbb{R}^d \setminus B_R$. We use the two inequalities

$$|f(z \circ \xi) - p_z(\xi)| \le [f]_{C_\ell^{2s+\alpha}} \|\xi\|^{2s+\alpha}, \tag{3-12}$$

$$|f(z \circ \xi \circ (0,0,w)) - p_{z \circ (0,0,w)}((0,0,w)^{-1} \circ \xi \circ (0,0,w))|$$
$$\le [f]_{C_\ell^{2s+\alpha}} \|(0,0,w)^{-1} \circ \xi \circ (0,0,w)\|^{2s+\alpha}. \tag{3-13}$$

The second one naturally requires some further analysis. We observe that

$$(0,0,w)^{-1} \circ \xi \circ (0,0,w) = (\xi_1, \xi_2 + \xi_1 w, \xi_3).$$

Therefore

$$\|(0,0,w)^{-1} \circ \xi \circ (0,0,w)\| \lesssim \|\xi\| + (|\xi_1| |w|)^{1/(1+2s)} \le \|\xi\| + \|\xi\|^{2s/(1+2s)} |w|^{1/(1+2s)}. \tag{3-14}$$

We will split the integral in $\mathbb{R}^d \setminus B_R$ as the sum of several terms:

$$\left| \int_{\mathbb{R}^d \setminus B_R} [\Delta_\xi \Delta_{(0,0,w)} f(z)] K(w) \, \mathrm{d}w \right| \le I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 := \int_{\mathbb{R}^d \setminus B_R} |f(z \circ \xi) - p_z(\xi)| K(w) \, \mathrm{d}w,$$

$$I_2 := \int_{\mathbb{R}^d \setminus B_R} |f(z \circ \xi \circ (0,0,w)) - p_{z \circ (0,0,w)}((0,0,w)^{-1} \circ \xi \circ (0,0,w))| K(w) \, \mathrm{d}w,$$

$$I_3 := \int_{\mathbb{R}^d \setminus B_R} |\delta_{(0,0,w)} p_z(\xi) - \Delta_{(0,0,w)} f(z)| K(w) \, \mathrm{d}w,$$

$$I_4 := \int_{\mathbb{R}^d \setminus B_R} |p_{z \circ (0,0,w)}((0,0,w)^{-1} \circ \xi \circ (0,0,w)) - p_{z \circ (0,0,w)}(\xi)| K(w) \, \mathrm{d}w.$$

We bound I_1 easily using (3-12) and (3-11):

$$I_1 \lesssim [f]_{C_\ell^{2s+\alpha}} \Lambda \|\xi\|^{2s+\alpha} R^{-\alpha}. \tag{3-15}$$

We bound I_2 following the procedure, but applying (3-14):

$$I_2 \lesssim [f]_{C_\ell^{2s+\alpha}} \left(\Lambda \|\xi\|^{2s+\alpha} R^{-\alpha} + \|\xi\|^{2s(2s+\alpha)/(1+2s)} \int_{\mathbb{R}^d \setminus B_R} |w|^{(2s+\alpha)/(1+2s)} K(w) \, dw \right)$$

$$\lesssim [f]_{C_\ell^{2s+\alpha}} \Lambda (\|\xi\|^{2s+\alpha} R^{-\alpha} + \|\xi\|^{2s(2s+\alpha)/(1+2s)} R^{(2s+\alpha)/(1+2s)-2s}). \tag{3-16}$$

For the analysis of I_3, we write p_z as a sum of monomials:

$$p_z(\xi) = \sum_{\deg_k m_j < 2s+\alpha} a_j(z) m_j(\xi).$$

Moreover $a_0(z) = f(z)$ and

$$\delta_{(0,0,w)} p_z(\xi) = \sum_{\deg_k m_j < 2s+\alpha} \Delta_{(0,0,w)} a_j(z) m_j(\xi).$$

From Lemma 2.7, we know that $[a_j]_{C_\ell^{2s+\alpha-\deg_k m_j}} \lesssim [f]_{C_\ell^{2s+\alpha}}$. Note that $2s + \alpha - \deg_k m_j < 1$ for any monomial such that $0 < \deg_k m_j < 2s + \alpha$. Thus,

$$|\delta_{(0,0,w)} p_z(\xi) - \Delta_{(0,0,w)} f(z)| \lesssim [f]_{C_\ell^{2s+\alpha}} \sum_{0 < \deg_k m_j < 2s+\alpha} |w|^{2s+\alpha-\deg_k m_j} \|\xi\|^{\deg_k m_j}.$$

Therefore,

$$I_3 \lesssim [f]_{C_\ell^{2s+\alpha}} \Lambda \sum_{0 < \deg_k m_j < 2s+\alpha} R^{\alpha - \deg_k m_j} \|\xi\|^{\deg_k m_j}.$$

Regarding I_4, note that since $2s + \alpha < 1 + 2s$, the polynomial p_z cannot have a term that involves its second component (x). Since $(0, 0, w)^{-1} \circ \xi \circ (0, 0, w)$ and ξ differ only on their second component, we actually have $I_4 = 0$.

When we choose $R = \|\xi\|$, the estimates of all terms are $\lesssim \Lambda [f]_{C_\ell^{2s+\alpha}} \|\xi\|^\alpha$, and therefore we conclude the proof. $\qquad \square$

We next derive a local estimate from the global one.

Lemma 3.7. *Let $\alpha = (2s/(1+2s))\gamma$ with $\gamma < \min(1, 2s)$. Then*

$$[\mathcal{L}_K f]_{C_\ell^\alpha(Q_{1/2})} \lesssim [f]_{C_\ell^{2s+\alpha}(Q_1)} + [f]_{C_\ell^\gamma((-1,0] \times B_1 \times \mathbb{R}^d)}.$$

Proof. It is enough to write $\mathcal{L}_K f(z) = \mathcal{L}_{\widetilde{K}} f(z) + C(z)$, where $\widetilde{K}(w) = \mathbf{1}_{B_\rho}(w) K(w)$ and $C(z)$ is $\mathcal{L}_{\widetilde{\widetilde{K}}} f(z)$ with $\widetilde{\widetilde{K}}(w) = (1 - \mathbf{1}_{B_\rho}) K(w)$ and ρ small. From the previous lemma, we have

$$[\mathcal{L}_{\widetilde{K}} f]_{C_\ell^\alpha(Q_{1/2})} \lesssim [f]_{C_\ell^{2s+\alpha}(Q_1)}.$$

