

Regularity for the Boltzmann equation conditional to macroscopic bounds

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Abstract. The Boltzmann equation is a nonlinear partial differential equation that plays a central role in statistical mechanics. From the mathematical point of view, the existence of global smooth solutions for arbitrary initial data is an outstanding open problem. In the present article, we review a program focused on the type of particle interactions known as *non-cutoff*. It is dedicated to the derivation of a priori estimates in C^∞ , depending only on physically meaningful conditions. We prove that the solution will stay uniformly smooth provided that its mass, energy and entropy densities remain bounded, and away from vacuum.

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1. Introduction

In this work, we study a priori estimates on the regularity of solutions to the Boltzmann equation,

$$\partial_t f + v \cdot \nabla_x f = Q(f, f). \quad (1.1)$$

The Boltzmann equation describes the dynamics of a dilute gas. The nonnegative function $f: [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ encodes the density of particles which at time t have position x and velocity v . The left hand side of the equation, pure transport, denotes the fact that particles travel in straight lines in the absence of external forces. The right-hand side, the Boltzmann collision operator $Q(f, f)$, corresponds to the fluctuations in velocity that result from particle interactions. Our regularity estimates apply to a type of collision operators known as *non-cutoff*, i.e. where long range grazing collisions play a crucial role.

The kinetic description of gas dynamics is intermediate between the microscopic scale, associated to the trajectory of each individual particle, and the macroscopic scale associated to fluid mechanics, such as the Euler or Navier–Stokes equations. In kinetic equations, the density function f encodes the state of a gas, the position and velocity of its particles for each time, but only in a statistical way.

The equation was initially studied by Maxwell and Boltzmann by the end of the 19th century. As one of the fundamental equations from statistical mechanics, it has received substantial interest from the mathematical community through the years. There are varied mathematical problems that result from the study of the Boltzmann equation. An important line of research is on the rigorous derivation of the equations of fluid mechanics from it. It is also natural to study the well posedness of the initial value problem, and it involves different problematics. In this program, we are concerned with regularity estimates for the solutions, which are intimately related to the classical solvability of the equation. As we discuss below, finding smooth solutions unconditionally, for any initial data, appears to be completely out of reach with current techniques. Our results are on a priori regularity estimates conditional to certain macroscopic bounds on physically meaningful quantities.

1.1. The collision operator. The Boltzmann collision operator has two terms: a gain term and a loss term. The loss term represents the fact that a particle with velocity v may collide with another one and change its velocity to something else. Conversely, the gain term accounts for collisions of other particles, with different velocities, that result in new particles with velocity v .

The Boltzmann collision operator is an integro-differential operator acting on the function $f(t, x, \cdot)$ for fixed values of t and x . Its most common form is

$$Q(f, f) = \iint_{\mathbb{R}^d \times \partial B_1} (f(v'_*)f(v') - f(v_*)f(v)) B(|v_* - v|, \cos \theta) d\sigma dv_*, \quad (1.2)$$

where $\cos \theta = \frac{v-v_*}{|v-v_*|} \cdot \sigma$. The nonnegative function B is referred to as the *collision kernel*.

The velocities v , v_* , v' and v'_* are pre- and post-collisional velocities. Particles with velocities v and v_* collide and immediately switch their velocities to v' and v'_* with a rate given by B . Conversely, particles with velocities v' and v'_* may collide and turn their velocities to v and v_* at the same rate. The kernel B denotes this rate by which velocities v and v_* deviate to v' and v'_* after a collision.

We consider elastic collisions that conserve momentum and energy. It is represented by the following two relations between pre- and post-collisional velocities.

$$\begin{aligned} v + v_* &= v' + v'_*, \\ |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2. \end{aligned} \quad (1.3)$$

Consequently, the segments $v v_*$ and $v' v'_*$ are two diameters of the same sphere in \mathbb{R}^d . For any given v , we can parametrize all possible values of v_* , v' and v'_* through the value of $v_* \in \mathbb{R}^d$ and a unit vector σ denoting the direction of v' from the center of the sphere,

$$\begin{aligned} v' &:= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \\ v'_* &:= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma. \end{aligned}$$

This explains the parametrization of the integral in (1.2).

Several different collision kernels B can be considered. A model where particles bounce each other like billiard balls leads to $B = c|v - v_*|$ for some constant $c > 0$. A model where particles repel each other by a power law potential when they are sufficiently close leads to a collision kernel B that is singular around $\theta = 0$. Moreover, in that case, B is not integrable with respect to σ . The integral in (1.2) still makes sense thanks to the cancellation in the factor $(f(v'_*)f(v') - f(v_*)f(v))$ as $v' \rightarrow v$. Note that the gain and loss terms, both integrate to $+\infty$. The non-integrability of B might be considered a *difficulty* in the analysis of the equation. It is common to tame this singularity by considering only collision kernels B that are integrable. This integrability condition for B takes the name *Grad's cutoff assumption*.

For the purpose of this work, it is essential that we *do not* make Grad's cutoff assumption. The singularity in B around $\theta = 0$ is what drives the regularization effects in the equation that we exploit to obtain our a priori estimates. We focus on the standard family of non-cutoff collision kernels, of the form

$$B(r, \cos \theta) = r^\gamma b(\cos \theta) \quad \text{with} \quad b(\cos \theta) \approx |\sin(\theta/2)|^{-(d-1)-2s} \quad (1.4)$$

with $\gamma > -d$ and $s \in (0, 1)$.

Our regularity results are restricted to the range of parameters $\gamma + 2s \in [0, 2]$. This includes cases usually referred to as hard potentials ($\gamma \geq 0$) and moderately soft potentials ($\gamma \in [-2s, 0)$). See Section 12.2 for a discussion of open problems outside this range.

1.2. Conserved quantities and entropy. The dynamics of the Boltzmann equation conserve the total mass, energy and momentum of the solution. That is

$$\begin{aligned}\langle \text{mass} \rangle &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \, dv \, dx = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, v) \, dv \, dx, \\ \langle \text{momentum} \rangle &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) v \, dv \, dx = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, v) v \, dv \, dx, \\ \langle \text{energy} \rangle &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) |v|^2 \, dv \, dx = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, v) |v|^2 \, dv \, dx,\end{aligned}$$

where $f_0(x, v) = f(0, x, v)$.

In addition to these three conserved quantities, a remarkable property of Boltzmann's collision operator is the fact that the total *entropy* of solutions decreases with time,

$$\langle \text{entropy} \rangle = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f \log f(t, x, v) \, dv \, dx \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0 \log f_0(x, v) \, dv \, dx. \quad (1.5)$$

1.3. The hydrodynamic limit. We define the mass, momentum, energy and entropy densities as the functions corresponding to the conserved quantities described above but without integrating in the space variable x . That is

$$\begin{aligned}\rho(t, x) &:= \int_{\mathbb{R}^d} f(t, x, v) \, dv, \\ \rho(t, x)u(t, x) &:= \int_{\mathbb{R}^d} f(t, x, v) v \, dv, \\ e(t, x) &:= \int_{\mathbb{R}^d} f(t, x, v) |v|^2 \, dv, \\ h(t, x) &:= \int_{\mathbb{R}^d} f \log f(t, x, v) \, dv.\end{aligned}$$

The temperature density $\theta(t, x)$ is usually defined as

$$\theta(t, x) := \frac{1}{3} \left(\frac{e}{\rho} - |u|^2 \right) = \frac{1}{3\rho} \int_{\mathbb{R}^d} f(t, x, v) |v - u|^2 \, dv.$$

The values of ρ , u , e , h and θ are the macroscopic quantities corresponding to a solution f to the Boltzmann equation. In certain asymptotic regimes, they formally converge to solutions to classical hydrodynamic equations like the Euler and Navier–Stokes (see [13]). More precisely, for a small parameter $\varepsilon > 0$, we consider the Boltzmann equation with enhanced collisions,

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f) \quad (1.6)$$

and a family of solutions f^ε to the equation (1.6) with the same initial data. As $\varepsilon \rightarrow 0$, the solutions are expected to converge to Maxwellian functions in v whose corresponding

hydrodynamic quantities satisfy the compressible Euler equations

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla (\rho \theta) &= 0, \\ \partial_t (\rho(|u|^2 + 3\theta)) + \nabla \cdot (\rho u(|u|^2 + 5\theta)) &= 0.\end{aligned}\tag{1.7}$$

Moreover, the limit of the entropy density satisfies $h = \rho \log(\rho^{2/3}/\theta)$ and

$$\partial_t h + \nabla \cdot (uh) \leq 0.$$

For $\varepsilon > 0$ small (but not quite zero), the hydrodynamic quantities solve certain local conservation laws. Up to a first order approximation, they solve a compressible Navier–Stokes system with a viscosity term depending on ε .

The kinetic function f provides a more detailed description of the state of the fluid than the hydrodynamic quantities alone. Yet, the hydrodynamic quantities are all that might be observable from a macroscopic point of view. Because the compressible Euler and Navier–Stokes equations arise as asymptotic regimes, one may wonder if any mechanism that produces a singularity in the flow of the hydrodynamic equations would be reproduced in the kinetic framework of the Boltzmann equation as well.

It is well known that, in the compressible Euler equations, shock singularities can emerge from the flow even if the initial data is smooth and away from vacuum. A *shock* singularity is a discontinuity in the solution of the equations, in a way that the values of all functions involved in the system stay bounded uniformly up to the singularity. The existence of shock singularities is classical and easy to verify in one dimensional profiles. The density also stays away from vacuum. In the three dimensional setting, a well known result by Sideris [92] shows the development of singularities for the compressible Euler equations. Even though his result does not describe the precise type of singularity that emerges, the proof suggests that it may be a shock. In recent years, the emergence of shock singularities for the compressible Euler equations and their stability has been studied in different scenarios (see [20, 21, 34, 76]). We will not see singularities in kinetic equations as a result of shocks, because the function f allows for different velocities to coexist at the same point in space. Likewise, shock singularities will not be observed in the Navier–Stokes equations since the diffusion would smooth out any bounded solution that stays away from vacuum.

There are other types of singularities for hydrodynamic equations where at least one of the values $\rho(t, x)$, $u(t, x)$ or $\theta(t, x)$ becomes unbounded in some finite time. Another possible singular behavior would be when ρ flows to zero (creating vacuum) or the temperature θ becomes zero somewhere (corresponding to unbounded entropy). There is a couple of very recent papers, [79] and [80], where the authors construct a stable *implosion* singularity for the compressible Euler and Navier–Stokes equations. Recall that the compressible Euler and Navier–Stokes equations correspond to the asymptotic behavior of (1.6) for small ε . Unlike the case of shock singularities, there is no apparent incompatibility between implosion singularities and the flow of the Boltzmann equation. It is conceivable that there may exist some solution to the Boltzmann equation that flows into a singularity, whose associated hydrodynamic quantities resemble the singular solution in [79] near the implosion point. It is currently difficult to assert whether this type of

singular solutions exist. Either a rigorous construction or an impossibility proof seems complicated, requires further work, and probably new ideas.

The study of hydrodynamic equations presents great mathematical difficulties. The Boltzmann equation keeps track of more information and is more complex than the compressible Euler or Navier–Stokes equations. It is to be expected that the study of the Boltzmann equation, its well posedness, regularity of solutions, and possible singularities, will include, at least, similar difficulties. Given a smooth initial data f_0 , away from vacuum, and with appropriate decay as $|v| \rightarrow \infty$, the question of existence of a smooth solution of the Boltzmann equation (1.1) is an outstanding open problem for any choice of a physical collision operator Q . Given our current understanding of the related hydrodynamic models, and the extra complexity of the Boltzmann equation, the question of well posedness¹ for (1.1) appears to be completely out of reach.

1.4. The conditional regularity program. Given the insurmountable difficulties of studying the existence of smooth solutions to (1.1) in general, it makes sense to study their *conditional regularity*. In the program reviewed in this note, we assume point-wise bounds on the hydrodynamic quantities: mass density, energy density and entropy density, described in the previous subsection. Conditional to these bounds, we prove C^∞ estimates for the solution f to the Boltzmann equation (1.1). By assuming that the hydrodynamic quantities stay bounded, we remove some of the difficulties of hydrodynamic equations, and we focus on the new regularization effects that are characteristic of the kinetic scale.

More precisely, we make the following assumption: there exist positive constants m_0, M_0, E_0, H_0 such that for all (t, x) ,

$$\left\{ \begin{array}{l} m_0 \leq \int_{\mathbb{R}^d} f(t, x, v) \, dv \leq M_0, \\ \int_{\mathbb{R}^d} f(t, x, v) |v|^2 \, dv \leq E_0, \\ \int_{\mathbb{R}^d} f \log f(t, x, v) \, dv \leq H_0. \end{array} \right. \quad (\text{H})$$

The lower bound m_0 on the mass density prevents the formation of vacuum regions. The upper bound H_0 on the entropy density prevents the concentration of the function f as a singular measure. In particular, the upper bound on the entropy density, together with the other upper bounds, prevents the temperature to reach absolute zero.

We stress that a justification that the assumption (H) holds for general solutions of (1.1) seems to be far out of reach at the moment. If an implosion singularity with a similar structure as in [80] was possible for the Boltzmann equation, then the assumption (H) may fail for such solutions.

We will see that as long as (H) is true, solutions of the Boltzmann equation *remain smooth*. It is the reason why we claim that a singularity is always macroscopically observable. We now state our main theorem, which we proved in [61] and whose proof relies on the series of articles [60, 62, 63].

¹Here, we mean unconditional well posedness far from equilibrium.

Theorem 1.1 (Global regularity estimates). *Let f be a solution to the Boltzmann equation (1.1). Assume that f is periodic in space, the collision kernel is of the form (1.4) and $\gamma + 2s \in [0, 2]$. If (H) holds, then for any multi-index $k \in \mathbb{N}^{1+2d}$, any time $\tau > 0$ and any decay rate $q > 0$,*

$$\|(1 + |v|)^q D^k f\|_{L^\infty([0, \tau] \times \mathbb{R}^d \times \mathbb{R}^d)} \leq C_{k, q, \tau}. \quad (1.8)$$

Here D^k is an arbitrary derivative of f of any order, in t , x and/or v .

When $\gamma > 0$, the constants $C_{k, q, \tau}$ depend only on k , q and τ , and the constants m_0 , M_0 , E_0 and H_0 from (H), and the parameters s , γ from the assumption (1.4) on the collision kernel, and dimension d .

When $\gamma \leq 0$, the constants $C_{k, q, \tau}$ depend in addition on the pointwise decay of the initial data f_0 . That is, on the constants N_r with $r \geq 0$, given by

$$N_r := \sup_{x, v} (1 + |v|)^r f_0(x, v) \quad \text{for each } r \geq 0. \quad (1.9)$$

Theorem 1.1 provides an a priori estimate on the smoothness and decay of the solutions of the Boltzmann equation without cutoff, provided that (H) holds.

Note the difference between the cases $\gamma > 0$ (hard potentials) and $\gamma \leq 0$ (soft potentials). In the case of hard potentials, all the bounds depend on the quantities in (H) and the parameters of the equation only. Both the smoothness and the decay estimates are self-generated for positive time. In the case of soft potentials, the equation does not force a fast decay at infinity, but it only propagates it. Our estimates depend on the (pointwise) decay of the initial data. Note also that in both cases our estimates remain uniform as $t \rightarrow \infty$.

The assumption that f is periodic in space is a convenient way for us to avoid the extra difficulties of analyzing the boundary behavior of kinetic equations in bounded domains. Our estimates do not depend on the size of the period. It would be straight forward to reproduce the estimates in Theorem 1.1 if instead of periodicity in x , we assume that $f(t, x, v)$ converges to a fixed Maxwellian as $|x| \rightarrow \infty$.

Theorem 1.1 provides a priori estimates for classical (smooth) solutions. In order to avoid gratuitous technical difficulties, we work with a very strong notion of solution. We start with a function f that is C^∞ with respect to all variables and, moreover, for every $q > 0$, $(1 + |v|)^q f(t, x, v)$ converges to zero as $|v| \rightarrow \infty$ uniformly in x and t . This is merely a qualitative assumption. The estimates in Theorem 1.1 are independent of any norm quantifying smoothness or decay for f . The question of whether the estimates of Theorem 1.1 would hold for any weaker notion of solution is discussed in Section 11.

1.5. Previously known regularity estimates. The first progress toward understanding the regularization effect of the Boltzmann equation without cutoff appeared in the form of entropy dissipation estimates. As we mentioned before, the total entropy is nonincreasing in time. The entropy dissipation equals minus its derivative

$$D(t) := -\frac{d}{dt} \iint_{\mathbb{R}^d \times \mathbb{R}^d} f \log f \, dv \, dx.$$

In [3], they showed that the entropy dissipation is bounded below by a weighted fractional Sobolev norm minus lower order terms.

$$D(t) \gtrsim \|\sqrt{f}\|_{L_x^2 H_v^s(|v|<R)}^2 - C\|f\|_{L_2^1}^2$$

where L_2^1 denotes the weighted norm $L^1(\mathbb{R}^d \times \mathbb{R}^d, (1 + |v|^2) dx dv)$. This entropy dissipation estimate effectively provides the regularity estimate $\sqrt{f} \in L_{t,x}^2 H_v^s$, at least for small velocities. It is the first indication of a regularization effect of the Boltzmann equation in the non-cutoff case.

In [3], they also obtain some form of a coercivity estimate for the Boltzmann collision operator $Q(f, f)$ (under rather restrictive assumptions on B), and the cancellation lemma which will be used later in this paper to compute the lower order term in (2.8).

The coercivity estimates for the Boltzmann collision operator were subsequently improved in several papers including [1, 2, 4, 5, 9, 32, 48, 54, 85].

The coercivity estimate is the main tool to obtain global smooth solutions for the non-cutoff Boltzmann equation in the *space homogeneous* regime. The first result of this kind was by Desvillettes and Wennberg [38] for a rather restrictive family of collision kernels B . Later on, He improves the result in [53] to include the natural non-cutoff collision kernels provided only $\gamma + 2s > 0$. See also [4, 5, 10, 32, 36, 57, 82, 83, 102] for other results in the space homogeneous regime. Note that due to the conservation of mass and energy, and the monotonicity of entropy, our condition (H) always holds true in the space homogeneous setting, for any initial data with finite mass, energy and entropy.

For the space in-homogeneous Boltzmann equation without cutoff, we only know the existence of smooth solutions if the initial data is sufficiently close to a Maxwellian with respect to a suitable norm. See [7–9, 47]. A sharp analysis on the asymptotic behavior as $|v| \rightarrow \infty$ for the coercivity estimate plays a key role in the proof of [47].

The well posedness of the equation for short time is obtained in [83] for initial data f_0 with Gaussian decay as $|v| \rightarrow \infty$. In [84] and [55], they develop the local well posedness theory for initial data f_0 that decays only algebraically as $|v| \rightarrow \infty$.

Regularity results for the inhomogeneous non-cutoff Boltzmann equation far from equilibrium and beyond the coercivity estimates are very scarce. The most relevant results in the literature are given in [6] and [33]. They prove C^∞ regularity estimates for any solution f to the Boltzmann equation (1.1) whose mass density is bounded below and with five derivatives (in all directions with respect to x and v) in a weighted L^2 space with infinite moments. Our condition (H) is naturally much less restrictive and arguably more physically meaningful.

Concerning *unconditional* regularity estimates for solutions to the non-cutoff Boltzmann equation (1.1), far from equilibrium, the only result we are aware of is by Arsenio and Masmoudi [12], in a fractional Sobolev space with a low order of differentiability. They show that $f/(1 + f) \in W_{t,x,v,\text{loc}}^{s,p}$, for every $p \in [1, d/(d - 1))$ and $s > 0$ depending on p .

1.6. Other related results. Here, we discuss briefly a few other results for the Boltzmann equations where the conditional regularity of solutions is applied.

There is a well known result by Desvillettes and Villani [37] about the long term behavior of solutions to the Boltzmann equation, in the non-cutoff case. It says that solutions converge to equilibrium as $t \rightarrow \infty$ provided that they stay uniformly smooth and

bounded below by a fixed Maxwellian function. Any conditional regularity result, like originally the one in [6], improves the result in [37] by reducing its regularity assumption. After our Theorem 1.1, the condition to apply the theorem of Desvillettes and Villani in [37] becomes simply that (H) holds. We discuss it further in Section 11.

There is a recent result by Duan, Liu, Sakamoto and Strain [41] where they construct global mild solutions to the Boltzmann equation without cutoff, whose initial data is close to equilibrium in a suitable norm. Interestingly, these solutions are not a priori very smooth, but one can verify that our assumption (H) holds. Thus, we can deduce the C^∞ smoothness and decay of the solutions in [41] as a consequence of Theorem 1.1. We also discuss this application in Section 11.

1.7. Notation. We use this section both to clarify our choices of notation, and also as a form of glossary of symbols. We list the places in which most symbols used throughout this paper are defined.

We typically use the letters $t \in \mathbb{R}$ for time, $x \in \mathbb{R}^d$ for space and $v \in \mathbb{R}^d$ for velocity. We also use $z \in \mathbb{R}^{1+2d}$ for $z = (t, x, v)$.

We write $a \lesssim b$ to denote that there exists a constant C (depending on the parameters that are appropriate for each scope) so that $a \leq Cb$. We write $a \approx b$ to express that $a \lesssim b$ and $a \gtrsim b$.

In the rest of this section we refer to objects that will be defined later on in this text. It makes little sense to read it linearly. It can be useful to come back here whenever the reader wants to remember the meaning of some symbol.

The integro-differential operator \mathcal{L}_K is defined in (4.2) with respect to some kernel $K(t, x, v, v')$. It is a fractional order diffusion operator in the velocity variable. It is equal to some integral involving the values of $f(t, x, v')$ and $K(t, x, v, v')$ for fixed t and x . The same formula makes sense for $f = f(v)$ and $K = K(v, v')$ independent of t and x .

The class of kinetic integro-differential equations of order $2s$ is invariant by a special scaling S_R and the Galilean Lie group structure $(\mathbb{R}^{1+2d}, \circ)$ defined in Section 5.1.

The Hölder norms \mathcal{C}_ℓ^α are invariant by the scaling S_R and the Galilean group structure. They are defined in Section 5.3.

The kinetic cylinders $Q_1 = (-1, 0] \times B_1 \times B_1$ and $Q_R(z_0) = \{z_0 \circ S_R(z) : z \in Q_1\}$ are defined in Section 5.2.

1.8. Organization of the article. After this introduction, in Section 2, we analyze the structure of the Boltzmann equation. We describe the collision operator $Q(f, f)$ as a nonlinear integro-differential operator, and we relate it to the study of nonlocal elliptic operators. We also compare the Boltzmann equation with some basic hypoelliptic equations.

In Section 3, we outline all the steps involved in the proof of Theorem 1.1. Section 4 sets up the foundation for working with kinetic integro-differential equations that will be needed later. Sections 6 to 10 provide some details for each of the ingredients listed before in Section 3. We provide references to the original papers in each case.

In Section 11, we describe two implications of Theorem 1.1: a continuation criteria and an improvement of the conditions for convergence to equilibrium. We also discuss its applicability to weak solutions. We finish the paper with a collection of related open problems in Section 12.

2. Structure of the diffusion

2.1. A hypoelliptic structure. The study of regularity properties of solutions of the Boltzmann equation necessarily starts with the study of the structure of the diffusion. The regularization effect of the equation is driven by the fact that the collision operator is *diffusive* with respect to the velocity variable. For non-cutoff collision kernels as in (1.4), the collision operator turns out to be a nonlinear integro-differential operator of order $2s$. The collision operator produces some regularization with respect to the velocity variable. This diffusion in “ v ” combined with the transport in “ x ” gives rise to a *hypoelliptic* structure that regularizes the solution in all variables.

In order to understand how the diffusion in velocity combined with the transport in space produces a regularization effect in all directions, it is better to start with a simpler toy model. If we replace the collision operator $Q(f, f)$ with the Laplacian of f with respect to v , we arrive at the following equation

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f. \quad (2.1)$$

In 1934, Kolmogorov computed explicitly in [71] the fundamental solution for this equation. This kernel is a C^∞ function with rapid decay at infinity both in v and x . It turns out that the *Kolmogorov equation* (2.1) enjoys similar regularization properties as the usual heat equation. Such an effect is a consequence of the combination of two mechanisms. The diffusion in the velocity variable v regularizes the solution in this variable and the *free streaming* operator $\partial_t + v \cdot \nabla_x$ transfers regularity in v into regularity in the (t, x) variables. This remarkable observation is the starting point of the hypoellipticity theory developed by Hörmander from 1967.

In the Boltzmann equation, the collision operator is an integro-differential nonlinear diffusion. Its closest linear analog would be the *fractional Kolmogorov equation*,

$$\partial_t f + v \cdot \nabla_x f + (-\Delta)_v^s f = 0. \quad (2.2)$$

The fractional Kolmogorov equation also enjoys similar regularizing properties. Its corresponding fundamental solution is C^∞ , but it decays polynomially at infinity, much like the fundamental solution corresponding to the fractional heat equation. The Boltzmann equation is a nonlinear variant of (2.2). From this point of view, it is natural to expect regularity estimates by using tools from hypoelliptic and integro-differential equations.

2.2. Non-local diffusions. Motivated by probabilistic models involving discontinuous stochastic processes, since the beginning of the 21st century there was an explosion of results in the area of nonlocal diffusions. Basically, a linear parabolic integro-differential equation is an equation of the form

$$\partial_t f(t, v) = \int_{\mathbb{R}^d} [f(t, w) - f(t, v)] K(t, v, w) dw. \quad (2.3)$$

Here K is a nonnegative kernel function satisfying some nondegeneracy and symmetry assumptions. This type of equations can be studied in the context of parabolic equations.

They satisfy similar characteristic properties: maximum principles, energy dissipation inequalities, regularization effects, etc. Some of the landmark results that make it possible to study the regularity of nonlinear parabolic equations were reproduced in the nonlocal setting. They include:

- the Harnack inequality of De Giorgi, Nash and Moser [14, 18, 23, 25, 44, 65, 68, 72];
- the Krylov–Safonov Harnack inequality [15–19, 24, 31, 66, 67, 75, 89, 93, 94, 97];
- the Schauder estimates [40, 58, 64, 81, 91].

The diversity of results in this area is explained in part by the richness of the family of equations. While a classical second order diffusion is characterized simply by a positive definite matrix of coefficients at each point, the integro-differential diffusion is defined in terms of a whole kernel function $K(t, v, \cdot)$, giving much more flexibility in terms of possible structural assumptions.

The primal example of an integro-differential operator is the fractional Laplacian $(-\Delta)^s$, which corresponds to the kernel

$$K(t, v, w) = c_{d,s} |v - w|^{-d-2s}$$

for some positive constant $c_{d,s}$ only depending on dimension and s . In the context of general integro-differential equations, one must start by making sense of the notions of uniform ellipticity, smoothness of coefficients, divergence form, non-divergence form, weak solutions, viscosity solutions, etc. The following dictionary provides a basic understanding of the different assumptions for integro-differential diffusions that correspond to common structural conditions for second order elliptic operators.

- *Uniform ellipticity* of order $2s$ corresponds to the bounds

$$\lambda |v - w|^{-d-2s} \leq K(t, v, w) \leq \Lambda |v - w|^{-d-2s}$$

for two positive constants λ, Λ . That is, the kernel should be comparable with that of the fractional Laplacian.

- Equations in *divergence* form correspond to the symmetry condition

$$K(t, v, w) = K(t, w, v).$$

In this case, the diffusion operator is self-adjoint in L^2 .

- Equations in *non-divergence* form correspond to the different symmetry condition

$$K(t, v, v + h) = K(t, v, v - h).$$

In this case, the diffusion operator evaluates to zero when applied to an affine function, and evaluates to a locally bounded value when applied to a smooth function.

These definitions are the starting point for understanding parabolic integro-differential equations. However, as we will see in the rest of this article, the conditions on the kernel K can be significantly relaxed for each of the three notions above.

2.3. The non-local diffusion of the Boltzmann equation. The collision operator $Q(f, f)$ in the Boltzmann equation is a nonlinear integro-differential diffusion in the velocity variable. However, looking at the expression (1.2), there is no apparent similarity with the equation (2.3). We resort to *Carleman coordinates* [28, 29] in order to rewrite $Q(f, f)$ as an integro-differential diffusion with an f -dependent kernel, plus a lower order term.

To avoid clutter, we omit writing explicitly the time t and space x variables in every formula. Every expression we write is evaluated for each fixed value of t and x .

We reparametrize the integral in (1.2) in terms of $w := v'_* - v$ and v' . The identities (1.3) are equivalent to

$$\begin{aligned} w &\perp (v' - v), \\ v_* &= v' + w. \end{aligned}$$

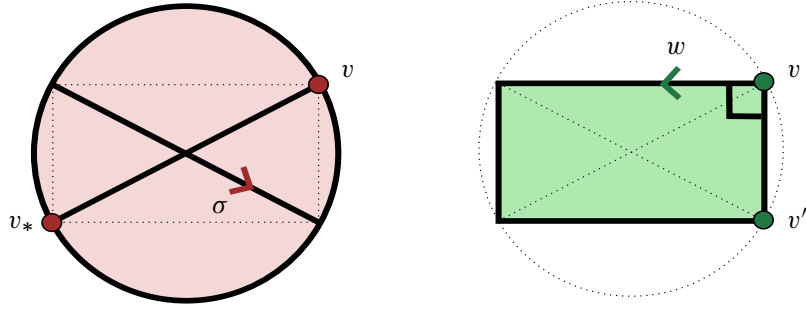


Figure 1. On the left hand side, pre- and post-collisional velocities are parametrized by $v, v_* \in \mathbb{R}^d$ and $\sigma \in \partial B_1$. On the right hand side, Carleman coordinates are shown: velocities are parametrized by $v, v' \in \mathbb{R}^d$ and $w \perp (v' - v)$.

In terms of these variables, and taking into account the Jacobian of this change of variables, the operator $Q(f, f)$ from (1.2) becomes

$$\begin{aligned} Q(f, f) &= \int_{\mathbb{R}^d} \left(\int_{w \perp \{v' - v\}} (f(v')f(v'_*) - f(v)f(v_*)) \right. \\ &\quad \cdot B(|v - v_*|, \cos \theta) \frac{2^{d-1}}{|v' - v|} |v - v_*|^{-d+2} dw \Big) dv'_* \\ &= \int_{\mathbb{R}^d} (f(v')K_f(v, v') - f(v)K_f(v', v)) dv', \end{aligned}$$

where the kernel K_f depends on the function f through the following explicit formula (see [95]),

$$K_f(v, v') = \frac{2^{d-1}}{|v' - v|} \int_{w \perp v' - v} f(v + w) B(r, \cos \theta) r^{-d+2} dw \quad (2.4)$$

with

$$\begin{cases} r^2 = |v' - v|^2 + |w|^2, \\ \cos \theta = \frac{w - (v - v')}{|w - (v - v')|} \cdot \frac{w + (v' - v)}{|w + (v' - v)|}. \end{cases}$$

Under the non-cutoff assumption (1.4), the kernel K_f satisfies,

$$K_f(v, v') \approx |v - v'|^{-d-2s} \left\{ \int_{w \perp v' - v} f(v + w) |w|^{\gamma+2s+1} dw \right\}. \quad (2.5)$$

The sign \approx means that the kernel K_f is bounded from below and from above by the right hand side, up to a constant only depending on B .

We observe in the formula (2.5) that if the factor inside the brackets on the right hand side is $\simeq 1$, that is to say is bounded from above and below by positive constants, then the kernel K_f is uniformly elliptic. However, we cannot ensure such a property based only on the hydrodynamic conditions in (H). Our kernel K_f , a priori, may be a lot more degenerate than those considered in the earlier literature on integro-differential equations. There is no pointwise lower or upper bound for K_f that can be deduced from (H).

The well known *cancellation lemma*, which appeared for the first time in [3], tells us that

$$\int_{\mathbb{R}^d} (K_f(v', v) - K_f(v, v')) dv' = c_b \int_{\mathbb{R}^d} f(v + w) |w|^\gamma dw$$

for some constant $c_b > 0$ only depending on the collision kernel B . The operator $Q(f, f)$ can thus be rewritten under the following form

$$Q(f, f) = \mathcal{L}_{K_f} f + f(f * c_b |\cdot|^\gamma)$$

with K_f given by (2.5) above. We write \mathcal{L}_K to denote an integro-differential operator in v associated to a kernel K ,

$$\mathcal{L}_K f(v) = \int_{\mathbb{R}^d} (f(v') - f(v)) K(v, v') dv'.$$

The term $\mathcal{L}_{K_f} f$ is a nonlinear (since K_f depends on f) integro-differential diffusion. The term $f(f * c_b |\cdot|^\gamma)$ is of lower order. The integro-differential diffusion leads the smoothing effect of the equation.

In order to apply ideas from the area of integro-differential equations in the context of the Boltzmann equation, there are several difficulties that we must overcome. In particular, we must answer the following questions.

- (1) In what way is the kernel K_f elliptic? Can we generalize regularity results for integro-differential equations to possibly degenerate kernels like the ones for the Boltzmann equation?
- (2) Is the Boltzmann kernel K_f in divergence or non-divergence form?
- (3) Is it possible to generalize the regularity results for integro-differential parabolic equations to the hypoelliptic setting of kinetic equations?

In the next subsection, we will discuss the precise way in which the kernel K_f is elliptic, only in terms of the parameters of (H). A first regularity result for parabolic integro-differential equations with such irregular kernels appeared in [89] in the form of a Krylov–Safonov type theorem. A version of De Giorgi–Nash–Moser theorem for kinetic integro-differential equations with possibly degenerate kernels appeared in [63].

The Boltzmann kernel K_f is naturally in non-divergence form, since the identity

$$K(v, v + h) = K(v, v - h)$$

is evident from its explicit formula. The difference $K(v, v') - K(v', v)$ satisfies cancellation conditions that allow us to also work with \mathcal{L}_K as an integro-differential operator in divergence form plus a lower order correction.

In [63] and [62], we develop integro-differential *kinetic* versions of the Giorgi–Nash–Moser theorem and of the Schauder estimates, making use of the hypoelliptic relationship between the integral diffusion and the transport terms. These are results for general kinetic integro-differential equations that were developed with the explicit purpose of applying them to our program of conditional regularity for the Boltzmann equation. We explain them in Sections 6 and 7

2.4. Non-degeneracy cones. The lower bound for the mass density of f in (H), combined with the upper bound for the energy density, tells us that there is certain amount of mass inside a ball $\{|v| < R\}$ (for R depending on m_0 and E_0). Moreover, the upper bound H_0 on entropy tells that this mass cannot concentrate in a set of measure zero. A quantification of this reasoning gives us that, for every value of t and x ,

$$\exists \ell, \mu, R > 0 \quad / \quad |\{f \geq \ell\} \cap \{|v| < R\}| \geq \mu.$$

In words, for every value of t and x , there exists a set of positive measure and localized around the origin where the function f is bounded from below. This set allows us to obtain a lower bound for the diffusion kernel K_f in some directions. Indeed, from any arbitrary point v , there is a symmetric cone of directions whose perpendicular planes will intersect the set $\{f \geq \ell\}$ on a set with \mathcal{H}^{d-1} positive Hausdorff measure. This is what we call the *nondegeneracy cone* of K_f at v (See Figure 2). This cone is characterized by the fact that $(v' - v)/|v' - v| \in A(v)$ for a certain subset $A(v)$ of the unit sphere S^{d-1} . We have

$$K_f(v, v') \geq \lambda(v) |v - v'|^{-d-2s} \quad \text{whenever } v' \in v + \mathbb{R}A(v), \quad (2.6)$$

for $\lambda(v) \approx (1 + |v|)^{\gamma+2s+1} > 0$ depending on v and the constants from (H). Note that the nondegeneracy cone depends on the function f and on the point $v \in \mathbb{R}^d$. The nondegeneracy cones rotate and stretch in some directions when we move the point v .

The lower bound $K(v, v') \geq \lambda(v) |v - v'|^{-d-2s}$, in the usual ellipticity condition for integro-differential equations, holds only when v' belongs to the nondegeneracy cone emanating from v . This is the only non-degeneracy condition that we can deduce from (H).