Let us prove that

$$[C]_{C_\ell^\alpha(Q_{1/2})} \lesssim [f]_{C_\ell^\gamma((-1,0] \times B_1 \times \mathbb{R}^d)}. \tag{3-17}$$

In order to do so, we write, for $z_1, z_2 \in Q_{1/2}$,

$$C(z_2) - C(z_1) = \int_{\mathbb{R}^d \setminus B_\rho} (f(z_2 \circ (0, 0, w)) - f(z_2) - f(z_1 \circ (0, 0, w)) + f(z_1)) K(w) \, dw$$

and we first prove that

$$|f(z_2 \circ (0,0,w)) - f(z_2) - f(z_1 \circ (0,0,w)) + f(z_1)| \lesssim [f]_{C_\ell^\gamma}(1 + |w|^{\gamma/(1+2s)}) d_\ell(z_1, z_2)^\alpha. \quad (3\text{-}18)$$

On the one hand, since $f \in C_\ell^\alpha(Q_{1/2})$ and $\alpha \le \gamma$, we have

$$|f(z_2) - f(z_1)| \lesssim [f]_{C_\ell^\gamma} d_\ell(z_1, z_2)^\alpha.$$

On the other hand, the C_ℓ^γ regularity of f yields

$$|f(z_2 \circ (0,0,w)) - f(z_1 \circ (0,0,w))| \le [f]_{C_\ell^\gamma} d_\ell(z_1 \circ (0,0,w), z_2 \circ (0,0,w))^\gamma$$

$$\lesssim [f]_{C_\ell^\gamma} \|(0,0,w)^{-1} \circ z_2^{-1} \circ z_1 \circ (0,0,w)\|^\gamma.$$

We now compute $(0,0,-w) \circ z_2^{-1} \circ z_1 \circ (0,0,w) = z_2^{-1} \circ z_1 - (0, (t_1 - t_2)w, 0)$ and get

$$\|(0,0,w)^{-1} \circ z_2^{-1} \circ z_1 \circ (0,0,w)\| \lesssim d_\ell(z_1, z_2) + |t_1 - t_2|^{1/(1+2s)} |w|^{1/(1+2s)}$$

$$\lesssim (1 + |w|^{1/(1+2s)}) d_\ell(z_1, z_2)^{2s/(1+2s)}.$$

Combining the three previous estimates yields (3-18). Since $\int_{\mathbb{R}^d \setminus B_\rho} (1 + |w|^{\gamma/(1+2s)}) K(w) \, dw \lesssim \Lambda C_\rho$, thanks to assumption (1-2) and the fact that $\gamma < 2s$, (3-18) implies (3-17). $\qquad \square$

3F. *The local Hölder estimate.* The symmetry condition $K(t, x, v, v+w) = K(t, x, v, v-w)$ corresponds to equations in *nondivergence form*, in the sense that the integrodifferential operator has a structure similar to that of elliptic equations of nondivergence form (as in $a_{ij}(t, x, v)\partial_{v_i v_j} f$). It is different from the symmetry condition that would make the operator self-adjoint, that is, $K(t, x, v, v') = K(t, x, v', v)$, and corresponds to equations in *divergence form*. The weak Harnack inequality, in the style of De Giorgi, obtained in [Imbert and Silvestre 2020] does not apply to (1-1) precisely because of this distinction of symmetry assumptions. Our kernels K do not satisfy the cancellation conditions (1.6) and (1.7) from [Imbert and Silvestre 2020].

The situation is simpler when we take a translation-invariant kernel $K(w)$ and consider the equation

$$f_t + v \cdot \nabla_x f = \mathcal{L} f + c(t, x, v). \quad (3\text{-}19)$$

It is an integrodifferential analog of an equation with constant coefficients. There is no distinction in this case between divergence and nondivergence form. The kernel $\widetilde{K}(t, x, v, v') = K(v' - v)$ satisfies the symmetry condition (and thus also the cancellation condition) $\widetilde{K}(t, x, v, v') = \widetilde{K}(t, x, v', v)$ for any kernel K in $\mathcal{K}$.

The regularity of the solution f to (3-19) is not important. It is straightforward to approximate any (weak/viscosity) solution to (3-19) with C^∞ solutions by mollification. Indeed, if f solves (3-19), then for any smooth compactly supported function $\varphi : \mathbb{R}^{1+2d} \to \mathbb{R}$, the function

$$\varphi *_k f(z) = \int_{\mathbb{R}^{1+2d}} \varphi(\xi) f(\xi \circ z) \, d\xi$$

also solves (3-19) (perhaps in a slightly smaller domain depending on the support of φ). Naturally, the function $\varphi *_k f \in C^\infty$ whenever $\varphi \in C^\infty$. Taking φ to be an approximation of the unit mass at the origin,

we approximate any solution of (3-19) by a smooth one. Therefore, we can safely assume, without loss of generality, that every function is C^∞ for the purposes of the results in this section.

We apply the main result from [Imbert and Silvestre 2020] to our setting.

Theorem 3.8 (local Hölder estimate). *Let $\mathcal{L}$ be an integral operator corresponding to a kernel in the class $\mathcal{K}$ (as in Definition 1.1). Let $f : (-1, 0] \times B_1 \times \mathbb{R}^d \to \mathbb{R}$ be a function that solves* (3-19) *in Q_1. Then, the following estimate holds:*

$$[f]_{C^\delta_\ell(Q_{1/2})} \leq C(\|f\|_{L^\infty((-1,0] \times B_1 \times \mathbb{R}^d)} + \|c\|_{L^\infty(Q_1)}).$$

Here $\delta > 0$ and C are constants depending only on dimension and the parameters λ and Λ of Definition 1.1.

Remark 3.9. Note that Theorem 3.8 and its corollaries below hold for several different choices of the distance function. See Section 2F.

Proof. The coercivity conditions in the definition of the class $\mathcal{K}$ of elliptic kernels slightly differ from the coercivity condition imposed in [Imbert and Silvestre 2020]. Let $f : \mathbb{R}^d \to \mathbb{R}$ be supported in some ball $B_{\bar{R}}$. Then

$$
\begin{aligned}
-\int f \mathcal{L} f \, dv &= \frac{1}{2} \iint (f' - f)^2 K(v, v') \, dv' \, dv \\
&\geq \frac{1}{2} \iint_{B_{4\bar{R}} \times B_{4\bar{R}}} (f' - f)^2 K(v, v') \, dv' \, dv \\
&\geq \frac{\lambda}{2} \iint_{B_{2\bar{R}} \times B_{2\bar{R}}} (f' - f)^2 |v - v'|^{-d-2s} \, dv' \, dv \\
&\geq \frac{\lambda}{2} \iint (f' - f)^2 |v - v'|^{-d-2s} \, dv' \, dv - \lambda \int_{B_{\bar{R}}} f^2(v) \left\{ \int_{\mathbb{R}^d \setminus B_{2\bar{R}}} |v' - v|^{-d-2s} \, dv' \right\} dv \\
&\geq \frac{\lambda}{2} \iint (f' - f)^2 |v - v'|^{-d-2s} \, dv' \, dv - \bar{\Lambda} \int f^2(v) \, dv,
\end{aligned}
$$

with $\bar{\Lambda} \simeq \bar{R}^{-2s}$. The other conditions from [Imbert and Silvestre 2020] are satisfied straightforwardly from our assumptions in Definition 1.1. $\qquad\square$

Note that the right-hand side depends on the L^∞ norm of f with respect to all values of $v \in \mathbb{R}^d$. This is a common inconvenience with nonlocal equations. The result can be easily improved to allow functions f that are unbounded as $|v| \to \infty$. Let ω_0 be the majorant function as in (3-3), centered at the origin. That is,

$$\omega_0(t, x, r) = \sup_{v \in B_{2r} \setminus B_{r/2}} f(t, x, v).$$

We derive the following improvement of Theorem 3.8.

Corollary 3.10. *Let $\mathcal{L}$ be an integral operator corresponding to a kernel in the class $\mathcal{K}$ (as in Definition 1.1). Let $f : (-1, 0] \times B_1 \times \mathbb{R}^d \to \mathbb{R}$ be a function that solves* (3-19) *in Q_1. Then, the following estimate holds:*

$$[f]_{C^\delta_\ell(Q_{1/2})} \leq C\left(\|f\|_{L^\infty(Q_1)} + \|c\|_{L^\infty(Q_1)} + \sup_{t \in (-1,0], x \in B_1} \int_{1/2}^\infty \omega_0(t, x, r) r^{-1-2s} \, dr \right).$$

Here $\delta > 0$ and C are constants depending only on dimension and the parameters λ and Λ of Definition 1.1.

Proof. We consider a C^∞ function $\eta : \mathbb{R}^d \to [0, 1]$ so that $\eta(v) = 1$ if $v \in B_{3/2}$ and $\eta(v) = 0$ when $v \notin B_2$. We apply Theorem 3.8 to the localized function $\tilde{f}(t, x, v) = f(t, x, v)\eta(v)$. We must analyze the equation that $\tilde{f}$ satisfies. We compute directly $\mathcal{L}\tilde{f}$ to get

$$\tilde{f}_t + v \cdot \nabla_x \tilde{f} - \mathcal{L}\tilde{f} = \tilde{c} - h(t, x, v) \quad \text{in } Q_1,$$

where $\tilde{c} = \eta c$ and

$$h(t, x, v) = \int_{\mathbb{R}^d \setminus B_{3/2}} f(t, x, v')(\eta(v') - 1)K(t, x, v, v')\,\mathrm{d}v'.$$

From Theorem 3.8, we have

$$\|f\|_{C_\ell^\delta(Q_{1/2})} = \|\tilde{f}\|_{C_\ell^\delta(Q_{1/2})} \le C(\|\tilde{f}\|_{L^\infty((-1,0]\times B_1 \times \mathbb{R}^d)} + \|c\|_{L^\infty(Q_1)} + \|h\|_{L^\infty(Q_1)}). \tag{3-20}$$

It is easy to see that

$$\|\tilde{f}\|_{L^\infty((-1,0]\times B_1 \times \mathbb{R}^d)} \le \|\tilde{f}\|_{L^\infty((-1,0]\times B_1 \times B_2)} \le \|f\|_{L^\infty(Q_1)} + C \sup_{t,x} \int_1^2 \omega_0(r)r^{-1-2s}\,\mathrm{d}r.$$

It only remains to prove that we have, for any $v \in B_1$,

$$|h(t, x, v)| \le \int_{|v'|>3/2} f(t, x, v')K(t, x, v, v')\,\mathrm{d}v' \lesssim \|f\|_{L^\infty(Q_1)} + C \int_{1/2}^\infty \omega_0(r)r^{-1-2s}\,\mathrm{d}r.$$

Let us justify this inequality. Arguing as in Lemma 3.1, we have

$$\int_{|v'|>3/2} f(t, x, v')K(t, x, v, v')\,\mathrm{d}v'$$

$$\le \int_{3/2<|v'|<7} f(t, x, v')K(t, x, v, v')\,\mathrm{d}v' + \int_{|v'|>7} f(t, x, v')K(t, x, v, v')\,\mathrm{d}v'$$

$$\le \Lambda\|f(t, x, \cdot)\|_{L^\infty(B_7 \setminus B_{3/2})} + \int_{|v'-v|>6} f(t, x, v')K(t, x, v, v')\,\mathrm{d}v'.$$

Moreover,

$$\int_{|v'-v|>6} f(t, x, v')K(t, x, v, v')\,\mathrm{d}v' \le \int_{|v'-v|>6} \Big\{ \inf_{3r/4<|v'-v|<3r/2} \omega_0(r) \Big\}K(t, x, v, v')\,\mathrm{d}v'$$

$$\lesssim \int_4^{+\infty} \frac{\omega_0(r)}{r^{1+2s}}\,\mathrm{d}r.$$

We also have

$$\|f(t, x, \cdot)\|_{L^\infty(B_7 \setminus B_{3/2})} \lesssim \int_{3/4}^{14} \omega_0(r)\,\mathrm{d}r \lesssim \int_{1/2}^{+\infty} \omega_0(r)r^{-1-2s}\,\mathrm{d}r. \qquad \square$$

For convenience, we also state the scaled version of the previous result.

Corollary 3.11. *Let $\mathcal{L}$ be an integral operator corresponding to a kernel in the class $\mathcal{K}$ (as in Definition 1.1). Let $f : (-R^{2s}, 0] \times B_{R^{1+2s}} \times \mathbb{R}^d$ be a function that solves* (3-19) *in Q_R. Then, the following estimate holds*:

$$[f]_{C_\ell^\delta(Q_{R/2})} \le C\Big(R^{-\delta}\|f\|_{L^\infty(Q_R)} + R^{2s-\delta}\|c\|_{L^\infty(Q_R)} + R^{2s-\delta} \sup_{t \in (-R^{2s},0], x \in B_{R^{1+2s}}} \int_{R/2}^\infty \omega_0(t, x, r)r^{-1-2s}\,\mathrm{d}r \Big).$$

Here $\delta > 0$ and C are constants depending only on dimension and the parameters λ and Λ of Definition 1.1.