The usual uniform ellipticity condition for integro-differential equations consists also of an upper bound for the form $K(v, v') \leq \Lambda |v - v'|^{-d-2s}$. A pointwise upper bound like

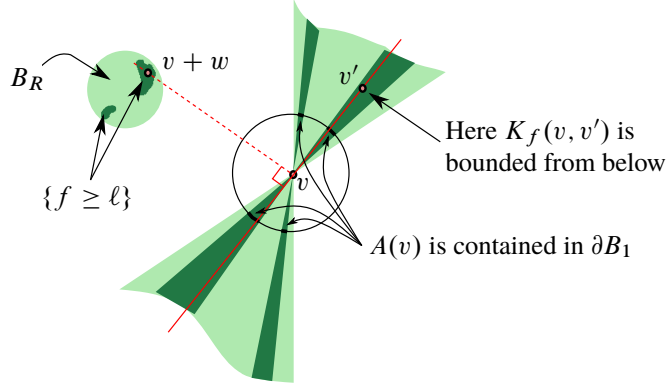


Figure 2. Non-degeneracy cone. It is made of lines passing through v of velocities v' such that $K_f(v, v')$ is bounded from below. In particular, it is centered at v . It is generated in terms of the dark green set $\{f \geq \ell\} \cap B_R$. A generic line of the cone (shown in red on the figure) is made of velocities v' such that the perpendicular plane $\{v + w : w \perp v' - v\}$ (shown in dashed red on the figure) significantly intersects the dark green set $\{f \geq \ell\} \cap B_R$. See (2.5). This figure is adapted from [95], see also [61].

that cannot be deduced from (H). Instead, we can deduce an upper bound in the following averaged sense. For every $r > 0$, we have

$$\int_{B_r} K(v, v + w) |w|^2 dw \leq \Lambda(v) r^{2-2s}. \quad (2.7)$$

Here, the value of $\Lambda(v) \approx (1 + |v|)^{\gamma+2s}$ depends only M_0 and E_0 in (H), provided that $\gamma + 2s \in [0, 2]$. Note that both parameters $\lambda(v)$ and $\Lambda(v)$ in (2.6) and (2.7) depend on v . They are locally uniformly bounded, but they do not stay bounded as $|v| \rightarrow \infty$.

The conditions (2.6) and (2.7) are weaker than the usual pointwise bounds

$$\lambda |v - v'|^{-d-2s} \leq K(v, v') \leq \Lambda |v - v'|^{-d-2s}$$

that appear in the earlier literature as a notion of uniform ellipticity appropriate for integro-differential equations.

2.5. Structure of the Boltzmann equation. The nonlinear integro-differential structure of the Boltzmann equation that we discussed so far can be summarized in the following formula,

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{free transport}} = \underbrace{\mathcal{L}_{K_f} f}_{\substack{\text{non-local diffusion,} \\ \text{non-degenerate} \\ \text{in several directions}}} + \underbrace{f(f * c_b |\cdot|^\gamma)}_{\text{lower order term}}. \quad (2.8)$$

We know that the diffusion is non-degenerate only in the cone of directions discussed in Section 2.4. As we will see, this cone of non-degeneracy is large enough to produce a regularizing effect in the velocity variable v . We stress the fact that it is Condition (H) that ensures the existence of such cones of non-degeneracy.

3. Global regularity estimates: a warm-up tour

The present section is dedicated to the presentation, in a few paragraphs, of the steps involved in the proof of Theorem 1.1. We describe the path from the assumption (H) all the way to the a priori estimates for all derivatives of f .

Our program combines several ingredients. We summarize them in the following list.

- Pointwise upper bounds that show an arbitrarily fast polynomial decay as $|v| \rightarrow \infty$.
- A weak Harnack inequality for kinetic integro-differential equations: it gives us local C^α estimates for some $\alpha > 0$ (possibly small).
- The Schauder estimates for kinetic integro-differential equations: they give us local $C^{2s+\alpha}$ estimates for some (small) $\alpha > 0$.
- A change of variables to adjust the ellipticity of the operators as $|v| \rightarrow \infty$: it turns our local Hölder and Schauder estimates into global ones.
- An iterative gain in regularity, by successively applying the Schauder estimates to derivatives of solutions, to obtain C^∞ estimates.

In this section, we briefly outline what each step does, and how they combine to produce a proof of Theorem 1.1. We give a more detailed explanation of each of these ingredients in later sections.

3.1. Pointwise decay estimates in the velocity variable. Before studying the derivatives of a solution, we are interested in its fast decay with respect to the velocity variable.

In [95], the second author of this paper showed that the condition (H) suffices to obtain an a priori estimate in L^∞ for the solution f , in terms of the constants from (H) only, provided that $\gamma + 2s \geq 0$ and $\gamma \leq 2$. For the range of parameters $\gamma + 2s \in [0, 2]$, in our joint work with Clément Mouhot [60], we improve the upper bound by proving that solutions decay at an arbitrary algebraic rate as $|v| \rightarrow \infty$. More precisely, given any solution f of (1.1), with a non-cutoff collision kernel as in (1.4), if (H) holds, then for any value of $q \geq 0$ and $\tau > 0$, there is a constant C_0 such that

$$\sup_{(t,x,v) \in [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d} (1 + |v|)^q |f(t, x, v)| \leq C_0. \quad (3.1)$$

The constant C_0 depends only on the parameters of (H) when $\gamma > 0$ (the so-called “hard potentials” case). In the case of $\gamma \leq 0$, it is necessary to impose that the initial datum $f_0(x, v) = f(0, x, v)$ enjoys a fast decay in v . The corresponding constant C_0 then also depends on the constants measuring this decay. In that sense, we say that the bounds (3.1) are self generated for hard potentials ($\gamma > 0$), and they are propagated from the initial data for (moderately) soft potentials ($\gamma \leq 0$).

The structure of the proof is inspired by the classical idea of barrier functions in elliptic PDEs. We set up a contradiction by evaluating the equation at the first point of contact between the solution f and an upper barrier. However, the upper barrier is not a super-solution of any particular equation. The contradiction is based on purely nonlocal considerations in the spirit of [93] or the nonlinear maximum principles in [35].

We give a more detailed description of this result in Section 8.

3.2. A Hölder modulus of continuity. Once the bound (3.1) is established, we are interested in estimating some Hölder modulus of continuity for the solution, with respect to all its variables. This will be done through a weak Harnack inequality, in the style of De Giorgi, for general kinetic integro-differential equations. In [63], we study equations of the form,

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}_K f + h \quad (3.2)$$

where

$$\mathcal{L}_K f(v) = \text{P. V.} \int_{\mathbb{R}^d} [f(t, x, v') - f(t, x, v)] K(t, x, v, v') dv'.$$

The kernel K needs to satisfy only some mild notion of ellipticity (implied by (2.6) and (2.7) in the case of Boltzmann equation) and cancellation conditions. The source function h needs to be bounded. The Hölder regularity of the solution u is obtained by following a variant of De Giorgi's method. Neither the kernel K , nor the source function h , need to be linked to the solution f by any additional formula. Because of that, a local Hölder regularity estimate for an equation of the form (3.2) is more general, and implies in particular a local Hölder regularity estimate for the Boltzmann equation (2.8).

Not long before our work in [63], the Harnack inequality, and consequential Hölder estimates, as in the theorem of De Giorgi, Nash and Moser, were obtained in the context of kinetic equations with second order diffusion with rough coefficients (see [46, 86, 100, 101]). They apply to equations of the form

$$\partial_t f + v \cdot \nabla_x f = \partial_{v_i} [a_{ij}(t, x, v) \partial_{v_j} f] + h, \quad (3.3)$$

where the coefficients a_{ij} are assumed to be uniformly elliptic (i.e. $\lambda I \leq \{a_{ij}\} \leq \Lambda I$ for some constants $\Lambda \geq \lambda > 0$), but they are not required to satisfy any smoothness condition.

The equation (3.3) would be hypoelliptic in the sense of Hörmander if its coefficients were smooth. In the present case, the coefficients a_{ij} are merely bounded and measurable with respect to t , x and v . Our result in [63] about an equation of the form (3.2) was motivated by these results for (3.3), and in particular by [46].

As we mentioned it in Section 2.2, there are several results available for integro-differential versions of the De Giorgi–Nash–Moser theorem. Our result in [63] can be described as an integro-differential version of the main result in [46]. However, we face some very specific difficulties.

- (1) We deal with kernels that are significantly more singular than in the earlier literature. The conditions (2.6) and (2.7) only provide a very mild form of ellipticity.
- (2) The integral operator in the Boltzmann equation has the symmetry structure characteristic of equations in non-divergence form. In order to adapt to that, we substitute the

natural symmetry condition,

$$K(t, x, v, v') = K(t, x, v', v),$$

by the following weaker cancellation condition. For all $r \in (0, 1)$

$$\left| \int_{B_r(v)} (K(t, x, v, v') - K(t, x, v', v)) dv' \right| \leq C r^{-2s} \quad (3.4)$$

$$\text{and} \quad \left| \int_{B_r(v)} (K(t, x, v, v') - K(t, x, v', v))(v' - v) dv' \right| \leq C r^{1-2s}. \quad (3.5)$$

In the context of the Boltzmann equation, the fact that the first inequality holds is a reformulation of the classical cancellation lemma from [3]. The second cancellation property seems to be new.

(3) The compactness argument which is at the core of the proof in [46] cannot possibly be adapted to integro-differential equations of order less than one. In that case, we construct special barrier functions and use some ideas that originated in the work of Krylov and Safonov for parabolic equations in non-divergence form. More precisely, we adapt their well known covering technique that they call *the growing ink spots lemma*.

In a later paper [98], Logan Stokols shows that a considerably simpler proof can be given when the kernels K are assumed to be symmetric (i.e. $K(t, x, v, v') = K(t, x, v', v)$) and uniformly elliptic in the most classical sense:

$$\lambda |v - v'|^{-d-2s} \leq K(t, x, v, v') \leq \Lambda |v - v'|^{-d-2s}.$$

Such a result, however, cannot be used to derive Hölder estimates for the Boltzmann equation.

The precise statement of our result, and more details on its proof, will be given in Section 6.

3.3. Gain of $2s$ derivatives through Schauder's approach. The classical Schauder theory gives us an estimate of the $C^{2+\alpha}$ norm of solutions to linear uniformly elliptic equations with C^α coefficients and C^α source terms.

Linear kinetic equations with second order diffusion are a particular case of the more general theory of ultraparabolic equations of Kolmogorov type. The Schauder theory for this type of equations was developed mostly in the late 1990s and early 2000s. See [39, 43, 77, 78, 87, 88], and the survey article [90]. In particular, it applies to equations of the form

$$\partial_t f + v \cdot \nabla_x f = a_{ij}(t, x, v) \partial_{v_i} \partial_{v_j} f + h, \quad (3.6)$$

when the coefficients a_{ij} are uniformly elliptic and these functions together with the source term h are Hölder continuous.

The Schauder estimates are a powerful tool to bootstrap higher regularity estimates once an initial Hölder estimate is established for a nonlinear parabolic equation. A nonlinear equation can often be written as a parabolic or elliptic equation whose coefficients depend on the solution itself. If we know a Hölder modulus of continuity for the solution,

it may imply a Hölder bound for the coefficients. The Schauder estimates then give us a $C^{2+\alpha}$ estimate for the solution. We thus know that the coefficients are even more regular, and we iterate. In other words, the Schauder estimate allows one to *gain two derivatives*: starting from the control of the modulus of continuity of the solution, we reach a control of the modulus of continuity of second order derivatives.

In [62], we obtain Schauder estimates for kinetic integro-differential equations of the form (3.2). The method of our proof is very different from the earlier work on ultraparabolic equations. Instead, we borrow ideas from the *blow-up* technique developed by Joaquim Serra [91] for integro-differential equations.

Classical Schauder theory for parabolic equations involves Hölder spaces whose definition encodes the parabolic scaling, through the introduction of a parabolic distance. Turning to kinetic equations with nonlocal diffusion in the velocity variable, a new scaling appears naturally that is different for time, space and velocity. The class of equations is not translation invariant anymore, but rather Galilean invariant. The Hölder spaces must take these invariances into consideration. The success of an appropriate Schauder theory depends a great deal on finding the right definition for kinetic Hölder spaces and an appropriate kinetic distance. These concepts, together with the precise formulation of the kinetic integro-differential Schauder estimates, will be given in Sections 5 and 7.

Applying this Schauder theory to the Boltzmann equation is then possible [62]. Like we described in the general framework above, the Schauder estimates are applied iteratively to gain higher regularity estimates starting from the initial Hölder estimate. In each application of the kinetic integro-differential Schauder estimates, we gain $2s$ derivatives in velocity, 1 derivative in time, and $2s/(1 + 2s)$ derivatives in space.

3.4. Bootstrap. The application of the kinetic integro-differential version of the De Giorgi–Nash–Moser theory developed in [63] gives us localized Hölder estimates for the solution of the Boltzmann equation (1.1). They apply provided that v is contained in some bounded ball B_R . This is because the ellipticity properties of the kernel K given in (2.6) and (2.7) degenerate as $|v| \rightarrow \infty$.

In order to improve this initial regularity estimate by successive application of the Schauder estimates, it is necessary to turn our local Hölder estimates into global ones. Inspired by an idea in [26], we devised in [61] a change of variables that transforms the kinetic integro-differential equation into one whose ellipticity parameters are uniform in v . This change of variables allows us to turn the local regularity estimates, from De Giorgi and Schauder theories, into global ones that hold for all velocities. It provides a key ingredient for the proof of Theorem 1.1 in [61].

The application of the Schauder estimates, together with the change of variables, gives us a global gain of regularity. We gain $2s$ derivatives in velocity, 1 derivative in time, and $2s/(1 + 2s)$ derivatives in space. Then we compute an equation for discrete incremental quotients of the solution, and apply the Schauder estimates again. An iteration of this procedure leads to the C^∞ estimates in Theorem 1.1. In each iteration, we gain a certain fractional number of derivatives in v , x and t . However, we also lose a decay power. More precisely, an upper bound for $(\partial_t)^m f$ that decays like $\lesssim (1 + |v|)^{-q}$ will depend on an earlier upper bound on $(\partial_t)^{m-1} f$ that decays like $\lesssim (1 + |v|)^{-\tilde{q}}$, for some $\tilde{q} > q$. Since we start with upper bounds that decay arbitrarily fast in (3.1), the iteration continues forever.

We explain the change of variables in Section 9 and the iteration procedure finalizing the proof of Theorem 1.1 in Section 10.

4. Generic kinetic equations with integral diffusion

The study of general kinetic equations with integral diffusion plays an important role in the derivation of the global regularity estimates stated in Theorem 1.1. A kinetic equation with integral diffusion takes the following form,

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}_K f + h, \quad (4.1)$$

where the integral diffusion \mathcal{L}_K depends on a kernel K and is given by the formula,

$$\mathcal{L}_K f(v) = \text{P. V.} \int_{\mathbb{R}^d} [f(t, x, v') - f(t, x, v)] K(t, x, v, v') dv'. \quad (4.2)$$

The integral diffusion operator \mathcal{L}_K acts similarly as a classical diffusion operator in the v variable but it is nonlocal, of fractional order. It can be compared with a second order operator in divergence form $\partial_{v_i}(a_{ij}(t, x, v)\partial_{v_j} f)$ or one in non-divergence form $a_{ij}(t, x, v)\partial_{v_j}\partial_{v_i} f$. Classical second order elliptic operators are studied using different tools depending on these two structures. Test functions and estimates in Sobolev spaces are typical of operators in divergence form, whereas barrier functions and comparison principles are typical of equations in non-divergence form. For integro-differential equations, the operators are always written with the same formula above. The divergence vs non-divergence structures are determined by two alternative symmetry assumptions on the kernel K :

- Divergence: $K(t, x, v, v') = K(t, x, v', v)$.
- Non-divergence: $K(t, x, v, v + w) = K(t, x, v, v - w)$.

The kernel K_f corresponding to the Boltzmann equation (as in (2.4)) is naturally in non-divergence form. In general, we have $K_f(t, x, v, v') \neq K_f(t, x, v', v)$. This is an obstruction in order to apply *divergence* techniques. However, the Boltzmann kernel satisfies the cancellation conditions (3.4) and (3.5), that are a weaker form of the *divergence* symmetry assumption. It turns out that these cancellation conditions suffice in order to derive the most crucial estimate for equations in divergence form: the De Giorgi–Nash–Moser (weak) Harnack inequality.

The reader should not be misled into thinking that any result for an equation of the form (4.1) would apply to linear equations only. A nonlinear kinetic equation, like the Boltzmann equation, also has the form (4.1) with the bonus piece of information that the kernel K and the source term h are related to the solution f by certain formulas. Any a priori estimate for solutions of (4.1), that depends on minor assumptions on K and h , will provide us with estimates in particular for the Boltzmann equation as part of a more general family of equations. The key is to analyze the equation (4.1) with minimalistic assumptions on the kernel K and the source term h , so that we can verify those assumptions in the case of the Boltzmann equation depending only on the hydrodynamic bounds in (H).

We develop two fundamental regularity techniques in the context of generic kinetic integral equations like (4.1). The weak Harnack inequality in the spirit of De Giorgi gives us Hölder continuity estimates for f in terms of quantitative bounds for K and h only. It applies to equations in *divergence form*, that in our context appears in the form of the cancellation conditions (3.4) and (3.5). The Schauder estimates give us regularity estimates in higher order Hölder spaces, once we have an initial Hölder continuity estimate for the source term and the kernel. Schauder estimates apply naturally to equations in non-divergence form.

4.1. Ellipticity conditions. The notion of ellipticity for integro-differential operators is more subtle than for second order differential equations. A classical second order operator involves a matrix of coefficients multiplying the second derivatives of a function, and could be written in divergence or non-divergence form. The ellipticity in the classical case reduces simply to upper and lower bounds on the matrix of coefficients. An integro-differential operator like \mathcal{L}_K is defined in terms of a nonnegative kernel K , which is typically singular around the origin. It should be thought of as a nonlocal diffusion operator in the variable v , which is applied for every fixed value of t and x . With this point of view, we state the ellipticity assumptions for kernels $K(v, v')$ depending on v and v' only. Ultimately, we will require our full kernel $K(t, x, v, v')$ to satisfy these conditions uniformly in t and x .