4. Liouville theorem

This section is devoted to the statement and the proof of a theorem of Liouville type.

Theorem 4.1 (Liouville). *Let $0 < \gamma < \min(1, 2s)$ and $\alpha = (2s/(1 + 2s))\gamma$. Assume that $\lfloor 2s + \alpha \rfloor <$ $2s + \alpha' < 2s + \alpha$ and $\alpha - \alpha' < \delta$, where δ is the constant from Theorem 3.8.*

Let $f \in C_{\ell,\mathrm{loc}}^{2s+\alpha'}((-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d)$ be a function that satisfies the following conditions:

(i) *There is a constant $C_1 > 0$ such that for all $R \geq 1$*

$$[f]_{C_\ell^\beta(Q_R)} \leq C_1 R^{2s+\alpha-\beta} \quad \text{for all } \beta \in [0, 2s + \alpha'].$$

(ii) *For any $\xi = (h, y, w) \in \mathbb{R}^{1+2d}$, with $h < 0$, we define $g(z) := f(\xi \circ z) - f(z)$ or equivalently, $g(t, x, v) := f(t + h, x + y + tw, v + w) - f(t, x, v)$. Then, g solves the equation*

$$g_t + v \cdot \nabla_x g = \mathcal{L}g \quad \text{in } (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d, \tag{4-1}$$

where $\mathcal{L}$ is the operator associated to some kernel $K \in \mathcal{K}$ as defined in (3-1).

Then f is a polynomial of kinetic degree smaller than $2s + \alpha$.

Remark 4.2. Note that the assumption (i) ensures that the tails of $\mathcal{L}g$ are integrable. Indeed, let us take $\beta = \gamma + \varepsilon < \min(1, 2s)$ for ε small. The assumption (i) tells us that

$$|g(z)| \leq C \|z^{-1} \circ \xi \circ z\|^\beta \|z\|^{2s+\alpha-\beta}. \tag{4-2}$$

Note that the condition $\beta < \min(1, 2s)$ ensures that the polynomial q_z in the definition of $[f]_{C_\ell^\alpha(z)}$ is the constant $q_z(\xi) = f(z)$.

Observe that for $z = (t, x, v)$ and $\xi = (h, y, w)$, we have

$$\|z^{-1} \circ \xi \circ z\| = \|(h, y + tw - hv, w)\| \leq \|\xi\| + |(tw - hv)|^{1/(1+2s)} \tag{4-3}$$

$$\leq C_\xi (1 + |t| + |v|)^{1/(1+2s)}.$$

Recalling that $\beta = \gamma + \varepsilon$, we get

$$g(t, x, v) \leq C_{t,x,\xi} (1 + |v|)^{(\gamma+\varepsilon)/(1+2s)+2s+\alpha-\gamma-\varepsilon}$$

$$= C_{t,x,\xi} (1 + |v|)^{2s(1-\varepsilon/(1+2s))} =: \omega(|v|). \tag{4-4}$$

The operator $\mathcal{L}g$ is well-defined because this function ω suffices to bound the expression (3-4). The constant $C_{t,x,\xi}$ depends on t, x, ξ, and the constant in the assumption (i) with $\beta = \gamma + \varepsilon$.

The tails of $\mathcal{L}f$ may not be integrable, and therefore we can only make sense of the equation for g, and not for f.

Remark 4.3. It is plausible that a version of this Liouville-type result holds also for higher values of α. In that case, for (4-1) to make sense, we would have to make g a higher-order incremental quotient of f.

We start with a simpler Liouville-type result that is a consequence of the Hölder estimate contained in Theorem 3.8.

Lemma 4.4 (Liouville). *Let δ be the constant from Theorem 3.8. Assume $\beta < \delta$ and f is a solution to* (3-19) *in $(-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d$ with $c = 0$. Assume further that for all $R \geq 1$*

$$\|f\|_{L^\infty(Q_R)} \leq C R^\beta.$$

Then f is constant.

Proof. We apply Corollary 3.11 in Q_R and make $R \to 0$. From our assumption on the growth of f, for all $(t, x) \in (-R^{2s}, 0] \times B_{R^{1+2s}}$ we have that $\omega(t, x, r) \lesssim r^\beta$. Thus, we have

$$\int_{R/2}^\infty \omega(t, x, r) r^{-1-2s} \, dr \lesssim R^{\beta - 2s}.$$

Then, Corollary 3.11 tells us that

$$[f]_{C^\delta_\ell(Q_{R/2})} \leq C R^{\beta - \delta}.$$

Taking $R \to \infty$, the seminorm $[f]_{C^\delta_\ell(Q_R)}$ converges to zero, and then the function f must be constant. $\square$

Proof of Theorem 4.1. We first claim that it is enough to prove the result assuming that $f \in C^\infty$. Indeed, if f is less regular, we can mollify it respecting the Lie group structure, and then apply the result to the approximate function and pass to the limit.

The remainder of the proof proceeds in several steps.

Step 1: f is constant in x. Let $y \in \mathbb{R}^d$ and $g(t, x, v) := f(t, x - y, v) - f(t, x, v)$. We apply the assumption (i) with $\beta = 2s + \alpha'$. Note that $2s + \alpha' < 1 + 2s$ by assumption. We get

$$\|g\|_{L^\infty(Q_R)} \leq C_1 |y|^{(2s+\alpha')/(1+2s)} R^{\alpha - \alpha'}. \tag{4-5}$$

Since we assume that $\alpha - \alpha' < \delta$, we can apply Lemma 4.4 and we get that g is constant. Therefore, f must be of the form

$$f(t, x, v) = a \cdot x + f(t, v)$$

for some constant $a \in \mathbb{R}^d$. However, the assumption (i) tells us that for all $R \geq 1$

$$\underset{Q_R}{\mathrm{osc}}\, f \leq 2\|f\|_{L^\infty(Q_R)} \leq C_1 R^{2s+\alpha}.$$

This is only possible if $a = 0$ (recall that $a \cdot x$ is a polynomial of order $1 + 2s > 2s + \alpha$). Thus, f is independent of x and from now on we write $f = f(t, v)$.

Step 2: f_t is constant. Observe that the kinetic order of ∂_t is $2s$. Therefore, f_t is well-defined since $f \in C^{2s+\alpha'}_{\ell,\mathrm{loc}}$. Moreover, from the assumption (i) and Lemma 2.7, we deduce that

$$\begin{cases} \|f_t\|_{L^\infty(Q_R)} \leq C_1 R^\alpha, \\ [f_t]_{C^{\tilde\beta}_\ell(Q_R)} \leq C_1 R^{\alpha - \tilde\beta} \quad \text{for all } \tilde\beta \in (0, \alpha']. \end{cases} \tag{4-6}$$

Since $f_t(t, v) = \lim_{h \to 0} (f(t + h, v) - f(t, v)))/h$, using (ii) we deduce that

$$\partial_t(f_t) = \mathcal{L} f_t.$$

We omitted the term $v \cdot \nabla_x f_t$ because it is identically zero.

Using the invariance of the equation by the Lie group action and the fact that f is independent of x, we have that, for any $(h, w) \in [0, \infty) \times \mathbb{R}^d$, the function

$$g := f_t(t + h, v + w) - f_t(t, v)$$

also solves

$$g_t = \mathcal{L}g.$$

Because of (4-6), with $\tilde{\beta} = \alpha'$, we get that

$$\|g\|_{L^\infty(Q_R)} \lesssim C_1 \|(h, 0, w)\|^{\alpha'} R^{\alpha - \alpha'}.$$

Thus, we obtain that g is constant applying Lemma 4.4. Therefore, f_t must be of the form $f_t(t, v) = a \cdot v + bt + c$. However, (4-6) implies that the kinetic degree of f_t cannot be more than α, and therefore f_t is a constant.

Since f is independent of x and f_t is constant, f has the form $f = at + \tilde{f}(v)$ for some constant a. The function $\tilde{f}$ satisfies

$$a = \mathcal{L}\tilde{f}.$$

We are left to prove that f is a polynomial in v.

Step 3: f is a polynomial in v. The third step is divided into three cases depending on the integer part of $2s + \alpha$. Indeed, the maximum number of terms in the polynomial $\tilde{f}(v)$ will depend on $2s + \alpha$ belonging to one of the three possible ranges $(0, 1)$, $(1, 2)$ and $(2, 3)$ (recall that $2s + \alpha$ is not an integer).

Let us start by assuming that $2s + \alpha \in (0, 1)$. Given any $w \in \mathbb{R}^d$, we set $g(v) := \tilde{f}(v + w) - \tilde{f}(v)$. Applying the assumption (i), with $\beta = 2s + \alpha'$ we get

$$\|g\|_{L^\infty(Q_R)} \le C_1 |w|^{\alpha'} R^{\alpha - \alpha'}.$$

Moreover, g solves

$$0 = \mathcal{L}g.$$

We apply Lemma 4.4 right away. We deduce that g is constant for any $w \in \mathbb{R}^d$. Therefore $\tilde{f}$ has the form $\tilde{f}(v) = b \cdot v + c$. However, the assumption (i) with $\beta = 0$ implies in this case that $b = 0$, so $\tilde{f}$ must be constant. Therefore, in the case $2s + \alpha \in (0, 1)$ we conclude that $f(t, x, v) = at + c$ for some constants a and c. Thus, f is a polynomial of degree at most $2s$.

In the case $2s + \alpha \in (1, 2)$ the function f must be differentiable in v because of the assumption (i) applied with $\beta \in (1, 2s + \alpha')$. Thus, if we let $f_j = \partial_{v_j} \tilde{f}$ we get that:

(i') There is a $C_1 > 0$ such that, for all $\beta \in [0, 2s + \alpha' - 1]$ and $R \ge 1$,

$$[f_j]_{C^\beta(B_R)} \le C_1 R^{2s + \alpha - 1 - \beta}.$$

(ii') For any $w \in \mathbb{R}^d$, we define

$$g(v) := f_j(v + w) - f_j(v).$$

Then, g solves

$$0 = \mathcal{L}g.$$

Therefore, we repeat the proof of the case $2s + \alpha \in (0, 1)$ for f_j instead of $\tilde{f}$ and get that each partial derivative f_j is constant. Therefore, in this case $\tilde{f}$ must be an affine function in v.

Likewise, in the case $2s + \alpha \in (2, 3)$, we apply the argument for $2s + \alpha \in [1, 2)$ to each partial derivative $f_j = \partial_{v_j} \tilde{f}$. In this case we obtain that each f_j is affine, and therefore $\tilde{f}$ must be a polynomial in v of degree at most 2. $\square$

5. Blowup argument

In this section, we prove that the Hölder exponent of a solution of a linear equation of the form (1-1) can be improved; moreover, the improvement is quantitative. The result is proved by blowup and compactness. It is first proved for equation with "constant coefficients" (Proposition 5.1) and then proved in the general case (Proposition 5.4).

Proposition 5.1 (improvement by blow up for "constant coefficients"). *Let α and α' be as in Theorem 4.1. Assume that*

$$f_t + v \cdot \nabla_x f - \mathcal{L}_K f = c_0(t, x, v) \quad in \ Q_1,$$

for some function $c_0 \in C_\ell^\alpha(Q_1)$ and some kernel $K \in \mathcal{K}$. Then

$$[f]_{C_\ell^{2s+\alpha}(Q_{1/2})} \leq C(\|c_0\|_{C_\ell^\alpha(Q_1)} + \|f\|_{C_\ell^{2s+\alpha'}((-1,0]\times B_1\times\mathbb{R}^d)}),$$

where C only depends on d, s, α, α', λ and Λ.

Before proving the proposition, we state a lemma corresponding to [Serra 2015a, Claim 3.2]. Its adaptation to kinetic Hölder spaces is straightforward.

Lemma 5.2 (from $C_\ell^{\beta'}$ to C_ℓ^β with $\beta > \beta'$). *Let $0 < \beta' < \beta$. Let ν be the maximum number in $\mathbb{N} + 2s\mathbb{N}$ such that $\nu < \beta$. Assume that $\nu < \beta' < \beta$. Let f be a continuous function in $C_\ell^{\beta'}((-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d)$ and let C_0 be such that*

$$\sup_{r>0} \sup_{z \in Q_{1/2}} r^{\beta'-\beta} [f]_{C_\ell^{\beta'}(Q_r(z))} \leq C_0.$$

Then $[f]_{C_\ell^\beta(Q_{1/2})} \leq C_0$. Moreover, if $\beta \notin \mathbb{N} + 2s\mathbb{N}$ (and assuming f smooth), the supremum is attained at some $r > 0$ and $z \in \bar{Q}_{1/2}$.