The first notion of ellipticity that appeared in the literature of nonlocal equations consist in a pointwise comparability between $K(v, v')$ and the kernel of the fractional Laplacian. The majority of the results on regularity of nonlocal equations that appeared between the years 2000 and 2015 depend on the assumption:

$$\lambda|v - v'|^{-d-2s} \leq K(v, v') \leq \Lambda|v - v'|^{-d-2s}.$$

In the case of the Boltzmann equation, there is no way to establish these bounds for the kernel K_f of (2.4) in terms of the parameters of (H) only. So, we are forced to consider more general ellipticity conditions, that are harder to work with.

Instead of the pointwise upper bound $K(v, v') \leq \Lambda|v - v'|^{-d-2s}$, we require that this bound holds *on average* only. That is, we require that for all $r > 0$,

$$\int_{B_r(v)} K(v, v')|v - v'|^2 dv' \leq \Lambda r^{2-2s}, \quad (4.3)$$

for every v in the domain of the equation.

In terms of the lower bound, as we described in Section 2.4, the pointwise lower bound for the Boltzmann equation holds on a symmetric cone of nondegeneracy emanating from each point v . That is, for every v in the domain of the equation, there exists a subset $A \subset S^{d-1}$ so that

$$|A|_{\mathcal{H}^{d-1}} \geq \mu, A = -A \quad K(v, v + w) \geq \lambda|w|^{-d-2s} \quad \text{if } w/|w| \in A. \quad (4.4)$$

One may argue that (4.4) does not seem to be such a weaker replacement of the lower bound $K(v, v') \geq \lambda|v - v'|^{-d-2s}$ as (4.3) is of the upper bound $K(v, v') \leq \Lambda|v - v'|^{-d-2s}$. Indeed, it is not clear what the sharpest notion of ellipticity should be. The important

feature of (4.3) and (4.4) is that they can be verified to hold for the Boltzmann kernel K_f of (2.4) with parameters $\lambda > 0$, Λ and $\mu > 0$ depending on the constants of (H) only.

Requiring that (4.3) and (4.4) hold with uniform constants $\mu > 0$, $\lambda > 0$ and Λ , for every (t, x, v) in the domain of the equation, is a notion of uniform ellipticity that is weaker than the usual assumptions in the literature. It is the assumption we will work with, in order to be able to apply our general estimates to the Boltzmann equation.

The ellipticity conditions (4.3) and (4.4) must be accompanied with symmetry assumptions depending on whether we want to apply methods from divergence or non-divergence equations. In the case of non-divergence equations, we would require the following symmetry condition

$$\text{non-divergence symmetry condition: } K(v, v + w) = K(v, v - w). \quad (4.5)$$

Naturally, the condition (4.5) should hold for every t, x and v in the domain of the equation. We recall that we are omitting writing the t and x dependence on K since the operator \mathcal{L}_K is applied for each fixed value of t and x .

Luckily, the expression (2.4) for the kernel of the Boltzmann equation satisfies the non-divergence symmetry condition (4.5). The Schauder estimate (described below in Section 7) requires this symmetry condition, and will apply to the Boltzmann equation.

The natural symmetry condition to apply methods for equations in divergence form would be $K(v, v') = K(v', v)$. Indeed, a characteristic property of second order operators in divergence form (of the form $\partial_i a_{ij} \partial_j f$) is that they are self-adjoint in L^2 . It is easy to see that \mathcal{L}_K will be self-adjoint if and only if $K(v, v') = K(v', v)$. Unfortunately, this symmetry condition does not hold for the Boltzmann kernel K_f of (2.4). Thus, we are forced to consider a more general condition, that is naturally harder to work with. To that aim, we state the following cancellation conditions: there exist a constant Λ so that for all v in the domain of the equation the following inequalities hold for all $r \in (0, 1)$,

$$\begin{aligned} & \left| \int_{B_r(v)} (K(v, v') - K(v', v)) dv' \right| \leq \Lambda r^{-2s}, \\ \text{if } s \geq 1/2, & \left| \int_{B_r(v)} (K(v, v') - K(v', v))(v - v') dv' \right| \leq \Lambda r^{1-2s}. \end{aligned} \quad (4.6)$$

Note that for $s \in (0, 1/2)$ there is only one cancellation condition, whereas for $s \in [1/2, 1)$ both inequalities are supposed to hold. In fact, for $s \in (0, 1/2)$, the second cancellation condition follows as a consequence of the first one, combined with (4.3).

There are a few extra inequalities that are a consequence of the ellipticity conditions (4.3) and (4.4) and the symmetry conditions (4.6). First of all, the upper bound (4.3) can be rephrased (by adjusting the constant Λ as necessary) in any of the following equivalent alternative formulations,

$$\begin{aligned} & \int_{B_{2r}(v) \setminus B_r(v)} K(v, v') dv' \leq \Lambda r^{-2s}, \\ & \int_{\mathbb{R}^d \setminus B_r(v)} K(v, v') dv' \leq \Lambda r^{-2s}. \end{aligned}$$

Combining (4.3) with the first inequality in (4.6), we get also, for all $r \in (0, 1)$,

$$\int_{B_r(v)} K(v', v) |v - v'|^2 dv' \leq \tilde{\Lambda} r^{2-2s},$$

for a constant $\tilde{\Lambda}$ depending on Λ .

The conditions (4.3) and (4.6) imply that the operator \mathcal{L}_K maps H^s into H^{-s} . The following result is proved in [63].

Proposition 4.1. *Assume K is a kernel for which (4.3) and (4.6) hold for every $v \in \mathbb{R}^d$. Then, for any pair of functions $f, g \in H^s(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} (\mathcal{L}_K f) g dv \leq C \|f\|_{H^s(\mathbb{R}^d)} \|g\|_{H^s(\mathbb{R}^d)},$$

for a constant C depending on Λ , s and dimension only.

Proposition 4.1 indicates that it is fair to think of the operator \mathcal{L}_K as a nonlocal operator of order $2s$. Its ellipticity is justified by the following proposition.

Proposition 4.2. *Assume K is a kernel for which (4.4) holds for every $v \in \mathbb{R}^d$. Then, for every $f \in H^s(\mathbb{R}^d)$, we have*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(v') - f(v)|^2 K(v, v') dv' dv \geq c \|f\|_{H^s(\mathbb{R}^d)}^2,$$

for a constant $c > 0$ depending on μ , λ , s and dimension only.

The following identity follows by a straight forward manipulation of the integral expression of (4.2) and applying Fubini's theorem.

$$\begin{aligned} - \int (\mathcal{L}_K f) f dv &= \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(v') - f(v)|^2 K(v, v') dv' dv \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K(v', v) - K(v, v') dv' \right) f(v)^2 dv. \end{aligned}$$

Note that the last term on the right hand side vanishes if the kernel satisfies the symmetry condition $K(v, v') = K(v', v)$. Otherwise, the cancellation condition (4.6) together with the upper bound (4.3) allow us to bound this term as $\lesssim \|f\|_{L^2}^2$. Thus, Proposition 4.2 provides a coercivity estimate for the operator \mathcal{L}_K in terms of $\|f\|_{\dot{H}^s}$ minus a lower order correction.

Proposition 4.2 is proved in [30] under more general conditions.

Naturally, when the conditions (4.3), (4.4) and (4.6) hold only on some subdomain of \mathbb{R}^d , then appropriately localized versions of Propositions 4.1 and 4.2 hold as well.

Note that Propositions 4.1 and 4.2 apply to generic integro-differential operators of the form 4.2. We can verify that when K is the kernel associated to the Boltzmann equation (as in (2.4)), the assumptions (4.3), (4.4) and (4.6) hold locally only in terms of the constants in (H), but they do not hold uniformly for all velocities in \mathbb{R}^d .

5. Cylinders and Hölder spaces

5.1. Invariant transformations. Here, we study the transformations that keep the class of equations of the form (4.1) invariant. We describe two types of transformations: scaling and Galilean translations.

We first describe the scaling of the equation. Given any $r > 0$, let us define

$$S_r: \mathbb{R}^{2d+1} \rightarrow \mathbb{R}^{2d+1}$$

by the following formula

$$S_r(t, x, v) = (r^{2s}t, r^{1+2s}x, rv).$$

Suppose that f is a solution of the equation (4.1) in some domain. Then, we verify by a direct computation that for any constants $a, r > 0$, the function

$$f_{a,r}(t, x, v) := af(S_r(t, x, v))$$

solves an equation of the same form in an appropriately scaled domain with the modified kernel

$$K_r(t, x, v, v') := r^{d+2s} K(r^{2s}t, r^{1+2s}x, rv, rv'),$$

and the modified source term

$$h_{a,r}(t, x, v) := ah(S_r(t, x, v)).$$

The importance of the choice of exponents in S_r is that if the kernel K satisfies the ellipticity conditions (4.3) and (4.4), then K_r also satisfies the same conditions with the same constants. In the space-homogeneous case (that is, when f does not depend on x), if $s = 1$, the scaling S_r coincides with the usual parabolic scaling $(t, v) \mapsto (r^2t, rv)$. The scaling exponent we write here is properly adjusted to operators of order $2s$ and kinetic equations.

The cancellation condition (4.6) is not exactly preserved by the scaling since the restriction $r \in (0, 1)$ in (4.6) would become $r \in (0, 1/\tilde{r})$ after the transformation $S_{\tilde{r}}$. In fact, the condition (4.6) is *subcritical* since it becomes stronger as we focus on small scales with $\tilde{r} \ll 1$.

Because of the v -dependence in the second term in (4.1), the class of equations is not translation invariant in the usual way. Instead, it is Galilean invariant. For a given $z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{1+2d}$, let us consider the Lie group operator

$$z_0 \circ (t, x, v) = (t_0 + t, x_0 + x + tv_0, v_0 + v).$$

The correction term tv_0 in the x variable accounts for the change of the position coordinate when passing from a motionless frame to another one moving at a constant speed v_0 . If f is a solution of the equation (4.1) in some domain, we consider its Galilean translation. The function

$$f_{z_0}(t, x, v) = f(z_0 \circ (t, x, v))$$

solves an equation of the same form with the modified kernel

$$K_{z_0}(t, x, v, v') := K(z_0 \circ (t, x, v), v_0 + v'),$$

and the modified source term

$$h_{z_0}(t, x, v) = h(z_0 \circ (t, x, v)).$$

Again, the domain of the equation has to be translated accordingly. When the kernel K satisfies any one of the conditions (4.3), (4.4), (4.5) and/or (4.6), the same holds for K_{z_0} .

5.2. Cylinders. Given $r > 0$ and $z_0 \in \mathbb{R}^{1+2d}$ and in view of the scaling $S_r: \mathbb{R}^{1+2d} \rightarrow \mathbb{R}^{1+2d}$ and the Galilean invariance $z \mapsto z_0 \circ z$, both defined above, it is natural to define *cylinders* $Q_r(z_0)$ centered at z_0 of radius $r > 0$ as follows,

$$Q_r(z_0) = \{z_0 \circ S_r(z) : z \in (-1, 0] \times B_1 \times B_1\}.$$

If $z_0 = (t_0, x_0, v_0)$, it is equivalent to the following definition

$$Q_r(z_0) = \{(t, x, v) \in \mathbb{R}^{1+2d} : -r^{2s} < t - t_0 \leq 0, |x - x_0 - (t - t_0)v_0| < r^{1+2s}, |v - v_0| < r\}.$$

Like in parabolic theory, the reference point z_0 for the cylinder is at the final time t_0 . The cylinder $Q_r(z_0)$ includes points at earlier times than t_0 but not on its future.

5.3. Kinetic Hölder spaces. Here, we describe an appropriate notion of Hölder space that is adapted to the scaling and Galilean invariance of the equations.

In order to provide a proper definition of Hölder spaces with any exponent $\alpha > 0$, we must start with a modified notion of degree for polynomials in $\mathbb{R}[t, x, v]$. Given a nonzero monomial $m \in \mathbb{R}[t, x, v]$, the kinetic degree $\deg_{\text{kin}} m$ is the number κ so that for all $z \in \mathbb{R}^{1+2d}$, $r > 0$, we have $m(S_r(z)) = r^\kappa m(z)$. For a general polynomial $p = \sum m_k$, the kinetic degree $\deg_{\text{kin}} p$ is defined as the maximal kinetic degree of the monomials m_k .

Roughly speaking, every exponent of t counts as $2s$, every exponent of x_i counts as $1 + 2s$ and every exponent of v_i counts as 1. A monomial

$$m(t, x, v) = at^{k_0} x_1^{k_1} \dots x_d^{k_d} v_1^{k_{d+1}} \dots v_d^{k_d}$$

has kinetic degree equal to

$$2sk_0 + (1 + 2s)(k_1 + \dots + k_d) + (k_{d+1} + \dots + k_{2d}).$$

We notice that the kinetic degree of a nonzero polynomial can be any number of the discrete set $\mathbb{N} + (2s)\mathbb{N}$. We adopt the convention that the kinetic degree of the zero polynomial equals $-\infty$.

With the definition of kinetic degree at hand, we can now define kinetic Hölder spaces.

Definition 5.1 (Hölder spaces). Given $\alpha \in (0, +\infty)$ and a open set $D \subset \mathbb{R}^{1+2d}$, we say that a function $f: D \rightarrow \mathbb{R}$ is α -Hölder continuous in D if there is some constant C so that for any kinetic cylinder $Q_r(z_0)$, there exists some polynomial p of kinetic degree strictly smaller than α such that

$$|f(z) - p(z)| \leq Cr^\alpha \quad \text{for all } z \in Q_r(z_0) \cap D.$$

The set of α -Hölder continuous functions $f: D \rightarrow \mathbb{R}$ is denoted by $\mathcal{C}_\ell^\alpha(D)$. The least positive constant C so that the inequality above holds is denoted by $[f]_{\mathcal{C}_\ell^\alpha(D)}$.

Note that with the definitions above, given any continuous function $f: D \rightarrow \mathbb{R}$, the seminorm $[f]_{\mathcal{C}_\ell^0(D)}$ is precisely the supremum norm:

$$[f]_{\mathcal{C}_\ell^0(D)} = \sup_D |f|.$$

We also define the norms

$$\|f\|_{\mathcal{C}_\ell^\alpha(D)} := [f]_{\mathcal{C}_\ell^\alpha(D)} + [f]_{\mathcal{C}_\ell^0(D)}.$$

We write the subindex ℓ in \mathcal{C}_ℓ^α to stress the fact that these norms are left-invariant by the action of the Galilean group, and not right-invariant.

The \mathcal{C}_ℓ^α semi-norms encode a Hölder continuity behavior that is compatible with the scaling and the Galilean invariances of the equation described above. For a function $f(v)$ that depends on v only, they would coincide with the usual C^α norms. For a function $f(t)$ that depends only on t , it would rather correspond to the $C^{\alpha/(2s)}$ norm. And for a function $f(x)$, depending only on x , it would correspond to the $C^{\alpha/(1+2s)}$ norm. The \mathcal{C}_ℓ^α norm is left-invariant by the action of the Galilean Lie group. It is *not* right-invariant.

Kinetic Hölder norms satisfy many of the same formal relationships as the usual Hölder norms. For example, the following interpolation inequality (proved in [62]) looks very much like the classical one.

Proposition 5.1 (Interpolation inequalities). *Given $0 \leq \alpha_1 < \alpha_2 < \alpha_3$ so that $\alpha_2 = \theta\alpha_1 + (1-\theta)\alpha_3$, a cylinder $Q_r(z_0)$ and a function $f \in \mathcal{C}_\ell^{\alpha_3}(Q_r(z_0))$,*

$$c[f]_{\mathcal{C}_\ell^{\alpha_2}(Q_r(z_0))} \leq [f]_{\mathcal{C}_\ell^{\alpha_1}(Q_r(z_0))}^\theta [f]_{\mathcal{C}_\ell^{\alpha_3}(Q_r(z_0))}^{1-\theta} + r^{-(\alpha_2-\alpha_1)} [f]_{\mathcal{C}_\ell^{\alpha_1}(Q_r(z_0))}$$

for some constant c only depending on dimension.

It is also true that if $f \in \mathcal{C}_\ell^\alpha$, then derivatives of f will belong to a Hölder space with a smaller exponent. In this case, we must account for differential operators that are left invariant by the Lie group, and their kinetic degree should be properly accounted for.

Proposition 5.2. *Let $f \in C^\alpha(Q)$ for some kinetic cylinder Q . Then:*

- If $\alpha \geq 2s$, $(\partial_t + v \cdot \nabla_x)f \in C^{\alpha-2s}$ and

$$[(\partial_t + v \cdot \nabla_x)f]_{C^{\alpha-2s}(Q)} \lesssim [f]_{C^\alpha(Q)}.$$

- If $\alpha \geq 1 + 2s$, $\partial_{x_i} f \in C^{\alpha-1-2s}$ and

$$[\partial_{x_1} f]_{C^{\alpha-1-2s}(Q)} \lesssim [f]_{C^\alpha(Q)}.$$

- If $\alpha \geq 1$, $\partial_{v_i} f \in C^{\alpha-1}$ and

$$[\partial_{v_1} f]_{C^{\alpha-1}(Q)} \lesssim [f]_{C^\alpha(Q)}.$$

Note that the statement of Proposition 5.2 involves the operator $(\partial_t + v \cdot \nabla_x)$ and not the plain time derivative ∂_t . The time derivative is not left-invariant by the Lie group structure (it is right-invariant). In practice, one can still compute an estimate for the Hölder norm of $\partial_t f$ in a bounded cylinder by combining the first two bullet points in Proposition 5.2 and the triangle inequality. However, such an estimate would depend on the size of the cylinder, and the Hölder exponent will not be better than $\alpha - 1 - 2s$.

These basic properties of kinetic Hölder spaces are stated and proved in [62]. Some further analysis of Hölder norms is continued in [61].

6. Hölder estimates via De Giorgi's method

In the study of regularity estimates for elliptic and parabolic quasilinear equations (away from equilibrium), the first step is usually to apply the method of De Giorgi, Nash and Moser. The key of this theorem is that it has no regularity requirement for the coefficients of the equation. In a quasilinear equation, the coefficients depend on the solution itself. Thus, at the beginning of our analysis we typically have little or no information on how smooth these coefficients might be. The Boltzmann equation, as described in (2.8), can be thought of as a quasilinear evolution equation. It is however not an ordinary parabolic equation, but a kinetic and nonlocal one.