We can now turn to the proof of Proposition 5.1.

Proof of Proposition 5.1. Without loss of generality, we normalize the problem so that

$$\|c_0\|_{C_\ell^\alpha(Q_1)} \leq 1 \quad \text{and} \quad \|f\|_{C_\ell^{2s+\alpha'}((-1,0]\times B_1\times\mathbb{R}^d)} \leq 1.$$

Under these conditions, we need to prove that $\|f\|_{C_\ell^{2s+\alpha}(Q_{1/2})} \lesssim 1$. We proceed by contradiction. Assuming the opposite, there would exist sequences f_j, $K_j \in \mathcal{K}$ and c_j such that

$$\|c_j\|_{C_\ell^\alpha(Q_1)} \leq 1, \tag{5-1}$$

$$\|f_j\|_{C_\ell^{2s+\alpha'}((-1,0]\times B_1\times\mathbb{R}^d)} \leq 1, \tag{5-2}$$

$$\partial_t f_j + v \cdot \nabla_x f_j - \mathcal{L}_j f_j = c_j \quad \text{in } Q_1, \tag{5-3}$$

$$\left(\sup_{r>0} \sup_{z \in Q_{1/2}} r^{\alpha'-\alpha} [f_j]_{C_\ell^{2s+\alpha'}(Q_r(z))} \right) \nearrow +\infty \quad \text{as } j \to \infty. \tag{5-4}$$

The last property holds since we cannot apply Lemma 5.2 uniformly. Thanks to Lemma 5.2, there exists $r_j > 0$ and $z_j \in \overline{Q}_{1/2}$ such that

$$F_j := \sup_{r>0} \sup_{z \in Q_{1/2}} r^{\alpha'-\alpha} [f_j]_{C_\ell^{2s+\alpha'}(Q_r(z))} = r_j^{\alpha'-\alpha} [f_j]_{C_\ell^{2s+\alpha'}(Q_{r_j}(z_j))}.$$

In particular, $F_j \to +\infty$ and $r_j \to 0^+$.

We define

$$\tilde{f}_j(z) := \frac{(f_j - q_j)(z_j \circ S_{r_j}(z))}{r_j^{2s+\alpha} F_j},$$

where q_j denotes the polynomial expansion of f_j at z_j as in the definition of $C_\ell^{2s+\alpha'}(Q_{r_j}(z_j))$. This sequence $\tilde{f}_j$ satisfies the following properties:

- Since $F_j = r_j^{\alpha'-\alpha} [f_j]_{C_\ell^{2s+\alpha'}(Q_{r_j}(z_j))}$, we have

$$[\tilde{f}_j]_{C_\ell^{2s+\alpha'}(Q_1)} = 1. \tag{5-5}$$

- Since $F_j \geq r^{\alpha'-\alpha} [f_j]_{C_\ell^{2s+\alpha'}(Q_r(z_j))}$, for all $r > r_j$ we have

$$[\tilde{f}_j]_{C_\ell^{2s+\alpha'}(Q_R)} \leq R^{\alpha-\alpha'} \quad \text{for all } R \geq 1.$$

- Because of the subtraction of the polynomial expansion q_j, we also have

$$\|\tilde{f}_j\|_{L^\infty(Q_R)} \leq R^{2s+\alpha} \quad \text{for all } R \geq 1.$$

- By interpolation (Proposition 2.10) between the last two items, we deduce for all $\beta \in (0, 2s + \alpha']$

$$[\tilde{f}_j]_{C_\ell^\beta(Q_R)} \lesssim R^{2s+\alpha-\beta} \quad \text{for all } R \geq 1. \tag{5-6}$$

For each choice of $\xi \in (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d$, we define

$$g_j(z) := \tilde{f}_j(\xi \circ z) - \tilde{f}_j(z).$$

Condition (5-6) allows us to bound the growth of $g_j(t, x, v)$ as $|v| \to \infty$ by a single majorant function $\omega(|v|)$ as in (4-4).

Because of (5-3), each function g_j satisfies for j large enough (see the choice of R_j below) the equation

$$\partial_t g_j + v \cdot \nabla_x g_j - \widetilde{\mathcal{L}}_j g_j = \tilde{c}_j(\xi \circ z) - \tilde{c}_j(z) \quad \text{in } Q_{R_j}(0), \tag{5-7}$$

where the operator $\widetilde{\mathcal{L}}_j$ corresponds to a scaled kernel $\widetilde{K}_j$ and the source term c_j is scaled too,

$$\widetilde{K}_j(w) = r_j^{-2s} K_j(r_j w),$$

$$\tilde{c}_j(z) = r_j^{-\alpha} F_j^{-1} c_j(z_j \circ S_{r_j}(z)).$$

In particular, we choose the radius $R_j \to +\infty$ such that $Q_{2R_jr_j}(z_j) \subset Q_1$ and $\xi \circ z \in Q_{2R_j}(0)$ for $z \in Q_{R_j}(0)$.

Because of (5-1), it is straightforward to verify that

$$\|\tilde{c}_j(\xi \circ z) - \tilde{c}_j(z)\|_{L^\infty(Q_R)} \lesssim F_j^{-1} d_\ell(\xi \circ z, z)^\alpha \lesssim F_j^{-1}.$$

Therefore, the right-hand side of (5-7) converges to zero over any compact set.

We observe that all the kernels $\tilde{K}_j$ belong to the class $\mathcal{K}$. Applying Lemma 3.4, they converge weakly-$*$ to a kernel K_∞ up to a subsequence. Because of Lemma 3.3, $K_\infty \in \mathcal{K}$.

We pick $\varepsilon > 0$ so that $\lfloor 2s + \alpha \rfloor < 2s + \varepsilon < 2s + \alpha'$. Because of the Arzelà–Ascoli theorem, we take a subsequence so that $\tilde{f}_j$ converges to some function $f : (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d$ locally in $C_\ell^{2s+\varepsilon}$. This function f_∞ also satisfies (5-6).

Since all the $\tilde{f}_j$ are controlled by a single majorizer ω, we can apply Lemma 3.5 and we have that the corresponding function $g_\infty = \lim \tilde{g}_j$ solves the limit equation from (5-7),

$$\partial_t g_\infty + v \cdot \nabla_x g_\infty - \mathcal{L}_\infty g_\infty = 0 \quad \text{in } (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d.$$

We are then able to apply the Liouville theorem, Theorem 4.1, and get that g_∞ is a polynomial. However, we subtracted the polynomial expansion of f_j in the definition of $\tilde{f}_j$, forcing $\tilde{f}_j$ to have a vanishing polynomial expansion at 0 of order up to $2s + \alpha'$. Since $\tilde{f}_j \to f_\infty$ in $C_\ell^{2s+\varepsilon}$ and $2s + \varepsilon > \lfloor 2s + \alpha \rfloor$, all derivatives of f_∞ at the origin must be zero and therefore $f_\infty = 0$.

This contradicts (5-5) and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.3. We would like to emphasize the importance of the majorant function ω. In order to do so, we point out that it is possible to follow the same outline of the proof by blowup to obtain the result of Lemma 5.2 under more general assumptions, at the expense of a more complicated proof. Again, we should not overlook the importance of the majorant function ω. For example, with a more complicated function ω it is possible to derive an estimate of the form

$$\|f\|_{C_\ell^{2s+\alpha}(Q_{1/2})} \le C(\|f\|_{C_\ell^{2s+\alpha'}(Q_1)} + \|f\|_{C_\ell^\gamma((-1,0]\times B_1\times\mathbb{R}^d)} + \|c\|_{C_\ell^\alpha((-1,0]\times B_1\times\mathbb{R}^d)}), \qquad (5\text{-}8)$$

provided that $\alpha \le 2s\gamma/(1 + 2s)$. However, (5-8) is not true for any $\gamma < \alpha$, even in the elliptic case. If we tried to reproduce the proof of Lemma 5.2 for $\gamma < \alpha$, we would still have scaled functions $\tilde{f}_j$ that converge locally uniformly to a function satisfying the assumption (i) in Theorem 4.1. We can still construct the functions $\tilde{g}_j$ that converge uniformly to a function g_∞. But unless we have an appropriate control on the tails of the original functions f_j, we cannot conclude that the limit function g_∞ will satisfy any equation.

We now extend the blowup lemma to the case of "variable coefficients". In order to do so, we are going to use Assumption 1.5 depending on a constant $A_0 > 0$.

Proposition 5.4 (improvement by blowup for variable coefficients). *Let α, α' be as in Theorem 4.1. Under the assumptions of Theorem 1.6, we have*

$$[f]_{C_\ell^{2s+\alpha}(Q_{1/4})} \le C(\|f\|_{C_\ell^{2s+\alpha'}(Q_1)} + A_0\|f\|_{C_\ell^{2s+\alpha}(Q_1)} + (1 + A_0)\|f\|_{C_\ell^\gamma((-1,0]\times B_1\times\mathbb{R}^d)} + \|c_0\|_{C_\ell^\alpha(Q_1)}),$$

where the constant C depends on dimension, s, α, α', λ and Λ.

Proof. Let $\eta \in C_c^\infty((-1, 0] \times B_1 \times \mathbb{R}^d)$. Assume that $\eta = 1$ in $Q_{3/4}$ and that $\eta(z) = 0$ whenever $z \notin Q_1$. Let $\tilde{f}(z) = \eta(z) f(z)$. Obviously, we have

$$\|\eta f\|_{C_\ell^{2s+\alpha'}((-1,0] \times B_1 \times \mathbb{R}^d)} \leq C \|f\|_{C_\ell^{2s+\alpha'}(Q_1)}.$$

We want to estimate the right-hand side that would make $\tilde{f}$ satisfy an equation as in (3-19). We *freeze coefficients* first,

$$K_0(w) = K(0, 0, 0, w).$$

and let $\mathcal{L}_0$ be the corresponding integrodifferential operator. A straightforward computation shows that, for all $z \in Q_{1/2}$,

$$\tilde{f}_t + v \cdot \nabla_x \tilde{f} - \mathcal{L}_0 \tilde{f} = c(z) + A(z) - B(z),$$

where

$$A(z) := \int_{\mathbb{R}^d} (f(t, x, v + w) - f(t, x, v))[K_z(w) - K_0(w)] \, dw,$$

$$B(z) := \int_{\mathbb{R}^d} (\eta(t, x, v + w) - \eta(t, x, v)) f(t, x, v + w) K_0(w) \, dw.$$

Lemma 5.5. $\qquad [A(z)]_{C_\ell^\alpha(Q_{1/2})} \lesssim A_0(\|f\|_{C_\ell^{2s+\alpha}(Q_1)} + \|f\|_{C_\ell^\gamma((-1,0] \times B_1 \times \mathbb{R}^d)}).$

Proof. We write $A(z_1) - A(z_2) = I_1 + I_2$, with

$$I_1 = \int (f(z_2 \circ (0, 0, w)) - f(z_2))(K_{z_1}(w) - K_{z_2}(w)) \, dw,$$

$$I_2 = \int (f(z_1 \circ (0, 0, w)) - f(z_1) - f(z_2 \circ (0, 0, w)) + f(z_2))(K_{z_1}(w) - K_0(w)) \, dw.$$

As far as I_1 is concerned, we write $I_1 = I_1^{\text{out}} + I_1^{\text{in}}$, with

$$I_1^{\text{out}} = \int_{|w| \geq 1} (f(z_2 \circ (0, 0, w)) - f(z_2))(K_{z_1}(w) - K_{z_2}(w)) \, dw,$$

$$I_1^{\text{in}} = \int_{|w| \leq 1} (f(z_2 \circ (0, 0, w)) - f(z_2))(K_{z_1}(w) - K_{z_2}(w)) \, dw.$$

For the nonsingular part of I_1, we simply write

$$|I_1^{\text{out}}| \leq \|f\|_{L^\infty((-1,0] \times B_1 \times \mathbb{R}^d)} \int_{|w| \geq 1} |K_{z_1}(w) - K_{z_2}(w)| \, dw$$

$$\lesssim A_0 \|f\|_{L^\infty((-1,0] \times B_1 \times \mathbb{R}^d)} d_\ell(z_1, z_2)^\alpha,$$

where we used (3-10), which is a consequence of Assumption 1.5.