In this section we describe a local Hölder estimate, in the style of the classical theorems of De Giorgi, Nash, and Moser, but this time for non-local kinetic equations. The precise statement is the following.

Theorem 6.1 (Local Hölder estimate). *Let f be a bounded function that solves (4.1) in $Q_1 = (-1, 0] \times B_1 \times B_1$. Assume that the kernel K is a nonnegative function defined in $(-1, 0] \times B_1 \times B_2 \times \mathbb{R}^d$ so that (4.3), (4.4) and (4.6) hold. Then, f is Hölder continuous in the cylinder $Q_{1/2}$ with*

$$[f]_{\mathcal{C}_\ell^\alpha(Q_{1/2})} \leq C (\|f\|_{L^\infty((-1,0] \times B_1 \times \mathbb{R}^d)} + \|h\|_{C^0(Q_1)}),$$

for some constants C and $\alpha > 0$ depending on dimension, (a lower bound for) s , and the ellipticity parameters μ , λ and Λ .

Note that the L^∞ norm of f in the right hand side is evaluated in the set $(-1, 0] \times B_1 \times \mathbb{R}^d$, even though the equation needs to hold in $Q_1 = (-1, 0] \times B_1 \times B_1$ only. This is a common theme in nonlocal equations. The operator \mathcal{L} takes into account the values of f for every $v' \in \mathbb{R}^d$. The first term in the right hand side of the inequality allows us to control the tails of the integral in the expression (4.2) for \mathcal{L}_K .

The symmetry condition (4.6) together with the upper bound on the kernel (4.3) allow us to properly understand the weak solutions of the equation in the sense of distributions via Proposition 4.1. In this case, it would not be a problem to state Theorem 6.2 for bounded weak solutions f .

Theorem 6.1 should be considered as a kinetic, integro-differential, version of the well known theorem by De Giorgi, Nash and Moser. We can also say that it is an integro-differential version of the more recent estimate in [46]. The cancellation condition (4.6) is a weaker form of the symmetry condition that corresponds to equations in divergence form.

Note that no smoothness assumption is imposed on either the kernel K or the source function h . The Hölder estimate for f depends only on quantitative conditions on K and h . As we discussed above (see Section 3), the kernel K_f of the Boltzmann equation given in (2.4) automatically satisfies (4.3), (4.4) and (4.6) for $v \in B_R$, for any bounded value of R , with parameters μ , λ and Λ depending on the constants in (H). As a consequence, Theorem 6.1 implies that any solution of the Boltzmann equation that satisfies the hydrodynamic condition (H) will be locally Hölder continuous. We need further work (described in Section 9) in order to obtain a Hölder estimate that holds uniformly for large velocities.

The rest of Section 6 is devoted to outline the steps of the proof of Theorem 6.1. It is somewhat technical and it may be difficult to follow for the readers that are unfamiliar with (at least) De Giorgi's method for parabolic equations [74]. A reader that is willing to take Theorem 6.1 for granted, can safely skip to Section 7 at this point.

6.1. The weak Harnack inequality. Theorem 6.1 is a consequence of the following weak Harnack inequality.

Theorem 6.2 (Weak Harnack inequality). *There exist radii $0 < r_0 < 1 < R_0$, only depending on dimension and s , and two positive constants ε (small) and C (large), only depending on dimension, s and ellipticity constants μ, λ, Λ , such that any non-negative super-solution f of (4.1) in $Q_{\text{ext}} := (-1, 0] \times B_{R_0^{1+2s}} \times B_{R_0}$,*

$$\partial_t f + v \cdot \nabla_x f \geq \mathcal{L}_K f + h \quad \text{in } Q_{\text{ext}},$$

where K satisfies (4.3), (4.4) and (4.6) and $h \in L^\infty(Q_{\text{ext}})$, satisfies

$$\left(\int_{Q^-} f^\varepsilon(z) \, dz \right)^{\frac{1}{\varepsilon}} \leq C_{\text{whi}} \left(\inf_{Q^+} f + \|h\|_{L^\infty(Q_{\text{ext}})} \right)$$

where $Q^+ = (-r_0^{2s}, 0] \times B_{r_0^{1+2s}} \times B_{r_0}$ and $Q^- = (-1, -1 + r_0^{2s}] \times B_{r_0^{1+2s}} \times B_{r_0}$.

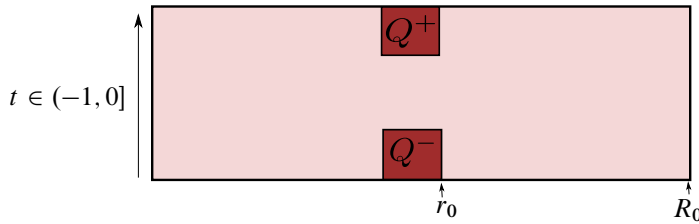


Figure 3. The geometric setting of the weak Harnack inequality.

It is well known, in different contexts, that the weak Harnack inequality implies the Hölder estimate of Theorem 6.1. The weak Harnack inequality is applied iteratively to obtain a decay of the oscillation of the function at a sequence of scales around a point. In this case, we have to adapt this classical procedure to the context of kinetic cylinders, and accounting for the nonlocality of the diffusion operator \mathcal{L}_K . There is no major new obstruction to prove that Theorem 6.2 implies Theorem 6.1. The difficult part is to prove Theorem 6.2.

De Giorgi's method has two main parts. The first one is typically presented as a control of the L^∞ norm of a solution to an elliptic or parabolic PDE in terms of its L^2 norm. The second part is an iterative improvement of oscillation leading to the Hölder continuity of that solution. Both parts will look very different in our context. One can still argue that the first part of our proof is inspired by the first part of De Giorgi's method. The second part of our proof, however, uses ideas from the proof of Krylov and Safonov for parabolic equations in non-divergence form [73].

6.2. The first lemma of De Giorgi. The very first ingredient that one needs for De Giorgi's proof is a relation known as *Cacciopoli's inequality*. It is essentially a localized version of the energy dissipation inequality that applies to nonnegative sub-solutions of the equation.

Let us start by analyzing the norms of f that we can control by the energy dissipation. If we multiply the equation (4.1) by f and integrate using the coercivity property of Proposition 4.2, we get

$$\sup_{t \in [0, T]} \|f(t, \cdot, \cdot)\|_{L^2}^2 + \int_0^T \|f(t, \cdot, \cdot)\|_{L_x^2 \dot{H}_v^s}^2 \leq \|f_0\|_{L^2}^2 + \langle \text{lower order terms} \rangle. \quad (6.1)$$

We were intentionally vague about the domains of the norms. They depend on the domain of the equation. Naturally, there are also some boundary terms involved when the equation (4.1) holds in a bounded domain. These boundary terms may be a complicated due to the nonlocality of the diffusion operator \mathcal{L}_K . Let us ignore them at this point.

The energy dissipation estimate (6.1) provides us with an improvement of differentiability with respect to the v -variable. Indeed, it involves the \dot{H}^s norm in this variable. In order to carry out De Giorgi's method, we will need to turn the estimate (6.1) into an estimate for $\|f\|_{L_{t,x,v}^p}$ for some $p > 2$. The estimate (6.1) does not suffice for that because it does not encode any regularization with respect to the variable x .

Now, we need to invoke the hypoelliptic nature of the equation (4.1). A standard idea in kinetic equations would be to apply averaging lemmas to achieve this (as in [98] or [46]). We take a more classical approach, which is inspired by [86]. We plug the function f into the fractional Kolmogorov equation (2.2).

$$\partial_t f + v \cdot \nabla_x f + (-\Delta)_v^s f = (-\Delta)_v^s f + \mathcal{L}_K f + h. \quad (6.2)$$

Admittedly, at first sight this idea looks artificial. There is no cancellation in the right hand side. The gain comes from the observation that the right hand side involves fractional differentiation with respect to the v variable only, and these are the directions where we got some regularity estimates from the energy dissipation (6.1). We can derive various

estimates thanks to our precise knowledge of the fractional Kolmogorov equation. It has a semi-explicit kernel from which we can compute exactly its regularization effects in all directions.

From the inequality (6.1) we get an estimate for $\|f\|_{L^2_{t,x}(H^s_v)}$. Combining this estimate with Proposition 4.1, we deduce that the right hand side in (6.2) is in $L^2_{t,x}(H^{-s}_v)$. Then we can estimate the $L^p_{t,x,v}$ norm of the convolution of this right hand side with the kernel of the fractional Kolmogorov equation. The fractional Kolmogorov equation is invariant by the same group of scaling and Galilean transformations as our class of kinetic nonlocal equations. We know an explicit formula for its solution. This formula encodes the hypoelliptic interaction between the kinetic transport terms and the fractional diffusion in the velocity variable.

Ultimately, the computation described above leads to an estimate of the $L^p_{t,x,v}$ norm of f , for some $p > 2$, in terms of its $L^2_{t,x,v}$ norm. Applying this estimate to proper truncations of the solution f and following De Giorgi's iteration leads to the following version of De Giorgi's first lemma.

Lemma 6.3. *Let $f: [-1, 0] \times B_1 \times \mathbb{R}^d \rightarrow [0, \infty)$ be a super-solution of the equation in Q_1 ,*

$$\partial_t f + v \cdot \nabla_x f - \mathcal{L}_K f \geq 0 \quad \text{in } Q_1.$$

There exists an $\varepsilon_0 > 0$ (depending only on dimension and the ellipticity parameters) so that if

$$|\{f < 2\} \cap Q_1| < \varepsilon_0,$$

then $f \geq 1$ in $Q_{1/2}$.

Lemma 6.3 is a simplified version of [63, Lemma 6.6]. Lemma 6.3 is a lower bound for nonnegative super-solutions of the equation. It differs from the classical presentation of De Giorgi's first lemma as an upper bound for sub-solutions. It would be possible to write an upper bound for sub-solutions under the stronger assumptions $K \approx |v - v'|^{-d-2s}$ (see [98]), but it is impossible under our less restrictive hypothesis on the kernel. This has already been observed in the context of parabolic integro-differential equations with degenerate kernels (see [42]).

Let us compare Lemma 6.3 with Theorem 6.2. Their geometric settings are of course different, but we can see some similarity when we state them in the following way.

- Lemma 6.3 says that if $f \leq 1$ at any point in $Q_{1/2}$, then

$$|\{f \geq 2\} \cap Q_1| \leq |Q_1| - \varepsilon_0.$$

- Theorem 6.2 (with $h = 0$) says that if $f \leq 1$ at any point in Q^+ , then

$$|\{f \geq A\} \cap Q^-| \lesssim A^{-1/\varepsilon}$$

for all $A > 0$.

Theorem 6.2 is effectively an upper bound for the measure of the level sets $|\{f \geq A\} \cap Q^-|$, for A large, for any super-solution of the equation so that $f \leq 1$ at some point in Q^- .

The second part of De Giorgi's proof consists of an estimate of the measure between two level sets of the solution f . Ultimately, it leads to a decay estimate for the

measure $|\{f \geq A\}|$ for large A 's. The classical method by De Giorgi involves an explicit computation relating the measure of level sets of a function with its H^1 norm. In [23], the authors follow an alternative idea for integro-differential equations using an estimate depending essentially on a lower bound on the kernel $K \gtrsim |v' - v|^{-d-2s}$. In [46], De Giorgi's original computation is replaced with an elegant compactness argument. None of those ideas apply in our context. The compactness argument in [46] can be applied to integro-differential equations, after working out several technical difficulties, but only in the case $s \geq 1/2$. For $s < 1/2$, we employ a completely different approach inspired by the ideas of Krylov and Safonov in [73]. Our method for $s < 1/2$ applies in the full range $s \in (0, 1)$ if we assume in addition the non-divergence symmetry condition (4.5).

The Boltzmann kernel K_f defined in (2.4) always satisfies (4.5). Thus, the method to prove Theorem 6.2 inspired by the ideas of Krylov and Safonov suffices for the whole range of parameters. In [63], we also describe the method inspired by the ideas in [46], that works only for $s \geq 1/2$, because it allows us to remove the assumption (4.5) in the statement of Theorem 6.2 for general kinetic integro-differential equations.

6.3. The propagation lemma. Lemma 6.3 involves a lower bound in the small kinetic cylinder $Q_{1/2}$. One can effortlessly scale Lemma 6.3 to relate the level set $\{f \geq 2\} \cap Q_r(z)$ with the minimum of f in $Q_{r/2}(z)$, provided that the equation holds in $Q_r(z)$ and f is nonnegative everywhere. Our next objective is to extend the set where we take the minimum of f to a larger kinetic cylinder than $Q_{r/2}(z)$. This is achieved through the use of explicit barrier functions described in the following lemma.

Lemma 6.4 (barrier functions). *Let $\tau > 0$, $R > 1$, and $T > 0$ be arbitrary parameters. There exist $\theta > 0$ and $R_1 > R > 0$ depending on these parameters, dimension, s and the ellipticity constants in (4.3), (4.4) so that the following statement is true.*

There exists a function $\varphi: [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, 1]$ satisfying the following properties.

- $\varphi \in C^{1,1}$. Moreover, φ is C^∞ in the set $\{\varphi > 0\}$.
- For any kernel K that satisfies (4.3) and (4.4) (and also (4.5) in the case $s \geq 1/2$), we have

$$\partial_t \varphi + v \cdot \nabla_x \varphi - \mathcal{L}_K \varphi \leq 0 \text{ in } [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

- $\varphi(0, x, v) > 0$ only if $(x, v) \in B_1 \times B_1$.
- $\varphi(t, x, v) \geq \theta$ if $(t, x, v) \in [\tau, T] \times B_{R^{1+2s}} \times B_R$.
- $\varphi(t, x, v) = 0$ if $t \in [0, T]$ and $(x, v) \notin B_{R_1^{1+2s}} \times B_{R_1}$.

The function φ is used as a lower barrier. It allows us to propagate a lower bound for a super-solution f on $\{0\} \times B_{r^{1+2s}} \times B_r$ to an arbitrarily large kinetic cylinder $[\tau, T] \times B_{(Rr)^{1+2s}} \times B_{Rr}$, provided that the equation holds in a suitable larger domain containing $[0, T] \times B_{(R_1 r)^{1+2s}} \times B_{R_1 r}$.

The proof of Lemma 6.4 consists of a more or less explicit computation. It is another proof where the hypoellipticity of the equation plays a role. This time, in a more crude and explicit manner. The computation leading to Lemma 6.4 is explained in [63, Section 7].

By scaling and translating this construction, we derive the following corollary.

Corollary 6.5. *Given $R_0 > 0$ and $r_0 > 0$ such that $R_0 - r_0 \geq R_1 r_0$, let Q_{ext} and Q^- be as in Theorem 6.2, see Figure 3. Let f be a super-solution*

$$\partial_t f + v \cdot \nabla_x f - \mathcal{L}_K f \geq 0 \quad \text{in } Q_{\text{ext}}.$$

Assume that K satisfies (4.3) and (4.4). We also assume (4.5) when $s \geq 1/2$. If $f \geq A$ in $Q_r(z_0)$ for some cylinder $Q_r(z_0)$ contained in Q^- , then for any arbitrarily large constants T and R as in Lemma 6.4, there exists $\theta > 0$ such that

$$f \geq \theta A \quad \text{in } [t_0, \min(t_0 + Tr^{2s}, 0)] \times B_{(Rr)^{1+2s}} \times B_{Rr}.$$

Moreover, the factor $\theta > 0$ depends on T , R , d , s and the ellipticity parameters of the kernel K .

We can further combine Corollary 6.5 with Lemma 6.3 and get

Corollary 6.6. *Let f be a super-solution*

$$\partial_t f + v \cdot \nabla_x f - \mathcal{L}_K f \geq 0 \quad \text{in } Q_{\text{ext}}.$$

Assume that K satisfies (4.3) and (4.4). We also assume (4.5) when $s \geq 1/2$. Assume also that for some cylinder $Q_r(z_0)$ contained in Q^- ,

$$|\{f \geq A\} \cap Q_r(z_0)| \geq (1 - \varepsilon_0)|Q_r|.$$

Then

$$f(t, x, v) \geq \theta A \quad \text{in } [t_0, t_0 + Tr^{2s}] \times B_{(Rr)^{1+2s}} \times B_{Rr}.$$

Here, θ , T , R , Q_{ext} and Q^- are as in Corollary 6.5.

Corollary 6.6 tells us a nontrivial relation between the level sets $\{f \geq A\}$ and $\{f \geq \theta A\}$. It implies Theorem 6.2. However, this implication is nontrivial. It depends on a special covering argument inspired in the crawling ink-spots lemma from [73], that we describe below.

Here is a further corollary that will be used in the proof of Theorem 6.2.

Corollary 6.7. *Let a kernel K satisfy (4.3) and (4.4) and, when $s \geq 1/2$, let it also satisfy (4.5). There exists a constant $r_1 > 0$, that only depends on dimension d , s and the ellipticity parameters of K , such that for all super-solution f of*

$$\partial_t f + v \cdot \nabla_x f - \mathcal{L}_K f \geq 0 \quad \text{in } Q_{\text{ext}}$$

such that $\min_{Q^+} f \leq 1$, if there exists some cylinder $Q_r(z_0)$ contained in Q^- such that,

$$|\{f \geq A\} \cap Q_r(z_0)| \geq (1 - \varepsilon_0)|Q_r|,$$

then $r < r_1$.

Corollary 6.7 is an immediate consequence of Corollary 6.6. Indeed, if r was large, Corollary 6.6 would imply that $f > 1$ in Q^+ .

6.4. Ink-spots. The following lemma is a type of covering result known as the ink-spots lemma.

Lemma 6.8 (Ink-spots). *Let $E \subset F \subset B_1$ be two measurable sets. Assume that for some constant $\delta > 0$,*

$$|E| \leq (1 - \delta)|B_1|$$

and whenever there is any ball $B \subset B_1$ such that

$$|E \cap B| \geq (1 - \delta)|B|,$$

we must have $B \subset F$. Then, the following inequality holds

$$|E| \leq (1 - c\delta)|F|$$

for some constant $c > 0$ depending on dimension only.

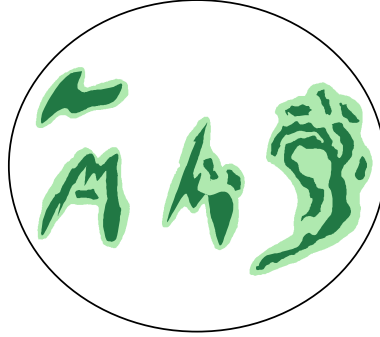


Figure 4. The set F (greenish color) looks like the result of a growing ink stain which started as E (dark color).

Lemma 6.8 is used to obtain the decay of level sets in the elliptic version of the weak Harnack inequality by Krylov and Safonov. It is relatively easy to prove Lemma 6.8 as a consequence of Vitali's covering lemma.

In [73], Krylov and Safonov describe a modification of Lemma 6.8 that is suitable for parabolic equations. They call it *the crawling ink-spots lemma*. In that case $E \subset F$ are sets in space-time. They assume that whenever there is a parabolic cylinder where E is very concentrated, then an enlarged version of that cylinder, that takes place later in time, is contained in F .

We need to further modify the covering lemma in [73] to fit the setting of kinetic equations. In order to state our kinetic version of the ink-spots lemma, we start with defining the *stacked* cylinder. Given any kinetic cylinder $Q = Q_r(z_0)$ (as in Section (5.2)), we define Q^m as

$$Q^m := \{(t, x, v) : 0 < t - t_0 < mr^{2s}, |v - v_0| < r, |x - x_0 - (t - t_0)v_0| < (m + 2)r^{1+2s}\}.$$

Note that t_0 is the final time for Q and the initial time for Q^m . The lapse of Q^m is m times the lapse of Q . Moreover, the space width of Q^m is enlarged by a factor $(m + 2)$ with respect to the space width of Q .

The following result is our kinetic version of the ink-spots lemma. It is proved in [63, Section 10].

Lemma 6.9 (Kinetic crawling ink-spots). *Let $E \subset F$ be two measurable sets. We make the following assumptions for some $\delta \in (0, 1)$ and some $r_1 > 0$.*

- $E \subset Q_1$.
- *Whenever a kinetic cylinder $Q \subset Q_1$ satisfies $|Q \cap E| \geq (1 - \delta)|Q|$, then $Q^m \subset F$ and also $Q = Q_r(z)$ for some $r < r_1$.*

Then

$$|E| \leq \frac{m+1}{m}(1 - c\delta)(|F \cap Q_1| + Cmr_1^{2s}).$$

With these ingredients, we can outline the proof of Theorem 6.2.

Sketch of proof of Theorem 6.2 for $h = 0$. Assume $f(z) \leq 1$ for some point $z \in Q^+$. We need to prove that

$$|\{f \geq A\} \cap Q^-| \lesssim A^{-1/\varepsilon}.$$

This decay follows by a simple iteration once we established the inequality

$$|\{f \geq A\} \cap Q^-| \leq (1 - c\varepsilon_0)|\{f \geq \theta A\} \cap Q^-|,$$

for $\theta A > 2$, θ and ε_0 as in Corollary 6.6, and c as in Lemma 6.9.

This inequality between level sets follows from Lemma 6.9 applied to

$$E = \{f \geq \theta A\} \cap Q^- \quad \text{and} \quad F = \{f \geq A\} \cap Q^-.$$

The assumptions of Lemma 6.9 are fulfilled thanks to Corollaries 6.6 and 6.7. \square

7. The Schauder estimate

This section is devoted to the Schauder theory for kinetic equations in non-divergence form with Hölder continuous coefficients. We consider a function f that solves (4.1) with a Hölder continuous source term h and a kernel K that satisfies the ellipticity conditions (4.3) and (4.4) together with the *non-divergence* symmetry condition (4.5).

We must first make sense of the notion of *Hölder continuous coefficients* for a kernel $K(t, x, v, v')$. We add the following assumption, which depends on a parameter $\alpha' \in (0, \min(1, 2s))$.

Assumption 7.1 (Hölder continuity of the kernel in (t, x, v)). There exists a positive constant A_0 such that whenever $z_1 = (t_1, x_1, v_1)$ and $z_2 = (t_2, x_2, v_2)$ belong to $Q_1 \cap Q_r(z_0)$, for any kinetic cylinder $Q_r(z_0)$, then

$$\forall \rho > 0, \quad \int_{B_\rho} |K(t_1, x_1, v_1, v_1 + w) - K(t_2, x_2, v_2, v_2 + w)| |w|^2 dw \leq A_0 \rho^{2-2s} r^{\alpha'}.$$

With this notion of $C^{\alpha'}$ Hölder coefficients, we are ready to state the Schauder estimates for general kinetic integro-differential equations.

Theorem 7.1 (The Schauder estimate). *Let $s \in (0, 1)$, $\alpha \in (0, \min(1, 2s))$ and $\alpha' = 2s\alpha/(1 + 2s)$. Let $K: Q_1 \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a nonnegative kernel such that (4.3), (4.4), (4.5) and Assumption 7.1 hold true. Let $h: Q_1 \rightarrow \mathbb{R}$ be α' -Hölder continuous. Assume further that $2s + \alpha' \notin \{1, 2\}$.*

If f satisfies (4.1) in Q_1 , then

$$[f]_{\mathcal{C}_\ell^{2s+\alpha'}(Q_{1/2})} \leq C(\|f\|_{\mathcal{C}_\ell^\alpha((-1,0] \times B_1 \times \mathbb{R}^d)} + \|h\|_{\mathcal{C}_\ell^{\alpha'}(Q_1)}).$$

The constant C only depends on dimension, the order $2s$ of the integral diffusion, ellipticity constants μ, λ, Λ and A_0 from Assumption 7.1.

A slightly more general version of Theorem 7.1 is obtained in [58].

Since we do not assume the cancellation condition (4.6), the notion of weak solutions in the sense of distributions does not make sense in the generality of Theorem 7.1. This is the same situation as in the classical Schauder estimates for elliptic PDEs in non-divergence form. The natural framework under which the Schauder estimates apply is that of viscosity solutions. Such a generalization would involve only some minor technical adjustments to our current proof. Likewise, if we assume in addition that the cancellation condition (4.6) holds (which is true in the case of the Boltzmann equation), then Theorem 7.1 would extend to weak solutions in the sense of distributions without any major additional difficulty.

To prove Theorem 7.1, we first analyze the simpler case where K depends only on $(v' - v)$. Then, we apply an interpolation inequality to account for the variations of the kernel, similarly as in the classical proof of the Schauder estimates. Note that if K depends only on $(v' - v)$, Assumption 7.1 automatically holds with $A_0 = 0$.

The case $K = K(v' - v)$ is proved using a blow-up technique, following closely the ideas by Serra [91]. We set up the requirements for an iterative proof of the regularity result. We proceed by contradiction by negating the main estimate. A *blow-up limit* leads to certain ancient solution of the equation. The contradiction is reached through a Liouville type result that rules out such a solution.

In the rest of this section, we outline the ideas involved in the proof of Theorem 7.1. The uninterested reader may move directly to Section 8.

7.1. A Liouville type result. A simple form of Liouville theorem that one can state for kinetic integro-differential equations is the following: every bounded solution of (4.1), with $h = 0$, in $(-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d$, under the assumptions of Theorem 6.1, must be constant. The proof of this statement is simply to apply Theorem 6.1 in Q_R for large R 's. We obtain that the Hölder seminorm of f in $Q_{R/2}$ is $\lesssim R^{-\alpha}$. We deduce that f is constant taking $R \rightarrow \infty$. A slightly more detailed analysis of the inequalities of this proof reveals that the *boundedness* hypothesis can be relaxed to a slow enough algebraic rate. Let us state it in the following proposition.

Proposition 7.2 (Liouville – I). *Let f be a solution of (4.1), with $h = 0$, in $Q_\infty = (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d$, where K satisfies (4.3), (4.4) and (4.6). Assume further that*

$$\|f\|_{C^0(Q_R)} \leq C(1 + R)^\delta,$$

for some constant C and some $\delta \geq 0$ smaller than the Hölder exponent α in Theorem 6.1. Then f is constant.

For the proof of Theorem 7.1, we need to allow more growth at infinity for the solution f . The precise form of the Liouville theorem that we use is the following. It applies to a kernel $K = K(v' - v)$ that depends on $(v' - v)$ only.

Proposition 7.3 (Liouville – II). *Let $\alpha \in (0, \min(1, 2s))$ and $\alpha' = \frac{2s}{1+2s}\alpha$ and β such that*

$$[2s + \alpha'] < 2s + \beta < 2s + \alpha' \quad \text{and} \quad \alpha' - \beta < \delta,$$

where $\delta > 0$ is the Hölder exponent in Theorem 6.1. Let f be a function in $\mathcal{C}_{\ell, \text{loc}}^{2s+\beta}((-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d)$ satisfying the following conditions.

(1) *There exists a constant $C_1 > 0$ such that for all $R > 0$,*

$$\forall \beta' \in [0, 2s + \beta], \quad [f]_{\mathcal{C}_{\ell}^{\beta'}(\mathcal{Q}_R)} \leq C_1 R^{2s+\alpha'-\beta'};$$

(2) *For any $\xi = (h, y, w) \in \mathbb{R}^{1+2d}$ with $h \leq 0$, the function $g(z) := f(\xi \circ z) - f(z)$ solves*

$$\partial_t g + v \cdot \nabla_x g - \mathcal{L}_K g = 0 \quad \text{in } (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d,$$

where $K = K(v - v')$ satisfies (4.3), (4.4), and (4.5).

Then f is a polynomial of kinetic degree smaller than $2s + \alpha'$.

One advantage of kernels that depend on $(v' - v)$ only is that they are in divergence and non-divergence form at the same time. Indeed, if $K = K(v' - v)$ satisfies (4.5), then it also satisfies (4.6) with $\Lambda = 0$. In this case, \mathcal{L}_K is comparable to a second order elliptic operator with constant coefficients.

Note that in the statement of Proposition 7.3, we do not require f to be the solution of an equation. The equation is imposed on its increments g . The growth at infinity for the function f would make the nonlocal operator $\mathcal{L}_K f$ undefined. Indeed, with the assumptions of Proposition 7.3, for any fixed value of t_0, x_0, v_0 , we only have

$$|f(t_0, x_0, v') - f(t_0, x_0, v_0)| \leq C_{t_0, x_0, v_0} (1 + |v'|)^{2s+\alpha'}.$$

With this growth at infinity, the tail of the integral in (4.2) diverges. We state the equation for g , and not for f , because the equation for f may not make sense.

The proof of Proposition 7.3 consists in applying Proposition 7.2 to various increments of f . The first step is to take increments in space $\xi = (0, y, 0)$. These values of ξ are in the center of the Galilean Lie group, so all the computations work like one would expect. Applying Proposition 7.2 to g , with $\xi = (0, y, 0)$, we conclude that the function f must be constant in x . Once we established that f must be constant in x , the statement of Proposition 7.3 reduces to a Liouville theorem for parabolic integro-differential equations. The rest of the proof continues by applying Proposition 7.2 to increments of f in t and v . The details of the proof are given in [62, Section 4].

7.2. The blow-up argument. In this paragraph, we briefly explain how to derive the Schauder estimate from the Liouville type result in the “constant coefficient case”, that is to say when the kernel K depends only on $(v' - v)$.

Sketch of the proof of Theorem 7.1 for $K = K(v' - v)$. Following Serra [91], the core of the proof of the Schauder estimate consists in proving that for some β smaller than α' , the following estimate holds true,

$$[f]_{\mathcal{C}_\ell^{2s+\alpha'}(Q_{1/2})} \lesssim (\|h\|_{\mathcal{C}_\ell^{\alpha'}(Q_1)} + \|f\|_{\mathcal{C}_\ell^{2s+\beta}((-1,0] \times B_1 \times \mathbb{R}^d)}). \quad (7.1)$$

Using the interpolation inequalities from Proposition 5.1 together with some control on the tails of the integral operator, yields the Schauder estimate as stated in Theorem 7.1. Let us focus on the derivation of the previous estimate.

Since we assumed $2s + \alpha'$ is not an integer, let us pick β so that

$$\lfloor 2s + \alpha' \rfloor \leq 2s + \beta \leq 2s + \alpha'.$$

Note that since $\alpha' = 2s\alpha/(1 + 2s)$ with $\alpha < \min(1, 2s)$, we know further that

$$(\mathbb{N} + 2s\mathbb{N}) \cap [2s + \beta, 2s + \alpha'] = \emptyset.$$

By an appropriate normalization, we assume without loss of generality that

$$\|h\|_{\mathcal{C}_\ell^{\alpha'}(Q_1)} + \|f\|_{\mathcal{C}_\ell^{2s+\beta}((-1,0] \times B_1 \times \mathbb{R}^d)} \leq 1,$$

and we aim at bounding $[f]_{\mathcal{C}_\ell^{2s+\alpha'}(Q_{1/2})}$ from above.

Since $(\mathbb{N} + 2s\mathbb{N}) \cap [2s + \beta, 2s + \alpha'] = \emptyset$, we are able to estimate the seminorm in $\mathcal{C}^{2s+\alpha'}$ if we establish the following inequality

$$[f]_{\mathcal{C}_\ell^{2s+\beta}(Q_r(z))} \leq C_0 r^{\alpha'-\beta} \quad \text{for all } z \in Q_{1/2}, r > 0 \text{ such that } Q_r(z) \subset Q_1.$$

Indeed, this inequality implies $[f]_{\mathcal{C}_\ell^{2s+\alpha'}(Q_{1/2})} \lesssim C_0$.

Since we work with a smooth² solution f , we know that the following maximum is achieved at some point $z \in \bar{Q}_{1/2}$ and $r \in (0, 1/2]$.

$$\max_{r>0, z \in \bar{Q}_{1/2}, Q_r(z) \subset Q_1} r^{\beta-\alpha'} [f]_{\mathcal{C}_\ell^{2s+\beta}(Q_r(z))} =: C_0.$$

We must prove that C_0 is bounded from above in terms of d, s and the ellipticity parameters only.

The proof proceeds by contradiction. We assume that there exist sequences $f_j \in \mathcal{C}_\ell^{2s+\beta}((-1, 0] \times B_1 \times \mathbb{R}^d)$, $h_j \in \mathcal{C}_\ell^{\alpha'}(Q_1)$, $K_j \in \mathcal{K}$ such that

$$\begin{aligned} \|h_j\|_{\mathcal{C}_\ell^{\alpha'}(Q_1)} + \|f_j\|_{\mathcal{C}_\ell^{2s+\beta}((-1,0] \times B_1 \times \mathbb{R}^d)} &\leq 1, \\ (\partial_t + v \cdot \nabla_x) f_j - \mathcal{L}_{K_j} f_j &= h_j, \\ \sup_{r>0, z \in \bar{Q}_{1/2}, Q_r(z) \subset Q_1} r^{\beta-\alpha'} [f_j]_{\mathcal{C}_\ell^{2s+\beta}(Q_r(z))} &\nearrow +\infty \quad \text{as } j \rightarrow +\infty. \end{aligned} \quad (7.2)$$

Above, we observed that, for each j , the supremum is reached at some $r_j > 0$ and $z_j \in \bar{Q}_{1/2}$. We can see that necessarily $r_j \rightarrow 0$ as $j \rightarrow +\infty$ in the above sequence.

²The qualitative smoothness assumption on f simplifies the proof but is not strictly necessary.

In order to derive a contradiction, we rescale the sequence of functions $\{f_j\}$ to map its values in $Q_{r_j}(z_j)$ to the values of \tilde{f}_j in Q_1 , to normalize the $\mathcal{C}_\ell^{2s+\beta}$ -norm of \tilde{f}_j in Q_1 , and by removing the polynomial part q_j (of kinetic degree strictly smaller than $2s + \beta$). Recalling that S_r denotes the natural scaling associated with the class of equations we work with, let \tilde{f}_j be defined for all $z \in Q_1$ by,

$$\tilde{f}_j(z) = \frac{(f_j - q_j)(z_j \circ S_{r_j}(z))}{r_j^{2s+\alpha'} F_j},$$

where

$$F_j := r_j^{\beta-\alpha'} [f_j]_{\mathcal{C}_\ell^{2s+\beta}(Q_{r_j}(z_j))}.$$

It then satisfies $\|\tilde{f}_j\|_{\mathcal{C}_\ell^{2s+\beta}(Q_1)} = 1$ and

$$\begin{aligned} \forall R \in [1, c_s r_j^{-1}], \quad & [\tilde{f}_j]_{\mathcal{C}_\ell^0(Q_R)} \leq R^{2s+\alpha'}, \\ \forall R \in [1, c_s r_j^{-1}], \quad & [\tilde{f}_j]_{\mathcal{C}_\ell^{2s+\beta}(Q_R)} \leq R^{2s+\alpha'-\beta}. \end{aligned}$$

The constant $c_s > 0$ in the previous estimates only depends on s . It is chosen so that if $z_j \in Q_{1/2}$ then $Q_{c_s}(z_j) \subset Q_1$.

The remainder of the proof consists in getting a limit f_∞ of the (sub)sequence of $\{\tilde{f}_j\}$ and to apply the Liouville theorem to the function f_∞ . We prove that the increments of f_∞ satisfy an equation, and Proposition 7.3 implies that the *blow-up* limit f_∞ has to be a polynomial of degree less than $2s + \alpha'$. However, the polynomial expansion at the origin is zero for each \tilde{f}_j by construction. We conclude that f_∞ has to vanish, contradicting $\|\tilde{f}_j\|_{\mathcal{C}_\ell^{2s+\beta}(Q_1)} = 1$. \square

8. Pointwise upper bounds

In this section we describe the pointwise upper bounds for the Boltzmann equation that we obtained in [60], and we briefly described in Section 3.1.