We now turn to I_1^{in}. We write

$$
|I_1^{\text{in}}| \leq \int_{|w|\leq 1} |f(z_2 \circ (0, 0, w)) - p_{z_2}((0, 0, w))||K_{z_1}(w) - K_{z_2}(w)| \, \mathrm{d}w
$$

$$
+ \frac{1}{2} \int_{|w|\leq 1} |p_{z_2}((0, 0, w)) + p_{z_2}((0, 0, -w)) - 2f(z_2)||K_{z_1}(w) - K_{z_2}(w)| \, \mathrm{d}w
$$

$$
\lesssim [f]_{C_\ell^{2s+\alpha}} \int_{|w|\leq 1} |w|^{2s+\alpha}|K_{z_1}(w) - K_{z_2}(w)| \, \mathrm{d}w
$$

$$
+ \|f\|_{C_\ell^{2s+\alpha}} \int_{|w|\leq 1} |w|^2 |K_{z_1}(w) - K_{z_2}(w)| \, \mathrm{d}w \quad \text{(only relevant if } 2s + \alpha > 2\text{)}
$$

$$
\lesssim A_0 \|f\|_{C_\ell^{2s+\alpha}} d_\ell(z_1, z_2)^\alpha.
$$

We used (3-9) which is a consequence of Assumption 1.5.

We now estimate I_2 thanks to Lemma 3.7. This achieves the proof of the lemma. $\qquad\square$

We now estimate the $C_\ell^\alpha(Q_{1/2})$ norm of B.

Lemma 5.6. $\qquad\qquad\qquad\qquad [B]_{C_\ell^\alpha(Q_{1/2})} \lesssim \|f\|_{C_\ell^\gamma((-1,0]\times B_1 \times \mathbb{R}^d)}.$

Proof. For $z_1, z_2 \in Q_{1/2}$, we compute,

$$
B(z_1) - B(z_2) = J_1 + J_2,
$$

with

$$
J_1 = \int_{|w|>1/4} (\eta(z_1 \circ (0, 0, w)) - \eta(z_1) - \eta(z_2 \circ (0, 0, w)) + \eta(z_2)) f(z_1 \circ (0, 0, w)) K_0(w) \, \mathrm{d}w,
$$

$$
J_2 = \int_{|w|>1/4} (\eta(z_2 \circ (0, 0, w)) - \eta(z_2))(f(z_1 \circ (0, 0, w)) - f(z_2 \circ (0, 0, w))) K_0(w) \, \mathrm{d}w.
$$

We turn to estimate J_1. Since η is smooth, and in particular C_ℓ^γ, we can apply Lemma 3.7 and get

$$
|J_1| \lesssim \|f\|_{L^\infty((-1,0]\times B_1 \times \mathbb{R}^d)} d_\ell(z_1, z_2)^\alpha.
$$

As far as J_2 is concerned, we get

$$
|J_2| \leq 2\|\eta\|_\infty [f]_{C_\ell^\gamma} \int_{|w|>1/4} d_\ell(z_1 \circ (0, 0, w), z_2 \circ (0, 0, w))^\gamma K_0(w) \, \mathrm{d}w,
$$

using that $\alpha = 2s\gamma/(1+2s)$,

$$
|J_2| \lesssim [f]_{C_\ell^\gamma} d_\ell(z_1, z_2)^{2s/(1+2s)\gamma} \lesssim [f]_{C_\ell^\gamma} d_\ell(z_1, z_2)^\alpha.
$$

This achieves the proof of the estimate for B. $\qquad\square$

Thanks to Lemmas 5.5 and 5.6, we conclude the proof of Proposition 5.4 by applying Proposition 5.1 to $\tilde{f}$ (with Q_1 replaced with $Q_{1/2}$). $\qquad\square$

Proof of Theorem 1.6. Without loss of generality, we perform an initial scaling to make sure that the constant A_0 is small (it is a subcritical parameter).

We will next prove the slightly stronger estimate

$$\|f\|_{C^{2s+\alpha}_{\ell,*}(Q_1)} \lesssim \|f\|_{C^{\gamma}_{\ell}([-1,0]\times B_1\times\mathbb{R}^d)} + \|c\|_{C^{\alpha}_{\ell,*}(Q_1)}. \tag{5-9}$$

Scaling and translating the estimate from Proposition 5.4, we get that for any cylinder $Q_\rho(z_0) \subset Q_1$

$$\begin{aligned}
\rho^{2s+\alpha}[f]_{C^{2s+\alpha}_{\ell}(Q_{\rho/4}(z_0))} \lesssim{}& \rho^{2s+\alpha'}[f]_{C^{2s+\alpha'}_{\ell}(Q_\rho(z_0))} + (\rho^{\alpha}A_0)\rho^{2s+\alpha}[f]_{C^{2s+\alpha}_{\ell}(Q_\rho(z_0))} \\
&+ \rho^{\gamma}[f]_{C^{\gamma}_{\ell}((-1,0]\times B_1\times\mathbb{R}^d)} + \|f\|_{L^{\infty}((-1,0]\times B_1\times\mathbb{R}^d)} \\
&+ \rho^{2s+\alpha}[c_0]_{C^{\alpha}_{\ell}(Q_\rho(z_0))} + \rho^{2s}\|c_0\|_{L^{\infty}(Q_\rho(z_0))}.
\end{aligned}$$

For any z_0 in Q_1, we choose $\rho = d_\ell(z_0, \partial Q_1) < 1$ so that $Q_\rho(z_0) \subset Q_1$.

Note that $A_0\rho^{\alpha} \leq A_0$ and $\rho^{\gamma}[f]_{C^{\gamma}_{\ell}((-1,0]\times B_1\times\mathbb{R}^d)} \leq [f]_{C^{\gamma}_{\ell}((-1,0]\times B_1\times\mathbb{R}^d)}$. We get

$$\|f\|_{C^{2s+\alpha}_{\ell,*}(Q_1)} \lesssim [f]_{C^{2s+\alpha'}_{\ell,*}(Q_1)} + A_0\|f\|_{C^{2s+\alpha}_{\ell,*}(Q_1)} + \|f\|_{C^{\gamma}_{\ell}([-1,0]\times B_1\times\mathbb{R}^d)} + \|c\|_{C^{\alpha}_{\ell,*}(Q_1)}.$$

We use the interpolation result of Proposition 2.10 (see Remark 2.11) to get

$$[f]_{C^{2s+\alpha'}_{\ell,*}(Q_1)} \leq \varepsilon[f]_{C^{2s+\alpha}_{\ell,*}(Q_1)} + C_\varepsilon\|f\|_{L^{\infty}(Q_1)}.$$

This achieves the proof of the main theorem. $\square$

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