Theorem 8.1 (Pointwise upper bounds). *Let $\gamma \in (-d, 1]$, $s \in (0, 1)$ such that $\gamma + 2s \in [0, 2]$ and let B be a collision kernel of the non-cutoff form (1.4). Let f be a solution of the Boltzmann equation (1.1) in $(0, T) \times \mathbb{T}^d \times \mathbb{R}^d$ such that $f(0, x, v) = f_0(x, v)$ in $\mathbb{T}^d \times \mathbb{R}^d$ and (H) holds. We obtain the following estimates.*

- (1) *Propagation of upper bounds. There exists a decay rate q_0 depending on s, γ, d and the parameters in (H) so that for any $q > q_0$, if $f_0 \leq C(1 + |v|)^{-q}$ for some $C > 0$, then there exists a constant N depending on C, d, s, d and the parameters in (H) such that*

$$\forall t \in [0, T], x \in \mathbb{T}^d, v \in \mathbb{R}^d, \quad f(t, x, v) \leq N(1 + |v|)^{-q}.$$

- (2) Generation of upper bounds for hard potentials. Assume $\gamma > 0$. For any $q \geq 0$, there exist constants N and $\beta > 0$, depending on d, s, d, q and the parameters in (H) only, such that

$$\forall t \in [0, T], x \in \mathbb{T}^d, v \in \mathbb{R}^d, \quad f(t, x, v) \leq N(1 + t^{-\beta})(1 + |v|)^{-q}.$$

- (3) Generation of upper bounds for soft potentials. Assume $\gamma \in [-2s, 0]$. There exists a constant N , depending on d, s, d and the parameters in (H) only, such that

$$\forall t \in [0, T], x \in \mathbb{T}^d, v \in \mathbb{R}^d, \quad f(t, x, v) \leq N(1 + t^{-\frac{d}{2s}})(1 + |v|)^{-d-1-\frac{d\gamma}{2s}}.$$

Unlike the Hölder estimates in Theorem 6.1, or the Schauder estimates in Theorem 7.1, the upper bounds in Theorem 8.1 do not apply to generic integro-differential equations. They are specific to the Boltzmann equation (1.1) with the collision operator $Q(f, f)$ and the non-cutoff kernel of the form (1.4), for solutions satisfying the assumption (H).

The proof of Theorem 8.1 has the basic structure of a classical barrier argument for parabolic PDEs. We postulate that

$$f(t, x, v) \leq U(t, v) := NA(t)(1 + |v|)^{-q}.$$

We pick $A(t)$ to be constant for the proof of the propagation of the upper bounds, and to be of the form $A(t) = (1 + t^{-\beta})$ for the generation of upper bounds. In either case, the inequality holds trivially at the initial time. If the conclusion of the theorem was false, there would be a first crossing point (t_0, x_0, v_0) so that

$$\begin{aligned} f(t_0, x_0, v_0) &= U(t_0, v_0), \\ f(t, x, v) &\leq U(t, v) \quad \text{whenever } t \leq t_0. \end{aligned}$$

At this first crossing point, the conditions above imply the following relations for the derivatives of f

$$\begin{aligned} \partial_t f(t_0, x_0, v_0) &\geq \partial_t U(t_0, v_0), \\ \nabla_x f(t_0, x_0, v_0) &= \nabla_x U(t_0, v_0) = 0. \end{aligned}$$

Unlike the classical method of barrier functions, there is no straight forward relationship between $Q(f, f)$ and $Q(U, U)$ at the first crossing point (t_0, x_0, v_0) . The inequality $\mathcal{L}_{K_f} U \geq \mathcal{L}_{K_f} f$ holds at (t_0, x_0, v_0) , but is not enough for the proof of our result. Instead, we shall use the inequality: $f(t_0, x_0, v) \leq U(t_0, v)$, for all $v \in \mathbb{R}^d$, together with the upper bounds on mass and energy in (H), and perform a delicate analysis of the quadratic integral operator $Q(f, f)$ to deduce a (negative) upper bound for $Q(f, f)(t_0, x_0, v_0)$. This is the key computation that leads to the proof of Theorem 8.1, and it is purely nonlocal. In particular, the same reasoning as in the proof of Theorem 8.1 does not work with the Landau equation (see [26] for a result on upper bounds for the Landau equation using different methods).

The precise computation for estimating the value of $Q(f, f)(t_0, x_0, v_0)$ relies on the integro-differential structure of the collision operator,

$$Q(f, f)(t_0, x_0, v_0) = \mathcal{L}_{K_f} f(t_0, x_0, v_0) + f(t_0, x_0, v_0)(f(t_0, x_0, \cdot) * c_b |\cdot|^\gamma)(v_0).$$

We take advantage of the cone of nondegeneracy described in Subection 2.4 and the upper bounds on mass and energy in (H). The integral structure of \mathcal{L}_{K_f} allows us to immediately transfer the information given by the bounds M_0 and E_0 in (H) to an estimate of the value of \mathcal{L}_{K_f} at the point (t_0, x_0, v_0) . There is no analog of this method for the usual, *local*, partial differential equations.

This is the only step in our program where we use the periodicity assumption of f with respect to the x variable. It is used for convenience and only to ensure the existence of the first crossing point (t_0, x_0, v_0) . There are many alternative structures for global solutions $f: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ that may also be considered. Note that our estimates do not depend on the length of the period.

From all the steps in our regularity program, Theorem 8.1 is the only one which would be hard to generalize to weak solutions. The idea of evaluating the equation at the first crossing point is not compatible with the notion of solution in the sense of distributions. The natural generalized framework for this method is that of *viscosity sub-solutions*, as developed by Crandall and Lions for first order, elliptic and parabolic equations. The notion of viscosity solutions have not attracted practically any attention in the context of the Boltzmann equations to date. The definition of viscosity super-solutions is given in [59]. The definition of viscosity sub-solution, is identical to the one of super-solution but with reversed inequalities. The definition would only make sense for a solution f that is at least locally bounded, so it would still be quite restrictive in terms of the qualitative properties of the solutions we start with.

9. The change of variables

After getting the upper bounds of Theorem 8.1, the Hölder estimate from Theorem 6.1 and the Schauder estimates from Theorem 7.1 imply local Hölder bounds for the Boltzmann equation as soon as we verify that their hypothesis hold for the Boltzmann kernel K_f of (2.4) in terms of the parameters of (H) only.

Like we explained in Section 2.4, the cone of nondegeneracy in condition (4.4) is automatically satisfied by the kernel K_f of the Boltzmann equation provided that we restrict our attention to some bounded set of velocities $v \in B_R$. In that case, we have that the set $A = A(v) \subset S^{d-1}$ is symmetric with respect to the origin and concentrated in a band of width $\lesssim (1 + |v|)^{-1}$ around the equator perpendicular to v . Its measure is $\approx (1 + |v|)^{-1}$, and

$$K_f(v, v') \geq \lambda(1 + |v|)^{1+\gamma+2s} |v' - v|^{-d-2s}, \quad \text{whenever } \frac{(v' - v)}{|v' - v|} \in A. \quad (9.1)$$

Naturally, the condition (4.4) is satisfied for K_f , but only if we restrict our analysis to some bounded set of velocities $v \in B_R$. We make the same observation for the other conditions (4.3) and (4.6). For example, the following inequality is proved in [95], which justifies that (4.3) holds for $v \in B_R$:

$$\int_{\mathbb{R}^d \setminus B_R} K(v, v') dv' \lesssim R^{-2s} \left(\int_{\mathbb{R}^d} f(v + w) |w|^{\gamma+2s} dw \right) \leq CR^{-2s} (1 + |v|)^{\gamma+2s},$$

for a constant C depending only on M_0 and E_0 in (H), provided that $\gamma + 2s \in [0, 2]$.

A similar inequality holds for (4.6). Therefore, Theorem 6.1 applies locally to any solution of the Boltzmann equation that satisfies (H). It tells us that any such solution is Hölder continuous in $(\tau, \infty) \times \mathbb{R}^d \times B_R$, for any positive time $\tau > 0$ and any bounded value for R . Now we would like to apply our Hölder estimate for f to deduce Assumption 7.1 for K_f , and in that way be able to apply the Schauder estimates of Theorem 7.1. Later on, we would want to keep applying the Schauder estimates to increments and derivatives of f to deduce higher regularity. However, in order to do so, it is essential to obtain global Hölder and Schauder estimates, that are not restricted to bounded velocities only. That difficulty is overcome by the change of variables described in this section.

For any value of $v_0 \notin B_1$, we define the linear transformation T_0 by the formula

$$T_{v_0}(av_0 + w) := \frac{a}{|v_0|}v_0 + w \quad \text{whenever } w \perp v_0.$$

Note that T_0 maps the unit ball B_1 into an ellipsoid that is flattened by the factor $1/|v_0|$ in the direction of v_0 . If $v_0 \in B_1$, we simply take T_{v_0} to be the identity. Given any $z_0 = (t_0, x_0, v_0)$, we further define

$$\mathcal{T}_{z_0} := z_0 \circ (|v_0|^{-\gamma-2s}t, |v_0|^{-\gamma-2s}T_{v_0}x, T_{v_0}v).$$

Here, \circ is the Galilean group operator in \mathbb{R}^{1+2d} . This transformation \mathcal{T}_{z_0} maps Q_1 into a neighborhood of z_0 that is scaled anisotropically, by flattening the direction of v_0 .

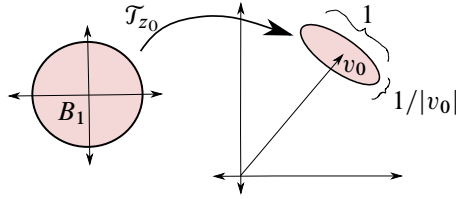


Figure 5. The linear transformation \mathcal{T}_{z_0} maps the velocities inside the unit ball B_1 to the velocities inside an ellipsoid centered at v_0 .

When f is a solution to the Boltzmann equation, then $\bar{f}(z) = f(\mathcal{T}_{z_0}(z))$ solves a modified equation in Q_1

$$\partial_t \bar{f} + v \cdot \nabla_x \bar{f} - \mathcal{L}_{\bar{K}_f} \bar{f} = \bar{h},$$

where

$$\bar{h}(z) = c|v_0|^{-\gamma-2s} \bar{f}(z) (f *_v |\cdot|^\gamma)(\mathcal{T}_{z_0}z),$$

and

$$\bar{K}_f(t, x, v, v') = |v_0|^{-\gamma-2s} K(\mathcal{T}_{z_0}z, v_0 + T_{v_0}v').$$

The benefit of this transformed equation is that the kernel \bar{K} satisfies the ellipticity conditions (4.3), (4.4) and (4.6) in Q_1 with constants depending on the parameters of (H) but independent of the value of $|v_0|$. Thus, Theorems 6.1 and 7.1 are applied to \bar{f} in Q_1 with fixed parameters. Then, from the explicit change of variables, we deduce explicit Hölder estimates that hold globally for $v \in \mathbb{R}^d$.

The formula for the change of variables is motivated by the nondegeneracy cone described above. Since the set A is concentrated in a band of width $\approx (1 + |v|)^{-1}$ around the equator perpendicular to v , the purpose of the linear transformation T_{v_0} is of course to stretch this band to make it of width ≈ 1 . Then, the factor $|v_0|^{-\gamma-2s}$ applied to the time and space variables normalizes the lower bound (9.1) by removing the factor that depends on $(1 + |v|)$. The change of variables \mathcal{T}_{z_0} , by design, transforms the estimate (9.1) into the condition (4.4) for \bar{K} in Q_1 , for values of μ and λ that are uniform with respect to $|v_0|$. It is remarkable that the same transformation \mathcal{T}_{z_0} also give us (4.3) and (4.6) for \bar{K} in Q_1 , in terms of M_0 and E_0 of (H), uniformly in $|v_0|$ as well.

Lemma 9.1. *Let $f: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ be a nonnegative function satisfying (H). Let \bar{K} be the kernel described above. Then \bar{K} satisfies (4.3), (4.4) and (4.6) with constants depending on the parameters in (H), s , γ and dimension, but **not** on z_0 .*

The proof of Lemma 9.1 is given in [61]. It involves relatively lengthy technical computations.

With Lemma 9.1 at hand, the Hölder estimate from Theorem 6.1, becomes an explicit global Hölder estimate for solutions of the Boltzmann equation. The Schauder estimates from Theorem 7.1 give us explicit higher order estimates for the solutions to the Boltzmann equation with an explicit (although somewhat complicated) asymptotic behavior for large velocities. The same logic can be applied to practically any local regularity estimate. For example combining Proposition 4.2 with the change of variables of Lemma 9.1, we recover the sharp coercivity estimate with respect to the anisotropic distance of Gressman and Strain [48].

10. The bootstrap argument

The last step in the proof of Theorem 1.1 is to iteratively obtain estimates in higher and higher order Hölder norms by applying the Schauder estimates to increments and derivatives of the solution f of equation (1.1).

In order to keep track of global Hölder norms that decay as $|v| \rightarrow \infty$, we introduce the following definition.

$$[f]_{C_{\ell,q}^\alpha([\tau,T] \times \mathbb{R}^d \times \mathbb{R}^d)} := \sup \left\{ (1 + |v|)^q [f]_{\mathcal{C}_\ell^\alpha(Q_r(z))} : \right. \\ \left. r \in (0, 1] \text{ and } Q_r(z) \subset [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d \right\}.$$

The seminorm $[\cdot]_{C_{\ell,q}^\alpha}$ encodes a decay of the form $(1 + |v|)^{-q}$ for the $[\cdot]_{\mathcal{C}_\ell^\alpha}$ norm as $|v| \rightarrow \infty$.

The upper bound of Theorem 8.1 gives us literally an a priori estimate for $[f]_{C_{\ell,q}^0([\tau,T] \times \mathbb{R}^d \times \mathbb{R}^d)}$ for every value of $q > 0$. Next, we apply the Hölder estimates of Theorem 6.1 combined with the change of variables of Section 9. We obtain the following conclusion (see [61, Proposition 7.1]).

Proposition 10.1. *Let $f: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ be a solution to (1.1) so that (H) holds. Then, for $q > d + \gamma + 2s$, we have,*

$$[f]_{C_{\ell,q-\gamma+}^\alpha} \leq C [f]_{C_{\ell,q}^0([\tau,T] \times \mathbb{R}^d \times \mathbb{R}^d)}$$

for some $\alpha > 0$ and C that depend on the parameters of (H), d , s and γ only.

Proposition 10.1 improves the upper bounds from Theorem 8.1 into a global Hölder estimate.

Since the kernel K_f depends on the function f through the formula (2.4), a regularity estimate for f translates into certain kind of regularity for K_f . The Hölder estimate of Proposition 10.1 implies that Assumption 7.1 holds (at least locally) for the kernel K_f .

The next step is to improve the smoothness of f by applying the Schauder estimate, first to f itself, and then to its increments and derivatives. The following proposition is the result of combining Theorem 7.1 with the change of variables of Section 9. We state it in terms of a generic function g because we will apply it to $g = f$ and also to g equal to several combinations of derivatives and increments of f . The next proposition is taken from [61, Proposition 7.5].

Proposition 10.2. *Let $f: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ be a solution to (1.1) so that (H) holds. Let $g, h: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ so that the following equation holds*

$$\partial_t g + v \cdot \nabla_x g - \mathcal{L}_{K_f} g = h.$$

Let $\tau > 0$, $\alpha \in (0, \min(1, 2s))$, and $\alpha' = 2s/(1 + 2s)\alpha$. The following inequality holds

$$\begin{aligned} [g]_{C_{\ell, q}^{2s+\alpha'}([\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d)} \\ \leq C \left(\|g\|_{C_{\ell, q+2s+\alpha}^\alpha((0, T] \times \mathbb{R}^d \times \mathbb{R}^d)} + \|h\|_{C_{\ell, q+2s+\alpha}^{\alpha'}((0, T] \times \mathbb{R}^d \times \mathbb{R}^d)} \right), \end{aligned}$$

where the constant C depends on the parameters in (H), d , s , γ and q .

Thus, we can summarize the first three steps of the proof of Theorem 1.1 as the following.

- (1) Apply Theorem 8.1 to obtain an a priori estimate for $\|f\|_{C_{\ell, q}^0([\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d)}$, for any values of $\tau > 0$ (arbitrarily small) and $q > 0$ (arbitrarily large).
- (2) Apply Proposition 10.1 to obtain an a priori estimate for $\|f\|_{C_{\ell, q}^\alpha([\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d)}$, for some small $\alpha > 0$ and for any values of $\tau > 0$ (arbitrarily small) and $q > 0$ (arbitrarily large).
- (3) Apply Proposition 10.2, with $g = f$ and $h = (f *_{\nu} |\cdot|^\gamma) f$, to obtain an a priori estimate for $\|f\|_{C_{\ell, q}^{2s+\alpha'}([\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d)}$, for an even smaller $\alpha' > 0$, and for any values of $\tau > 0$ (arbitrarily small) and $q > 0$ (arbitrarily large).

The next steps in order to complete the proof of Theorem 1.1 is to apply Proposition 10.2 with g equal to incremental quotients and derivatives of the solution f . To that end, we have to compute the equation satisfied by each of these functions. In each case there are a number of error terms that are absorbed into the right hand side h . The precise computations are carried out in [61], together with several technical estimates relating the Hölder norms $C_{\ell, q}^\alpha$ with the Boltzmann collision operator \mathcal{Q} , derivatives and incremental quotients.

11. Possible generalizations and implications

In this section, we review some of the natural implications of Theorem 1.1 and discuss their possible generalizations.

11.1. A continuation criteria. A continuation criteria can easily be derived by combining Theorem 1.1 with a compatible short time existence result. Indeed, the short-time existence result says that a smooth solution exists for some period of time depending on some regularity norm of the initial data. Theorem 1.1 says that for as long as (H) holds, then no regularity norm of the solution will blow up. Consequently, the smooth solution can be extended indefinitely.

The existence of smooth solutions for a short period of time for the non-cutoff Boltzmann equation was first established in [6] for initial data with five derivatives in L^2_{loc} and Gaussian decay (in the L^2 sense). This result is not compatible with our Theorem 1.1 because we cannot ensure the persistence of the Gaussian decay. Our upper bounds in Theorem 8.1 ensure the persistence of arbitrarily large algebraic decay rate, but not precisely the Gaussian decay.

The first short time existence result that requires an algebraic decay rate for the initial data was given in [84] for $s \in (0, 1/2)$ and $\gamma \in (-3/2, 0]$. More recently, the result was extended to the full range of parameters in [55]. These results are compatible with Theorem 1.1. Thus, we effectively get the following continuation condition.

Corollary 11.1. *Assume that $f: (0, T) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ is a smooth function, periodic in x , so that $(1 + |v|)^q f(t, x, v)$ is bounded for all $q > 0$, and f satisfies the Boltzmann equation (1.1). If such a solution f cannot be continued to a larger time interval, then one of the following events must occur:*

$$\begin{aligned} \lim_{t \rightarrow T} \inf_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} f(t, x, v) dv &= 0, \\ \lim_{t \rightarrow T} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} f(t, x, v) dv &= +\infty, \\ \lim_{t \rightarrow T} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |v|^2 f(t, x, v) dv &= +\infty, \\ \lim_{t \rightarrow T} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} f \log f(t, x, v) dv &= +\infty. \end{aligned}$$

Our Theorem 1.1 gives us the extra piece of information that the estimate (1.8) holds uniformly for all times (away from zero). Thus, any solution f for which (H) holds is uniformly smooth as $t \rightarrow \infty$.

In a very recent preprint [56], Henderson, Snelson and Tarfulea relax the continuation criteria of Corollary 11.1 to only an upper bound for the mass or energy densities. They assume that the initial value f_0 is bounded below in a solid ball somewhere (which is always true if f_0 is continuous and not identically zero). By a barrier argument, they propagate this lower bound to all positive times and all other points in space. In this way, they prove that a lower bound for the mass density holds for all time, even though it might degenerate as $t \rightarrow \infty$. They also use this lower barrier to establish the cone of

nondegeneracy for the kernel K_f as in (2.6). They observe that the only place where we crucially use the upper bound for the entropy in our program was in establishing this cone of nondegeneracy. In this way, they are able to improve Corollary 11.1 and deduce that if a singularity ever occurs at time T for the Boltzmann equation without cutoff, then one of the following two events must occur:

$$\lim_{t \rightarrow T} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} f(t, x, v) dv = +\infty,$$

$$\lim_{t \rightarrow T} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |v|^2 f(t, x, v) dv = +\infty.$$

Note that using the methods in [56], we cannot obtain global regularity estimates that are uniform for all time, depending only on an upper bound for the mass and energy estimates. Their estimates hold in any interval of time $(t, x, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, but they degenerate as $T \rightarrow \infty$.

11.2. Convergence to equilibrium. A very well known result by L. Desvillettes and C. Villani [37] says that the solution f to the Boltzmann equation (1.1) converges to a Maxwellian provided that the following two conditions hold.

- (1) The function f is C^∞ with uniform bounds as $t \rightarrow \infty$.
- (2) The function f is bounded below by a fixed Maxwellian.

Our Theorem 1.1 says that the first condition of the Theorem of Desvillettes and Villani holds as soon as (H) holds. The Maxwellian lower bound is obtained also as a consequence of (H) in our joint work with Clément Mouhot [59]. Thus, combining all these results, (H) becomes the only condition necessary to deduce the convergence to equilibrium of the solution f as $t \rightarrow \infty$.

11.3. Weak solutions. Our Theorem 1.1 is stated as an a priori estimate for smooth classical solutions. It is natural to wonder if the same regularity estimates would hold for weak solutions as well. But what is a *weak solution* exactly?

The term *weak solution* typically means that the function f is not necessarily smooth, and the equation is to be understood in the sense of distributions. This notion of solution, in the sense of distributions, works very well for many linear equations (like the Laplace, heat or wave equations). However, it fails miserably for most nonlinear equations. For example, it is well known that solutions in the sense of distributions have very undesirable properties for nonlinear conservation law equations, including non-uniqueness. *Entropy solutions* is the right notion of solution for scalar conservation laws. Distributional solutions are also completely unsuitable for the study of the Hamilton-Jacobi equations, or fully nonlinear parabolic equations. In these latter cases, the right notion of solution is in the *viscosity sense*.

It is currently very unclear what kind of generalized solution we should study for the Boltzmann equation (1.1). There is no notion of weak solution for which we can prove both existence and uniqueness in fair generality.

The only global existence result, far from equilibrium, known so far for the non-cutoff Boltzmann equation (1.1) is that of Alexandre and Villani [11]. They prove the existence

of a renormalized solution with defect measure. The uniqueness of solutions of this kind is not known, and arguably not expected.

It is still an open problem to determine whether the result of Theorem 1.1 holds for the class of solutions defined by Alexandre and Villani. What we mean is that if f is a (not necessarily smooth) renormalized solution with defect measure so that (H) holds almost everywhere, then one might expect f to be necessarily smooth and satisfy the estimates in Theorem 1.1.

Using entropy dissipation estimates from [3], we see that for any renormalized solution f of (1.1) for which the assumption (H) holds almost everywhere, we have $\sqrt{f} \in L^2_{t,x} H^s_{v,\text{loc}}$. The main difficulty for moving forward with our regularity program is in establishing the upper bounds of Theorem 8.1. In its proof, we evaluate the equation at a *single point*, where the function f would first cross certain upper barriers.

Except from the upper bounds of Theorem 8.1, the other steps of the proof are relatively easy to adapt to weak solutions. We commented in Sections 6 and 7 how the Hölder and Schauder estimates would be extended to weak solutions.

It is debatable whether it is interesting to explore the applicability of Theorem 1.1 to an intermediate kind of solution. That is, to a notion of solution stronger than that of Alexandre and Villani, but weaker than classical smooth solutions. In terms of our proofs, there seems to be no major obstruction to extend Theorem 1.1 to hold for solutions in the sense of distributions that are bounded with a decay of the form $f \leq C(1 + |v|)^{q_0}$ for a q_0 large enough as described in [60, Section 5]. Naturally, the regularity estimates would not depend on this constant C .

11.4. Mild solutions. Even though we work with a very strong notion of solution, Theorem 1.1 can be applied for any class of solution where we can establish *stability*. We mean any class of solutions where we can approximate the initial data with a smooth function and pass to the limit. A great example is the global mild solutions to the Boltzmann equation (1.1) without cutoff constructed by Duan, Liu, Sakamoto and Strain in [41].

These solutions stay close to a Maxwellian in a suitable norm that does not impose any regularity with respect to the v variable. In this perturbative regime, they establish the global well posedness of the problem. A priori, these solutions could be very rough, but it can be verified by a direct computation that they satisfy our assumption (H). We explain here how to apply Theorem 1.1 to conclude that they are actually C^∞ .

If the initial data f_0 is smooth and rapidly decaying as $|v| \rightarrow \infty$, in addition to satisfying the hypothesis in [41], the continuation criteria tells us that it can only blow up when one of the conditions in Corollary 11.1 holds. The smooth solution with initial data f_0 must coincide with the solution constructed in [41] for as long as this smooth solution exists, since uniqueness holds in both regimes. But the solution constructed in [41] satisfies (H) globally, so it can never blow up. Thus, it will be a global smooth solution and the estimates in Theorem 1.1 will hold.

If the initial data f_0 satisfies the hypothesis of [41] but is not smooth, we can approximate it with a sequence f_0^ε of smooth ones, and rapidly decaying. For each f_0^ε , the solution f^ε is C^∞ and the result of our Theorem 1.1 holds. Our estimates for f do not depend quantitatively on the smoothness of f_0^ε . Thus, we get uniform regularity estimates

that pass to the limit and apply for any (non-smooth) initial data f_0 . We can legitimately take the limit as $\varepsilon \rightarrow 0$ thanks to the well posedness result in [41].

We should point out that when $\gamma \leq 0$, the assumptions in [41] do not imply our decay condition (1.9). Thus, for $\gamma \leq 0$, we would only apply the result of Theorem 1.1 to the solutions in [41] if the initial data satisfies (1.9) in addition to their assumptions.

Remark 11.1. It is conceivable that under the strong moment estimates from [41], pointwise upper bounds should follow as in Theorem 8.1 without any additional hypothesis on f_0 even for $\gamma \leq 0$. However, it does not follow directly from any result currently in the literature.

12. Open problems

Some open questions were already discussed in the previous section, including the possible extension of the estimates in Theorem 1.1 to renormalized solutions with defect measure. In this last section, we propose several other open problems related to the material in this survey.

12.1. Control on the hydrodynamic quantities. The most obvious open problem after Theorem 1.1 is whether the hypothesis in (H) can be ensured in any way. This would imply the unconditional global solvability of the Boltzmann equation. It is a remarkable open problem. In the first chapter of [99], Cédric Villani recounts a lively discussion with Clément Mouhot from several years ago about this issue. Based on the discussion included in the introduction of this survey, we believe that the global existence of smooth solutions to the non-cutoff Boltzmann equation is an open problem that, if true, would be harder to prove than the global solvability of the Navier–Stokes equations (also if true). The latter is one of the famous Millennium problems.

A more plausible project in the near term would be to remove or weaken some of the conditions in (H). The well known Prodi–Serrin condition for the Navier–Stokes equation suggests that maybe only an upper bound in some suitable L^p space would suffice for the mass, energy and/or entropy densities. Perhaps only a subset of the inequalities from (H) suffice to obtain Theorem 1.1. There are several possibilities. We do not have any precise conjecture in this direction.

12.2. Very soft potentials. The non-cutoff collision kernel (1.4) makes sense for $s \in (0, 1)$ and $\gamma > -d$. Yet, we only present our results in the range $\gamma + 2s \in [0, 2]$. The case $\gamma + 2s > 2$ is covered in [27]. The case $\gamma + 2s < 0$ remains open and would require new ideas.

The case $\gamma + 2s < 0$ is commonly referred to as *very soft potentials*. In particular, $\gamma = -3$ and $s \rightarrow 1$ corresponds to the Landau–Coulomb equation in three dimensions which is most relevant for the study of plasma dynamics. Our proof of the upper bounds in Theorem 8.1 fails in this range.

When $\gamma + 2s < 0$, regularity estimates fail even for the space-homogeneous Boltzmann equation. We do not currently know any global L^∞ estimate for the solution f of the

non-cutoff Boltzmann equation when $\gamma + 2s < 0$, even in the space-homogeneous case and for initial data f_0 in the Schwartz class.

The main difficulty for the very soft potential range is to control the lower order term in (2.8), that is

$$f(v) \left(\int_{\mathbb{R}^d} f(v-w) |w|^\gamma dw \right),$$

with the diffusion term $\mathcal{L}_{K_f} f$. Naturally, the more negative γ is, the more singular the lower order term in (1.1) becomes. The cone of nondegeneracy described in Section 2.4, together with the upper bounds on mass, energy and entropy, allows us to control this lower order term with $\mathcal{L}_{K_f} f$ only when $\gamma + 2s \geq 0$. It seems that in order to succeed in proving an L^∞ estimate for $\gamma + 2s < 0$, we would need some further understanding on $\mathcal{L}_{K_f} f$ sharper than the information we get from its cone of nondegeneracy.

The problem of L^∞ bounds for very soft potentials is also open in the context of the space homogenous Landau equation. See for example [45, 49, 96].

12.3. Bounded domains.. We stated Theorem 1.1 for periodic boundary conditions in the space variable x . It would be straight forward to extend the result to other variants of space domains without boundary. For example, we may assume that as $|x| \rightarrow \infty$ the solution $f(t, x, v)$ converges uniformly to a fixed Maxwellian $M(v)$. Our proof would work *mutatis mutandis* under this alternative formulation.

The case of domains with boundary is of course of physical relevance. Several types of boundary conditions are considered in the literature of kinetic equations: diffuse reflection, specular reflection and bounce back reflection. An extension of Theorem 1.1 for solutions f in a bounded domain in space, with any of these boundary conditions, requires further work. There are several subtleties involved on the effects of the boundary on the regularity of f . See for example [50–52, 69, 70], for analysis of boundary effects on solutions of the Boltzmann equation. No analysis has been made yet concerning the possibility to extend some form of Theorem 1.1 to any domain with boundary, for any of the physical boundary conditions.

Another possibility would be to extend the result of Theorem 1.1 as an interior regularity condition. That is, if a function $f: [0, T] \times \Omega \times \mathbb{R}^d \rightarrow [0, \infty)$ solves the non-cutoff Boltzmann equation (1.1) and satisfies (H), we would expect the same estimates of Theorem 1.1 to hold in any subdomain of the form $[\tau, T] \times K \times \mathbb{R}^d$ for any $\tau > 0$ and K compactly contained in Ω . This is an open problem in a bounded domain that is independent of the subtleties involved in the analysis of physical boundary conditions.

12.4. Hölder estimates for kinetic equations with diffusion in non-divergence form.

Our theorem 6.1 is a kinetic non-local version of the classical result of De Giorgi, Nash and Moser. There is no regularity assumption on the kernel K . The cancellation condition (4.6) is a nonlocal form of the divergence structure of elliptic operators. A natural question is: can we replace the cancellation condition (4.6) by the *non-divergence* symmetry condition (4.5)?

The Boltzmann kernel K_f naturally satisfies the symmetry condition (4.5). It also satisfies the cancellation condition (4.6), but it takes more trouble to verify it. The reason why we presented Theorem 6.1 under the cancellation condition (4.6) instead of the

symmetry condition (4.5) is simply because that is what we are able to prove. It does not mean that the alternative is false. In fact, we believe it is probably true as well.

The reason why we succeed to prove Theorem 6.1 under the cancellation condition (4.6) but not under the symmetry condition (4.5) has its roots in the study of kinetic equations with second order diffusion.

Let us recall the setting of the classical Hölder estimates for parabolic equations with rough coefficients. The theorem of De Giorgi, Nash and Moser gives us estimates in Hölder spaces for solutions of an equation of the form

$$u_t - \partial_{x_i}(a_{ij}(t, x)\partial_{x_j}u) = 0,$$

only under the uniform ellipticity assumption $\lambda I \leq \{a_{ij}\} \leq \Lambda I$. There is also an analogous result by Krylov and Safonov for parabolic equations in non-divergence form

$$u_t - a_{ij}(t, x)\partial_{x_i x_j}u = 0.$$

The techniques involved in the proofs of the theorem of De Giorgi, Nash and Moser are very different from those involved in the proof of the theorem by Krylov and Safonov. A kinetic version of these two types of equations would be

$$\begin{cases} \text{(divergence form)} & \partial_t f + v \cdot \nabla_x f - \partial_{v_i}(a_{ij}(t, x, v)\partial_{v_j}f) = 0, \\ \text{(non-divergence form)} & \partial_t f + v \cdot \nabla_x f - a_{ij}(t, x, v)\partial_{v_i v_j}f = 0. \end{cases}$$

In either case, we should assume the uniform ellipticity condition:

$$\lambda I \leq \{a_{ij}(t, x, v)\} \leq \Lambda I.$$

No further smoothness assumptions should be made on the coefficients. One would expect that if the equation holds in a kinetic cylinder \mathcal{Q}_1 , then the following estimate holds

$$\|f\|_{C_\ell^\alpha(\mathcal{Q}_{1/2})} \leq C \|f\|_{C_\ell^0(\mathcal{Q}_1)}.$$

Such an estimate is known to be true in the divergence case (see [46, 100, 101]). However, it is still an open problem for the non-divergence case.

12.5. Coercivity estimates for integro-differential operators. In Proposition 4.2, we describe a coercivity estimate for the Boltzmann collision operator. This coercivity condition is well known and has a long history in the Boltzmann literature. The upper bound on the entropy in (H) can be replaced by a lower bound on the temperature tensor, or a more general condition described in [48]. However, these results do not follow from the general condition in [30] and rely on the specific structure of the Boltzmann kernel K_f described in (2.4).

The result in [30] is a coercivity estimate for general integro-differential operators of the form \mathcal{L}_K as in (4.2), not necessarily related to the Boltzmann equation. The question is to determine simple sufficient conditions on a kernel $K(v, v')$ so that the following inequality holds for some constant $c > 0$.

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (f(v) - f(v'))^2 K(v, v') dv' dv \geq c \|f\|_{\dot{H}^s}^2. \quad (12.1)$$

It is obvious that (12.1) holds when $K(v, v') \gtrsim |v' - v|^{-d-2s}$. Proposition 4.2 says that (12.1) follows as a consequence of (4.4). The assumptions in [30] are more general, but there are still some simple cases where (12.1) holds even though the assumptions in [30] do not apply. The simplest example, in two dimensions, is to consider

$$\mathcal{L}f = -((-\partial_{11})^s + (-\partial_{22})^s)f.$$

This operator corresponds to a kernel K consisting of a singular measure concentrated on $v_1 - v'_1 = 0$ and $v_2 - v'_2 = 0$. It is easy to verify that in this case

$$-\int_{\mathbb{R}^d} (\mathcal{L}f)f \, dv \gtrsim \|f\|_{\dot{H}^s}^2,$$

however, the kernel in \mathcal{L} is a singular measure. It is equal to zero almost everywhere.

A general condition that has been suggested is the following: there exists $\lambda > 0$ so that for every $v \in \mathbb{R}^d$, $r > 0$ and $e \in S^{d-1}$ we have

$$\int_{B_r(v)} [(v' - v) \cdot e]_+^2 K(v, v') \, dv' \geq \lambda r^{2-2s}.$$

This condition is satisfied by every kernel that is currently known to satisfy (12.1). Whether it actually implies (12.1) is still an open problem. See [42] and [22] for more references concerning this question.

12.6. Regularity estimates for moderately soft potentials whose initial data does not decay. In the case of moderately soft potentials ($\gamma \in [-2s, 0]$), our estimates in Theorem 1.1 depend on the decay of the initial data f_0 through the values of N_r . We require (1.9) to hold for every $r \geq 0$.

We do not expect to have any gain of moments in the case of soft potentials. Without the assumption (1.9), we only expect a moderate decay for large velocities at positive time, as described in the third case of Theorem 8.1.

It is natural to wonder whether C^∞ regularity estimates may hold in the case $\gamma \leq 0$, for solutions that do not enjoy a fast decay for large velocities. The iteration leading to our proof of Theorem 1.1 does not suffice to prove it. In each step we gain some differentiation at the expense of some decay power. If we do not start by a fast decay as in (1.9), our iteration would stop after finitely many steps.

12.7. Conditional propagation of regularity for the cutoff Boltzmann equation. With the cutoff assumption on the collision kernel B , there is no regularization effect in the Boltzmann equation. However, it is still plausible to expect propagation of regularity. If the initial data f_0 is smooth and rapidly decaying as $|v| \rightarrow \infty$, it is conceivable that the solution of the Boltzmann equation (1.1) *with cutoff* stays smooth for as long as (H) holds, at least in the case of hard potentials.

This problem is of a very different nature compared with the regularization estimates studied in these notes. We have not done any work on the cutoff case so far.

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